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BAIRE SPACES AND VIETORIS HYPERSPACES

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Abstract. We prove that if the Vietoris hyperspace $CL(X)$ of all nonempty closed subsets of a space X is Baire, then all finite powers of X must be Baire spaces. In particular, there exists a metrizable Baire space X whose Vietoris hyperspace $CL(X)$ is not Baire. This settles an open problem of R. A. McCoy stated in 1975.

1. Introduction

In this paper, all topological spaces are assumed to be infinite and at least Hausdorff. Also, all product spaces are endowed with the Tychonoff product topology. A topological space X is called *Baire* [HM] if the intersection of any sequence of dense open subsets of X is dense in X, or equivalently, if all nonempty open subsets of X are of second category. Baire spaces have numerous applications in analysis and topology, such as the open mapping and closed graph theorems, and the Banach-Steinhaus theorem [Con]. For some other applications, see [E, Za1] or [Za2]. A fundamental treatise on Baire spaces in general topology is [HM], and several open problems on Baire and related spaces are discussed in [AL].

For a space X, let $CL(X)$ be the collection of all nonempty closed subsets of X endowed with the *Vietoris topology* τ_V. Recall that a canonical base for τ_V is given by all subsets of $CL(X)$ of the form

$$\langle \mathcal{V} \rangle = \left\{ F \in CL(X) : F \subset \bigcup \mathcal{V},\ F \cap V \neq \varnothing \text{ for any } V \in \mathcal{V} \right\},$$

where $\mathcal{V}$ runs over the finite families of nonempty open subsets of X. In the sequel, any subset $\mathcal{D} \subset CL(X)$ will carry the relative Vietoris topology τ_V as a subspace of $(CL(X), \tau_V)$. McCoy [Mc] has studied Baire Vietoris hyperspaces and obtained various sufficient and necessary conditions: for example, if X is a second countable, regular Baire space, then $CL(X)$ is Baire, too; on the other side, if $CL(X)$ is a Baire space, then so is X. Moreover, if $K(X)$ (the hyperspace of all nonempty compact subsets of X) is a Baire space, then X^2 is also Baire and hence, if X is

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a Baire space with a non-Baire square, then $K(X)$ is not Baire. In this context, McCoy posed the following natural problem (see page 140 of [Mc]).

Problem 1.1. Let X be a metrizable Baire space such that X^2 is not Baire. Must $CL(X)$ be Baire?

For the proper understanding of the above problem, let us explicitly mention that there exists a Baire space X whose square X^2 is not Baire. The first space with such properties, constructed under the Continuum Hypothesis, is due to Oxtoby [O]. Then, the example was improved to an absolute one by Cohen [Coh] relying on forcing. Finally, Fleissner and Kunen [FK] constructed a metrizable Baire space X whose square X^2 is not Baire in **ZFC** by direct combinatorial arguments.

There are several recent results concerning hyperspaces of Baire spaces (see, for instance, [BHZ, CG], or [Zs]), but we were unable to find any reference about possible progress in the solution of Problem 1.1. The main purpose of the present paper is to provide the negative solution of this problem by proving the following theorem.

Theorem 1.2. *If $CL(X)$ is a Baire space, then all finite powers of X must be Baire spaces.*

By Theorem 1.2 and the previous remarks, we have the following consequence.

Corollary 1.3. *There exists a metrizable Baire space X such that $CL(X)$ is not Baire.*

Another possible application of Theorem 1.2 regards Volterra spaces. Recall that a space X is *Volterra* if the intersection of any two dense G_δ-subsets of X is dense in X [GP]. Clearly, any Baire space is Volterra. In fact, a space X which contains a dense metrizable subspace is Baire if and only if it is Volterra [GL].

Corollary 1.4. *If X is a metrizable space such that $CL(X)$ is Volterra, then all finite powers of X must be Baire.*

Proof. Note that $K(X)$ forms a metrizable subspace of $CL(X)$ ([Mi]), which is also dense in $CL(X)$, so the above-mentioned result of [GL] and Theorem 1.2 apply. □

The proof of Theorem 1.2 along with some auxiliary material shall be given in the next section. Also, Theorem 2.1 may be of some independent interest.

2. Proof of Theorem 1.2

For a space X and $n \geq 1$, let $\mathcal{F}_n(X) = \{S \in CL(X) : |S| \leq n\}$. Note that $\mathcal{F}_n(X)$ is always a closed subset of $CL(X)$ and $\mathcal{F}_1(X)$ is naturally homeomorphic to X (which means that the Vietoris topology is *admissible* [Mi]). We may look at $\mathcal{F}_n(X)$ as an "unordered" version of X^n, so the following first step towards proving Theorem 1.2 is not surprising.

Theorem 2.1. *For each $n \geq 1$, $\mathcal{F}_n(X)$ is a Baire space if and only if X^n is a Baire space.*

The proof of this theorem is based on the following observation (see [HM] and [N] for generalizations).

Proposition 2.2. *Let $f : X \to Y$ be a finite-to-one open continuous surjection. Then X is a Baire space if and only if Y is a Baire space.*

Proof. If X is a Baire space, then so is Y because f is an open and continuous surjection. Conversely, suppose that Y is a Baire space and $G = \bigcap\{V_k : k < \omega\}$ for some decreasing sequence of open dense subsets $V_k \subset X$, $k < \omega$. Also, let $W \subset X$ be a nonempty open set. Then, each $U_k = f(V_k \cap W)$, $k < \omega$, is open and dense in $H = f(W)$ because $g = f \restriction W : W \to H$ is open and continuous. Hence, $D = \bigcap\{U_k : k < \omega\}$ is dense in H, because H is open in Y, so there exists some $y \in D$. Now, let us observe that $\{g^{-1}(y) \cap V_k : k < \omega\}$ is a decreasing sequence of nonempty finite subsets of X, since g is finite-to-one, and therefore it has a nonempty intersection. This implies that $W \cap G \neq \varnothing$. □

Proof of Theorem 2.1. The map $f : X^n \to \mathcal{F}_n(X)$, defined as

$$f((x_1, ..., x_n)) = \{x_1, ..., x_n\},$$

is clearly finite-to-one. Further, f is open, since if $U_1, ..., U_n$ are open subsets of X, then $f(U_1 \times \cdots \times U_n) = \langle\{U_1, ..., U_n\}\rangle$. To show that f is continuous, take an $(x_1, ..., x_n) \in X^n$ and a finite collection σ of nonempty open subsets of X such that $f((x_1, ..., x_n)) \in \langle\sigma\rangle$. Next, define $U_i = \bigcap\{V \in \sigma : x_i \in V\}$ for each $1 \leq i \leq n$. Then $(x_1, ..., x_n) \in U_1 \times \cdots \times U_n$ and $f(U_1 \times \cdots \times U_n) = \langle\{U_1, ..., U_n\}\rangle \subset \langle\sigma\rangle$. The theorem now follows from Proposition 2.2. □

If σ and γ are families of nonempty subsets of X, then σ is called a *refinement* of γ if any element of σ is a subset of some element of γ; furthermore, σ is a *strong refinement* of γ if in addition to being a refinement of γ, each element of γ contains some element of σ. Observe that if σ and γ are finite families of nonempty open subsets of X and σ is a strong refinement of γ, then $\langle\sigma\rangle \subset \langle\gamma\rangle$. Motivated by this, to any finite family γ of nonempty open subsets of X we will associate the set $\mathcal{SR}(\gamma)$ of all strong refinements σ of γ such that σ consists of nonempty open subsets of X and $|\sigma| = |\gamma|$. Note that if γ is pairwise disjoint, then any $\sigma \in \mathcal{SR}(\gamma)$ is also pairwise disjoint, while $\langle\gamma\rangle \neq \varnothing$ implies $\varnothing \neq \langle\sigma\rangle \subset \langle\gamma\rangle$ for every $\sigma \in \mathcal{SR}(\gamma)$. Finally, if $n \geq 1$, the symbol $[S]^n$ will stand for the collection of all n-element subsets of the set S. The next lemma will be our main tool in working with strong refinements.

Lemma 2.3. *Let γ be a finite family of pairwise disjoint nonempty open subsets of X with $|\gamma| \geq n \geq 1$, and let $\mathcal{V} \in \tau_V$ be dense in $[X]^n$. Then there exists a $\sigma \in \mathcal{SR}(\gamma)$ such that $\langle\tau\rangle \subset \mathcal{V}$ for every $\tau \in [\sigma]^n$.*

Proof. Let $[\gamma]^n = \{\gamma_1, ..., \gamma_m\}$ for some $m < \omega$. Also, for convenience, let $\gamma_0 = \gamma$. For each $0 \leq k \leq m$ and each $V \in \gamma$, define by induction a nonempty open subset $V_k \subset V$ such that

(i) $V_k \subset V_\ell$, whenever $0 \leq \ell \leq k$;
(ii) $\langle\{V_k : V \in \gamma_k\}\rangle \subset \mathcal{V}$, if $k \geq 1$.

To see how this can be done, if $k = 0$, then we merely let $V_0 = V$ for each $V \in \gamma_0$. Suppose that the construction has been done up to some $0 \leq k < m$. Then $|\gamma_{k+1}| = n$, while $\mathcal{V}$ is open (in $CL(X)$) and dense in $[X]^n$, so for each $V \in \gamma_{k+1}$, there exists a nonempty open subset $V_{k+1} \subset V_k$ such that $\langle\{V_{k+1} : V \in \gamma_{k+1}\}\rangle \subset \mathcal{V}$. To complete the construction, let $V_{k+1} = V_k$ for every $V \in \gamma \setminus \gamma_{k+1}$. Clearly, (i) and (ii) hold for this particular $k + 1$, which completes the induction.

Finally, let us show that $\sigma = \{V_m : V \in \gamma\}$ is as required. Suppose that $\tau \in [\sigma]^n$. According to (i), σ is a strong refinement of γ, hence we can define $\gamma_\tau \in [\gamma]^n$ by letting $\gamma_\tau = \{V \in \gamma : V_m \in \tau\}$. Now, on the one hand, $\gamma_\tau = \gamma_k$ for some $k \leq m$, with $k \geq 1$, so, by (ii), $\langle\{V_k : V \in \gamma_\tau\}\rangle \subset \mathcal{V}$. On the other hand, by (i), τ is a strong refinement of $\{V_k : V \in \gamma_\tau\}$, therefore $\langle\tau\rangle \subset \langle\{V_k : V \in \gamma_\tau\}\rangle \subset \mathcal{V}$. This completes the proof. □

Proof of Theorem 1.2. According to Theorem 2.1, it suffices to show that $\mathcal{F}_n(X)$ is a Baire space for each $n \geq 1$. Take a countable family $\{\mathcal{V}_k : k < \omega\}$ of τ_V-open subsets of $CL(X)$ which are dense in $\mathcal{F}_n(X)$, and let us show that

$$\mathcal{G} = \bigcap\{\mathcal{V}_k : k < \omega\}$$

is dense in $\mathcal{F}_n(X)$. To this end, let λ be a finite family of nonempty open subsets of X such that $\langle\lambda\rangle \cap \mathcal{F}_n(X) \neq \varnothing$. In case that $\bigcup\lambda$ is finite, then $\langle\lambda\rangle$ consists only of isolated points of $\mathcal{F}(X)$, hence there exists an $S \in \langle\lambda\rangle \cap \mathcal{F}_n(X)$ which is an isolated point of $\mathcal{F}(X)$. Clearly, in this case $S \in \mathcal{G}$. If $\bigcup\lambda$ is infinite, then $\langle\lambda\rangle \cap [X]^n \neq \varnothing$, and there is a finite family μ consisting of pairwise disjoint nonempty open subsets of X such that $|\mu| = n$ and $\langle\mu\rangle \subset \langle\lambda\rangle$. For every $k < \omega$, consider the collection Σ_k of all finite families σ consisting of pairwise disjoint nonempty open subsets of X such that

(2.1) $\qquad \sigma$ is a strong refiniment of μ and $\langle\tau\rangle \subset \mathcal{V}_k$ for every $\tau \in [\sigma]^n$.

Next, we consider the τ_V-open sets

$$\mathcal{U}_k = \bigcup\{\langle\sigma\rangle : \sigma \in \Sigma_k\}, \quad k < \omega,$$

in $CL(X)$, and we are going to show that they are dense in $\langle\mu\rangle$. Take $k < \omega$ and a finite family ν of open subsets of X such that $\langle\nu\rangle \cap \langle\mu\rangle \neq \varnothing$. Since ν is finite and $|\mu| = n$, there now exists a finite family γ consisting of pairwise disjoint nonempty open subsets of X such that $|\gamma| \geq n$, and $\langle\gamma\rangle \subset \langle\nu\rangle \cap \langle\mu\rangle$. Since $\mathcal{V}_k$ is dense in $[X]^n$, Lemma 2.3 implies the existence of a $\sigma \in \mathcal{SR}(\gamma)$ such that $\langle\tau\rangle \subset \mathcal{V}_k$ for every $\tau \in [\sigma]^n$; thus, $\sigma \in \Sigma_k$ and $\varnothing \neq \langle\sigma\rangle \subset \langle\gamma\rangle \cap \mathcal{U}_k$, so $\mathcal{U}_k$ is dense in $\langle\mu\rangle$. As a result, we get that $\mathcal{D} = \bigcap\{\mathcal{U}_k : k < \omega\}$ is a τ_V-dense subset of $\langle\mu\rangle$ because $\langle\mu\rangle$ is itself a Baire space, being a τ_V-open subset of $CL(X)$, so there exists an $F \in \langle\mu\rangle \cap \mathcal{D}$. For every $W \in \mu$, fix a point $x_W \in F \cap W$ and define $T = \{x_W : W \in \mu\}$. Note that $|T| = |\mu| = n$, and, in particular, $T \in \mathcal{F}_n(X)$. Now, on the one hand, for every $k < \omega$ we can find a $\sigma_k \in \Sigma_k$ with $F \in \langle\sigma_k\rangle$; on the other hand, we can define a special subfamily of σ_k by letting $\tau_k = \{S \in \sigma_k : S \cap T \neq \varnothing\}$. Then $|\tau_k| = |T| = n$ because σ_k is a pairwise disjoint strong refinement of μ, while $|T \cap W| = 1$ for every $W \in \mu$. According to (2.1), this implies that $T \in \langle\tau_k\rangle \subset \mathcal{V}_k$ for every $k < \omega$, so $T \in \langle\mu\rangle \cap \mathcal{G} \subset \langle\lambda\rangle \cap \mathcal{G}$, which completes the proof. □

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