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Connecting the Two Worlds: Well-partial-orders and Ordinal Notation Systems

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Das Unendliche hat wie keine andere Frage von jeher so tief das Gemüt des Menschen bewegt; das Unendliche hat wie kaum eine andere Idee auf den Verstand so anregend und fruchtbar gewirkt; das Unendliche ist aber auch wie kein anderer Begriff so der Aufklärung bedürftig.

From time immemorial, the infinite has stirred men's emotions more than any other question. Hardly any other idea has stimulated the mind so fruitfully. Yet, no other concept needs clarification more than it does.

- David Hilbert, *Über das Unendliche* (*On the infinite*) [39]

Preface

Kruskal claims in his now-classical 1972 paper [47] that well-partial-orders are among the most frequently rediscovered mathematical objects. Well-partial-orders have applications in many fields outside the theory of orders: computer science, proof theory, reverse mathematics, algebra, combinatorics, etc.

The maximal order type of a well-partial-order characterizes that order's strength. Moreover, in many natural cases, a well-partial-order's maximal order type can be represented by an ordinal notation system. However, there are a number of natural well-partial-orders whose maximal order types and corresponding ordinal notation systems remain unknown. Prominent examples are Friedman's well-partial-orders of trees with the gap-embeddability relation [76].

The main goal of this dissertation is to investigate a conjecture of Weiermann [86], thereby addressing the problem of the unknown maximal order types and corresponding ordinal notation systems for Friedman's well-partial-orders [76]. Weiermann's conjecture concerns a class of structures, a typical member of which is denoted by $\mathcal{T}(W)$, each are ordered by a certain gap-embeddability relation. The conjecture indicates a possible approach towards determining the maximal order types of the structures $\mathcal{T}(W)$. Specifically, Weiermann conjectures that the collapsing functions ϑ_i correspond to maximal linear extensions of these well-partial-orders $\mathcal{T}(W)$, hence also that these collapsing functions correspond to maximal linear extensions of Friedman's famous well-partial-orders.

For a more *detailed overview and summary* of the dissertation, we refer to the introductory Sections 1.1 and 1.3.

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Chapter 1

Introduction

1.1 Historical background

In order that the reader can situate this dissertation in its context, we present a brief overview of ordinal notation systems and well-partial-orders. One can find more complete surveys and overviews of these subjects in the literature (e.g. [20, 47]).

1.1.1 Ordinal notation systems

Well-orders and *ordinal notation systems* have been studied for their own order-theoretic and combinatorial interests. Additionally, also their applications in proof-theoretic investigations of formal systems are studied [13, 31, 59, 60, 71, 79]. In this dissertation, we especially focus on the orderings themselves.

At the end of the 19^{th} century, Cantor extended the natural numbers into the transfinite by defining ordinals (also called *ordinal numbers*). It enabled him to study the order of such infinite numbers. In 1908, Veblen [84] introduced new fast growing functions on the class of ordinals by his techniques of derivation (i.e. enumerating fixed points of monotonic increasing continuous functions) and iteration. Veblen's techniques of derivation and iteration can be seen as a generalization of Cantor's normal form. Veblen's work gives rise to ordinal representation systems for specific ordinals. Hence, in some way, Veblen's (or Cantor's) work can be considered to be the starting point of ordinal notation systems and also of the notorious *natural well-ordering problem* [20], which is known to be extremely difficult. The natural well-ordering problem is a conceptual question about when a representation of a well-ordering is considered *natural*. Veblen's article [84] yields the Veblen hierarchy, which is nowadays well-known among most proof-theorists. It consists of a family of functions φ_α on ordinals, where α is also an ordinal number. The function φ_α is defined as the enumeration function of the common fixed points of all φ_δ with $\delta < \alpha$. $\varphi_\alpha\beta$ is interpreted as a binary function $\varphi\alpha\beta$. Using the binary function φ, one can define an ordinal notation system for the ordinal Γ_0, the limit of predicativity. Veblen extended this idea not only to an arbitrary finite number of arguments but also to transfinitely many arguments. In the finite case, this led to a notation system for the *small Veblen number* or *small Veblen ordinal number*, denoted in this dissertation by $\vartheta\Omega^\omega$. In the case of transfinitely many arguments, i.e. $\varphi(\alpha_0, \ldots, \alpha_\beta)$, where only a finite number of the arguments α_γ (with $\gamma < \beta$) are non-zero, this led to a notation system for the *big Veblen number* or *big Veblen ordinal number*, which is in this dissertation denoted by $\vartheta\Omega^\Omega$.

Bachmann [5] used Veblen's method of defining a hierarchy of functions. He added an additional procedure of diagonalization for constructing new functions. Bachmann's new idea was the systematic use of uncountable ordinals to keep track of systems of fundamental sequences. Later, Bachmann's work became very important for proof-theorists. Howard used Bachmann's work to classify the proof-theoretic strength of Bar recursion of type 0 by an ordinal which later became known as the *Howard-Bachmann ordinal*. Bachmann's system became the standard source of ordinal notations needed in proof theory.

Bachmann's approach for ordinal notation systems uses fundamental sequences, hence the systems can be very technical and involved if one goes beyond the Howard-Bachmann ordinal. More and more layers of fundamental sequences are needed to describe larger and larger ordinals. Feferman [24] succeeded in devising a notion of autonomous closure operations. This can be used in favor of Bachmann's complicated use of fundamental sequences to construct ordinal notation systems for large ordinals. This in turn led to the introduction of the *collapsing functions* $\theta_\alpha : On \to On$ $(\alpha \in On)$, where On is the class of ordinals, that extend the usual Veblen hierarchy. Later, this approach was modified by Buchholz [8, 13] to obtain the functions $\overline{\theta}_\alpha$, which are better suited to proof-theoretic applications. After a few years of polishing, a smooth theory of ordinal notation systems was built up by extending Feferman's idea of autonomous closure (see e.g. [12, 43, 58]). In [9, 14] Buchholz simplified $\overline{\theta}_\alpha$ by introducing the ψ_ν functions. Additionally, Gordeev [34] introduced different collapsing functions D_ν based on Buchholz's ψ-functions, that have the following appealing behavior with respect to iterated collapsing: $D_\mu D_\nu \alpha = D_{\min\{\mu,\nu\}}\alpha$, where equality means here that the normal forms of the terms are the same. The correspondence of Gordeev's D_ν functions with Buchholz's Ψ_ν functions is established in [34] and [83]. Next to all these collapsing functions, other functions ϑ_n, with $n < \omega$, were introduced (see e.g. [91–93]). The functions ϑ_n are not so different from θ_α and $\overline{\theta}_\alpha$ because e.g. $\overline{\theta}_\beta 0$ corresponds to $\vartheta_0(\Omega \cdot \beta)$. In this dissertation, we use the collapsing functions ϑ_i. For a discussion about the connection between ϑ, which is in some sense ϑ_0, and ψ, we refer the reader to [66]. In this context, it is also definitely worth mentioning Rathjen's results on extending ordinal notation systems by using weakly Mahlo cardinals, weakly compact cardinals, and even larger cardinals. He used these cardinals in the notation system for his ordinal analysis of Π^1_2-comprehension [64].

Takeuti [79] developed a different approach to ordinal notation systems using *ordinal diagrams*. Later, this approach has been extended to far reaching ordinal notation systems by Arai (e.g. see [3]). The connection of the ordinal notation systems of the Japanese school with the traditional use of collapsing functions is not yet entirely understood, although some work has already been done (e.g., by Levitz [49] and Buchholz [12]).

In general, a notation system for ordinals is defined using (possibly partial) recursive functions: an ordinal notation system T is a term representation system consisting of a (least) set T and an ordering $<_T$ such that $0 \in T$ and $f(t_1, \ldots, t_n) \in T$ provided that $t_1, \ldots, t_n$ are already in T, where f is a constructor symbol from a given signature. The constructor symbols could be function symbols, but more general operations may be allowed. In practice, a notation system T is represented by a set of ordinals. More specifically, T is the least set such that $0 \in T$ and if $\alpha_1, \ldots, \alpha_n \in T$, then $f(\alpha_1, \ldots, \alpha_n) \in T$, where the symbols f now represent functions on the class of ordinals. The ordering $<_T$ is the usual order relation between the ordinals. The notation system $\mathrm{T} = (T, <_T)$ then represents the least ordinal α such that $\alpha \notin T$. This ordinal α is also called the *closure ordinal* of T. The ordinal α is equal to the largest initial segment in T that is downwards closed in the ordinals, and that is why we also call α the order type of T. A standard example is the notation system for the ordinal ε_0, which is the least ordinal that cannot be described using 0 and the binary monotonic increasing function $\xi, \eta \mapsto \omega^\xi + \eta$ (see Sections 1.2.2 and 1.2.3).

Of course, one cannot say much about the order type of T in the general case, but the situation changes somewhat surprisingly if we require conditions like increasingness, i.e. $t_i \leq_T f(t_1, \ldots, t_n)$ and monotonicity, i.e. $f(t_1, \ldots, t_n) \leq_T f(t'_1, \ldots, t'_n)$ provided that $t_i \leq_T t'_i$ for all $i \leq n$. Diana Schmidt characterized completely the order types which could be generated from the ordinal 0 by applying monotonic increasing functions [68]. A monotonic increasing binary function generates from 0 no order type larger than ε_0. (See an unpublished paper of de Jongh. This is also mentioned in [68].) Functions of bigger arities easily produce ordinals bigger than Γ_0, and in fact they produce ordinals of size comparable to the small Veblen number $\vartheta\Omega^\omega$ [68].

An important facet of these investigations is the role of associated well-quasi-orders or well-partial-orders. This research program goes back to Diana Schmidt. In her Habilitationsschrift [69], Diana Schmidt showed that studying bounds on closure ordinals can best be achieved by determining the maximal order types of well-partial-orderings that reflect the monotonicity properties of the functions in question. She commented that by moving to well-partial-orderings, she had been able to prove stronger results, with sometimes even simpler proofs. For example, she classified the maximal order types of various classes of structured labeled trees [69]. The basic idea is to take the notation system T in question and to restrict the ordering between terms to those cases that are syntactically justified, i.e. justified by the monotonicity and increasingness conditions (the subterm property). The new ordering becomes a well-partial-order, and its maximal linear extension provides an upper bound for the order type of the original ordinal notation system T. It is interesting that for several examples of natural well-orderings, the order type of T coincides with the maximal order type of the underlying well-partial-order. So in some sense, natural well-orderings produce the maximal possible order type out of the syntactical material given for defining the corresponding notation system. This dissertation is written in line with the continuation of Diana Schmidt's research program, as suggested by Weiermann. We investigate the connections between well-partial-orders and ordinal notation systems.

This kind of research has already been taken up in [89], where Weiermann extended Schmidt's approach to transfinite arities. More specifically, motivated by order-theoretic properties of the functions considered by Veblen and Schütte (see e.g., [70, 84] for further details), he investigated a well-partial-order, denoted in this dissertation by $\mathcal{T}(M^\diamond(\tau \times \cdot))$, and showed that the maximal order type is bounded by $\vartheta\Omega^\tau$, thus giving rise to an ordinal notation system for $\vartheta\Omega^\tau$. The underlying set of this well-partial-order was introduced as follows: let $\underline{0}$ and $\underline{\psi}$ be two distinct symbols. For a countable ordinal τ let $\mathcal{T}(M^\diamond(\tau \times \cdot))$ be the least set T such that $\underline{0} \in T$ and such that if $\xi_1 < \cdots < \xi_n < \tau$ and $t_1, \ldots, t_n \in T$, then $\langle \underline{\psi}, \langle \xi_1, t_1 \rangle, \ldots, \langle \xi_n, t_n \rangle \rangle \in T$.

Let the underlying ordering $\leq_\tau$ be the least binary reflexive and transitive relation on $\mathcal{T}(M^\diamond(\tau \times \cdot))$ such that

1. $t_i \leq_\tau \langle \underline{\psi}, \langle \xi_1, t_1 \rangle, \dots, \langle \xi_n, t_n \rangle \rangle$ $(1 \leq i \leq n)$,
2. if $h : \{1, \dots, m\} \to \{1, \dots, n\}$ is a one-to-one mapping and if $\xi_i \leq \xi'_{h(i)}$ and $t_i \leq_\tau t'_{h(i)}$ for all $i = 1, \dots, m$, then

$$\langle \underline{\psi}, \langle \xi_1, t_1 \rangle, \dots, \langle \xi_m, t_m \rangle \rangle \leq_\tau \langle \underline{\psi}, \langle \xi'_1, t'_1 \rangle, \dots, \langle \xi'_n, t'_n \rangle \rangle.$$

Note that in the last condition the comparison is based on comparing multisets of pairs consisting of ordinals (the ordinal addresses) and previously defined terms. In [89] it is shown that the maximal order type of $\mathcal{T}(M^\diamond(\tau \times \cdot))$ is bounded by $\vartheta\Omega^\tau$ so that it can give rise to an ordinal notation system for $\vartheta\Omega^\tau$. Furthermore (by allowing the case $\tau = \Omega$), it has been indicated in [89] that the order type $\mathcal{T}(M^\diamond(\Omega \times \cdot))$ is bounded by the big Veblen number $\vartheta\Omega^\Omega$. In some sense, these results are not fully satisfying since they refer (what the ordinal valued addresses in the terms concerns) to an underlying structure of ordinals and not to terms of the corresponding ordinal notation system. Therefore, the representation of $\vartheta\Omega^\tau$ using $\mathcal{T}(M^\diamond(\tau \times \cdot))$ provides an ordinal notation system which can only be developed if we have an a priori effective term description for the segment τ. And in the case of $\mathcal{T}(M^\diamond(\Omega \times \cdot))$ it is even more difficult to use this set to built up a constructive notation system. Chapter 3 improves these results by replacing τ by *previously defined terms*. This produces an order-theoretic characterization of the big Veblen number $\vartheta\Omega^\Omega$. More specifically, we show that the maximal order type of $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ is equal to $\vartheta\Omega^\Omega$. In Chapter 3, we also study what would happen if we replace the multisets by sequences. To this end, we investigate the well-partial-order $\mathcal{T}((\cdot \times \cdot)^*)$, which is based on finite sequences of pairs of previously defined terms. Weiermann wondered in the 90's if any ordinal notation system which respects the construction of finite *sequences* of pairs of terms is bounded in order type by $\vartheta\Omega^\Omega$. Somewhat surprisingly we show that the relevant order type is equal to $\vartheta\Omega^{\Omega^\Omega}$, which is considerably bigger than the big Veblen number.

In [88], Weiermann showed that the Howard-Bachmann ordinal could be characterized as a closure ordinal of so-called essentially monotonic increasing functions. Since then, it has been an open question whether a corresponding order-theoretic characterization in terms of maximal order types is possible. In Chapter 4, we answer this question positively by investigating a subordering of Friedman's famous well-partial-order which is defined using the so-called gap-embeddability relation [76].

1.1.2 Well-partial-orderings

The notion of *well-partial-orders* or well-partial-orderings appeared in Vázsonyi's conjecture (around 1940), stating that in an infinite collection of finite trees, there are two trees such that one is homeomorphically embeddable into the other one. Additionally, the concept of well-partial-orders occurred in a problem about the natural numbers proposed by Erdös [22]. And in [56], some indications of well-partial-order-theory were mentioned.

The first explicit use of well-partial-orders appeared simultaneously in two independent articles in 1952. One was due to Higman [38], where he called the determining property of a well-partial-order the *finite basis property*. Nowadays, his results are still very useful, e.g. in commutative algebra. The other paper where one uses well-partial-orders is that of Erdös and Rado [23]. They published an answer to the problem proposed by Erdös in [22]. Since then, the concept of well-partial-orders occurred in more and more different contexts, often independently. In [47], Kruskal claimed that well-partial-orders are frequently rediscovered objects. For a nice overview of the history of well-partial-orders, we also refer to that article. Nowadays, more refined notions like α-well-quasi-orders [50, 61] and better-quasi-orders [62, 75] exist.

Well-partial-orders are very useful for constructing independence results. A famous example is the gap-embeddability relation (or gap-ordering) between labeled rooted trees, invented by Harvey Friedman [76] in 1985. This ordering generates ordinals of size $\psi_0\Omega_\omega$, the proof-theoretical ordinal of Π^1_1-CA_0. These well-partial-orders, denoted by $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$ when the trees under consideration have n labels, yield a spectacular independence result for Π^1_1-CA_0, the strongest theory of the *Big Five* in reverse mathematics:

$$\Pi^1_1\text{-}\mathsf{CA}_0 \nvdash \forall n < \omega \text{ `}(\mathbb{T}_n, \leq_{gap}) \text{ is a well-partial-order'}.$$

$wgap$ in $\mathbb{T}_n^{wgap}$ stands for the *weak gap-embeddability relation* and $sgap$ in $\mathbb{T}_n^{sgap}$ stands for the *strong gap-embeddability relation* (see Definition 1.81). It is an interesting problem to identify a natural sub-ordering of Friedman's order which matches the Howard-Bachmann ordinal. This will be answered in Chapter 4 as mentioned above. In this context, it is worth mentioning that a *symmetric gap-condition* on the set of labeled rooted trees was investigated by Gordeev in [35, 36] and that Kříž [45] generalized Friedman's partial order to trees with labels that are ordinal numbers.

Classifying the strength of Friedman's assertion for the case that the set of the labels on the trees consists of n elements where n is fixed from the outside, is still an open problem. That is, the maximal order types of these famous well-partial-orders $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$ are still unknown. In [76], it is only shown that the maximal order type is bounded from below by $\psi_0\Omega_n$. In this dissertation, we address this problem by investigating a conjecture of Weiermann. Weiermann introduced in [86] special tree-classes $\mathcal{T}(W)$, all of them equipped with a sort of gap-embeddability relation. For each specific parameter W, $\mathcal{T}(W)$ corresponds to subclasses of Friedman's partial orders. Weiermann's conjecture describes what the maximal order type of $\mathcal{T}(W)$ looks like. More specifically, the conjecture states that one can read off a *maximal linear extension* of $\mathcal{T}(W)$ by looking at the related ordinal notation system based on the ϑ-functions. This involves defining a maximal linear extension of $\mathbb{T}_n^{wgap}$ by a straightforward use of the collapsing functions ϑ_i. From this, it follows that the maximal order type of the *countable* well-partial-order $\mathbb{T}_n^{wgap}$ can be described by the n^{th} *uncountable* cardinal number Ω_n. The origin of this connection seems magical.

1.2 Preliminaries

1.2.1 Notations

If s is a finite sequence, let $lh(s)$, respectively s_i, denote the length, respectively the i^{th} element, of the sequence, where we start counting at zero. Hence,

$$s = (s_0, \ldots, s_{lh(s)-1}).$$

If $s = (s_0, \ldots, s_{lh(s)-1})$ and $t = (t_0, \ldots, t_{lh(t)-1})$ are two finite sequences, define the *concatenation* $s^\frown t$ as the finite sequence

$$(s_0, \ldots, s_{lh(s)-1}, t_0, \ldots, t_{lh(t)-1}).$$

If X and Y are two sets, the *disjoint union* of X and Y is the set $X + Y := \{(x, 0) : x \in X\} \cup \{(y, 1) : y \in Y\}$ and the *cartesian product* $X \times Y$ is the set $\{(x, y) : x \in X, y \in Y\}$. Let $f : X \to Y$ be a function and $A \subseteq X$. Define $f(A)$ as $\{y \in Y : \exists a \in A(f(a) = y)\}$.

1.2.2 Partial orders, linear orders, well-orders and ordinals

A *quasi order* or *quasi ordering* $\leq$ on a set X is a binary relation $\leq \subseteq X \times X$ that is reflexive ($x \leq x$) and transitive ($x \leq y \wedge y \leq z \to x \leq z$). A *partial* *order* or *partial ordering* is a quasi order that is anti-symmetric ($x \leq y \wedge y \leq x \rightarrow x = y$). If X and Y are two partial orders, then X is *order-isomorphic* to Y (denoted by $X \cong Y$) if there exists a bijective function f from X to Y such that $x \leq_X x' \Leftrightarrow f(x) \leq_Y f(x')$ for every $x, x' \in X$.

A *linear order* or *linear ordering* is a partial order that is total ($x < y \vee x = y \vee x > y$). If X is a linear order, define $X + 1$ as the linear order that adds one extra element bigger than any other element of X.

A partial order X is *well-founded* if every subset of X has at least one minimal element. Define a *well-order* as a well-founded linear order. Cantor defined the *order type* of a well-order as the equivalence class of all well-orders equivalent to it under $\cong$. Nowadays, an order type of a well-ordering X is defined as the ordinal which is isomorphic to X. This ordinal is denoted by *otype*(X).

Ordinals

Ordinals are generalizations of the natural numbers developed by Cantor in the 19th century. The original definition of the ordinals, as the representatives of the equivalence classes of well-orders under $\cong$, might give trouble when used in the framework of an arbitrary set theory, as the equivalence class of an ordering is not necessarily a set. In ZFC, one avoids this problem by defining an ordinal as a transitive set ($z \in y \in X \rightarrow z \in X$) which is well-ordered under $\in$. Denote the class of ordinals by On.

Intuitively, ordinals start with

$$0, 1, 2, 3, 4, \ldots$$

ω denotes the supremum of this sequence, i.e. the order type of $\mathbb{N}$. It is also the least infinite ordinal. Cantor's theory allows us to continue counting

$$\omega + 1, \omega + 2, \ldots$$

and further

$$\omega + \omega = \omega \cdot 2, \ldots, \omega \cdot 3, \ldots, \omega \cdot \omega = \omega^2, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots$$

The limit of $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$ is denoted by ε_0. After this point, we can start counting again: $\varepsilon_0 + 1, \ldots$

For $\alpha \in On$, define $\alpha + 1$ as the least ordinal strictly above α. Ordinals of the form $\alpha + 1$ are called *successors*, the remaining non-zero ordinals are called *limits*. Note that if β is a limit and $\alpha < \beta$, then $\alpha + 1 < \beta$.

In this dissertation, we use the principles *transfinite induction* and *transfinite recursion*. Transfinite induction is the following scheme for formulas F:

$$\forall\alpha((\forall\beta<\alpha)F(\beta)\to F(\alpha))\to\forall\alpha F(\alpha).$$

Transfinite recursion states that given a class-function G, there is an H with domain On such that

$$H(\alpha)=G(H\restriction\{\beta:\beta<\alpha\},\alpha).$$

Using transfinite recursion, we can define addition, multiplication and exponentiation on the class of ordinals (see e.g. [59]).

If α and β are two ordinals with $\beta>1$ and $\alpha>0$, then there exist unique ordinals $\alpha_1,\dots,\alpha_n,\delta_1,\dots,\delta_n$ such that

$$\alpha=\beta^{\alpha_1}\delta_1+\cdots+\beta^{\alpha_n}\delta_n$$

with $\alpha\geq\alpha_1>\cdots>\alpha_n$ and $0<\delta_1,\dots,\delta_n<\beta$. If β is the ordinal ω, then $\delta_1,\dots,\delta_n$ are finite ordinals, i.e. natural numbers. So

$$\alpha=\omega^{\alpha_1}+\cdots+\omega^{\alpha_n}$$

with $\alpha_1\geq\cdots\geq\alpha_n$. We call this representation of α (using the base ω) its *Cantor normal form* (or in short *normal form*) and we sometimes write $=_{NF}$ or $=_{CNF}$ to make this clear (see e.g. [71]).

Define P (in the literature also known as $\mathbb{H}$) as the class of *additively closed* ordinals, i.e.

$$P=\{\alpha\in On:\forall\beta,\gamma\in On(\beta,\gamma<\alpha\to\beta+\gamma<\alpha)\}.$$

One can prove that P is equal to $\{\omega^\alpha:\alpha\in On\}$. An ordinal α is said to be multiplicatively closed if $\forall\beta,\gamma<\alpha(\beta\cdot\gamma<\alpha)$. One can prove that an ordinal is *multiplicatively closed* iff $\alpha=\omega^{\omega^\beta}$ for some $\beta\in On$.

Define the *epsilon numbers* as the ordinals that are closed under ω-exponentiation, i.e. $\forall\beta<\alpha(\omega^\beta<\alpha)$. Let E be the class of all epsilon numbers. The notation $\omega_n[\alpha]$ is defined as $\omega_0[\alpha]=\alpha$ and $\omega_{n+1}[\alpha]=\omega^{\omega_n[\alpha]}$. We abbreviate $\omega_n[1]$ by ω_n. Let ε_α denote the enumeration function of the class E. For example, ε_0 is the first ordinal that is closed under ω-exponentiation, hence

$$\varepsilon_0=\sup_n\omega_n.$$

Note that every ordinal below ε_0 has a unique Cantor normal form. Furthermore, we point out that $\varepsilon_\Omega=\Omega$, where Ω is the first uncountable ordinal.

If $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and $\beta =_{NF} \omega^{\alpha_{n+1}} + \cdots + \omega^{\alpha_{n+m}}$, we define the *natural sum* of α and β, denoted by $\alpha \oplus \beta$ as

$$\omega^{\alpha_{\pi(1)}} + \cdots + \omega^{\alpha_{\pi(n+m)}},$$

where π is a permutation on $\{1, \ldots, n+m\}$ such that $\alpha_{\pi(1)} \geq \cdots \geq \alpha_{\pi(n+m)}$. We define the *natural product* $\alpha \otimes \beta$ of α and β as

$$\bigoplus_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \omega^{\alpha_i \oplus \alpha_{n+j}}.$$

For a set of ordinals A and an ordinal α, we write $A < \alpha$ if $\forall \beta \in A(\beta < \alpha)$ and $\alpha < A$ if $\exists \beta \in A(\alpha < \beta)$. More details on ordinals can be found in e.g., [2, 44, 57, 59, 60].

1.2.3 Ordinal notation systems below Γ_0

As mentioned before, an ordinal notation system T is in general a term representation system consisting of a (least) set T and an ordering $<_T$ such that $0 \in T$ and $f(t_1, \ldots, t_n) \in T$ provided that $t_1, \ldots, t_n$ were already in T, where f is a constructor symbol from a given signature. In practice, a notation system T is represented by a set of ordinals, the constructors f are functions on ordinals and $<_T$ is the order relation between the ordinals. We refer to Section 1.1.1 for an overview about ordinal notation systems. We describe here the notation systems for ε_0 and Γ_0. Both systems are based on ordinal functions mentioned in the previous sections. In later subsections (Subsection 1.2.4 and 1.2.5), we discuss several other examples.

The epsilon number ε_0

This is the ordinal notation system that appears if we use the binary function $f : \xi, \eta \mapsto \omega^\xi + \eta$. Using this ordinal notation system, some ordinals below ε_0 have infinitely many (term) representations in this system. To achieve uniqueness of the terms, a subset OT is defined using the property of Cantor's normal form.

- $0 \in OT$,
- If $\alpha_0, \ldots, \alpha_{n-1} \in OT$ and $\alpha_0 \geq_T \cdots \geq_T \alpha_{n-1}$, then $\omega^{\alpha_0} + \cdots + \omega^{\alpha_{n-1}} \in OT$,

where $<_T$ represents the ordering relation of the original ordinal notation system. $(OT, <_T)$ yields a unique representation for all ordinals below ε_0.

The limit of predicativity Γ_0

The famous ordinal Γ_0 is the proof theoretic ordinal of the formal theory ATR_0 [29]. Γ_0 is commonly called the *the limit of predicativity* because it cannot be reached by a bootstrapping procedure, but every ordinal $\xi < \Gamma_0$ can (see e.g. [57]). The underlying idea is to call a notion predicative if it can be defined without referring to itself. Γ_0 is also called the Schütte-Feferman ordinal and it is normally defined as the least strongly critical ordinal [57].

The notation system is based on Veblen's hierarchy $\varphi\alpha\beta$ [84], where $\varphi 0\alpha$ is ω^α and $\cdot \mapsto \varphi\alpha\cdot$ is an enumeration of the commonly fixed points of all functions $\cdot \mapsto \varphi\beta\cdot$ with $\beta < \alpha$. So $\varphi 1\alpha = \varepsilon_\alpha$ for all $\alpha \in On$. In this context, $\Gamma_0 = \min\{\beta : \varphi\beta 0 = \beta\}$. More specifically, Γ_α is an enumeration function of the set $\{\beta : \varphi\beta 0 = \beta\}$. The key in the development of the representation system is the following lemma (see e.g. [60]).

Lemma 1.1. *For every ordinal $\alpha \in P$ such that $\alpha < \varphi\alpha 0$, there are uniquely determined ordinals ξ and η such that $\alpha = \varphi\xi\eta$ and $\xi, \eta < \alpha$.*

Therefore, using Cantor's normal form, every ordinal α strictly below Γ_0 can uniquely be written as $\varphi\xi_1\eta_1 + \cdots + \varphi\xi_n\eta_n$ such that $\xi_i, \eta_i < \alpha$. The ordinal Γ_0 is therefore the least ordinal that cannot be defined using φ and $+$ without referring to itself.

To state it clearly, we define the representation system.

- $0 \in T$,
- If $\alpha_0, \ldots, \alpha_{n-1}, \beta_0, \ldots, \beta_n$ are all in T, $\varphi\alpha_0\beta_0 \geq_T \cdots \geq_T \varphi\alpha_{n-1}\beta_{n-1} >_T 0$ and $\alpha_i, \beta_i < \varphi\alpha_i\beta_i$ for all i, then $\varphi\alpha_0\beta_0 + \cdots + \varphi\alpha_{n-1}\beta_{n-1} \in T$,
- $0 <_T \alpha$, for all $\alpha \in T\backslash\{0\}$,
- $\varphi\alpha\beta <_T \varphi\gamma\delta$ with $\varphi\alpha\beta, \varphi\gamma\delta \in T$ iff one of the following conditions is satisfied
  - $\alpha <_T \gamma$ and $\beta <_T \varphi\gamma\delta$,
  - $\alpha = \gamma$ and $\beta <_T \delta$,
  - $\alpha >_T \gamma$ and $\varphi\alpha\beta <_T \delta$,

- $\varphi\alpha_0\beta_0 + \cdots + \varphi\alpha_{n-1}\beta_{n-1} <_T \varphi\gamma_0\delta_0 + \cdots + \varphi\gamma_{m-1}\delta_{m-1}$ with $\varphi\alpha_0\beta_0 + \cdots + \varphi\alpha_{n-1}\beta_{n-1}, \varphi\gamma_0\delta_0 + \cdots + \varphi\gamma_{m-1}\delta_{m-1} \in T$ iff one of the following conditions is valid
  - $\varphi\alpha_0\beta_0 <_T \varphi\gamma_0\delta_0$, or
  - $\varphi\alpha_0\beta_0 = \varphi\gamma_0\delta_0$ and $\varphi\alpha_1\beta_1 + \cdots + \varphi\alpha_{n-1}\beta_{n-1} <_T \varphi\gamma_1\delta_1 + \cdots + \varphi\gamma_{m-1}\delta_{m-1}$.

This yields $\varphi\alpha_0\beta_0 + \cdots + \varphi\alpha_{n-1}\beta_{n-1} <_T \varphi\gamma_0\delta_0$ if $\forall i < n(\varphi\alpha_i\beta_i <_T \varphi\gamma_0\delta_0)$. In this system, every ordinal below Γ_0 has a unique representation.

1.2.4 Ordinal notation systems going beyond the limit of predicativity

There are various ways to describe a notation system for bigger ordinals. For a short, but not complete, history about this subject, we refer to Subsection 1.1.1. In this dissertation, we work with the ϑ-functions. They can be defined in two different ways. One definition uses the closure sets $C(\alpha, \beta)$ to define ϑ_m with $m < \omega$ (see e.g. [91–93]). The other one defines ϑ_m directly without referring to the closure sets $C(\alpha, \beta)$. In this dissertation, we will use the second approach. In [10, 11] it is proved that these approaches are equivalent.

Up to Howard-Bachmann

In this subsection, we discuss an example for an ordinal representation system for the ordinals less than or equal to the Howard-Bachmann ordinal η_0. The ordinal η_0 (which is also denoted by $\psi\varepsilon_{\Omega+1}$, $\vartheta\varepsilon_{\Omega+1}$, $\theta_{\varepsilon_{\Omega+1}}0$, $d\varepsilon_{\Omega+1}$) belongs to the most well-established arsenal of proof-theoretic ordinals of natural theories for developing significant parts of (impredicative) mathematics. η_0 is the proof-theoretic ordinal of the first order theory ID_1, which extends PA by schemes for smallest fixed points of non-iterated positive inductive definitions. The ordinal η_0 is also the proof-theoretic ordinal of the theory $KP\omega$ which formalizes an admissible universe containing ω, and η_0 is also the proof theoretic ordinal of $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$ which formalizes lightface Π^1_1-comprehension and of the theory $\mathsf{RCA}_0 + \mathsf{BI}$ which extends RCA_0 by the scheme of bar induction.

First, we introduce an ordinal function ϑ that is a *collapsing function*, i.e. it maps uncountable ordinals to countable ordinals. This function is used in Chapters 2, 3, 4. Then, we introduce an ordinal representation system

$OT(\vartheta)$. This is used in Chapter 6 and is needed to work more rigorously with these ordinals in a formal theory. In Chapter 6, we also use a different notation system $OT'(\vartheta)$ to represent the ordinal $\vartheta(\Omega^\omega)$.

Before we give the definition of ϑ, we should note that we use this collapsing function and not other ones from the literature because for our purpose, the ϑ-function is more closely related to a tree-structure: it has the subterm property, i.e. increasingness and monotonicity. Increasingness here means $k\alpha < \vartheta\alpha$ (see below) and monotonicity means $\vartheta\alpha < \vartheta\alpha'$ where α' is constructed from α by replacing one element of $k\alpha$ in α by a bigger one. Both properties also hold if we work with trees. Other collapsing functions from the literature, e.g. Buchholz's ψ (see [9, 14]), do not necessarily have both properties.

Definition 1.2. *Let Ω denote the first uncountable ordinal. Every ordinal $0 < \alpha < \varepsilon_{\Omega+1}$ can uniquely be written as $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ with $0 < \beta_i < \Omega$ and $\alpha > \alpha_1 > \cdots > \alpha_n$. Define the set of coefficients recursively as $K\alpha = \{\beta_1, \ldots, \beta_n\} \cup K\alpha_1 \cup \cdots \cup K\alpha_n$. Let $K(0)$ be $\{0\}$. Define then $k\alpha$ as the ordinal $\max(K\alpha)$.*

Definition 1.3. *For an ordinal α, define $\Omega_0[\alpha]$ as α and $\Omega_{n+1}[\alpha]$ as $\Omega^{\Omega_n[\alpha]}$.*

Definition 1.4. *Let P denote the set of the additively closed ordinal numbers $\{\omega^\alpha : \alpha \in On\}$. For every ordinal $\alpha < \varepsilon_{\Omega+1}$, define $\vartheta\alpha$ as*

$$\min\{\zeta \in P : k\alpha < \zeta \text{ and } \forall\beta < \alpha\,(k\beta < \zeta \to \vartheta\beta < \zeta)\}.$$

The Howard-Bachmann ordinal number is defined as

$$\eta_0 = \vartheta(\varepsilon_{\Omega+1}) = \sup_n \left(\vartheta\left(\Omega_n[1]\right)\right).$$

It can be shown by a cardinality argument that $\vartheta\alpha < \Omega$ (see for example [11] or Lemma 2.2 in [10]). The definition of ϑ yields easily that the ordering between ϑ-terms can be described as follows.

Lemma 1.5. $\vartheta\alpha < \vartheta\beta \iff \begin{cases} \alpha < \beta \text{ and } k\alpha < \vartheta\beta \\ \beta < \alpha \text{ and } \vartheta\alpha \leq k\beta. \end{cases}$

Proof. Assume $\vartheta\alpha < \vartheta\beta$. If $\alpha < \beta$, then $k\alpha < \vartheta\alpha < \vartheta\beta$. Assume $\alpha > \beta$. If $\vartheta\alpha > k\beta$, then the definition of $\vartheta\alpha$ yields $\vartheta\alpha > \vartheta\beta$, a contradiction. This finishes the left-to-right direction.

Assume $\alpha < \beta$ and $k\alpha < \vartheta\beta$. The definition of $\vartheta\beta$ yields $\vartheta\alpha < \vartheta\beta$. Assume $\beta < \alpha$ and $\vartheta\alpha \leq k\beta$. Then $\vartheta\alpha \leq k\beta < \vartheta\beta$. This finishes the right-to-left direction. □

Lemma 1.6. *Assume $\alpha \in P \cap \Omega$, but $\alpha \notin Im(\vartheta)$. If for $\beta < \varepsilon_{\Omega+1}$*

$$k(\beta) < \alpha < \vartheta(\beta),$$

we obtain an ordinal $\gamma < \beta$ such that

$$k(\gamma) < \alpha < \vartheta(\gamma).$$

Proof. Using the definition of ϑ, the inequalities $k(\beta) < \alpha < \vartheta(\beta)$ yield

$$\neg (\forall \xi < \beta (k(\xi) < \alpha \rightarrow \vartheta(\xi) < \alpha)).$$

Hence, there exists an ordinal $\gamma < \beta$ such that $k(\gamma) < \alpha$ and $\alpha \leq \vartheta(\gamma)$. $\alpha \notin Im(\vartheta)$ implies

$$k(\gamma) < \alpha < \vartheta(\gamma).$$

□

Corollary 1.7. *Assume $\alpha \in P \cap \Omega$. If there is an ordinal $\beta < \varepsilon_{\Omega+1}$ such that*

$$k(\beta) < \alpha < \vartheta(\beta),$$

then $\alpha \in Im(\vartheta)$.

Proof. If not, one can construct an infinite strictly decreasing sequence of ordinals, hence a contradiction. □

Hence, if $\alpha \in P$ and $\alpha < \vartheta(\beta)$ for a certain β, but $\alpha \notin Im(\vartheta)$, then $\alpha \leq k(\beta)$.

Corollary 1.8. *If $\alpha < \vartheta(\varepsilon_{\Omega+1})$ and $\alpha \in P$, then there exists a unique $\beta < \varepsilon_{\Omega+1}$ such that $\alpha = \vartheta(\beta)$. More specifically, $\beta = 0$ or*

$$\beta =_{NF} \Omega^{\beta_1}\gamma_1 + \cdots + \Omega^{\beta_n}\gamma_n < \varepsilon_{\Omega+1},$$

where $\beta > \beta_1 > \cdots > \beta_n$ and $0 < \gamma_i < \Omega$ and $k(\beta) < \alpha$.

Proof. If $\alpha < \vartheta(\varepsilon_{\Omega+1})$, then $\alpha < \vartheta(\Omega_n[1])$ for a certain n. Moreover, $k(\Omega_n[1]) = 1$, hence if $\alpha > 1$, then $\alpha \in Im(\vartheta)$. Furthermore, 1 is in $Im(\vartheta)$, because $1 = \vartheta(0)$. □

This does not yield that every ordinal in $P \cap \Omega$ is reached by ϑ: for example there exists no $\xi < \varepsilon_{\Omega+1}$ such that $\vartheta(\xi) = \vartheta(\varepsilon_{\Omega+1}) \in P \cap \Omega$.

The interested reader can find this definition of the ϑ-function in [10, 11]. Sometimes, we denote our definition of ϑ by ϑ^P to indicate very clearly that $\vartheta\alpha$ is defined as a least additively closed ordinal number. In [66], the authors approached ϑ slightly different: Rathjen and Weiermann used the closure sets $C(\alpha, \beta)$ and they assumed that this set was closed under ω-exponentiation. In [10] or paragraph 4 of [11], it is proved that this is the same as defining $\vartheta\alpha$ as $\min\{\zeta \in E : k\alpha < \zeta \text{ and } \forall \beta < \alpha\, (k\beta < \zeta \rightarrow \vartheta\beta < \zeta)\}$. Denote this definition of ϑ by ϑ^E. Rathjen's and Weiermann's approach looks dissimilar with our definition, but actually it is not: both ϑ-functions coincide for ordinals bigger than $\Omega \cdot \omega$. Note that Rathjen and Weiermann also used a different $k\alpha$, but this does not make a difference.

Lemma 1.9. *Let ϑ^P be our definition of ϑ and ϑ^E be the version of Rathjen and Weiermann. Then $\vartheta^P(\Omega + \alpha) = \vartheta^E(\alpha)$ for every $\alpha < \varepsilon_{\Omega+1}$.*

Proof. By induction on α. □

For our purpose, it does not matter if we use ϑ^P or ϑ^E because most of the time we work with ordinals bigger than Ω^2. We will use the version over P as in Definition 1.4 for notational ease. We need the following additional lemmas. Both of them can be proved by direct calculations.

Lemma 1.10. *Suppose α and β are ordinals beneath $\varepsilon_{\Omega+1}$. Then*

$$\begin{aligned} k(\alpha \oplus \beta) &\leq k(\alpha) \oplus k(\beta), \\ k(\alpha \otimes \beta) &\leq \max\{k(\alpha) \oplus k(\beta), k(\alpha) \otimes k(\beta) \otimes \omega\}, \\ k(\omega^\alpha) &\leq \omega^{k(\alpha)}. \end{aligned}$$

Furthermore, $k(\alpha), k(\beta) \leq k(\alpha \oplus \beta)$ and $k(\alpha) \leq k(\alpha \otimes \beta)$ if $\beta > 0$.

Lemma 1.11. *Suppose $\alpha_n, \ldots, \alpha_0$ are countable ordinal numbers with $\alpha_i < \gamma$ for an epsilon number γ. Then $k(o((\Omega^n \alpha_n + \cdots + \Omega\alpha_1 + \alpha_0)^*)) < \gamma$.*

Up to now, we have defined ϑ as a function on ordinals and we will use it look this in Chapters 2, 3 and 4. However, if one wants to formalize the proofs in these chapters in a formal theory (like we will do in Chapter 6), we need a real ordinal notation system, i.e. a formal set of terms with an order relation on it, but the details become then quite messy. We introduce $OT(\vartheta)$, which is still a set of ordinals, but it indicates very clearly how to deal with the considered ordinals as terms using 0, $\Omega^x \cdot y + z$, $x + y$ and ϑx, where x, y and z are placeholders. We will use $OT(\vartheta)$ in Chapter 6.

Definition 1.12. *Define inductively a set $OT(\vartheta)$ of ordinals and a natural number $G_\vartheta \alpha$ for $\alpha \in OT(\vartheta)$ as follows:*

1. $0 \in OT(\vartheta)$ *and* $G_\vartheta(0) := 0$,
2. *if* $\alpha = \Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ *with* $n \geq 1$, $\alpha_1 > \cdots > \alpha_n$ *and* $\Omega > \beta_1, \ldots, \beta_n > 0$, *then*
   (a) *if* ($n > 1$ *or* $\alpha_1 > 0$) *and* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in OT(\vartheta)$, *then* $\alpha \in OT(\vartheta)$ *and* $G_\vartheta \alpha := \max\{G_\vartheta(\alpha_1), \ldots, G_\vartheta(\alpha_n), G_\vartheta(\beta_1), \ldots, G_\vartheta(\beta_n)\} + 1$,
   (b) *if* $n = 1$, $\alpha_1 = 0$ *and* $\alpha = \delta_1 + \cdots + \delta_m > \delta_1 \geq \cdots \geq \delta_m > 0$ *with* $m \geq 2$ *and* $\delta_1, \ldots, \delta_m \in OT(\vartheta) \cap P$, *then* $\alpha \in OT(\vartheta)$ *and* $G_\vartheta \alpha := \max\{G_\vartheta(\delta_1), \ldots, G_\vartheta(\delta_m)\} + 1$,
3. *if* $\alpha = \vartheta\beta$ *and* $\beta \in OT(\vartheta)$, *then* $\alpha \in OT(\vartheta)$ *and* $G_\vartheta \alpha := G_\vartheta \beta + 1$.

Because $\vartheta\beta$ is always additively closed and ϑ is injective, the function G_ϑ is well-defined. Furthermore, we note that $\Omega \in OT(\vartheta)$ because $\Omega = \Omega^1 \cdot 1$ and $1 = \vartheta(0)$. The previous definition needs maybe a bit more explanation. If α is uncountable, then $\alpha = \Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ with $n > 1$ or $\alpha_1 > 0$, hence we are always in case 2.(a). If α is countable, then we first split α in additively closed ordinals δ_i using case 2.(b), meaning $\alpha = \delta_1 + \cdots + \delta_m$. Then we apply case 3. on the terms δ_i. This is the reason why we make an, at first sight strange, distinction in case 2. There is a straightforward translation of the ordinals in $OT(\vartheta)$ to terms over the symbols 0, $\Omega^x \cdot y + z$, $x + y$ and ϑx, where x, y and z are placeholders. For example, $\vartheta(\Omega^\Omega \cdot 2)$ is

$$\vartheta(\Omega^{\Omega^1 \cdot 1 + 0} \cdot (1 + 1) + 0),$$

where 1 is interpreted as $\vartheta 0$.

Lemma 1.13. *If $\xi \in OT(\vartheta)$, then $K(\xi) \subseteq OT(\vartheta)$. Furthermore, $G_\vartheta(k(\xi)) \leq G_\vartheta(\xi)$ for all ξ in $OT(\vartheta)$.*

Proof. We prove this by induction on $G_\vartheta(\xi)$. If $\xi = 0$, then this trivially holds. If $\xi = \vartheta(\xi')$, then $K(\xi) = \{\xi\}$, hence this also trivially holds. Assume $\xi = \Omega^{\xi_1}\beta_1 + \cdots + \Omega^{\xi_n}\beta_n$ with $n \geq 1$, $\xi_1 > \cdots > \xi_n$ and $\Omega > \xi_1, \ldots, \xi_n > 0$. If $n = 1$ and $\xi_1 = 0$, then also $K(\xi) = \{\xi\}$, hence the assertion holds. Let now $n > 1$ or $\xi_1 > 0$. Then $K(\xi) = \{\beta_1, \ldots, \beta_n\} \cup K(\xi_1) \cup \cdots \cup K(\xi_n)$. The induction hypothesis yields $K(\xi) \subseteq OT(\vartheta)$. Furthermore, $G_\vartheta(k(\xi_i)) \leq G_\vartheta(\xi_i) <$

$G_\vartheta(\xi)$. Therefore, the strict inequality $G_\vartheta(k(\xi)) = G_\vartheta(\max_i\{k(\xi_i), \beta_i\}) < G_\vartheta(\xi)$ holds. □

Each ordinal $\alpha \in OT(\vartheta)$ has a unique normal form using the symbols $0, \Omega, +$ and ϑ. The relation $\alpha < \beta$ can be expressed using the ordinals appearing in their normal form (by Lemma 1.5), which have strictly smaller G_ϑ-values by the previous Lemma 1.13. Hence, the following lemmas hold.

Lemma 1.14. *There exists a specific coding of $(OT(\vartheta), <_{OT(\vartheta)})$ in the natural numbers such that $(OT(\vartheta) \cap \Omega, <_{OT(\vartheta)})$ can be interpreted as a primitive recursive ordinal notation system for the ordinals less than $\vartheta(\varepsilon_{\Omega+1})$. Furthermore, one can choose this coding in such a way that $\forall \xi \in K(\alpha)(\xi \leq_{\mathbb{N}} \alpha)$.*

Lemma 1.15. $OT(\vartheta) \cap \alpha = \alpha$ *for* $\alpha \in OT(\vartheta) \cap \Omega$.

This is the ordinal notation system that we will use in Chapter 6 if we work in the theory ACA_0 augmented with a comprehension scheme (namely lightface Π^1_1-comprehension). In the theories RCA_0 and RCA_0^* (augmented with the same comprehension scheme), we use a different ordinal notation system $OT'(\vartheta)$. We postpone the definition of $OT'(\vartheta)$ to Chapter 6.

Beyond Howard-Bachmann

We introduce the collapsing functions ϑ_i with i strictly less than a fixed n. The collapsing functions describe ordinals bigger than the Howard-Bachmann number $\vartheta\varepsilon_{\Omega+1}$. In [92, 93], Wilken introduced ϑ_i with closure sets $C_i^j(\alpha, \beta)$. Later, Weiermann and Wilken [91] showed the correspondence of these functions with the simultaneously defined theta-functions $\overline{\vartheta}_i$. In this dissertation, we introduce them, like the ϑ-function, directly as functions on ordinals without referring to closure sets. Following [10] or paragraph 4 of [11], one can prove that our functions coincide with the ones considered by Wilken after a certain small ordinal.

Fix a natural number $n \geq 1$. We define the collapsing functions ϑ_i for $i < n$. Let $\Omega_0 := 1$ and if $i > 0$ let Ω_i be the i^{th} regular ordinal number strictly above ω. So Ω_1 is the first uncountable cardinal number, which previously was denoted by Ω. Define Ω_ω as $\sup_i \Omega_i$.

Definition 1.16. *Let $\alpha < \varepsilon_{\Omega_n+1}$. Then $\alpha = 0$ or $\alpha = \Omega_n^{\alpha_1}\beta_1 + \cdots + \Omega_n^{\alpha_m}\beta_m > \alpha_1 > \cdots > \alpha_m$ with $0 < \beta_1, \ldots, \beta_m < \Omega_n$. Define $K_{n-1}\alpha$ as $\bigcup_i K_{n-1}\alpha_i \cup \{\beta_1, \ldots, \beta_n\}$. Let $k_{n-1}\alpha = \max(K_{n-1}\alpha)$.*

Definition 1.17. *Let E be the set of epsilon numbers. For every $\alpha < \varepsilon_{\Omega_n+1}$ define $\vartheta_{n-1}\alpha$ as*

$$\min\{\zeta \in E : \Omega_{n-1} \leq \zeta \text{ and } k_{n-1}\alpha < \zeta \text{ and } \forall\beta < \alpha(k_{n-1}\beta < \zeta \to \vartheta_{n-1}\beta < \zeta)\}.$$

Definition 1.18. *Pick $j < n-1$ and assume $\vartheta_i : \vartheta_{i+1} \dots \vartheta_{n-1}\varepsilon_{\Omega_n+1} \to [\Omega_i, \Omega_{i+1}[$ and K_i are defined for every $n > i > j$. Define for every $\alpha < \vartheta_{j+1} \dots \vartheta_{n-1}\varepsilon_{\Omega_n+1}$, the functions K_j and ϑ_j as follows.*

$$\begin{array}{ll}
K_j 0 = \{0\} & \\
K_j\alpha = K_j\alpha_1 \cup K_j\alpha_2 & \text{if } \alpha = \alpha_1 \oplus \alpha_2 \text{ with } \alpha_1, \alpha_2 \neq 0 \\
K_j\alpha = K_j\beta & \text{if } \alpha = \omega^\beta > \beta \\
K_j\alpha = \{\alpha\} & \text{if } \alpha \in E \cap \Omega_{j+1} \\
K_j\alpha = K_j(K_{j+1}(\beta)) & \text{if } \alpha = \vartheta_{j+1}\beta \geq \Omega_{j+1}
\end{array}$$

and let $\vartheta_j\alpha$ be

$$\min\{\zeta \in E : \Omega_j \leq \zeta \text{ and } k_j\alpha < \zeta \text{ and } \forall\beta < \alpha(k_j\beta < \zeta \to \vartheta_j\beta < \zeta)\},$$

where $k_j\alpha := \max(K_j\alpha)$.

Note that if $n = 1$, Definition 1.18 is useless and we only have the function ϑ_0. The collapsing function ϑ_0 is the same as ϑ^E (see just above Lemma 1.9 for the definition of ϑ^E). That is why we sometimes denote our ϑ_i's by ϑ_i^E. We could also define $\vartheta_i\alpha$ as a least ordinal over P, i.e.

$$\min\{\zeta \in P : \Omega_i \leq \zeta \text{ and } k_i\alpha < \zeta \text{ and } \forall\beta < \alpha(k_i\beta < \zeta \to \vartheta_i\beta < \zeta)\}.$$

We denote these collapsing functions as ϑ_i^P (the version of Wilken [92, 93] is with P). Like in the ϑ-case, ϑ_i^E and ϑ_i^P are the same after a certain point, i.e. both definitions of ϑ_i coincide after $\Omega_{i+1} \cdot \omega$.

Definition 1.19. *If α is an ordinal number, then*

$$-1 + \alpha := \begin{cases} \alpha - 1 & \text{if } 0 < \alpha < \omega, \\ \alpha & \text{otherwise.} \end{cases}$$

Lemma 1.20. *Let ϑ_i^E be as in Definitions 1.17 and 1.18. Assume ϑ_i^P is the same as ϑ_i^E, but we define ϑ_i over P instead of over E. Then for every $0 < i < n$ and for every $0 < \alpha < \vartheta_{i+1}^E \dots \vartheta_{n-1}^E\varepsilon_{\Omega_n+1}$, we have $\vartheta_i^E(0) = \vartheta_i^P(0) = \Omega_i$ and*

$$\vartheta_i^E(\alpha) = \vartheta_i^P(\Omega_{i+1} + (-1 + \alpha)).$$

If $i = 0$, then for every $\alpha < \vartheta_1^E \dots \vartheta_{n-1}^E\varepsilon_{\Omega_n+1}$, we have

$$\vartheta_0^E(\alpha) = \vartheta_0^P(\Omega_1 + \alpha).$$

Proof. By induction on α. Note that $\Omega_0 = 1$. □

For our purpose, we could choose either ϑ_i^E or ϑ_i^P because we work with bigger ordinals than $\Omega_{i+1} \cdot \omega$. In Section 5.2, we use ϑ_i^E as defined in Definitions 1.17 and 1.18. However, in Section 5.3, we use ϑ_i^P because this makes the proofs in that section easier.

Similarly as in the ϑ-case (see Corollary 1.8), one can prove that ϑ_i is surjective on E in some sense. We need two more additional lemmas. First, fix a natural number $n \geq 2$ and assume that we have defined $\vartheta_0, \ldots, \vartheta_{n-1}$ and $K_0, \ldots, K_{n-1}$.

Lemma 1.21. *Suppose that $\alpha, \beta > 0$, then for $j < n - 1$*

$$K_j(\alpha \otimes \beta) = K_j(\alpha) \cup K_j(\beta).$$

Proof. By a straightforward calculation. □

Lemma 1.22. *For $j < n - 1$, if $\alpha < \beta < \Omega_{j+1}$, then $k_j(\alpha) \leq k_j(\beta)$ and $k_j(\alpha) \leq \alpha$.*

Proof. By induction on $\alpha \oplus \beta$ and respectively on α. □

We denote the corresponding ordinal notation system by $OT(\vartheta_i)$. To recapitulate, if we work below the Howard-Bachmann ordinal, we use $\vartheta = \vartheta^P$. If we work beyond the Howard-Bachmann ordinal, we work with either ϑ_i^E (Section 5.2) or ϑ_i^P (Section 5.3).

1.2.5 Ordinal notation systems without addition

In this subsection, we define several ordinal notation systems for ordinals smaller than or equal to ε_0. All of them do *not* use the addition operator. In this dissertation, we investigate the last one in Chapter 7, i.e. the notation system based on the collapsing functions ϑ_i.

The Veblen hierarchy

Assume that $(T, <)$ is a notation system with $otype(T) \in \varepsilon_0 \backslash \{0\}$. Define the representation system $\varphi_T 0$ recursively as follows.

Definition 1.23.
- $0 \in \varphi_T 0$,

- *if* $\alpha \in \varphi_T 0$ *and* $t \in T$*, then* $\varphi_t \alpha \in \varphi_T 0$.

Define on $\varphi_T 0$ the following total order

Definition 1.24. *For* $\alpha, \beta \in \varphi_T 0$*,* $\alpha < \beta$ *is valid if*

- $\alpha = 0$ *and* $\beta \neq 0$*,*
- $\alpha = \varphi_{t_1} \alpha'$*,* $\beta = \varphi_{t_2} \beta'$ *and one of the following cases holds:*
  1. $t_1 < t_2$ *and* $\alpha' < \beta$*,*
  2. $t_1 = t_2$ *and* $\alpha' < \beta'$*,*
  3. $t_1 > t_2$ *and* $\alpha \leq \beta'$*.*

Recall Definition 1.19.

Theorem 1.25. *Assume otype*$(T) = \alpha \in \varepsilon_0 \backslash \{0\}$*. Then* $(\varphi_T 0, <)$ *is a notation system for the ordinal* $\omega^{\omega^{-1+\alpha}}$*.*

Proof. For a proof, we refer to theorem 16 in [48]. □

Using the π_i-collapsing functions

We use an ordinal notation system that employs the π_i-collapsing functions. These functions are based on Buchholz's ψ_i-functions (see [9, 14]), but now addition is avoided. They are investigated by Schütte and Simpson in [72] to prove an independence result concerning sequences with the gap-embeddability relation. We state some basic facts that the reader can find in [9] and [72]. Recall $\Omega_0 = 1$ and if $i > 0$, Ω_i is the i^{th} regular ordinal number strictly above ω. Also remember $\Omega_\omega = \sup_n \Omega_n$.

Define the sets $B_i^m(\alpha)$ and $B_i(\alpha)$ and the ordinal numbers $\pi_i \alpha$ inductively as follows (see [72]).

Definition 1.26. *Take an ordinal number* α *and* $i < \omega$*. Assume that we have defined* $B_j(\beta)$ *and* $\pi_j \beta$ *for every* $\beta < \alpha$ *and every* $j < \omega$*. Define* $B_i^m(\alpha)$ *and* $B_i(\alpha)$ *as the least set of ordinals such that*

- *if* $\gamma = 0$ *or* $\gamma < \Omega_i$*, then* $\gamma \in B_i^m(\alpha)$*,*
- *if* $i \leq j$*,* $\beta < \alpha$*,* $\beta \in B_j(\beta)$ *and* $\beta \in B_i^m(\alpha)$*, then* $\pi_j \beta \in B_i^{m+1}(\alpha)$*,*
- *define* $B_i(\alpha)$ *as* $\bigcup_{m<\omega} B_i^m(\alpha)$*.*

Define $\pi_i\alpha$ *as* $\min\{\eta : \eta \notin B_i(\alpha)\}$.

Lemma 1.27. *1. If* $i \leq j$ *and* $\alpha \leq \beta$*, then* $B_i(\alpha) \subseteq B_j(\beta)$ *and* $\pi_i\alpha \leq \pi_j\beta$*,*

2. $\Omega_i \leq \pi_i\alpha < \Omega_{i+1}$*,*
3. $\pi_i 0 = \Omega_i$*,*
4. $\alpha \in B_i(\alpha)$ *and* $\alpha < \beta$ *yield* $\pi_i\alpha < \pi_i\beta$*,*
5. $\alpha \in B_i(\alpha)$*,* $\beta \in B_i(\beta)$ *and* $\pi_i\alpha = \pi_i\beta$ *yield* $\alpha = \beta$*.*

Proof. See lemmas 1.1,1.2 and 1.4 in [72]. □

Definition 1.28. *For ordinals* $\alpha \in B_0(\Omega_\omega)$*, define* $G_i(\pi_j\alpha)$ *as*

$$\begin{cases} \emptyset & \text{if } j < i, \\ G_i\alpha \cup \{\alpha\} & \text{otherwise.} \end{cases}$$

Define $G_i(0)$ *as* $\emptyset$*.*

This is well-defined because one can prove $\pi_j\alpha \in B_0(\Omega_\omega)$ yields $\alpha \in B_0(\Omega_\omega)$.

Lemma 1.29. *If* $\alpha \in B_0(\Omega_\omega)$*, then* $G_i(\alpha) < \beta$ *iff* $\alpha \in B_i(\beta)$*.*

Proof. We prove this by induction on the length of construction of α. If $\alpha = 0$ or $\alpha = \pi_j\delta$ with $j < i$, then this is trivial. Assume $\alpha = \pi_j\delta$ with $j \geq i$. $\alpha = \pi_j\delta \in B_0(\Omega_\omega)$ yields $\delta \in B_j(\delta)$. Now, $G_i(\alpha) < \beta$ is valid iff $G_i(\delta) < \beta$ and $\delta < \beta$. By the induction hypothesis, this is equivalent with $\delta \in B_i(\beta)$ and $\delta < \beta$, which is equivalent with $\alpha = \pi_j\delta \in B_i(\beta)$ because $\delta \in B_j(\delta)$. □

Now we define the ordinal notation systems $\pi(\omega)$ and $\pi(n)$, but first, we define a set of terms $\pi(\omega)'$ and $\pi(n)'$.

Definition 1.30.
- $0 \in \pi(\omega)'$ *and* $0 \in \pi(n)'$*,*
- *if* $\alpha \in \pi(\omega)'$*, then* $D_j\alpha \in \pi(\omega)'$*,*
- *if* $\alpha \in \pi(n)'$ *and* $j < n$*, then* $D_j\alpha \in \pi(n)'$*.*

Definition 1.31. *Let* $\alpha, \beta \in \pi(\omega)'$ *or* $\alpha, \beta \in \pi(n)'$*. Then define* $\alpha < \beta$ *if*

1. $\alpha = 0$ *and* $\beta \neq 0$*,*
2. $\alpha = D_j\alpha'$*,* $\beta = D_k\beta'$ *and* $i < j$ *or* $i = j$ *and* $\alpha' < \beta'$*.*

Lemma 1.32. *$<$ is a linear order on $\pi(\omega)'$ and $\pi(n)'$.*

Proof. Similar as Lemma 2.1 in [9]. □

Definition 1.33. *For $\alpha \in \pi(\omega)', \pi(n)'$, define $G_i(\alpha)$ as follows.*

1. $G_i(0) = \emptyset$,
2. $G_i(D_j\alpha') := \begin{cases} G_i(\alpha') \cup \{\alpha'\} & \text{if } i \leq j, \\ \emptyset & \text{if } i > j. \end{cases}$

Now, we are ready to define the ordinal notation systems $\pi(\omega) \subseteq \pi(\omega)'$ and $\pi(n) \subseteq \pi(n)'$.

Definition 1.34. *$\pi(\omega)$ and $\pi(n)$ are the least sets such that*

1. $0 \in \pi(\omega)$, $0 \in \pi(n)$,
2. *if* $\alpha \in \pi(\omega)$ *and* $G_i(\alpha) < \alpha$, *then* $D_i\alpha \in \pi(\omega)$,
3. *if* $\alpha \in \pi(n)$, $i < n$ *and* $G_i(\alpha) < \alpha$, *then* $D_i\alpha \in \pi(n)$.

Apparently, the $D_j\alpha$'s correspond to the ordinal functions $\pi_j\alpha$:

Definition 1.35. *For $\alpha \in \pi(\omega)$ and $\pi(n)$, define*

1. $o(0) := 0$,
2. $o(D_j\alpha') := \pi_j(o(\alpha'))$.

Lemma 1.36. *For $\alpha, \beta \in \pi(\omega)$ or $\pi(n)$, we have:*

1. $o(\alpha) \in B_0(\Omega_\omega)$,
2. $G_i(o(\alpha)) = \{o(x) : x \in G_i(\alpha)\}$,
3. $\alpha < \beta \to o(\alpha) < o(\beta)$.

Proof. A similar proof is lemma 2.2 in [9]. □

Lemma 1.37.

1. $\{o(x) : x \in \pi(\omega)\} = B_0(\Omega_\omega)$,
2. $\{o(x) : x \in \pi(\omega)$ *and* $x < D_1 0\} = \pi_0\Omega_\omega$,
3. $\{o(x) : x \in \pi(n)$ *and* $x < D_1 0\} = \pi_0\Omega_n$ *if* $n > 0$.

Proof. A similar proof is lemma 2.3 in [9]. □

Define $\pi(\omega) \cap D_1 0$ as $\pi_0(\omega)$ and $\pi(n) \cap D_1 0$ as $\pi_0(n)$. It is important to notice that that we work with two different contexts: one context is at the level of ordinals, i.e. if we use the π_i's. The other context at the syntactical level, i.e. if we use the D_i's (because it is an ordinal notation system). The previous results actually indicate that D_i and π_i play the same role and for notational convenience, we will identify these two notations: from now on, we write π_i instead of D_i. The context will make clear what we mean. If we use Ω_i in the ordinal context, it is interpreted as usual, i.e. as the i^{th} regular ordinal number strictly above ω for $i > 0$. In the other context, at the level of ordinal notation systems, we define Ω_i as $D_i 0$ (which is now also denoted by $\pi_i 0$). We could also have defined $\pi(\omega)$ in the following equivalent way.

Definition 1.38. *Define $\pi(\omega)$ as the least set of ordinals such that*

1. $0 \in \pi(\omega)$,
2. *If $\alpha \in \pi(\omega)$ and $\alpha \in B_i(\alpha)$, then $\pi_i\alpha \in \pi(\omega)$.*

Define $\pi(n)$ in the same manner, but with the restriction that $i < n$.

In [72] (lemma 2.11), the following theorem is shown. Therefore, one can interpret $\pi_0(n)$ as a notation system for $\omega_n[1]$ if $n > 0$ and $\pi_0(\omega)$ as a system for ε_0.

Theorem 1.39.
1. $\pi_0\Omega_n = \omega_n[1]$ *if* $n > 0$,
2. $\pi_0\Omega_\omega = \varepsilon_0$.

Using the ϑ_i-collapsing functions

In this subsection, we give ordinal representation systems that are based on the ϑ_i-functions (see section 1.2.4). We introduce them without the addition-operator.

Definition 1.40. *Define T and the function S simultaneously as follows. T is the least set such that $0 \in T$, where $S(0) := -1$, and if $\alpha \in T$ with $S(\alpha) \leq i + 1$, then $\vartheta_i\alpha \in T$ and $S(\vartheta_i\alpha) := i$. We call the number of occurrences of symbols ϑ_j in $\alpha \in T$ the* ***length*** *of α and denote this by $lh(\alpha)$. Furthermore, let $\Omega_i := \vartheta_i 0$.*

So in this context Ω_i is *not* equal to the i^{th} uncountable regular cardinal, but it can be interpreted as it, like in the D_i-case. $S(\alpha)$ represents the index i of the first occurring ϑ_i in α if $\alpha \neq 0$.

Definition 1.41. *Let $n < \omega$. Define T_n as the set of elements α in T such that for all ϑ_j which occur in α, we have $j < n$. Let $T[m]$ be the set of elements α in T such that $S(\alpha) \leq m$. Define $T_n[m]$ accordingly.*

For example $T_1 = T_1[0] = \{0, \vartheta_0 0, \vartheta_0\vartheta_0 0, \dots\}$. For every element α in T, we define its *coefficients*. The definition is based on the usual definition of the coefficients in a notation system *with* addition.

Definition 1.42. *Let $\alpha \in T$. If $\alpha = 0$, then $k_i(0) := 0$. Assume $\alpha = \vartheta_j(\beta)$. Let $k_i(\alpha)$ then be*

$$\begin{cases} \alpha & \text{if } j \leq i, \\ k_i(\beta) & \text{if } j > i. \end{cases}$$

Using this definition, we introduce a well-ordering on T (and its substructures). This ordering is based on the usual ordering between the ϑ_i-functions defined with addition.

Definition 1.43.

1. *If $\alpha \neq 0$, then $0 < \alpha$,*
2. *if $i < j$, then $\vartheta_i\alpha < \vartheta_j\beta$,*
3. *if $\alpha < \beta$ and $k_i\alpha < \vartheta_i\beta$, then $\vartheta_i\alpha < \vartheta_i\beta$,*
4. *if $\alpha > \beta$ and $\vartheta_i\alpha \leq k_i\beta$, then $\vartheta_i\alpha < \vartheta_i\beta$.*

Notation 1.44. *If $\alpha, \gamma \in T$ and $\gamma < \Omega_1$, let $\alpha[\gamma]$ be the element of T where the last zero in α is replaced by γ.*

The following lemma gives some useful properties of this ordinal notation system.

Lemma 1.45. *For all α, β and γ in T and for all $i < \omega$,*

1. *$k_i(\alpha) \leq \alpha$,*
2. *if $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} t$ with $j_1, \dots, j_n \geq i$ and ($t = 0$ or $t = \vartheta_k t'$ with $k \leq i$), then $t < \vartheta_i(\alpha)$,*
3. *$k_i(\alpha) < \vartheta_i\alpha$,*
4. *$k_i(\alpha)[\gamma] = k_i(\alpha[\gamma])$ for $\gamma < \Omega_1$,*
5. *if $\gamma < \Omega_1$, then $\gamma \leq \beta[\gamma]$ and there is only equality if $\beta = 0$,*
6. *if $\alpha < \beta$ and $\gamma < \Omega_1$, then $\alpha[\gamma] < \beta[\gamma]$.*

Proof. 1. The first assertion is easy to see.

2. By induction on $lh(\alpha)$ and sub-induction on $lh(t)$. If $\alpha = 0$, then the claim is trivial. Assume from now on $\alpha > 0$. If $t = 0$ or $t = \vartheta_k t'$ with $k < i$, then this is trivial. Assume $t = \vartheta_i t'$. Then $t = \vartheta_i \vartheta_{l_1} \dots \vartheta_{l_m} k_i(t')$ with $l_1, \dots, l_m > i$. The sub-induction hypothesis, $lh(k_i(t')) < lh(t)$ and $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} \vartheta_i \vartheta_{l_1} \dots \vartheta_{l_m} k_i(t')$ yield $k_i(t') < \vartheta_i \alpha$. If $t' < \alpha$, then $t = \vartheta_i t' < \vartheta_i \alpha$. Assume $t' > \alpha$. Note that equality is impossible because t' is a strict subterm of α. We claim that $t = \vartheta_i t' \leq k_i(\alpha)$, hence we are done. We know $k_i(\alpha) = \vartheta_{j_p} \dots \vartheta_{j_n} \vartheta_i t'$ for a certain p with $j_p = i$ or $k_i(\alpha) = \vartheta_i t'$. In the latter case, the claim is trivial. In the former case, the main induction hypothesis on $\vartheta_{j_{p+1}} \dots \vartheta_{j_n} \vartheta_i t'$ yields $t < \vartheta_i \vartheta_{j_{p+1}} \dots \vartheta_{j_n} \vartheta_i t' = k_i(\alpha)$.

3. This follows easily from the second assertion because $\alpha = \vartheta_{j_1} \dots \vartheta_{j_n} k_i(\alpha)$ with $j_1, \dots, j_n > i$.

4. Follows easily by induction on $lh(\alpha)$.

5. By induction on $lh(\gamma)$ and sub-induction on $lh(\beta)$. If $\gamma = 0$, the statement is trivial to see. From now on, let $\gamma = \vartheta_0 \gamma'$. If $\beta = 0$ or $\beta = \vartheta_i \beta'$ with $i > 0$, the statement also easily follows. Assume $\beta = \vartheta_0 \beta'$. We see $\beta[\gamma] = \vartheta_0(\beta'[\gamma])$. Suppose $\gamma' < \beta'[\gamma]$. Assume $\gamma' = \vartheta_{j_1} \dots \vartheta_{j_k} k_0(\gamma')$ with $j_1, \dots, j_k > 0$ and define $\overline{\beta}$ as $\beta[\vartheta_0 \vartheta_{j_1} \dots \vartheta_{j_k} 0]$. The main induction hypothesis yields $k_0(\gamma') \leq \overline{\beta}[k_0(\gamma')] = \beta[\gamma] = \vartheta_0(\beta'[\gamma])$. Note that equality is not possible because $k_0(\gamma')$ is a strict subterm of $\overline{\beta}[k_0(\gamma')]$, hence $\gamma = \vartheta_0 \gamma' < \vartheta_0(\beta'[\gamma]) = \beta[\gamma]$. Assume $\gamma' > \beta'[\gamma]$. The sub-induction hypothesis yields $\gamma \leq k_0(\beta')[\gamma] \overset{\gamma \leq \Omega_1}{=} k_0(\beta'[\gamma])$. Hence, $\gamma \leq k_0(\beta'[\gamma]) < \vartheta_0(\beta'[\gamma]) = \beta[\gamma]$.

6. By induction on $lh(\alpha) + lh(\beta)$. If $\alpha = 0$ and $\beta \neq 0$, then the previous assertion yields $\alpha[\gamma] = \gamma < \beta[\gamma]$. Assume $\alpha = \vartheta_i \alpha' < \vartheta_j \beta' = \beta$. If $i < j$, then also $\alpha[\gamma] < \beta[\gamma]$. Suppose $i = j$. Then either $\alpha' < \beta'$ and $k_i(\alpha') < \vartheta_j \beta'$, or $\alpha \leq k_j(\beta')$. In the former case, the induction hypothesis yields $\alpha'[\gamma] < \beta'[\gamma]$ and $k_i(\alpha'[\gamma]) \overset{\gamma \leq \Omega_1}{=} k_i(\alpha')[\gamma] < (\vartheta_j \beta')[\gamma] = \vartheta_j(\beta'[\gamma])$. Hence, $\alpha[\gamma] = (\vartheta_i \alpha')[\gamma] = \vartheta_i(\alpha'[\gamma]) < \vartheta_j(\beta'[\gamma]) = (\vartheta_j \beta')[\gamma] = \beta[\gamma]$. In the latter case, the induction hypothesis yields $\alpha[\gamma] \leq k_j(\beta')[\gamma] \overset{\gamma \leq \Omega_1}{=} k_j(\beta'[\gamma]) < \vartheta_j(\beta'[\gamma]) = (\vartheta_j \beta')[\gamma] = \beta[\gamma]$.

□

1.2.6 Theories and reverse mathematics

One of the most fundamental notions in proof theory is the *proof-theoretic ordinal* of a logical system T. Roughly speaking, the sub-discipline ordinal analysis assigns ordinals in a given representation system to formal theories of logic [1, 59, 63]. Following Gentzen in [32], we nowadays define the proof-theoretic ordinal of a formal theory T to be the supremum of the order types of primitive recursive well-orders on the natural numbers whose well-foundedness can be derived from the axioms of T. This ordinal is denoted by $|T|$ and is also called the Π^1_1-ordinal of the theory (see e.g. [63]).

The most standard first order theory is the one named after Giuseppe Peano: *Peano Arithmetic*, denoted by PA. The theory PA consists of the axioms needed for 0 and for the functions of summation and multiplication between the natural numbers. Furthermore, the theory PA contains an induction scheme over the natural numbers:

$$[F(0) \wedge \forall n(F(n) \to F(n+1))] \to \forall n F(n),$$

where F is an arbitrary formula in the language of arithmetic. If we restrict the number of allowed unbounded quantifiers in F, let us say bounded by n, we obtain the theories called $I\Sigma_n$. The union of these theories is of course the original theory PA. The proof-theoretic ordinal of $I\Sigma_n$ is equal to ω_{n+1}[1]. The ordinal of PA is the supremum of these ordinals, namely ε_0.

Reverse mathematics

Reverse mathematics is a program in the foundations of mathematics introduced by Friedman [26], which aims to classify the theorems of ordinary mathematics by their proof-theoretic strengths. Theorem φ is considered to be stronger than theorem ψ if φ requires stronger axioms to prove than ψ does. Or, equivalently, if φ implies ψ but not conversely over some fixed weak base theory.

A remarkable phenomenon is that there are only five theories (called the 'Big Five': RCA_0, WKL_0, ACA_0, ATR_0 and Π^1_1-CA_0) such that almost all ordinary theorems φ are equivalent with one of these five over RCA_0. The name 'reverse mathematics' comes from the fact that one not only looks at the question if $T \vdash \varphi$, but also that one searches for the minimal axioms needed to prove φ, i.e. that also $\mathsf{RCA}_0 + \varphi \vdash T$ holds. The literature of reverse mathematics is too vast to oversee, but we do mention the study of ordinals [40, 42, 77] and the study of well-partial-orders [17, 30, 52, 54], especially Kruskal's theorem [66]. We refer to [77] for a more detailed and comprehensive overview about reverse mathematics.

Before we can introduce these big five systems of reverse mathematics, we need to know over what language we define everything. Without stepping into much details (for more details see [77]), the *language of second order arithmetic* L_2 consists of two distinct sorts of variables. The first one (denoted by small letters like x, y) intends to range over the natural numbers and the second one (denoted by capital letters like X, Y) over all subsets of the natural numbers. The axioms of second order arithmetic consist of the *basic axioms*, stating that $\mathbb{N}$ is a discretely ordered commutative semi-ring with a unit. More precisely, the basic axioms are the sentences

$$\begin{aligned}
&\forall m(m+1 \neq 0)\\
&\forall m \forall n(m+1 = n+1 \rightarrow m = n)\\
&\forall m(m+0 = m)\\
&\forall m \forall n(m+(n+1) = (m+n)+1)\\
&\forall m(m \cdot 0 = 0)\\
&\forall m \forall n(m \cdot (n+1) = m \cdot n + m)\\
&\forall m \neg(m < 0)\\
&\forall m \forall n(m < n+1 \leftrightarrow m < n \vee m = n)
\end{aligned}$$

Furthermore, second order arithmetic contains the *induction axiom*

$$\forall X[(0 \in X \wedge \forall n(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n(n \in X)]$$

and the *comprehension scheme*

$$\exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where we take the universal closure of the last formula. φ is an arbitrary formula in the language of second order arithmetic.

The axioms of RCA_0 (Recursive Comprehension Axiom) consist of the basic axioms, the Σ^0_1-induction scheme and the Δ^0_1-comprehension scheme. More precisely, it contains the universal closures of all formulas of the form

$$[\varphi(0) \wedge \forall n(\varphi(n) \rightarrow \varphi(n+1))] \rightarrow \forall n \varphi(n),$$

where φ is Σ^0_1, and the universal closures of all formulas of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where φ is Σ^0_1, ψ is Π^0_1, and X is not free in φ. Despite that RCA_0 is being a weak system, several interesting and familiar theorems are provable in RCA_0. For example RCA_0 proves that $\mathbb{N}^m$ is a well-partial-order, where $\mathbb{N}^m$ is defined with the cartesian product (see Sections 1.2.7 and 1.2.8 for the notions of well-partial-orders and cartesian products). For more examples of provable statements in RCA_0, see e.g. [77]. It can be proved that the proof-theoretical ordinal of RCA_0 is the ordinal ω^ω (see e.g. [77]).

As mentioned before, reverse mathematics deals with equivalences of formulas with theories over a weak base theory and quite often, we use RCA_0 as that base theory. RCA_0 can be seen as a foundation of *what a normal computer can handle*: is deals with *recursive* sets. Sometimes, people weaken the base theory (e.g. by restricting the induction scheme) and investigate if the equivalences still hold. An example of such a weaker base theory is RCA_0^*. The language of RCA_0^* is that of RCA_0 but it is augmented by a binary operation symbol *exp* that denotes the exponential function. This language is denoted by $L_2(exp)$. The axioms of RCA_0^* consist of the basic axioms, the exponentiation axioms, the Δ^0_1-comprehension scheme and the Σ^0_0-induction scheme. The groundwork for this has been laid in [78].

The theory WKL_0 is RCA_0 augmented with the principle WKL (Weak König's Lemma) stating that every infinite subtree of 2^* has an infinite path. WKL_0 has the same proof-theoretic ordinal as RCA_0, namely ω^ω (see e.g. [77]).

The axioms of ACA_0 (Arithmetic Comprehension Axiom) are those of RCA_0 augmented with the arithmetical comprehension scheme, which consists of the universal closures of all formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where φ is an arithmetical formula (meaning no set-quantifiers) in which X is not free. Many familiar theorems are equivalent to ACA_0 over RCA_0 (see e.g. [77]). An example in the context of this dissertation is Higman's lemma (see Lemma 1.63). The ordinal of the theory ACA_0 is the first epsilon number, namely ε_0 (see e.g. [71, 79]). One can prove that over RCA_0, ACA_0 is equivalent with

For every set X, the n^{th} *Turing jump* of X exists,

where n is fixed from the outside [77]. Define ACA_0' as the theory ACA_0 augmented with the axiom

$$\forall n (\text{for every set } X, \text{ the } n^{\text{th}} \text{ Turing jump of } X \text{ exists}),$$

The proof-theoretic ordinal of ACA_0' is ε_ω, the ω^{th} epsilon number.

The theory ATR_0 (Arithmetical Transfinite Recursion) is equal to ACA_0 together with the assertion that arithmetic comprehension can be iterated along any countable well-ordering. We do not formalize ATR_0 because we do not need it in this dissertation. ATR_0 is strong enough to develop a good theory of countable ordinals. For example, Friedman showed that ATR_0 is equivalent over RCA_0 to the assertion about the comparability of well-ordering, i.e. it states that for every two well-order X and Y, either X is (isomorphic to) an initial segment of Y, or Y is (isomorphic to) an initial segment of X (see e.g. [27]). The proof-theoretic ordinal of ATR_0 is Γ_0, the limit of predicativity (see e.g. [29]).

The axioms of Π^1_1-CA_0 (Π^1_1-Comprehension Axiom) are those of ACA_0 augmented by the Π^1_1-comprehension scheme, i.e. all universal closures of the formulas of the form

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where φ is now a Π^1_1-formula in which X is not free. The ordinal of Π^1_1-CA_0 is $\sup_n \vartheta_0 \dots \vartheta_{n-1}\Omega_n$ (see e.g. [13, 14, 71, 79]).

1.2.7 Well-partial-orderings

In [47], well-partial-orderings are claimed to be among the most frequently rediscovered objects. In logic, well-partial-orders are used for studying ordinal notation systems and independence results. Furthermore, they are used in computer science to prove the termination of rewrite systems [87] and to calculate running times. Additionally, they have applications in transitions systems [7]. In algebra, more specifically computer algebra, these structures are used for investigations on Gröbner bases [6], Braid groups [16] and commutative algebra [4]. In this section we recall some basic facts from the theory of well-partial-orderings.

Definition 1.46. *A* ***well-partial-ordering***, ***well-partial-order*** *or* $\mathbb{wpo}$ *is a partial order* $(X, \leq_X)$ *such that for every infinite sequence* $(x_i)_{i=1}^{+\infty}$ *of elements of* X*, there exist two indices* i *and* j *such that* $i < j$ *and* $x_i \leq_X x_j$*. We denote the* $\mathbb{wpo}$ $(X, \leq_X)$ *by* X *if the ordering is clear from the context.*

Definition 1.47. *A sequence* $(x_i)_{i<\alpha}$ *of elements of a partial order* X *(with* $\alpha \leq \omega$*) is* ***bad*** *if for all* $i < j < \alpha$, $x_i \not\leq_X x_j$ *holds. If a sequence is not bad we call it good.*

Definition 1.48. *For a partial order X, let* $\mathrm{Bad}(X)$ *be the tree of finite bad sequences ordered by inclusion: $s \leq t$ iff $s \sqsubseteq t$ (i.e. $lh(s) \leq lh(t)$ and $\forall i(i < lh(s) \to s_i = t_i)$). This implies that the empty sequence is the root (= the smallest element) of the tree* $\mathrm{Bad}(X)$.

In the literature one frequently encounters the similar notion of a well-*quasi*-ordering (which is the same as a well-partial-ordering, but it lacks anti-symmetry). This notion does not differ a lot from a wpo because every wpo is a well-quasi-order and after an obvious factorization, every well-quasi-order is a wpo.

There are other equivalent definitions of well-partial-orderings, however they are all not computably equivalent. For an overview of the equivalent definitions in a reverse mathematical setting, we refer the reader to [17, 30].

Lemma 1.49. *Let X be a partial order. Then the following are equivalent*

1. *X is a well-partial-order (i.e. it does not contain infinite bad sequences),*
2. *The tree* $\mathrm{Bad}(X)$ *is well-founded (i.e. it does not contain an infinite path through the tree),*
3. *X is well-founded and does not admit infinite anti-chains (which is a set of two-by-two incomparable elements).*
4. *Every extension of the partial order X to a linear ordering on X is a well-ordering.*
5. *Every extension of the partial order X to a partial ordering on X is well-founded.*
6. *Every infinite sequence of elements of X contains a weakly increasing subsequence.*
7. *X has the finite basis property (i.e. for every $S \subseteq X$, there exists a finite $F \subseteq S$ such that $\forall x \in S \exists y \in F(y \leq_X x)$).*

In a groundbreaking paper, de Jongh and Parikh [21] have been able to isolate a mathematical invariant of well-partial-orderings which is crucial in determining the proof-theoretic strength of well-partial-orderings.

Definition 1.50. *The* ***maximal order type*** *of the* wpo $(X, \leq_X)$ *is equal to*

$$\sup\{\alpha\colon \leq_X \subseteq \preceq,\ \preceq \text{ is a well-ordering on } X \text{ and } otype(X, \preceq) = \alpha\}.$$

We denote this ordinal by $o(X, \leq_X)$ or by $o(X)$ if the ordering is clear from the context.

The following theorem by de Jongh and Parikh [21] shows that this supremum is actually a maximum.

Theorem 1.51 (de Jongh and Parikh [21]). *Assume that $(X, \leq_X)$ is a $\mathbb{wpo}$. Then there exists a well-ordering $\preceq$ on X which is an extension of $\leq_X$ such that otype$(X, \preceq) = o(X, \leq_X)$.*

Definition 1.52. *A **maximal linear extension** of a $\mathbb{wpo}$ X is a well-ordering $\preceq$ on X that satisfies Theorem 1.51.*

The maximal order type is given by a set-theoretic definition. In case of concretely given well-partial-orderings, it is in quite a few times possible to calculate these ordinals more explicitly. To do so, it turns out to be useful to approximate well-partial-orderings by suitable subsets, the so-called 'left-sets' of elements.

Definition 1.53. *Let $(X, \leq_X)$ be a $\mathbb{wpo}$ and $x \in X$. Define the left-set $L(x)$ as the set $\{y \in X : x \not\leq_X y\}$ and $l(x) := o(L(x), \leq_X \restriction L(x))$.*

The role of these sets becomes clear by the following structural theorem.

Theorem 1.54 (de Jongh and Parikh [21]). *Assume that X is a partial order. If $L(x)$ is a $\mathbb{wpo}$ for every $x \in X$, then X is a $\mathbb{wpo}$. (The converse is trivially true.) In this case, $o(X) = \sup\{l(x) + 1 : x \in X\}$.*

Therefore, the maximal order type of a $\mathbb{wpo}$ X is equal to the height of the root of the tree of bad sequences $\mathrm{Bad}(X)$, and so in nice cases, the maximal order type can be calculated in a recursive way. To obtain bounds on maximal order types, it turns out to be useful to consider mappings which preserve well-partial-orderedness. We call these mappings quasi-embeddings.

Definition 1.55. *Let X and Y be two partial orders. A map $e : X \to Y$ is called a **quasi-embedding** if for all $x, x' \in X$ with $e(x) \leq_Y e(x')$, we have $x \leq_X x'$.*

This definition looks artificial at first sight but it turns out to be the appropriate notion to work with, as is indicated by the next lemma.

Lemma 1.56. *If X and Y are partial orders and $e : X \to Y$ is a quasi-embedding and Y is a $\mathbb{wpo}$, then X is a $\mathbb{wpo}$ and $o(X) \leq o(Y)$.*

A different method to obtain bounds uses reifications.

Definition 1.57. *A **reification** from a partial order X to a linear order α is a mapping $f : \mathrm{Bad}(X) \to \alpha + 1$ such that $f(s^\frown(u)) < f(s)$ for all $s, s^\frown(u) \in \mathrm{Bad}(X)$.*

Lemma 1.58. *A partial order X is a $\mathbb{wpo}$ if and only if there exists a reification f from $\mathrm{Bad}(X)$ to $\alpha+1$, where α is an ordinal number. Furthermore, in this case $o(X) \leq \alpha$.*

Proof. If there exists a reification $f : \mathrm{Bad}(X) \to \alpha + 1$ with α an ordinal number, then $\mathrm{Bad}(X)$ is well-founded and hence X is a $\mathbb{wpo}$. Furthermore, the height of the root of the tree $\mathrm{Bad}(X)$ is bounded above by α, hence $o(X) \leq \alpha$.

The other direction follows from defining $f(x_0 \dots x_n)$ as $o(\{y \in X : x_0 \not\leq_X y \wedge \dots \wedge x_n \not\leq_X y\})$ for $(x_0, \dots, x_n) \in \mathrm{Bad}(X)$. □

1.2.8 Examples of and constructors on well-partial-orderings

In this subsection, we discuss important $\mathbb{wpo}$'s and constructors on $\mathbb{wpo}$'s. From one or two fixed well-partial-orders, one can construct a dozen of other $\mathbb{wpo}$'s. We will not yet mention the orders with the gap-embeddability relation in this subsection because they are so important in this dissertation that we will discuss them in a separate section (see Section 1.2.9).

Definition 1.59. *Let X_0 and X_1 be two $\mathbb{wpo}$'s. Define the **disjoint sum** $X_0 + X_1$ as the set $\{(x, 0) : x \in X_0\} \cup \{(y, 1) : y \in X_1\}$ with the ordering:*

$$(x, i) \leq (y, j) \Longleftrightarrow i = j \text{ and } x \leq_{X_i} y.$$

The underlying set without the ordering is called the disjoint union*. For an arbitrary element (x, i) in $X_0 + X_1$, we omit the second coordinate i if it is clear from the context to which set the element x belongs to.*

*Define the **cartesian product** $X_0 \times X_1$ as the set $\{(x, y) : x \in X_0, y \in X_1\}$ with the ordering:*

$$(x, y) \leq (x', y') \Longleftrightarrow x \leq_{X_0} x' \text{ and } y \leq_{X_1} y'.$$

Recall that if we talk about the disjoint *union* of X_0 and X_1, we only speak about the set $\{(x, 0) : x \in X_0\} \cup \{(y, 1) : y \in X_1\}$.

Definition 1.60. *Let X^* be the set of* ***finite sequences*** *over X ordered by*

$$\begin{aligned} &(x_1,\ldots,x_n) \leq^*_X (y_1,\ldots,y_m)\\ \iff &(\exists 1 \leq i_1 < \cdots < i_n \leq m)(\forall j \in \{1,\ldots,n\})(x_j \leq_X y_{i_j}). \end{aligned}$$

If the underlying ordering on X is clear from the context, we write $\leq^*$ instead of $\leq^*_X$. The order $\leq^*$ is called the ***Higman order*** *on X^*. Sometimes, we even write X^* if we mean the partial order $(X^*, \leq^*)$. The context will indicate what we mean.*

Higman's original paper [38] studied the partial order $(X^*, \leq^*_X)$. He showed that it has the finite basis property, hence that it is a wpo, whenever X is a wpo. This partial order is a very important example in well-partial-order theory and is extensively studied in other subjects, e.g. in [25, 37, 74]. De Jongh, Parikh and Schmidt provided precise bounds for the maximal order types of these well-partial-orderings.

Theorem 1.61 (Higman [38], de Jongh and Parikh [21], Schmidt [69])**.** *If X_0, X_1 and X are wpo's, then $X_0 + X_1$, $X_0 \times X_1$ and X^* are still wpo's, and*

$$\begin{aligned} o(X_0 + X_1) &= o(X_0) \oplus o(X_1),\\ o(X_0 \times X_1) &= o(X_0) \otimes o(X_1), \end{aligned}$$

and

$$o(X^*) = \begin{cases} \omega^{\omega^{o(X)-1}} & \text{if } o(X) \text{ is finite,}\\ \omega^{\omega^{o(X)+1}} & \text{if } o(X) = \varepsilon + n, \text{ with } \varepsilon \text{ an epsilon number and } n < \omega,\\ \omega^{\omega^{o(X)}} & \text{otherwise.} \end{cases}$$

Furthermore, there are known results concerning the reverse mathematical strength of Higman's lemma. First, we define ω^X following [53].

Definition 1.62. *Let X be a linear order. Define ω^X as the subset of X^* such that $(x_0,\ldots,x_{n-1}) \in \omega^X$ if $x_0 \geq_X \cdots \geq_X x_{n-1}$. Define the ordering on ω^X as the lexicographic one: $(x_0,\ldots,x_{n-1}) \leq_{\omega^X} (y_0,\ldots,y_{m-1})$ if either $n \leq m$ and $x_i = y_i$ for all $i \leq n$, or there exists a $j < \min\{n,m\}$ such that $x_j <_X y_j$ and $x_i = y_i$ for all $i < j$.*

The idea behind the previous definition is the Cantor normal form. Every ordinal $\alpha > 0$ is equal to

$$\alpha =_{CNF} \omega^{\alpha_0} + \cdots + \omega^{\alpha_{n-1}},$$

with $\alpha_0 \geq \cdots \geq \alpha_{n-1} \geq 0$.

Lemma 1.63. *Over* RCA_0*, the following are equivalent*

1. ACA_0,
2. $\forall X(X$ *is a well-order* $\to \omega^X$ *is a well-order),*
3. *Higman's lemma, i.e.* $\forall X(X$ *is a* $\mathbb{wpo} \to X^*$ *is a* $\mathbb{wpo})$*,*
4. $\forall X(X$ *is a well-quasi-order* $\to X^*$ *is a well-quasi-order).*

Proof. A proof of the equivalence of (1.) and (2.) goes back to Gentzen (e.g. see [59, 71]). In [33] and [40], one can find a proof of the fact that ACA_0 is equivalent over RCA_0 with $\forall X(X$ is a well-order $\to 2^X$ is a well-order), where 2^X is the sub-ordering of ω^X of *strictly* decreasing sequences over X. One can prove that this last statement is equivalent with (2.).

One can also prove that the statement $\forall X(X$ is a well-order $\to 2^X$ is a well-order) is equivalent with Higman's lemma. For a detailed version see [19].

It is trivial to show that (3.) and (4.) are equivalent. □

We now consider two different embeddings on multisets. The first one is called the *term ordering* by Aschenbrenner and Pong [4]. The second one is named the *multiset ordering* in the term rewriting community. We define the set of multisets $Multi(X)$ as the set X^* together with the straightforward equivalence relation induced by permutability. Most of the time, we denote an element of $Multi(X)$ by an element of X^* and not as an equivalence class. If a property P is valid for the multiset m, we mean that it is valid for every m' equivalent with m. In the next definitions, if $m \leq m'$, then $\overline{m} \leq \overline{m}'$ for every $\overline{m}$ equivalent with m and every $\overline{m}'$ equivalent with m'. So this actually means $[m] \leq [m']$, where $[\cdot]$ is the corresponding equivalence class.

Definition 1.64. *Define the partial order* $M^\diamond(X)$ *as* $(Multi(X), \leq_X^\diamond)$*, where*

$$m \leq_X^\diamond m' \iff (\exists f : m \hookrightarrow m')(\forall x \in m)[x \leq_X f(x)].$$

The symbol $\hookrightarrow$ *means that* f *is an injective function. We also denote* $\leq_X^\diamond$ *by* $\leq^\diamond$ *if the underlying ordering on* X *is clear from the context.* $M^\diamond(X)$ *is the* ***term ordering****.*

Definition 1.65. *Define the partial order* $M(X)$ *as* $(Multi(X), \underline{\leq_X \leq_X})$*, where*

$$m \,\underline{\leq_X \leq_X}\, m' \iff m = m' \text{ or } (\forall x \in m \backslash (m \cap m'))(\exists y \in m' \backslash (m \cap m'))(x \leq_X y),$$

where $\setminus$ and $\cap$ refer to multiset operations. We sometimes denote $\underline{<_X<_X}$ by $\underline{<<}$ if the underlying ordering on X is clear from the context. $M(X)$ is called the ***multiset ordering***

These two multiset-constructors on well-partial-orderings produce again $\mathbb{wpo}$'s (since there is a quasi-embedding to Higman's X^*) and their maximal order types in terms of $o(X)$ are known.

Theorem 1.66. *Let $(X, \leq_X)$ be a $\mathbb{wpo}$. Then $M(X)$ is also a $\mathbb{wpo}$ and $o(M(X)) = \omega^{o(X)}$.*

Proof. In [87], Andreas Weiermann proved that $M(X)$ is a $\mathbb{wpo}$ (under the assumption that X is well-partial-ordered) and $o(M(X)) \leq \omega^{o(X)}$. A proof of the lower bound can be found in our joint article (Theorem 4 in [82]). □

For describing the maximal order type of $M^\diamond(X)$, we need some additional notations.

Definition 1.67. *Let α be an ordinal. Define α' by*

$$\alpha' := \begin{cases} \alpha + 1 & \textit{if } \alpha = \varepsilon + n, \textit{ with } \varepsilon \textit{ an epsilon number and } n < \omega, \\ \alpha & \textit{otherwise.} \end{cases}$$

Notation 1.68. *Let $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ be an ordinal. We use the notation $\widehat{\alpha}$ for the ordinal $\omega^{\alpha_1'} + \cdots + \omega^{\alpha_n'}$. Note that $\widehat{\alpha \oplus \beta} = \widehat{\alpha} \oplus \widehat{\beta}$ and that $\alpha < \beta$ implies $\widehat{\alpha} < \widehat{\beta}$. Also $\widehat{0} := 0$.*

Theorem 1.69. *Let $(X, \leq_X)$ be a well-partial-ordering. Then $M^\diamond(X)$ is also a $\mathbb{wpo}$ and $o(M^\diamond(X)) = \omega^{\widehat{o(X)}}$.*

Proof. The original proof of Theorem 1.69 can be found in [90]. However, that proof contains a small error for some exceptional cases. The author, Weiermann, already sketched a correction. In our joint article, the interested reader can find a completely corrected proof (Theorem 5 in [82]). □

Kruskal's tree theorem states that the set of finite trees over a well-quasi-ordered set of labels is itself well-quasi-ordered (under homeomorphic embedding). The theorem was conjectured by Vázsonyi in 1937 and proved later in 1960 by Kruskal [46]. Independently, a proof of the same theorem was announced by Tarkowski [80]. In 1963, Nash-Williams gave a different, but short and beautiful proof in [55] based on minimal bad sequences. The minimal bad sequence argument makes the proofs more appealing and it is a method that possesses a lot of strength. More precisely, Marcone [51] showed that the general version of that argument has the strength of Π^1_1-CA_0.

Definition 1.70. *A* ***rooted tree*** *is a partial order* $(T, \leq)$ *such that it has one minimal element (called the root) and for all* $t \in T$*, the set* $\{s \in T : s \leq t\}$ *is linearly ordered. The elements of* T *are called the nodes or vertices of* T *and most of the time, we write* T *instead of* $(T, \leq)$*. The* successors *of a vertex* t *of* T *are the elements* s *such that* $t < s$*. The* immediate successors *are the successors* s *such that there is no other successor* s' *such that* $t < s' < s$*. The* predecessor *of* t*, when* t *is not the root, is defined as the unique element* $s < t$ *where* t *is an immediate successor of* s*. A* leaf *of a tree is a node with no successors. If* $s \in T$*, define* T_s *as the subtree* $\{t : t \geq s\}$*. If* s *is an immediate successor of the root of* T*, then* T_s *is called an* immediate subtree *of* T*. A* path *through a rooted tree* T *from* a *to* b *(*$T \ni a < b \in T$*), is a function* $f : \{0, \dots, k\} \to T$ *with* $k \leq \omega$ *such that* $f(0) = a$ *and* $f(i+1)$ *is an immediate successor of* $f(i)$ *if* $f(i) \neq b$ *such that* $b \in T_{f(i+1)}$*. If* $f(i) = b$*, we do not define* $f(i+1)$*. A path through a rooted tree* T *starting from* a *(*$a \in T$*), is a function* $f : \{0, \dots, k\} \to T$ *with* $k \leq \omega$ *such that* $f(0) = a$ *and* $f(i+1)$ *is an immediate successor of* $f(i)$ *if* $f(i)$ *is not a leaf. If* $f(i)$ *is a leaf, we do not define* $f(i+1)$*. A tree is* well-founded *if there is no infinite path through the tree, i.e. if* $f : \{0, \dots, k\} \to T$ *is a path, then* $k < \omega$*. A* ***labeled rooted tree*** *with labels in* X *is a rooted tree* T *together with a labeling function* l *from* T *to* X*. Most of the time, we do not mention this labeling function explicitly. A* ***structured rooted tree*** *is a rooted tree* T *together with a relation which, for each vertex* t *of* T*, well-orders the set of* t*'s immediate successors. An unstructured rooted tree is just a normal rooted tree.*

Notation 1.71. *Let* $\mathbb{T}$ *be the set of the finite rooted trees and* $\mathbb{T}^{\leq m}$ *the set of rooted trees such that every node in the trees has at most* m *many immediate successors. Let* $\mathbb{T}(X)$ *be the set of finite rooted trees with labels in* X*. Define* $\mathbb{T}^{\leq m}(X)$ *in a similar way.*

Let $\mathbb{T}(X_0, \dots, X_n)$ *be the set of finite rooted trees with labels in the disjoint union* $X_0 + \dots + X_n$ *such that every node* t *in a tree* $T \in \mathbb{T}(X_0, \dots, X_n)$ *with exactly* i *immediate successors has a label in* X_i*. This yields* $\mathbb{T}(X_0, \dots, X_n) \subseteq \mathbb{T}^{\leq n}(X_0 + \dots + X_n)$*.*

Let $\mathbb{T}\begin{pmatrix} X_0 & \dots & X_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ *with* $0 < \alpha_0 < \dots < \alpha_n \leq \omega$ *and* $n \geq 0$ *be the set of finite rooted trees with labels in* $X_0 + \dots + X_n$ *such that for every node* t *in* *a tree* $T \in \mathbb{T}\begin{pmatrix} X_0 & \dots & X_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ *the following holds: if the label of* t *is in the set* X_i*, then* t *has strictly less than* α_i *immediate successors.*

If we talk about structured trees, we will add a superscript 's', e.g. $\mathbb{T}^s(X_0, \dots, X_n)$.

Notation 1.72. *Let* $\mathbb{T}_n$ *denote the set* $\mathbb{T}(\{0, \dots, n-1\})$ *of finite rooted trees with labels in* $\{0, \dots, n-1\}$*. Accordingly, define* $\mathbb{T}_n^{\leq m}$ *in a similar way.*

Note that $\mathbb{T} \cong \mathbb{T}(\{0\}) = \mathbb{T}_1$.

Definition 1.73. *Let* T_1 *and* T_2 *be two rooted trees. Then* T_1 *is* ***(homeomorphically) embeddable*** *into* T_2 *(we write this as* $T_1 \leq T_2$*) if there is an injective function* h *from* T_1 *to* T_2 *such that*

1. *h preserves the ordering:* $s <_{T_1} t$ *iff* $h(s) <_{T_2} h(t)$ *for all* $s, t \in T_1$*,*
2. *h preserves infima: for any* $t \in T_1$ *and distinct immediate successors* s_1 *and* s_2 *of* t*, the paths from* $h(t)$ *to* $h(s_1)$ *and* $h(t)$ *to* $h(s_2)$ *do not have any vertices in common, except for* $h(t)$ *itself.*

We call the function h *the* ***(homeomorphic) embedding****. This embedding relation yields a partial order on* $\mathbb{T}$ *and* $\mathbb{T}^{\leq m}$.

Assume that we add labels to our trees: suppose X *is a partial order and let* T_1, T_2 *be both in* $\mathbb{T}(X)$*. Assume that* l_i *is the labeling function from* T_i *to* X*. Then* $T_1 \leq T_2$ *if there is a function* h *that satisfies the above conditions such that* $\forall t \in T_1(l_1(t) \leq_X l_2(h(t)))$*. One can also define this natural embedding relation on* $\mathbb{T}(X_0, \dots, X_n)$ *and* $\mathbb{T}\begin{pmatrix} X_0 & \dots & X_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ *in a similar way, by looking at the labeling functions from the trees to* $X_0 + \dots + X_n$*, where* $X_0, \dots, X_n$ *are partial orders.*

If we consider structured trees, then h *also needs to fulfill one extra property: it has to preserve the horizontal relation, i.e. if* $t \in T_1$ *and* s_1 *and* s_2 *are immediate successors of* t *with* $s_1 \prec s_2$*, where* $\prec$ *is the well-order on* t*'s immediate successors, then* $u_1 \prec' u_2$ *where*

- u_i *is the immediate successor of* $h(t)$ *on the path from* $h(t)$ *to* $h(s_i)$*,*
- $\prec'$ *is the well-order on* $h(t)$*'s immediate successors.*

Most of the time in this dissertation, we work with finite trees.

Remark 1.74. Assume that we work with a finite rooted tree. Often, we do not use this tree as a partial order, but we interpret this tree as a term where one can see its immediate subtrees. The term $\cdot$ represents the tree consisting of one node. The term $\cdot(t_1, \dots, t_n)$ represents the tree with immediate subtrees $T_1, \dots, T_n$, where T_i is the tree that is represented by the term t_i. If we work with unstructured trees, the terms $\cdot(t_1, t_2, \dots, t_n)$ and $\cdot(t_2, t_1, \dots, t_n)$ represent the same tree. Therefore, we see $\mathbb{T}$ as this set of terms together with a specific, but obvious, equivalence relation and if we work with an unstructured tree, we mean that we work with a term up to this equivalence relation. If we work with structured trees, the terms $\cdot(t_1, t_2, \dots, t_n)$ and $\cdot(t_2, t_1, \dots, t_n)$ represent different trees, unless $t_1 = t_2$. Therefore, we define $\mathbb{T}^s$ as this set of terms where the equivalence relation is the identity. If the root has a label x, then we denote this by $\cdot_x$ and $\cdot_x(t_1, \dots, t_n)$. Hence, one can also see $\mathbb{T}(X)$, $\mathbb{T}^s(X)$, etc. as sets of terms with an equivalence relation (that is the identity in the case of structured trees).

This interpretation yields the following lemma.

Lemma 1.75. *Let T and T' be two finite rooted trees. Then $T \leq T'$ iff either T is embeddable in an immediate subtree of T' or there exists an injective function f from the set of immediate subtrees of T to the set of immediate subtrees of T', such that for every immediate subtree S from T, $S \leq f(S)$ holds.*

With regard to this interpretation, it is useful to talk about the height of a tree T.

Definition 1.76. *Let T be a rooted well-founded tree. For every $s \in T$, define $ht(s) := \sup\{ht(t) + 1 : s < t\}$. Define the height of a tree T as $ht(root(T))$ and denote it by $ht(T)$.*

Now we have enough equipment to state Kruskal's theorem.

Theorem 1.77 (Kruskal [46])**.** *If X is a* wpo*, then $(\mathbb{T}(X), \leq)$ is also a* wpo*.*

Proof. A beautiful non-constructive proof of this theorem, based on the *minimal bad sequence* argument, can be found in [55]. On the other hand, Kruskal's original proof in his thesis is constructive. □

Note that Higman's lemma mentioned in Lemmas 1.61 and 1.63, follows from the previous theorem since a sequence can be seen as a linear tree (i.e. every node has zero or one immediate successor(s)). In some sense, a generalization from trees to arbitrary graphs is given by the Robertson-Seymour theorem, also known as the graph minor theorem [28, 67], although the restriction of this order relation (which is the minor relation) is not the same as Kruskal's usual relation on trees. Before Robertson and Seymour proved it, it was known as Wagner's conjecture [85].

In her Habilitationsschrift, Diana Schmidt considered the tree-classes $\mathbb{T}^s(X)$, $\mathbb{T}^s(X_0, \dots, X_n)$ and $\mathbb{T}^s\begin{pmatrix} X_0 & \dots & X_n \\ \alpha_0 & \dots & \alpha_n \end{pmatrix}$ and calculated their maximal order types.

Definition 1.78. *If α is an ordinal number, then*

$$\overline{\alpha} := \begin{cases} \alpha - 1 & \text{if } \alpha < \omega, \\ \alpha + 1 & \text{if } \alpha = \varphi_2\beta + n \text{ with } n \text{ a natural number,} \\ \alpha & \text{otherwise.} \end{cases}$$

Recall Definition 1.19.

Theorem 1.79 (Diana Schmidt [69])**.** *If $X, X_0, \dots, X_n$ are* wpo*'s and $0 \leq \alpha_0 < \dots < \alpha_n \leq \omega$, then $\mathbb{T}^s(X)$, $\mathbb{T}^s(X_0, \dots, X_n)$ and $\mathbb{T}^s\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}$ are also* wpo*'s and if $X, X_0, \dots, X_n$ are countable, then*

$$o(\mathbb{T}^s(X_0, \dots, X_n)) \leq \vartheta(\Omega^n \cdot o(X_n) + \dots + \Omega \cdot o(X_1) + (-1 + o(X_0))),$$

$$o(\mathbb{T}^s(X, \emptyset, \{0\})) = \varepsilon_{\overline{o(X)}},$$

$$o\left(\mathbb{T}^s\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}\right) \leq \vartheta(\Omega^{\alpha_n} \cdot o(X_n) + \dots + \Omega^{\alpha_0} \cdot o(X_0)),$$

where the first inequality is an equality if

$$\Omega^n \cdot o(X_n) + \dots + \Omega \cdot o(X_1) + (-1 + o(X_0)) \geq \Omega^3.$$

Her results yield $o(\mathbb{T}^s(X)) \leq \vartheta(\Omega^\omega \cdot o(X))$ and one can prove easily that this is an equality. We note that she does not mention if $\vartheta(\Omega^{\alpha_n} \cdot o(X_n) + \dots + \Omega^{\alpha_0} \cdot o(X_0))$ is a lower bound for $o\left(\mathbb{T}^s\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}\right)$.

Diana Schmidt did not restrict herself to only countable wpo's. Her results are also valid for arbitrary well-partial-orderings because she used a different ordinal notation system than the ϑ-function, namely the one based on Schütte's Klammer symbols. Note that the right-hand sides of the inequalities in Theorem 1.79 can be undefined if the X_i's are not necessarily countable wpo's.

In Chapter 2, we prove that the unstructured versions of some mentioned tree-classes have the same maximal order types as the structured versions.

In this context, it is also worth to mention that Kruskal's theorem is not provable in predicative analysis.

Theorem 1.80 (Friedman [76]). $\mathsf{ATR}_0 \nvdash$ '$\mathbb{T}$ *is a* $\mathbb{wpo}$'.

The previous theorem is shown by giving a primitive recursive mapping o from $\mathbb{T}$ to Γ_0 (the ordinal of ATR_0) that is order-preserving and surjective. In [66] Rathjen and Weiermann investigated the reverse mathematical strength of Kruskal's theorem without labels.

1.2.9 Well-partial-orders with gap-condition

In 1982, Harvey Friedman introduced a well-partial-ordering, where the ordering is called the gap-embeddability relation or in short gap-ordering, on the set of finite rooted trees with labels in $\{0, \ldots, n-1\}$. This was later published by Simpson in [76]. This $\mathbb{wpo}$ is very important because it led to the first *natural* example of a statement not provable in the strongest theory of the *Big Five* in reverse mathematics, Π^1_1-CA_0.

Definition 1.81 (Friedman [76]). *On the set* $\mathbb{T}_n$*, define the following ordering, known as the gap-embeddability relation. Let* $T_1, T_2 \in \mathbb{T}_n$ *and assume that* l_i *is the labeling function of* T_i*. Then* $T_1 \leq^w_{gap} T_2$ *if there exists a homeomorphic embedding* h *from* T_1 *to* T_2 *such that*

1. $\forall t \in T_1$*, we have* $l_1(t) = l_2(f(t))$*.*
2. $\forall t \in T_1$ *and for all immediate successors* $t' \in T_1$ *of* t*, we have that if* $\bar{t} \in T_2$ *and* $f(t) < \bar{t} < f(t')$*, then* $l_2(\bar{t}) \geq l_2(f(t')) = l_1(t')$*.*

This ordering on $\mathbb{T}_n$ *is called the* ***weak gap-embeddability relation****. The partial ordering* $(\mathbb{T}_n, \leq^w_{gap})$ *is also denoted by* $\mathbb{T}^{wgap}_n$*. The* ***strong gap-embeddability relation*** *fulfills the extra condition*

3. *for all* $t' < f(root(T_1))$*, we have* $l_2(t') \geq l_2(f(root(T_1))) = l_1(root(T_1))$*.*

to the definition of $\leq^w_{gap}$*. The latter ordering on* $\mathbb{T}_n$ *is denoted by* $\leq^s_{gap}$ *We also write* $\mathbb{T}^{sgap}_n$ *for the partial order* $(\mathbb{T}_n, \leq^s_{gap})$*. If we restrict ourselves to structured rooted trees, then we denote this by* $\mathbb{T}^{s,wgap}_n$ *and* $\mathbb{T}^{s,sgap}_n$*.*

If we do not mention the word 'gap' in sub- or superscript, we mean that we work with the normal homeomorphic embeddability relation without the gap-condition. If we write $\leq_{gap}$, we mean that the theorem is valid for both $\leq^s_{gap}$ and $\leq^w_{gap}$.

Theorem 1.82 (Friedman [76]). *For all n, $(\mathbb{T}_n, \leq_{gap})$ is a $\mathbb{wpo}$ and $\Pi^1_1\text{-}\mathsf{CA}_0 \nvdash \forall n < \omega$ '$(\mathbb{T}_n, \leq_{gap})$ is a $\mathbb{wpo}$'.*

The previous theorem is proved by showing that ACA_0 proves for all n the following:

$$(\mathbb{T}_n, \leq^w_{gap}) \text{ is a } \mathbb{wpo} \rightarrow \psi_0(\Omega_n) \text{ is a well-order.}$$

The proof shows that the maximal order type of $(\mathbb{T}_n, \leq_{gap})$ is bounded from below by this ordinal. Theorem 1.82 was used in [28] to prove that the graph minor theorem is not provable in $\Pi^1_1\text{-}\mathsf{CA}_0$.

The linearized version has been studied extensively by Schütte and Simpson [72].

Definition 1.83 (Schütte-Simpson [72]). *Define $\mathbb{S}$ as the set $\mathbb{N}^*$ and $\mathbb{S}_n$ as $\{0, \dots, n-1\}^*$. We say that $s = s_0 \dots s_{k-1} \leq^w_{gap} s'_0 \dots s'_{l-1} = s'$ if there exists a strictly increasing function $f : \{0, \dots, k-1\} \rightarrow \{0, \dots, l-1\}$ such that*

1. *for all $0 \leq i \leq k-1$, we have $s_i = s'_{f(i)}$,*
2. *for all $0 \leq i < k-1$ and all j between $f(i)$ and $f(i+1)$, the inequality $s'_j \geq s'_{f(i+1)} = s_{i+1}$ holds.*

*This ordering on $\mathbb{S}_n$ is called the **weak gap-embeddability relation**. The partial ordering $(\mathbb{S}_n, \leq^w_{gap})$ is also denoted by $\mathbb{S}^{wgap}_n$. The **strong gap-embeddability relation** fulfills the extra condition*

3. *for all $j < f(0)$, we have $s'_j \geq s'_{f(0)} = s_0$.*

to the definition of $\leq^w_{gap}$. The latter ordering on $\mathbb{S}_n$ is denoted by $\leq^s_{gap}$ We also write $\mathbb{S}^{sgap}_n$ for the partial order $(\mathbb{S}_n, \leq^s_{gap})$.

Theorem 1.84 (Schütte-Simpson [72], Friedman [76]). *For all n, $(\mathbb{S}_n, \leq_{gap})$ is a $\mathbb{wpo}$.*

Theorem 1.85 (Schütte-Simpson [72]).
$\mathsf{ACA}_0 \nvdash \forall n < \omega$ *'$(\mathbb{S}_n, \leq_{gap})$ is a $\mathbb{wpo}$'.*

Theorem 1.86 (Schütte-Simpson [72]).
For all n, $\mathsf{ACA}_0 \vdash$ '$(\mathbb{S}_n, \leq_{gap})$ is a $\mathbb{wpo}$'.

We postpone the further discussion on the linearized version to Chapter 7.

1.2.10 Tree-constructors and Weiermann's conjecture

The main goal of this dissertation is to capture the maximal order types of structures with the gap-embeddability relation, especially for Friedman's trees. In order to address this problem, we introduce special tree-classes $\mathcal{T}(W)$ (originally in [86]), where W is a function symbol for a constructor on $\mathbb{wpo}$'s. Intuitively, W can be seen as a function from the class of partial orders to itself, such that if we restrict ourselves to $\mathbb{wpo}$'s, the image under W is still a $\mathbb{wpo}$. And if we restrain ourselves to the countable world, we maintain to stay in this world. However, instead of working with W as a function, we adapt an approach from universal algebra to define W as a *function symbol.* This idea resembles the use of primitive recursive function symbols in studying primitive recursive functions [18]. This subsection will introduce rigorously the notions that we need to work with $\mathcal{T}(W)$. The notions are defined in this way to completely delete any hand-waving. However, in latter chapters we will sometimes use a slightly different notation for our convenience.

Definition 1.87. *Assume that $Y_1, \dots, Y_k$ are fixed partial orderings. Define $\mathcal{W}(Y_1, \dots, Y_k)$ as the following set of function symbols and define $|\cdot|$ as a measure of the complexity of the symbols.*

- *For any $i = 1, \dots, k$, let $\mathcal{C}_{Y_i} \in \mathcal{W}(Y_1, \dots, Y_k)$ and $|\mathcal{C}_{Y_i}| = 0$,*
- *$Id \in \mathcal{W}(Y_1, \dots, Y_k)$ and $|Id| = 0$,*
- *If $W, V, W_0, \dots, W_n \in \mathcal{W}(Y_1, \dots, Y_k)$, then*
  - *$W \odot V$ is in $\mathcal{W}(Y_1, \dots, Y_k)$ and $|W \odot V| = \max\{|W|, |V|\} + 1$ (where $\odot \in \{+, \times\}$),*
  - *$M^\diamond(W)$, $M(W)$ and W^* are in $\mathcal{W}(Y_1, \dots, Y_k)$ and $|M^\diamond(W)| = |M(W)| = |W^*| = |W| + 1$,*
  - *$\mathbb{T}(W)$, $\mathbb{T}^s(W)$, $\mathbb{T}^{\leq m}(W)$ and $\mathbb{T}^{s, \leq m}(W)$ are in $\mathcal{W}(Y_1, \dots, Y_k)$ and $|\mathbb{T}^{(s),(\leq m)}(W)| = |W| + 1$,*
  - *$\mathbb{T}(W_0, \dots, W_n)$ and $\mathbb{T}^s(W_0, \dots, W_n)$ are in $\mathcal{W}(Y_1, \dots, Y_k)$ and $|\mathbb{T}^{(s)}(W_0, \dots, W_n)| = \max\{|W_0|, \dots, |W_n|\} + 1$.*
  - *If $0 < \alpha_1 < \dots < \alpha_n \leq \omega$, then $\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$ and*

$$\mathbb{T}^s \begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix} \text{ are in } \mathcal{W}(Y_1, \dots, Y_k) \text{ and}$$
$$\left| \mathbb{T}^{(s)} \begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix} \right| = \max\{|W_1|, \dots, |W_n|\} + 1.$$

The elements of $\mathcal{W}(Y_1, \dots, Y_k)$ are syntactical symbols such that we can work with it as a formal object. Every element of $\mathcal{W}(Y_1, \dots, Y_k)$ corresponds to a function from the class of partial orders to the class of partial orders. For obtaining this correspondence, we give a natural interpretation function I.

Definition 1.88. *For any $W \in \mathcal{W}(Y_1, \dots, Y_k)$, we define $I(W)$ as a function from the class of partial orders to the class of partial orders inductively as follows. Let X be a partial order. Then $I(W)(X)$ is*

- Y_i *if* $W = \mathcal{C}_{Y_i}$,
- X *if* $W = Id$.

Furthermore,

- $I(W \odot V)(X) := I(W)(X) \odot I(V)(X)$, *where* $\odot \in \{+, \times\}$,
- $I(M^\diamond(W))(X) := M^\diamond(I(W)(X))$, $I(M(W))(X) := M(I(W)(X))$ *and* $I(W^*)(X) := (I(W)(X))^*$,
- $I(\mathbb{T}(W))(X) := \mathbb{T}(I(W)(X))$ *and* $I(\mathbb{T}^s(W))(X) := \mathbb{T}^s(I(W)(X))$,
- $I(\mathbb{T}^{\leq m}(W))(X) := \mathbb{T}^{\leq m}(I(W)(X))$ *and* $I(\mathbb{T}^{s,\leq m}(W))(X) := \mathbb{T}^{s,\leq m}(I(W)(X))$,
- $I(\mathbb{T}(W_0, \dots, W_n))(X) := \mathbb{T}(I(W_0)(X), \dots, I(W_n)(X))$ *and* $I(\mathbb{T}^s(W_0, \dots, W_n))(X) := \mathbb{T}^s(I(W_0)(X), \dots, I(W_n)(X))$,
- $I\left(\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}\right)(X) := \mathbb{T}\begin{pmatrix} I(W_1)(X) & \dots & I(W_n)(X) \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$ *and* $I\left(\mathbb{T}^s\begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}\right)(X) := \mathbb{T}^s\begin{pmatrix} I(W_1)(X) & \dots & I(W_n)(X) \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$.

Note that if $W \in \mathcal{W}(Y_1, \dots, Y_k)$ and if $Y_1, \dots, Y_k, X$ are $\mathbb{wpo}$'s, then $I(W)(X)$ is also a $\mathbb{wpo}$. Additionally, if $Y_1, \dots, Y_k, X$ are countable, then $I(W)(X)$ is also countable. Now, we establish the notion of *elements of* W. A crucial fact is that every element of $I(W)(X)$ can be described using a term in finitely many elements of X. For example, if $W = (\cdot^* \times \mathcal{C}_{Y_1})$, then the element $((x_1, \dots, x_n), \mathbb{y}_1) \in I(W)(X) = (X^* \times Y_1)$ can be described by a concrete term using the elements $x_1, \dots, x_n \in X$. Therefore, an arbitrary element of $I(W)(X)$ can be represented as $w(x_1, \dots, x_n)$, with a term w and

$x_i \in X$. By abstracting elements away, an *element of* W can be represented as $w(\cdot, \ldots, \cdot)$. E.g. for our element, this is equal to $((\cdot, \ldots, \cdot), \mathtt{y}_1)$. This element of the mapping W maps elements of the partial ordering X to an element of the partial ordering $I(W)(X)$. This informal idea is mentioned in [86]. We now formally define this notion.

Definition 1.89. *Assume $W \in \mathcal{W}(Y_1, \ldots, Y_k)$. We inductively define a set of* terms T_W *and we call the elements of T_W the* elements of W. *Additionally, we define the length $|w|$ for every term $w \in T_W$.*

- *If $W = \mathcal{C}_{Y_i}$, then $T_W := Y_i$ which is considered as a set of constants, and define $|y| = 0$, for all $y \in T_W$.*
- *If $W = Id$, let $T_W = Var = \{x_i : i < \omega\}$, a set of variables, and define $|x_i| = 0$, for all $x_i \in T_W$.*
- *If $W = V_1 + V_2$, then $T_W = \{(v, i) : v \in T_{V_i}, i = 1, 2\}$, and define $|(v, i)| := |v| + 1$.*
- *If $W = V_1 \times V_2$, then $T_W = \{(v_1, v_2) : v_i \in T_{V_i}\}$, and define $|(v_1, v_2)| := \max\{|v_1|, |v_2|\} + 1$.*
- *If $W = M^\diamond(V)$, then $T_W = \{[v_1, \ldots, v_n] : v_i \in T_V$ and $n < \omega\}$, and define $|[v_1, \ldots, v_n]| := \max\{|v_1|, \ldots, |v_n|\} + 1$.*
- *If $W = M(V)$, then $T_W = \{[v_1, \ldots, v_n] : v_i \in T_V$ and $n < \omega\}$, and define $|[v_1, \ldots, v_n]| := \max\{|v_1|, \ldots, |v_n|\} + 1$.*
- *If $W = V^*$, then $T_W = \{(v_1, \ldots, v_n) : v_i \in T_V$ and $n < \omega\}$, and define $|(v_1, \ldots, v_n)| := \max\{|v_1|, \ldots, |v_n|\} + 1$.*
- *If $W = \mathbb{T}(V)$, then $T_W = \{tree(v_1, \ldots, v_n) : v_i \in T_V$ and $tree(v_1, \ldots, v_n)$ is an arbitrary finite rooted tree with labels $v_1, \ldots, v_n\}$. More specifically, $T_W := \bigcup_{i<\omega} T^i_W$, where we define T^i_W as the least sets such that (using Remark 1.74 for interpreting trees as terms)*
  - *$\cdot_v \in T^i_W$ for all $v \in T_V$ and let $| \cdot_v |$ be $|v| + 1$,*
  - *If $T_1, \ldots, T_n \in T^i_W$, then $\cdot_v(T_1, \ldots, T_n) \in T^{i+1}_W$ for all $v \in T_V$ and let $| \cdot_v (T_1, \ldots, T_n)|$ be $\max\{|v|, |T_1|, \ldots, |T_n|\} + 1$.*
- *Similarly for $W = \mathbb{T}^s(V), \mathbb{T}^{\leq m}(V), \mathbb{T}^{s,\leq m}(V), \mathbb{T}(V_0, \ldots, V_n)$, $\mathbb{T}^s(V_0, \ldots, V_n)$, $\mathbb{T}\begin{pmatrix} V_1 & \ldots & V_n \\ \alpha_1 & \ldots & \alpha_n \end{pmatrix}$ and $\mathbb{T}^s\begin{pmatrix} V_1 & \ldots & V_n \\ \alpha_1 & \ldots & \alpha_n \end{pmatrix}$.*

Remark 1.90. *1.* The elements of T_W are terms over the language y_i, x_i, $a+b$, $a \times b$, $[a,\dots,b]$, $(a,\dots,b)$, $\cdot_a$ and $\cdot_a(b,\dots,c)$, where $y_i \in Y_i$, x_i is a variable and a, b and c are place holders.

2. An element of $T_{V\times V}$ can be interpreted wrongly as an element of T_{V^*}. This can be solved by using two kinds of brackets to make the difference between (v,v) in $T_{V\times V}$ and T_{V^*} clear. This is however not needed because the context will make clear what we mean. Hence, we did not do this for notational convenience.

3. Similarly, elements of $T_{M^\diamond(V)}$ and $T_{M(V)}$ are defined in the same way, which is actually not allowed. Again, this can be solved by using two kinds of brackets to see the difference between the two. This is however not needed because the context will make clear what we mean. Hence, we did not do this for notational convenience.

4. If $W = M^\diamond(V)$, $M(V)$, $\mathbb{T}(V)$, $\mathbb{T}^{\leq m}(V)$, $\mathbb{T}(V_0,\dots,V_n)$ or $\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}$, we know that elements of $I(W)(X)$ are defined up to an equivalence relation. Similarly as in that setting, we define T_W also up to a specific equivalence relation. E.g., if $W = M(V)$, we see the elements $[v_1, v_2, \dots, v_n]$ and $[v_2, v_1, \dots, v_n]$ as two different terms that are equivalent. If we prove a property of the first term, we also mean that it is valid for the second one.

T_W is later needed to define the partial order $\mathcal{T}(W)$. We can inductively define the set of variables and the set of constants occurring in $w \in T_W$.

Definition 1.91. *If $w \in T_W$, define $Var(w)$ as the set of variables occurring in w. If $Var(w) = \{x_{i_1},\dots,x_{i_k}\}$, we often write w as $w(x_{i_1},\dots,x_{i_k})$.*

Definition 1.92. *If $w \in T_W$, define $Con(w)$ as the set of elements $y \in T_{\mathcal{C}_{Y_i}} = Y_i$ occurring in w.*

Lemma 1.93. *Assume $W \in \mathcal{W}(Y_0,\dots,Y_k)$. If $w(x_1,\dots,x_n) \in T_W$, then $w(x_{i_1},\dots,x_{i_n})$ is also in T_W. $w(x_{i_1},\dots,x_{i_n})$ means that you substitute x_{i_j} in x_j for every j.*

Proof. This can be proved by a straightforward induction on $|w|$. A similar proof can be found in Lemma 1.94. □

The following lemmas show that $I(W)(X)$ is actually T_W if we replace the variables of the terms in T_W by elements of X.

Lemma 1.94. *Assume that $W \in \mathcal{W}(Y_1, \ldots, Y_k)$ and X is an arbitrary partial order. If s is an element of $I(W)(X)$, then there exists a term $w(x_1, \ldots, x_n) \in T_W$, where $x_1, \ldots, x_n$ are distinct variables, and elements $\mathbb{x}_1, \ldots, \mathbb{x}_n$ of X such that $s = w(\mathbb{x}_1, \ldots, \mathbb{x}_n)$, where $w(\mathbb{x}_1, \ldots, \mathbb{x}_n)$ means that every occurrence of x_i in w is replaced by $\mathbb{x}_i$.*

Proof. We prove this by main induction on $|W|$. If $W = \mathcal{C}_{Y_i}$, then this is trivial. If $W = Id$, then $I(W)(X) = X$, hence $s = \mathbb{x} \in X$. Define the term w as x_1, hence $w = w(x_1)$ and let $\mathbb{x}_1$ be $\mathbb{x}$.

Assume $W = M^\diamond(V)$ and assume $s \in I(W)(X) = M^\diamond(I(V)(X))$. So $s = [s_1, \ldots, s_n]$, with $s_i = I(V)(X)$. The induction hypothesis yields $s_i = w_i(\mathbb{x}^i_1, \ldots, \mathbb{x}^i_{m_i})$ for all i with $w_i(x^i_1, \ldots, x^i_{m_i}) \in T_V$ and $\mathbb{x}^i_1, \ldots, \mathbb{x}^i_{m_i} \in X$. By Lemma 1.93, one can assume that all variables x^i_j are distinct. Define $w = w(x^1_1, \ldots, x^n_{m_n})$ as the term $[w_1, \ldots, w_n]$. Then $s = w(\mathbb{x}^1_1, \ldots, \mathbb{x}^n_{m_n})$ trivially holds.

Assume $W = \mathbb{T}(V)$. Then $s \in \mathbb{T}(I(V)(X))$ is a tree with labels in $I(V)(X)$. We prove by sub-induction on $ht(s)$ that the lemma holds. If $ht(s) = 0$, then $s = \cdot_{s'}$, with $s' \in I(V)(X)$. The main induction hypothesis yields the existence of a term $w'(x_1, \ldots, x_n) \in T_V$ and elements $\mathbb{x}_1, \ldots, \mathbb{x}_n$ in X such that $s' = w'(\mathbb{x}_1, \ldots, \mathbb{x}_n)$. Define w as $\cdot_{w'(x_1,\ldots,x_n)}$. Then $s = w(\mathbb{x}_1, \ldots, \mathbb{x}_n)$.

If $ht(s) > 0$, then $s = \cdot_{s'}(s_1, \ldots, s_n)$, with $s' \in I(V)(X)$ and $s_i \in I(W)(X)$. The main induction hypothesis yields again the existence of a term $w'(x_1, \ldots, x_k) \in T_V$ and elements $\mathbb{x}_1, \ldots, \mathbb{x}_k$ in X such that $s' = w'(\mathbb{x}_1, \ldots, \mathbb{x}_k)$. The sub-induction hypothesis yields the existence of $w_i(x^i_1, \ldots, x^i_{m_i}) \in T_W$ and elements $\mathbb{x}^i_1, \ldots, \mathbb{x}^i_{m_i}$ in X such that $s_i = w_i(\mathbb{x}^i_1, \ldots, \mathbb{x}^i_{m_i})$. Define $w = w(x_1, \ldots, x_k, x^1_1, \ldots, x^n_{m_n})$ as $\cdot_{w'}(w_1, \ldots, w_n)$. Then $s = w(\mathbb{x}_1, \ldots, \mathbb{x}_k, \mathbb{x}^1_1, \ldots, \mathbb{x}^n_{m_n})$.

Note that the exact equality $s = w(\mathbb{x}_1, \ldots, \mathbb{x}_n)$ is valid because we use the notation of a tree in Remark 1.74.

The other cases can be treated in a similar way. □

Definition 1.95. *Let X be an* arbitrary *partial order. Following Definition 1.91, define for $w(\mathbb{x}_1, \ldots, \mathbb{x}_n)$ in the partial order $I(W)(X)$, the leaf-set $Leaves(w(\mathbb{x}_1, \ldots, \mathbb{x}_n))$ as the set $\{\mathbb{x}_1, \ldots, \mathbb{x}_n\}$.*

Lemma 1.96. *Assume $W \in \mathcal{W}(Y_1, \ldots, Y_k)$ and X is an arbitrary partial order. If $w(x_1, \ldots, x_n) \in T_W$ and $\mathbb{x}_1, \ldots, \mathbb{x}_n \in X$, then $w(\mathbb{x}_1, \ldots, \mathbb{x}_n) \in I(W)(X)$.*

Proof. This can be proved in a similar way by induction on $|w|$. □

The previous lemmas indicate that T_W is equal to the partial order $I(W)(X)$ as a *set* when X is the partial order $(Var, =)$. However, using the extra information of the ordering on $I(W)(X)$, one can also define a natural ordering relation on T_W.

Definition 1.97. *For term $w, w' \in T_W$, define $w \leq w'$ if $w \leq_{I(W)(X)} w'$ if X is the partial order $(Var, =)$.*

In practice, we only need T_W as a set. Before we go on, we give an example.

Example 1.98. *Let $W = Id^*$ and X be an arbitrary partial order. Then $I(W)(X) = X^*$ and $T_W = \{(v_1, \ldots, v_n) : v_i \in T_{Id}\} = \{(x_1, \ldots, x_n) : x_i \in Var\}$. Hence, if $s \in I(W)(X) = X^*$, then $s = (\mathbb{x}_1, \ldots, \mathbb{x}_n)$ and $w = (x_1, \ldots, x_n) \in T_W$ like in Lemma 1.94.*

A very crucial lemma in this dissertation is the so-called Lifting Lemma. The proof is rather easy and straightforward, but somewhat technical. It is used to prove that if we know

$$w(q(\mathbb{y}_1), \ldots, q(\mathbb{y}_n)) \leq_{I(W)(Z)} v(q(\mathbb{y}'_1), \ldots, q(\mathbb{y}'_m)),$$

with q a quasi-embedding from Y to Z, then one can delete this q, i.e.

$$w(\mathbb{y}_1, \ldots, \mathbb{y}_n) \leq_{I(W)(Y)} v(\mathbb{y}'_1, \ldots, \mathbb{y}'_m).$$

Hence, in some sense the Lifting Lemma says that one can *lift* the ordering on X to the ordering on $I(W)(X)$.

Lemma 1.99 (Lifting Lemma)**.** *Assume $W \in \mathcal{W}(Y_1, \ldots, Y_k)$ and $w(y_1, \ldots, y_n)$, $v(y'_1, \ldots, y'_m) \in T_W$. If q is a quasi-embedding from partial ordering Y to partial ordering Z, then for every $\mathbb{y}_1, \ldots, \mathbb{y}_n$, $\mathbb{y}'_1, \ldots, \mathbb{y}'_m \in Y$, we have if $w(q(\mathbb{y}_1), \ldots, q(\mathbb{y}_n)) \leq_{I(W)(Z)} v(q(\mathbb{y}'_1), \ldots, q(\mathbb{y}'_m))$, then $w(\mathbb{y}_1, \ldots, \mathbb{y}_n) \leq_{I(W)(Y)} v(\mathbb{y}'_1, \ldots, \mathbb{y}'_m)$.*

Proof. This can be proved by main induction on $|W|$. If $|W| = 0$, then $W = Id$ or $W = \mathcal{C}_{Y_i}$ for a certain i. If $W = \mathcal{C}_{Y_i}$, then the lemma trivially holds. If $W = Id$, then $w = w(y_1) = y_1$ and $v = v(y'_1) = y'_1$, where y_1, y'_1 are variables and $w(q(\mathbb{y}_1)) \leq_{I(W)(Z)} v(q(\mathbb{y}'_1))$ yields $q(\mathbb{y}_1) \leq q(\mathbb{y}'_1)$, hence $\mathbb{y}_1 \leq \mathbb{y}'_1$. So, $w(\mathbb{y}_1) \leq_{I(W)(Y)} v(\mathbb{y}'_1)$.

Assume $|W| > 0$. For example, let $W = W_1 \times W_2$. Let $w_1(y_1, \ldots, y_{n_1})$, $v_1(y_{n_1+1}, \ldots, y_{n_1+n_2})$ be elements of T_{W_1} and $w_2(y'_1, \ldots, y'_{m_1})$, $v_2(y'_{m_1+1}, \ldots, y'_{m_1+m_2})$ in T_{W_2}. Assume q is a quasi-embedding from Y to Z. Pick $\mathbb{y}_1, \ldots, \mathbb{y}_{n_1}$,

$\mathtt{y}_{n_1+1}, \ldots, \mathtt{y}_{n_1+n_2}$ and $\mathtt{y}'_1, \ldots, \mathtt{y}'_{m_1}$, $\mathtt{y}'_{m_1+1}, \ldots, \mathtt{y}'_{m_1+m_2}$ arbitrarily from Y and assume

$$(w_1(q(\mathtt{y}_1), \ldots, q(\mathtt{y}_{n_1})), w_2(q(\mathtt{y}'_1), \ldots, q(\mathtt{y}'_{m_1})))$$
$$\leq_{I(W)(Z)} (v_1(q(\mathtt{y}_{n_1+1}), \ldots, q(\mathtt{y}_{n_1+n_2})), v_2(q(\mathtt{y}'_{m_1+1}), \ldots, q(\mathtt{y}'_{m_1+m_2}))).$$

This inequality yields

$$w_1(q(\mathtt{y}_1), \ldots, q(\mathtt{y}_{n_1})) \leq_{I(W_1)(Z)} v_1(q(\mathtt{y}_{n_1+1}), \ldots, q(\mathtt{y}_{n_1+n_2}))$$

and

$$w_2(q(\mathtt{y}'_1), \ldots, q(\mathtt{y}'_{m_1})) \leq_{I(W_2)(Z)} v_2(q(\mathtt{y}'_{m_1+1}), \ldots, q(\mathtt{y}'_{m_1+m_2})).$$

By the induction hypothesis, we obtain

$$w_1(\mathtt{y}_1, \ldots, \mathtt{y}_{n_1}) \leq_{I(W_1)(Y)} v_1(\mathtt{y}_{n_1+1}, \ldots, \mathtt{y}_{n_1+n_2}),$$
$$w_2(\mathtt{y}'_1, \ldots, \mathtt{y}'_{m_1}) \leq_{I(W_2)(Y)} v_2(\mathtt{y}'_{m_1+1}, \ldots, \mathtt{y}'_{m_1+m_2}),$$

and therefore,

$$(w_1(\mathtt{y}_1, \ldots, \mathtt{y}_{n_1}), w_2(\mathtt{y}'_1, \ldots, \mathtt{y}'_{m_1}))$$
$$\leq_{I(W)(Y)} (v_1(\mathtt{y}_{n_1+1}, \ldots, \mathtt{y}_{n_1+n_2}), v_2(\mathtt{y}'_{m_1+1}, \ldots, \mathtt{y}'_{m_1+m_2})).$$

Another example is $W = \mathbb{T}(V)$. The other cases can be treated in a similar way. Assume that the Lifting Lemma is valid for V. Take $w(y_1, \ldots, y_n)$, $v(y'_1, \ldots, y'_m)$ in $T_W = I(W)(Var) = \mathbb{T}(I(V)(Var))$. Hence, w, respectively v, is a tree T_1, respectively T_2, with labels in $I(V)(Var)$. Let q be a quasi-embedding from Y to Z and assume $\mathtt{y}_1, \ldots, \mathtt{y}_n, \mathtt{y}'_1, \ldots, \mathtt{y}'_m \in Y$.

We prove the lemma by sub-induction on the height of T_2. If $ht(T_2) = 0$, then

$$w(q(\mathtt{y}_1), \ldots, q(\mathtt{y}_n)) \leq_{I(W)(Z)} v(q(\mathtt{y}'_1), \ldots, q(\mathtt{y}'_m))$$

yields $ht(T_1) = 0$. So assume $w = \cdot_{w'(y_1,\ldots,y_n)}$ and $v = \cdot_{v'(y'_1,\ldots,y'_m)}$, with $w'(y_1, \ldots, y_n), v'(y'_1, \ldots, y'_m) \in I(V)(Var) = T_V$. From

$$w(q(\mathtt{y}_1), \ldots, q(\mathtt{y}_n)) \leq_{I(W)(Z)} v(q(\mathtt{y}'_1), \ldots, q(\mathtt{y}'_m)),$$

we obtain $w'(q(\mathtt{y}_1), \ldots, q(\mathtt{y}_n)) \leq_{I(V)(Z)} v'(q(\mathtt{y}'_1), \ldots, q(\mathtt{y}'_m))$, hence the main induction hypothesis implies

$$w'(\mathtt{y}_1, \ldots, \mathtt{y}_n) \leq_{I(V)(Y)} v'(\mathtt{y}'_1, \ldots, \mathtt{y}'_m),$$

so

$$w(\mathtt{y}_1, \ldots, \mathtt{y}_n) \leq_{I(W)(Y)} v(\mathtt{y}'_1, \ldots, \mathtt{y}'_m).$$

Assume $ht(T_2) > 0$. Then

$$v = \cdot_{v'(y'_{i_1},\dots,y'_{i_{k'}})}(T_2^1(y'^1_1,\dots,y'^1_{m_1}),\dots,T_2^{l'}(y'^{l'}_1,\dots,y'^{l'}_{m_{l'}})),$$

with $v'(y'_{i_1},\dots,y'_{i_{k'}}) \in I(V)(Var) = T_V$ and $T_2^i \in T_W$.

We know $w(q(\mathtt{y}_1),\dots,q(\mathtt{y}_n)) \leq_{I(W)(Z)} v(q(\mathtt{y}'_1),\dots,q(\mathtt{y}'_m))$. If

$$w(q(\mathtt{y}_1),\dots,q(\mathtt{y}_n)) \leq_{I(W)(Z)} T_2^i(\mathtt{y}'^i_1,\dots,\mathtt{y}'^i_{m_i})$$

for a certain i, the lemma follows in a straightforward way from the sub-induction hypothesis.

If $w(q(\mathtt{y}_1),\dots,q(\mathtt{y}_n)) \not\leq_{I(W)(Z)} T_2^i(\mathtt{y}'^i_1,\dots,\mathtt{y}'^i_{m_i})$ for every i, then

$$w = \cdot_{w'(y_{j_1},\dots,y_{j_k})}(T_1^1(y^1_1,\dots,y^1_{n_1}),\dots,T_1^l(y^l_1,\dots,y^l_{n_l})),$$

with $w'(y_{j_1},\dots,y_{j_k}) \in I(V)(Var) = T_V$ and $T_1^i \in T_W$. In this case,

$$w(q(\mathtt{y}_1),\dots,q(\mathtt{y}_n)) \leq_{I(W)(Z)} v(q(\mathtt{y}'_1),\dots,q(\mathtt{y}'_m))$$

implies

$$w'(q(\mathtt{y}_{j_1}),\dots,q(\mathtt{y}_{j_k})) \leq_{I(V)(Z)} v'(q(\mathtt{y}'_{i_1}),\dots,q(\mathtt{y}'_{i_{k'}}))$$

and the existence of l distinct elements $p_1,\dots,p_l$ in $\{1,\dots,l'\}$ such that

$$T_1^j(q(\mathtt{y}^j_1),\dots,q(\mathtt{y}^j_{n_j})) \leq_{I(W)(Z)} T_2^{p_j}(q(\mathtt{y}'^{p_j}_1),\dots,q(\mathtt{y}'^{p_j}_{m_{p_j}}))$$

for every j. The main and sub-induction hypothesis yield

$$w'(\mathtt{y}_{j_1},\dots,\mathtt{y}_{j_k}) \leq_{I(V)(Y)} v'(\mathtt{y}'_{i_1},\dots,\mathtt{y}'_{i_{k'}})$$

and

$$T_1^j(\mathtt{y}^j_1,\dots,\mathtt{y}^j_{n_j}) \leq_{I(W)(Y)} T_2^{p_j}(\mathtt{y}'^{p_j}_1,\dots,\mathtt{y}'^{p_j}_{m_{p_j}})$$

for every j, hence $w(\mathtt{y}_1,\dots,\mathtt{y}_n) \leq_{I(W)(Y)} v(\mathtt{y}'_1,\dots,\mathtt{y}'_m)$. □

Before we give the definition of $\mathcal{T}(W)$, we list some crucial but straightforward properties of $W \in \mathcal{W}(Y_1,\dots,Y_k)$.

Lemma 1.100. *1. If Y and Z are two partial orders such that $Y \subseteq Z$ and $\leq_Y \subseteq \leq_Z$, then $I(W)(Y) \subseteq I(W)(Z)$ and $\leq_{I(W)(Y)} \subseteq \leq_{I(W)(Z)}$.*

2. If X is a partial order and $w(\mathtt{x}_1,\dots,\mathtt{x}_n) \leq_{I(W)(X)} w'(\mathtt{x}'_1,\dots,\mathtt{x}'_m)$, then for every $i \in \{1,\dots,n\}$, there exists a $j \in \{1,\dots,m\}$ such that $\mathtt{x}_i \leq_X \mathtt{x}'_j$.

Proof. The assertions are both straightforward and can be proved by induction on $|W|$. To make things clear, we prove the second assertion for some cases.

If $|W| = 0$, then $W = Id$ or $W = \mathcal{C}_{Y_i}$ for a certain i. If $W = \mathcal{C}_{Y_i}$, then the lemma trivially holds because then $n = 0$. Assume $W = Id$. Then $w(\mathbb{x}_1) = \mathbb{x}_1$ and $w'(\mathbb{x}'_1) = \mathbb{x}'_1$, hence the assertion is trivial.

Assume $|W| > 0$. For example, let $W = W_1 \times W_2$. Assume

$$\begin{aligned}&(w_1(\mathbb{x}_1, \ldots, \mathbb{x}_{n_1}), w_2(\mathbb{x}_{n_1+1}, \ldots, \mathbb{x}_{n_1+n_2}))\\ \leq_{I(W)(X)}&(w'_1(\mathbb{x}'_1, \ldots, \mathbb{x}'_{m_1}), w'_2(\mathbb{x}'_{m_1+1}, \ldots, \mathbb{x}'_{m_1+m_2})).\end{aligned}$$

This inequality yields

$$\begin{aligned} w_1(\mathbb{x}_1, \ldots, \mathbb{x}_{n_1}) &\leq_{I(W_1)(X)} w'_1(\mathbb{x}'_1, \ldots, \mathbb{x}'_{m_1})\\ w_2(\mathbb{x}_{n_1+1}, \ldots, \mathbb{x}_{n_1+n_2}) &\leq_{I(W_2)(X)} w'_2(\mathbb{x}'_{m_1+1}, \ldots, \mathbb{x}'_{m_1+m_2}).\end{aligned}$$

Pick an arbitrary $\mathbb{x}_i$, for example $\mathbb{x}_{n_1}$. The induction hypothesis on W_1 yields the existence of an index $j \in \{1, \ldots, m_1\} \subseteq \{1, \ldots, m_1 + m_2\}$ such that $\mathbb{x}_i \leq_X \mathbb{x}'_j$. Hence, the assertion follows. All the other cases can be treated in a similar way. If $W = \mathbb{T}(V)$, we also need an induction argument on the height of the trees. □

Now, we are ready to define $\mathcal{T}(W)$ and its ordering relation for the function symbols W in $\mathcal{W}(Y_1, \ldots, Y_k)$. From now on, we assume that $Y_1, \ldots, Y_k$ are fixed $\mathbb{wpo}$'s. First, we give a formal definition $\mathcal{T}(W)$. We present an informal description after Lemma 1.107. $\mathcal{T}(W)$ is a formal set of terms over the language $\circ$, [,] and T_W, where the variables in the elements of T_W are occupied by *previously defined terms.*

Definition 1.101. *We define $\mathcal{T}^i(W)$ as follows*

1. $\mathcal{T}^0(W) := \{\circ\}$
2. *Assume that $\mathcal{T}^i(W)$ is defined. Then*
   (a) $\circ \in \mathcal{T}^{i+1}(W)$,
   (b) *if $s \in I(W)(\mathcal{T}^i(W))$, hence $s = w(t_1, \ldots, t_n)$ with $w \in T_W$ and $t_1, \ldots, t_n \in \mathcal{T}^i(W)$, then $\circ[w(t_1, \ldots, t_n)] \in \mathcal{T}^{i+1}(W)$.*

Define $\leq^i \subseteq \mathcal{T}^i(W) \times \mathcal{T}^i(W)$ as follows

1. $\circ \leq^0 \circ$,

2. *Assume that* $\leq^i$ *is defined. Then*
   (a) $\circ \leq^{i+1} t$ *for all* $t \in \mathcal{T}^{i+1}(W)$*,*
   (b) *if* $s \leq^i t_j$*, then* $s \leq^{i+1} \circ[w(t_1, \ldots, t_n)]$*,*
   (c) *if* $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^i(W))} w'(t'_1, \ldots, t'_m)$*, then*
   $$\circ[w(t_1, \ldots, t_n)] \leq^{i+1} \circ[w'(t'_1, \ldots, t'_m)].$$

Notation 1.102. *Often, we write* $\circ w(t_1, \ldots, t_n)$ *instead of* $\circ[w(t_1, \ldots, t_n)]$ *because* $w(t_1, \ldots, t_n)$ *has most of the time already enough brackets.*

Lemma 1.103. *For every* i*,* $\mathcal{T}^i(W) \subseteq \mathcal{T}^{i+1}(W)$ *and* $\leq^i \subseteq \leq^{i+1}$*.*

Proof. $\mathcal{T}^i(W) \subseteq \mathcal{T}^{i+1}(W)$ follows from a straightforward induction on i. Additionally, $\leq^i \subseteq \leq^{i+1}$ also follows from a straightforward induction. In the crucial case, one actually needs that $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^{i-1}(W))} w'(t'_1, \ldots, t'_m)$ implies $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^i(W))} w'(t'_1, \ldots, t'_m)$ if $i > 0$ knowing that $\leq^{i-1} \subseteq \leq^i$. But this follows from Lemma 1.100. □

Definition 1.104. *Define* $(\mathcal{T}(W), \leq_{\mathcal{T}(W)})$ *as* $\mathcal{T}(W) = \bigcup_i \mathcal{T}^i(W)$ *and* $\leq_{\mathcal{T}(W)} = \bigcup_i \leq^i$*. We write* $\leq$ *instead of* $\leq_{\mathcal{T}(W)}$ *if the ordering is clear from the context.*

Definition 1.105. *If* $t \in \mathcal{T}(W)$*, define* $C(t)$ *as the least* i *such that* $t \in \mathcal{T}^i(W)$*.* C *stands for 'Complexity'.*

If $t = \circ[w(t_1, \ldots, t_n)]$, then $C(t_i) < C(t)$.

Lemma 1.106. *if* t, t', t'' *are in* $\mathcal{T}(W)$*, then* $t \leq t'$ *and* $t' \leq t''$ *yield* $t \leq t''$*.*

Proof. We prove this by induction on $C(t) + C(t') + C(t'')$. If t, t' or t'' are $\circ$, this is trivial. So assume $t = \circ[w(t_1, \ldots, t_n)]$, $t' = \circ[w'(t'_1, \ldots, t'_m)]$ and $t'' = \circ[w''(t''_1, \ldots, t''_k)]$. Assume $t \leq^i t'$ and $t' \leq^j t''$. Then $t' \leq^{j-1} t''_l$ for a certain l or $w'(t'_1, \ldots, t'_m) \leq_{I(W)(\mathcal{T}^{j-1}(W))} w''(t''_1, \ldots, t''_k)$.

The induction hypothesis in the former case implies $t \leq t''_l$. Hence, $t \leq t''$.

Assume that $w'(t'_1, \ldots, t'_m) \leq_{I(W)(\mathcal{T}^{j-1}(W))} w''(t''_1, \ldots, t''_k)$. $t \leq^i t'$ yields either $t \leq^{i-1} t'_p$ for a certain p or $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^{i-1}(W))} w'(t'_1, \ldots, t'_m)$.

Assume $t \leq^{i-1} t'_p$ for a certain p. Then

$$w'(t'_1, \ldots, t'_m) \leq_{I(W)(\mathcal{T}^{j-1}(W))} w''(t''_1, \ldots, t''_k)$$

and Lemma 1.100 yield $t'_p \leq t''_q$ for a certain q. Hence, $t \leq t''_q$, so $t \leq t''$. Now assume $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^{i-1}(W))} w'(t'_1, \ldots, t'_m)$. Together with $w'(t'_1, \ldots, t'_m) \leq_{I(W)(\mathcal{T}^{j-1}(W))} w''(t''_1, \ldots, t''_k)$, we obtain

$$w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}^{\max\{i,j\}-1}(W))} w''(t''_1, \ldots, t''_k).$$

So $t \leq t''$. □

Lemma 1.107. *$(\mathcal{T}(W), \leq_{\mathcal{T}(W)})$ is a partial ordering.*

Proof. Lemma 1.103 proves that $(\mathcal{T}(W), \leq_{\mathcal{T}(W)})$ is well-defined and Lemma 1.106 shows that $(\mathcal{T}(W), \leq_{\mathcal{T}(W)})$ is transitive.

Reflexivity: by induction on $C(t)$. If $C(t) = 0$, then trivially $t \leq t$. Assume $t = \circ[w(t_1, \ldots, t_n)]$. The induction hypothesis yields $t_i \leq t_i$ for all i. Hence, $w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}(W))} w(t_1, \ldots, t_n)$, so

$$\circ[w(t_1, \ldots, t_n)] \leq_{I(W)(\mathcal{T}(W))} \circ[w(t_1, \ldots, t_n)].$$

Anti-symmetry: by induction on $C(t) + C(t')$, we prove $t \leq t'$ and $t' \leq t$ imply $t = t'$. If t or t' are $\circ$, this is trivial. Assume not. Then $t = \circ[w(t_1, \ldots, t_n)]$ and $t' = \circ[w'(t'_1, \ldots, t'_m)]$. If $t \leq t'_j$ for a certain j, then $t' \leq t'_j$, a contradiction. (One can prove that this is a contradiction by induction on t'.) Similarly, $t' \leq t_j$ for a certain j yields also a contradiction. So we can assume $t \not\leq t'_j$ and $t' \not\leq t_j$ for every j. Therefore, $w(t_1, \ldots, t_n) \leq w'(t'_1, \ldots, t'_m)$ and $w(t_1, \ldots, t_n) \geq w'(t'_1, \ldots, t'_m)$. $w(t_1, \ldots, t_n), w'(t'_1, \ldots, t'_m)$ are elements of $I(W)(\mathcal{T}^{C(t)+C(t')-1}(W))$. By the induction hypothesis, $\leq_{\mathcal{T}(W)}$ restricted to $\mathcal{T}^{C(t)+C(t')-1}(W)$ is a partial order, hence $I(W)(\mathcal{T}^{C(t)+C(t')-1}(W))$ is also a partial order. So $w(t_1, \ldots, t_n) = w'(t'_1, \ldots, t'_m)$ and $t = t'$. □

Remark 1.108. For notational convenience, we sometimes use the following rule: write $\cdot$ instead of Id. For example, $\mathcal{T}(M^\diamond(Id \times Id))$ can also be written as $\mathcal{T}(M^\diamond(\cdot \times \cdot))$. Additionally, we identify W and $I(W)$, meaning that we sometimes write W instead of $I(W)$. The context will make clear what we mean. We sometimes define W by saying what $W(X)$ (actually $I(W)(X)$) is, e.g. if $W = Id^*$, we denote this by $W(X) = X^*$.

The definition of $\mathcal{T}(W)$ is rather technical and hence an informal description is needed to clarify the notions. As mentioned in the beginning of the subsection, $\mathcal{T}(W)$ is introduced by Weiermann in [86] to investigate structures with the gap-embeddability relation. For every W, the partial order $\mathcal{T}(W)$ can be seen as such a structure.

Elements of $\mathcal{T}(W)$ can be interpreted as trees. An element of $\mathcal{T}(W)$ is of the form $\circ[w(t_1,\ldots,t_n)]$, where $t_1,\ldots,t_n$ are also in $\mathcal{T}(W)$. Here, $\circ$ is interpreted as the root of the tree and $t_1,\ldots,t_n$ are interpreted as subtrees, not necessarily immediate subtrees. How the subtrees $t_1,\ldots,t_n$ and the root $\circ$ are connected with each other is determined by the term $w \in T_W$.

In the tree-class $\mathbb{T}^s$, a tree T consists of a root and immediate subtrees $T_1,\ldots,T_n$. The subtrees are connected as a finite sequence: for two structured trees T and T' with immediate subtrees $T_1,\ldots,T_n$ and $T'_1,\ldots,T'_m$, T is embeddable into T' if T is embeddable into T'_j for a certain j or

$$(T_1,\ldots,T_n) \leq^* (T'_1,\ldots,T'_m).$$

Here, one really sees that the subtrees should be connected by using finite sequences. The next lemma formally proves this.

Theorem 1.109. *If* $W(X) = X^*\backslash\{()\}$, *then* $\mathcal{T}(W) \cong \mathbb{T}^s$.

Proof. Elements of $W(X)$ are the finite sequences of length strictly bigger than 0. Intuitively, $\mathcal{T}(W) \cong \mathbb{T}^s$ follows from the interpretation of a structured tree as a root together with a finite sequence of immediate subtrees. We give a precise order-isomorphism f between $\mathcal{T}(W)$ and $\mathbb{T}^s$. Define $f(\circ)$ as the tree consisting of one node. Define $f(\circ[(t_1,\ldots,t_n)])$ with $n \geq 1$ as the structured tree that has a root and immediate subtrees $f(t_1),\ldots,f(t_n)$. Trivially, f is surjective. Therefore, one can interpret the symbol $\circ$ in $\circ[(t_1,\ldots,t_n)]$ as the root of a tree.

Now we prove that $t \leq t' \Leftrightarrow f(t) \leq f(t')$ for all $t,t' \in \mathcal{T}(W)$ by induction on the complexity of t and t'. If t or t' are $\circ$, this is trivial. Hence, we can assume that $t = \circ(t_1,\ldots,t_n)$ and $t' = \circ(t'_1,\ldots,t'_m)$.

If $t \leq t'$, then $t \leq t'_j$ for a certain j or $(t_1,\ldots,t_n) \leq^* (t'_1,\ldots,t'_m)$. The induction hypothesis yields in the former case $f(t) \leq f(t'_j)$, hence $f(t) \leq f(t')$. In the latter case, the induction hypothesis yields $(f(t_1),\ldots,f(t_n)) \leq^* (f(t'_1),\ldots,f(t'_m))$. Hence, the structured tree with immediate subtrees $f(t_1)$, $\ldots$, $f(t_n)$ is embeddable in the structured with immediate subtrees $f(t'_1)$, $\ldots$, $f(t'_m)$. So $f(t) \leq f(t')$.

Now, assume $f(t) \leq f(t')$. This yields either that $f(t)$ is embeddable in an immediate subtree of $f(t')$, say $f(t'_j)$, or

$$(f(t_1),\ldots,f(t_n)) \leq^* (f(t'_1),\ldots,f(t'_m)).$$

Again, one can easily conclude in both cases that $t \leq t'$ using the induction hypothesis. □

Note that $\mathcal{T}((\cdot)^*) \neq \mathbb{T}^s$ because $\mathcal{T}((\cdot)^*)$ has two trees with no immediate subtrees, namely $\circ$ and $\circ[()]$.

In a similar way, unstructured trees are trees T consisting of a root and immediate subtrees $T_1, \ldots, T_n$ that are ordered using finite multisets. Hence,

$$\mathcal{T}(M^\diamond(\cdot)\backslash\{[]\}) \cong \mathbb{T}.$$

The partial order $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ is studied in Chapter 3. In order to have even a better understanding of $\mathcal{T}(W)$, we give the tree-structure corresponding to this partial order. First of all, $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ can be seen as a couple $(T, \leq_T)$ such that T and $\leq_T$ are chosen in the least possible way satisfying

- $\circ \in T$,
- if $t_1, \ldots, t_{2n} \in T$, then $\circ\left[[(t_1, t_2), \ldots, (t_{2n-1}, t_{2n})]\right] \in T$, which we denote by $\circ[(t_1, t_2), \ldots, (t_{2n-1}, t_{2n})]$ as mentioned in Notation 1.102,

and

- $\circ \leq_T t$ for every $t \in T$,
- if $s \leq_T t_j$ for a certain $j \in \{1, \ldots, 2n\}$, then $s \leq_T \circ[(t_1, t_2), \ldots, (t_{2n-1}, t_{2n})]$,
- if $[(t_1, t_2), \ldots, (t_{2n-1}, t_{2n})] \leq_{M^\diamond(T\times T)} [(t'_1, t'_2), \ldots, (t'_{2m-1}, t'_{2m})]$, then $\circ[(t_1, t_2), \ldots, (t_{2n-1}, t_{2n})] \leq_T \circ[(t'_1, t'_2), \ldots, (t'_{2m-1}, t'_{2m})]$.

Let $\widehat{\mathbb{T}}$ be the tree-structure corresponding to this partial order. We now fully describe the partial order $\widehat{\mathbb{T}}$. First of all, it has a tree with one node (which corresponds to the element $\circ \in T$). Additionally, $\widehat{\mathbb{T}}$ has another tree with one node (which corresponds to the element $\circ[] \in T$) which is bigger than the previous one (corresponding with the $\leq_T$-relation). To avoid the problem of having two trees with one node, we could have used $M^\diamond(\cdot \times \cdot)\backslash\{[]\}$ instead of $M^\diamond(\cdot \times \cdot)$ in the beginning.

All the other trees S in $\widehat{\mathbb{T}}$ have a root and an even number of immediate subtrees $S_1, \ldots, S_{2n}$, where these subtrees are glued together in pairs: S_1 is always sticked together with S_2, S_3 with S_4, and so on. We denote S by $\circ[(S_1, S_2), \ldots, (S_{2n-1}, S_{2n})]$, where $\circ$ denotes the root. One can see now the correspondence with $(T, \leq_T)$. We note that the pairs are ordered, e.g., $\circ[(S_1, S_2), (S_3, S_4)]$ is a different tree than $\circ[(S_2, S_1), (S_3, S_4)]$. Furthermore, the collection of pairs are unordered, e.g., $\circ[(S_1, S_2), (S_3, S_4)]$ is the same tree as $\circ[(S_3, S_4), (S_1, S_2)]$. The embeddability relation on $\widehat{\mathbb{T}}$ maintains these subdivisions in pairs: if T and T' are trees in $\widehat{\mathbb{T}}$ with immediate subtrees $T_1, \ldots, T_{2n}$ and $T'_1, \ldots, T'_{2m}$, then T is embeddable in T' if T is embeddable in an immediate subtree T'_j or

- there exists an odd index i_1 such that the pair T_1, T_2 is mapped into T'_{i_1}, T'_{i_1+1}. Furthermore, T_1 is mapped into T'_{i_1} and T_2 into T'_{i_1+1} according to the $\leq_{\widehat{\mathbb{T}}}$-relation,
- and there exists an odd index $i_2 \neq i_1$ such that the pair T_3, T_4 is mapped into T'_{i_2}, T'_{i_2+1}. Furthermore, T_3 is mapped into T'_{i_2} and T_4 into T'_{i_2+1} according to the $\leq_{\widehat{\mathbb{T}}}$-relation,
- ...

This actually means that

$$[(T_1, T_2), \ldots, (T_{2n-1}, T_{2n})] \leq_{M^\diamond(\widehat{\mathbb{T}} \times \widehat{\mathbb{T}})} [(T'_1, T'_2), \ldots, (T'_{2m-1}, T'_{2m})],$$

which corresponds to the definition of $\leq_T$. These examples clearly indicate how one can interpret the element of $\mathcal{T}(W)$ as (special) trees.

We already mentioned that all the partial orders $\mathcal{T}(W)$ are structures with a certain gap-embeddability relation. In order to clarify this fact, let us first investigate the partial order $\mathbb{T}_2^{wgap}$. Assume $t, t' \in \mathbb{T}_2^{wgap}$ and $t \leq_{gap}^{w} t'$. A path of nodes with label 1 in t has to be mapped into a path of nodes with label 1 in t'. It is not allowed that there is a node with label 0 between two nodes that are in the image of that path. More informally, call a block of adjacent nodes with label 1 a 1-block. Then, a 1-block in t has to be mapped into only one 1-block in t': it cannot be spread out over two or more 1-blocks. So the gap-embeddability relation of $\mathbb{T}_2^{wgap}$ implies that the 1's are sticked together and cannot be separated by a homeomorphic embedding.

The order relation $\leq_{\mathcal{T}(W)}$ implies the same property: we can see the term $w \in T_W$ in $t = \circ[w(t_1, \ldots, t_n)] \in \mathcal{T}(W)$ as a 1-block because it cannot be separated by $\leq_{\mathcal{T}(W)}$. (If W is a tree-class like $\mathbb{T}(\cdot)$, the notion of 1-blocks in $t \in \mathcal{T}(W)$ does even make more sense). $\leq_{\mathcal{T}(W)}$ yields that 1-blocks of $t \in \mathcal{T}(W)$ has to be mapped into only one 1-block. It is not allowed that it is mapped into two or more 1-blocks. More specifically, if

$$w(t_1, \ldots, t_n) \leq_{I(W)(\mathcal{T}(W))} w'(t'_1, \ldots, t'_m),$$

then for every replacement of $t_1, \ldots, t_n, t'_1, \ldots, t'_m$ by elements $s_1, \ldots, s_n, s'_1, \ldots, s'_m$ in $\mathcal{T}(W)$ that behave in a similar way as $t_1, \ldots, t_n, t'_1, \ldots, t'_m$ among each other, i.e. $t_i \leq_{\mathcal{T}(W)} t_j \Leftrightarrow s_i \leq_{\mathcal{T}(W)} s_j$, etc., we also have that

$$w(s_1, \ldots, s_n) \leq_{I(W)(\mathcal{T}(W))} w'(s'_1, \ldots, s'_m).$$

So the term w is only mapped into w' and not on the terms in $t'_1, \dots, t'_m$.

We state some specific examples of W's such that $\mathcal{T}(W)$ is a known partial order with a gap-embeddability relation.

- If $W(\cdot) = \mathbb{B}(\cdot) := \mathbb{T}(\cdot, \emptyset, \{0\})$, i.e. $W(X)$ is equal to the set of binary trees with leaf-labels in X, then $\mathcal{T}(W)$ is a subset of $\mathbb{T}_2^{wgap}$. For more information on this example, see Notation 4.1, Definition 4.2 and Lemma 4.3.
- If $W = M^{\diamond}(\mathbb{T}^{leaf}(\cdot))\backslash\{[]\}$, then $\mathcal{T}(W)$ is $\mathbb{T}'^{wgap}_2[0]$. For more information on $\mathbb{T}^{leaf}(\cdot)$ and this example, we refer to Definitions 5.1, 5.3 and Lemma 5.16.

Notation 1.110. *If $t = \circ[w(t_1, \dots, t_n)] \in \mathcal{T}(W)$, then $\times t$ denotes the element $w(t_1, \dots, t_n) \in I(W)(\mathcal{T}(W))$. Informally, this is the same as deleting the root of the tree corresponding to t. Note that $w(x_1, \dots, x_n)$ is an element of T_W.*

We are ready to state Weiermann's conjecture [86]. Informally, the conjecture of Weiermann states that $\mathcal{T}(W)$ is a wpo and describes (an upper bound for) the maximal order type of it. It even characterizes a maximal linear extension of $\leq_{\mathcal{T}(W)}$ on $\mathcal{T}(W)$ using the collapsing function ϑ. So it can be used in order to address the problem of determining the maximal order type of Friedman's famous wpo of trees with gap-condition. In [86], the conjecture is stated for every W that has the following properties.

1. If X is a countable wpo, then $W(X)$ is also a countable wpo.
2. Elements of $W(X)$ can be described as generalized terms in which the variables are replaced by constants for the elements of X.
3. The ordering between elements of $W(X)$ is induced effectively by the ordering from X.
4. We have an explicit knowledge on $o(W(X))$ if X is a wpo such that $o(W(X)) = o(W(o(X)))$ and such that this equality can be proved using an effective reification like in [66].

We cannot formally define properties 3. and 4. because of the informal notion *effective*. Informally, 3. means that the ordering on $W(X)$ can easily be described using the ordering on X and 4. means that the calculation of the maximal order type of $W(X)$ is in some sense well-understood. Every basic construction, i.e. every $W \in \mathcal{W}(Y_1, \dots, Y_k)$, has these informal properties. That is one reason why we restrict ourselves to only these W's in

$\mathcal{W}(Y_1, \ldots, Y_k)$ in the following conjecture. Another reason would be that we can use an easier induction argument (we can use an induction argument on $|W|$).

Conjecture 1.111 (Weiermann [86]). *If $Y_1, \ldots, Y_k$ are countable* $\mathbb{wpo}$*'s, then for every $W \in \mathcal{W}(Y_1, \ldots, Y_k)$, the partial order $\mathcal{T}(W)$ is a* $\mathbb{wpo}$ *and if $o(W(\Omega)) \leq \varepsilon_{\Omega+1}$, then its maximal order type is bounded above by $\vartheta(o(W(\Omega)))$. Furthermore, this upper bound is also a lower bound if $o(W(\Omega)) \geq \Omega^3$.*

We note that Ω is a $\mathbb{wpo}$, hence $W(\Omega)$ is also a $\mathbb{wpo}$ and $o(W(\Omega))$ is the maximal order type of this well-partial-order. The conjecture says that in almost all cases, $o(\mathcal{T}(W))$ is equal to $\vartheta(o(W(\Omega)))$. Moreover, using the conjecture we can in some sense read off a maximal linear extension of $\mathcal{T}(W)$: in most cases one can embed the partial order $\mathcal{T}(W)$ into $\vartheta(o(W(\Omega)))$, i.e. there exists a function f from $\mathcal{T}(W)$ to this ordinal that is order-preserving. This yields that (a subset of) $\vartheta(o(W(\Omega)))$ can be seen as a *natural* linear extension of $\mathcal{T}(W)$. Using the conjecture, one directly obtains that this is a maximal linear extension (if the subset of $\vartheta(o(W(\Omega)))$ has the same order type as $\vartheta(o(W(\Omega)))$). Correspondingly, the fact that $\vartheta(o(W(\Omega)))$ is a lower bound for the maximal order type of $\mathcal{T}(W)$ is quite often proved by defining a quasi-embedding from $\vartheta(o(W(\Omega)))$ into $\mathcal{T}(W)$. Hence, one can really embed this ordinal in the tree-class $\mathcal{T}(W)$.

Let us explain this with an example. As we have mentioned before (see Theorem 1.109), if $W(X) = X^*\backslash\{()\}$, then $\mathcal{T}(W) \cong \mathbb{T}^s$, hence $o(\mathcal{T}(W)) = o(\mathbb{T}^s)$. Weiermann's conjecture is true for this specific case: Diana Schmidt's results (see Theorem 1.79) yield $o(\mathbb{T}^s) = \vartheta(\Omega^\omega)$ and for $W(X) = X^*\backslash\{()\}$, $o(W(\Omega)) = o(\Omega^*) = \omega^{\omega^{\Omega+1}} = \Omega^\omega$. Here, we used the fact that $o(X^*\backslash\{()\}) = -1 + o(X^*)$.

There exists a natural embedding f from $\mathcal{T}(W)$ into $\vartheta(o(W(\Omega)))$ for this W: let $f(\circ)$ be 0 and define $f(\circ[(t_1, \ldots, t_n)])$ as

$$\vartheta(\Omega^{n-1} \cdot (f(t_1) + 1) + \cdots + \Omega^0 \cdot (f(t_n) + 1)).$$

One can prove that f is order-preserving and that the order type of the image of f is equal to $\vartheta(o(W(\Omega)))$. Hence, (a subset of) $\vartheta(o(W(\Omega)))$ corresponds to a linear extension of $\mathcal{T}(W)$. Weiermann's conjecture even implies that this is a *maximal* linear extension.

In this dissertation, we prove that Weiermann's conjecture is true for some specific examples: for W equal to $M^\diamond(\cdot \times \cdot)$, $(\cdot \times \cdot)^*$ and $\mathbb{T}(\cdot, \emptyset, \{0\})$ (also denoted by $\mathbb{B}(\cdot)$). Of course, we will prove Weiermann's conjecture for other W's as well because we need them to prove the conjecture for the above mentioned cases. In order to generalize the conjecture about the maximal order type of $\mathcal{T}(W)$ to other tree-classes, e.g. if $W = \mathbb{T}(\cdot)$, one has to change its formulation so that it also fits nicely if $o(W(\Omega)) \geq \varepsilon_{\Omega+1}$. Then one has to use the collapsing functions $(\vartheta_i)_{i<\omega}$ (which was already indicated in [86]). We refer the reader to Chapter 5 for more information, however the original conjecture already indicates what one should expect for these kind of tree-classes. In Chapter 5 we prove that $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ is equal to $o(\mathcal{T}(W))$ if $W = M^\diamond(\mathbb{T}^{leaf}(\cdot))^* \backslash \{[]\}$ (see Chapter 5 for the definition of $\mathbb{T}^{leaf}(\cdot)$). Using Theorem 5.16, this actually implies an exact characterization of $\mathbb{T}_2^{wgap}$.

If $\mathbb{X}_n, \ldots, \mathbb{X}_0$ are countable $\mathbb{wpo}$'s, then the conjecture for $W(X) = X^n \times \mathbb{X}_n + \cdots + X \times \mathbb{X}_1 + \mathbb{X}'_0$ corresponds to Diana Schmidt's result on $o(\mathbb{T}^s(\mathbb{X}_0, \ldots, \mathbb{X}_n))$ (see Theorem 1.79). Here, $\mathbb{X}'_0$ is defined as $\mathbb{X}_0$ minus a minimal element of $\mathbb{X}_0$ (such an element exists because $\mathbb{X}_0$ is well-founded). Note that $o(\mathbb{X}'_0) = -1 + o(\mathbb{X}_0)$.

Another specific case for which the conjecture is true is the next theorem. This example is needed to make a solid induction basis later on.

Theorem 1.112. *Assume $Y_1, \ldots, Y_k$ are countable $\mathbb{wpo}$'s. If Id does not appear in $W \in \mathcal{W}(Y_1, \ldots, Y_k)$, then $\mathcal{T}(W)$ is a $\mathbb{wpo}$ and $o(\mathcal{T}(W)) < \vartheta(o(W(\Omega)))$.*

Proof. We prove that there exists an order-isomorphism f between $\mathcal{T}(W)$ and $\mathbb{X} := \{0\} \cup I(W)(\emptyset)$, where the ordering on $\mathbb{X}$ is that on $I(W)(\emptyset)$ and every element of $I(W)(\emptyset)$ is strictly above 0. Note that $I(W)(\emptyset) = I(W)(X)$ for every partial order X.

Define $f(\circ)$ as 0 in $\mathbb{X}$. Take $t = \circ[w(t_1, \ldots, t_n)] \in \mathcal{T}(W)$. Then we know that $w(t_1, \ldots, t_n) \in I(W)(\mathcal{T}(W)) = I(W)(\emptyset)$, hence $n = 0$ and we denote $w(t_1, \ldots, t_n)$ by w. Define $f(w)$ as $w \in \mathbb{X}$. One can prove in a straightforward way that f is an order-isomorphism.

We know that $I(W)(\emptyset)$ is a $\mathbb{wpo}$, hence $\mathbb{X}$ is also a $\mathbb{wpo}$, so $\mathcal{T}(W)$ is a $\mathbb{wpo}$. Furthermore, $o(\mathcal{T}(W)) = o(\mathbb{X}) \leq o(I(W)(\emptyset)) + 1 = o(I(W)(\Omega)) + 1$. Because $I(W)(\Omega)$ is a countable $\mathbb{wpo}$, $o(I(W)(\Omega)) = k(o(I(W)(\Omega))) < \vartheta(o(I(W)(\Omega)))$. So, $o(\mathcal{T}(W)) < \vartheta(o(I(W)(\Omega))) = \vartheta(o(W(\Omega)))$. □

It should also be noted that all the (partial) proofs of the conjecture are of a constructive nature: they can be translated into a constructive proof using reifications (for more information see [66] and [73]). This would allow us to do an extraction of the computational content of the proofs.

1.3 Overview and summary of the dissertation and possible applications

This first chapter introduces the preliminaries. Chapters 2, 3, 4 and 5 are all dedicated to Diana Schmidt's research program mentioned in Section 1.1.1 and to Weiermann's conjecture: they all prove that for specific W's, Weiermann's conjecture is true.

Chapter 2 explores the unstructured trees. In her Habilitationsschrift [69], Diana Schmidt investigated the maximal order types of structured tree-classes. However, she did not discuss the unstructured version. We show that the ordinals of the unstructured tree-classes are equal to the ordinals of the corresponding structured tree-classes.

Chapter 3 improves the results of Weiermann in [89] by replacing τ in $\mathcal{T}(M^\diamond(\tau \times \cdot))$ by previously defined terms. We show that this gives rise to a representation of the big Veblen number. Moreover, we investigate what happens if we replace the unordered multisets by ordered finite sequences. Apparently, this yields a notation system for a much bigger ordinal, namely $\vartheta\Omega^{\Omega^\Omega}$.

In Chapter 4 we give a sub-ordering of one of Friedman's famous well-partial-orders with the Howard-Bachmann ordinal as maximal order type. More specifically, we show that $o(\mathcal{T}(\mathbb{B}(\cdot))) = o(\mathcal{T}(\mathbb{B}^s(\cdot))) = \vartheta(\varepsilon_{\Omega+1})$, where $\mathbb{B}(\cdot)$ is defined as $\mathbb{T}(\cdot, \emptyset, \{0\})$ and $\mathbb{B}^s(\cdot)$ is defined as $\mathbb{T}^s(\cdot, \emptyset, \{0\})$

Chapter 5 is about the maximal order type of $\mathbb{T}_2^{wgap}$. The overall conclusion reached in Chapter 5 is that $\mathbb{T}_2^{wgap}$ represents the ordinal $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$. It is possible to read Chapter 5 immediately without reading Chapters 2, 3 and 4. However, the underlying idea behind Chapter 5 is built up in these intermediate chapters.

Chapter 6 explores independence results arising from the structures $\mathcal{T}(W)$. Friedman's well-partial-orders yield an independence result for Π^1_1-CA_0: in [76] it is proved that the supremum over n of the maximal order type of $\mathbb{T}_n^{wgap}$ is equal to the proof-theoretic ordinal of Π^1_1-CA_0. We believe that applying the Π^1_1-comprehension scheme n many times yields the provability of the well-partial-orderedness of $\mathbb{T}_n^{wgap}$. But if we only allow $n-1$ applications of the Π^1_1-comprehension scheme, then the well-partial-orderedness of $\mathbb{T}_n^{wgap}$ is unprovable. Chapter 6 investigates light-face Π^1_1-comprehension, i.e. restricting Π^1_1-comprehension to one application. We show some results from the joint article with Michael Rathjen and Andreas Weiermann [81] about the proof-theoretical ordinals of $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$ and $\mathsf{RCA}_0^* + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$. Furthermore, we prove the following theorems.

Theorem 1.113.

1. $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^- \nvdash$ '$\mathcal{T}(\mathbb{B}(\cdot))$ *is a well-partial-order',*
2. *For every natural n,* $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^- \vdash$ '$\mathcal{T}(\cdot^{\overbrace{* \cdots *}^{n}})$ *is a well-partial-order'.*

Theorem 1.114.

1. $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \nvdash$ '$\mathcal{T}(\cdot^*)$ *is a well-partial-order',*
2. *For every natural n,* $\mathsf{RCA}_0 + \mathsf{CAC} + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \vdash$ '$\mathcal{T}(\cdot^n)$ *is a well-partial-order'.*

Chapter 7 explores whether the general conjecture of the correspondence between a maximal linear extension of $\mathbb{T}_n^{wgap}$ and the ϑ_i collapsing functions is still valid in the sequential version. We show that the correspondence is true if $n = 1$ or $n = 2$, but from the moment that $n > 2$, it is surprisingly not anymore the case. More specifically, we show that

$$\vartheta_0 \dots \vartheta_n \Omega_{n+1} = \omega_{n+2},$$

if $n \geq 1$, where ϑ_i are the usual collapsing functions, but now defined without the addition-operator. Furthermore, in Section 7.4, we show that a statement, proposed by Keita Yokoyama, about the sequences with the gap-embeddability relation has reverse mathematical strength equal to ACA_0'. The appendix contains a Dutch summary.

We note that the ordinal representations of the tree-classes $\mathbb{T}_n^{wgap}$ for $n > 2$ are not discussed in this dissertation. We believe that only small progress will be needed to obtain the expected results. In Chapter 5 we indicate how this could possibly be achieved.

In this context, we want to make a comment on possible *applications* of the results obtained in this dissertation. First of all, it can lead to an exact classification of Friedman's $\mathbb{wpo}$'s $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$. This can in turn imply independence results for theories lying between ATR_0 and $\Pi^1_1\text{-}\mathsf{CA}_0$. Secondly, following [73], the proof of the correctness of Weiermann's conjecture for specific W's can lead to a constructive well-partial-orderedness proof of $\mathcal{T}(W)$ by reifications (for more information see also [66]). This can imply a constructive proof of the well-partial-orderedness of Friedman's $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$ and maybe of the related graph minor theorem (see [28] for more information on the last theorem). To mention one more possible application, it can be used in term rewriting systems: $\mathbb{wpo}$'s are used in computer science to prove that certain term rewriting systems terminates and to obtain bounds on the *length of termination*. These bounds can be described using the notion of the maximal order type of the used well-partial-order.

Chapter 2

Unstructured trees

2.1 Introduction

Diana Schmidt's results [69] (see Theorem 1.79) give a characterization of well-partial-orders on the set of finite *structured* rooted trees. However, an order-theoretic characterization of the *unstructured* trees has never been established. In this chapter, we are going to solve this problem.

Trivially, the identity is a quasi-embedding from the unstructured trees to the structured ones. More specifically, this quasi-embedding maps the unstructured tree $[t]$ to t (using Remark 1.74), where $[t]$ is an equivalence class on the set of unstructured trees. This function actually picks one fixed representative from an equivalence class. Therefore, we have the following obvious results (see Notation 1.79 for the definition of the tree-classes).

1. $o(\mathbb{T}(X)) \leq o(\mathbb{T}^s(X))$,
2. $o(\mathbb{T}^{\leq m}(X)) \leq o(\mathbb{T}^{s,\leq m}(X))$,
3. $o(\mathbb{T}(X_0, \dots, X_n)) \leq o(\mathbb{T}^s(X_0, \dots, X_n))$,
4. $o\left(\mathbb{T}\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}\right) \leq o\left(\mathbb{T}^s\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}\right)$.

One might conjecture that these inequalities are all equalities, however one cannot find complete proofs of these facts in the literature. Harvey Friedman and Andreas Weiermann worked on binary unstructured trees with one or two labels, i.e. the partial orders $\mathbb{T}(\{0\}, \emptyset, \{0\})$ and $\mathbb{T}(\{0,1\}, \emptyset, \{0,1\})$.

They proved that the maximal order type in this case is the same as for the structured versions, i.e. ε_0 and Γ_0 respectively (unpublished preprint). They can also generalize their techniques to obtain a full classification of all binary trees, meaning they can calculate the maximal order type of $\mathbb{T}(X_0, X_1, X_2)$ where X_i are arbitrary wpo's. In later chapters, we only need the following statement.

Theorem 2.1. $o(\mathbb{T}(X, \emptyset, \{0\})) = o(\mathbb{T}^s(X, \emptyset, \{0\})) = \varepsilon_{\overline{o(X)}}$.

In this chapter, a proof about the equality of the maximal order types of some not-yet studied structured and unstructured versions is given. More specifically, we will show that

1. $o(\mathbb{T}^{\leq n}) = o(\mathbb{T}^{s,\leq n}) = \vartheta(\Omega^n)$ if $n \geq 3$,
2. $o(\mathbb{T}) = o(\mathbb{T}^s) = \vartheta(\Omega^\omega)$.

These results are a contribution to Weiermann's conjecture: similarly as the proof of Theorem 1.109, the reader can show $\mathbb{T} \cong \mathcal{T}(W)$ if $W(X) = M^\diamond(X)\backslash\{[]\}$. Hence, our results would yield that the conjecture is true for this specific W.

Notation 2.2. *By $Multi_n(X)$ we denote the subset of $Multi(X)$ where every multiset has length n.*

Definition 2.3. *Let α be an ordinal. Define $\overline{<}$ on $Multi(\alpha)$ as follows.*

$$[]\overline{<}[m_1, \dots, m_p]$$

if $p \geq 1$. And

$$[n_1, \dots, n_l]\overline{<}[m_1, \dots, m_p],$$

with $n_1 \leq \dots \leq n_l$ and $m_1 \leq \dots \leq m_p$ iff

$$n_1 < m_1 \text{ or } (n_1 = m_1 \text{ and } [n_2, \dots, n_l]\overline{<}[m_2, \dots, m_p]).$$

We write $m\overline{\leq}n$ if $m\overline{<}n$ or $m = n$.

Lemma 2.4. *The ordering $\overline{\leq}$ is a linear extension of $\leq^\diamond$ on $Multi(\alpha)$.*

Proof. Straightforward. □

Lemma 2.5. *Suppose that* $\alpha_i, \beta_i < \Omega$. *Then*

$$\begin{aligned}
&\Omega^{n-1}\alpha_{n-1} + \cdots + \Omega^0\alpha_0 \leq \Omega^{n-1}\beta_{n-1} + \cdots + \Omega^0\beta_0 \\
\iff &[\alpha_{n-1}, \alpha_{n-1} \oplus \alpha_{n-2}, \ldots, \alpha_{n-1} \oplus \cdots \oplus \alpha_0] \\
&\overline{\leq} [\beta_{n-1}, \beta_{n-1} \oplus \beta_{n-2}, \ldots, \beta_{n-1} \oplus \cdots \oplus \beta_0] \\
\iff &\Omega^{n-1}\alpha_{n-1} + \Omega^{n-2}(\alpha_{n-1} \oplus \alpha_{n-2}) + \cdots + \Omega^0(\alpha_{n-1} \oplus \cdots \oplus \alpha_0) \\
&\leq \Omega^{n-1}\beta_{n-1} + \Omega^{n-2}(\beta_{n-1} \oplus \beta_{n-2}) + \cdots + \Omega^0(\beta_{n-1} \oplus \cdots \oplus \beta_0).
\end{aligned}$$

Proof. Straightforward. □

2.2 Lower bound

As indicated in the previous section and using the results of Diana Schmidt in Theorem 1.79, we have the following results

$$\begin{aligned}
o(\mathbb{T}^{\leq n}) &\leq o(\mathbb{T}^{s,\leq n}) = \vartheta(\Omega^n), \\
o(\mathbb{T}) &\leq o(\mathbb{T}^s) = \vartheta(\Omega^\omega).
\end{aligned}$$

Therefore, if we can prove that $\vartheta(\Omega^n)$ is a lower bound for $o(\mathbb{T}^{\leq n})$, we obtain the desired results:

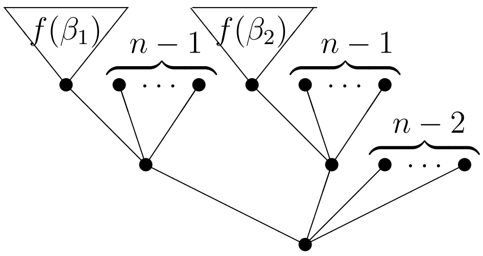
$$\begin{aligned}
o(\mathbb{T}^{\leq n}) &= o(\mathbb{T}^{s,\leq n}) = \vartheta(\Omega^n), \\
o(\mathbb{T}) &= o(\mathbb{T}^s) = \vartheta(\Omega^\omega).
\end{aligned}$$

Actually, we will show that $\vartheta(\Omega^n)$ is a lower bound for the maximal order type of $\mathbb{T}^{=n}$, where we define $\mathbb{T}^{=n}$ as $\mathbb{T}(\{0\}, \overbrace{\emptyset, \ldots, \emptyset}^{n-2}, \{0\})$, the set of trees where every node is either a leaf or has exactly n many successors. One can interpret $\mathbb{T}^{=n}$ in a natural way as a subset of the partial order $\mathbb{T}^{\leq n}$, hence this would yield

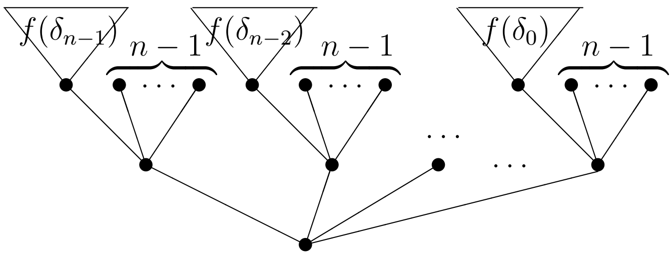
$$\vartheta(\Omega^n) \leq o(\mathbb{T}^{=n}) \leq o(\mathbb{T}^{\leq n}).$$

Theorem 2.6. $\vartheta(\Omega^n) \leq o(\mathbb{T}^{=n})$ *for* $n \geq 3$.

Proof. We do not write the labels in a tree from $\mathbb{T}^{=n} = \mathbb{T}(\{0\}, \emptyset, \ldots, \emptyset, \{0\})$ for notational convenience. We construct a quasi-embedding f from $\vartheta(\Omega^n)$ to $\mathbb{T}^{=n}$. Define $f(0)$ as the tree with one single node. Assume that $\beta = \beta_1 \oplus \beta_2 > \beta_1, \beta_2 > 0$ and $\beta_1 \in P$. Define $f(\beta)$ as



Assume $\beta \in P$. Then $\beta = \vartheta(\Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0) > \beta_i$ for all β_i. Because $\beta \in P$, we know that $\beta_{n-1} \oplus \cdots \oplus \beta_i < \beta$. Define $f(\beta)$ as



where $\delta_i = \beta_{n-1} \oplus \cdots \oplus \beta_i$. We now prove that f is a quasi-embedding. Suppose that $f(\alpha) \leq f(\beta)$. We prove by induction on α and β that $\alpha \leq \beta$. If α and/or β is equal to zero, then this is trivial. Assume $\alpha, \beta > 0$.

i) $\alpha = \alpha_1 \oplus \alpha_2$ *and* $\beta = \beta_1 \oplus \beta_2$
If the root of $f(\alpha)$ is embedded in $f(\beta_1)$ or $f(\beta_2)$, then the claim follows from the induction hypothesis. If the root of $f(\alpha)$ is not mapped into $f(\beta_1)$ and $f(\beta_2)$, then it should be mapped on the root of $f(\beta)$ due to embeddability constraints. From this, we obtain $\alpha_1 \leq \beta_1$ and $\alpha_2 \leq \beta_2$ or $\alpha_1 \leq \beta_2$ and $\alpha_2 \leq \beta_1$. Hence $\alpha \leq \beta$.

ii) $\alpha = \alpha_1 \oplus \alpha_2$ *and* $\beta = \vartheta(\Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0)$
$f(\alpha) \leq f(\beta)$ yields $f(\alpha_1), f(\alpha_2) < f(\beta)$. Hence the induction hypothesis yields $\alpha_1, \alpha_2 < \beta$, so $\alpha < \beta$ because $\beta \in P$.

iii) $\alpha = \vartheta(\Omega^{n-1}\alpha_{n-1} \oplus \cdots \oplus \alpha_0)$ *and* $\beta = \beta_1 \oplus \beta_2$
Define γ_i as $\alpha_{n-1} \oplus \cdots \oplus \alpha_i$. If the root of $f(\alpha)$ is embedded in $f(\beta_1)$ or $f(\beta_2)$, then the claim follows from the induction hypothesis. If the root of $f(\alpha)$ is not mapped into $f(\beta_1)$ and $f(\beta_2)$, then it should be mapped on the root of $f(\beta)$. But this is impossible because $f(\alpha)$ has strictly more than

two immediate subtrees of height bigger than zero and $f(\beta)$ has only two immediate subtrees of height bigger than zero.

iv) $\alpha = \vartheta(\Omega^{n-1}\alpha_{n-1} \oplus \cdots \oplus \alpha_0)$ *and* $\beta = \vartheta(\Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0)$
Define γ_i as $\alpha_{n-1} \oplus \cdots \oplus \alpha_i$ and δ_i as $\beta_{n-1} \oplus \cdots \oplus \beta_i$. If the root of $f(\alpha)$ is embedded in $f(\delta_i)$ for a certain i, then the claim follows from the induction hypothesis. If the root of $f(\alpha)$ is not mapped into $f(\delta_i)$ for all i, then it should be mapped on the root of $f(\beta)$. From this, it follows that

$$[f(\gamma_{n-1}), \dots, f(\gamma_0)] \leq^{\diamond} [f(\delta_{n-1}), \dots, f(\delta_0)].$$

By the induction hypothesis, we obtain

$$[\gamma_{n-1}, \dots, \gamma_0] \leq^{\diamond} [\delta_{n-1}, \dots, \delta_0]. \tag{2.1}$$

By Lemma 2.4, we see that

$$[\gamma_{n-1}, \dots, \gamma_0] \trianglelefteq [\delta_{n-1}, \dots, \delta_0].$$

Hence, by Lemma 2.5, we obtain

$$\Omega^{n-1}\alpha_{n-1} \oplus \cdots \oplus \alpha_0 \leq \Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0$$

Also, by inequality (2.1), we obtain that $\alpha_i \leq \gamma_i \leq \delta_{j_i} < \vartheta(\Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0) = \beta$ for all i because β is additively closed. Hence,

$$\alpha = \vartheta(\Omega^{n-1}\alpha_{n-1} \oplus \cdots \oplus \alpha_0) \leq \vartheta(\Omega^{n-1}\beta_{n-1} \oplus \cdots \oplus \beta_0) = \beta.$$

We can conclude that f is a quasi-embedding. □

Theorem 2.7. *If* $n \geq 3$, *then* $o(\mathbb{T}^{=n}) = o(\mathbb{T}^{\leq n}) = \vartheta(\Omega^n)$ *and* $o(\mathbb{T}) = \vartheta(\Omega^\omega)$.

Proof. This follows from the discussion just above Theorem 2.6. □

We believe that

$$\vartheta(\Omega^n \cdot o(X_n) + \cdots + \Omega \cdot o(X_1) + (-1 + o(X_0))) \leq o(\mathbb{T}(X_0, \dots, X_n))$$

and

$$\vartheta(\Omega^{\alpha_n} \cdot o(X_n) + \cdots + \Omega^{\alpha_0} \cdot o(X_0)) \leq o\left(\mathbb{T}\begin{pmatrix} X_0 & \dots & X_n \\ 1+\alpha_0 & \dots & 1+\alpha_n \end{pmatrix}\right)$$

can be proved in a similar way for $n \geq 3$.

Chapter 3

Capturing the big Veblen number

3.1 Introduction

In 1908, Veblen [84] introduced techniques of iteration and diagonalization to provide fast growing functions. His work led to the well-known *Veblen hierarchy* $\varphi_\alpha\beta$, where φ_α is the enumeration function of all common fixed points of φ_δ with $\delta < \alpha$ and $\varphi_0\beta := \omega^\beta$. This hierarchy can be used for a representation system for the limit of predicativity Γ_0 (see Section 1.2.3). In his paper Veblen also extended the binary Veblen function φ to finitely and transfinitely many argument. More specifically, he considered $\varphi(\alpha_0, \ldots, \alpha_\beta)$, where only finitely many arguments are non-zero. This led to a notation system for the *big Veblen number* $E(1)$ (in Veblen's notation). Schütte and others later denoted this ordinal by $E(0)$, where Schütte [70] developed his ordinal notation system for this ordinal by using his Klammer-symbols

$$\begin{pmatrix} \alpha_0 & \ldots & \alpha_n \\ \beta_0 & \ldots & \beta_n \end{pmatrix}.$$

In this dissertation, we use the ϑ-function (see Section 1.2.4) to denote this ordinal by $\vartheta\Omega^\Omega$.

As mentioned in the introductory Section 1.1, a way to study ordinal notation systems has been devised by Diana Schmidt in 1979 (see [69]). She showed that studying bounds on closure ordinals can best be achieved by determining maximal order types of wpo's which reflect monotonicity properties of the functions in question. If one takes an ordinal notation system T and restricts the ordering between the terms to those cases which are justified by the monotonicity and increasingness condition, one gets a well-partial-order and the maximal linear extension provides quite often an upper bound for the order type of the original set T. Apparently, in several examples of natural well-orderings the order type coincides with the maximal order type of the underlying well-partial-order.

As said in Subsection 1.1.1, this research has been taken up by Andreas Weiermann in [89]. He investigated an order-theoretic characterization in terms of well-partial-orders of the Schütte-Veblen hierarchy based on Klammer-symbols. He proved that the maximal order type of a wpo (in our notation the partial order $\mathcal{T}(M^\diamond(\tau \times \cdot)))$ is bounded by $\vartheta(\Omega^\tau)$, where τ is a fixed countable ordinal. Unfortunately, these results are not fully satisfying since they refer to an underlying structure of ordinals (τ) and not to terms of the corresponding ordinal notation system. Therefore, the representation of $\vartheta\Omega^\tau$ using $\mathcal{T}(M^\diamond(\tau \times \cdot))$ provides an ordinal notation system which can only be developed if we have an a priori effective term description for the segment τ (which is in general not allowed for an ordinal notation system). In this chapter, we continue this investigation and improve these results by replacing τ by *previously defined terms*, i.e. we do not need an a priori given segment of ordinals to describe such large ordinals. This produces an order-theoretic characterization of the big Veblen number $\vartheta\Omega^\Omega$. More specifically, we show that the maximal order type of $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ is equal to $\vartheta\Omega^\Omega$. This will be dealt with in Section 3.2. In the follow-up section (Section 3.3), we investigate what would happen if we replace the multisets by (ordered) sequences. Somewhat surprisingly, this wpo creates a much bigger ordinal, namely $\vartheta(\Omega^{\Omega^\Omega})$. The results in this chapter are also available in the article [82].

3.2 Finite multisets of pairs

Before we start, we fix some definitions and notations.

Definition 3.1. *Let α be an ordinal. Define $\breve{\alpha}$ by*

$$\breve{\alpha} := \begin{cases} \alpha + 1 & \text{if } \alpha < \omega, \\ \alpha & \text{otherwise.} \end{cases}$$

Notation 3.2. *Let* $\alpha =_{CNF} \omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ *be an ordinal. We use the notation* $\widetilde{\alpha}$ *for the ordinal* $\omega^{\breve{\alpha}_1} + \dots + \omega^{\breve{\alpha}_n}$.

Some easy consequences (recall Definition 1.67 and Notation 1.68).

Lemma 3.3. *1.* $\widetilde{\alpha}$ *is always a limit ordinal,*

2. $\alpha < \beta$ *implies* $\widetilde{\alpha} < \widetilde{\beta}$,

3. $\omega^{\widehat{\Omega\alpha}} = \Omega^{\widetilde{\alpha}}$ *for every countable ordinal* α.

Notation 3.4. *Let* γ *be an ordinal number. Define* $\gamma^{\overline{1}}$ *as* γ *and* $\gamma^{\overline{n+1}}$ *as* $\gamma^{\overline{n}} \otimes \gamma$.

The following theorem is needed for Theorem 3.6.

Theorem 3.5. *Assume* $W \in \mathcal{W}(\dots, \mathbb{X}_{k,l}, \dots, \mathbb{Y}_{k,l}, \dots)$ *with* $\mathbb{X}_{k,l}$ *and* $\mathbb{Y}_{k,l}$ *countable* wpo*'s* ($k \in \{0, \dots, K\}$ *and* $l \in \{0, \dots, L\}$). *Let* $W(X)$ *be*

$$\sum_{k=0}^{K} \sum_{l=0}^{L} M^{\diamond}(X \times \mathbb{X}_{k,l}) \times X^l \times \mathbb{Y}_{k,l},$$

where X^l *denotes the product* $X \times \dots \times X$ *with* l X*'s. Then* $\mathcal{T}(W)$ *is a* wpo *and* $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.

Proof. We prove the theorem by main induction on the ordinal $o(W(\Omega))$. Without loss of generality, we can assume that $\mathbb{Y}_{k,l}$ are nonempty wpo's, otherwise we can delete the corresponding term. If $o(W(\Omega)) < \Omega$, then $L = 0$ and $\mathbb{X}_{k,l} = \emptyset$ for every k and l. Hence $W(X) \cong \sum_{k=0}^{K} \mathbb{Y}_{k,0} =: \mathbb{Y}$ for every X. So $\mathcal{T}(W) \cong \mathcal{T}(\mathbb{Y})$. Theorem 1.112 then yields $\mathcal{T}(W)$ is a wpo and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.

If $o(W(\Omega)) \geq \Omega$, in other words X really occurs in $W(X)$, then $\vartheta(o(W(\Omega)))$ is an epsilon number. We want to prove that $L(t)$ is a wpo and $l(t) < \vartheta(o(W(\Omega)))$ for every t in $\mathcal{T}(W)$. We do this by induction on the complexity $C(t)$ of t. The theorem then follows from Theorem 1.54. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(o(W(\Omega)))$. Assume that

$$t = \circ([(t_1^1, x_1), \dots, (t_n^1, x_n)], (t_1^2, \dots, t_b^2), y)$$

with $L(t_j^i)$ wpo's and $l(t_j^i) < \vartheta(o(W(\Omega)))$, $x_i \in \mathbb{X}_{a,b}$ and $y \in \mathbb{Y}_{a,b}$. We show that $L(t)$ is a wpo and $l(t) < \vartheta(o(W(\Omega)))$. We note that if $\mathbb{X}_{a,b} = \emptyset$, then $n = 0$.

Suppose that $s = \circ([(s_1^1, \overline{x_1}), \ldots, (s_m^1, \overline{x_m})], (s_1^2, \ldots, s_{\overline{b}}^2), \overline{y})$, with $\overline{x_i} \in \mathbb{X}_{\overline{a},\overline{b}}$ and $\overline{y} \in \mathbb{Y}_{\overline{a},\overline{b}}$. $t \leq_{\mathcal{T}(W)} s$ is valid iff $t \leq_{\mathcal{T}(W)} s_i^j$ for a certain i and j or $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, and

$$(t_1^2, \ldots, t_b^2) \leq (s_1^2, \ldots, s_{\overline{b}}^2),$$
$$[(t_1^1, x_1), \ldots, (t_n^1, x_n)] \leq^{\diamond} [(s_1^1, \overline{x_1}), \ldots, (s_m^1, \overline{x_m})].$$

Therefore, $s \in L(t)$ iff $s_i^j \in L(t)$ for every i and j and one of the following holds

1. $a \neq \overline{a}$,
2. $a = \overline{a}$, $b \neq \overline{b}$,
3. $a = \overline{a}$, $b = \overline{b}$, $y \not\leq \overline{y}$,
4. $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, $(t_1^2, \ldots, t_b^2) \not\leq (s_1^2, \ldots, s_{\overline{b}}^2)$,
5. $a = \overline{a}$, $b = \overline{b}$, $y \leq \overline{y}$, $(t_1^2, \ldots, t_b^2) \leq (s_1^2, \ldots, s_{\overline{b}}^2)$, $[(t_1^1, x_1), \ldots, (t_n^1, x_n)] \not\leq^{\diamond} [(s_1^1, \overline{x_1}), \ldots, (s_m^1, \overline{x_m})]$.

We assume that if we talk about multisets, we have a fixed representation (note that a multiset is formally defined as an equivalence class). If (4.) holds, there must be a minimal index $l(s)$ such that

$$t_1^2 \leq s_1^2, \ldots, t_{l(s)-1}^2 \leq s_{l(s)-1}^2, t_{l(s)}^2 \not\leq s_{l(s)}^2.$$

If (5.) holds, we must be in one of the following groups

1. $(t_1^1, x_1) \not\leq (s_i^1, \overline{x_i})$ for every i,
2. there exists i_1 such that $(t_1^1, x_1) \leq (s_{i_1}^1, \overline{x_{i_1}})$ and $(t_2^1, x_2) \not\leq (s_i^1, \overline{x_i})$ for every $i \neq i_1$ (choose i_1 minimal),
3. there exist distinct indices i_1 and i_2 such that $(t_1^1, x_1) \leq (s_{i_1}^1, \overline{x_{i_1}})$, $(t_2^1, x_2) \leq (s_{i_2}^1, \overline{x_{i_2}})$ and $(t_3^1, x_3) \not\leq (s_i^1, \overline{x_i})$ for every $i \neq i_1, i_2$ (choose i_1, i_2 minimal with respect to the lexicographic ordering on the couples (i_1, i_2) for which this holds),

   ...

n. there exist distinct indices $i_1, \ldots, i_{n-1}$ such that $(t_1^1, x_1) \leq (s_{i_1}^1, \overline{x_{i_1}})$, $(t_2^1, x_2) \leq (s_{i_2}^1, \overline{x_{i_2}})$, $\ldots$, $(t_{n-1}^1, x_{n-1}) \leq (s_{i_{n-1}}^1, \overline{x_{i_{n-1}}})$ and $(t_n^1, x_n) \not\leq (s_i^1, \overline{x_i})$ for every $i \neq i_1, \ldots, i_{n-1}$ (choose $i_1, \ldots, i_{n-1}$ minimal with respect to the lexicographic ordering on the $(n-1)$-tuples $(i_1, i_2, \ldots, i_{n-1})$ for which this holds).

It is easy to see that $(t_i^1, x_i) \not\leq (s_j^1, \overline{x_j})$ is equivalent with the formula $s_j^1 \in L(t_i^1) \vee (t_i^1 \leq s_j^1 \wedge \overline{x_j} \in L_{\mathbb{X}_{a,b}}(x_i))$. Define $W'(X)$ as

$$\sum_{k=0,k\neq a}^{K} \sum_{l=0}^{L} M^\diamond(X \times \mathbb{X}_{k,l}) \times X^l \times \mathbb{Y}_{k,l}$$
$$+ \sum_{l=0,l\neq b}^{L} M^\diamond(X \times \mathbb{X}_{a,l}) \times X^l \times \mathbb{Y}_{a,l}$$
$$+ M^\diamond(X \times \mathbb{X}_{a,b}) \times X^b \times L_{\mathbb{Y}_{a,b}}(y)$$
$$+ \sum_{l=1}^{b} M^\diamond(X \times \mathbb{X}_{a,b}) \times X^{b-1} \times L_{\mathcal{T}(W)}(t_l^2) \times \mathbb{Y}_{a,b}$$
$$+ \sum_{k=1}^{n} (X \times \mathbb{X}_{a,b})^{k-1} \times M^\diamond(L_{\mathcal{T}(W)}(t_k^1) \times \mathbb{X}_{a,b}) \times M^\diamond(X \times L_{\mathbb{X}_{a,b}}(x_k)) \times X^b \times \mathbb{Y}_{a,b}$$

W' is an element of $\mathcal{W}(\mathbb{X}_{k,l}, \mathbb{Y}_{k,l}, L_{\mathbb{Y}_{a,b}}(y), L_{\mathbb{X}_{a,b}}(x_k), L(t_j^i))$. Note that all $\mathbb{X}_{k,l}, \mathbb{Y}_{k,l}, L_{\mathbb{Y}_{a,b}}(y), L_{\mathbb{X}_{a,b}}(x_k), L(t_j^i)$ are countable $\mathbb{wpo}$'s by assumption.

The five terms separated by + correspond to the five groups (1. - 5.) in which s can lie in. The index l in the fourth line corresponds to $l(s)$. The index k in the fifth line corresponds to which case (1. - n.) we are at that moment.
So recall that if $s \in L(t)$, then $s_i^j \in L(t)$ for every i and j and $\times t \not\leq_{W(\mathcal{T}(W))} \times s$, where we characterized $\times t \not\leq_{W(\mathcal{T}(W))} \times s$ by the five cases 1. - 5., hence it is characterized by W'. Therefore, $\times s$ (with $s \in L(t)$) can be interpreted as an element $w'(s_1, \ldots, s_r)$ of $W'(L(t))$ with every s_k equal to a certain $s_i^j \in L(t)$ and $w'(x_1, \ldots, x_r) \in W'$. Let $w'(s_1, \ldots, s_r)$ be this interpretation of $\times s$ and $w''(s'_1, \ldots, s'_{r'})$ the interpretation of $\times s'$ for an arbitrary $s' \in L(t)$. It can be proved in a straightforward way that the inequality $w'(s_1, \ldots, s_r) \leq_{W'(\mathcal{T}(W))} w''(s'_1, \ldots s'_{r'})$ implies $\times s \leq_{W(\mathcal{T}(W))} \times s'$, hence $s \leq_{\mathcal{T}(W)} s'$. Because a similar argument (completely written out) can be found in the proof of Theorem 3.6, we skip the detailed verification of this fact.

There exists a quasi-embedding f from $L(t)$ in $\mathcal{T}(W')$: define $f(\circ)$ as $\circ$. Assume

$$s = \circ([(s_1^1, \overline{x_1}), \ldots, (s_m^1, \overline{x_m})], (s_1^2, \ldots, s_{\overline{b}}^2), \overline{y}) \in L(t)$$

and suppose that $f(s_i^j)$ is already defined. Let $w'(s_1, \ldots, s_r)$ be the interpretation of $\times s$ in $W'(L(t))$. Then $\{s_1, \ldots, s_r\} \subseteq \{s_1^1, \ldots, s_m^1, s_1^2, \ldots, s_{\overline{b}}^2\}$ and define $f(s)$ as the element $\circ[w'(f(s_1), \ldots, f(s_r))]$ in $\mathcal{T}(W')$.

We show that f is a quasi-embedding. We prove, by induction on the sum of the complexities of s and s', that $f(s) \leq_{\mathcal{T}(W')} f(s')$ implies $s \leq_{\mathcal{T}(W)} s'$. If either s or s' is equal to $\circ$, this is trivial. Suppose $f(s) \leq_{\mathcal{T}(W')} f(s')$ with $f(s) = \circ[w'(f(s_1), \dots, f(s_r))]$ and $f(s') = \circ[w''(f(s'_1), \dots, f(s'_{r'}))]$. Then $f(s) \leq_{\mathcal{T}(W')} f(s'_i)$ for a certain i or

$$w'(f(s_1), \dots, f(s_r)) \leq_{W'(\mathcal{T}(W'))} w''(f(s'_1), \dots, f(s'_{r'})).$$

In the former case, we obtain by the induction hypothesis, that $s \leq_{\mathcal{T}(W)} s'_i \leq_{\mathcal{T}(W)} s'$. In the latter case, f is a quasi-embedding of the set $S = \{s^1_1, \dots, s^2_b\} \cup \{s'^1_1, \dots, s'^2_{b'}\} \subseteq \mathcal{T}(W)$ into $f(S) \subseteq \mathcal{T}(W')$ by the induction hypothesis. Therefore, by the Lifting Lemma

$$w'(s_1, \dots, s_r) \leq_{W'(\mathcal{T}(W))} w''(s'_1, \dots s'_{r'}).$$

Hence, $s \leq_{\mathcal{T}(W)} s'$.

Because of Lemma 1.56, we obtain $o(L(t)) \leq o(\mathcal{T}(W'))$. If we can prove the inequalities $o(W'(\Omega)) < o(W(\Omega))$ and $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$, the main induction hypothesis yields that

$$\begin{aligned} o(L(t)) &\leq o(\mathcal{T}(W')) \\ &\leq \vartheta(o(W'(\Omega))) \\ &< \vartheta(o(W(\Omega))), \end{aligned}$$

and that $L(t)$ is a $\mathbb{wpo}$ by Lemma 1.56. Hence, we are done.

a) $o(W'(\Omega)) < o(W(\Omega))$.
For notational convenience, we write sometimes $\mathbb{Y}$ instead of $o(\mathbb{Y})$ for $\mathbb{wpo}$'s $\mathbb{Y}$. $o(W'(\Omega)) < o(W(\Omega))$ is equivalent with (using Theorem 1.61 and 1.69)

$$\begin{aligned} &\omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes l_{\mathbb{Y}_{a,b}}(y) \\ &\oplus \bigoplus_{l=1}^{b} \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^{b-1} \otimes l_{\mathcal{T}(W)}(t^2_l) \otimes \mathbb{Y}_{a,b} \\ &\oplus \bigoplus_{k=1}^{n} (\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t^1_k) \otimes \mathbb{X}_{a,b}}} \otimes \omega^{\widehat{\Omega \otimes l_{\mathbb{X}_{a,b}}(x_k)}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} \\ < {}& \omega^{\widehat{\Omega \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b}. \end{aligned}$$

It is easy to see that there exists a finite N such that for every $k \in \{0, \dots, K\}$,

$$(\Omega \otimes \mathbb{X}_{a,b})^{\overline{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t^1_k) \otimes \mathbb{X}_{a,b}}} \otimes \Omega^b \otimes \mathbb{Y}_{a,b} < \Omega^N.$$

Note that all occurring wpo's are countable. Furthermore,

$$\omega^{\widehat{\Omega\otimes l_{\mathbb{X}_{a,b}}(x_k)}} = \Omega^{\widetilde{l_{\mathbb{X}_{a,b}}(x_k)}},$$
$$\omega^{\widehat{\Omega\otimes \mathbb{X}_{a,b}}} = \Omega^{\widetilde{\mathbb{X}_{a,b}}},$$

using Notations 1.68 and 3.2. Because $\widetilde{\mathbb{X}_{a,b}}$ is a limit ordinal and $\widetilde{l_{\mathbb{X}_{a,b}}(x_k)} < \widetilde{\mathbb{X}_{a,b}}$, we obtain

$$\begin{aligned}
&(\Omega\otimes\mathbb{X}_{a,b})^{\overline{k-1}}\otimes\omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1)\otimes\mathbb{X}_{a,b}}}\otimes\omega^{\widehat{\Omega\otimes l_{\mathbb{X}_{a,b}}(x_k)}}\otimes\Omega^b\otimes\mathbb{Y}_{a,b}\\
&<\Omega^{\widetilde{l_{\mathbb{X}_{a,b}}(x_k)}+N}\\
&<\Omega^{\widetilde{\mathbb{X}_{a,b}}}\\
&=\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}.
\end{aligned}$$

The last ordinal number is additively closed, hence

$$\bigoplus_{k=1}^{n}(\Omega\otimes\mathbb{X}_{a,b})^{\overline{k-1}}\otimes\omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1)\otimes\mathbb{X}_{a,b}}}\otimes\omega^{\widehat{\Omega\otimes l_{\mathbb{X}_{a,b}}(x_k)}}\otimes\Omega^b\otimes\mathbb{Y}_{a,b}<\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}.$$

Similarly,

$$\left(\bigoplus_{l=1}^{b}\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^{b-1}\otimes l_{\mathcal{T}(W)}(t_l^2)\otimes\mathbb{Y}_{a,b}\right)\oplus\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\le\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b,$$

because $\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b$ is additively closed, and from which we can conclude

$$\begin{aligned}
&\left(\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b\otimes l_{\mathbb{Y}_{a,b}}(y)\right)\oplus\left(\bigoplus_{l=1}^{b}\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^{b-1}\otimes l_{\mathcal{T}(W)}(t_l^2)\otimes\mathbb{Y}_{a,b}\right)\\
&\oplus\left(\bigoplus_{k=1}^{n}(\Omega\otimes\mathbb{X}_{a,b})^{\overline{k-1}}\otimes\omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1)\otimes\mathbb{X}_{a,b}}}\otimes\omega^{\widehat{\Omega\otimes l_{\mathbb{X}_{a,b}}(x_k)}}\otimes\Omega^b\otimes\mathbb{Y}_{a,b}\right)\\
<\ &\left(\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b\otimes l_{\mathbb{Y}_{a,b}}(y)\right)\oplus\left(\bigoplus_{l=1}^{b}\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^{b-1}\otimes l_{\mathcal{T}(W)}(t_l^2)\otimes\mathbb{Y}_{a,b}\right)\\
&\oplus\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\\
\le\ &\left(\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b\otimes l_{\mathbb{Y}_{a,b}}(y)\right)\oplus\left(\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b\right)\\
\le\ &\omega^{\widehat{\Omega\otimes\mathbb{X}_{a,b}}}\otimes\Omega^b\otimes\mathbb{Y}_{a,b}.
\end{aligned}$$

This strict inequality also holds in the exceptional cases $b=0$ and $n=0$.

b) $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.

$$o(W(\Omega)) = \bigoplus_{k=0}^{K}\bigoplus_{l=0}^{L} \omega^{\widehat{\Omega\otimes\mathbb{X}_{k,l}}} \otimes \Omega^{l} \otimes \mathbb{Y}_{k,l} = \bigoplus_{k=0}^{K}\bigoplus_{l=0}^{L} \Omega^{\widetilde{\mathbb{X}_{k,l}}\oplus l} \otimes \mathbb{Y}_{k,l},$$

from which we obtain

$$\mathbb{Y}_{k,l}, \widetilde{\mathbb{X}_{k,l}} \oplus l \leq k(o(W(\Omega))) < \vartheta(o(W(\Omega))). \tag{3.1}$$

Furthermore, $\mathbb{X}_{k,l} \leq \widetilde{\mathbb{X}_{k,l}}$. Now, $o(W'(\Omega))$ is equal to

$$\begin{aligned}
&\bigoplus_{k=0,k\neq a}^{K}\ \bigoplus_{l=0}^{L} \Omega^{\widetilde{\mathbb{X}_{k,l}}\oplus l} \otimes \mathbb{Y}_{k,l}\\
\oplus &\bigoplus_{l=0,l\neq b}^{L} \Omega^{\widetilde{\mathbb{X}_{a,l}}\oplus l} \otimes \mathbb{Y}_{a,l}\\
\oplus\ &\Omega^{\widetilde{\mathbb{X}_{a,b}}\oplus b} \otimes l_{\mathbb{Y}_{a,b}}(y)\\
\oplus &\bigoplus_{l=1}^{b} \Omega^{\widetilde{\mathbb{X}_{a,b}}\oplus (b-1)} \otimes l_{\mathcal{T}(W)}(t_l^2) \otimes \mathbb{Y}_{a,b}\\
\oplus &\bigoplus_{k=1}^{n} \Omega^{(k-1)\oplus \widetilde{l_{\mathbb{X}_{a,b}}(x_k)}\oplus b} \otimes \overline{\mathbb{X}_{a,b}^{k-1}} \otimes \omega^{\widehat{l_{\mathcal{T}(W)}(t_k^1)\otimes \mathbb{X}_{a,b}}} \otimes \mathbb{Y}_{a,b}.
\end{aligned}$$

Hence, $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$ by Lemma 1.10, inequality (3.1), $\widehat{o(\mathbb{X})} \leq o(\mathbb{X}) \otimes \omega$, $\widetilde{o(\mathbb{X})} \leq o(\mathbb{X}) \otimes \omega$, $l_{\mathcal{T}(W)}(t_j^i) < \vartheta(o(W(\Omega)))$ and the fact that $\vartheta(o(W(\Omega)))$ is an epsilon number. □

Theorem 3.6. *$\mathcal{T}(M^\diamond(\cdot \times \cdot))$ is a wpo and $o(\mathcal{T}(M^\diamond(\cdot \times \cdot))) \leq \vartheta(\Omega^\Omega)$.*

Proof. To prove the inequality $o(\mathcal{T}(M^\diamond(\cdot\times\cdot))) \leq \vartheta(\Omega^\Omega)$ and the well-partial-orderedness of $\mathcal{T}(M^\diamond(\cdot \times \cdot))$, we show that $L(t)$ is a wpo and $l(t) < \vartheta(\Omega^\Omega)$ for every t in $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ by induction on the complexity $C(t)$ of t. The theorem then follows from Theorem 1.54. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(\Omega^\Omega)$. Assume $t = \circ[(s_1, t_1) \ldots, (s_n, t_n)]$ with $L(t_i)$, $L(s_i)$ wpo's and $l(t_i), l(s_i) < \vartheta(\Omega^\Omega)$.

Take an arbitrary $v = \circ[(u_1, v_1), \ldots, (u_m, v_m)]$ in $\mathcal{T}(M^\diamond(\cdot \times \cdot))$. Then $t \leq v$ iff

$t \leq u_i$ or $t \leq v_i$ for a certain i
or $(\exists f : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, m\})(\forall i \in \{1, \ldots, n\})((s_i, t_i) \leq (u_{f(i)}, v_{f(i)}))$.

Hence $v = \circ[(u_1, v_1), \dots, (u_m, v_m)]$ is an element of $L(t)$ iff $u_i, v_i \in L(t)$ for every i and

$$(\forall f : \{1, \dots, n\} \hookrightarrow \{1, \dots, m\})(\exists i \in \{1, \dots, n\})((s_i, t_i) \not\leq (u_{f(i)}, v_{f(i)})).$$

Therefore, if $v \neq \circ$ and $v \in L(t)$, then $u_i, v_i \in L(t)$ for every i and one of the following conditions is satisfied:

1. $(s_1, t_1) \not\leq (u_i, v_i)$ for every i,
2. there exists i_1 such that $(s_1, t_1) \leq (u_{i_1}, v_{i_1})$ and $(s_2, t_2) \not\leq (u_i, v_i)$ for every $i \neq i_1$ (i_1 minimal chosen),
3. there exist distinct indices i_1 and i_2 such that $(s_1, t_1) \leq (u_{i_1}, v_{i_1})$; $(s_2, t_2) \leq (u_{i_2}, v_{i_2})$ and $(s_3, t_3) \not\leq (u_i, v_i)$ for every $i \neq i_1, i_2$ (choose i_1, i_2 minimal with respect to the lexicographic ordering on the couples (i_1, i_2) for which this holds),

   ...

n. there exist distinct indices $i_1, \dots, i_{n-1}$ such that $(s_1, t_1) \leq (u_{i_1}, v_{i_1})$; $(s_2, t_2) \leq (u_{i_2}, v_{i_2})$; ... $(s_{n-1}, t_{n-1}) \leq (u_{i_{n-1}}, v_{i_{n-1}})$ and $(s_n, t_n) \not\leq (u_i, v_i)$ for every $i \neq i_1, \dots, i_{n-1}$ (pick the indices $i_1, \dots, i_{n-1}$ minimal with respect to the lexicographic ordering on the $(n-1)$-tuples $(i_1, i_2, \dots, i_{n-1})$ for which this holds).

Also note that $\circ \in L(t)$. Define

$$W'(X) := \sum_{k=1}^{n} (X \times X)^{k-1} \times M^{\diamond}\left((L(s_k) \times X) + (X \times L(t_k))\right),$$

which we identify with $\sum_{k=1}^{n} (X \times X)^{k-1} \times M^{\diamond}\left((L(s_k) + L(t_k)) \times X\right)$. k represents which case (1.-n.) holds.

W' is an element of $\mathcal{W}(L(s_1) + L(t_1), \dots, L(s_n) + L(t_n))$. By assumption, $L(s_i) + L(t_i)$ are wpo's, hence Theorem 3.6 yields that $\mathcal{T}(W')$ is a wpo and $o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega)))$.

Define the map $f : L(t) \to \mathcal{T}(W')$ recursively as follows. First, let $f(\circ)$ be $\circ$. Secondly, suppose that $v = \circ[(u_1, v_1), \dots, (u_m, v_m)] \in L(t)$ and that $f(u_i)$, $f(v_i)$ are already defined. Assume that v lies in group k (hence we have indices $i_1, \dots, i_{k-1}$) and take $\{j_1, \dots, j_l\}$ as the subset of

$$\{1, \dots, m\} \backslash \{i_1, \dots, i_{k-1}\}$$

such that $u_{j_p} \in L(s_k)$ for every p. Define $\{r_1, \dots, r_t\}$ as

$$\{1, \dots, m\} \backslash \{i_1, \dots, i_{k-1}, j_1, \dots, j_l\}.$$

Note that $v_{r_p} \in L(t_k)$ for every p. Let $f(v)$ be the following element of $\mathcal{T}(W')$:

$$\circ \Big((f(u_{i_1}), f(v_{i_1})), \dots, (f(u_{i_{k-1}}), f(v_{i_{k-1}})), \\ [(u_{j_1}, f(v_{j_1})), \dots, (u_{j_l}, f(v_{j_l})), (f(u_{r_1}), v_{r_1}) \dots, (f(u_{r_t}), v_{r_t})] \Big). \quad (3.2)$$

If f is a quasi-embedding, then Lemma 1.56 yields that $L(t)$ is a wpo and

$$o(L(t)) \leq o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega))) = \vartheta\left(\bigoplus_{k=1}^{n} \Omega^{2k-2} \omega^{\widehat{\Omega \otimes (l(s_k) \oplus l(t_k))}}\right).$$

Seeing that

$$l(s_k) \oplus l(t_k) < \vartheta(\Omega^\Omega),$$

it can be shown in a similar way as in the proof of Theorem 3.5 that

$$\vartheta\left(\bigoplus_{k=1}^{n} \Omega^{2k-2} \omega^{\widehat{\Omega \otimes (l(s_k) \oplus l(t_k))}}\right) < \vartheta(\Omega^\Omega),$$

hence $o(L(t)) < \vartheta(\Omega^\Omega)$.

We still have to prove that f is a quasi-embedding. We show that $f(v) \leq f(v')$ implies $v \leq v'$ by induction on the complexity of v'. If $f(v) \leq f(\circ) = \circ$, then $v = \circ \leq v'$. Assume $v' = \circ[(u'_1, v'_1), \dots, (u'_{m'}, v'_{m'})] \in L(t)$ with $f(v')$ defined as

$$\circ \Big((f(u'_{i'_1}), f(v'_{i'_1})), \dots, (f(u'_{i'_{k'-1}}), f(v'_{i'_{k'-1}})), \\ [(u'_{j'_1}, f(v'_{j'_1})), \dots, (u'_{j'_{l'}}, f(v'_{j'_{l'}})), (f(u'_{r'_1}), v'_{r'_1}) \dots, (f(u'_{r'_{t'}}), v'_{r'_{t'}})] \Big)$$

and suppose $f(v) \leq f(v')$. We show that $v \leq v'$ holds. If $v = \circ$, this is trivial. Assume $v \neq \circ$ and say that $f(v)$ is defined as in (3.2). Because $f(v) \leq f(v')$, we obtain $f(v) \leq f(u'_p)$ or $f(v) \leq f(v'_p)$ for a certain p or $k = k'$ and

$$(f(u_{i_1}), f(v_{i_1})) \leq (f(u'_{i'_1}), f(v'_{i'_1})),$$

$$\dots$$

$$(f(u_{i_{k-1}}), f(v_{i_{k-1}})) \leq (f(u'_{i'_{k'-1}}), f(v'_{i'_{k'-1}})),$$

and

$$\begin{aligned}&[(u_{j_1}, f(v_{j_1})), \dots, (u_{j_l}, f(v_{j_l})), (f(u_{r_1}), v_{r_1}) \dots, (f(u_{r_t}), v_{r_t})]\\ \leq^\diamond &[(u'_{j'_1}, f(v'_{j'_1})), \dots, (u'_{j'_{l'}}, f(v'_{j'_{l'}})), (f(u'_{r'_1}), v'_{r'_1}) \dots, (f(u'_{r'_{t'}}), v'_{r'_{t'}})].\end{aligned}$$

In the two former cases, we obtain by the induction hypothesis that $v \leq u'_p$ or $v \leq v'_p$, hence $v \leq v'$. In the latter case, the induction hypothesis yields $(u_{i_p}, v_{i_p}) \leq (u'_{i'_p}, v'_{i'_p})$ for every $p = 1, \dots, k-1$. Furthermore, there exists an injective function $g : \{j_1, \dots, j_l, r_1, \dots, r_t\} \to \{j'_1, \dots, j'_{l'}, r'_1, \dots, r'_{t'}\}$ such that $g(j_p) = j'_l$ for a certain l and $g(r_p) = r'_l$ for a certain l with

$$(u_{j_p}, f(v_{j_p})) \leq (u'_{g(j_p)}, f(v'_{g(j_p)}))$$

and

$$(f(u_{r_p}), v_{r_p}) \leq (f(u'_{g(r_p)}), v'_{g(r_p)})$$

for every p. This is because $(u_{j_p}, f(v_{j_p}))$ is only comparable with a certain element $(u'_{j'_l}, f(v'_{j'_l}))$ and never with an element $(f(u'_{r'_l}), v'_{r'_l})$, since it follows the order on $W'(\mathcal{T}(W'))$. And $(f(u_{r_p}), v_{r_p})$ is only comparable with a certain $(f(u'_{r'_l}), v'_{r'_l})$ and never with a $(u'_{j'_l}, f(v'_{j'_l}))$. Using the induction hypothesis, we obtain

$$(u_{j_p}, v_{j_p}) \leq (u'_{g(j_p)}, v'_{g(j_p)})$$

and

$$(u_{r_p}, v_{r_p}) \leq (u'_{g(r_p)}, v'_{g(r_p)}).$$

Therefore

$$\begin{aligned}&[(u_{j_1}, v_{j_1}), \dots, (u_{j_l}, v_{j_l}), (u_{r_1}, v_{r_1}) \dots, (u_{r_t}, v_{r_t})]\\ \leq^\diamond &[(u'_{j'_1}, v'_{j'_1}), \dots, (u'_{j'_{l'}}, v'_{j'_{l'}}), (u'_{r'_1}, v'_{r'_1}) \dots, (u'_{r'_{t'}}, v'_{r'_{t'}})].\end{aligned}$$

Together with $(u_{i_p}, v_{i_p}) \leq (u'_{i'_p}, v'_{i'_p})$ for every $p = 1, \dots, k = k'$, we can conclude that

$$[(u_1, v_1), \dots, (u_m, v_m)] \leq^\diamond [(u'_1, v'_1), \dots, (u'_{m'}, v'_{m'})],$$

hence $v \leq v'$. □

The previous proof allows a constructive well-partial-orderedness proof by reifications (for more information see [66] and [73]). Now, we show that $\vartheta(\Omega^\Omega)$ is also a lower bound, but first we prove an additional lemma.

Lemma 3.7. *Suppose* $1 < \alpha < \vartheta(\Omega^\Omega)$ *and* $\alpha \in P$*, the set of additive closed ordinal numbers. Then there exists unique* $0 < \beta_i < \Omega$ *and* $\alpha_i < \Omega$ *such that* $\alpha = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n)$*,* $\alpha_1 > \cdots > \alpha_n$*.*

Proof. Using Corollary 1.8, we obtain a unique $\xi < \varepsilon_{\Omega+1}$ such that $\alpha = \vartheta(\xi)$. Denote ξ by $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n$ with $0 < \beta_i < \Omega$ and $\alpha_1 > \cdots > \alpha_n$. We only have to show that $\alpha_1 < \Omega$. If $\alpha_1 \geq \Omega$, then $\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \geq \Omega^\Omega$. So, $\alpha < \vartheta(\Omega^\Omega)$ can only hold if $\alpha \leq k(\Omega^\Omega)$. But $k(\Omega^\Omega) = 1 < \alpha$, hence α_1 has to be smaller than Ω. □

Theorem 3.8. $o(\mathcal{T}(M^\diamond(\cdot \times \cdot))) \geq \vartheta(\Omega^\Omega)$.

Proof. Define

$$\begin{aligned} g : \vartheta(\Omega^\Omega) &\to \mathcal{T}(M^\diamond(\cdot \times \cdot)), \\ 0 &\mapsto \circ, \\ 1 &\mapsto \circ[(\circ, \circ)], \\ \alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n},\ n \geq 2 &\mapsto \circ[(g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ)], \\ \alpha = \omega^\beta = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n) > 1 &\mapsto \circ[(g(\alpha_1), g(\beta_1)), \ldots, (g(\alpha_n), g(\beta_n))]. \end{aligned}$$

In this definition we assume that $\beta_i > 0$ as in Lemma 3.7. Obviously we see that $\beta = 0$ iff $g(\beta) = \circ$ and $\beta = 1$ iff $g(\beta) = \circ[(\circ, \circ)]$. If we prove that g is a quasi-embedding, we can conclude the theorem by Lemma 1.56. We show that $g(\alpha) \leq g(\alpha')$ implies $\alpha \leq \alpha'$ by induction on $\alpha \oplus \alpha'$. The cases α and/or α' equal to 0 or 1 are trivial, so we may assume that $\alpha, \alpha' > 1$.

a) $\alpha' =_{CNF} \omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}$**,** $m \geq 2$**.**

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$*,* $n \geq 2$*.*

If

$$g(\alpha) = \circ[(g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ)] \leq \circ[(g(\alpha'_1), \circ), \ldots, (g(\alpha'_m), \circ)] = g(\alpha').$$

then $g(\alpha) \leq g(\alpha'_i)$ for a certain i or

$$[(g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ)] \leq^\diamond [(g(\alpha'_1), \circ), \ldots, (g(\alpha'_m), \circ)]. \tag{3.3}$$

In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha'_i < \alpha'$. In the latter case, there exists an injective function f from $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$ such that $(g(\alpha_i), \circ) \leq (g(\alpha'_{f(i)}), \circ)$ for every i. Hence $g(\alpha_i) \leq$

$g(\alpha'_{f(i)})$, so the induction hypothesis yields $\alpha_i \leq \alpha'_{f(i)}$ for every i. Because $\alpha_1 \geq \cdots \geq \alpha_n$ and $\alpha'_1 \geq \cdots \geq \alpha'_m$, this implies $\alpha \leq \alpha'$.

ii) $1 < \alpha = \omega^\beta = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n)$.
$\beta_i > 0$, hence $g(\beta_i) \neq \circ$. Assume $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\alpha'_i)$ for a certain i or $[(g(\alpha_1), g(\beta_1)), \ldots, (g(\alpha_n), g(\beta_n))] \leq^\diamond [(g(\alpha'_1), \circ), \ldots, (g(\alpha'_m), \circ)]$. In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha'_i < \alpha'$. The latter case is impossible because $g(\beta_i) \not\leq \circ$.

b) $1 < \alpha' = \omega^{\beta'} = \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m)$.
We know that $g(\beta'_i) \neq \circ$.

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, $n \geq 2$.
Suppose $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\alpha'_i)$ for a certain i, or $g(\alpha) \leq g(\beta'_i)$ for a certain i, or

$$[(g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ)] \leq^\diamond [(g(\alpha'_1), g(\beta'_1)), \ldots, (g(\alpha'_m), g(\beta'_m))].$$

In the two former cases, we obtain by the induction hypothesis that $\alpha \leq \alpha'_i$ or $\alpha \leq \beta'_i$. In both cases, $\alpha \leq k(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m) < \vartheta(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m) = \alpha'$. If

$$[(g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ)] \leq^\diamond [(g(\alpha'_1), g(\beta'_1)), \ldots, (g(\alpha'_m), g(\beta'_m))] \tag{3.4}$$

holds, then for every i there exists a j such that $g(\alpha_i) \leq g(\alpha'_j)$. By the induction hypothesis, we obtain $\alpha_i \leq \alpha'_j < \alpha'$. If $\alpha'_1 > 0$, then α' is an epsilon number, so $\alpha < \alpha'$. Suppose $\alpha' = \vartheta(\Omega^0\beta_1)$ with $\beta_1 > 0$. Then $g(\alpha') = \circ[(\circ, g(\beta'_1))]$ and inequality 3.4 yield a contradiction because $n \geq 2$.

ii) $1 < \alpha = \omega^\beta = \vartheta(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n)$.
If

$$\begin{aligned} g(\alpha) &= \circ[(g(\alpha_1), g(\beta_1)), \ldots, (g(\alpha_n), g(\beta_n))] \\ &\leq g(\alpha') = \circ[(g(\alpha'_1), g(\beta'_1)), \ldots, (g(\alpha'_m), g(\beta'_m))], \end{aligned}$$

then either $g(\alpha) \leq g(\alpha'_i)$ or $g(\alpha) \leq g(\beta'_i)$ for a certain i or

$$\begin{aligned} &[(g(\alpha_1), g(\beta_1)), \ldots, (g(\alpha_n), g(\beta_n))] \\ &\leq^\diamond [(g(\alpha'_1), g(\beta'_1)), \ldots, (g(\alpha'_m), g(\beta'_m))]. \end{aligned} \tag{3.5}$$

In the former cases, $\alpha \leq \alpha'_i < \alpha'$ or $\alpha \leq \beta'_i < \alpha'$ by the induction hypothesis. In the latter case, there exists an injective function f from $\{1, \ldots, n\}$ to

$\{1, \ldots, m\}$ such that $(g(\alpha_i), g(\beta_i)) \leq (g(\alpha'_{f(i)}), g(\beta'_{f(i)}))$ for every i. This yields $(\alpha_i, \beta_i) \leq (\alpha'_{f(i)}, \beta'_{f(i)})$. Hence,

$$\begin{aligned} & k\left(\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n\right) \\ = & \max\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\} \\ \leq & \max\{\alpha'_1, \ldots, \alpha'_m, \beta'_1, \ldots, \beta'_m\} \\ = & k\left(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m\right) \\ < & \vartheta\left(\Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m\right). \end{aligned}$$

So if we can prove

$$\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \leq \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m,$$

we can finish the proof.

If $f(i) = i$ for every $i = 1, \ldots, n$, then $\alpha_i \leq \alpha'_i$ and $\beta_i \leq \beta'_i$ for all i. This yields

$$\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \leq \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m.$$

If $f(i)$ is not equal to i for every i, then choose the least i such that $f(i) \neq i$. Hence, $\alpha_j \leq \alpha'_j$ and $\beta_j \leq \beta'_j$ for all $j < i$, so

$$\Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_{i-1}}\beta_{i-1} \leq \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_{i-1}}\beta'_{i-1}.$$

Additionally $f(i) \neq i$ yields $f(i) > i$, so $\alpha_i \leq \alpha'_{f(i)} < \alpha'_i$. Hence, $\Omega^{\alpha_i}\beta_i + \cdots + \Omega^{\alpha_n}\beta_n < \Omega^{\alpha'_i}$. Therefore,

$$\begin{aligned} & \Omega^{\alpha_1}\beta_1 + \cdots + \Omega^{\alpha_n}\beta_n \\ < & \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_{i-1}}\beta'_{i-1} + \Omega^{\alpha'_i} \\ \leq & \Omega^{\alpha'_1}\beta'_1 + \cdots + \Omega^{\alpha'_m}\beta'_m. \end{aligned}$$

□

Corollary 3.9. $o(\mathcal{T}(M^{\diamond}(\cdot \times \cdot))) = \vartheta(\Omega^{\Omega})$.

Proof. Follows from Theorems 3.6 and 3.8. □

In [82], we also investigated the partial order $\mathcal{T}(M(\cdot \times \cdot))$. It turned out that this is also a wpo with the same maximal order type $\vartheta(\Omega^{\Omega})$.

3.3 Finite sequences of pairs

In this section, we show that using finite sequences instead of finite multisets implies a wpo that has a maximal order type bigger the the big Veblen number. The following theorem is needed for proving Theorem 3.11.

Theorem 3.10. *Let $\mathbb{Y}^i_j$, $\mathbb{Z}^i_j$ and $\mathbb{Z}_i$ be countable* wpo*'s and n_i and m_i be natural numbers. Assume $W \in \mathcal{W}(\dots, \mathbb{Y}^i_j, \dots, \mathbb{Z}^i_j, \dots, \mathbb{Z}_i, \dots)$. If*

$$W(X) = \sum_{i=0}^{N} \left(\left(\mathbb{Y}^i_1 \times X + \mathbb{Z}^i_1 \right)^* \times \dots \times \left(\mathbb{Y}^i_{n_i} \times X + \mathbb{Z}^i_{n_i} \right)^* \times X^{m_i} \times \mathbb{Z}_i \right),$$

then $\mathcal{T}(W)$ is a wpo *and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.*

Proof. We prove the theorem by main induction on the ordinal $o(W(\Omega))$. Without loss of generality, we may assume that $\mathbb{Y}^i_j$ and $\mathbb{Z}_i$ are non-empty wpo's (unless $W(X) \cong \emptyset$), otherwise we can rewrite or delete the corresponding term. If $o(W(\Omega)) < \Omega$, then $W(X)$ does not contain X (or W does not contain $\cdot$) and it is equal to a countable wpo $\mathbb{Z}$: in this case, $n_i = m_i = 0$ for all i. Therefore, $W(X) \cong \sum_{i=0}^{N} \mathbb{Z}_i$, which we call $\mathbb{Z}$. Hence $\mathcal{T}(W) \cong \mathcal{T}(\mathbb{Z})$, so Theorem 1.112 then yields $\mathcal{T}(W)$ is a wpo and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.

If $o(W(\Omega)) \geq \Omega$, in other words X really occurs in $W(X)$, then $\vartheta(o(W(\Omega)))$ is an epsilon number. We want to prove that $L(t)$ is a wpo and $l(t) < \vartheta(o(W(\Omega)))$ for every t in $\mathcal{T}(W)$, by induction on the complexity of t. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(o(W(\Omega)))$. Assume

$$t = \circ((\overline{t_1}, \dots, \overline{t_{n_k}}), (t_1, \dots, t_{m_k}), z)$$

with $\overline{t_j} = ((\overline{t_j})_1, \dots, (\overline{t_j})_{p_j})$ and either $(\overline{t_j})_i = z^j_i$ or $(\overline{t_j})_i = (y^j_i, t^j_i)$ with $L(t_i)$, $L(t^j_i)$ wpo's and $l(t_i), l(t^j_i) < \vartheta(o(W(\Omega)))$, $y^j_i \in \mathbb{Y}^k_j$, $z^j_i \in \mathbb{Z}^k_j$ and $z \in \mathbb{Z}_k$. Suppose s is an arbitrary element of $\mathcal{T}(W)$, different from $\circ$. Then

$$\begin{aligned} s &= \circ((\overline{s_1}, \dots, \overline{s_{n_l}}), (s_1, \dots, s_{m_l}), z'), \\ \overline{s_j} &= ((\overline{s_j})_1, \dots, (\overline{s_j})_{q_j}), \\ (\overline{s_j})_i &= z'^j_i \text{ or } (y'^j_i, s^j_i) \end{aligned} \tag{3.6}$$

with $z' \in \mathbb{Z}_l$, $y'^j_i \in \mathbb{Y}^l_j$ and $z'^j_i \in \mathbb{Z}^l_j$. $s \in L(t)$ is valid iff $s_i \in L(t)$, $s^j_i \in L(t)$ and one of the following holds:

1. $k \neq l$,

2. $k = l$, $z' \in L_{\mathbb{Z}_k}(z)$,
3. $k = l$, $z \leq_{\mathbb{Z}_k} z'$, $(t_1, \ldots, t_{m_k}) \not\leq (s_1, \ldots, s_{m_k})$,
4. $k = l$, $z \leq_{\mathbb{Z}_k} z'$, $(t_1, \ldots, t_{m_k}) \leq (s_1, \ldots, s_{m_k})$, $(\overline{t_1}, \ldots, \overline{t_{n_k}}) \not\leq (\overline{s_1}, \ldots, \overline{s_{n_k}})$.

If (3.) holds, there must be a minimal index $l(s)$ such that

$$t_1 \leq s_1, \ldots, t_{l(s)-1} \leq s_{l(s)-1}, t_{l(s)} \not\leq s_{l(s)}.$$

If (4.) holds, there must be a minimal index $k(s)$ such that

$$\overline{t_1} \leq \overline{s_1}, \ldots, \overline{t_{k(s)-1}} \leq \overline{s_{k(s)-1}}, \overline{t_{k(s)}} \not\leq \overline{s_{k(s)}}.$$

In this case

$$\overline{t_{k(s)}} = ((\overline{t_{k(s)}})_1, \ldots, (\overline{t_{k(s)}})_{p_{k(s)}}) \not\leq ((\overline{s_{k(s)}})_1, \ldots, (\overline{s_{k(s)}})_{q_{k(s)}}) = \overline{s_{k(s)}}$$

is valid iff one of the following cases holds

1. $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every j,
2. there exists an index j_1 such that $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every $j < j_1$, $(\overline{t_{k(s)}})_1 \leq (\overline{s_{k(s)}})_{j_1}$ and $(\overline{t_{k(s)}})_2 \not\leq (\overline{s_{k(s)}})_j$ for every $j > j_1$,

   ...

$p_{k(s)}$. there exist $p_{k(s)} - 1$ indices $j_1 < \cdots < j_{p_{k(s)}-1}$ such that $(\overline{t_{k(s)}})_1 \not\leq (\overline{s_{k(s)}})_j$ for every $j < j_1$, $(\overline{t_{k(s)}})_1 \leq (\overline{s_{k(s)}})_{j_1}$, $(\overline{t_{k(s)}})_2 \not\leq (\overline{s_{k(s)}})_j$ for every $j_2 > j > j_1$, ..., $(\overline{t_{k(s)}})_{p_{k(s)}-1} \leq (\overline{s_{k(s)}})_{j_{p_{k(s)}-1}}$ and $(\overline{t_{k(s)}})_{p_{k(s)}} \not\leq (\overline{s_{k(s)}})_j$ for every $j > j_{p_{k(s)}-1}$.

If $(\overline{t_j})_i = z_i^j$, define L_i^j as $\mathbb{Y}_j^k \times X + L_{\mathbb{Z}_j^k}(z_i^j)$. If $(\overline{t_j})_i = (y_i^j, t_i^j)$, define L_i^j as $(L_{\mathbb{Y}_j^k}(y_i^j) \times X) + (\mathbb{Y}_j^k \times L(t_i^j)) + \mathbb{Z}_j^k$. Define $W'(X)$ as follows

$$\begin{aligned}
&\sum_{i=0, i\neq k}^{N} \left(\left(\mathbb{Y}_1^i \times X + \mathbb{Z}_1^i\right)^* \times \cdots \times \left(\mathbb{Y}_{n_i}^i \times X + \mathbb{Z}_{n_i}^i\right)^* \times X^{m_i} \times \mathbb{Z}_i\right) \\
&+ \left(\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k} \times L_{\mathbb{Z}_k}(z)\right) \\
&+ \sum_{i=1}^{m_k} \left(\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k-1} \times L(t_i) \times \mathbb{Z}_k\right) \\
&+ \sum_{j=1}^{n_k} \sum_{i=1}^{p_j} \left[\left(\mathbb{Y}_1^k \times X + \mathbb{Z}_1^k\right)^* \times \cdots \times \left(\mathbb{Y}_{j-1}^k \times X + \mathbb{Z}_{j-1}^k\right)^*\right. \\
&\quad \times (\mathbb{Y}_j^k \times X + \mathbb{Z}_j^k)^{i-1} \times (L_1^j)^* \times \cdots \times (L_i^j)^* \\
&\quad \left.\times \left(\mathbb{Y}_{j+1}^k \times X + \mathbb{Z}_{j+1}^k\right)^* \times \cdots \times \left(\mathbb{Y}_{n_k}^k \times X + \mathbb{Z}_{n_k}^k\right)^* \times X^{m_k} \times \mathbb{Z}_k\right].
\end{aligned}$$

The four cases separated by a + represents the four groups in which s can lie in. The index i in the third term represents $l(s)$. The index j, respectively i, in the fourth term represents $k(s)$, respectively case 1. - $p_{k(s)}$. We can interpret $\times s$ with $s \in L(t)$ as an element of $W'(L(t))$ like in the proof of Theorems 3.5 and 3.6. With this in mind, we can define a map $f : L(t) \to \mathcal{T}(W')$ as follows. First define $f(\circ)$ as $\circ$. Then, assuming s as in (3.6) and assuming that $f(s_i)$ and $f(s_i^j)$ are already defined, let $f(s)$ be $\circ[w(f(s'_1), \ldots, f(s'_r))]$, where $w(s'_1, \ldots, s'_r)$ is the interpretation of $\times s$ as an element of $W'(L(t))$ and $\{s'_1, \ldots, s'_r\} \subseteq \{s_1, \ldots, s_{m_l}, s_1^1, \ldots, s_{q_{n_l}}^{n_l}\}$.
It can be proved in a similar way as in Theorem 3.6 that f is a quasi-embedding.

By Lemma 1.56 we obtain

$$o(L(t)) \leq o(\mathcal{T}(W')).$$

If $o(W'(\Omega)) < o(W(\Omega))$, we obtain that $\mathcal{T}(W')$ is a wpo (hence $L(t)$ is a wpo) and $o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega)))$ by the main induction hypothesis. If additionally the inequality $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$ holds, then

$$o(L(t)) \leq o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega))) < \vartheta(o(W(\Omega))).$$

So the only two remaining things we have to prove are the inequalities $o(W'(\Omega)) < o(W(\Omega))$ and $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.

a) $o(W'(\Omega)) < o(W(\Omega))$.
For notational convenience, we write sometimes $\mathbb{Y}$, respectively $\mathbb{Z}^*$, instead of $o(\mathbb{Y})$, respectively $o(\mathbb{Z}^*)$, for wpo's $\mathbb{Y}$ and $\mathbb{Z}$. It is easy to see that $\mathbb{Y}_j^k \otimes \Omega \oplus l_{\mathbb{Z}_j^k}(z_i^j) < \mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k$ and $(l_{\mathbb{Y}_j^k}(y_i^j) \otimes \Omega) \oplus (\mathbb{Y}_j^k \otimes l(t_i^j)) \oplus \mathbb{Z}_j^k < \mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k$ and $\Omega^{m_k} \otimes \mathbb{Z}_k < \Omega^\omega \leq \left(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\right)^*$, hence

$$\left(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\right)^{\overline{i-1}} \otimes o((L_1^j)^*) \otimes \cdots \otimes o((L_i^j)^*) \otimes \Omega^{m_k} \otimes \mathbb{Z}_k < \left(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\right)^*,$$

because $\left(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\right)^*$ is multiplicatively closed. We obtain

$$\begin{aligned} &\bigoplus_{j=1}^{n_k}\bigoplus_{i=1}^{p_j}\Big[\left(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\right)^* \otimes \cdots \otimes \left(\mathbb{Y}_{j-1}^k \otimes \Omega \oplus \mathbb{Z}_{j-1}^k\right)^* \otimes \left(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\right)^{\overline{i-1}} \\ &\otimes o((L_1^j)^*) \otimes \cdots \otimes o((L_i^j)^*) \otimes \left(\mathbb{Y}_{j+1}^k \otimes \Omega \oplus \mathbb{Z}_{j+1}^k\right)^* \otimes \cdots \otimes \left(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\right)^* \\ &\otimes \Omega^{m_k} \otimes \mathbb{Z}_k\Big] \\ < \ &\left(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\right)^* \otimes \cdots \otimes \left(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\right)^*, \end{aligned}$$

hence

$$
\begin{aligned}
& \big(\big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k} \otimes L_{\mathbb{Z}_k}(z)\big) \\
& \oplus \bigoplus_{i=1}^{m_k} \big(\big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k-1} \otimes L(t_i) \otimes \mathbb{Z}_k\big) \\
& \oplus \bigoplus_{j=1}^{n_k} \bigoplus_{i=1}^{p_j} \big[\big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{j-1}^k \otimes \Omega \oplus \mathbb{Z}_{j-1}^k\big)^* \\
& \quad \otimes \big(\mathbb{Y}_j^k \otimes \Omega \oplus \mathbb{Z}_j^k\big)^{\overline{i-1}} \otimes o((L_1^j)^*) \otimes \cdots \otimes o((L_i^j)^*) \\
& \quad \otimes \big(\mathbb{Y}_{j+1}^k \otimes \Omega \oplus \mathbb{Z}_{j+1}^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k} \otimes \mathbb{Z}_k\big] \\
< \ & \big(\big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k} \otimes L_{\mathbb{Z}_k}(z)\big) \\
& \oplus \ \big(\big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k}\big) \\
\leq \ & \big(\mathbb{Y}_1^k \otimes \Omega \oplus \mathbb{Z}_1^k\big)^* \otimes \cdots \otimes \big(\mathbb{Y}_{n_k}^k \otimes \Omega \oplus \mathbb{Z}_{n_k}^k\big)^* \otimes \Omega^{m_k} \otimes \mathbb{Z}_k.
\end{aligned}
$$

This inequality yields $o(W'(\Omega)) < o(W(\Omega))$.

b) $k(o(W'(\Omega))) < \vartheta(o(W(\Omega)))$.
This can be proved similarly as in Theorem 3.5. □

Theorem 3.11. *$\mathcal{T}((\cdot \times \cdot)^*)$ is a wpo and $o(\mathcal{T}((\cdot \times \cdot)^*)) \leq \vartheta(\Omega^{\Omega^\Omega})$.*

Proof. We show that $L(t)$ is a wpo and $l(t) < \vartheta(\Omega^{\Omega^\Omega})$ hold for every t in $\mathcal{T}((\cdot\times\cdot)^*)$ by induction on the complexity of t. The theorem then follows from Theorem 1.54. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(\Omega^{\Omega^\Omega})$. Assume $t = \circ((t_1^1, t_2^1), \dots, (t_1^k, t_2^k))$ with $L(t_i^j)$ wpo's and $l(t_i^j) < \vartheta(\Omega^{\Omega^\Omega})$ and suppose that $s = \circ((s_1^1, s_2^1), \dots, (s_1^l, s_2^l))$. Then $t \leq s$ iff $t \leq s_i^j$ for certain i and j or

$$((t_1^1, t_2^1), \dots, (t_1^k, t_2^k)) \leq ((s_1^1, s_2^1), \dots, (s_1^l, s_2^l)).$$

Hence, $s \in L(t)$ yields $s_i^j \in L(t)$ for every i and j and one of the following holds

1. $(t_1^1, t_2^1) \not\leq (s_1^i, s_2^i)$ for every i,
2. there exists an index l_1 such that $(t_1^1, t_2^1) \not\leq (s_1^i, s_2^i)$ for every $i < l_1$, $(t_1^1, t_2^1) \leq (s_1^{l_1}, s_2^{l_1})$ and $(t_1^2, t_2^2) \not\leq (s_1^i, s_2^i)$ for every $l_1 < i$,

   ...

k. there exist indices $l_1 < \cdots < l_{k-1}$ such that $(t_1^1, t_2^1) \not\leq (s_1^i, s_2^i)$ for every $i < l_1$, $(t_1^1, t_2^1) \leq (s_1^{l_1}, s_2^{l_1})$, $(t_1^2, t_2^2) \not\leq (s_1^i, s_2^i)$ for every $l_1 < i < l_2$,

$(t_1^2, t_2^2) \leq (s_1^{l_2}, s_2^{l_2})$, ..., $(t_1^{k-1}, t_2^{k-1}) \leq (s_1^{l_{k-1}}, s_2^{l_{k-1}})$ and $(t_1^k, t_2^k) \not\leq (s_1^i, s_2^i)$ for every $l_{k-1} < i$.

Also $(t_1^i, t_2^i) \not\leq (s_1^j, s_2^j)$ is valid if one of the following is satisfied:

a. $s_1^j \in L(t_1^i)$,

b. $t_1^i \leq s_1^j$ and $s_2^j \in L(t_2^i)$.

Let $W'(X)$ be

$$\sum_{j=1}^{k} \left(\prod_{p=1}^{j-1} \left(Y_p \times X^2 \right) \right) \times Y_j$$

with Y_q for $q = 1, \ldots, k$ defined as

$$((L(t_1^q) \times X) + (X \times L(t_2^q)))^* .$$

Define the mapping $f : L(t) \to \mathcal{T}(W')$ recursively as follows. First let $f(\circ)$ be $\circ$. Assume $s = \circ((s_1^1, s_2^1), \ldots, (s_1^l, s_2^l)) \in L(t)$ and $f(s_j^i)$ is already defined for every i and j. We only consider that 2. and always b. hold. We use the same indices as there. The other cases can be treated in a similar way. Define $f(s)$ then as

$$\circ \big(((s_1^1, f(s_2^1)), \ldots, (s_1^{l_1-1}, f(s_2^{l_1-1}))), (f(s_1^{l_1}), f(s_2^{l_1})), \\ ((s_1^{l_1+1}, f(s_2^{l_1+1})), \ldots, (s_1^l, f(s_2^l))) \big) .$$

One can prove that f is a quasi-embedding in the same manner as Theorem 3.6. By Lemma 1.56 and Theorem 3.10 we obtain that $L(t)$ is a $\mathbb{wpo}$ and

$$l(t) \leq o(\mathcal{T}(W')) \leq \vartheta(o(W'(\Omega))).$$

The only remaining thing that needs a proof is $\vartheta(o(W'(\Omega))) < \vartheta(\Omega^{\Omega^\Omega})$. By the induction hypothesis it is known that $l(t_i^j) < \vartheta(\Omega^{\Omega^\Omega}) < \Omega$, hence $(l(t_1^j) \otimes \Omega) \oplus (\Omega \otimes l(t_2^j)) + 1 < \Omega^2$. We obtain

$$\left((l(t_1^j) \otimes \Omega) \oplus (\Omega \otimes l(t_2^j)) \right)^* < \omega^{\omega^{\Omega^2}} = \Omega^{\Omega^\Omega},$$

hence $o(W'(\Omega)) < \Omega^{\Omega^\Omega}$. Furthermore, $o(W'(\Omega))$ is equal to

$$\bigoplus_{j=1}^{k} \Big(\left((l(t_1^1) \otimes \Omega) \oplus (\Omega \otimes l(t_2^1)) \right)^* \otimes \Omega^2 \otimes \ldots \\ \otimes \Omega^2 \otimes \left((l(t_1^j) \otimes \Omega) \oplus (\Omega \otimes l(t_2^j)) \right)^* \Big)$$

$$= \bigoplus_{j=1}^{k} \left(\Omega^{2(j-1)} \otimes \left(\Omega \otimes (l(t_1^1) \oplus l(t_2^1)) \right)^* \otimes \cdots \otimes \left(\Omega \otimes (l(t_1^j) \oplus l(t_2^j)) \right)^* \right) .$$

Because $l(t_i^j) < \vartheta(\Omega^{\Omega^\Omega})$ and $\vartheta(\Omega^{\Omega^\Omega})$ is an epsilon number, we have $l(t_1^j) \oplus l(t_2^j) < \vartheta(\Omega^{\Omega^\Omega})$. Using Lemma 1.11, we see that $k\big((\Omega \otimes (l(t_1^j) \oplus l(t_2^j)))^*\big)$ is strictly smaller than $\vartheta(\Omega^{\Omega^\Omega})$. Furthermore, from Lemma 1.10 it follows that the coefficients of $o(W'(\Omega))$ are strictly smaller than $\vartheta(\Omega^{\Omega^\Omega})$. □

Again, the previous proof allows a constructive well-partial-orderedness proof by reifications (for more information see [66] and [73]). Now, we show that $\vartheta(\Omega^{\Omega^\Omega})$ is also a lower bound for the maximal order type of the wpo $\mathcal{T}((\cdot\times\cdot)^*)$.

Theorem 3.12. *If $W(X) = (X \times X)^*$, then $o(\mathcal{T}(W)) \geq \vartheta(\Omega^{\Omega^\Omega})$.*

Proof. We define a quasi-embedding g from $\vartheta(\Omega^{\Omega^\Omega})$ to $\mathcal{T}((\cdot \times \cdot)^*)$ in the following recursive way: let $g(0)$ be $\circ$. If $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ with $n \geq 2$, define $g(\alpha)$ as $\circ\,((g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ))$.

For every ordinal $\delta < \Omega^{\Omega^\Omega} = \omega^{\omega^{\Omega^2}}$ with $\delta \geq \omega$, there exists unique ordinals $k < \omega$, $\overline{\delta} < \Omega^2$, $\delta_0, \ldots, \delta_k < \omega^{\omega^{\overline{\delta}}}$ with $\delta_k > 0$ such that

$$\delta = \omega^{\omega^{\overline{\delta}}\cdot k}\delta_k + \cdots + \omega^{\omega^{\overline{\delta}}\cdot 1}\delta_1 + \delta_0. \tag{3.7}$$

Note that $k > 0$ because otherwise $\delta < \omega^{\omega^0} = \omega$. From $\overline{\delta} < \Omega^2$, we obtain two unique countable ordinals $\overline{\delta_1}$ and $\overline{\delta_2}$ such that $\overline{\delta} = \Omega\overline{\delta_1} + \overline{\delta_2}$. Now, define $f(\delta) \in W(\Omega) = (\Omega \times \Omega)^*$ for every $\delta < \omega^{\omega^{\Omega^2}}$ recursively as follows. If $\delta = n < \omega$, let $f(\delta)$ be $((0,0), \ldots, (0,0))$, where $(0,0)$ occurs $n+1$ times. If $\delta \geq \omega$, write δ as in (3.7) and let $f(\delta)$ be

$$f(\delta_k)^\frown((1+\overline{\delta_1}, 1+\overline{\delta_2}))^\frown f(\delta_{k-1})\ldots((1+\overline{\delta_1}, 1+\overline{\delta_2}))^\frown f(\delta_0),$$

where $^\frown$ represents the concatenation of the strings. Note that the length of the finite sequence $f(\delta)$ with $\delta > 0$ is strictly bigger than 1. Before we give the definition of $g(\alpha) = g(\vartheta(\beta))$, we first want to prove that the largest countable ordinal occurring in $f(\delta) \in (\Omega\times\Omega)^*$, call it $Max(f(\delta))$, is less than or equal to $k(\delta)+\omega$. Furthermore, we want to prove $k(\delta) < \omega^{\omega^{Max(f(\delta))+1}}$. We prove both inequalities by induction on δ. If $\delta < \omega$, they are trivial. Assume $\delta \geq \omega$. Then, as in (3.7),

$$\begin{aligned}\delta &= \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot k}\delta_k + \cdots + \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0\\ &= \Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot k}\delta_k + \cdots + \Omega^{\Omega^{-1+\overline{\delta_1}}\cdot\omega^{\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0.\end{aligned}$$

From the induction hypothesis, we can conclude that

$$Max(f(\delta)) \leq \max\left\{1+\overline{\delta_1}, 1+\overline{\delta_2}, k(\delta_0)+\omega, \ldots, k(\delta_k)+\omega\right\}$$

and $k(\delta_i) < \omega^{\omega^{Max(f(\delta_i))+1}}$ for all i. Using the second part of Lemma 1.10, we see that $k(\delta_0), \ldots, k(\delta_k) \leq k(\delta)$ and $k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot k}) \leq k(\delta)$. The latter implies that

$$k(\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot k) = \max\{-1+\overline{\delta_1}, \omega^{\overline{\delta_2}} \cdot k\} \leq k(\delta).$$

Hence, $1+\overline{\delta_1} \leq k(\delta)+\omega$ and $1+\overline{\delta_2} \leq k(\delta)+\omega$. We conclude that $Max(f(\delta)) \leq k(\delta)+\omega$. Using the first part of Lemma 1.10, we obtain

$$\begin{aligned}
& k(\delta) \\
\leq\ & k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot k}\delta_k) \oplus \cdots \oplus k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot 1}\delta_1) \oplus k(\delta_0) \\
\leq\ & \max\{k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot k}) \oplus k(\delta_k), k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot k}) \otimes k(\delta_k) \otimes \omega\} \\
& \oplus \ldots \\
& \oplus \max\{k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot 1}) \oplus k(\delta_1), k(\Omega^{\Omega^{-1+\overline{\delta_1}} \cdot \omega^{\overline{\delta_2}} \cdot 1}) \otimes k(\delta_1) \otimes \omega\} \\
& \oplus k(\delta_0) \\
\leq\ & \max\{k(-1+\overline{\delta_1}) \oplus k(\delta_k), k(\omega^{\overline{\delta_2}} \cdot k) \oplus k(\delta_k), \\
& \qquad k(-1+\overline{\delta_1}) \otimes k(\delta_k) \otimes \omega, k(\omega^{\overline{\delta_2}} \cdot k) \otimes k(\delta_k) \otimes \omega\} \\
& \oplus \ldots \\
& \oplus \max\{k(-1+\overline{\delta_1}) \oplus k(\delta_1), k(\omega^{\overline{\delta_2}} \cdot 1) \oplus k(\delta_1), \\
& \qquad k(-1+\overline{\delta_1}) \otimes k(\delta_1) \otimes \omega, k(\omega^{\overline{\delta_2}} \cdot 1) \otimes k(\delta_1) \otimes \omega\} \\
& \oplus k(\delta_0).
\end{aligned}$$

Because $k(\delta_i) < \omega^{\omega^{Max(f(\delta_i))+1}} \leq \omega^{\omega^{Max(f(\delta))+1}}$ and $k(1+\overline{\delta_1}) = 1+\overline{\delta_1} \leq Max(f(\delta)) < \omega^{\omega^{Max(f(\delta))+1}}$ and $k(\omega^{\overline{\delta_2}} \cdot i) = \omega^{\overline{\delta_2}} \cdot i \leq \omega^{Max(f(\delta))} \cdot i < \omega^{\omega^{Max(f(\delta))+1}}$ and $\omega^{\omega^{Max(f(\delta))+1}}$ is an additive and multiplicative closed ordinal number, we can conclude that $k(\delta) < \omega^{\omega^{Max(f(\delta))+1}}$.

We still want to prove one more thing, before we give the definition of $g(\alpha) = g(\vartheta(\beta))$: if $\delta < \omega^{\omega^{\Omega\zeta+\eta}}$ for certain countable ordinals ζ and η, then for all pairs (δ_1^i, δ_2^i) occurring in $f(\delta)$ we have $(\delta_1^i, \delta_2^i) <_{lex} (1+\zeta, 1+\eta)$, where $<_{lex}$ is the lexicographical ordering between pairs. We prove this by induction on δ. If $\delta < \omega$, then this is trivial. Assume that $\omega \leq \delta < \omega^{\omega^{\Omega\zeta+\eta}}$.

Write δ as in (3.7). Then $\delta_0,\dots,\delta_k < \omega^{\omega^{\Omega\zeta+\eta}}$, hence for all pairs (δ_1^i,δ_2^i) occurring in $f(\delta_0),\dots,f(\delta_k)$, we have $(\delta_1^i,\delta_2^i) <_{lex} (1+\zeta,1+\eta)$. Furthermore, from $\delta < \omega^{\omega^{\Omega\zeta+\eta}}$ we obtain the strict inequality $(\overline{\delta_1},\overline{\delta_2}) <_{lex} (\zeta,\eta)$. Hence, $(1+\overline{\delta_1},1+\overline{\delta_2}) <_{lex} (1+\zeta,1+\eta)$. Therefore, for all pairs (δ_1^i,δ_2^i) occurring in $f(\delta)$ we have $(\delta_1^i,\delta_2^i) <_{lex} (1+\zeta,1+\eta)$.

Now we are ready to define $g(\alpha)$ for $\alpha = \vartheta(\beta)$ with $\beta < \Omega^{\Omega^\Omega} = \omega^{\omega^{\Omega^2}}$. If $\beta < \Omega$, define $g(\alpha)$ as $\circ((g(\beta),g(\beta)))$. Hence $g(1) = \circ((\circ,\circ))$. Assume $\beta \geq \Omega$ and suppose $f(\beta) = ((\beta_1^1,\beta_2^1),\dots,(\beta_1^n,\beta_2^n)) \in W(\Omega)$. Then define $g(\alpha)$ as

$$\circ\left((g(\beta_1^1),g(\beta_2^1)),\dots,(g(\beta_1^n),g(\beta_2^n))\right).$$

$g(\alpha)$ is well-defined because for every i and j, $\beta_j^i \leq Max(f(\beta)) \leq k(\beta)+\omega < \vartheta(\beta) = \alpha$.

Obviously, it follows from the definition of g that $\alpha = 0$ iff $g(\alpha) = \circ$ and $\alpha = n < \omega$ iff $g(\alpha) = \circ((\circ,\circ),\dots,(\circ,\circ))$, where $(\circ,\circ)$ occurs n times.

The last part of this theorem consists of proving that g is a quasi-embedding: from Lemma 1.56 we can then conclude this theorem. We show that $g(\alpha) \leq g(\alpha')$ implies $\alpha \leq \alpha'$ for all $\alpha,\alpha' < \vartheta(\Omega^{\Omega^\Omega})$ by induction on $\alpha \oplus \alpha'$. If α or α' is equal to 0, this is trivial. So we may assume that $\alpha,\alpha' > 0$.

a) $\alpha' =_{CNF} \omega^{\alpha'_1}+\dots+\omega^{\alpha'_m}$**,** $m \geq 2$**.**

i) $\alpha =_{CNF} \omega^{\alpha_1}+\dots+\omega^{\alpha_n}$, $n \geq 2$.

If

$$g(\alpha) = \circ\left((g(\alpha_1),\circ),\dots,(g(\alpha_n),\circ)\right) \leq \circ\left((g(\alpha'_1),\circ),\dots,(g(\alpha'_m),\circ)\right) = g(\alpha').$$

then $g(\alpha) \leq g(\alpha'_i)$ for a certain i or

$$((g(\alpha_1),\circ),\dots,(g(\alpha_n),\circ)) \leq^* ((g(\alpha'_1),\circ),\dots,(g(\alpha'_m),\circ)).$$

In the former case, we obtain from the induction hypothesis that $\alpha \leq \alpha'_i < \alpha'$. In the latter case, there exist indices $1 \leq i_1 < \dots < i_n \leq m$ such that $g(\alpha_j) \leq g(\alpha'_{i_j})$. By the induction hypothesis, we obtain that $\alpha_j \leq \alpha'_{i_j}$ for every j. Hence $\alpha \leq \alpha'$.

ii) $\alpha = \vartheta(\beta)$.

If $\beta = 0$, then $\alpha = 1 \leq \alpha'$. Assume that $0 < \beta < \Omega$, then $g(\alpha) =$

$\circ((g(\beta), g(\beta)))$. Hence, $g(\alpha) \leq g(\alpha') = \circ((g(\alpha'_1), \circ), \ldots, (g(\alpha'_m), \circ))$ implies $g(\alpha) \leq g(\alpha'_i)$ for a certain i because $g(\beta) \not\leq \circ$. The induction hypothesis implies $\alpha \leq \alpha'_i < \alpha'$. Now suppose that $\beta \geq \Omega$ and $f(\beta) = ((\beta_1^1, \beta_2^1), \ldots, (\beta_1^n, \beta_2^n))$. Looking at the definition of $f(\beta)$ for $\beta \geq \Omega > \omega$, one can see that at least one β_2^i is strictly bigger than 0, so $g(\alpha) \leq g(\alpha')$ implies $g(\alpha) \leq g(\alpha'_i)$ for a certain i. Therefore, $\alpha \leq \alpha'_i < \alpha'$ like before.

b) $\alpha' = \vartheta(\beta')$.
If $\beta' < \Omega$, then $g(\alpha) \leq g(\alpha') = \circ((g(\beta'), g(\beta')))$ implies $g(\alpha) \leq g(\beta')$ or that $\alpha = \vartheta(\beta)$ with $\beta < \Omega$ and $g(\beta) \leq g(\beta')$. The other cases are simply not possible because in these cases, the length of the corresponding finite sequence of $g(\alpha)$ is always strictly bigger than 1. We can conclude that $\alpha \leq \alpha'$. Assume from now on that $\beta' \geq \Omega$ and $f(\beta') = ((\beta_1'^1, \beta_2'^1), \ldots, (\beta_1'^m, \beta_2'^m))$.

i) $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, $n \geq 2$.
Suppose $g(\alpha) \leq g(\alpha')$. Then either $g(\alpha) \leq g(\beta_j'^i)$ for certain i and j or

$$\begin{aligned}&((g(\alpha_1), \circ), \ldots, (g(\alpha_n), \circ))\\ \leq^* &((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))).\end{aligned}$$

The induction hypothesis and the fact that every ordinal in $f(\beta')$ is less than or equal to $k(\beta') + \omega < \alpha'$ implies in the first case $\alpha \leq \beta_j'^i < \alpha'$, what we want, and in the latter case

$$\begin{aligned}&((\alpha_1, \circ), \ldots, (\alpha_n, \circ))\\ \leq^* &((\beta_1'^1, \beta_2'^1), \ldots, (\beta_1'^m, \beta_2'^m)).\end{aligned}$$

Hence, for every i there exists an index j such that $\alpha_i \leq \beta_1'^j \leq k(\beta') + \omega < \alpha'$. We know that α' is an epsilon number because $\beta' \geq \Omega$. So, $\alpha < \alpha'$.

ii) $\alpha = \vartheta(\beta)$.
If $\beta < \Omega$, then

$$\begin{aligned}&g(\alpha) = \circ((g(\beta), g(\beta)))\\ \leq\ &g(\alpha') = \circ\left((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))\right)\end{aligned}$$

implies either $g(\alpha) \leq g(\beta_j'^i)$ for certain i and j or

$$((g(\beta), g(\beta))) \leq^* \left((g(\beta_1'^1), g(\beta_2'^1)), \ldots, (g(\beta_1'^m), g(\beta_2'^m))\right).$$

The induction hypothesis in the former case implies $\alpha \leq \beta'^i_j < \alpha'$ and in the latter case, it implies $\beta \leq \beta'^r_s < \alpha' = \vartheta(\beta')$ for certain r and s. Hence, in the latter case $\alpha = \vartheta(\beta) \leq \vartheta(\beta') = \alpha'$ because $\beta < \Omega \leq \beta'$ and $k(\beta) = \beta < \vartheta(\beta')$.

Assume now that $\beta \geq \Omega$ and $f(\beta) = ((\beta^1_1, \beta^1_2), \dots, (\beta^n_1, \beta^n_2))$. $g(\alpha) \leq g(\alpha')$ then either implies $g(\alpha) \leq g(\beta'^i_j)$ for certain i and j or

$$\left((g(\beta^1_1), g(\beta^1_2)), \dots, (g(\beta^n_1), g(\beta^n_2))\right) \leq^* \left((g(\beta'^1_1), g(\beta'^1_2)), \dots, (g(\beta'^m_1), g(\beta'^m_2))\right).$$

In the former case, the induction hypothesis implies $\alpha \leq \beta'^i_j < \alpha'$. In the latter case, it implies

$$f(\beta) = \left((\beta^1_1, \beta^1_2), \dots, (\beta^n_1, \beta^n_2)\right) \leq^* \left((\beta'^1_1, \beta'^1_2), \dots, (\beta'^m_1, \beta'^m_2)\right) = f(\beta'). \quad (3.8)$$

Therefore, for every i and j, there exist r and s such that $\beta^i_j \leq \beta'^r_s < \vartheta(\beta') = \alpha'$. Hence $k(\beta) \leq \omega^{\omega^{Max(f(\beta))+1}} = \omega^{\omega^{\left(Max_{i,j}\beta^i_j\right)+1}} < \alpha'$ because α' is an epsilon number. If we now could prove that $f(\delta) \leq f(\delta')$ implies $\delta \leq \delta'$ for all $\delta, \delta' < \omega^{\omega^{\Omega^2}}$, we are done because (3.8) then implies $\beta \leq \beta'$. Hence, $\alpha = \vartheta(\beta) \leq \vartheta(\beta') = \alpha'$.

So assume that $f(\delta) \leq^*_{\Omega\times\Omega} f(\delta')$. We prove by induction on $\delta \oplus \delta'$ that $\delta \leq \delta'$. Assume that $\delta' < \omega$. Then $f(\delta) \leq^* f(\delta') = ((0,0), \dots, (0,0))$, where $(0,0)$ occurs $\delta' + 1$ many times. Hence $f(\delta)$ is also of the form $((0,0), \dots, (0,0))$, so $\delta < \omega$ and $\delta \leq \delta'$. Assume that $\delta' \geq \omega$. If $\delta < \omega$, then $\delta \leq \delta'$ trivially holds. Assume that $\delta \geq \omega$. Like in (3.7), there exist unique ordinals $k, l < \omega$, $\overline{\delta_1}, \overline{\delta_2}, \overline{\delta'_1}, \overline{\delta'_2} < \Omega$, $\delta_0, \dots, \delta_k < \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}}$ with $\delta_k > 0$, $\delta'_0, \dots, \delta'_l < \omega^{\omega^{\Omega\overline{\delta'_1}+\overline{\delta'_2}}}$ with $\delta'_l > 0$ such that

$$\delta = \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot k}\delta_k + \dots + \omega^{\omega^{\Omega\overline{\delta_1}+\overline{\delta_2}}\cdot 1}\delta_1 + \delta_0 \quad (3.9)$$

$$\delta' = \omega^{\omega^{\Omega\overline{\delta'_1}+\overline{\delta'_2}}\cdot l}\delta'_l + \dots + \omega^{\omega^{\Omega\overline{\delta'_1}+\overline{\delta'_2}}\cdot 1}\delta'_1 + \delta'_0. \quad (3.10)$$

$f(\delta) \leq f(\delta')$ then implies

$$\begin{aligned} & f(\delta_k)^\frown((1+\overline{\delta_1}, 1+\overline{\delta_2}))^\frown f(\delta_{k-1}) \dots ((1+\overline{\delta_1}, 1+\overline{\delta_2}))^\frown f(\delta_0) \\ \leq^*_{\Omega\times\Omega} \; & f(\delta'_l)^\frown((1+\overline{\delta'_1}, 1+\overline{\delta'_2}))^\frown f(\delta'_{l-1}) \dots ((1+\overline{\delta'_1}, 1+\overline{\delta'_2}))^\frown f(\delta'_0). \end{aligned} \quad (3.11)$$

Because $\delta'_i < \omega^{\omega^{\Omega\overline{\delta'_1}+\overline{\delta'_2}}}$, all pairs occurring in $f(\delta'_i)$ are lexicographically strictly smaller than $(1+\overline{\delta'_1}, 1+\overline{\delta'_2})$. So if a certain $(1+\overline{\delta_1}, 1+\overline{\delta_2})$ occurring in $f(\delta)$ would *not* be mapped onto $(1+\overline{\delta'_1}, 1+\overline{\delta'_2})$ according to inequality (3.11), then $(1+\overline{\delta_1}, 1+\overline{\delta_2})$ is lexicographically smaller than a pair in $f(\delta'_i)$ for a certain i, hence $(1+\overline{\delta_1}, 1+\overline{\delta_2}) <_{lex} (1+\overline{\delta'_1}, 1+\overline{\delta'_2})$. Therefore, $\delta < \delta'$. Assume now that every $(1+\overline{\delta_1}, 1+\overline{\delta_2})$ occurring in $f(\delta)$ is mapped onto a $(1+\overline{\delta'_1}, 1+\overline{\delta'_2})$ in $f(\delta')$ according to inequality (3.11). Hence $(1+\overline{\delta_1}, 1+\overline{\delta_2}) \leq (1+\overline{\delta'_1}, 1+\overline{\delta'_2})$, so $(1+\overline{\delta_1}, 1+\overline{\delta_2}) \leq_{lex} (1+\overline{\delta'_1}, 1+\overline{\delta'_2})$. If $(1+\overline{\delta_1}, 1+\overline{\delta_2}) <_{lex} (1+\overline{\delta'_1}, 1+\overline{\delta'_2})$, then $\delta < \delta'$. Assume $(1+\overline{\delta_1}, 1+\overline{\delta_2}) = (1+\overline{\delta'_1}, 1+\overline{\delta'_2})$. If $k < l$, then $\delta < \delta'$, so assume from now on that $k = l$. Then, inequality (3.11) implies $f(\delta_i) \leq f(\delta'_i)$ for all $i = 1, \dots, k$. From the induction hypothesis, this implies $\delta_i \leq \delta'_i$. We can conclude that $\delta \leq \delta'$. This finishes the proof. □

Corollary 3.13. $o(\mathcal{T}((\cdot \times \cdot)^*)) = \vartheta(\Omega^{\Omega^\Omega})$.

Proof. This follows from Theorems 3.11 and 3.12. □

Chapter 4

Using one uncountability: the Howard-Bachmann number

4.1 Introduction

The famous Howard-Bachmann ordinal $\vartheta(\varepsilon_{\Omega+1})$ belongs to the most well-established arsenal of proof-theoretic ordinals of natural theories for developing significant parts of (impredicative) mathematics. Following Chapter 3, we give here an order-theoretic characterization of $\vartheta(\varepsilon_{\Omega+1})$ in terms of well-partial-orders. In [88], Andreas Weiermann showed that the Howard-Bachmann number can be characterized as a closure ordinal of so-called essentially monotonic increasing functions, but it is unknown if such a characterization was possible in terms of $\mathbb{wpo}$'s. Friedman's famous $\mathbb{wpo}$'s $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$ [76] have ordinals that are much bigger than $\vartheta(\varepsilon_{\Omega+1})$. Therefore, it seems plausible to single out a natural sub-ordering of this $\mathbb{wpo}$'s which exactly matches with the Howard-Bachmann ordinal.

Weiermann's conjecture 1.111 indicates that for $W(X) = \mathbb{T}(X, \emptyset, \{0\})$, denoted in this chapter by $W(X) = \mathbb{B}(X)$, the partial order $\mathcal{T}(W)$ is a $\mathbb{wpo}$ with maximal order type $\vartheta(\varepsilon_{\Omega+1})$. This is in fact true and is shown in Corollary 4.9. The ordering on $\mathcal{T}(W)$ is some kind of gap-ordering, hence for this W, $\mathcal{T}(W)$ could be interpreted as a natural sub-ordering of $\mathbb{T}_n^{wgap}$. We show in Lemma 4.3 that this is indeed the case.

This chapter is based on the joint article with Michael Rathjen and Andreas Weiermann [81]. However, in that article we considered only the $\mathbb{wpo}$

$\mathcal{T}(\mathbb{B}^s(\cdot))$. Now, we also investigate the unstructured version, i.e. $\mathcal{T}(\mathbb{B}(\cdot))$ because this corresponds better to a subset of the unstructured trees $\mathbb{T}_2$.

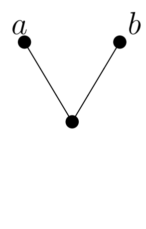
Notation 4.1. *Define $\mathbb{B}(X)$ as the tree-class $\mathbb{T}(X, \emptyset, \{0\})$, i.e. all binary trees where only the leaves have labels in X. Using a later definition (see Definition 5.3), we could also denote this by $\mathbb{T}^{leaf,=2}(X)$ or $\mathbb{T}_1^{leaf,=2}(X)$. Let $\mathbb{B}^s(X)$ be the structured version of this tree-class.*

Recall $o(\mathbb{B}(X)) = o(\mathbb{B}^s(X)) = \varepsilon_{\overline{o(X)}}$ (see Theorem 2.1). We show that $\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo with maximal order type equal to the Howard-Bachmann ordinal $\vartheta(\varepsilon_{\Omega+1})$. This wpo can be seen as an explicit subset of Friedman's wpo with gap-condition with the same ordering.

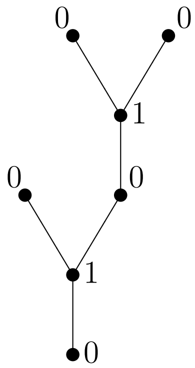
Definition 4.2. *Define the partial ordering $\underline{\mathbb{T}}_2$ as the subset of $(\mathbb{T}_2, \leq^w_{gap})$ which consists of all finite rooted trees such that nodes with label 0 have zero or one immediate successor and nodes with label 1 have exactly two immediate successors. Furthermore, for every tree in $\underline{\mathbb{T}}_2$, the root-label is 0.*

Lemma 4.3. *The partial-ordering $\mathcal{T}(\mathbb{B}(\cdot))$ is order-isomorphic to the partial ordering $\underline{\mathbb{T}}_2$.*

Proof. Define $g : \mathcal{T}(\mathbb{B}(\cdot)) \to \underline{\mathbb{T}}_2$ as follows. Let $g(\circ)$ be the tree which consists of one node with label 0. Take $t = \circ[B(t_1, \ldots, t_n)]$ with $B(t_1, \ldots, t_n)$ a binary tree with leaf-labels in the set $\{t_1, \ldots, t_n\}$ and assume that $g(t_1), \ldots, g(t_n)$ are already defined. Set $g(t)$ then as the tree consisting of a root with label 0, that root connected with an edge to the root of $B(t_1, \ldots, t_n)$. Give all the internal nodes of B label 1 and plug for every i, $g(t_i)$ in those leaves of $B(t_1, \ldots, t_n)$ which have the label t_i. For example, if $t = \circ[B(\circ, \circ[B(\circ, \circ)])]$, with $B(a, b)$ equal to



Then $g(t)$ is



It is easy to see that g is surjective. If we can prove

$$t \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t' \Leftrightarrow g(t) \leq_{\mathbb{I}_2} g(t'),$$

then we are done. We prove this by induction on the sum of complexities of t and t'. If $t = \circ$ or $t' = \circ$, then this is trivial. Assume that both t and t' are different from $\circ$. Let $t = \circ[B(t_1, \ldots, t_n)]$ and $t' = \circ[B'(t'_1, \ldots, t'_m)]$.

Assume $g(t) \leq_{\mathbb{I}_2} g(t')$. We know that the root of $g(t)$, which has label 0, is mapped on a node with label 0. If it is not mapped onto the root of $g(t')$, then it is mapped onto a node with label 0 in $g(t'_i)$ for a certain i. Hence $g(t) \leq_{\mathbb{I}_2} g(t'_i)$, so $t \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t'_i \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t'$. Now assume that the root of $g(t)$ is mapped onto the root of $g(t')$. If we can prove

$$B(g(t_1), \ldots, g(t_n)) \leq_{\mathbb{B}(\mathbb{I}_2)} B'(g(t'_1), \ldots, g(t'_m)),$$

then we can finish the proof because the main induction hypothesis yields that g is a quasi-embedding from the set $\{t_1, \ldots, t_n, t'_1, \ldots, t'_m\}$ to $\mathbb{I}_2$. So the Lifting Lemma implies

$$B(t_1, \ldots, t_n) \leq_{\mathbb{B}(\mathcal{T}(\mathbb{B}(\cdot)))} B'(t'_1, \ldots, t'_m).$$

Hence, $t \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t'$.

If $ht(B) = 0$, then $B(g(t_1), \ldots, g(t_n))$ is a single node with label $g(t_i)$ for a certain i. Then $g(t)$ is the tree with one immediate subtree $g(t_i)$ and root-label 0. $g(t) \leq_{\mathbb{I}_2} g(t')$ yields $g(t_i) \leq_{\mathbb{I}_2} g(t'_j)$ for a certain j because the first nodes of $g(t')$ with label 0 above the root are the roots of $g(t'_j)$. This yields

$$B(g(t_1), \ldots, g(t_n)) \leq_{\mathbb{B}(\mathbb{I}_2)} B'(g(t'_1), \ldots, g(t'_m)).$$

Assume $ht(B) > 0$. $g(t) \leq_{\mathbb{I}_2} g(t')$ yields that every internal node a of B must be mapped on an internal node of B' because otherwise the internal node a

of B is mapped into a $g(t_i)$ for a certain i, but then the root-label 0 of $g(t_i)$ gives a contradiction with the gap-embeddability relation. Furthermore, the leaves of B must be mapped onto the leaves of B'. We can conclude that

$$B(g(t_1), \dots, g(t_n)) \leq_{\mathbb{B}(\mathbb{I}_2)} B'(g(t'_1), \dots, g(t'_m)).$$

One can make this reasoning more rigorously by doing an induction argument on $ht(B) + ht(B')$.

If $t \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t'$, then either $t \leq_{\mathcal{T}(\mathbb{B}(\cdot))} t'_i$ for a certain i or $B(t_1, \dots, t_n) \leq_{\mathbb{B}(\mathcal{T}(\mathbb{B}(\cdot)))} B'(t'_1, \dots, t'_m)$. In both cases, the induction hypothesis yields $g(t) \leq_{\mathbb{I}_2} g(t')$ in a similar way as the other direction. □

From the previous lemma, one can actually already conclude that $\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo. Therefore, one can think that the well-partial-orderedness proof of $\mathcal{T}(\mathbb{B}(\cdot))$ in Theorem 4.7 is superfluous. However, this well-partial-orderedness proof does not need an extra argument: it follows from the calculation of the upper bound for the maximal order type of $\mathcal{T}(\mathbb{B}(\cdot))$. Therefore, we do not really waste efforts by stating it in Theorem 4.7. Also, it allows a constructive well-partial-orderedness proof by reifications.

4.2 Approaching Howard-Bachmann

The main goal of this chapter is to show that $\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo with maximal order type $\vartheta(\varepsilon_{\Omega+1})$. This separate section is needed to approximate this wpo from below. The results in this section are generalizations of Theorems 3.10 and 3.11 in Section 3.3. The proofs follow the same procedure as there, however they are more involved. We will not give the proofs of the theorems in this section. The interested reader can find them in [81].

Theorem 4.4. *Suppose $\mathbb{Y}_{i,j,k}$ and $\mathbb{Z}_i$ are countable wpo's for all indices and $W \in \mathcal{W}(\dots, \mathbb{Y}_{i,j,k}, \dots, \mathbb{Z}_i, \dots)$. Let $W(X)$ be*

$$\sum_{i=0}^{N} \left(\left(\sum_{j=0}^{k_{i,1}} \mathbb{Y}_{i,j,1} \times X^j \right)^* \times \cdots \times \left(\sum_{j=0}^{k_{i,n_i}} \mathbb{Y}_{i,j,n_i} \times X^j \right)^* \times X^{m_i} \times \mathbb{Z}_i \right).$$

Then $\mathcal{T}(W)$ is a wpo and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.

Proof. This is a generalization of Theorems 3.10 and 3.11. For a detailed proof, see [81]. □

Theorem 4.5. *If $W(X) = X^{**}$, then $\mathcal{T}(W)$ is a* wpo *and* $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega))) = \vartheta\left(\Omega^{\Omega^{\Omega^\omega}}\right)$.

Proof. This is a generalization of Theorems 3.10 and 3.11. For a detailed proof, see [81]. □

Theorem 4.6. *Suppose that $W \in \mathcal{W}(\mathbb{Y}_0, \ldots, \mathbb{Y}_k)$ consists only of Id, $+$, $\times$, * and countable* wpo*'s $\mathbb{Y}_i$. Then $\mathcal{T}(W)$ is a* wpo *and $o(\mathcal{T}(W)) \leq \vartheta(o(W(\Omega)))$.*

Proof. This is a generalization of Theorems 4.4 and 4.5. □

4.3 Obtaining Howard-Bachmann

The previous section yields

$$o(\mathcal{T}(\cdot^{\overbrace{*\cdots*}^{n}})) \leq \vartheta(\Omega_{2n-1}[\omega]).$$

$W = \cdot^{\overbrace{*\cdots*}^{n}}$ is defined by applying the Higman-operator * n many times. As indicated in the conjecture of Weiermann (see Conjecture 1.111), we believe that one can prove that $\vartheta(\Omega_{2n-1}[\omega])$ is also a lower bound. Therefore, the tree-structures $\mathcal{T}(\cdot^{\overbrace{*\cdots*}^{n}})$ give rise to representations of countable ordinals strictly below the Howard-Bachmann ordinal and the *limit* of these structures give a representation system for this famous ordinal. But what do we mean by the *limit* of these structures? In some sense, the set of binary trees is the limit of an iteration of the $*$-operator. Hence, one can expect that $o(\mathcal{T}(\mathbb{B}(\cdot))) = \sup_{n<\omega} \vartheta(\Omega_{2n-1}[\omega]) = \vartheta(\varepsilon_{\Omega+1})$. In this section, we prove that this is indeed the case. This result yields that the Howard-Bachmann ordinal can be represented as a tree-structure using binary trees, or more specifically, as the wpo $(\mathbb{I}_2, \leq^w_{gap})$. We also prove that $\mathcal{T}(\mathbb{B}^s(\cdot))$ represents the same ordinal. As usual, there exists a natural quasi-embedding from $\mathcal{T}(\mathbb{B}(\cdot))$ to $\mathcal{T}(\mathbb{B}^s(\cdot))$, hence we have to show that $o(\mathcal{T}(\mathbb{B}^s(\cdot))) \leq \vartheta(\varepsilon_{\Omega+1})$ and $\vartheta(\varepsilon_{\Omega+1}) \leq o(\mathcal{T}(\mathbb{B}(\cdot)))$.

Theorem 4.7. *$\mathcal{T}(\mathbb{B}^s(\cdot))$ is a* wpo *and $o(\mathcal{T}(\mathbb{B}^s(\cdot))) \leq \vartheta(\varepsilon_{\Omega+1})$.*

Proof. We prove that $L(t)$ is a wpo and $l(t) < \vartheta(\varepsilon_{\Omega+1})$ for every t in $\mathcal{T}(\mathbb{B}^s(\cdot))$ by induction on the complexity of t. The theorem then follows from Theorem 1.54. If $t = \circ$, then $L(t)$ is the empty wpo and $l(t) = 0 < \vartheta(\varepsilon_{\Omega+1})$.

Let $B(t_1,\dots,t_n)$ be an element of $\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))$. If we write $B(t_1,\dots,t_n)$, we mean that the leaf-labels are elements of $\{t_1,\dots,t_n\}$. If it is clear from the context, we sometimes write B instead of $B(t_1,\dots,t_n)$. If $B(t_1,\dots,t_n)$ is a tree of height zero with leaf-label t_i, define $W_B(X)$ as the partial ordering $\mathbb{B}^s(L_{\mathcal{T}(\mathbb{B}^s(\cdot))}(t_i))$. Note that Id does not occur in W_B. If $B(t_1,\dots,t_n)$ is a tree with immediate subtrees B_1 and B_2, define $W_B(X) = W_{B(t_1,\dots,t_n)}(X)$ as

$$(W_{B_1}(X) + W_{B_2}(X))^* \times X.$$

We prove by induction on the height of the tree B that there exists a mapping g_B from L_B, which is defined as

$$\{D(d_1,\dots,d_k) \in \mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot))) : B(t_1,\dots,t_n) \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D(d_1,\dots,d_k)\},$$

to the partial ordering $W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))$ such that g_B is a quasi-embedding and if $g_B(D) = w(d'_1,\dots,d'_m)$, with w a term in T_{W_B} and $d'_1,\dots,d'_m \in \mathcal{T}(\mathbb{B}^s(\cdot))$, then $\{d'_1,\dots,d'_m\} \subseteq \{d_1,\dots,d_k\}$, where $D = D(d_1,\dots,d_k)$.

i) height$(B) = 0$.
Let $B(t_1,\dots,t_n)$ be a tree with one node and leaf-label t_i. Then $D(d_1,\dots,d_k)$ is in L_B iff $d_j \in L(t_i)$ for every j. Define then the element $g_B(D(d_1,\dots,d_k))$ as $D(d_1,\dots,d_k) \in \mathbb{B}^s(L_{\mathcal{T}(\mathbb{B}^s(\cdot))}(t_i)) = W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))$. If we write $g_B(D)$ as $w(d'_1,\dots,d'_m)$, then $m = 0$, hence one can show easily that the desired properties of g_B are valid.

ii) height$(B) > 0$.
Let $B(t_1,\dots,t_n)$ be a binary tree with immediate subtrees B_1 and B_2. By the induction hypothesis, there exist functions g_{B_1} and g_{B_2} with the desired properties. Now, pick an arbitrary $D(d_1,\dots,d_k) \in \mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))$. Then $B(t_1,\dots,t_n) \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D(d_1,\dots,d_k)$ is valid iff one of the following holds

1. $D(d_1,\dots,d_k)$ is a binary tree of height 0 with label d_i,
2. $D(d_1,\dots,d_k)$ is a tree of height strictly larger than 0 with immediate subtrees D_1 and D_2, $B(t_1,\dots,t_n) \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D_i$ for $i = 1,2$ and one of the following occurs
   (a) $B_1 \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D_1$,
   (b) $B_1 \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D_1$ and $B_2 \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D_2$.

Because $B(t_1,\dots,t_n) \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} D_i$ for $i = 1,2$, we can also use the above case-study for the trees D_1 and D_2. This leads us to the following definition. Choose an arbitrary $D(d_1,\dots,d_k) \in L_B$. Define E_0 and F_0 as $D(d_1,\dots,d_k)$.

Assume that we have E_i and F_i for a certain i as elements of $\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))$. If F_i is a tree of height strictly larger than 0, we want to define E_{i+1} and F_{i+1}. Suppose that F_i^1 and F_i^2 are the immediate subtrees of F_i. Then define E_{i+1} as

$$\begin{cases} F_i^1 & \text{in case that 2.}(a) \text{ holds if we look to the condition } B \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} F_i \\ F_i^2 & \text{in case that 2.}(b) \text{ holds if we look to the condition } B \not\leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} F_i. \end{cases} \tag{4.1}$$

Let x_{i+1} be the number j such that $E_{i+1} = F_i^j$ and let F_{i+1} be $F_i^{3-x_{i+1}}$, the *other* immediate subtree of F_i. From this definition, we obtain a finite sequence $E_0, E_1, \ldots, E_p, F_p$ with $E_0 = D$ and F_p a tree of height 0. Therefore, F_p consists of only one node with a label, let us say, s. Note that s is also a leaf-label of the tree D. Define now $g_B(D)$ as follows using the fact that we have g_{B_1} and g_{B_2}:

$$g_B(D) := ((g_{B_{x_1}}(E_1), \ldots, g_{B_{x_p}}(E_p)), s).$$

Note that $B_{x_i} \not\leq E_i$, which means that $E_i \in L_{B_{x_i}}$. So $g_{B_{x_i}}(E_i)$ is well-defined, hence $g_B(D) \in W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))$. Does g satisfy the desired properties?

First, we already noted that s is a leaf-label of $D(d_1, \ldots, d_k)$. Secondly, if $g_{B_{x_i}}(E_i)$ is equal to $w_i(s_1^i, \ldots, s_{n_i}^i)$, then by the induction hypothesis and the fact that E_i is a subtree of D, we obtain $\{s_1^i, \ldots, s_{n_i}^i\} \subseteq \{d_1, \ldots, d_k\}$. Hence, if $g_B(D) = w(d'_1, \ldots, d'_m) \in W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))$, then $\{d'_1, \ldots, d'_m\} \subseteq \{d_1, \ldots, d_k\}$. Now we want to prove that g is a quasi-embedding. Let $\overline{E_0}, \overline{E_1}, \ldots, \overline{E_q}, \overline{F_q}$ and $y_1, \ldots, y_q$ be the finite sequences forthcoming from definition (4.1), the definitions of x_{i+1} and F_{i+1}, but now starting with $\overline{D}(\overline{d_1}, \ldots, \overline{d_l}) \in L_B$. Denote the label of the tree $\overline{F_q}$ of height zero by $\overline{s}$. Then

$$g_B(\overline{D}) = ((g_{B_{y_1}}(\overline{E_1}), \ldots, g_{B_{y_q}}(\overline{E_q})), \overline{s}).$$

Assume furthermore that

$$\begin{aligned} g_B(D) &= ((g_{B_{x_1}}(E_1), \ldots, g_{B_{x_p}}(E_p)), s) \\ \leq_{W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))} g_B(\overline{D}) &= ((g_{B_{y_1}}(\overline{E_1}), \ldots, g_{B_{y_q}}(\overline{E_q})), \overline{s}). \end{aligned} \tag{4.2}$$

We show that inequality $D(d_1, \ldots, d_k) \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{D}(\overline{d_1}, \ldots, \overline{d_l})$ holds by induction on q.

From (4.2), we obtain $s \leq_{\mathcal{T}(\mathbb{B}^s(\cdot))} \overline{s}$. Furthermore, there exist indices $1 \leq i_1 < \cdots < i_p \leq q$ such that $g_{B_{x_j}}(E_j) \leq g_{B_{y_{i_j}}}(\overline{E_{i_j}})$. Because the left

hand side of this inequality is in $W_{B_{x_j}}(\mathcal{T}(\mathbb{B}^s(\cdot)))$ and the right hand side in $W_{B_{y_{i_j}}}(\mathcal{T}(\mathbb{B}^s(\cdot)))$, we obtain $x_j = y_{i_j}$ for every j. Furthermore, $E_j \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{E_{i_j}}$ for every j because $g_{B_{x_j}}$ is a quasi-embedding.
If $q = 0$, then also $p = 0$. Therefore, D is a tree of height zero with leaf-label s and $\overline{D}$ is a tree of the same height with leaf-label $\overline{s}$. Hence $D \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{D}$.
Now let $q > 0$. By construction,

$$\begin{aligned} g_B(F_1) &= ((g_{B_{x_2}}(E_2), \dots, g_{B_{x_p}}(E_p)), s), \\ g_B(\overline{F_{i_1}}) &= ((g_{B_{y_{i_1+1}}}(\overline{E_{i_1+1}}), \dots, g_{B_{y_q}}(\overline{E_q})), \overline{s}). \end{aligned}$$

Because $g_B(D) \leq_{W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))} g_B(\overline{D})$ and $g_{B_{x_1}}(E_1)$ is mapped onto $g_{B_{y_{i_1}}}(\overline{E_{i_1}})$, we obtain $g_B(F_1) \leq_{W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))} g_B(\overline{F_{i_1}})$. Hence, by the sub-induction hypothesis on q, we have the inequality $F_1 \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{F_{i_1}}$. We also know that $E_1 \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{E_{i_1}}$ and $x_1 = y_{i_1}$. If $x_1 = 1$, then E_1 is the left-immediate subtree of $F_0 = D$ and $\overline{E_{i_1}}$ is the left-immediate subtree of $\overline{F_{i_1-1}}$. Furthermore, F_1 is the right-immediate subtree of $F_0 = D$ and $\overline{F_{i_1}}$ is the right-immediate subtree of $\overline{F_{i_1-1}}$. We conclude that $D = F_0 \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{F_{i_1-1}} \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{F_0} = \overline{D}$. The same argument holds for $x_1 = 2$. Therefore, g_B is a quasi-embedding.

Assume $t = \circ[B(t_1, \dots, t_n)] \in \mathcal{T}(\mathbb{B}^s(\cdot))$ with $B(t_1, \dots, t_n)$ a binary tree in the partial ordering $\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))$ and assume that $L(t_i)$ are wpo's and $l(t_i) < \vartheta(\varepsilon_{\Omega+1})$. We want to prove that $L(t)$ is a wpo and $l(t) < \vartheta(\varepsilon_{\Omega+1})$.
First of all, we define a quasi-embedding f from $L(t)$ into $\mathcal{T}(W_B)$. First note that $d = \circ[D(d_1, \dots, d_k)] \in L(t)$ iff $d_i \in L(t)$ and $D(d_1, \dots, d_k) \in L_B$. Define $f(\circ)$ as $\circ$. Suppose $d = \circ[D(d_1, \dots, d_k)] \in L(t)$ and that $f(d_1), \dots, f(d_k)$ are already defined. If $g_B(D) = w(d'_1, \dots, d'_m) \in W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))$ with $\{d'_1, \dots, d'_m\} \subseteq \{d_1, \dots, d_k\}$, define $f(d)$ as $\circ[w(f(d'_1), \dots, f(d'_m))] \in \mathcal{T}(W_B)$. Now we want to prove that f is a quasi-embedding.

Assume $f(d) \leq_{\mathcal{T}(W_B)} f(\overline{d})$. We prove that this implies $d \leq_{\mathcal{T}(\mathbb{B}^s(\cdot))} \overline{d}$ by induction on the sum of the complexities of d and $\overline{d}$. If d or $\overline{d}$ is equal to $\circ$, then this is trivial. Assume $g_B(D(d_1, \dots, d_k)) = w(d'_1, \dots, d'_m)$, $g_B(\overline{D}(\overline{d_1}, \dots, \overline{d_l})) = \overline{w}(\overline{d'_1}, \dots, \overline{d'_p})$ with $\{d'_1, \dots, d'_m\} \subseteq \{d_1, \dots, d_k\}$ and $\{\overline{d'_1}, \dots, \overline{d'_p}\} \subseteq \{\overline{d_1}, \dots, \overline{d_l}\}$ and

$$\begin{aligned} f(d) &= f(\circ[D(d_1, \dots, d_k)]) = \circ[w(f(d'_1), \dots, f(d'_m))] \\ \leq_{\mathcal{T}(W_B)} f(\overline{d}) &= f(\circ[\overline{D}(\overline{d_1}, \dots, \overline{d_l})]) = \circ[\overline{w}(f(\overline{d'_1}), \dots, f(\overline{d'_p}))]. \end{aligned}$$

This implies either $f(d) \leq_{\mathcal{T}(W_B)} f(\overline{d'_i})$ for some i or

$$w(f(d'_1), \dots, f(d'_m)) \leq_{W_B(\mathcal{T}(W_B))} \overline{w}(f(\overline{d'_1}), \dots, f(\overline{d'_p})).$$

In the former case, the induction hypothesis yields $d \leq_{\mathcal{T}(\mathbb{B}^s(\cdot))} \overline{d'_i} \leq_{\mathcal{T}(\mathbb{B}^s(\cdot))} \overline{d}$. In the latter case, we observe that the induction hypothesis implies that f is a quasi-embedding from $\{d_1, \ldots, d_k, \overline{d_1}, \ldots, \overline{d_l}\}$ to $\mathcal{T}(W_B)$. Therefore, the Lifting Lemma brings the inequality

$$g_B(D) = w(d'_1, \ldots, d'_m) \leq_{W_B(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{w}(\overline{d'_1}, \ldots, \overline{d'_p}) = g_B(\overline{D}).$$

Hence we have $D(d_1, \ldots, d_k) \leq_{\mathbb{B}^s(\mathcal{T}(\mathbb{B}^s(\cdot)))} \overline{D}(\overline{d_1}, \ldots, \overline{d_l})$ and $d \leq_{\mathcal{T}(\mathbb{B}^s(\cdot))} \overline{d}$. We conclude that f is a quasi-embedding.

By Lemma 1.56 and Theorem 4.6 we obtain that $L(t)$ is a wpo and

$$l(t) \leq o(\mathcal{T}(W_B)) \leq \vartheta(W_B(\Omega)).$$

So if $\vartheta(W_B(\Omega)) < \vartheta(\varepsilon_{\Omega+1})$, the proof is finished. Let $m \geq 1$ be the least natural number such that for every i, $l(t_i) < \vartheta(\Omega_m[1])$. We prove simultaneously by induction on the height of the tree B that

$$W_B(\Omega) < \Omega_{m+1+3\cdot ht(B)}[1],$$
$$k(W_B(\Omega)) < \vartheta(\Omega_{m+1+3\cdot ht(B)}[1]),$$

Note that we write $W_B(\Omega)$ instead of $o(W_B(\Omega))$ for notational convenience. These strict inequalities yield $\vartheta(W_B(\Omega)) < \vartheta(\varepsilon_{\Omega+1})$.

If the height of the tree B is zero, we defined $W_B(X)$ as $\mathbb{B}^s(L_{\mathcal{T}(\mathbb{B}^s(\cdot))}(t_i))$. Hence

$$\begin{aligned} k(W_B(\Omega)) = k(o(\mathbb{B}^s(L(t_i)))) = o(\mathbb{B}^s(L(t_i))) \leq \varepsilon_{l(t_i)+1} \\ < \varepsilon_{\vartheta(\Omega_m[1])+1} = \vartheta(\Omega + \vartheta(\Omega_m[1])) \leq \vartheta(\Omega_{m+1+3\cdot ht(B)}[1]) \end{aligned}$$

and

$$W_B(\Omega) \leq \varepsilon_{l(t_i)+1} < \Omega < \Omega_{m+1+3\cdot ht(B)}[1].$$

Assume that the height of B is strictly larger than zero such that B_1 and B_2 are immediate subtrees of B. Because of the induction hypothesis, we know that there exist natural numbers k and l such that

$$W_{B_1}(\Omega) < \Omega_k[1],$$
$$k(W_{B_1}(\Omega)) < \vartheta(\Omega_k[1]),$$
$$W_{B_2}(\Omega) < \Omega_l[1],$$
$$k(W_{B_2}(\Omega)) < \vartheta(\Omega_l[1]).$$

We prove that $W_B(\Omega) < \Omega_{\max\{k,l\}+3}[1]$ and $k(W_B(\Omega)) < \vartheta(\Omega_{\max\{k,l\}+3}[1])$. We defined $W_B(X)$ as $(W_{B_1}(X) + W_{B_2}(X))^* \times X$, hence

$$\begin{aligned} W_B(\Omega) &< \omega^{\omega^{\Omega_k[1]\oplus\Omega_l[1]\oplus 1}} \otimes \Omega = \omega^{\omega^{\Omega_k[1]}\cdot\omega^{\Omega_l[1]}\cdot\omega} \otimes \Omega \\ &\leq \omega^{\Omega_{k+1}[1]\cdot\Omega_{l+1}[1]\cdot\omega} \otimes \Omega = (\omega^{\Omega_{k+1}[1]})^{\Omega_{l+1}[1]\cdot\omega} \otimes \Omega \\ &= (\Omega_{k+2}[1])^{\Omega_{l+1}[1]\cdot\omega} \otimes \Omega = (\Omega^{\Omega_{k+1}[1]})^{\Omega_{l+1}[1]\cdot\omega} \otimes \Omega \\ &= \Omega^{\Omega_{k+1}[1]\cdot\Omega_{l+1}[1]\cdot\omega} \otimes \Omega < \Omega_{\max\{k,l\}+3}[1] \end{aligned}$$

Furthermore, by Lemma 1.10

$$\begin{aligned} &k(W_B(\Omega)) \\ &= k\left(\omega^{\omega^{W_{B_1}(\Omega)\oplus W_{B_2}(\Omega)(\pm 1)}} \otimes \Omega\right) \\ &\leq \max\{k(\omega^{\omega^{W_{B_1}(\Omega)\oplus W_{B_2}(\Omega)(\pm 1)}}) \oplus k(\Omega), k(\omega^{\omega^{W_{B_1}(\Omega)\oplus W_{B_2}(\Omega)(\pm 1)}}) \otimes k(\Omega) \otimes \omega\} \\ &\leq \max\{\omega^{\omega^{k(W_{B_1}(\Omega))\oplus k(W_{B_2}(\Omega))\oplus 1}} + 1, \omega^{\omega^{k(W_{B_1}(\Omega))\oplus k(W_{B_2}(\Omega))\oplus 1}} \cdot \omega\} \\ &\leq \max\{\omega^{\omega^{\vartheta(\Omega_k[1])\oplus\vartheta(\Omega_l[1])\oplus 1}} + 1, \omega^{\omega^{\vartheta(\Omega_k[1])\oplus\vartheta(\Omega_l[1])\oplus 1}} \cdot \omega\} \\ &< \vartheta(\Omega_{\max\{k,l\}+1}[1]) \\ &< \vartheta(\Omega_{\max\{k,l\}+3}[1]). \end{aligned}$$

□

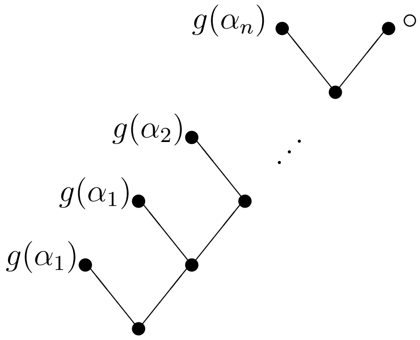
As mentioned before, there exists a quasi-embedding from $\mathcal{T}(\mathbb{B}(\cdot))$ to $\mathcal{T}(\mathbb{B}^s(\cdot))$. Hence, the previous theorem also yields that $\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo with $o(\mathcal{T}(\mathbb{B}(\cdot))) \leq o(\mathcal{T}(\mathbb{B}^s(\cdot))) \leq \vartheta(\varepsilon_{\Omega+1})$. Now we prove that $\vartheta(\varepsilon_{\Omega+1}) \leq o(\mathcal{T}(\mathbb{B}(\cdot)))$. If you look closer at the proof of the next theorem, one can see how every ordinal below $\vartheta(\varepsilon_{\Omega+1})$ can be represented as an element of $\mathcal{T}(\mathbb{B}(\cdot))$. Note that this proof can be carried out in ACA_0 if we have a predefined primitive recursive ordinal notation system for $\vartheta(\varepsilon_{\Omega+1})$. One cannot find the next theorem in our joint article [81] because now we consider the wpo $\mathcal{T}(\mathbb{B}(\cdot))$ instead of $\mathcal{T}(\mathbb{B}^s(\cdot))$.

Theorem 4.8. *$o(\mathcal{T}(\mathbb{B}(\cdot))) \geq \vartheta(\varepsilon_{\Omega+1})$.*

Proof. Define

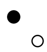
$$g : \vartheta(\varepsilon_{\Omega+1}) \to \mathcal{T}(\mathbb{B}(\cdot))$$

in the following recursive way. Let $g(0)$ be $\circ$. Pick an arbitrary $\alpha < \vartheta(\varepsilon_{\Omega+1})$ and assume that $g(\beta)$ is already defined for every $\beta < \alpha$. If $\alpha =_{CNF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ with $n \geq 2$, define $g(\alpha)$ as $\circ[B]$ with B the following unstructured binary tree:

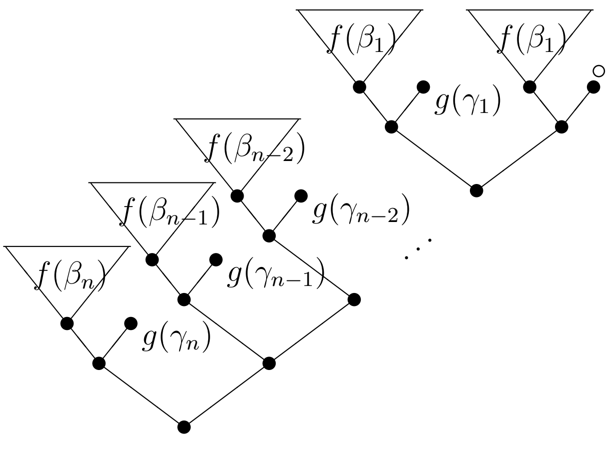


Because $n \geq 2$, the height of this tree is at least three. If $\alpha < \vartheta(\varepsilon_{\Omega+1})$ and $\alpha \in P$, we can write α as $\vartheta(\beta)$ as in Corollary 1.8. Because every element of $K(\beta)$ is strictly smaller than α, we can assume that $g(\gamma)$ is defined for every $\gamma \in K(\beta)$. Define $g(\alpha)$ as $\circ[f(\beta)]$, where we define the binary tree $f(\beta)$ in the following recursive way.

Let $f(0)$ be the binary tree with one node and leaf-label $\circ$:



Now, let $\beta = \Omega^{\beta_1}\gamma_1 + \cdots + \Omega^{\beta_n}\gamma_n > \beta_1 > \cdots > \beta_n \geq 0$. Then define $f(\beta)$ as



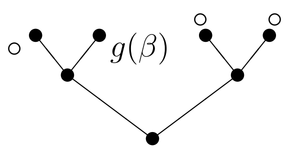
Note that all labels in the tree $f(\beta)$ are elements of $g(K(\beta) \cup \{0\})$. Additionally, every element of $g(K(\beta) \cup \{0\})$ is a label in the tree $f(\beta)$.

Is g a quasi-embedding? We show by induction on α' that $g(\alpha) \leq g(\alpha')$ implies $\alpha \leq \alpha'$. If α or α' is equal to zero, then this is trivial, hence we may assume that both α and α' are different from zero. There are now four cases left:

a) $g(\omega^{\alpha_1} + \cdots + \omega^{\alpha_n}) \leq g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m})$.
Then either $g(\alpha) \leq g(\alpha'_i)$ or $(g(\alpha_1), \ldots, g(\alpha_m)) \leq^* (g(\alpha'_1), \ldots, g(\alpha'_m))$. In both cases the induction hypothesis yields $\omega^{\alpha_1} + \cdots + \omega^{\alpha_n} \leq \omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}$.

b) $g(\vartheta(\beta)) \leq g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m})$.
If $\beta = 0$, then it is trivial. Assume $\beta \neq 0$. Then $g(\vartheta(\beta)) \leq g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m})$ is only possible if $g(\vartheta(\beta)) \leq g(\alpha'_i)$ for a certain i. The induction hypothesis yields $\vartheta(\beta) \leq \omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}$.

c) $g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}) \leq g(\vartheta(\beta))$.
It is impossible that $\beta = 0$ because $m \geq 2$. If $0 < \beta < \Omega$, then $g(\vartheta(\beta))$ is equal to

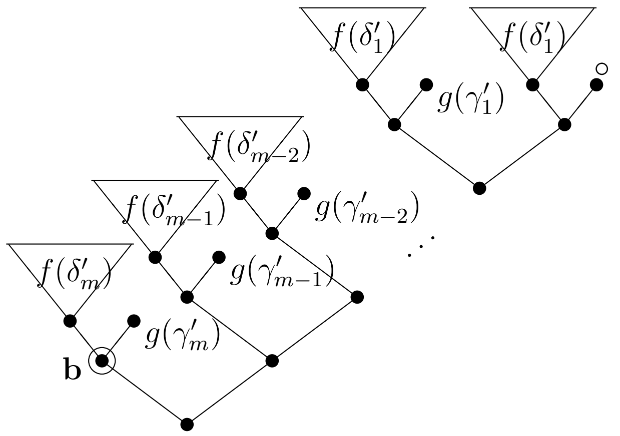


So $g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}) \leq g(\vartheta(\beta))$ can only occur if $g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}) \leq g(\beta)$ because the height of $g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m})$ is at least three. The induction hypothesis yields $\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m} \leq \beta < \vartheta(\beta)$.

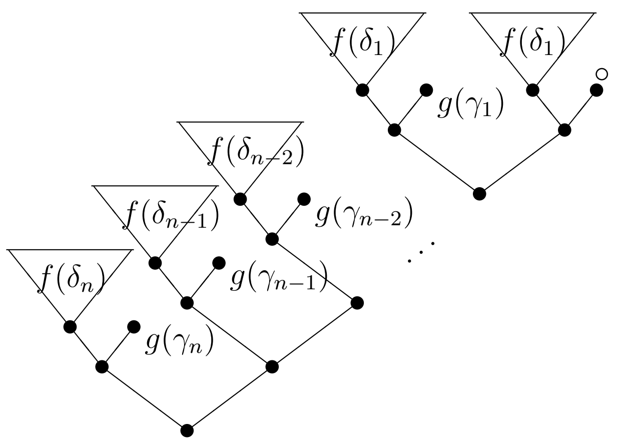
In the case that $\beta \geq \Omega$, we have that $g(\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m}) \leq g(\gamma)$ for a certain $\gamma \in K(\beta) \cup \{0\}$ or for every α'_i, there exists a $\gamma_i \in K(\beta) \cup \{0\}$ such that $g(\alpha'_i) \leq g(\gamma_i)$. In the former case, we obtain that $\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m} \leq \gamma \leq k(\beta) < \vartheta(\beta)$. In the latter case, we obtain that $\alpha'_i \leq \gamma_i \leq k(\beta) < \vartheta(\beta)$. Because $\vartheta(\beta)$ is an epsilon number ($\beta \geq \Omega$), we get that $\omega^{\alpha'_1} + \cdots + \omega^{\alpha'_m} < \vartheta(\beta)$.

d) $g(\vartheta(\beta)) \leq g(\vartheta(\beta'))$
If $\beta' = 0$, then β must also be zero, hence $\vartheta(\beta) \leq \vartheta(\beta')$. Now assume that $\beta' > 0$. $g(\vartheta(\beta)) \leq g(\vartheta(\beta'))$ is possible if $g(\vartheta(\beta)) \leq g(\gamma')$ for a certain $\gamma' \in K(\beta') \cup \{0\}$ or if $f(\beta) \leq f(\beta')$. In the former case, we obtain by the induction hypothesis that $\vartheta(\beta) \leq \gamma' \leq k(\beta') < \vartheta(\beta')$. If the latter case occurs, then for every $\gamma \in K(\beta)$, there exists a $\gamma' \in K(\beta') \cup \{0\}$ such that $g(\gamma) \leq g(\gamma')$. Hence, $k(\beta) \leq k(\beta') < \vartheta(\beta')$. For ending the proof of $\vartheta(\beta) \leq \vartheta(\beta')$, we need to show that $\beta \leq \beta'$.

We prove by induction on δ' that $f(\delta) \leq f(\delta')$ implies $\delta \leq \delta'$ for every δ with $K(\delta) \subseteq K(\beta)$ and every δ' with $K(\delta') \subseteq K(\beta')$. If this is true, we can conclude that $\beta \leq \beta'$.
If $\delta' = 0$ or $\delta = 0$, then this is trivial. Hence we may assume that both δ and δ' are different from zero. Assume $\delta' =_{NF} \Omega^{\delta'_1}\gamma'_1 + \cdots + \Omega^{\delta'_m}\gamma'_m > 0$. Then $f(\delta')$ is equal to



Let $\delta =_{NF} \Omega^{\delta_1}\gamma_1 + \cdots + \Omega^{\delta_n}\gamma_n$ and assume $f(\delta)$ is



1. *The root of $f(\delta)$ is mapped into the left-drawn immediate subtree of $f(\delta')$.*

   (a) $m = 1$**:** Then immediately $f(\delta_1) < f(\delta'_1)$. Hence, by the induction hypothesis $\delta_1 < \delta'_1$, so $\delta < \delta$.

(b) $m > 1$**:** Then $f(\delta_1) \le f(\delta'_m)$. Hence, by the induction hypothesis $\delta_1 \le \delta'_m < \delta'_1$. So $\delta < \delta'$.

2. *The root of $f(\delta)$ is mapped into the right-drawn immediate subtree of $f(\delta')$.*

   (a) $m = 1$**:** Similarly as before.

   (b) $m > 1$**:** Then $f(\delta) \le f(\delta')$ yields $f(\delta) \le f(\Omega^{\delta'_1}\gamma'_1 + \cdots + \Omega^{\delta'_{m-1}}\gamma'_{m-1})$, hence the induction hypothesis yields $\delta \le \Omega^{\delta'_1}\gamma'_1 + \cdots + \Omega^{\delta'_{m-1}}\gamma'_{m-1} < \delta'$.

3. *The root of $f(\delta)$ is mapped on the root of $f(\delta')$.*

   (a) $m = 1$**:** $f(\delta_1)$ is mapped into $f(\delta'_1)$ or onto the single node with label $g(\gamma'_1)$ or $\circ$.

   If we are in the latter case, then $\delta_1 = 0$, hence $\delta_1 \le \delta'_1$. If $\delta'_1 > 0$, then $\delta < \delta'$. If $\delta'_1 = 0$, then $n = 1$ and $f(\delta) \le f(\delta')$ yield $g(\gamma_1) \le g(\gamma'_1)$. Hence, $\gamma_1 \le \gamma'_1$, so $\delta \le \delta'$.

   So we can assume that we are in the former case, meaning that $f(\delta_1)$ is mapped into $f(\delta'_1)$. Hence $\delta_1 \le \delta'_1$ by the main induction hypothesis. If $\delta_1 < \delta'_1$, then $\delta < \delta'$ and we are done. Assume $\delta_1 = \delta'_1$. But then $n = 1$ again, so $f(\delta) \le f(\delta')$ yields $g(\gamma_1) \le g(\gamma'_1)$. So $\gamma_1 \le \gamma'_1$, hence $\delta \le \delta'$.

   (b) $m > 1$**:** If the right-drawn immediate subtree of $f(\delta)$ is mapped into the left-drawn immediate subtree of $f(\delta')$, then the tree $f(\delta_1)$ is mapped into $f(\delta'_m)$ or on the tree consisting of a single node with label $g(\gamma'_m)$. In both cases, we obtain $\delta_1 \le \delta'_m$ (in the latter case, we even have $\delta_1 = 0$). Hence $\delta_1 \le \delta'_m < \delta'_1$, so $\delta < \delta'$.

   So we can assume that the right-drawn, resp. left-drawn, immediate subtree of $f(\delta)$ is mapped into the right-drawn, resp. left-drawn, immediate subtree of $f(\delta')$. If $n = 1$, then again $\delta_1 \le \delta'_m$ for the same reasons. So $\delta < \delta'$. From now on, assume $n > 1$. $f(\delta) \le f(\delta')$ yields

$$f(\Omega^{\delta_1}\gamma_1 + \cdots + \Omega^{\delta_{n-1}}\gamma_{n-1}) \le f(\Omega^{\delta'_1}\gamma'_1 + \cdots + \Omega^{\delta'_{m-1}}\gamma'_{m-1}),$$

   so $\Omega^{\delta_1}\gamma_1 + \cdots + \Omega^{\delta_{n-1}}\gamma_{n-1} \le \Omega^{\delta'_1}\gamma'_1 + \cdots + \Omega^{\delta'_{m-1}}\gamma'_{m-1}$ by the induction hypothesis. If the root of the left-drawn immediate subtree of $f(\delta)$ is mapped into $f(\delta'_m)$, then $\delta_n < \delta_m$, so $\delta < \delta'$. If the root of the left-drawn immediate subtree of $f(\delta)$ is mapped onto node b, then $f(\delta_n)$ is mapped into $f(\delta'_m)$ or onto a single node with label

$g(\gamma'_m)$. In both cases, $\delta_n \leq \delta'_m$ (in the last case is δ_n even zero). If $\delta_n < \delta'_m$, then $\delta < \delta'$. Assume $\delta_n = \delta'_m$. Then $g(\gamma_n) \leq g(\gamma'_m)$, Hence $\gamma_n \leq \gamma'_m$, so $\delta \leq \delta'$.

□

Corollary 4.9. $o(\mathcal{T}(\mathbb{B}(\cdot))) = o(\mathcal{T}(\mathbb{B}^s(\cdot))) = o(\underline{\mathbb{I}}_2) = \vartheta(\varepsilon_{\Omega+1})$.

Proof. Follows from Lemma 4.3 and Theorems 4.7 and 4.8 and the fact that there exists a natural quasi-embedding from $\mathcal{T}(\mathbb{B}(\cdot))$ to $\mathcal{T}(\mathbb{B}^s(\cdot))$. □

Chapter 5

Capturing the gap-trees with two labels

5.1 Introduction

In 1982, Harvey Friedman proved that Kruskal's theorem is not provable in predicative analysis. Additionally, he constructed a new ordering, the gap-embeddability relation, on $\mathbb{T}_n$ to create a statement stronger than Π^1_1-CA_0 [76]. Since then, the exact proof-theoretical strengths of $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$ are unknown. In this chapter, we answer this problem partially by showing that $o(\mathbb{T}_2^{wgap}) = \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ and $o(\mathbb{T}_2^{sgap}) = \vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$. Furthermore, we indicate how one can generalize the procedure to arbitrary n.

We use the ordinal collapsing functions $\vartheta_0, \ldots, \vartheta_{n-1}$ defined in Definitions 1.17 and 1.18, where the number n is the same n as in $\mathbb{T}_n^{wgap}$. So in Sections 5.2 and 5.3, where we study $\mathbb{T}_2^{wgap}$, this n is equal to 2. Note that we defined $\vartheta_i\alpha$ as epsilon numbers. Note $k_{n-1}\alpha = \alpha$ if $\alpha < \Omega_n$, but $k_i\beta$ is not necessarily equal to β for $i < n-1$ and $\beta < \Omega_{i+1}$.

Definition 5.1. *Define $\mathbb{T}_n[m]$ as the set of trees T in $\mathbb{T}_n$ such that the root has a label smaller than or equal to m. Define $\mathbb{T}'_n$, respectively $\mathbb{T}'_n[m]$, as the subset of $\mathbb{T}_n$, respectively $\mathbb{T}_n[m]$, such that all the leaves have label 0. Denote respectively $(\mathbb{T}_n[m], \leq^w_{gap})$ and $(\mathbb{T}_n[m], \leq^s_{gap})$ by respectively $\mathbb{T}_n^{wgap}[m]$ and $\mathbb{T}_n^{sgap}[m]$. Similarly for $\mathbb{T}'_n$ and $\mathbb{T}'_n[m]$.*

Trivially, $\mathbb{T}_n^{wgap}[0] = \mathbb{T}_n^{sgap}[0]$.

Lemma 5.2. *1.* $o(\mathbb{T}_n^{wgap}) = o(\mathbb{T}_n^{wgap}[0])$,

2. $o(\mathbb{T}_n^{wgap}) = o(\mathbb{T}'^{wgap}_n) = o(\mathbb{T}'^{wgap}_n[0])$,

3. $o(\mathbb{T}_n^{sgap}) = o(\mathbb{T}'^{sgap}_n)$.

Proof. It is easy to see that $o(\mathbb{T}_n^{wgap}[0]) \leq o(\mathbb{T}_n^{wgap})$ because $\mathbb{T}_n[0] \subseteq \mathbb{T}_n$. Now, define a mapping e from $\mathbb{T}_n^{wgap}$ to $\mathbb{T}_n^{wgap}[0]$ as follows: $e(T)$ is the tree consisting of a root with label 0 and exactly one immediate subtree, namely T. Trivially, e is a quasi-embedding, hence $o(\mathbb{T}_n^{wgap}) \leq o(\mathbb{T}_n^{wgap}[0])$. This finishes the proof of the first assertion. The second equality of the second assertion can be proved in a similar way.

Now, we prove $o(\mathbb{T}_n^{wgap}) = o(\mathbb{T}'^{wgap}_n)$. The third assertion can be proved in a similar way. Trivially, $\mathbb{T}'^{wgap}_n \subseteq \mathbb{T}_n^{wgap}$, hence $o(\mathbb{T}'^{wgap}_n) \leq o(\mathbb{T}_n^{wgap})$. Define a mapping g from $\mathbb{T}_n^{wgap}$ to $\mathbb{T}'^{wgap}_n$ in the following straightforward way: $g(T)$ is the tree where we add one extra node with label 0 to every leaf of T. This means that the leaves of T are not leaves anymore in $g(T)$. One can prove easily that g is a quasi-embedding. □

Following [76], we introduce $\mathbb{T}_n^{leaf}(X)$.

Definition 5.3. *Define $\mathbb{T}_n^{leaf}(X)$ as the set of trees where the internal nodes have labels in $\{0, \dots, n-1\}$ and the leaf nodes have labels in X. Define $\mathbb{T}_n^{leaf}[m](X)$ in a similar way. Let $\mathbb{T}^{leaf}(X)$ be the set of trees where the internal nodes do not have labels and the leaf nodes have labels in X.*

$\mathbb{T}_n^{leaf}(X)$ and $\mathbb{T}_n^{leaf}[m](X)$ are subsets of the partial order

$$\mathbb{T}\begin{pmatrix} X & \{0,\dots,n-1\} \\ 1 & \omega \end{pmatrix},$$

hence we can talk about the (natural) homeomorphic embedding on them. If we write $\mathbb{T}_n^{leaf}(X)$ or $\mathbb{T}_n^{leaf}[m](X)$, we often mean the partial order instead of only looking at it as a set of trees. The context will make clear what we mean. On $\mathbb{T}^{leaf}(X)$, there exists also a natural homeomorphic embedding between the elements, if we interpret $\mathbb{T}^{leaf}(X)$ as $\mathbb{T}_1^{leaf}(X)$. Similarly, we often write $\mathbb{T}^{leaf}(X)$ if we talk about it as a partial order.

As we have mentioned, $\mathbb{T}_n^{leaf}(X)$ can be interpreted as a subset of

$$\mathbb{T}\begin{pmatrix} X & \{0,\dots,n-1\} \\ 1 & \omega \end{pmatrix}.$$

Therefore, $o(\mathbb{T}_n^{leaf}(X)) \leq \vartheta(\Omega^\omega \cdot n + o(X))$ if X is a countable wpo, where ϑ is the collapsing function from Definition 1.4. This not necessarily a lower bound as we will prove here.

Lemma 5.4. *If X is a countable* wpo*, then $o(\mathbb{T}_n^{leaf}(X)) \leq \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$.*

Proof. Of course, we assume that $n \geq 1$ and $o(X) \geq 1$.

We prove this by induction on $o(X)$. The left-sets $L(t)$ for $t \in \mathbb{T}_n^{leaf}(X)$ will be very crucial: we prove that $l(t) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$ for every $t \in \mathbb{T}_n^{leaf}(X)$ by induction on the height of t. A similar idea can be found in [69]. If t is a tree consisting of one node with label x, then there is a quasi-embedding from $L(t)$ to $\mathbb{T}_n^{leaf}(L_X(x))$. If $l_X(x) = 0$, then $l(t) = 0$, hence $l(t) < \vartheta(\Omega^\omega \cdot n + o(X))$. Assume $0 < l_X(x) < o(X)$. Then $-1 + l_X(x) < -1 + o(X)$, hence $l(t) \leq o(\mathbb{T}_n^{leaf}(L_X(x))) \leq \vartheta(\Omega^\omega \cdot n + (-1 + l_X(x))) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$.

Assume t is a tree with immediate subtrees $t_1, \ldots, t_m$ and label i and suppose that $l(t_i) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$ for every i. We show that there exists a quasi-embedding f from $L(t)$ to

$$\mathbb{T}\begin{pmatrix} X & \{0, \ldots, n-1\} \times L & \{0, \ldots, i-1\} \\ 1 & m & \omega \end{pmatrix},$$

where

$$L = \{0\} + \sum_{l=1}^{m} M^\diamond(L(t_l)).$$

$\{0\}$ just stands for the wpo consisting of one element. How does one define f? If $s \in L(t)$ is a tree of height zero, define $f(s)$ as s. If $s \in L(t)$ is a tree with root-label j and immediate subtrees $s_1, \ldots, s_k$, then $s_1, \ldots, s_k \in L(t)$ and one of the following conditions are satisfied

(a) $j < i$, or

(b) $j \geq i$ and $[t_1, \ldots, t_m] \not\leq^\diamond [s_1, \ldots, s_k]$.

If $[t_1, \ldots, t_m] \not\leq^\diamond [s_1, \ldots, s_k]$, then we have one of the next properties.

1. $k < m$, or
2. $k \geq m$ and $s_i \in L(t_1)$ for all i, or
3. $k \geq m$ and there is an index i_1 such that $t_1 \leq s_{i_1}$ and for all $i \neq i_1$, we have $s_i \in L(t_2)$, or

...

m+1. $k \geq m$ and there are distinct indices $i_1, \ldots, i_{m-1}$ such that $t_j \leq s_{i_j}$ for every j and for all $i \neq i_1, \ldots, i_{m-1}$, we have $s_i \in L(t_m)$.

So assume $s \in L(t)$ is a tree with root-label j and immediate subtrees $s_1, \ldots, s_k$. If we are in case (a), define $f(s)$ as the tree with root-label j and immediate subtrees $f(s_1), \ldots, f(s_k)$. If we are in case (b), then we have the following sub-cases.

1. If 1. holds, define $f(s)$ as the tree with root-label $(j, 0) \in \{0, \ldots, n-1\} \times \{0\}$ and immediate subtrees $f(s_1), \ldots, f(s_k)$.

2. If 2. holds, define $f(s)$ as the single node with root-label $(j, [s_1, \ldots, s_k])$.

3. If 3. holds, define $f(s)$ as the tree with immediate subtree $f(s_{i_1})$ and root-label $(j, [s_1, \ldots, s_{i_1-1}, s_{i_1+1}, \ldots, s_k])$.

...

m+1. If m+1. holds, define $f(s)$ as the tree with immediate subtrees $f(s_{i_1})$, $\ldots$, $f(s_{i_{m-1}})$ and root-label $(j, multi)$, where $multi$ is the multiset consisting of the s_i such that $i \neq i_1, \ldots, i_{m-1}$.

One can prove that f is a quasi-embedding in a straightforward way (one can find similar proofs in previous chapters). Therefore

$$\begin{aligned} l(t) &\leq o\left(\mathbb{T}\begin{pmatrix} X & \{0, \ldots, n-1\} \times L & \{0, \ldots, i-1\} \\ 1 & m & \omega \end{pmatrix}\right) \\ &\leq \vartheta\left(\Omega^\omega \cdot i + \Omega^{m-1} \cdot (o(L) \otimes n) + o(X)\right) \\ &< \vartheta(\Omega^\omega \cdot n + (-1 + o(X))). \end{aligned}$$

In the previous inequalities we used Theorem 1.79, $i < n$, $o(L) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$ (because $l(t_i) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X))))$ and $o(X) < \vartheta(\Omega^\omega \cdot n + (-1 + o(X)))$. □

Definition 5.5. *On the set $\mathbb{T}_n^{leaf}(X)$ define the weak gap-embeddability relation $T_1 \leq_{gap}^w T_2$ if there exists a homeomorphic embedding h from T_1 to T_2 such that*

1. *For all leaves t of T_1, $f(t)$ is a leaf of T_2 and $l_1(t) \leq_X l_2(f(t))$.*

2. *For all internal nodes t of T_1, we have that $f(t)$ is an internal node of T_2 and $l_1(t) = l_2(f(t))$.*

3. *$\forall t \in T_1$ and for all immediate successors $t' \in T_1$ of t such that t and t' are internal nodes, we have that if $\bar{t} \in T_2$ and $f(t) < \bar{t} < f(t')$, then $l_2(\bar{t}) \geq l_2(f(t')) = l_1(t')$.*

In this context, l_i is the labeling function of T_i. The strong gap-embeddability relation ($\leq^s_{gap}$) fulfills the extra condition

4. *For all $t' < f(root(T_1))$, we have $l_2(t') \geq l_2(f(root(T_1))) = l_1(root(T_1))$.*

to the definition of $\leq^w_{gap}$. We write $\mathbb{T}^{wgap}_n(X)$, respectively $\mathbb{T}^{sgap}_n(X)$, for the partial order $(\mathbb{T}^{leaf}_n(X), \leq^w_{gap})$, respectively $(\mathbb{T}^{leaf}_n(X), \leq^s_{gap})$. Note that the notation $\mathbb{T}^{wgap}_n(X)$ cannot be misunderstood about which nodes have labels in X because $\mathbb{T}^{wgap}_n(X)$ indicates two label sets, namely $\{0, \ldots, n-1\}$ and X. If we restrict ourselves to structured rooted trees, then we denote this by $\mathbb{T}^{s,wgap}_n(X)$ and $\mathbb{T}^{s,sgap}_n(X)$. Similarly for $\mathbb{T}^{leaf}_n[m](X)$.

Following paragraph 4 in [76], we have the next lemma.

Lemma 5.6. $\mathbb{T}^{sgap}_{n+1} \cong \mathbb{T}^{sgap}_n(\mathbb{T}^{wgap}_{n+1})$.

Corollary 5.7. $\mathbb{T}^{sgap}_{n+1}[m] \cong \mathbb{T}^{sgap}_n[m-1](\mathbb{T}^{wgap}_{n+1})$.

So if we can show that $o(\mathbb{T}^{wgap}_2) = \vartheta_0(\vartheta_1(\Omega^\omega_2)^\omega)$, then we can also prove (see Corollary 5.19)

$$o(\mathbb{T}^{sgap}_2) = \vartheta_0(\Omega^\omega_1 + \vartheta_0(\vartheta_1(\Omega^\omega_2)^\omega)),$$

by using $\mathbb{T}^{sgap}_1(X) \cong \mathbb{T}^{leaf}_1(X)$.

We will obtain the maximal order type of $\mathbb{T}^{wgap}_2$ by calculating the maximal order type of $\mathcal{T}(W)$ for $W = M^\diamond(\mathbb{T}^{leaf}(\cdot))\backslash\{[]\}$: Lemma 5.16 indicates that for this symbol W, the partial order $\mathcal{T}(W)$ is equal to $\mathbb{T}'^{wgap}_2[0]$, and Lemma 5.2 yields that the maximal order type of this wpo is also equal to the maximal order type of $\mathbb{T}^{wgap}_2$.

Weiermann's conjecture 1.111 only indicates the maximal order type of $\mathcal{T}(W)$ if $o(W(\Omega)) \leq \varepsilon_{\Omega+1}$. If $o(W(\Omega)) > \varepsilon_{\Omega+1}$, one has to rephrase the conjecture using the collapsing functions $(\vartheta_i)_{i<\omega}$.

Conjecture 5.8. *If $Y_1, \ldots, Y_k$ are countable wpo's, then for every $W \in \mathcal{W}(Y_1, \ldots, Y_k)$, the partial order $\mathcal{T}(W)$ is a wpo and its maximal order type is bounded above by $\vartheta_0(f(W))$.*

$f(W)$ is not the same as $o(W(\Omega_1))$, but has some little differences. See Definition 5.10. We also want to note that not for every W, the upper bound $\vartheta_0(f(W))$ of $o(\mathcal{T}(\mathcal{W}))$ will be optimal (e.g., $W = \mathcal{C}_{Y_i}$).

Before we go on, we modify the definition of $\mathcal{W}(Y_1,\dots,Y_k)$ a bit for a smoother proof. We use an auxiliary set of symbols $\mathcal{W}'(Y_1,\dots,Y_k)$.

Definition 5.9. *Assume that $Y_1,\dots,Y_k$ are fixed partial orderings. Define $\mathcal{W}(Y_1,\dots,Y_k)$ and $\mathcal{W}'(Y_1,\dots,Y_k)$ as the following set of function symbols.*

- *For any $i=1,\dots,k$, let $\mathcal{C}_{Y_i} \in \mathcal{W}(Y_1,\dots,Y_k)$,*
- *$Id \in \mathcal{W}(Y_1,\dots,Y_k)$,*
- *$\mathbb{T}^{leaf}(Id) \in \mathcal{W}(Y_1,\dots,Y_k)$,*
- *If $W_{i,j} \in \mathcal{W}'(Y_1,\dots,Y_k)$ for all i and j and (if $n=0$, then $m_0>0$), then $\sum_{i=0}^{n}\prod_{j=0}^{m_i} W_{i,j} \in \mathcal{W}(Y_1,\dots,Y_k)$*
- *If $W, W_1,\dots,W_n \in \mathcal{W}(Y_1,\dots,Y_k)$, then*
  - *$M^\diamond(W)$ and $M^\diamond(W)\backslash\{[]\}$ are in $\mathcal{W}(Y_1,\dots,Y_k)$ and $\mathcal{W}'(Y_1,\dots,Y_k)$,*
  - *If $0 \le \alpha_1 < \dots < \alpha_n \le \omega$, then $\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}$ is in $\mathcal{W}(Y_1,\dots,Y_k)$ and $\mathcal{W}'(Y_1,\dots,Y_k)$.*

Define the complexity $|W|$ for symbols $W \in \mathcal{W}(Y_1,\dots,Y_k)$ as usual, using $|\sum_{i=0}^{n}\prod_{j=0}^{m_i} W_{i,j}| := \max_{i,j}(|W_{i,j}|)+1$ and $|\mathbb{T}^{leaf}(Id)| := 0$. $I(\mathbb{T}^{leaf}(Id))(X)$ is defined as $\mathbb{T}^{leaf}(I(Id)(X)) = \mathbb{T}^{leaf}(X)$ and let $I(\sum_{i=0}^{n}\prod_{j=0}^{m_i} W_{i,j})(X)$ be $\sum_{i=0}^{n}\prod_{j=0}^{m_i} I(W_{i,j})(X)$. We can also redefine the other definitions and reprove the statements in Section 1.2.10 in a straightforward way. Especially, we will need the Lifting Lemma.

The auxiliary set $\mathcal{W}'(Y_1,\dots,Y_k)$ is defined for obtaining symbols

$$\textstyle\sum_{i=0}^{n}\prod_{j=0}^{m} W_{i,j},$$

where $W_{i,j}$ is *not* of the same form. This construction of $+$ and $\times$ does not yield a completely new set $\mathcal{W}(Y_1,\dots,Y_k)$ because $(A+B)\times(C+D) \cong (A\times C)+(A\times D)+(B\times C)+(B\times D)$ and $(A\times B)+(A\times C) \cong A\times(B+C)$. We assign an ordinal $f(W)$ to every symbol in $\mathcal{W}(Y_1,\dots,Y_k)$ and $\mathcal{W}'(Y_1,\dots,Y_k)$.

Definition 5.10. *Let $Y_1,\dots,Y_k$ be $\mathbb{wpo}$'s, then*

- *$f(\mathcal{C}_{Y_i}) := o(Y_i)+2$,*
- *$f(Id) := \Omega_1$,*
- *$f(\mathbb{T}^{leaf}(Id)) := \vartheta_1(\Omega_2^\omega)$.*
- *$f(\sum_{i=0}^{n}\prod_{j=0}^{m} W_{i,j}) = \bigoplus_{i=0}^{n}\bigotimes_{j=0}^{m} f(W_{i,j})$,*

- $f(M^\diamond(W)) := \omega^{\omega^{f(W)+1}}$,
- $f(M^\diamond(W)\backslash\{[]\}) := \omega^{\omega^{f(W)+1}}$,
- $f\left(\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}\right) = \vartheta_1(\Omega_2^{\alpha_n} f(W_n) + \dots + \Omega_2^{\alpha_1} f(W_1))$.

Lemma 5.11. $K_0(f(\sum_{i=0}^n \prod_{j=0}^{m_i} W_{i,j})) = \bigcup_{i,j} K_0(f(W_{i,j}))$.

Proof. By Lemma 1.21. □

For every $W \in \mathcal{W}(Y_0, \dots, Y_k)$, we define its sub-symbols.

Definition 5.12.
- $sub(\mathcal{C}_{Y_i}) := \{\mathcal{C}_{Y_i}\}$,
- $sub(Id) := \{Id\}$,
- $sub(\mathbb{T}^{leaf}(Id)) := \{\mathbb{T}^{leaf}(Id), Id\}$,
- $sub(\sum_{i=0}^n \prod_{j=0}^m W_{i,j}) := \bigcup_{i,j}\{W_{i,j}\} \cup \{\sum_{i=0}^n \prod_{j=0}^m W_{i,j}\}$,
- $sub(M^\diamond(W)) := \{M^\diamond(W)\} \cup sub(W)$,
- $sub(M^\diamond(W)\backslash\{[]\}) := \{M^\diamond(W)\backslash\{[]\}\} \cup sub(W)$,
- $sub\left(\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}\right) = \left\{\mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}\right\} \cup \bigcup_i sub(W_i)$.

Note that $W \in sub(W)$.

Lemma 5.13. *For all $W, W' \in \mathcal{W}(Y_1, \dots, Y_k)$,*

1. $1 < f(W) < \Omega_2$,
2. *If $W' \in sub(W)\backslash\{W\}$, then $f(W') < f(W)$ and $k_0(f(W')) \leq k_0(f(W))$.*

Proof. The first assertion is trivial. We prove the second one by induction on $|W|$. If $|W| = 0$, then the assertion is trivial. Assume $|W| > 0$. If $W = \sum_{i=0}^n \prod_{j=0}^m W_{i,j}$, then $f(W) = \oplus_{i=0}^n \otimes_{j=0}^m f(W_{i,j})$. Because $f(W_{i,j}) > 1$, it is easy to see that $f(W_{i,j}) < f(W)$. The lemma then easily follows. Furthermore, $k_0(f(W)) = \max_{i,j} k_0(f(W_{i,j}))$

If $W = M^\diamond(V)$ or $M^\diamond(V)\backslash\{[]\}$, then $f(W) = \omega^{\omega^{f(V)+1}}$ and the lemma also in this case easily follows.

If $W = \mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}$, then $f(W) = \vartheta_1(\Omega_2^{\alpha_n} f(W_n) + \cdots + \Omega_2^{\alpha_1} f(W_1))$, so

$$f(W_i) = k_1(f(W_i)) < \vartheta_1(\Omega_2^{\alpha_n} f(W_n) + \cdots + \Omega_2^{\alpha_1} f(W_1)) = f(W)$$

and

$$\begin{aligned} K_0(f(W_i)) &\subseteq K_0(K_1(\Omega_2^{\alpha_n} f(W_n) + \cdots + \Omega_2^{\alpha_1} f(W_1))) \\ &= K_0(\vartheta_1(\Omega_2^{\alpha_n} f(W_n) + \cdots + \Omega_2^{\alpha_1} f(W_1))) \\ &= K_0(f(W)). \end{aligned}$$

□

5.2 Upper bound for $o(\mathbb{T}_2^{wgap})$

The next lemma is one of the most crucial lemmas in this dissertation. The techniques are based on Diana Schmidt's proofs of the maximal order types of the tree-classes (see [69]). Recall that $\vartheta_0\alpha$ and $\vartheta_1\alpha$ are epsilon numbers and $k_1\beta = \beta$ if $\beta < \Omega_2$.

Lemma 5.14. *For all symbols W, W' in $\mathcal{W}(Y_1, \dots, Y_k)$, where Y_i are countable* wpo*'s, and for all $s \in W'(\mathcal{T}(W))$. If*

- *$W' \in sub(W)$,*
- *$s = w'(t_1, \dots, t_n)$ with $w' \in T_{W'}$ and $t_i \in \mathcal{T}(W)$,*
- *for all i, $L(t_i)$ are* wpo*'s and $l(t_i) < \vartheta_0(f(W))$,*

then there exists a symbol $W_s \in \mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$[1] *and there exists a quasi-embedding e_s from $\{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$ to $W_s(\mathcal{T}(W))$ with the properties*

a. $Leaves(e_s(u)) \subseteq Leaves(u)$ for all $u \in \{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$,

b. $f(W_s) < f(W')$,

c. $k_0(f(W_s)) < \vartheta_0(f(W))$.

Proof. We prove this lemma by induction on $|W'|$ (see Definition 1.87) and subsidiary induction on $|w'|$ (see Definition 1.89).

[1] where i runs over $1, \dots, k$; j over $1, \dots, n$ and y over all symbols in Y_i such that $y \in Con(w')$ (see Definition 1.92).

1. If $W' = \mathcal{C}_{Y_i}$, then $s = y \in Y_i$ and $n = 0$. Define W_s as $\mathcal{C}_{L_{Y_i}(y)} \in \mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$ and e_s as the identity. Property a. is trivial. Property b. follows from $o(L_{Y_i}(y)) < o(Y_i)$ and property c. from

$$\begin{array}{rcl} k_0(f(W_s)) & = & k_0(o(L_{Y_i}(y))) \\ & \overset{\text{Lemma 1.22}}{\leq} & k_0(o(Y_i)) \\ & = & k_0(f(W')) \\ & \overset{\text{Lemma 5.13}}{\leq} & k_0(f(W)) \\ & < & \vartheta_0(f(W)). \end{array}$$

2. If $W' = Id$, then $s = t_1$ with $t_1 \in \mathcal{T}(W)$, $L(t_1)$ a wpo and $l(t_1) < \vartheta_0(f(W))$. Define W_s as $\mathcal{C}_{L(t_1)}$ and e_s as the natural quasi-embedding from $\{u \in W'(\mathcal{T}(W)) : t_1 \not\leq u\}$ to $W_s(\mathcal{T}(W)) = W_s = L(t_1)$. e_s is in some sense the identity, but the elements of the domain and the range are from a different context: in the domain, the elements lie in $W'(\mathcal{T}(W)) = \mathcal{T}(W)$, whereas the elements of the range are seen as constants in $\mathcal{C}_{L_{\mathcal{T}(W)}(t_1)}$. $Leaves(e_s(u)) = \emptyset$, hence property a. trivially holds. Property b. follows easily. Property c. is valid because $k_0(f(W_s)) = k_0(o(L(t_1))) \overset{\text{Lemma 1.22}}{\leq} o(L(t_1)) < \vartheta_0(f(W))$.

3. If $W' = \mathbb{T}^{leaf}(Id)$, then $s \in W'(\mathcal{T}(W))$ is a tree with leaf-labels in $\mathcal{T}(W)$. We prove the lemma by induction on the height of s.

   Case a.
   If the height is zero, s is the tree that consists of a single node with label $t_1 \in T(W)$. By assumption, $L(t_1)$ is a wpo and $l(t_1) < \vartheta_0(f(W))$. Define W_s as $\mathcal{C}_Y$ with $Y = \mathbb{T}^{leaf}(L(t_1))$ and e_s as the natural embedding from $\{u \in \mathbb{T}^{leaf}(\mathcal{T}(W)) : s \not\leq u\}$ to $\mathbb{T}^{leaf}(L(t_1))$. Again, like in case 2., e_s is the identity, but the elements of the domain and the range of e_s are from a different context/set. Property a. trivially holds. Property b. follows from $f(W_s) = o(\mathbb{T}^{leaf}(L(t_1))) + 2 = \vartheta_0(\Omega_1^\omega + l(t_1)) + 2 < \Omega_1 \leq f(W')$ and property c. from $k_0(f(W_s)) \overset{\text{Lemma 1.22}}{\leq} f(W_s) = \vartheta_0(\Omega_1^\omega + l(t_1)) + 2 < \vartheta_0(f(W))$, where the last inequality is valid because $k_0(\Omega_1^\omega + l(t_1)) = k_0(l(t_1)) \leq l(t_1) < \vartheta_0(f(W))$ and $\Omega_1^\omega + l(t_1) < \Omega_1^{\omega+1} \leq \vartheta_1(\Omega_2^\omega) = f(W') \overset{\text{Lemma 5.13}}{\leq} f(W)$.

   Case b.
   If the height of s is strictly larger than zero, then s has immediate subtrees $s_1, \ldots, s_m$ with $m \geq 1$. The sub-induction hypothesis

yields symbols W_{s_i} and quasi-embeddings e_{s_i} for every i such that $Leaves(e_{s_i}(u)) \subseteq Leaves(u)$ and the inequalities $f(W_{s_i}) < f(W')$ and $k_0(f(W_{s_i})) < \vartheta_0(f(W))$ hold for all i.

Now, if v is an arbitrary element of $A := \{u \in W'(\mathcal{T}(W)) : s \not\leq_{W'(\mathcal{T}(W))} u\}$, then one of the following conditions is satisfied,

1. v is a tree consisting of a single node with label in $\mathcal{T}(W)$.
2. v is a tree with immediate subtrees $v_1, \ldots, v_k$ with $1 \leq k < m$ and $v_i \in A$.
3. v is a tree with immediate subtrees $v_1, \ldots, v_k$ with $k \geq m$ and $v_1, \ldots, v_k \in \{u \in W'(\mathcal{T}(W)) : s_1 \not\leq_{W'(\mathcal{T}(W))} u\}$
4. v is a tree with immediate subtrees $v_1, \ldots, v_k$ with $k \geq m$ and there exists a j_1 such that $v_{j_1} \in \{u \in W'(\mathcal{T}(W)) : s_1 \leq_{W'(\mathcal{T}(W))} u\}$, $v_{j_1} \in A$ and $v_1, \ldots, v_{j_1 - 1}, v_{j_1 + 1}, \ldots, v_k \in \{u \in W'(\mathcal{T}(W)) : s_2 \not\leq_{W'(\mathcal{T}(W))} u\}$,

   . . .

m+2. v is a tree with immediate subtrees $v_1, \ldots, v_k$ with $k \geq m$ and there exist distinct $j_1, \ldots, j_{m-1}$ such that $v_{j_l} \in \{u \in W'(\mathcal{T}(W)) : s_l \leq_{W'(\mathcal{T}(W))} u\}$ and $v_{j_l} \in A$ for every l and for $l \neq j_1, \ldots, j_{m-1}$, we have $v_l \in \{v \in W'(\mathcal{T}(W)) : s_m \not\leq_{W'(\mathcal{T}(W))} v\}$.

If $m > 1$, define W_s as

$$\mathbb{T}\begin{pmatrix} X & \{0\} + M^{\diamond}(W_{s_1}) + \cdots + M^{\diamond}(W_{s_m}) \\ 1 & m \end{pmatrix}$$

and if $m = 1$, define W_s as

$$\mathbb{T}\begin{pmatrix} X + M^{\diamond}(W_{s_1}) + \cdots + M^{\diamond}(W_{s_m}) \\ m \end{pmatrix}.$$

Let $e_s : A \to W_s(\mathcal{T}(W))$ be the natural resulting map, namely:

1. If case 1. holds, define $e_s(v)$ as v.
2. If case 2. holds, then $m > 1$ and define $e_s(v)$ as the tree with root-label 0 and immediate subtrees $e_s(v_1), \ldots, e_s(v_k)$.
3. If case 3. holds, then $k \geq m$ and define $e_s(v)$ as the tree consisting of one node with root-label $[e_{s_1}(v_1), \ldots, e_{s_1}(v_k)]$.

4. If case 4. holds, then $k \geq m$ and define $e_s(v)$ as the tree with root-label $[e_{s_2}(v_1), \ldots, e_{s_2}(v_{j_1-1}), e_{s_2}(v_{j_1+1}), \ldots, e_{s_2}(v_k)]$ and immediate subtree $e_s(v_{j_1})$.

   ...

m+2. If case $m+2$ holds, then $k \geq m$ and define $e_s(v)$ as the tree with root-label equal to the multiset consisting of the elements $e_{s_m}(v_l)$ with $l \neq j_1, \ldots j_{m-1}$ and immediate subtrees $e_s(v_{j_1}), \ldots, e_s(v_{j_{m-1}})$.

That $e_s(v) \leq e_s(v')$ implies $v \leq v'$ can be proved in a straightforward way using induction on $ht(v) + ht(v')$: if v and v' are in two different cases, then this can be proved by an easy induction argument. For example, if for v case 4. and for v' case 5. holds, then the root-labels of $e_s(v)$ and $e_s(v')$ are incomparable, hence $e_s(v) \leq e_s(v')$ yields $e_s(v) \leq e_s(v'_{j'_1})$ or $e_s(v) \leq e_s(v'_{j'_2})$. The induction hypothesis yields $v \leq v'_{j'_l} \leq v'$. If for both v and v' the same case holds, then one also needs the properties of e_{s_i}. For example, assume that for v and v' case 4. is valid. Then $e_s(v) \leq e_s(v')$ yields $e_s(v) \leq e_s(v'_{j'_1})$ or

$$\begin{aligned}&[e_{s_2}(v_1), \ldots, e_{s_2}(v_{j_1-1}), e_{s_2}(v_{j_1+1}), \ldots, e_{s_2}(v_k)]\\ \leq^\diamond &[e_{s_2}(v'_1), \ldots, e_{s_2}(v'_{j'_1-1}), e_{s_2}(v'_{j'_1+1}), \ldots, e_{s_2}(v'_{k'})]\end{aligned}$$

and $e_s(v_{j_1}) \leq e_s(v'_{j'_1})$. The former case easily yields $v \leq v'_{j'_1} \leq v'$. In the latter case, the induction hypothesis yields $v_{j_1} \leq v'_{j'_1}$ and the properties of e_{s_2} implies

$$[v_1, \ldots, v_{j_1-1}, v_{j_1+1}, \ldots, v_k] \leq^\diamond [v'_1, \ldots, v'_{j'_1-1}, v'_{j'_1+1}, \ldots, v'_{k'}].$$

This yields $v \leq v'$.

Property $a.$ can be proved by induction on the height of the tree. One has to consider every case separately. For example in case 4.,

$$\begin{aligned}Leaves(e_s(v)) &= \bigcup_{i \neq j_1} Leaves(e_{s_2}(v_i)) \cup Leaves(e_s(v_{j_1}))\\ &\subseteq \bigcup_{i \neq j_1} Leaves(v_i) \cup Leaves(v_{j_1})\\ &= \bigcup_i Leaves(v_i) = Leaves(v).\end{aligned}$$

If $m > 1$, then $f(W_s) = \vartheta_1(\Omega_2^{m-1} f(W_1) + f(W_2))$ with

$$W_1 = \{0\} + M^\diamond(W_{s_1}) + \cdots + M^\diamond(W_{s_m})$$

and $W_2 = X$. Hence $f(W_s) < f(W') = \vartheta_1(\Omega_2^\omega)$ follows from

$$\Omega_2^{m-1} f(W_1) + f(W_2) < \Omega_2^\omega$$

and $k_1(\Omega_2^{m-1} f(W_1) + f(W_2)) = \max\{f(W_1), f(W_2)\} < f(W')$ using $f(W_{s_i}) < f(W')$. So property $b.$ is also valid. Property $b.$ for the case $m = 1$ can be proved in a similar way.

$k_0(f(W_s)) = \max\{k_0(f(W_1)), k_0(f(W_2))\} < \vartheta_0(f(W))$ using $k_0(f(W_{s_i})) < \vartheta_0(f(W))$, which yields property $c.$

4. If $W' = \sum_{i=0}^n \prod_{j=0}^{m_i} W'_{i,j}$, then $s = (s_0, \dots, s_{m_i})$ with $s_j \in W'_{i,j}(\mathcal{T}(W))$ for a certain i and $0 \leq j \leq m_i$. Without lose of generality, we can assume that $i = 0$. Because $|W'_{0,j}| < |W'|$, there exist symbols W_{s_j} with $f(W_{s_j}) < f(W'_{0,j})$ and $k_0(f(W_{s_j})) < \theta_0(f(W))$ and corresponding mappings e_{s_j}. Define W_s as

$$\sum_{i=1}^{n} \prod_{j=0}^{m_i} W'_{i,j} + \sum_{j=0}^{m} \left(W_{s_j} \times \prod_{k=0, k \neq j}^{m_0} W'_{0,k} \right).$$

Pick $u \in \{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$. Then there exists an index i' such that $u \in \prod_{j=0}^{m_{i'}} W'_{i',j}$. If $i' \neq 0$, define $e_s(u)$ as $u \in W_s(\mathcal{T}(W))$. If $i' = 0$, then $u = (u_0, \dots, u_{m_0})$ and there exists a least index j' such that $s_{j'} \not\leq u_{j'}$. Define then $e_s(u)$ as $(e_{s_{j'}}(u_{j'}), (u_0, \dots, u_{j'-1}, u_{j'+1}, \dots, u_{m_0})) \in W_s(\mathcal{T}(W))$. It is trivial to see that e_s is a quasi-embedding using the fact that the e_{s_j} are quasi-embeddings. If $i' \neq 0$, then $Leaves(e_s(u)) = Leaves(u)$ and if $i' = 0$, then

$$\begin{aligned} Leaves(e_s(u)) &\subseteq Leaves(u) \cup Leaves(e_{s_{j'}}(u_{j'})) \\ &\subseteq Leaves(u) \cup Leaves(u_{j'}) \\ &= Leaves(u). \end{aligned}$$

So property $a.$ is valid. Property $b.$ holds if

$$f\left(\sum_{j=0}^{m} \left(W_{s_j} \times \prod_{k=0, k \neq j}^{m_0} W'_{0,k} \right) \right) < f\left(\prod_{j=0}^{m_0} W'_{0,j} \right)$$

$f(W'_{0,j})$ is always an additive closed ordinal number by construction of W', hence $f\left(\prod_{j=0}^{m_0} W'_{0,j}\right)$ is this as well. Therefore, the strict inequality is valid if $f(W_{s_j}) < f(W'_{0,j})$ for all j, but we know that this is true. Property $c.$ follows from Lemma 5.11, $k_0(f(W_{s_j})) < \vartheta_0(f(W))$ and the fact that $k_0(f(W')) < \vartheta_0(f(W))$ using Lemma 5.13.

5. If $W' = M^{\diamond}(V)$, then $s = [s_1, \ldots, s_n]$ with $s_i \in V(\mathcal{T}(W))$. Because $|V| < |W'|$, we obtain symbols $W_{s_i} \in \mathcal{W}(Y_i, L_{Y_i}(y), L(t_j))$ and quasi-embeddings e_{s_i} that satisfy properties $a.$, $b.$ and $c.$ We prove the lemma by sub-induction on n.

   (a) If $n = 0$, then s is equal to $[]$. The set $\{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$ is then empty. Define W_s as $\mathcal{C}_\emptyset$ and e_s as the trivial mapping. Properties $a.$, $b.$ and $c.$ easily follow.

   (b) Let $n > 0$. s is equal to $[s_1, \ldots, s_n]$ and $s_i \in V(\mathcal{T}(W))$ for every i. Take an element $u = [u_1, \ldots, u_m] \in \{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$. Then $s_n \not\leq_{V(\mathcal{T}(W))} u_j$ for all j or there exists a least index j_0 such that $s_n \leq_{V(\mathcal{T}(W))} u_{j_0}$ and $[s_1, \ldots, s_{n-1}] \not\leq_{W'(\mathcal{T}(W))} [u_1, \ldots, u_{j_0-1}, u_{j_0+1}, \ldots, u_m]$. The sub-induction hypothesis yields a symbol $W_{[s_1,\ldots,s_{n-1}]}$ and a quasi-embedding $e_{[s_1,\ldots,s_{n-1}]}$ that satisfies properties $a.$, $b.$ and $c.$ Define W_s as

   $$M^{\diamond}(W_{s_n}) + V \times W_{[s_1,\ldots,s_{n-1}]}.$$

   By an obvious translation, this can be seen as a symbol in $\mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$.

   Take an element $u = [u_1, \ldots, u_m]$ in $\{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$. If $s_n \not\leq_{V(\mathcal{T}(W))} u_j$ for all j, define $e_s(u)$ as $[e_{s_n}(u_1), \ldots, e_{s_n}(u_n)]$. If there exists a least index j_0 such that $s_n \leq_{V(\mathcal{T}(W))} u_{j_0}$ and $[s_1, \ldots, s_{n-1}] \not\leq_{W'(\mathcal{T}(W))} [u_1, \ldots, u_{j_0-1}, u_{j_0+1}, \ldots, u_m]$, define $e_s(u)$ as

   $$(u_{j_0}, e_{[s_1,\ldots,s_{n-1}]}([u_1, \ldots, u_{j_0-1}, u_{j_0+1}, \ldots, u_m])).$$

   Obviously, e_s is a quasi-embedding if e_{s_n} and $e_{[s_1,\ldots,s_{n-1}]}$ are quasi-embeddings.

   Now, we check property $a.$ for e_s. If $s_n \not\leq_{V(\mathcal{T}(W))} u_j$ for all j, then $Leaves(e_s(u)) = \bigcup_i Leaves(e_{s_n}(u_i)) \subseteq \bigcup_i Leaves(u_i) = Leaves(u)$. If not $s_n \not\leq_{V(\mathcal{T}(W))} u_j$ for all j, then $Leaves(e_s(u)) = Leaves(u_{j_0}) \cup Leaves(e_{[s_1,\ldots,s_{n-1}]}([u_1, \ldots, u_{j_0} - 1, u_{j_0+1}, \ldots, u_m])) \subseteq Leaves(u_{j_0}) \cup Leaves([u_1, \ldots, u_{j_0} - 1, u_{j_0+1}, \ldots, u_m]) = Leaves(u)$. Property $b.$ follows if

   $$f\left(M^{\diamond}(W_{s_n}) + V \times W_{[s_1,\ldots,s_{n-1}]}\right) < f(W').$$

   We know $f(W_{[s_1,\ldots,s_{n-1}]}) < f(W')$, $f(W_{s_n}) < f(V)$ and $f(W') = \omega^{\omega^{f(V)+1}}$. The strict inequality follows because $f(W')$ is multiplicatively closed. Property $c.$ is valid because $K_0(f(M^{\diamond}(W_{s_n}))) =$

$K_0(f(W_{s_n}))$, $K_0(f(V)) \subseteq K_0(f(W'))$, $k_0(f(W_{s_n})) < \vartheta_0(f(W))$, $k_0(f(W')) < \vartheta_0(f(W))$ (using Lemma 5.13) and $k_0(f(W_{[s_1,\dots,s_{n-1}]}))$ $< \vartheta_0(f(W))$.

6. One can prove the lemma for $W' = M^{\diamond}(V)\backslash\{[]\}$ in a similar way.

7. Let $W' = \mathbb{T}\begin{pmatrix} W_1 & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_n \end{pmatrix}$ with $0 \leq \alpha_1 < \dots < \alpha_n \leq \omega$. This case is similar as $W' = \mathbb{T}^{leaf}(Id)$. $W' \in Sub(W)$ yields $k_0(f(W')) \leq k_0(f(W)) < \vartheta_0(f(W))$, hence $k_0(f(W_i)) < \vartheta_0(f(W))$ for all i. Assume $s \in W'(\mathcal{T}(W))$. Then s is a tree and we prove the lemma by sub-induction on the height of s, i.e. in some sense by sub-induction on $|w'|$.

   Case a.
   s is a tree with height zero and root-label s' with $s' \in W_i(\mathcal{T}(W))$ for a certain i. Because $|W_i| < |W'|$, we obtain a symbol $W_{s'} \in \mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$ and a quasi-embedding $e_{s'}$ that satisfy properties $a.$, $b.$ and $c.$ Let u be an arbitrary element $\{u \in W'(\mathcal{T}(W)) : s \not\leq_{W'(\mathcal{T}(W))} u\}$. Then, u can be interpreted as an element of

$$(W_s)\,(\mathcal{T}(W))$$

   with W_s equal to

$$\mathbb{T}\begin{pmatrix} W_1 & \dots & W_{i-1} & W_{s'} & W_{i+1} & \dots & W_n \\ 1+\alpha_1 & \dots & 1+\alpha_{i-1} & 1+\alpha_i & 1+\alpha_{i+1} & \dots & 1+\alpha_n \end{pmatrix}.$$

   Define e_s as follows. Take $v \in \{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$ and assume that v has root-label $v' \in W_j(\mathcal{T}(W))$ and immediate subtrees $v_1, \dots, v_k \in \{u \in W'(\mathcal{T}(W)) : s \not\leq u\}$ with $k < 1+\alpha_j$ for a certain j. If $j \neq i$, define $e_s(v)$ as the tree with root-label v' and immediate subtrees $e_s(v_1), \dots, e_s(v_k)$. If $j = i$, define $e_s(v)$ as the tree with root-label $e_{s'}(v')$ and immediate subtrees $e_s(v_1), \dots, e_s(v_k)$.

   e_s is a quasi-embedding and $Leaves(e_s(u)) \subseteq Leaves(u)$ are straightforward verifications.

$$k_0(f(W_s)) = \max\{k_0(f(W_1)), \dots, k_0(f(W_{i-1})), k_0(f(W_{i+1})), \dots, k_0(f(W_n)), k_0(f(W_{s'}))\},$$

   so property $c.$ follows because $k_0(f(W_{s'})) < \vartheta_0(f(W))$. From $f(W_{s'}) < f(W_i)$ one can easily prove $f(W_s) < f(W')$.

   Case b.
   s is a tree with immediate subtrees $s_1, \dots, s_m$ and root-label $s' \in$

$W_i(\mathcal{T}(W))$. Hence $1 \le m < 1 + \alpha_i$, so $\alpha_i \ge 1$. By the main induction and sub-induction hypothesis, there exist symbols $W_{s'}$ and W_{s_i} in $\mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$ and quasi-embeddings $e_{s'}$, e_{s_i} for every i such that $Leaves(e_{s'}(u)) \subseteq Leaves(u)$, $Leaves(e_{s_i}(u)) \subseteq Leaves(u)$, $f(W_{s'}) < f(W_i)$, $f(W_{s_i}) < f(W')$, $k_0(f(W_{s'})) < \vartheta_0(f(W))$ and $k_0(f(W_{s_i})) < \vartheta_0(f(W))$ for all i.

Now, if v is an arbitrary element of $A := \{u \in W'(\mathcal{T}(W)) : s \not\le_{W'(\mathcal{T}(W))} u\}$, then one of the following conditions is satisfied.

1. v is a tree with root-label a in $W_1, \dots, W_{i-1}, W_{i+1}, \dots, W_{n-1}$ or W_n and immediate subtrees $v_1, \dots, v_k \in A$ (possibly $k = 0$),
2. v is a tree with root-label a in $\{u \in W_i(\mathcal{T}(W)) : s' \not\le_{W_i(\mathcal{T}(W))} u\}$ and immediate subtrees $v_1, \dots, v_k \in A$ (possibly $k = 0$),
3. v is a tree with root-label a in $\{u \in W_i(\mathcal{T}(W)) : s' \le_{W_i(\mathcal{T}(W))} u\}$ and immediate subtrees $v_1, \dots, v_k \in A$ and $k < m$ (possibly $k = 0$),
4. v is a tree with root-label a in $\{u \in W_i(\mathcal{T}(W)) : s' \le_{W_i(\mathcal{T}(W))} u\}$ and immediate subtrees $v_1, \dots, v_k \in A$ with $k \ge m$. Furthermore, $v_1, \dots, v_k \in \{u \in W'(\mathcal{T}(W)) : s_1 \not\le_{W'(\mathcal{T}(W))} u\}$,
5. v is a tree with root-label a in $\{u \in W_i(\mathcal{T}(W)) : s' \le_{W_i(\mathcal{T}(W))} u\}$ and immediate subtrees $v_1, \dots, v_k \in A$ with $k \ge m$. Furthermore, there exists an index j_1 such that $v_1, \dots, v_{j_1-1}, v_{j_1+1}, \dots, v_k \in \{u \in W'(\mathcal{T}(W)) : s_2 \not\le_{W'(\mathcal{T}(W))} u\}$ and $v_{j_1} \in \{u \in W'(\mathcal{T}(W)) : s_1 \le_{W'(\mathcal{T}(W))} u\}$,

   ...

m+3. v is a tree with root-label a in $\{u \in W_i(\mathcal{T}(W)) : s' \le_{W_i(\mathcal{T}(W))} u\}$ and immediate subtrees $v_1, \dots, v_k \in A$ with $k \ge m$. Furthermore, there exist distinct $j_1, \dots, j_{m-1}$ such that for $l \ne j_1, \dots, j_{m-1}$, we have $v_l \in \{u \in W'(\mathcal{T}(W)) : s_m \not\le_{W'(\mathcal{T}(W))} u\}$ and for $l' = 1, \dots, m-1$, we have $v_{j_{l'}} \in \{u \in W'(\mathcal{T}(W)) : s_{l'} \le_{W'(\mathcal{T}(W))} u\}$.

Note that the case 'v is a tree consisting of one single node' (i.e. $k = 0$) is allocated among cases 1., 2. and 3. If $m = 1 + \alpha_j$, define W_s as

$$\mathbb{T}\begin{pmatrix} W_1 & \dots & L & \dots & W_{s'} & \dots & W_n \\ 1+\alpha_1 & \dots & m & \dots & 1+\alpha_i & \dots & 1+\alpha_n \end{pmatrix}$$

with

$$L = W_j + W_i + W_i \times (M^\diamond(W_{s_1}) + M^\diamond(W_{s_2}) + \dots + M^\diamond(W_{s_m}))$$

and if $m \neq 1 + \alpha_j$ for every j, define W_s as

$$\mathbb{T}\begin{pmatrix} W_1 & \dots & L' & \dots & W_{s'} & \dots & W_n \\ 1+\alpha_1 & \dots & m & \dots & 1+\alpha_i & \dots & 1+\alpha_n \end{pmatrix}$$

with

$$L' = W_i + W_i \times (M^\diamond(W_{s_1}) + M^\diamond(W_{s_2}) + \dots + M^\diamond(W_{s_m})).$$

Let $e_s : A \to W_s(\mathcal{T}(W))$ be the natural resulting map, namely:

1. If case 1. holds, define $e_s(v)$ as the tree with the same root-label as v and with immediate subtrees $e_s(v_1), \dots, e_s(v_k)$.
2. If case 2. holds, define $e_s(v)$ as the tree with root-label $e_{s'}(a)$ and immediate subtrees $e_s(v_1), \dots, e_s(v_k)$.
3. If case 3. holds, define $e_s(v)$ as the tree with the same root-label as v and with immediate subtrees $e_s(v_1), \dots, e_s(v_k)$.
4. If case 4. holds, define $e_s(v)$ as the tree consisting of a single node with label $(a, [e_{s_1}(v_1), \dots, e_{s_1}(v_k)])$.
5. If case 5. holds, define $e_s(v)$ as the tree with root-label

$$(a, [e_{s_2}(v_1), \dots, e_{s_2}(v_{j_1-1}), e_{s_2}(v_{j_1+1}), \dots, e_{s_2}(v_k)])$$

and immediate subtree $e_s(v_{j_1})$.

...

m+3. If case $m+3$. holds, define $e_s(v)$ as the tree with root-label (a, m) and with immediate subtrees $e_s(v_{j_1}), \dots, e_s(v_{j_m})$, where m is the multiset consisting of the elements $e_{s_m}(v_l)$ with $l \neq j_1, \dots, j_{m-1}$.

$e_s(v) \leq e_s(v')$ implies $v \leq v'$ can be proved in a straightforward way using induction on $ht(v) + ht(v')$: if v and v' are in two different cases, then this follows easily from the induction hypothesis. For example, if v is in case 5. and v' in case 3., then the root-labels of $e_s(v)$ and $e_s(v')$ are incomparable, hence $e_s(v) \leq e_s(v')$ yields $e_s(v) \leq e_s(v'_i)$ for a certain i. The induction hypothesis implies $v \leq v'_i \leq v'$. If for both v and v' the same case holds, then one also needs the properties of $e_{s'}$ and e_{s_i}. For example, assume that v and v' are in case 5. Then $e_s(v) \leq e_s(v')$ yields either $e_s(v) \leq e_s(v'_{j'_1})$ or

$$\begin{aligned}&(a, [e_{s_2}(v_1), \dots, e_{s_2}(v_{j_1-1}), e_{s_2}(v_{j_1+1}), \dots, e_{s_2}(v_k)]) \\ \leq &(a', [e_{s_2}(v'_1), \dots, e_{s_2}(v'_{j'_1-1}), e_{s_2}(v'_{j'_1+1}), \dots, e_{s_2}(v'_{k'})])\end{aligned}$$

and $e_s(v_{j_1}) \leq e_s(v'_{j'_1})$. If $e_s(v) \leq e_s(v'_{j'_1})$, then the induction hypothesis yields $v \leq v'_{j'_1} \leq v'$. In the other case, the induction hypothesis and the properties of e_{s_2} imply

$$\begin{aligned}&(a, [v_1, \dots, v_{j_1-1}, v_{j_1+1}, \dots, v_k]) \\ \leq&(a', [v'_1, \dots, v'_{j'_1-1}, v'_{j'_1+1}, \dots, v'_{k'}])\end{aligned}$$

and $v_{j_1} \leq v'_{j'_1}$, hence $v \leq v'$.

To prove property *a.*, one has to consider every case separately. For example in case 5.,

$$\begin{aligned}Leaves(e_s(v)) =& \bigcup_{i \neq j_1} Leaves(e_{s_2}(v_i)) \cup Leaves(e_s(v_{j_1})) \cup Leaves(a) \\ \subseteq& \bigcup_{i \neq j_1} Leaves(v_i) \cup Leaves(v_{j_1}) \cup Leaves(a) \\ =& Leaves(v).\end{aligned}$$

$f(W_s) < f(W')$ follows from $f(W_{s'}) < f(W_i)$ and $f(W_{s'}), f(L), f(L') < f(W')$. These inequalities follow from $f(W_i), f(M^\diamond(W_{s_j})) < f(W')$, where the last inequality is implied by $f(W_{s_j}) < f(W')$. Property *c.* follows from

$$\begin{aligned}k_0(f(W_s)) =& \max\{k_0(f(W_1)), \dots, k_0(f(W_n)), k_0(f(W_{s'})), \\ & k_0(f(W_{s_1})), \dots, k_0(f(W_{s_m}))\} \\ <& \vartheta_0(f(W)).\end{aligned}$$

This finishes the proof of the lemma. □

Theorem 5.15. *Let $W \in \mathcal{W}(Y_1, \dots, Y_k)$ and $Y_1, \dots, Y_k$ be countable* wpo*'s. Then $\mathcal{T}(W)$ is a* wpo *and*

$$o(\mathcal{T}(W)) \leq \theta_0(f(W)).$$

Proof. We prove this by induction on $\theta_0(f(W))$. Pick an arbitrary $W \in \mathcal{W}(Y_1, \dots, Y_k)$ and $t \in \mathcal{T}(W)$. We want to prove that $L_{\mathcal{T}(W)}(t)$ is a wpo and $l_{\mathcal{T}(W)}(t) < \theta_0(f(W))$ and we do this by subsidiary induction on the complexity $C(t)$. If $t = \circ$, then this is trivial. Suppose not, then $t = \circ[w(t_1, \dots, t_n)]$, with $s = w(t_1, \dots, t_n) \in W(\mathcal{T}(W))$. We know $C(t_i) < C(t)$, hence the sub-induction hypothesis yields $L_{\mathcal{T}(W)}(t_i)$ are wpo's and $l_{\mathcal{T}(W)}(t_i) < \theta_0(f(W))$ for every i.

Lemma 5.14 yields the existence of a symbol

$$W_s \in \mathcal{W}(\emptyset, \{0\}, Y_i, L_{Y_i}(y), L(t_j), \mathbb{T}^{leaf}(L(t_j)))$$

and a quasi-embedding e_s from $\{u \in W(\mathcal{T}(W)) : s \not\leq u\}$ to $W_s(\mathcal{T}(W))$ with the properties

a. $Leaves(e_s(u)) \subseteq Leaves(u)$ for all $u \in \{u \in W(\mathcal{T}(W)) : s \not\leq u\}$,

b. $f(W_s) < f(W)$,

c. $k_0(f(W_s)) < \vartheta_0(f(W))$.

Properties *b.* and *c.* yield $\vartheta_0(f(W_s)) < \vartheta_0(f(W))$, hence the main induction hypothesis yields that $\mathcal{T}(W_s)$ is a wpo and $o(\mathcal{T}(W_s)) \leq \vartheta_0(f(W_s)) < \vartheta_0(f(W))$. If we can find a quasi-embedding e from $L_{\mathcal{T}(W)}(t)$ to $\mathcal{T}(W_s)$, we can finish the proof by Lemma 1.56.

We construct e by induction on the complexity. Define $e(\circ)$ as $\circ$. Take an arbitrary element $v = \circ[u(v_1, \dots, v_m)]$ in $L_{\mathcal{T}(W)}(t)$ and assume that $e(v_i)$ are already defined. Then $v_i \in L_{\mathcal{T}(W)}(t)$ for every i and $s = w(t_1, \dots, t_n) \not\leq_{W(\mathcal{T}(W))} u(v_1, \dots, v_m)$. We know that $e_s(u(v_1, \dots, v_n)) \in W_s(\mathcal{T}(W))$, hence using Lemma 1.94, there exists a term $\overline{u} \in T_{W_s}$ and elements $\overline{v}_1, \dots, \overline{v}_l$ in $\mathcal{T}(W)$ such that $e_s(u(v_1, \dots, v_m)) = \overline{u}(\overline{v}_1, \dots, \overline{v}_l)$. Actually,

$$\{\overline{v}_1, \dots, \overline{v}_l\} = Leaves(e_s(u)) \subseteq Leaves(u) = \{v_1, \dots, v_m\},$$

hence we can assume that $e(\overline{v}_i)$ are defined as elements of $\mathcal{T}(W_s)$ because $e(v_i)$ are already defined. Define

$$e(v) := \circ[\overline{u}(e(\overline{v}_1), \dots, e(\overline{v}_l))].$$

This is a well-defined element of $\mathcal{T}(W_s)$. We prove that e is a quasi-embedding and show by induction on $C(v) + C(v')$ that $e(v) \leq e(v')$ yields $v \leq v'$. If v or v' are $\circ$, then this is trivial. Assume $v = \circ[u(v_1, \dots, v_m)]$, $v' = \circ[u'(v'_1, \dots, v'_{m'})]$, $e_s(u(v_1, \dots, v_m)) = \overline{u}(\overline{v}_1, \dots, \overline{v}_l)$ and $e_s(u'(v'_1, \dots, v'_{m'})) = \overline{u'}(\overline{v'}_1, \dots, \overline{v'}_{l'})$. Then

$$e(v) = \circ[\overline{u}(e(\overline{v}_1), \dots, e(\overline{v}_l))] \leq \circ[\overline{u'}(e(\overline{v'}_1), \dots, e(\overline{v'}_{l'}))] = e(v')$$

yields $e(v) \leq e(\overline{v'}_i)$ for a certain i or $\overline{u}(e(\overline{v}_1), \dots, e(\overline{v}_l)) \leq \overline{u'}(e(\overline{v'}_1), \dots, e(\overline{v'}_{l'}))$. The induction hypothesis in the former case yields $v \leq \overline{v'}_i = v'_j$ for a certain j, hence $v \leq v'$. In the latter case, the induction hypothesis yields that e restricted to $\{\overline{v}_1, \dots, \overline{v}_l, \overline{v'}_1, \dots, \overline{v'}_{l'}\}$ is a quasi-embedding. The Lifting Lemma 1.99 then yields $\overline{u}(\overline{v}_1, \dots, \overline{v}_l) \leq \overline{u'}(\overline{v'}_1, \dots, \overline{v'}_{l'})$, hence $e_s(u(v_1, \dots, v_m)) \leq e_s(u'(v'_1, \dots, v'_{m'}))$. Because e_s is a quasi-embedding, this implies $u(v_1, \dots, v_m) \leq u'(v'_1, \dots, v'_{m'})$, so $v \leq v'$. □

Lemma 5.16. *If* $W = M^\diamond(\mathbb{T}^{leaf}(\cdot))\backslash\{[]\}$*, then* $\mathcal{T}(W) \cong \mathbb{T}'^{wgap}_2[0]$.

Proof. The proof is similar as the proofs of Theorem 1.109 and Lemma 4.3. We define an order-isomorphism g from $\mathcal{T}(W)$ to $\mathbb{T}'^{wgap}_2[0]$, where $W = M^\diamond(\mathbb{T}^{leaf}(\cdot))\backslash\{[]\}$. Assume $t \in \mathcal{T}(W)$. If $t = \circ$, define $g(t)$ as the tree consisting of a single node with label 0. Let

$$t = \circ[T_1(t^1_1, \ldots, t^1_{m_1}), \ldots, T_n(t^n_1, \ldots, t^n_{m_n})],$$

where $T_i(t^i_1, \ldots, t^i_{m_i})$ are trees in $\mathbb{T}^{leaf}(\mathcal{T}(W))$ such that the leaf-labels are among $\{t^i_1, \ldots, t^i_{m_i}\}$. Assume that $g(t^i_j)$ are already defined. Define $g(t)$ are the tree with root-label 0 and immediate subtrees $f(T_1), \ldots, f(T_n)$, where we define a mapping f from $\mathbb{T}^{leaf}(\{t^1_1, \ldots, t^n_{m_n}\})$ to $\mathbb{T}'^{wgap}_2[0]$ using g as follows:

1. If T is a tree in $\mathbb{T}^{leaf}(\{t^1_1, \ldots, t^n_{m_n}\})$ of height zero with leaf-label t^i_j, define $f(T)$ as $g(t^i_j)$.
2. If T is a tree in $\mathbb{T}^{leaf}(\{t^1_1, \ldots, t^n_{m_n}\})$ with immediate subtrees $T_1, \ldots, T_k$, define $f(T)$ as the tree with immediate subtrees $f(T_1), \ldots, f(T_k)$ and root-label 1.

Claim 1: g is surjective.
Take an arbitrary $T \in \mathbb{T}'^{wgap}_2[0]$. If T is the tree consisting of one node with label 0, then trivially T can be reached by g. Assume that the height of T is strictly bigger than 0. T is a tree with root-label 0 and immediate subtrees $T_1, \ldots, T_n$ $(n \geq 1)$. Define T^*_i as the tree which results when we convert each node t of T_i that has label 0 and that every node below t has label 1, to a leaf node of T^*_i with label $(T_i)_t$ (see Definition 1.70). Delete all the labels of the internal nodes (which are necessarily equal to 1) of T^*_i. So $T^s_i = T'_i(T^i_1, \ldots, T^i_{m_i}) \in \mathbb{T}^{leaf}(\mathbb{T}'^{wgap}_2[0])$, where $T^i_j \in \mathbb{T}'^{wgap}_2[0]$. The height of T^i_j is strictly smaller than $ht(T)$, so by an induction argument we can assume that there exist $t^i_j \in \mathcal{T}(W)$ such that $g(t^i_j) = T^i_j$. Note that T'_i is officially a term in $T_{\mathbb{T}^{leaf}(\cdot)}$. Define t as

$$\circ[T'_1(t^1_1, \ldots, t^1_{m_1}), \ldots, T'_n(t^n_1, \ldots, t^n_{m_n})].$$

One can prove that $g(t) = T$.

Claim 2: g is order-isomorphic.
$t \leq t' \Rightarrow g(t) \leq g(t')$ can be proved by an induction argument on $C(t)+C(t')$. Additionally, the direction $T = g(t) \leq g(t') = T' \Rightarrow t \leq t'$ can be obtained by induction on $ht(T) + ht(T')$ and using the interpretations of T and T' as in Claim 1. Both directions are like in Lemma 4.3. □

Corollary 5.17. $o(\mathbb{T}_2^{wgap}) \leq \vartheta_0((\vartheta_1(\Omega_2^\omega))^\omega) = \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$.

Proof. Lemmas 5.2 and 5.16 yield

$$o(\mathbb{T}_2^{wgap}) = o(\mathbb{T}'^{wgap}_2[0]) = o(\mathcal{T}(M^\diamond(\mathbb{T}^{leaf}(Id))\backslash\{[]\})).$$

Hence, Theorem 5.15 implies

$$\begin{aligned} o(\mathbb{T}_2^{wgap}) \leq & \vartheta_0(f(M^\diamond(\mathbb{T}^{leaf}(Id))\backslash\{[]\})) \\ =& \vartheta_0(\omega^{\omega^{f(\mathbb{T}^{leaf}(Id))+1}}) \\ =& \vartheta_0(\omega^{\omega^{\vartheta_1(\Omega_2^\omega)+1}}) \\ =& \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega). \end{aligned}$$

□

Like before, Theorem 5.15 allows a constructive well-partial-orderedness proof of $\mathbb{T}_2^{wgap}$ by reifications (for more information see [66] and [73]).

5.3 Lower bound for $o(\mathbb{T}_2^{wgap})$

In this section, we prove that $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ is a lower bound for $o(\mathbb{T}_2^{wgap})$. However, it is easier if the collapsing functions ϑ_i^P are used (mentioned in Lemma 1.20) instead of $\vartheta_i = \vartheta_i^E$. This does not make a difference because Lemma 1.20 yields $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega) = \vartheta_0^P(\vartheta_1^P(\Omega_2^\omega)^\omega)$. For notational ease, we write ϑ_i instead of ϑ_i^P. Note that we use the same K_1, k_1, K_0 and k_0 as before.

Theorem 5.18. $o(\mathbb{T}_2^{wgap}) \geq \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$.

Proof. We define a quasi-embedding g from $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ to $\mathbb{T}_2^{wgap}[0]$ as follows. $g(0)$ is the single node with label 0. If $\alpha = \alpha_1 \oplus \alpha_2 > \alpha_1, \alpha_2 > 0$ with $\alpha_1 \in P$, define $g(0)$ as the tree with immediate subtrees $g(\alpha_1)$ and $g(\alpha_2)$ and root-label 0. If $\alpha \in P$, then $\alpha = \vartheta_0\beta$. $\alpha < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ yields $\beta < \vartheta_1(\Omega_2^\omega)^\omega$ and $k_0\beta < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ or $\alpha \leq k_0(\vartheta_1(\Omega_2^\omega)^\omega) = k_0\left(\omega^{\omega^{k_1(\Omega_2^\omega)+1}}\right) = 0$. Hence

$$\beta = (\vartheta_1(\Omega_2^\omega))^n \cdot \beta_n + \cdots + \vartheta_1(\Omega_2^\omega) \cdot \beta_1 + \beta_0,$$

with $n < \omega$, $\beta_i < \vartheta_1(\Omega_2^\omega)$ and $\beta_n > 0$. If $\beta_i =_{NF} \omega^{\beta_i^1} + \cdots + \omega^{\beta_i^{n_i}}$, then

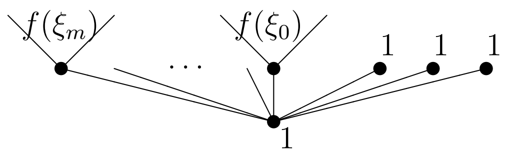
$$\begin{aligned} k_0\beta &= k_0\left(\vartheta_1(\Omega_2^\omega)^n\beta_n + \cdots + \vartheta_1(\Omega_2^\omega)\beta_1 + \beta_0\right) \\ &= \max_i k_0\left(\omega^{\vartheta_1(\Omega_2^\omega)\cdot i}\cdot\beta_i\right) \\ &= \max_{i,j} k_0\left(\omega^{\vartheta_1(\Omega_2^\omega)\cdot i+\beta_i^j}\right) \\ &= \max\{k_0(\vartheta_1(\Omega_2^\omega)), \max_{i,j} k_0(\beta_i^j)\} \\ &= \max_i k_0(\beta_i). \end{aligned}$$

Therefore, $k_0\beta < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ is equivalent with $k_0\beta_i < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ for every i. Before we define $g(\alpha)$, we introduce an intermediate function f.

For every $\beta < \vartheta_1(\Omega_2^\omega)$ with $k_0(\beta) < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ and the assumption that $g(\delta)$ is already defined for every $\delta \leq k_0(\beta)$, we define a tree $f(\beta)$ by recursion on β.
If $\beta = 0$, define $f(\beta)$ as the single node with label 1. If $\beta = \beta_1 \oplus \beta_2 > \beta_1, \beta_2 > 0$ with $\beta_1 \in P$, define $f(\beta)$ as the tree with immediate subtrees $f(\beta_1)$ and $f(\beta_2)$ and root-label 1. Assume $\beta = \omega^{\beta_1} > \beta_1$. Define $f(\beta)$ as the tree with the three immediate subtrees $f(\beta_1)$, $f(0)$ and $f(0)$ and root-label 1. Assume from now on that $\beta \in E$.[2] If $\beta < \Omega_1$, define $f(\beta)$ as $g(\beta)$. Note that $g(\beta)$ is well-defined because $\beta = k_0\beta$ in this case. Assume $\beta \geq \Omega_1$. Then $\beta = \vartheta_1\gamma$ with $\gamma < \Omega_2^\omega$ and $k_1(\gamma) < \beta < \vartheta_1(\Omega_2^\omega)$ or $\beta \leq k_1(\Omega_2^\omega) = \omega$. Note that the latter case is impossible. $\gamma_1 < \Omega_2^\omega$ yields

$$\gamma = \Omega_2^m \cdot \gamma_m + \cdots + \Omega_2 \cdot \gamma_1 + \gamma_0,$$

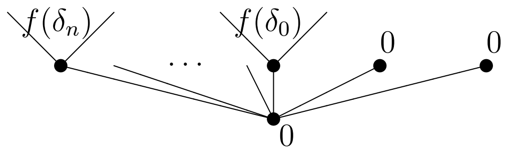
with $\gamma_m > 0$, $m < \omega$ and $\gamma_i < \beta < \vartheta_1(\Omega_2^\omega)$. Now, $k_0(\gamma_m \oplus \cdots \oplus \gamma_i) = \max_i k_0(\gamma_i) \leq k_0(\beta) < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$, so we can assume that $f(\gamma_m \oplus \cdots \oplus \gamma_i)$ is already defined for every i. Define $f(\beta)$ then as the tree



where $\xi_i := \gamma_m \oplus \cdots \oplus \gamma_i$.

[2]We note that although we use the ϑ_i-functions defined over P, we define only for $\beta \in E$ and not for $\beta \in P$, $f(\beta)$ as $g(\beta)$. This is because of the specific definitions of k_0 and k_1.

Now, we can define $g(\alpha)$: it is the tree



where $\delta_i := \beta_n \oplus \cdots \oplus \beta_i$. Note that $\delta_i < \vartheta_1(\Omega_2^\omega)$, $k_0(\delta_i) \leq k_0(\beta) < \vartheta_0\beta = \alpha < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ and that we can assume that $g(\delta)$ is defined for every $\delta \leq k_0(\delta_i)$, hence $g(\alpha)$ is well-defined. Note that *every* $g(\alpha)$ is a tree with root-label zero.

Now we prove that g is a quasi-embedding: we prove by induction on $\alpha \oplus \alpha'$ that $g(\alpha) \leq g(\alpha')$ yields $\alpha \leq \alpha'$. If α or α' are 0, then this is trivial. So we can assume that both α and α' are strictly larger than zero. There are four cases

1. $\alpha = \alpha_1 \oplus \alpha_2$ and $\alpha' = \alpha_1' \oplus \alpha_2'$. If the root of $g(\alpha)$ is mapped into an immediate subtree of $g(\alpha')$, then $g(\alpha) \leq g(\alpha_i')$, so $\alpha \leq \alpha_i' < \alpha'$. If the root of $g(\alpha)$ is mapped onto the root of $g(\alpha')$ then $[g(\alpha_1), g(\alpha_2)] \leq^\diamond [g(\alpha_1'), g(\alpha_2')]$. The induction hypothesis yields $\alpha = \alpha_1 \oplus \alpha_2 \leq \alpha_1' \oplus \alpha_2' = \alpha'$.

2. $\alpha = \vartheta_0\beta$ and $\alpha' = \alpha_1' \oplus \alpha_2'$. If the root of $g(\alpha)$ is mapped into an immediate subtree of $g(\alpha')$, then $g(\alpha) \leq g(\alpha_i')$, so $\alpha \leq \alpha_i' < \alpha'$. It is impossible that the root of $g(\alpha)$ is mapped onto the root of $g(\alpha')$.

3. $\alpha = \alpha_1 \oplus \alpha_2$ and $\alpha' = \vartheta_0\beta'$. $g(\alpha) \leq g(\alpha')$ yields $g(\alpha_i) < g(\alpha')$, so $\alpha_i < \alpha'$. Hence $\alpha = \alpha_1 \oplus \alpha_2 < \alpha'$.

4. $\alpha = \vartheta_0(\vartheta_1(\Omega_2^\omega)^n\beta_n + \cdots + \vartheta_1(\Omega_2^\omega)\beta_1 + \beta_0)$ and $\alpha' = \vartheta_0(\vartheta_1(\Omega_2^\omega)^k\beta_k' + \cdots + \vartheta_1(\Omega_2^\omega)\beta_1' + \beta_0')$. Let $\delta_i = \beta_n \oplus \cdots \oplus \beta_i$ and $\delta_i' := \beta_k' \oplus \cdots \oplus \beta_i'$. First we prove two claims

   1. **If T is a subtree of $f(\beta)$ (possibly $T = f(\beta)$) with root-label 0, then $T = g(\delta)$ for a certain $\delta \leq k_0(\beta)$.**

   This can be proved in a straightforward way from the construction of f.

   2. **For every β and β' with $k_0(\beta) < \alpha$ and $k_0(\beta) < \alpha'$, we have that $f(\beta) \leq^{sgap} f(\beta')$ yields $\beta \leq \beta'$.**

   We prove this claim by induction on $\beta \oplus \beta'$. If $\beta = 0$, this is trivial. If $\beta' = 0$, then $f(\beta) \leq^{sgap} f(\beta')$ is not possible unless β is also zero. Assume $\beta, \beta' > 0$. There are a few cases left.

Case 1: $\beta = \beta_1 \oplus \beta_2$.

(a) The case $\beta' = \beta'_1 \oplus \beta'_2$ can be treated in a similar way as we did for the function g.

(b) If $\beta' = \omega^{\beta'_1}$ or ($\beta' \in E$ and $\beta' \geq \Omega_1$), then $f(\beta) \leq^{sgap} f(\beta')$ yields $f(\beta_i) <^{sgap} f(\beta')$, hence $\beta = \beta_1 \oplus \beta_2 < \beta'$.

(c) If $\beta' \in E \cap \Omega_1$, then $f(\beta) \leq^{sgap} f(\beta') = g(\beta')$ is impossible because $g(\beta')$ is a tree with root-label zero.

Case 2: $\beta = \omega^{\beta_1}$.

(a) The case $\beta' = \beta'_1 \oplus \beta'_2$ yields $f(\beta) \leq^{sgap} f(\beta'_i)$, hence $\beta \leq \beta'_i < \beta'$.

(b) If $\beta' = \omega^{\beta'_1}$, then $f(\beta) \leq^{sgap} f(\beta')$ yields $f(\beta_1) \leq^{sgap} f(\beta'_1)$, hence $\beta \leq \beta'$.

(c) If $\beta' \in E$ and $\beta' \geq \Omega_1$, then $f(\beta) \leq^{sgap} f(\beta')$ yields $f(\beta_1) <^{sgap} f(\beta')$, hence $\beta < \beta'$.

(d) If $\beta' \in E \cap \Omega_1$, then $f(\beta) \leq^{sgap} f(\beta') = g(\beta')$ is impossible because $g(\beta')$ is a tree with root-label zero.

Case 3: $\beta \in E \cap \Omega_1$.
Then $f(\beta) = g(\beta)$ is embeddable in a subtree T of $f(\beta')$ with root-label 0. Using the first claim, T is equal to $g(\delta')$ for a $\delta' \leq k_0(\beta')$. Hence, $g(\beta) \leq g(\delta')$. Note that $\beta \oplus \delta' \leq k_0(\beta) \oplus k_0(\beta') < \alpha \oplus \alpha'$, hence the induction hypothesis on g yields $\beta \leq \delta' \leq k_0(\beta') \leq \beta'$.

Case 4: $\beta = \vartheta_1 \gamma$. Then $\gamma = \Omega_2^m \cdot \gamma_m + \cdots + \Omega_2 \cdot \gamma_1 + \gamma_0$ with $\gamma_i < \beta < \vartheta_1(\Omega_2^\omega)$ and $k_0(\gamma_i) < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$. Let $\xi_i = \gamma_m \oplus \cdots \oplus \gamma_i$.

(a) If $\beta' = \beta'_1 \oplus \beta'_2$ or $\beta' = \omega^{\beta'_1}$, then $f(\beta) \leq^{sgap} f(\beta')$ yields $f(\beta) \leq^{sgap} f(\beta'_i)$, hence $\beta \leq \beta'_i < \beta'$.

(b) If $\beta' \in E \cap \Omega_1$, then $f(\beta) \leq^{sgap} f(\beta') = g(\beta')$ is impossible because $g(\beta')$ is a tree with root-label 0 and $f(\beta)$ is a tree with root-label 1.

(c) If $\beta' \in \vartheta_1(\gamma')$, then $\gamma' = \Omega_2^l \cdot \gamma'_l + \cdots + \Omega_2 \cdot \gamma'_1 + \gamma'_0$ with $\gamma'_i < \beta' < \vartheta_1(\Omega_2^\omega)$ and $k_0(\gamma'_i) < \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$. Let $\xi'_i = \gamma'_l \oplus \cdots \oplus \gamma'_i$. $f(\beta) \leq^{sgap} f(\beta')$ yields $f(\beta) \leq^{sgap} f(\xi'_i)$ for a certain i or

$$[f(\xi_m), \ldots, f(\xi_0)](\leq^{sgap})^\diamond[f(\xi'_l), \ldots, f(\xi'_0)].$$

In the former case, $\beta \leq \xi'_i < \beta'$. Assume that we are in the latter case. The induction hypothesis then yields

$$[\xi_m, \dots, \xi_0](\leq^{sgap})^\diamond[\xi'_l, \dots, \xi'_0],$$

meaning

$$[\gamma_m, \dots, \gamma_m \oplus \dots \oplus \gamma_0] \leq^\diamond [\gamma'_l, \dots, \gamma'_l \oplus \dots \oplus \gamma'_0].$$

If $m < l$, then $\gamma < \gamma'$. So assume $m = l$. Similarly as in Theorem 2.6, this yields $\gamma \leq \gamma'$. Furthermore, $k_1(\gamma) = \max_i \gamma_i \leq \max_i \xi_i \leq \max_i \xi'_i < \vartheta_1(\gamma')$. We conclude that $\beta \leq \beta'$.

This finishes the proof of the second claim.

So we were in the case

$$\alpha = \vartheta_0(\vartheta_1(\Omega_2^\omega)^n \beta_n + \dots + \vartheta_1(\Omega_2^\omega)\beta_1 + \beta_0)$$

and

$$\alpha' = \vartheta_0(\vartheta_1(\Omega_2^\omega)^k \beta'_k + \dots + \vartheta_1(\Omega_2^\omega)\beta'_1 + \beta'_0).$$

Let $\delta_i = \beta_n \oplus \dots \oplus \beta_i$ and $\delta'_i = \beta'_k \oplus \dots \oplus \beta'_i$. $g(\alpha) \leq g(\alpha')$ yields either that $g(\alpha)$ is mapped into a subtree of $f(\delta'_i)$ for a certain i or that the root of $g(\alpha)$ is mapped onto the root of $g(\alpha')$. If we are in the former case, then the first claim yields $g(\alpha) \leq g(\delta')$ with $\delta' \leq k_0(\delta'_i)$. Hence, $\alpha \leq \delta' \leq k_0(\delta'_i) < \alpha'$. Assume that we are in the latter case. Then $g(\alpha) \leq g(\alpha')$ implies

$$[f(\delta_n), \dots, f(\delta_0)](\leq^{sgap})^\diamond[f(\delta'_k), \dots, f(\delta'_0)].$$

The second claim yields

$$[\delta_n, \dots, \delta_0] \leq^\diamond [\delta'_k, \dots, \delta'_0].$$

Hence,

$$\begin{aligned} \beta &= \vartheta_1(\Omega_2^\omega)^n \beta_n + \dots + \vartheta_1(\Omega_2^\omega)\beta_1 + \beta_0 \\ &\leq \vartheta_1(\Omega_2^\omega)^k \beta'_k + \dots + \vartheta_1(\Omega_2^\omega)\beta'_1 + \beta'_0 = \beta'. \end{aligned}$$

Furthermore,

$$\begin{aligned} k_0(\beta) &= \max_i k_0(\beta_i) = \max_i k_0(\delta_i) \\ &\leq \max_i k_0(\delta'_i) = \max_i k_0(\beta'_i) = k_0(\beta') \\ &< \vartheta_0\beta'. \end{aligned}$$

This yields $\alpha = \vartheta_0\beta \leq \vartheta_0\beta' = \alpha'$.

□

Corollary 5.19. *1.* $o(\mathbb{T}_2^{wgap}) = \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$,

2. $o(\mathbb{T}_2^{sgap}) = \vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$.

Proof. The first assertion follows from Corollary 5.17 and Theorem 5.18.

By Lemma 5.6, we know that $o(\mathbb{T}_2^{sgap}) = o(\mathbb{T}^{leaf}(\mathbb{T}_2^{wgap}))$. Using Lemma 5.4, we obtain $o(\mathbb{T}_2^{sgap}) \leq \vartheta(\Omega^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$. By comparing the ϑ and ϑ_0 functions, we obtain $o(\mathbb{T}_2^{sgap}) \leq \vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$.

We show that there exists a quasi-embedding from $\vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$ to $\mathbb{T}^{leaf}(\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$, from which $o(\mathbb{T}_2^{sgap}) \geq \vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega))$ follows. We abbreviate the ordinal $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$ by α. We recall that we use the ϑ_i-functions defined on P.

Define for every $\delta < \vartheta_0(\Omega_1^\omega + \alpha)$ an element $f(\delta) \in \mathbb{T}^{leaf}(\alpha)$ as follows.

If $\delta = 0$, define $f(\delta)$ as the tree consisting of a single node with label 0. If $\delta = \delta_1 \oplus \delta_2 > \delta_1, \delta_2 > 0$ with $\delta_1 \in P$, define $f(\delta)$ as the tree with the two immediate subtrees $f(\delta_1)$ and $f(\delta_2)$. Assume $\delta \in P$. If $\delta < \alpha$, define $f(\delta)$ as the tree consisting of a single node with label δ. If $\delta = \alpha$, define $f(\delta)$ as tree with three immediate subtrees $f(0)$. Assume $\delta > \alpha$. Because $k_0(\alpha) \leq \alpha < \delta < \vartheta_0(\alpha)$, we know that $\delta \in Im(\vartheta_0)$ (like in Corollary 1.7). Then $\delta = \vartheta_0(\xi)$ with either $\xi = \Omega_1^n \cdot \delta_n + \cdots + \Omega_1 \cdot \delta_1 + \delta_0$ with $K_0(\delta_i) < \delta$ and $\delta_n > 0$, or $\xi = \Omega_1^\omega + \delta'$ with $\delta' < \alpha$, or $\xi > \Omega_1^\omega + \alpha$. First, we prove that the two last cases cannot occur.

If $\xi = \Omega_1^\omega + \delta'$ with $\delta' < \alpha$, then $\xi < \Omega_1^\omega + \alpha = \Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega) < \vartheta_1(\Omega_2^\omega)^\omega$. Furthermore, $k_0(\xi) = k_0(\Omega_1^\omega + \delta') = k_0(\delta') \leq \delta' < \alpha = \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$, hence $\delta = \vartheta_0(\xi) < \alpha$, a contradiction.

If $\xi > \Omega_1^\omega + \alpha$, then $\delta = \vartheta_0(\xi) < \vartheta_0(\Omega_1^\omega + \alpha)$ yields $\delta \leq k_0(\Omega_1^\omega + \alpha) = k_0(\alpha) \leq \alpha$, again a contradiction.

In the other case, we have $\delta = \vartheta_0(\Omega_1^n \cdot \delta_n + \cdots + \Omega_1 \cdot \delta_1 + \delta_0)$ with $K_0(\delta_i) < \delta$ and $\delta_n > 0$.

Actually, we know that $\delta_i < \delta$ for every i: if $n > 0$, then δ is an epsilon number, so $K_0(\delta_i) < \delta$ yields $\delta_i < \delta$. If $n = 0$, then $\delta = \vartheta_0(\delta_0)$ is the least element of P that is strictly bigger than $k_0(\delta_0)$. If $\delta_0 \in E$, then $k_0(\delta_0) = \delta_0$, hence $\delta_0 < \vartheta_0(\delta_0) = \delta$. If $\delta_0 \notin E$, then $\delta = \vartheta_0(\delta_0) \geq \omega^{\delta_0} > \delta_0$. The first inequality can be proved by induction on $\delta_0 < \Omega_1$.

Therefore, we know that $f(\delta_n \oplus \cdots \oplus \delta_i \oplus 1)$ is well-defined for every i. Define $f(\delta)$ as the tree with $n+3$ immediate subtrees: $f(\delta_n \oplus 1), \ldots, f(\delta_n \oplus \cdots \oplus \delta_0 \oplus 1)$ and two times the tree $f(0)$.

One can prove in a similar way as in other theorems of this dissertation that f is a quasi-embedding. □

We believe that we can also prove

$$\vartheta(\Omega^\omega + (-1 + o(X))) \leq o(\mathbb{T}^{leaf}(X)),$$

for every countable $\mathbb{wpo}$ X.

We can redo these entire arguments in the previous two sections to prove that for the structured version, the maximal order type of $\mathbb{T}_2^{s,wgap}$ is $\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)$. This is because $\mathcal{T}(W) \cong \mathbb{T}'^{s,wgap}_2[0]$ for $W = (\mathbb{T}^{s,leaf}(\cdot))^* \backslash \{()\}$. We skip the detailed version of these proofs as they are completely similar.

5.4 Maximal order type of $\mathbb{T}_n^{gap}$

In this section, we indicate how one should prove the maximal order type of a characteristic subset of $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$. We believe that the proof will become highly technical and it is not our purpose here to be complete or entirely correct. First of all, we define the mentioned subset of $\mathbb{T}_n$.

Definition 5.20. *Define $\overline{\mathbb{T}}_n$ as the set of trees T in $\mathbb{T}_n$ such that for all nodes t in T, if t has label $m < n$, then all its successors have label at most $m+1$. Define $\overline{\mathbb{T}}_n[m]$, $\overline{\mathbb{T}'}_n$ and $\overline{\mathbb{T}'}_n[m]$ in a similar way. We add a superscript* wgap *or* sgap *if we work with the weak or strong gap-embeddability relation on the corresponding set.*

We conjecture the following

$$o(\overline{\mathbb{T}}_n^{wgap}) = \vartheta_0(\vartheta_1(\ldots(\vartheta_{n-1}(\Omega_n^\omega)^\omega)\ldots)^\omega).$$

Here, we use the $(\vartheta_i)_{i<m}$-functions from Definitions 1.17 and 1.18, where m is equal to the number of labels in $\overline{\mathbb{T}}_n$. I.e., n is equal to m. Notice that $\overline{\mathbb{T}}_2 = \mathbb{T}_2$, hence the conjecture is true for $n = 1$ and $n = 2$. We could also state conjectures about the maximal order types of $\mathbb{T}_n^{wgap}$ and $\mathbb{T}_n^{sgap}$, but then the corresponding ordinals become more involved and less smooth. That is why we restrict ourselves to the subsets $\overline{\mathbb{T}}_n$.

In order to obtain the maximal order type of $\overline{\mathbb{T}}_n^{wgap}$, we calculate the maximal order type of $\overline{\mathbb{T}'}_n^{wgap}[0]$ like we did before for the $n = 2$ case. The main key in order to obtain the last mentioned maximal order type is to redefine $\mathcal{W}(Y_0, \dots, Y_k)$ and $\mathcal{T}(W)$ for $W \in \mathcal{W}(Y_0, \dots, Y_k)$ and to reprove the main lemma, which is Lemma 5.14.

Definition 5.21. *Assume that $Y_1, \dots, Y_k$ are fixed partial orderings. Define $\mathcal{W}(Y_1, \dots, Y_k)$ and $\mathcal{W}'(Y_1, \dots, Y_k)$ as the following set of function symbols.*

- *For $1 \leq i \leq k$, let $\mathcal{C}_{Y_i} \in \mathcal{W}(Y_1, \dots, Y_k)$.*
- *For $i < n$, let $Id_i \in \mathcal{W}(Y_1, \dots, Y_k)$.*
- *If $W_{i,j} \in \mathcal{W}'(Y_1, \dots, Y_k)$ for all i and j, then $\sum_{i=0}^{n} \prod_{j=0}^{m_i} W_{i,j} \in \mathcal{W}(Y_1, \dots, Y_k)$.*
- *If $W \in \mathcal{W}(Y_1, \dots, Y_k)$, then $M^\diamond(W)$ and $M^\diamond(W) \backslash \{[]\}$ are in $\mathcal{W}(Y_1, \dots, Y_k)$.*
- *If $W \in \mathcal{W}(Y_1, \dots, Y_k)$ and $Id_j \in Sub(W)$ such that $j > 0$ is maximal among all k's for which $Id_k \in sub(W)$, then $\mathcal{T}_j(W) \in \mathcal{W}(Y_1, \dots, Y_k)$.*

Recall that it is not our purpose to be entirely correct and we know that the previous definition needs some refinements. Here the symbol $\mathcal{T}_j(W)$ (with now $j \geq 0$) has the following interpretation:

Definition 5.22. *Define $\mathcal{T}_j(W)$ as the least set such that*

1. *$\circ \in \mathcal{T}_j(W)$,*
2. *If $s = w(t_1, \dots, t_n) \in W(\mathcal{T}_j(W))$ with $t_1, \dots, t_n$ are plugged in the variables that match with Id_j, then $\circ[w(t_1, \dots, t_n)] \in \mathcal{T}_j(W)$.*

Define an ordering on $\mathcal{T}_j(W)$ in a similar way as on $\mathcal{T}(W)$.

This definition should yield that

$$\mathbb{T'}_2^{wgap}[0] \cong \mathcal{T}_0(M^\diamond(X_0 + \mathcal{T}_1(M^\diamond(X_1 + X_0))))$$

and that $\overline{\mathbb{T}'}_n^{wgap}[0]$ is isomorphic

$$\mathcal{T}_0(M^\diamond(\mathcal{T}_1(M^\diamond(\dots(\mathcal{T}_{n-1}(M^\diamond(X_{n-1} + \dots + X_1 + X_0))\dots)\dots X_1 + X_0)) + X_0)).$$

Using these definitions, it should then be possible to reprove Lemma 5.14.

Chapter 6

Independence results

6.1 Introduction

This chapter explores independence and provability results concerning the studied $\mathbb{wpo}$'s $\mathcal{T}(W)$. In general, if one has a natural well-partial-order X and a natural theory T, then

$$T \vdash \text{`}X \text{ is a well-partial-order'} \iff |T| > o(X),$$

where $|T|$ is the proof-theoretical ordinal of the theory T.

A nice and smooth technique to prove the well-partial-orderedness of a partial order X is the *minimal bad sequence argument* developed by Nash-Williams [55]. The reverse mathematical strength of this method is investigated by Marcone [51]. He showed that the general version of the minimal bad sequence argument has the strength of Π^1_1-CA_0, which formalizes Π^1_1-comprehension.

We conjecture that allowing to apply the Π^1_1-comprehension scheme up to n times yields the well-partial-orderedness of $\mathbb{T}^{wgap}_n$. But if we only allow to apply the Π^1_1-comprehension scheme at most $n-1$ times, then the well-partial-orderedness of $\mathbb{T}^{wgap}_n$ becomes unprovable. In this conjecture, we assume that we use a fixed base theory that is strong enough, e.g. ACA_0.

To explain it intuitively, assume $n = 2$. Following Section 5.4, the well-partial-orderedness of $\mathbb{T}^{wgap}_2$ follows from the well-partial-orderedness of

$$\mathcal{T}_0(M^\diamond(X_0 + \mathcal{T}_1(M^\diamond(X_1 + X_0)))).$$

Or following Section 5.2, it also implied by the well-partial-orderedness of $\mathcal{T}(M^\diamond(\mathbb{T}^{leaf}(\cdot)))$. The last partial orders consist of two nested Kruskal's tree-classes $\mathbb{T}(X)$. Kruskal's theorem '$\mathbb{T}$ is a well-partial-order' is provable, using the minimal bad sequence argument, if we apply the Π^1_1-comprehension scheme exactly once over the base theory ACA_0 (see a similar result in Theorem 6.30). If we are allowed to apply the Π^1_1-comprehension scheme exactly two times, we are allowed to use the minimal bad sequence argument two (nested) times. Hence, this yields that well-partial-orderedness of $\mathcal{T}_0(M^\diamond(X_0 + \mathcal{T}_1(M^\diamond(X_1 + X_0))))$: applying the Π^1_1-comprehension scheme a first time, reduces the two-times nested tree-class $\mathcal{T}_0(M^\diamond(X_0 + \mathcal{T}_1(M^\diamond(X_1 + X_0))))$ to a one-nested tree-class (meaning it is not nested anymore). The second application is needed to prove the well-partial-orderedness of this one-nested tree-class.

Before one should prove the above conjecture, one first has to obtain the maximal order types of $\mathbb{T}^{wgap}_n$ (see Section 5.4). Therefore, we restrict ourselves in this chapter to light-face Π^1_1-comprehension, i.e. applying the Π^1_1-comprehension scheme exactly once. First, we prove bounds on the ordinals of theories using light-face Π^1_1-comprehension (Section 6.2). In the next section 6.3, we investigate independence results and provability results. We want to note that most of the results in Section 6.2, especially the results on the upper bounds for the proof-theoretic ordinals, are due to Michael Rathjen, so we sometimes skip the proofs and refer to our joint article [81]. The well-orderedness proof (Theorem 6.17) is due to Andreas Weiermann.

6.2 Impredicative theories

As we will see later, the theories $\mathsf{RCA}_0 + (\Pi^1_1{-}CA_0)^-$ and $\mathsf{ACA}_0 + (\Pi^1_1{-}CA_0)^-$ will be used to obtain provability and unprovability results concerning the $\mathbb{wpo}$'s $\mathcal{T}(W)$. For the unprovability part, we need (upper bounds on) the proof-theoretic ordinals of both theories. It is known that the ordinal of the theory $\mathsf{ACA}_0 + (\Pi^1_1{-}CA_0)^-$ is equal to the Howard-Bachmann ordinal $\vartheta(\varepsilon_{\Omega+1})$ (see e.g. [13]), so we only have to concentrate on the theory $\mathsf{RCA}_0 + (\Pi^1_1{-}CA_0)^-$. As a side question, we were wondering what would happen with the ordinal if we replace RCA_0 by RCA_0^*. We could not determine the exact proof-theoretic ordinal of these theories, but we could do it for restricted versions. Fortunately, these restricted theories are strong enough to obtain the intended independence and provability results.

Definition 6.1. *Let $n \leq \omega$. A $\Pi^1_1(\Pi^0_n)$-formula is a formula of the form $\forall X B(X)$, where $B(X)$ is Π^0_n. B can contain set and numerical parameters. Note that a Π^0_ω formula is the same as an arithmetical formula, hence a $\Pi^1_1(\Pi^0_\omega)$ is a standard Π^1_1-formula. A $(\Pi^1_1(\Pi^0_n))^-$-formula is a formula of the form $\forall X B(X)$, where $B(X)$ is Π^0_n and $\forall X B(X)$ contains no free set parameters. It is allowed that B contains numerical parameters.*

Definition 6.2. *Let $\mathcal{F}$-CA_0 be the following well-known comprehension scheme*

$$\exists Z \forall n (n \in Z \leftrightarrow A(n)),$$

where $A(n)$ is a formula in the class $\mathcal{F}$. If $\mathcal{F} = (\Pi^1_1(\Pi^0_n))^-$, we denote $\mathcal{F}$-CA_0 by $(\Pi^1_1(\Pi^0_n)\text{-}\mathsf{CA}_0)^-$

We show the following results.

Theorem 6.3.

1. $|\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| = \vartheta(\Omega^\omega)$,
2. $|\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| \geq \varphi\omega 0 = \vartheta(\Omega \cdot \omega)$.

In order to prove these results, we use the bar-induction theories Π^1_n-BI_0.

Definition 6.4.

$$\begin{aligned} Prog(\prec, F) &:= \forall x(\forall y(y \prec x \to F(y)) \to F(x)), \\ TI(\prec, F) &:= Prog(\prec, F) \to \forall x F(x), \\ WF(\prec) &:= \forall X TI(\prec, X), \end{aligned}$$

where the formula $Prog(\prec, F)$ stands for progressiveness, $TI(\prec, F)$ for transfinite induction and $WF(\prec)$ for well-foundedness. $F(x)$ is an arbitrary $L_2(exp)$-formula if we work in RCA^*_0 or an arbitrary L_2-formula if we work in RCA_0 or ACA_0. For an element $\alpha \in OT(\vartheta)$, the formula $WF(\alpha)$ stands for '$<_{OT(\vartheta)}$ restricted to $\{\beta \in OT(\vartheta) : \beta < \alpha\}$ is well-founded'. We also denote this by $WF(<\restriction \alpha)$. One can define a similar definition for the notation system $OT'(\vartheta)$ defined in Definition 6.7.

Definition 6.5. *The theory Π^1_n-BI_0 is defined as ACA_0 augmented with the Π^1_n bar induction scheme:*

$$WF(\prec) \to TI(\prec, F),$$

where $\prec$ is a well-ordering that is Π^0_1-definable and F is a Π^1_n-formula.

The exact proof-theoretical strength of the theories Π^1_n-BI_0 are already established.

Theorem 6.6.

1. *If $n \geq 2$, the proof-theoretic ordinal of the theory Π^1_n-BI_0 is $\vartheta(\Omega_{n-1}[\omega])$,*
2. *The proof-theoretic ordinal of the theory Π^1_1-BI_0 is $\varphi\omega 0$.*

Proof. The first assertion is proved in [66].

Π^1_1-BI_0 is equivalent with Σ^1_1-$\mathsf{DC_0}$ over ACA_0: see Theorem VIII.5.12 in [77]. Note that they use a different notation for bar induction, namely TI. The theory Σ^1_1-$\mathsf{DC_0}$ has $\varphi\omega 0$ as proof-theoretic ordinal (see e.g. [15]). □

6.2.1 Lower bounds

In Section 1.2.4, we defined a primitive recursive ordinal notation system $OT(\vartheta)$ to represent the ordinal $\vartheta(\varepsilon_{\Omega+1})$. More specifically, the set $OT(\vartheta)\cap\Omega$ represents the Howard-Bachmann ordinal. Therefore, this ordinal notation system could also be used to represent to small Veblen number $\vartheta\Omega^\omega$, the expected proof-theoretic ordinal of $\mathsf{RCA}_0+(\Pi^1_1(\Pi^0_3)$-$\mathsf{CA}_0)^-$. More specifically, we could use $OT(\vartheta)\cap\vartheta(\Omega^\omega)$ to represent the ordinal $\vartheta(\Omega^\omega)$. We slightly redefine $OT(\vartheta)$ and call it $OT'(\vartheta)$, although this is not a huge difference. We define this new ordinal notation system because it will make some proofs easier. Similarly as for $OT(\vartheta)$, we have make two distinctions in case 2.

Definition 6.7. *Define inductively a set $OT'(\vartheta)$ of ordinals and a natural number $G'_\vartheta\alpha$ for $\alpha\in OT'(\vartheta)$ as follows:*

1. *$0\in OT'(\vartheta)$ and $G'_\vartheta(0):=0$,*
2. *if $\alpha=\Omega^n\beta_n+\cdots+\Omega^0\beta_0$, with $\Omega>\beta_n>0$ and $\Omega>\beta_0,\ldots,\beta_{n-1}$, then*
   (a) *if $n>0$ and $\beta_0,\ldots,\beta_n\in OT'(\vartheta)$, then $\alpha\in OT'(\vartheta)$ and $G'_\vartheta\alpha:=\max\{G'_\vartheta(\beta_0),\ldots,G'_\vartheta(\beta_n)\}+1$,*
   (b) *if $n=0$ and $\alpha=_{NF}\delta_1+\cdots+\delta_m$ with $m\geq 2$ and $\delta_1,\ldots,\delta_m\in OT'(\vartheta)$, then $\alpha\in OT'(\vartheta)$ and $G'_\vartheta\alpha:=\max\{G'_\vartheta(\delta_1),\ldots,G'_\vartheta(\delta_m)\}+1$,*
3. *if $\alpha=\vartheta\beta$ and $\beta\in OT'(\vartheta)$, then $\alpha\in OT'(\vartheta)$ and $G'_\vartheta\alpha:=G'_\vartheta\beta+1$.*

Similar as for $OT(\vartheta)$, the function G'_ϑ is well-defined.

Lemma 6.8. *If $\xi \in OT'(\vartheta)$, then $K(\xi) \subseteq OT'(\vartheta)$. Furthermore, $G'_\vartheta(k(\xi)) \leq G'_\vartheta(\xi)$ for all ξ in $OT'(\vartheta)$.*

Proof. If $\xi = 0$, then this trivially holds. If $\xi = \vartheta(\xi')$, then $K(\xi) = \{\xi\}$, hence this also trivially holds. Assume $\xi = \Omega^n \xi_n + \cdots + \Omega^0 \xi_0$ with $\Omega > \xi_n > 0$ and $\Omega > \xi_1, \ldots, \xi_{n-1}$. If $n = 0$, then also $K(\xi) = \{\xi\}$, hence the proof is valid. Let now $n > 0$. Then $K(\xi) \subseteq \{n, \ldots, 0, \xi_0, \ldots, \xi_n\}$. We know $\xi_0, \ldots, \xi_n \in OT'(\vartheta)$. Additionally, it is trivial to show $\{n, \ldots, 0\} \subseteq OT'(\vartheta)$. Hence, $K(\xi) \subseteq OT'(\vartheta)$. It is also trivial to show that $G'_\vartheta(m) \leq 2$ for all natural numbers m. Therefore, $G'_\vartheta(n) \leq G'_\vartheta(\xi)$ because $G'_\vartheta(\xi_n) \geq 1$. Also, $G'_\vartheta(\xi_i) < G'_\vartheta(\xi)$. Hence $G'_\vartheta(k(\xi)) = G'_\vartheta(\max_i\{n, \xi_i\}) < G'_\vartheta(\xi)$. □

Lemma 6.9. *There exists a specific coding of $(OT'(\vartheta), <_{OT'(\vartheta)})$ in the natural numbers such that $(OT'(\vartheta) \cap \Omega, <_{OT'(\vartheta)})$ can be interpreted as a primitive recursive ordinal notation system for ordinals less than $\vartheta(\Omega^\omega)$. Furthermore, one can choose this coding in such a way that $\forall \xi \in K(\alpha)(\xi \leq_{\mathbb{N}} \alpha)$.*

If we mention ACA_0 in the beginning of a theorem, we assume that we work in the system $OT(\vartheta)$. Accordingly, if we mention RCA_0 or RCA_0^* in the beginning of a theorem, we assume that we work in the system $OT'(\vartheta)$. Similar ideas in this subsection are used in [66].

Definition 6.10. *1. $Acc := \{\alpha < \Omega : WF(<\restriction \alpha)\}$,*

2. $M := \{\alpha : K(\alpha) \subseteq Acc\}$,

3. $\alpha <_\Omega \beta \Leftrightarrow \alpha, \beta \in M \wedge \alpha < \beta$.

The next lemma shows that Acc, M and $<_\Omega$ can be expressed by a $(\Pi^1_1(\Pi^0_3))^-$-formula.

Lemma 6.11. *Acc, M and $<_\Omega$ are expressible by a $(\Pi^1_1(\Pi^0_3))^-$-formula.*

Proof. The proof is the same for the ordinal notation systems $OT(\vartheta)$ and $OT'(\vartheta)$.

$$WF(\alpha) = \forall X \left(\forall x(\forall y(y \prec x \to y \in X) \to x \in X) \to \forall x(x \in X)\right),$$

where $\prec$ is $<\restriction \alpha$. It is easy to see that the prenex normal form of the formula $WF(\alpha)$ is $(\Pi^1_1(\Pi^0_3))^-$, hence Acc can be expressed by such a formula. M can be represented by the formula $\forall \xi \leq_{\mathbb{N}} \alpha(\xi \in K(\alpha) \to \xi \in Acc)$. Because $\xi \in K(\alpha)$ is elementary recursive, both M and $<_\Omega$ are also expressible by a $(\Pi^1_1(\Pi^0_3))^-$-formula. □

Lemma 6.12. (RCA_0^*) *α, $\beta \in Acc \Longrightarrow \alpha + \beta \in Acc$.*

Proof. Straightforward. □

Definition 6.13. *Let $Prog_\Omega(X)$ be the formula*

$$(\forall \alpha \in M)\,[(\forall \beta <_\Omega \alpha)(\beta \in X) \to \alpha \in X]\,.$$

Let Acc_Ω be the set $\{\alpha \in M : \vartheta(\alpha) \in Acc\}$.

Lemma 6.14. (RCA_0^*) *$Prog_\Omega(Acc_\Omega)$.*

Proof. We work in $OT'(\vartheta)$. Assume $\alpha \in M$ and $(\forall \beta <_\Omega \alpha)(\vartheta(\beta) \in Acc)$. We want to proof that $\vartheta(\alpha) \in Acc$. We show that $(\forall \xi < \vartheta(\alpha))(\xi \in Acc)$ by induction on $G'_\vartheta \xi$, from which the lemma follows. If $\xi = 0$, then this trivially holds. So assume $\xi > 0$.

a) Assume $\xi \notin P$
Because $\xi < \Omega$, we have $\xi =_{NF} \xi_1 + \cdots + \xi_n > \xi_1 \geq \cdots \geq \xi_n$ $(n \geq 2)$. The induction hypothesis yields $\xi_i \in Acc$. Hence from Lemma 6.12, we obtain $\xi \in Acc$.

b) Assume $\xi = \vartheta(\xi')$
From $\vartheta(\xi') < \vartheta(\alpha)$, we obtain either $\xi' < \alpha$ and $k(\xi') < \vartheta(\alpha)$ or $\vartheta(\xi') \leq k(\alpha)$. In the former case, $G'_\vartheta(k(\xi')) \leq G'_\vartheta(\xi') < G'_\vartheta(\xi)$ and the induction hypothesis implies $k(\xi') \in Acc$, hence $K(\xi') \subseteq Acc$. So $\xi' \in M$, from which it follows $\xi' <_\Omega \alpha$, hence $\vartheta(\xi') \in Acc$. In the latter case, we know that $k(\alpha) \in Acc$ because $\alpha \in M$. Therefore, $\vartheta(\xi') \in Acc$. □

Lemma 6.15. *Let $A(a)$ be a $(\Pi^1_1(\Pi^0_3))^-$-formula. Define A_k as*

$$\forall \alpha[(\forall \beta <_\Omega \alpha)A(\beta) \to (\forall \beta <_\Omega \alpha + \Omega^k)A(\beta)].$$

Then $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$ proves $Prog_\Omega(\{\xi : A(\xi)\}) \to A_k$.

Proof. We prove the assertion by outer induction on k. First note that $\{a : A(a)\}$ can be expressed by a set in $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$. Assume that $Prog_\Omega(\{\xi : A(\xi)\})$ and pick an arbitrary α such that $(\forall \beta <_\Omega \alpha)A(\beta)$. If $\alpha \notin M$, the assertion trivially follows. Assume $\alpha \in M$. If $k = 0$, the proof follows easily from $Prog_\Omega(\{\xi : A(\xi)\})$. Assume $k = l + 1$ and suppose $(\forall \beta <_\Omega \alpha)A(\beta)$. We want to prove that $(\forall \beta <_\Omega \alpha + \Omega^{l+1})A(\beta)$. Take an arbitrary $\beta <_\Omega \alpha + \Omega^{l+1}$. RCA_0 proves that there exists a $\xi \in Acc$ such that

$\beta <_\Omega \alpha+\Omega^l\xi$ (by induction on the construction of β in $OT'(\vartheta)$, one can show that one can take $\xi = k(\beta)+1$). Let $B(\zeta)$ be

$$(\forall \beta <_\Omega \alpha + \Omega^l \zeta) A(\beta).$$

$B(\zeta)$ is a Π^0_1-formula in A by Lemma 6.11. It is known (Lemma 6 in [41]) that $\mathsf{RCA}_0 \vdash \forall X(WO(X) \to TI_X(\psi))$ for all Π^0_1-formulas ψ. Because $\xi \in Acc$, we have that $\xi+1$ is well-ordered, hence we know that $TI_{\xi+1}(B)$ is true. This means

$$\forall x \leq \xi[\forall y \leq \xi(y < x \to B(y)) \to B(x)] \to \forall x \leq \xi B(x).$$

The theorem follows from $B(\xi)$, hence we only have to prove the progressiveness of B along $\xi+1$.

Assume that $x \leq \xi$. Then $x \in Acc$. If $x = 0$, then $B(0)$ follows from the assumption. Assume that x is a limit. If $\beta <_\Omega \alpha + \Omega^l x$, then there exists a $y < x$ such that $\beta <_\Omega \alpha + \Omega^l y$. Because $B(y)$ is valid, one obtains $A(\beta)$. Assume that $x = x'+1 \in Acc$. From $x' < x$, one obtains $(\forall \beta <_\Omega \alpha + \Omega^l x')A(\beta)$. Because A_l is valid, we get $(\forall \beta <_\Omega \alpha + \Omega^l x' + \Omega^l)A(\beta)$, hence $(\forall \beta <_\Omega \alpha + \Omega^l(x'+1))A(\beta) = B(x)$. □

Theorem 6.16. $|\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| \geq \vartheta(\Omega^\omega)$.

Proof. This follows from Lemmas 6.14, 6.15 and the fact that Acc_Ω is expressible by a $(\Pi^1_1(\Pi^0_3))^-$-formula. □

Lemma 6.17. *Let $A(a)$ be a $(\Pi^1_1(\Pi^0_3))^-$-formula. Define A_k as*

$$(\forall \beta <_\Omega \Omega \cdot k) A(\beta).$$

Then $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$ proves $Prog_\Omega(\{\xi : A(\xi)\}) \to A_k$.

Proof. We prove the assertion by outer induction on k. First note that $A(a)$ can be expressed by a set in $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$. It is easy to see that the case $k = 0$ holds. Assume $k = l+1$ and $Prog_\Omega(\{\xi : A(\xi)\})$. Then we know $(\forall \beta <_\Omega \Omega \cdot l)A(\beta)$. Pick an arbitrary $\beta <_\Omega \Omega \cdot (l+1)$. If $\beta < \Omega \cdot l$, we obtain $A(\beta)$. Hence assume that $\beta \geq \Omega \cdot l$. There exists a $\xi < \Omega$ such that $\beta = \Omega \cdot l + \xi$. Let $B(\zeta)$ be $A(\Omega \cdot l + \zeta)$. $B(\zeta)$ is a Π^0_0-formula in A. We prove by induction on ζ that $(\forall \zeta \in Acc)B(\zeta)$ is true. From this, the theorem follows.
If $\zeta = 0$, then $B(\zeta)$ is true because $(\forall \beta <_\Omega \Omega \cdot l)A(\beta)$ and $Prog_\Omega(\{\xi : A(\xi)\})$ imply $A(\Omega \cdot l)$. Assume $\zeta \in Acc$ is a limit and assume $(\forall \zeta' <_\Omega \zeta)B(\zeta)$. From $Prog_\Omega(\{\xi : A(\xi)\})$, we obtain $B(\zeta)$. Let $\zeta = \zeta'+1 \in Acc$. Then $B(\zeta)$ follows from $B(\zeta')$ and $Prog_\Omega(\{\xi : A(\xi)\})$. □

Theorem 6.18. $|\mathsf{RCA}_0^* + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| \geq \vartheta(\Omega \cdot \omega) = \varphi\omega 0.$

Proof. This follows from Lemmas 6.14, 6.17 and the fact that Acc_Ω is expressible by a $(\Pi^1_1(\Pi^0_3))^-$-formula. □

6.2.2 Upper bounds

In this subsection, we give an upper bound for the proof-theoretic ordinal of $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$. For this purpose, we use the theory $\Pi^1_2\text{-}\mathsf{BI}_0$ that has the same proof-theoretic ordinal (see Theorem 6.6).

Theorem 6.22 needs a specific normal form of an arithmetical formula. This will be developed in the next lemmas. It can be seen as an adaptation of the normal form theorem V.1.4 in [77].

Lemma 6.19. *For every arithmetical formula $B(X)$ with all free set variables indicated, there is a Δ_0-formula $R(x, X, f)$ such that*

1. $\mathsf{ACA}_0 \vdash B(X) \to \exists f \forall x R(x, X, f)$
2. *If T is a theory with $\mathsf{RCA}_0 \subseteq T$ and $\mathcal{F}(x, y)$ is an arbitrary formula such that,*

$$T \vdash \forall x \exists! y \mathcal{F}(x, y) \wedge \forall x \exists z (lh(z) = x \wedge \forall i < x \mathcal{F}(i, (z)_i)),$$

then $T \vdash \forall x R(x, X, \mathcal{F}) \to B(X)$, where $R(x, X, \mathcal{F})$ results from $R(x, X, f)$ by replacing subformulae of the form $f(t) = s$ by $\mathcal{F}(t, s)$.

Proof. See [81]. □

By adapting the previous lemma, we get the following corollary.

Corollary 6.20. *If in addition to the conditions of Lemma 6.19(2.), T also satisfies*

$$T \vdash \forall x \exists! y \mathcal{G}(x, y) \wedge \forall x \exists z (lh(z) = x \wedge \forall i < x \mathcal{G}(i, (z)_i)),$$

then $T \vdash \forall x R(x, \mathcal{G}, \mathcal{F}) \to B(\mathcal{G})$, where $R(x, \mathcal{G}, \mathcal{F})$ results from $R(x, X, \mathcal{F})$ by replacing $t \in X$ by $\mathcal{G}(t, 0)$.

Lemma 6.21. *Assume that the conditions for $\mathcal{F}$ and $\mathcal{G}$ from Lemma 6.19 and Corollary 6.20 are satisfied. Then there exists a Δ^0_1-formula P such that*

$$T \vdash \forall x R(x, \mathcal{G}, \mathcal{F}) \leftrightarrow \forall x P(\mathcal{G}[x], \mathcal{F}[x]),$$

where $\mathcal{G}[x]$ is the finite sequence s with length x such that $\forall i < x(\mathcal{G}(i,(s)_i))$. Similarly for $\mathcal{F}[x]$.

Proof. See [81]. □

Theorem 6.22. *Assume that $B(X,n)$ is a Π^0_3-formula, then*

$$\Pi^1_2\text{-}\mathsf{BI}_0 \vdash \forall n[\forall A \in Rec((\Pi^1_1)^-)B(A(\cdot),n) \leftrightarrow (\forall X \subseteq \omega)B(X,n)],$$

where $B(A(\cdot),n)$ results from $B(X,n)$ by replacing $t \in X$ by $A(t)$.

Proof. See [81]. □

Corollary 6.23. $|\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| = \vartheta(\Omega^\omega)$.

Proof. See [81]. □

In a similar way, one can also show the following lemma.

Lemma 6.24. *Assume that $B(X,n)$ is a Π^0_2-formula, then*

$$\Pi^1_1\text{-}\mathsf{BI}_0 \vdash \forall n[\forall A \in Rec((\Pi^1_1)^-)B(A(\cdot),n) \leftrightarrow (\forall X \subseteq \omega)B(X,n)].$$

Therefore, $|\Pi^1_1\text{-}\mathsf{BI}_0| = \varphi\omega 0$ (see Theorem 6.6) yields that the ordinal of $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_2)\text{-}\mathsf{CA}_0)^-$ is bounded above by $\varphi\omega 0$. Actually, one can prove that the ordinal of the theory $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_2)\text{-}\mathsf{CA}_0)^-$ is even much lower (We want to thank Leszek Kołodziejczyk, who reminded us on these facts). Using WKL_0 and the normal form theorem II.2.7 in [77], one can prove that every $(\Pi^1_1(\Pi^0_2))^-$-formula is equivalent with a $(\Pi^0_2)^-$-formula (and if the original formula has extra parameters, then the equivalent one will also have those parameters). The intuitive idea behind this is that WKL_0 proves that the projection of closed set is a closed set, meaning that $\exists X\forall x \dots$ can be reduced to $\forall z \dots$ (a closed set can be seen in some sense as a Π^0_1-formula). Furthermore, one can prove the following lemmas.

Lemma 6.25. *Let $\mathcal{F}$ be $(\Pi^1_1(\Pi^0_n))^-$ or $(\Pi^0_n)^-$. Then the first order part of the theory $\mathsf{WKL}_0 + \mathcal{F}\text{-}\mathsf{CA}_0$ is the same as that of $\mathsf{RCA}_0 + \mathcal{F}\text{-}\mathsf{CA}_0$.*

Proof. Can be proved by little adaptations of paragraph IX.2 in [77]. □

Lemma 6.26. *The first order part of $\mathsf{RCA}_0 + (\Pi^0_2)^-\text{-}\mathsf{IND}$ is $I\Sigma_3$.*

Proof. Can be proved by adaptations of paragraph IX.1 in [77]. □

Hence, the proof-theoretic ordinal of $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_2)\text{-}\mathsf{CA}_0)^-$ is equal to the ordinal of the theory $I\Sigma_3$, which is known to be $\omega^{\omega^{\omega^{\omega}}}$. In a similar but more technical way, one can prove that $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_2)\text{-}\mathsf{CA}_0)^-$ is Π^0_4-conservative over $I\Sigma_2$. Hence, the ordinal of $\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_2)\text{-}\mathsf{CA}_0)^-$ is $\omega^{\omega^{\omega}}$.

Some questions remain unsolved. We state them here as conjectures.

Conjecture 6.27.

1. $|\mathsf{RCA}^*_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-| = \varphi\omega 0$,
2. $|\mathsf{RCA}^*_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-| = \varphi\omega 0$,
3. $|\mathsf{RCA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-| = \vartheta(\Omega^\omega)$.

6.3 Independence results

6.3.1 Independence results for $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$

In this section, we state provability and unprovability statements concerning the wpo's $\mathcal{T}(W)$.

Theorem 6.28. $\mathsf{ACA}_0 \vdash$ *'$\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo'* $\to WF(\vartheta(\varepsilon_{\Omega+1}))$.

Proof. Theorem 4.8 can be carried out in ACA_0 if we use the primitive recursive ordinal notation system $OT(\vartheta) \cap \Omega$ for $\vartheta(\varepsilon_{\Omega+1})$. □

Therefore, one obtains the following theorem.

Theorem 6.29. $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^- \nvdash$ *'$\mathcal{T}(\mathbb{B}(\cdot))$ is a wpo'.*

Proof. Follows from Theorem 6.28 and the fact that the proof-theoretic ordinal of the theory $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$ is $\vartheta(\varepsilon_{\Omega+1})$ (see e.g. [13]). □

The wpo $\mathcal{T}(\mathbb{B}(\cdot))$ can be seen as the *limit* of $\mathcal{T}(\cdot\overbrace{^{*\cdots *}}^{n})$, where $\cdot\overbrace{^{*\cdots *}}^{n}$ is defined by applying the Higman-operator * n many times, because a binary tree can be generated as an iteration of the *-operator. For example, one can interpret an element of $\{a\}^{**}$ as a binary tree which goes only one time to the left and every node has label a. So, it would be interesting if $\mathsf{ACA}_0 +$

$(\Pi^1_1\text{-}\mathsf{CA}_0)^- \vdash$ '$\mathcal{T}(\cdot\overbrace{* \cdots *}^{n})$ is a wpo' for every natural number n because this theory does not prove the *limit*.

The maximal order type of $\mathcal{T}(\cdot^{*\cdots*})$ is strictly below the proof-theoretical ordinal of $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$. Therefore, one can really expect this provability assertion because $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$ proves the well-foundedness of $o(\mathcal{T}(\cdot^{*\cdots*}))$. In Theorem 6.30, we prove that this is really the case and we use a minimal bad sequence argument. We also want to note that we believe that a constructive proof of the well-partial-orderedness of $\mathcal{T}(\cdot^{*\cdots*})$ can be obtained from the proof of the upper bound for the maximal order type of $\mathcal{T}(\cdot^{*\cdots*})$ by reifications (for more information see [66] and [73]).

Theorem 6.30. *For all natural numbers n, the theory* $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$ *can prove the well-partial-orderedness of* $\mathcal{T}(\cdot\overbrace{* \cdots *}^{n})$.

Proof. Fix a natural number n. We reason in $\mathsf{ACA}_0 + (\Pi^1_1\text{-}\mathsf{CA}_0)^-$. Code the elements of $\mathcal{T}(\cdot\overbrace{* \cdots *}^{n})$ as natural numbers such that $\circ$ has code 0 and the code of t_i is strictly smaller than the code of $t = \circ[w(t_1, \ldots, t_i, \ldots, t_n)]$. This coding can be done primitive recursively. The leaves of $\circ[w(t_1, \ldots, t_i, \ldots, t_n)]$ are $\{t_1, \ldots, t_n\}$ and we assume that there is a primitive recursive relation '... is a leaf of ...'.

If σ is a code for a finite sequence, then this sequence is equal to

$$((\sigma)_0, \ldots, (\sigma)_{lh(\sigma)-1}),$$

where $lh(\sigma)$ is the length and $(\sigma)_i$ is the i-th element of the sequence. An infinite sequence $(\sigma_i)_{i<\omega}$ is decoded by a set $\{(i, \sigma_i) : i < \omega\}$ and $(\{(i, \sigma_i) : i < \omega\})_i$ stands for the element σ_i. Note that one can recursively construct the set $\{(\sigma_0, \ldots, \sigma_i) : i < \omega\}$ from the original set $\{(i, \sigma_i) : i < \omega\}$, where the finite sequence $(\sigma_0, \ldots, \sigma_i)$ is decoded by a natural number. Furthermore, if one has the set $\{(\sigma_0, \ldots, \sigma_i) : i < \omega\}$, one can reconstruct the original set $\{(i, \sigma_i) : i < \omega\}$ from it in a recursive way.

Now, assume that $\mathcal{T}(\cdot^{*\cdots*})$ is not a well-partial-order. Then there exists an infinite sequence $(t_i)_{i<\omega}$ in $\mathcal{T}(\cdot^{*\cdots*})$ such that $\forall i, j(i < j \to t_i \not\leq t_j)$. Define $\chi(\sigma)$ as

$$\begin{aligned} &\sigma \text{ is a finite sequence of elements of } \mathcal{T}(\cdot^{*\cdots*}) \\ \text{and } &\exists Z(Z \text{ is an infinite bad sequence in } \mathcal{T}(\cdot^{*\cdots*}) \wedge \forall i < lh(\sigma)((\sigma)_i = (Z)_i)), \end{aligned}$$

and $\psi(\sigma)$ as

$$\begin{aligned} &\sigma \text{ is a finite sequence of elements of } \mathcal{T}(\cdot^{*\cdots *}) \\ &\text{and } \forall Y\, [(Y \text{ is an infinite bad sequence in } \mathcal{T}(\cdot^{*\cdots *})) \\ &\quad \to \forall i < lh(\sigma)\, [\forall k < i\, ((\sigma)_k = (Y)_k) \to (\sigma)_i \leq (Y)_i]]\,, \end{aligned}$$

Note that $(\sigma)_i \leq (Y)_i$ is interpreted as the inequality relation between natural numbers and not between elements of $\mathcal{T}(\cdot^{*\cdots *})$. Using $(\Pi^1_1\text{-}\mathsf{CA}_0)^-$, there exists a set S such that $\sigma \in S \leftrightarrow \chi(\sigma) \wedge \psi(\sigma)$. Choose now two arbitrary elements s, s' in S. We want to prove that either s is an initial segment of s' or s' an initial segment of s. Assume that there is an index $i < \min\{lh(s), lh(s')\}$ such that $(s)_i < (s')_i$ and $\forall k < i$, $(s)_k = (s')_k$. The case $(s)_i > (s')_i$ can be handled in a similar way. Note that $(s)_i < (s')_i$ is seen as an inequality between natural numbers and not between elements of $\mathcal{T}(\cdot^{*\cdots *})$. Because s is in S, s can be extended to an infinite bad sequence Y in $\mathcal{T}(\cdot^{*\cdots *})$. This, however, contradicts $\psi(s')$ because $(Y)_k = (s)_k = (s')_k$ for all $k < i$, but $(Y)_i = (s)_i < (s')_i$.

If $s \in S$, one can show by minimization in RCA_0, that there is a $z \in \mathcal{T}(\cdot^{*\cdots *})$ such that $s^\frown(z) \in S$. Therefore, there exists an infinite sequence $(s_i)_{i<\omega}$ in $\mathcal{T}(\cdot^{*\cdots *})$ such that $S = \{(s_0, \ldots, s_i) : i < \omega\}$.

Now, define $subS$ as the set of all pairs (i, t) such that t is a leaf of s_i. Note that $subS$ is definable in RCA_0 because

$$\begin{aligned} (i,t) \in subS &\Leftrightarrow \exists \sigma(\sigma \in S \text{ and } lh(\sigma) = i+1 \text{ and } t \text{ is a leave of } \sigma_i) \\ &\Leftrightarrow \forall \sigma((\sigma \in S \text{ and } lh(\sigma) = i+1) \to t \text{ is a leave of } \sigma_i). \end{aligned}$$

On $subS$, we define the following ordering: $(i,t) \leq (j,t') \Leftrightarrow t \leq_{\mathcal{T}(\cdot^{*\cdots *})} t'$. With this ordering $subS$ is a quasi-order. We want to prove that it is a well-quasi-order.

Assume that this is not true. This implies the existence of an infinite bad sequence $((n_i, s'_i))_{i<\omega}$ in $subS$. This implies $s'_i \leq s_{n_i}$ for all i. Construct now an infinite subsequence H such that $(n_i)_{i<\omega}$ is strictly increasing and the first element of H is (n_0, s'_0). So $H = \{(i, (n_{j_i}, s'_{j_i})) : i < \omega\}$ with $j_0 = 0$. This is possible in RCA_0 because the number of leaves of an element of $\mathcal{T}(\cdot^{*\cdots *})$ is finite.
Construct now recursively a set S' such that all elements that lie in S' have the form $(\sigma_0, \ldots, \sigma_i)$ such that if $i < n_0$, the element σ_i is equal to s_i and if $i \geq n_0$, then σ_i is equal to $s'_{j_{i-n_0}}$. The existence of S' is however in contradiction with the definition of S. This is because $\psi((s_0, \ldots, s_{n_0}))$, $s'_{j_0} =$

$s'_0 < s_{n_0}$ as a natural number and $(\sigma_i)_{i<\omega}$ is an infinite bad sequence. Take for example $k < l < \omega$. If $k < l < n_0$, then $\sigma_k \not\leq \sigma_l$ follows from $s_k \not\leq s_l$. If $n_0 \leq k < l$, then $\sigma_k \not\leq \sigma_l$ follows from $s'_{j_{k-n_0}} \not\leq s'_{j_{l-n_0}}$. Assume $k < n_0 \leq l$. Then $\sigma_k \not\leq \sigma_l$ follows from the fact that otherwise $s_k \leq s'_{j_{l-n_0}} \leq s_{n_{j_{l-n_0}}}$ with $k < n_0 = n_{j_0} \leq n_{j_{l-n_0}}$, a contradiction. So we conclude that *subS* is indeed a well-quasi-order.

By Lemma 1.63, we obtain that $(subS)^{*\cdots *}$ is a well-quasi-ordering. Now, look at the infinite sequence $(s_i)_{i<\omega}$ in $\mathcal{T}(\cdot^{*\cdots *})$. Rewrite every element $\times s_i$ to an element of $(subS)^{*\cdots *}$ and call it $\overline{s_i}$. For example, if

$$s_i = \circ\left(\left(s^1_1, \ldots, s^1_{n_1}\right), \ldots, \left(s^k_1, \ldots, s^k_{n_k}\right)\right),$$

then $\overline{s_i}$ is equal to

$$\left(\left((i, s^1_1), \ldots, (i, s^1_{n_1})\right), \ldots, \left((i, s^k_1), \ldots, (i, s^k_{n_k})\right)\right).$$

Because we know that $(subS)^{*\cdots *}$ is a well-quasi-order, there exist two indices $i < j$ such that $\overline{s_i} \leq_{(subS)^{*\cdots *}} \overline{s_j}$. Therefore, $s_i \leq_{\mathcal{T}(\cdot^{*\cdots *})} s_j$, a contradiction. □

6.3.2 A general approach

Following the previous theorem (Theorem 6.30), we can state the following.

Theorem 6.31. *If T is a theory such that*

$$T \vdash \forall X(X \text{ is a } \mathbb{wpo} \to W(X) \text{ is a } \mathbb{wpo}),$$

then

$$\mathsf{RCA}_0 + T + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \vdash \mathcal{T}(W) \text{ is a } \mathbb{wpo},$$

and even

$$\mathsf{RCA}_0 + T + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \vdash \mathcal{T}(W_n) \text{ is a } \mathbb{wpo},$$

where $W_n(X) := W(W(\ldots(W(X))\ldots))$ with W applying n many times on X.

Furthermore, assume W' is in some sense the *limit* of W, i.e. one can prove $\forall X(X$ is a $\mathbb{wpo} \to W'(X)$ is a $\mathbb{wpo})$ from finitely, but unbounded many iterations of the statement $\forall X(X$ is a $\mathbb{wpo} \to W(X)$ is a $\mathbb{wpo})$, meaning one can prove it from $\forall n \forall X(X$ is a $\mathbb{wpo} \to W_n(X)$ is a $\mathbb{wpo})$, but one cannot prove $\forall X(X$ is a $\mathbb{wpo} \to W'(X)$ is a $\mathbb{wpo})$ from $\forall X(X$ is a $\mathbb{wpo} \to W(X)$ is a $\mathbb{wpo})$. Then in general we predict

$$\mathsf{RCA}_0 + T + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \nvdash \mathcal{T}(W') \text{ is a } \mathbb{wpo}.$$

6.3.3 Independence results for $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$

The proof-theoretic ordinal of $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$ is the small Veblen number $\vartheta(\Omega^\omega)$. Furthermore, one can easily show

$$\mathsf{RCA}_0 \vdash \mathcal{T}(X^*) \text{ is a } \mathbb{wpo} \to WF(\vartheta(\Omega^\omega))$$

by giving a quasi-embedding from $\vartheta(\Omega^\omega)$ to $\mathcal{T}(X^*)$. This can be done is a similar way as in Chapter 2. Therefore, the theory $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^-$ cannot prove the well-partial-orderedness of $\mathcal{T}(X^*)$. Note that the maximal order type of $\mathcal{T}(X^*)$ is equal to $\vartheta(\Omega^\omega)$.

Again, like in the ACA_0-case, one can search for provability results: $W'(X) = X^*$ can be seen as the *limit* of $W(X) = X^n$. Therefore, it would be interesting if $\mathsf{RCA}_0 + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \vdash$ '$\mathcal{T}(X^n)$ is a $\mathbb{wpo}$'. However, the theory RCA_0 does not prove $\forall X$('X is a $\mathbb{wpo}$' $\to$ 'X^n is a $\mathbb{wpo}$'), but it is provable in $\mathsf{RCA}_0 + \mathsf{CAC}$ (see [17] for more information). CAC is the principle saying that every infinite sequence in a partial order has a subsequence that is either a chain or an antichain. This implies that $\mathsf{RCA}_0 + \mathsf{CAC} + (\Pi^1_1(\Pi^0_3)\text{-}\mathsf{CA}_0)^- \vdash$ '$\mathcal{T}(X^n)$ is a $\mathbb{wpo}$'. We believe that we can get rid of CAC by using reifications like in [66].

Chapter 7

The linearized version

7.1 Introduction

We assume that the reader of this chapter is familiar with the preliminaries in Section 1.2.5. This chapter is based on the joint article [65].

The previous chapters indicated that there is a close connection between the tree-classes with Friedman's gap-embeddability relation and the theta-functions. More explicitly, one can define a maximal linear extension of $\mathcal{T}(W)$ using these collapsing functions in a straightforward way (see the discussion after Conjecture 1.111). To recall the discussion, if $W = X^*\backslash\{()\}$ (then $\mathcal{T}(W) \cong \mathbb{T}^s$), define a maximal linear extension $\prec$ on $\mathcal{T}(W)$ as follows. Assume $T \in \mathcal{T}(W)$. If $T = \circ$, define $f(T)$ as zero and if $T = \circ[(T_1, \ldots, T_n)]$, then define $f(T)$ as $\vartheta(\Omega^{n-1}(f(T_1) + 1) + \cdots + \Omega^0(f(T_n) + 1))$. Take two trees $T, T' \in \mathcal{T}(W)$. Let $T \prec T'$ iff $f(T) < f(T')$. This is a maximal linear extension on $\mathcal{T}(W)$. One can use the same kind of argument for the W such that $\mathcal{T}(W) \cong \mathbb{T}'^{wgap}_2[0]$, to define straightforwardly a maximal linear extension of this well-partial-order using the collapsing functions ϑ_0 and ϑ_1.

This chapter investigates whether this connection remains true for the sequential version. More explicitly, let T_n be the linearized version of the usual ordinal notation system $OT(\vartheta_i)$, i.e. the ordinal notation system consisting of the collapsing functions ϑ_i, but now defined without addition. For a formal definition of T_n, see Definition1.41. Let $\overline{\mathbb{S}}^{wgap}_n$ be the linearized version of $\overline{\mathbb{T}}^{wgap}_n$ (see Definitions 5.20 and 7.1), hence it is a subset of $\mathbb{S}_n$ with the weak gap-embeddability relation (see Section 1.2.9). The reformulation of

Weiermann's conjecture in terms of sequences is:

Is the maximal order type of $\overline{\mathbb{S}}_n^{wgap}[0]$ equal to the order type of $T_n[0]$?

Note that we restrict ourselves to the sequences, respectively to terms in T_n, that start with 0, respectively ϑ_0, like we did for the tree-case. Furthermore, note that we use $\overline{\mathbb{S}}_n^{wgap}$ instead of $\mathbb{S}_n^{wgap}$ because of the specific form of the defined terms in T_n (see Definitions 1.40 and 1.41).

We also want to mention that T_n is the right counterpart of the ordinal notation system $OT(\vartheta_i)$ that is needed for the linearized version: a sequence over $\{0, \dots, n-1\}$ is (isomorphic to) a tree with labels in $\{0, \dots, n-1\}$ where every node has at most one immediate successor. Hence, we do not summation like in $f(T) = \vartheta(\Omega^{n-1}(f(T_1)+1) + \cdots + \Omega^0(f(T_n)+1))$ because $n = 1$ (the successor operator $+1$ can be dealt with in a different way).

In contrast with the tree-case, the maximal order type of $\overline{\mathbb{S}}_n^{wgap}[0]$ *is* known. However, now the order type of the ordinal notation system $T_n[0]$ is unknown. So to address the above question, we have to calculate the order type of the notation system $T_n[0]$ and check whether it is equal to the maximal order type of $\overline{\mathbb{S}}_n^{wgap}[0]$. This maximal order type is ω_{2n-1}, as will be shown in Lemma 7.10. Before we go on, let us formally define $\overline{\mathbb{S}}_n^{wgap}$.

Definition 7.1. *Denote the subset of $\mathbb{S}_n$ of elements $s_0 \dots s_k$ that fulfill the extra condition $s_0 \leq i$ by $\mathbb{S}_n[i]$. Like in Definition 1.83, $(\mathbb{S}_n[i], \leq_{gap}^w)$, respectively $(\mathbb{S}_n[i], \leq_{gap}^s)$, is denoted by $\mathbb{S}_n^{wgap}[i]$, respectively $\mathbb{S}_n^{sgap}[i]$.*

Let $\overline{\mathbb{S}}_n$ be the subset of $\mathbb{S}_n$ which consists of all sequences $s_0 \dots s_{k-1}$ in $\mathbb{S}_n$ such that for all $i < k-1$, $s_i - s_{i+1} \geq -1$. This means that if $s_i = j$, then s_{i+1} is an element of $\{0, \dots, j+1\}$. For example $02 \notin \overline{\mathbb{S}}_3$. Like in Definition 1.83, we denote the subset of $\overline{\mathbb{S}}_n$ that has the extra condition $s_0 \leq i$ by $\overline{\mathbb{S}}_n[i]$. We denote $(\overline{\mathbb{S}}_n, \leq_{gap}^w)$ by $\overline{\mathbb{S}}_n^{wgap}$, $(\overline{\mathbb{S}}_n, \leq_{gap}^s)$ by $\overline{\mathbb{S}}_n^{sgap}$, $(\overline{\mathbb{S}}_n[i], \leq_{gap}^w)$ by $\overline{\mathbb{S}}_n^{wgap}[i]$ and $(\overline{\mathbb{S}}_n[i], \leq_{gap}^s)$ by $\overline{\mathbb{S}}_n^{sgap}[i]$.

This definition corresponds to Definition 5.20. On T and its substructures, we define the following partial order $\trianglelefteq$, which can be seen as a natural sub-order of the ordering $<$ on T. T_n together with this natural partial order is actually equal to $\overline{\mathbb{S}}_n^{sgap}$.

Definition 7.2. *1. $0 \trianglelefteq \alpha$,*

2. if $\alpha \trianglelefteq k_i\beta$, then $\alpha \trianglelefteq \vartheta_i\beta$,

3. if $\alpha \trianglelefteq \beta$, then $\vartheta_i\alpha \trianglelefteq \vartheta_i\beta$.

Lemma 7.3. $(T_n, \trianglelefteq) \cong (\overline{\mathbb{S}}_n, \leq^s_{gap})$.

Proof. Define $e : T_n \to \overline{\mathbb{S}}_n$ as follows. $e(0)$ is the empty sequence ε. Let $e(\vartheta_i\alpha)$ be $(i)^\frown e(\alpha)$. For example $e(\vartheta_2\vartheta_1 0)$ is the finite sequence 21. It is trivial to see that e is a bijection. So the only thing we still need to show is that for all α and β in T_n, $e(\alpha) \leq^s_{gap} e(\beta)$ if and only if $\alpha \trianglelefteq \beta$. We show this by induction on the sum of the lengths of α and β. If α or β are equal to 0, then this is trivial. Assume α and β are different from 0. Hence, $\alpha = \vartheta_i\alpha'$ and $\beta = \vartheta_j\beta'$. Assume $\alpha \trianglelefteq \beta$. Then $\alpha \trianglelefteq k_j\beta'$ or $i = j$ and $\alpha' \trianglelefteq \beta'$. In the latter case, the induction hypothesis yields $e(\alpha') \leq^s_{gap} e(\beta')$, hence $e(\alpha) = (i)^\frown e(\alpha') \leq^s_{gap} (i)^\frown e(\beta') = e(\beta)$. In the former case, assume $\beta' = \vartheta_{l_1} \ldots \vartheta_{l_k}\beta''$, with $l_1, \ldots, l_k > j$ and $S(\beta'') \leq j$ such that $k_j(\beta') = \beta''$. The induction hypothesis yields $e(\alpha) \leq^s_{gap} e(\beta'')$. From the strong gap-embeddability relation we obtain $i \leq S(\beta'') \leq j$, hence $e(\alpha) \leq^s_{gap} (jl_1, \ldots l_k)^\frown e(\beta'')$ because $j, l_1, \ldots, l_k \geq i$. The reverse direction can be proved in a similar way. □

The previous proof also yields $(T_n[0], \trianglelefteq) \cong (\overline{\mathbb{S}}_n[0], \leq^s_{gap}) = (\overline{\mathbb{S}}_n[0], \leq^w_{gap})$. We prove that the linear order $<$ on T_n is a linear extension of $\triangleleft$. Let $\alpha \triangleleft \beta$ if $\alpha \trianglelefteq \beta$ and $\alpha \neq \beta$.

Lemma 7.4. *If $\alpha \trianglelefteq \beta$, then $\alpha \leq \beta$.*

Proof. We prove this by induction on the sum of the lengths of α and β Assume $\alpha \trianglelefteq \beta$. If $\alpha = 0$, then trivially $\alpha \leq \beta$. Assume $\alpha = \vartheta_i\alpha'$. $\alpha \trianglelefteq \beta$ yields $\beta = \vartheta_i\beta'$ and either $\alpha \trianglelefteq k_i\beta'$ or $\alpha' \trianglelefteq \beta'$. In the first case, the induction hypothesis yields $\alpha \leq k_i\beta' < \vartheta_i\beta' = \beta$. Assume that $\alpha' \trianglelefteq \beta'$. The induction hypothesis yields $\alpha' \leq \beta'$. if $\alpha' = \beta'$, we can finish the proof, so assume $\alpha' < \beta'$. We want to prove that $k_i\alpha' < \beta$. Using the induction hypothesis, it is sufficient to prove that $k_i\alpha' \triangleleft \beta$. This follows from $\alpha = \vartheta_i\vartheta_{j_1} \ldots \vartheta_{j_l} k_i\alpha' \trianglelefteq \beta$ (with $j_1, \ldots, j_l > i$) and Lemma 7.3. □

So the linear ordering on $T_n[0]$ is a linear extension of $\overline{\mathbb{S}}_n[0]$ with the strong (and weak) gap-embeddability relation and furthermore,

$$o(T_n[0], \trianglelefteq) = o(\overline{\mathbb{S}}^{sgap}_n[0]) = o(\overline{\mathbb{S}}^{wgap}_n[0]) = o(\overline{\mathbb{S}}^{wgap}_n).$$

We want to investigate whether this is a *maximal linear extension.* This means that we want to prove, as mentioned before, whether the order type of the notation system $T_n[0]$ is equal to ω_{2n-1}. Quite surprisingly, this is not that case for every n. If $n = 1$ or $n = 2$, this is true. But if $n > 2$, then it does not hold anymore. More explicitly, we show that the order type of $(T_n[0], <)$ is ω_{n+1} if $n \geq 2$. This indicates that the general belief about the correspondence of the theta-functions and a maximal linear extension of structures with the gap-embeddability relation is not true for the sequential version. One really needs trees in order to make the belief true. We show $(T_n[0], <) = \omega_{n+1}$ for $n \geq 2$ in Section 7.3. In Section 7.4, we look to the sequences with the gap-embeddability relation from a reverse mathematical point of view.

7.2 Known results

In this subsection, we state the results of Schütte and Simpson [72] about the maximal order types of $\mathbb{S}_n^{wgap}$ and $\mathbb{S}_n^{sgap}$. Sometimes, we abbreviate $\mathbb{S}_n^{wgap}$ and $\mathbb{S}_n^{sgap}$ by $\mathbb{S}_n^w$ and $\mathbb{S}_n^s$ (now there is no misunderstanding of the meaning of s: in this setting s never means *structured*).

Theorem 7.5 (Schütte-Simpson[72], Simpson/Friedman[76]). *For every natural number n, $(\mathbb{S}_n, \leq_{gap}^w)$ and $(\mathbb{S}_n, \leq_{gap}^s)$ are* wpo*'s.*

Theorem 7.6 (Schütte-Simpson[72]).
$\mathsf{ACA}_0 \nvdash \forall n < \omega$ *'$(\mathbb{S}_n, \leq_{gap}^w)$ is a* wpo*',*
$\mathsf{ACA}_0 \nvdash \forall n < \omega$ *'$(\mathbb{S}_n, \leq_{gap}^s)$ is a* wpo*'.*

Theorem 7.7 (Schütte-Simpson[72]).
For all n, $\mathsf{ACA}_0 \vdash$ *'$(\mathbb{S}_n, \leq_{gap}^w)$ is a* wpo*',*
For all n, $\mathsf{ACA}_0 \vdash$ *'$(\mathbb{S}_n, \leq_{gap}^s)$ is a* wpo*'.*

If one takes a closer look at the proofs of Schütte and Simpson [72], one can also find results on the maximal order types of the sequences with the gap-embeddability relation (see Lemma 5.5. in [72]). There is a small error in that proof, although we believe that this is actually more like a typo. For clarity reasons, the proof is given here.

Theorem 7.8 (Schütte-Simpson[72]).
$\mathbb{S}_{n+1}^s \cong \mathbb{S}_n^s \times (\mathbb{S}_n^s)^*$.

Proof. Assume $n \geq 0$. We define an order-preserving bijection h_n from $\mathbb{S}_{n+1}^s$ to $\mathbb{S}_n^s \times (\mathbb{S}_n^s)^*$. Let $h_n(\varepsilon)$ be $(\varepsilon, ())$. Take an arbitrary element $s \in \mathbb{S}_{n+1}^s \backslash \{\varepsilon\}$. Then $s = s_0^+ 0 \dots 0 s_k^+$, with $s_i \in \mathbb{S}_n^s$ for all i and s_i^+ is the result of replacing every number j in s_i by $j+1$. Define then $h_n(s)$ as $(s_0, (s_1, \dots, s_k))$. Note that for the sequence $s = 0$, $k \geq 1$. In other words, k represents the number of 0's occurring in s. It is easy to see that h_n is a bijection.

We know prove that $s < s'$ yields $h_n(s) < h_n(s')$ by induction on $lh(s)+lh(s')$. If s or s' is ε, then this is trivial. So assume $s = s_0^+ 0 \dots 0 s_k^+$ and $s' = {s'}_0^+ 0 \dots 0 {s'}_l^+$. If $k = 0$, then $s < s'$ yields $l = 0$ and $s_0^+ < {s'}_0^+$, or $l > 0$ and $s_0^+ \leq {s'}_0^+$. In both cases, $h_n(s) < h_n(s')$. Assume $k, l > 0$. $s < s'$ yields $s_0^+ \leq {s'}_0^+$ and $s_1^+ 0 \dots 0 s_k^+ \leq {s'}_j^+ 0 \dots 0 {s'}_l^+$ for a certain $j \geq 1$. From $s_1^+ 0 \dots 0 s_k^+ \leq {s'}_j^+ 0 \dots 0 {s'}_l^+$, one can prove as before (or by an additional induction argument on k) that $s_1^+ \leq {s'}_j^+$ and $s_2^+ 0 \dots 0 s_k^+ \leq {s'}_{j_2}^+ 0 \dots 0 {s'}_l^+$ for a certain $j_2 \geq j+1$. In the end, we have $s_0^+ \leq {s'}_0^+$ and $(s_1^+, \dots, s_k^+) \leq^* ({s'}_1^+, \dots, {s'}_l^+)$. This yields $h_n(s) < h_n(s')$. The reverse direction $h_n(s) < h_n(s') \to s < s'$ can be proved in a similar way. □

Therefore, from the maximal order type of $\mathbb{S}_1^s$, which is ω, one can calculate the maximal order types of all $\mathbb{S}_n^s$. Following the same template, one also obtains the following lemma.

Lemma 7.9. *1.* $o(\mathbb{S}_{n+1}^s) = o(\mathbb{S}_n^s) \otimes o((\mathbb{S}_n^s)^*)$,
2. $o(\mathbb{S}_{n+1}^w) = o(\mathbb{S}_{n+1}^w[0]) = o(\mathbb{S}_{n+1}^s[0]) = o((\mathbb{S}_n^s)^*)$.

Proof. The first equality follows from Theorem 7.8. The equality $o(\mathbb{S}_{n+1}^s[0]) = o((\mathbb{S}_n^s)^*)$ also follows from the same theorem. $o(\mathbb{S}_{n+1}^w[0]) = o(\mathbb{S}_{n+1}^s[0])$ is trivial as they refer to the same ordering. To prove $o(\mathbb{S}_{n+1}^w) = o(\mathbb{S}_{n+1}^w[0])$, note that $\mathbb{S}_{n+1}^w[0] \subseteq \mathbb{S}_{n+1}^w$, hence $o(\mathbb{S}_n^w[0]) \leq o(\mathbb{S}_n^w)$. Furthermore, the mapping e which plots $s_0 \dots s_{k-1}$ to $0s_0 \dots s_{k-1}$ is a quasi-embedding from $\mathbb{S}_n^w$ to $\mathbb{S}_n^w[0]$. Hence, $o(\mathbb{S}_n^w) \leq o(\mathbb{S}_n^w[0])$. □

For example, we have $o(\mathbb{S}_2^w) = \omega^{\omega^\omega}$. We are especially interested in substructures of these wpo's with maximal order types exactly equal to an exact ω-tower $\omega^{\omega^{\cdot^{\cdot^{\cdot^\omega}}}}$ (without any '+1'). Luckily, this corresponds to $\overline{\mathbb{S}}_n$.

Lemma 7.10. $o(\overline{\mathbb{S}}_{n+1}^w) = o(\overline{\mathbb{S}}_{n+1}^w[0]) = o(\overline{\mathbb{S}}_{n+1}^{sgap}[0]) = o((\overline{\mathbb{S}}_n^{sgap}[0])^*)$
$= o((\overline{\mathbb{S}}_n^w[0])^*) = o((\overline{\mathbb{S}}_n^w)^*)$.

Proof. Similar as in Theorem 7.8 and Lemma 7.9. □

Corollary 7.11. *For all n,* $o(\overline{\mathbb{S}}_n^w) = \omega_{2n-1}$.

7.3 From an order-theoretical view

7.3.1 Maximal linear extension of gap-sequences with one and two labels

It is trivial to show that the order type of $(T_1[0], <)$ is equal to ω, hence $(T_1[0], <)$ corresponds to a maximal linear extension of $\overline{\mathbb{S}}_1^{sgap}[0]$. So we can concentrate on the case of $T_2[0]$. We show that the order type of $(T_2[0], <)$ is equal to ω^{ω^ω}. This implies that the ordinal notation system $(T_2[0], <)$ corresponds to a maximal linear extension of $\overline{\mathbb{S}}_2^w[0]$ and that the order type of $(T_2[0], <)$ is equal to $o(\overline{\mathbb{S}}_2^w)$. More specifically, we show that

$$\sup_{n_1,\dots,n_k} \vartheta_0\vartheta_1^{n_1}\dots\theta_0\vartheta_1^{n_k}(0) = \omega^{\omega^\omega}.$$

The supremum is equal to $\vartheta_0\vartheta_1\vartheta_2(0)$. Knowing that Ω_i is defined as $\vartheta_i(0)$, we thus want to show that

$$\vartheta_0\vartheta_1\Omega_2 = \omega^{\omega^\omega}.$$

Theorem 7.12. $\vartheta_0\vartheta_1\Omega_2 = \omega^{\omega^\omega}$.

Proof. φ_ω is the ordinal notation system defined without addition based on Veblen's hierarchy (see Subsection 1.2.5). We present a order-preserving bijection from $\varphi_\omega 0$ to $\vartheta_0\vartheta_1\Omega_2$. Lemma 1.25 then yields the assertion.

Define $\chi 0 := 0$ and $\chi\varphi_n\alpha := \vartheta_0\vartheta_1^n\chi\alpha$. Then χ is order preserving. Indeed, let us show $\alpha < \beta \to \chi\alpha < \chi\beta$ by induction on $lh(\alpha) + lh(\beta)$. If $\alpha = 0$ and $\beta \neq 0$, then trivially $\chi\alpha < \chi\beta$. Let $\alpha = \varphi_n\alpha' < \beta = \varphi_m\beta'$. If $\alpha' < \beta$ and $n < m$ then the induction hypothesis yields $\chi\alpha' < \vartheta_0\vartheta_1^m\chi\beta'$ and then $n < m$ yields $\chi\alpha = \vartheta_0\vartheta_1^n\chi\alpha' < \vartheta_0\vartheta_1^m\chi\beta' = \chi\beta$. If $n = m$ and $\alpha' < \beta'$ then $\chi\alpha = \vartheta_0\vartheta_1^n\chi\alpha' < \vartheta_0\vartheta_1^n\chi\beta' = \chi\beta$. If $\alpha \leq \beta'$, then $\chi\alpha \leq \chi\beta' < \vartheta_0\vartheta_1^m\chi\beta'$. □

It might be instructive, although it is in fact superfluous, to redo the argument for the standard representation for ω^{ω^ω}. First, we need an additional lemma.

Lemma 7.13. *Let α, β and γ be elements of T.*

1. *$\alpha < \beta < \Omega_1$ and $l_i < n$, $k_i > 0$ for all $i \leq r$ yield* $\vartheta_0^{k_0}\vartheta_1^{l_1}\vartheta_0^{k_1}\dots\vartheta_1^{l_r}\vartheta_0^{k_r}\vartheta_1^n\alpha < \vartheta_0\vartheta_1^n\beta$,

2. *$\alpha < \beta < \Omega_1$ and $l_{ij} < n$, $k_{ij} > 0$ for all i, j yield*
   $\vartheta_0^{k_{00}}\vartheta_1^{l_{01}}\vartheta_0^{k_{01}}\ldots\vartheta_1^{l_{0m_0}}\vartheta_0^{k_{0m_0}}\vartheta_1^{n}\ldots\vartheta_0^{k_{r0}}\vartheta_1^{l_{r1}}\vartheta_0^{k_{r1}}\ldots\vartheta_1^{l_{rm_r}}\vartheta_0^{k_{rm_r}}\vartheta_1^{n}\alpha <$
   $\vartheta_0^{p_{00}}\vartheta_1^{q_{01}}\vartheta_0^{p_{01}}\ldots\vartheta_1^{q_{0s_0}}\vartheta_0^{p_{0s_0}}\vartheta_1^{n}\ldots\vartheta_0^{p_{r0}}\vartheta_1^{q_{r1}}\vartheta_0^{p_{r1}}\ldots\vartheta_1^{q_{rs_r}}\vartheta_0^{p_{rs_r}}\vartheta_1^{n}\beta$*,*

3. *$l_i < n$ and $k_i > 0$ for all $i \leq r$ yield $\vartheta_0^{k_0}\vartheta_1^{l_1}\vartheta_0^{k_1}\ldots\vartheta_1^{l_r}\vartheta_0^{k_r}0 < \vartheta_0\vartheta_1^{n}0$.*

Proof. The first assertion follows by induction on r: If $r = 0$, then $\vartheta_0^{k_0}\vartheta_1^{n}\alpha < \vartheta_0\vartheta_1^{n}\beta$ follows by induction on k_0. If $r > 0$, then the induction hypothesis yields $\xi = \vartheta_0^{k_1}\ldots\vartheta_1^{l_r}\vartheta_0^{k_r}\vartheta_1^{n}\alpha < \vartheta_0\vartheta_1^{n}\beta$. We have $\xi < \vartheta_1^{n-l_1}\beta$ because $k_1 > 0$, and thus $\vartheta_1^{l_1}\xi < \vartheta_1^{n}\beta$. We prove $\vartheta_0^{k_0}\vartheta_1^{l_1}\xi < \vartheta_0\vartheta_1^{n}\beta$ by induction on k_0. First note that we know $k_0(\vartheta_1^{l_1}\xi) = \xi < \vartheta_0\vartheta_1^{n}\beta$, hence the induction base $k_0 = 1$ easily follows. The induction step is straightforward.
The second statement follows from the first by induction on the number of involved blocks.
The third assertion follows by induction on r. □

Theorem 7.14. $\omega^{\omega^{\omega}} = \vartheta_0\vartheta_1\Omega_2$

Proof. Define $\chi : \omega^{\omega^{\omega}} \to \vartheta_0\vartheta_1\Omega_2$ as follows. Take $\alpha < \omega^{\omega^{\omega}}$. Let n be the least number such that $\alpha < \omega^{\omega^{n}}$. Let m then be minimal such that

$$\alpha = \omega^{\omega^{n-1}\cdot m}\cdot\alpha_m + \cdots + \omega^{\omega^{n-1}\cdot 0}\cdot\alpha_0,$$

with $\alpha_m \neq 0$ and $\alpha_0, \ldots, \alpha_m < \omega^{\omega^{n-1}}$. Put $\chi\alpha$ as the element

$$\vartheta_0\vartheta_1^{n}\chi(\alpha_0)\cdots\vartheta_0\vartheta_1^{n}\chi(\alpha_m).$$

It is trivial to see that χ is surjective. We claim that $\alpha < \beta$ yields $\chi(\alpha) < \chi(\beta)$. We prove the claim by induction on $lh(\alpha) + lh(\beta)$.
Let $\alpha = \omega^{\omega^{n-1}\cdot m}\cdot\alpha' + \tilde{\alpha}$ and $\beta = \omega^{\omega^{n'-1}\cdot m'}\cdot\beta' + \tilde{\beta}$ with $\alpha', \beta' > 0$, $\tilde{\alpha} < \omega^{\omega^{n-1}\cdot m}$ and $\tilde{\beta} < \omega^{\omega^{n'-1}\cdot m'}$. If $n < n'$, then $\chi(\beta)$ contains a consecutive sequence of $\vartheta_1^{n'}$ which has no counterpart in $\chi(\alpha)$. Hence, $\chi\alpha < \chi\beta$ follows from a combination of the second and third assertion of the previous lemma. If $n = n'$ and $m < m'$ then $\chi(\beta)$ contains at least one more consecutive sequence of ϑ_1^{n} than the ones occurring in $\chi(\alpha)$. Thus again $\chi\alpha < \chi\beta$ using the second and third assertion of the previous lemma. If $n = n'$ and $m = m'$ and $\alpha' < \beta'$ then the induction hypothesis yields $\chi(\alpha') < \chi(\beta')$. We know $\chi(\alpha) = \chi(\tilde{\alpha})\vartheta_0\vartheta_1^{n}\chi(\alpha')$ and $\chi(\beta) = \chi(\tilde{\beta})\vartheta_0\vartheta_1^{n}\chi(\beta')$. So, the second assertion of the previous lemma yields the assertion. If $n = n'$ and $m = m'$ and $\alpha' = \beta'$ then $\tilde{\alpha} < \tilde{\beta}$ and the induction hypothesis yield $\chi(\tilde{\alpha}) < \chi(\tilde{\beta})$ and $\chi(\alpha) = \chi(\tilde{\alpha})\vartheta_0\vartheta_1^{n}\chi(\alpha')$ and $\chi(\beta) = \chi(\tilde{\beta})\vartheta_0\vartheta_1^{n}\chi(\beta')$. The assertion follows from the sixth assertion of Lemma 1.45. □

7.3.2 The order type of $(T_n[0], <)$ with $n > 2$

As mentioned in the introduction Section 7.1, we first believed that the order type of $T_3[0]$ is equal to the maximal order type of $\overline{\mathbb{S}}_3^{wgap}$, meaning $\omega^{\omega^{\omega^{\omega^{\omega}}}} = \vartheta_0\vartheta_1\vartheta_2\Omega_3$. Surprisingly, calculations of the author showed that this is actually wrong. These calculations were later turned into a formal proof by Weiermann. We now show that $\omega_{n+2} = \vartheta_0\vartheta_1\vartheta_2 \dots \vartheta_n\Omega_{n+1}$, where the latter is the order type of $(T_{n+1}[0], <)$. One can also find these proofs in the joint article [65].

Lower bound

In this subsection, we prove $\omega_{n+2} \le \vartheta_0\vartheta_1\vartheta_2 \dots \vartheta_n\Omega_{n+1}$, where $n \ge 1$.

Definition 7.15. *1. If $\alpha \in T$, define*

$$d_i\alpha := \begin{cases} \vartheta_i\alpha & \text{if } S\alpha \le i, \\ \vartheta_i d_{i+1}\alpha & \text{otherwise.} \end{cases}$$

2. For ordinals in $\pi(\omega)$, define $\bar{\cdot}$ as follows:

- $\overline{0} := 0$,
- $\overline{\pi_i\alpha} := d_{i+1}\overline{\alpha}$.

3. On T, define $0[\beta] := \beta$ and $(\vartheta_i\alpha)[\beta] := \vartheta_i(\alpha[\beta])$.

4. Let ψ be the function from $\varphi_{\pi_0(n)}0$ to T which is defined as follows:

- $\psi 0 := 0$,
- $\psi\varphi_{\pi_0\alpha}\beta := d_0\overline{\alpha}[\psi\beta]$.

It is easy to see that the image of ψ lies in $T_{n+1}[0]$. We show that the function ψ is order-preserving in order to obtain a lower bound for the order type of $T_{n+1}[0]$.

Lemma 7.16. *Let α, β be elements of $\pi(\omega)$ and γ, δ elements of T.*

1. *$\alpha < \beta$ and $\gamma, \delta < \Omega$ yield $\overline{\alpha}[\gamma] < \overline{\beta}[\delta]$,*
2. *$\gamma < \delta < \Omega$ yields $\overline{\alpha}[\gamma] < \overline{\alpha}[\delta]$,*
3. *$G_k\alpha < \beta$ and $\gamma, \delta < \Omega$ yield $k_{k+1}\overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$,*

4. $\alpha < \beta$, $G_k\alpha < \beta$ *and* $\gamma, \delta < \Omega$ *yield* $d_{k+1}\overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$,

5. If $\zeta, \eta \in \varphi_{\pi_0(n)}0$, *then* $\zeta < \eta$ *yields* $\psi\zeta < \psi\eta$.

Proof. We prove assertions 1.–4. simultaneously by induction on $lh(\alpha)$. If $\alpha = 0$, then 1. and 2. are trivial. Assertion 3. is also easy to see because $k_{k+1}\overline{\alpha}[\gamma] = \gamma < \Omega \leq d_{k+1}\overline{\beta}[\delta]$. In assertion 4., $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\gamma$. Now, $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}\zeta$ for a certain $\zeta \geq \Omega$. Therefore, $\gamma < \zeta$ and $k_{k+1}\gamma = \gamma < d_{k+1}\overline{\beta}[\delta]$, which yields $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\gamma < d_{k+1}\overline{\beta}[\delta]$.

From now on, assume $\alpha = \pi_i\alpha'$.

Assertion 1.: $\alpha < \beta$ yields $\beta = \pi_j\beta'$ with $i \leq j$. If $i < j$, then the assertion follows. Assume $i = j$. Then $\alpha' < \beta'$. We know that $G_i(\alpha') < \alpha'$ because $\pi_i\alpha' \in \pi(\omega)$. Assertion 4. and $\alpha' < \beta'$ yield $d_{i+1}\overline{\alpha'}[\gamma] < d_{i+1}\overline{\beta'}[\delta]$, which is $\overline{\alpha}[\gamma] < \overline{\beta}[\delta]$.

Assertion 2.: We know that $G_i(\alpha') < \alpha'$, hence $G_l(\alpha') < \alpha'$ for all $l \geq i$. Assertion 3. then yields $k_{l+1}\overline{\alpha'}[\gamma] < d_{l+1}\overline{\alpha'}[\delta]$ for all $l \geq i$. If $\alpha' = 0$, then assertion 2. easily follows from $\gamma < \delta$. Assume $\alpha' \neq 0$.

If $S(\overline{\alpha'}) \leq i+1$, then $\overline{\alpha}[\gamma] = d_{i+1}\overline{\alpha'}[\gamma] = \vartheta_{i+1}\overline{\alpha'}[\gamma]$. Therefore, assertion 2. follows if $\overline{\alpha'}[\gamma] < \overline{\alpha'}[\delta]$ and $k_{i+1}\overline{\alpha'}[\gamma] < \vartheta_{i+1}\overline{\alpha'}[\delta] = d_{i+1}\overline{\alpha'}[\delta]$. We already know that the second inequality is valid. The first inequality follows from the main induction hypothesis.

Assume now $S(\overline{\alpha'}) > i+1$. We claim that $d_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ for all $j \in \{i+1, \dots, S(\overline{\alpha'})\}$. Assertion 2. then follows from $j = i+1$. We prove our claim by induction on $l = S(\overline{\alpha'}) - j \in \{0, \dots, S(\overline{\alpha'}) - i - 1\}$. If $l = 0$, then $j = S(\overline{\alpha'}) > i+1$. Then the claim follows if $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ and $\overline{\alpha'}[\gamma] < \overline{\alpha'}[\delta]$. The first inequality follows from assertion 3. and the fact that $G_{j-1}(\alpha') < \alpha'$. The second inequality follows from the main induction hypothesis. Now, assume that the claim is true for l. We want to prove that it is true for $l+1 = S(\overline{\alpha'}) - j$. Hence, $l = S(\overline{\alpha'}) - (j+1)$. The induction hypothesis yields $d_{j+1}\overline{\alpha'}[\gamma] < d_{j+1}\overline{\alpha'}[\delta]$. We also see that $j \geq i+1$, so $j-1 \geq i$, hence $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$. Because $S(\overline{\alpha'}) - j = l+1 > 0$, we have $S(\overline{\alpha'}) > j$. Hence, $d_j\overline{\alpha'}[\gamma] = \vartheta_j d_{j+1}\overline{\alpha'}[\gamma]$. The claim follows if $k_j\overline{\alpha'}[\gamma] < d_j\overline{\alpha'}[\delta]$ and $d_{j+1}\overline{\alpha'}[\gamma] < d_{j+1}\overline{\alpha'}[\delta]$, but we already know that both inequalities are true.

Assertion 3.: If $i < k$, then $k_{k+1}\overline{\alpha}[\gamma] = \overline{\alpha}[\gamma] < d_{k+1}\overline{\beta}[\delta]$ because $S(\overline{\alpha}[\gamma]) = i+1 < k+1$.
If $i > k$, then $k_{k+1}\overline{\alpha}[\gamma] = k_{k+1}\overline{\alpha'}[\gamma]$. Therefore, $G_k(\alpha) = G_k(\alpha') \cup \{\alpha'\} < \beta$ and the induction hypothesis yield the assertion.

Assume that $i = k$. Then $k_{k+1}\overline{\alpha}[\gamma] = \overline{\alpha}[\gamma] = d_{k+1}\overline{\alpha'}[\gamma]$ and $G_k(\alpha) = G_k(\alpha') \cup \{\alpha'\} < \beta$. The induction hypothesis on assertion 4. yields $d_{k+1}\overline{\alpha'}[\gamma] < d_{k+1}\overline{\beta}[\delta]$, from which we can conclude the assertion.

Assertion 4.: $\alpha < \beta$ yields $\beta = \pi_j\beta'$ with $i \leq j$.

If $i + 1 = S(\overline{\alpha}) \leq k + 1$, then $d_{k+1}\overline{\alpha}[\gamma] = \vartheta_{k+1}\overline{\alpha}[\gamma]$. There are two subcases: either $j + 1 = \overline{\beta}[\delta] \leq k + 1$ or not. In the former case, we obtain $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}\overline{\beta}[\delta]$. Assertion 4. then follows from assertions 1. and 3. and the induction hypothesis. In the latter case, we have $d_{k+1}\overline{\beta}[\delta] = \vartheta_{k+1}d_{k+2}\overline{\beta}[\delta]$. Assertion 4. follows from $\overline{\alpha}[\gamma] < d_{k+2}\overline{\beta}[\delta]$ and assertion 3. The previous strict inequality is valid because $S(\overline{\alpha}[\gamma]) = i + 1 \leq k + 1 < k + 2$.

From now on assume that $i + 1 = S(\overline{\alpha}) > k + 1$. Actually, we only assume that $S(\overline{\alpha}) \geq k$.

$G_k\alpha < \beta$ yields $G_l\alpha < \beta$ for all $l \geq k$. We claim that $d_{j+1}\overline{\alpha}[\gamma] < d_{j+1}\overline{\beta}[\delta]$ for all $j \in \{k, \ldots, S(\overline{\alpha})\}$ and show this by induction on $l = S(\overline{\alpha}) - j \in \{0, \ldots, S(\overline{\alpha}) - k\}$. The assertion then follows from taking $l = S(\overline{\alpha}) - k$.

If $l = 0$ or $l = 1$, then $S(\overline{\alpha}) = k$ or equals $k + 1$, hence the claim follows from the case $S(\overline{\alpha}) \leq k + 1$. Assume that the claim is true for $l \geq 1$. We want to prove that this is also true for $l + 1 = S(\overline{\alpha}) - j$. The induction hypothesis on $l = S(\overline{\alpha}) - (j + 1)$ yields $d_{j+2}\overline{\alpha}[\gamma] < d_{j+2}\overline{\beta}[\delta]$. Now because $l \geq 1$, we have $S(\overline{\beta}) \geq S(\overline{\alpha}) \geq j + 2 > j + 1$. So, $d_{j+1}\overline{\alpha}[\gamma] = \vartheta_{j+1}d_{j+2}\overline{\alpha}[\gamma]$ and $d_{j+1}\overline{\beta}[\delta] = \vartheta_{j+1}d_{j+2}\overline{\beta}[\delta]$. Then the claim is valid if $d_{j+2}\overline{\alpha}[\gamma] < d_{j+2}\overline{\beta}[\delta]$ and $k_{j+1}\overline{\alpha}[\gamma] < d_{j+1}\overline{\beta}[\delta]$. We already know the first strict inequality. The second one follows from assertion 3. and $j \geq k$.

Assertion 5.: We prove this by induction on $lh(\zeta) + lh(\eta)$. Assume $\zeta = \varphi_{\pi_0\alpha}\gamma < \varphi_{\pi_0\beta}\delta = \eta$. There are three cases.

Case 1: $\pi_0\alpha < \pi_0\beta$ and $\gamma < \eta$. The induction hypothesis yields $\psi(\gamma) < \psi(\eta)$. Furthermore, we know that $\alpha < \beta$. If $\alpha = 0$, then $d_0\overline{\alpha}[\psi(\gamma)] = \vartheta_0\psi(\gamma)$. We want to check if this is strictly smaller than $\psi(\eta) = d_0\overline{\beta}[\psi(\delta)] = \vartheta_0 d_1\overline{\beta}[\psi(\delta)]$. Trivially $\psi(\gamma) < d_1\overline{\beta}[\psi(\delta)]$. Furthermore, $k_0(\psi(\gamma)) = \psi(\gamma) < \psi(\eta)$. Hence $\psi(\zeta) = \vartheta_0\psi(\gamma) < \vartheta_0 d_1\overline{\beta}[\psi(\delta)] = \psi(\eta)$. Assume now $0 < \alpha < \beta$. We want to prove that

$$\begin{aligned} d_0\overline{\alpha}[\psi(\gamma)] &= \vartheta_0 d_1\overline{\alpha}[\psi(\gamma)] \\ &< d_0\overline{\beta}[\psi(\delta)] = \vartheta_0 d_1\overline{\beta}[\psi(\delta)]. \end{aligned}$$

Assertion 4., $\alpha < \beta$ and $G_0(\alpha) < \alpha < \beta$ yield $d_1\overline{\alpha}[\psi(\gamma)] < d_1\overline{\beta}[\psi(\delta)]$. Additionally,

$$k_0 d_1 \overline{\alpha}[\psi(\gamma)] = \psi(\gamma) < \psi(\eta) = \vartheta_0 d_1 \overline{\beta}[\psi(\delta)],$$

hence $d_0\overline{\alpha}[\psi(\gamma)] < d_0\overline{\beta}[\psi(\delta)]$.

Case 2: $\pi_0\alpha = \pi_0\beta$ and $\gamma < \delta$. The induction hypothesis yields $\psi(\gamma) < \psi(\delta)$. Assertion 2. on $\pi_0\alpha$ then yields $\overline{\pi_0\alpha}[\psi(\gamma)] < \overline{\pi_0\alpha}[\psi(\delta)]$. Hence, $d_1\overline{\alpha}[\psi(\gamma)] < d_1\overline{\alpha}[\psi(\delta)] = d_1\overline{\beta}[\psi(\delta)]$. Additionally,

$$k_0 d_1 \overline{\alpha}[\psi(\gamma)] = \psi(\gamma) < \psi(\delta) = k_0(d_1\overline{\beta}[\psi(\delta)]) \leq \vartheta_0(d_1\overline{\beta}[\psi(\delta)]),$$

hence $d_0\overline{\alpha}[\psi(\gamma)] < d_0\overline{\beta}[\psi(\delta)]$.

Case 3.: $\pi_0\alpha > \pi_0\beta$ and $\zeta < \delta$. Then $\psi(\zeta) < \psi(\delta) \leq k_0(d_1\overline{\beta}[\psi(\delta)]) \leq \vartheta_0(d_1\overline{\beta}[\psi(\delta)]) = \psi(\eta)$. □

Corollary 7.17. $\omega_{n+2} \leq \vartheta_0\vartheta_1 \ldots \vartheta_n\Omega_{n+1}$

Proof. From Theorems 1.25 and 1.39, we know that the order type of $\varphi_{\pi_0(n)}0$ is ω_{n+2}. Therefore, using assertion 5 in Lemma 7.16, we obtain $\omega_{n+2} \leq otype(T_{n+1}[0]) = \vartheta_0 \ldots \vartheta_n\Omega_{n+1}$. □

Upper bound

In this subsection, we prove $\vartheta_0\vartheta_1\vartheta_2 \ldots \vartheta_n\Omega_{n+1} = otype(T_{n+1}[0]) \leq \omega_{n+2}$. For this purpose, we introduce a new notation system T'_n with the same order type as T_n.

Definition 7.18. *Let $n < \omega$. Define T'_{n+1} as the least subset of T_{n+1} such that*

- $0 \in T'_{n+1}$,
- *if $\alpha \in T'_{n+1}$, $S\alpha = i+1$ and $i < n$, then $\vartheta_i\alpha \in T'_{n+1}$,*
- *if $\alpha \in T'_{n+1}$, then $\vartheta_n\alpha \in T'_{n+1}$.*

Note that for all $\alpha \in T'_{n+1}$, we have $S\alpha \leq n$. Let T'_0 be $\{0\}$ and define $T'_n[m]$ accordingly as $T_n[m]$.

Lemma 7.19. *The order types of T'_n and T_n are equal.*

Proof. Trivially, $T'_n \subseteq T_n$, hence $otype(T'_n) \leq otype(T_n)$. Now, we give an order-preserving function ψ from T_n to T'_n. If $n = 0$, this function appears trivially. So assume $n = m + 1 > 0$.

$$\begin{array}{rlll} \psi: & T_{m+1} & \to & T'_{m+1}, \\ & 0 & \mapsto & 0, \\ & \vartheta_i\alpha & \mapsto & \vartheta_i\vartheta_{i+1}\dots\vartheta_m\psi(\alpha). \end{array}$$

Let us first prove the following claim: for all $i \leq m$, if $\psi(\xi) < \psi(\zeta) < \Omega_{i+1} = \vartheta_{i+1}0$, then $\psi(\vartheta_i\xi) < \psi(\vartheta_i\zeta)$. We prove this claim by induction on $m-i$. $i = m$, then $\psi(\vartheta_m\xi) = \vartheta_m\psi(\xi)$ and $\psi(\vartheta_m\zeta) = \vartheta_m\psi(\zeta)$. Hence, $\psi(\vartheta_m\xi) < \psi(\vartheta_m\zeta)$ easily follows because $k_m(\psi(\xi)) = \psi(\xi) < \psi(\zeta) = k_m(\psi(\zeta)) < \vartheta_m(\psi(\zeta))$. Let $i < m$. Then

$$\psi(\vartheta_i\xi) = \vartheta_i\dots\vartheta_m\psi(\xi),$$
$$\psi(\vartheta_i\zeta) = \vartheta_i\dots\vartheta_m\psi(\zeta).$$

Using the induction hypothesis, we obtain $\psi(\vartheta_{i+1}\xi) = \vartheta_{i+1}\dots\vartheta_m\psi(\xi) < \psi(\vartheta_{i+1}\zeta) = \vartheta_{i+1}\dots\vartheta_m\psi(\zeta)$. Furthermore, $k_i(\vartheta_{i+1}\dots\vartheta_m\psi(\xi)) = k_i(\psi(\xi)) = \psi(\xi) < \psi(\zeta) = k_i(\psi(\zeta)) = k_i(\vartheta_{i+1}\dots\vartheta_m\psi(\zeta)) < \vartheta_i(\vartheta_{i+1}\dots\vartheta_m\psi(\zeta))$. Hence, $\psi(\vartheta_i\xi) = \vartheta_i\dots\vartheta_m\psi(\xi) < \psi(\vartheta_i\zeta) = \vartheta_i\dots\vartheta_m\psi(\zeta)$. This finishes the proof of the claim.

Now we prove by main induction on $lh(\alpha) + lh(\beta)$ that $\alpha < \beta$ yields $\psi(\alpha) < \psi(\beta)$. If $\alpha = 0$, then the claim trivially holds. Assume $0 < \alpha < \beta$. Then $\alpha = \vartheta_i\alpha'$ and $\beta = \vartheta_j\beta'$. If $i < j$, then $\psi(\alpha) < \psi(\beta)$ is also trivial. Assume $i = j \leq m$ and let $\alpha' = \vartheta_{j_1}\dots\vartheta_{j_k}k_i\alpha'$ and $\beta' = \vartheta_{n_1}\dots\vartheta_{n_l}k_i\beta'$ with $j_1,\dots,j_k,n_1,\dots,n_l > i$. $\alpha < \beta$ either yields $\alpha \leq k_i\beta'$ or $\alpha' < \beta'$ and $k_i\alpha' < \beta$. In the former case, the induction hypothesis yields $\psi(\alpha) \leq \psi(k_i\beta') = k_i(\psi(\vartheta_{n_1}\dots\vartheta_{n_l}k_i\beta')) = k_i(\psi(\beta')) = k_i(\vartheta_{i+1}\dots\vartheta_m\psi(\beta'))$
$< \vartheta_i(\vartheta_{i+1}\dots\vartheta_m\psi(\beta')) = \psi(\beta)$.

Assume that we are in the latter case, meaning $\alpha' < \beta'$ and $k_i\alpha' < \beta$. The induction hypothesis yields $\psi\alpha' < \psi\beta'$ and $\psi(k_i\alpha') < \psi\beta$. Like before, we attain $\psi(k_i\alpha') = k_i(\vartheta_{i+1}\dots\vartheta_m\psi(\alpha')) < \psi\beta = \vartheta_i(\vartheta_{i+1}\dots\vartheta_m\psi(\beta'))$. So if we can prove $\vartheta_{i+1}\dots\vartheta_m\psi(\alpha') < \vartheta_{i+1}\dots\vartheta_m\psi(\beta')$, we are done. But this follows from the claim: if $i = j < m$, then $S(\alpha'), S(\beta') \leq i+1 \leq m$, hence $\psi(\alpha') < \psi(\beta') < \Omega_{i+2}$, so $\vartheta_{i+1}\dots\vartheta_m\psi(\alpha') = \psi(\vartheta_{i+1}\alpha') < \psi(\vartheta_{i+1}\beta') = \vartheta_{i+1}\dots\vartheta_m\psi(\beta')$. If $i = j = m$, then $\vartheta_{i+1}\dots\vartheta_m\psi(\alpha')$ and $\vartheta_{i+1}\dots\vartheta_m\psi(\beta')$ are actually $\psi(\alpha')$ and $\psi(\beta')$ and we know that $\psi(\alpha') < \psi(\beta')$ holds. □

The previous proof also yields that the order types of $T'_n[m]$ and $T_n[m]$ are equal.

The instructive part of the upper bound: $\vartheta_0\vartheta_1\vartheta_2\Omega_3 \le \omega^{\omega^{\omega^{\omega}}}$

In this subsection, we prove that $\omega^{\omega^{\omega^{\omega}}}$ is an upper bound for $\vartheta_0\vartheta_1\vartheta_2\Omega_3$ as an instructive instance for the general case

$$\vartheta_0\vartheta_1\vartheta_2\ldots\vartheta_n\Omega_{n+1} = otype(T_{n+1}[0]) \le \omega_{n+2}.$$

We will show this by proving that $otype(T_3'[0]) \le \omega^{\omega^{\omega^{\omega}}}$. We start with two simple lemmata, where we interpret Ω_i as usual as the i^{th} uncountable cardinal number for $i > 0$.

Lemma 7.20. *If $\Omega_2 \cdot \alpha + \beta < \Omega_2 \cdot \gamma + \delta$ and $\alpha, \gamma < \varepsilon_0$ and $\beta, \delta < \Omega_2$ and if $\beta = \xi \cdot \beta'$ where $\beta' < \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$ and $\xi < \omega^{\omega^\gamma}$, then $\Omega_1 \cdot \omega^\alpha + \omega^{\omega^\alpha} \cdot \beta < \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$.*

Proof. Note that it is possible that $\beta, \delta \ge \Omega_1$. If $\alpha = \gamma$ then $\beta < \delta$ and the assertion is obvious. So assume $\alpha < \gamma$. $\beta' < \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$ yields $\beta = \xi\beta' < \xi(\Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta) = \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$ since Ω_1 and ω^{ω^γ} are multiplicatively closed. By the same argument $\omega^{\omega^\alpha}\beta < \omega^{\omega^\alpha}(\Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta) = \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$. Finally, $\Omega_1 \cdot \omega^\alpha + \omega^{\omega^\alpha} \cdot \beta < \Omega_1 \cdot \omega^\alpha + \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta = \Omega_1 \cdot \omega^\gamma + \omega^{\omega^\gamma} \cdot \delta$. □

Lemma 7.21. *If $\Omega_1 \cdot \alpha + \beta < \Omega_1 \cdot \gamma + \delta$ and $\alpha, \gamma < \varepsilon_0$ and $\beta, \delta < \Omega_1$ and if $\beta < \omega^{\omega^\gamma} \cdot \delta$, then $\omega^{\omega^\alpha} \cdot \beta < \omega^{\omega^\gamma} \cdot \delta$.*

Proof. If $\alpha = \gamma$, then $\beta < \delta$ and the assertion is obvious. So assume $\alpha < \gamma$. Then $\omega^{\omega^\alpha} \cdot \beta < \omega^{\omega^\alpha}\omega^{\omega^\gamma} \cdot \delta = \omega^{\omega^\gamma} \cdot \delta$. □

The last two lemmas indicate how one might replace iteratively terms in ϑ_i (starting with the highest level i) by terms in $\omega, +, \Omega_i$ in an order-preserving way such that terms of level 0 are smaller than ϵ_0.

Definition 7.22. *Define E as the least set such that*

- $0 \in E$,
- *if $\alpha \in E$, then $\omega^\alpha \in E$,*
- *if $\alpha, \beta \in E$, then $\alpha + \beta \in E$.*

Define the subset P of E as the set of all elements of the form ω^α for $\alpha \in E$. This actually means that P is the set of the additively closed ordinals strictly below ε_0.

A crucial role is played by the following function f.

Definition 7.23. *Let* $f(0) := 0$ *and* $f(\omega^{\alpha_1} + \alpha_2) := \omega^{\alpha_1} + f(\alpha_1) + f(\alpha_2)$.

This definition even works (by magic) also for non Cantor normal forms. So if $\omega^{\alpha_1} + \alpha_2 = \alpha_2$ we still have $f(\omega^{\alpha_1} + \alpha_2) = \omega^{\alpha_1} + f(\alpha_1) + f(\alpha_2)[= f(\alpha_2)]$. The function f is easily shown to be order-preserving. Moreover, one finds $\omega^{\alpha_1} \leq f(\omega^{\alpha_1} + \alpha_2) < \omega^{\alpha_1+1}$ if $\alpha_2 < \omega^{\alpha_1+1}$.

Fix a natural number n. We formally work with 4-tuples $(\alpha, \beta, \gamma, \delta) \in E \times T[n-1] \times P \times E$ with $\alpha, \delta \in E$, $\gamma \in P$, $\beta \in T[n-1]$ and $\delta < \gamma$. Let $T[-1] := \{0\}$. We order these tuples lexicographically. Intuitively, we interpret such a tuple as the ordinal

$$\Omega_n \cdot \alpha + \gamma \cdot \beta + \delta,$$

where Ω_i is as usual the i^{th} uncountable ordinal for $i > 0$, but now Ω_0 is interpreted as 0.

We note that the interpretation of $(\alpha, \beta, \gamma, \delta)$ as an ordinal number is not entirely correct: the lexicographic order on the tuples is not the same as the induced order by the ordering on the class of ordinals On. But in almost all applications, we know that $\gamma = \omega^{f(\alpha)}$. And if this true, we know that the order induced by the ordering on On is the same as the defined lexicographic one. Additionally, the encountered cases where $\gamma \neq \omega^{f(\alpha)}$, we know that if we compare two tuples $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ such that $\alpha = \alpha'$, then we already know that $\gamma = \gamma'$. Hence, the order induced by the ordering on On between these terms is also the same as the lexicographic one.

β is either 0 or of the form $\vartheta_j \beta'$ with $j < n$, hence we can interpret that $\beta < \Omega_n$ for $n > 0$. Assume that $\zeta \in P$. Then we know that $\zeta \cdot \Omega_n = \Omega_n$. Hence using all of these interpretations, $\zeta \cdot (\alpha, \beta, \gamma, \delta)$ is still a 4-tuple, namely it is equal to $(\alpha, \beta, \zeta \cdot \gamma, \zeta \cdot \delta)$. We can also define the sum between 4-tuples: assume $n > 0$. If $\alpha' > 0$, then

$$\begin{aligned}
(\alpha, \beta, \gamma, \delta) + (\alpha', \beta', \gamma', \delta') &= \Omega_n \cdot \alpha + \gamma \cdot \beta + \delta + \Omega_n \cdot \alpha' + \gamma' \cdot \beta' + \delta' \\
&= \Omega_n \cdot (\alpha + \alpha') + \gamma' \cdot \beta' + \delta' \\
&= (\alpha + \alpha', \beta', \gamma', \delta')
\end{aligned}$$

If $\alpha' = 0$ and $\beta' = 0$, then

$$\begin{aligned}
(\alpha, \beta, \gamma, \delta) + (\alpha', \beta', \gamma', \delta') &= \Omega_n \cdot \alpha + \gamma \cdot \beta + \delta + \Omega_n \cdot \alpha' + \gamma' \cdot \beta' + \delta' \\
&= \Omega_n \alpha + \gamma \cdot \beta + (\delta + \delta') \\
&= (\alpha, \beta, \gamma, \delta + \delta')
\end{aligned}$$

We do not need the case $\alpha' = 0$ and $\beta' \neq 0$. If $n = 0$, then

$$\begin{aligned}(\alpha, \beta, \gamma, \delta) + (\alpha', \beta', \gamma', \delta') &= \Omega_n \cdot \alpha + \gamma \cdot \beta + \delta + \Omega_n \cdot \alpha' + \gamma' \cdot \beta' + \delta' \\ &= \delta + \delta' \\ &= (0, 0, 0, \delta + \delta')\end{aligned}$$

From now on, we write

$$\Omega_n \cdot \alpha + \gamma \cdot \beta + \delta,$$

instead of the 4-tuple $(\alpha, \beta, \gamma, \delta)$, although we know that the induced order by the ordering on On is not entirely the same as the lexicographic one.

Definition 7.24. *Define T_n^{all} as the set consisting of $\Omega_n \cdot \alpha + \omega^{f(\alpha)} \cdot \delta + \gamma$, where $\alpha, \gamma \in E$ with $\gamma < \omega^{f(\alpha)}$ and $\delta \in T[n-1]$.*

Note that after an obvious interpretation, $T_0^{all} = E$ and $T_n \subseteq T[n-1] \subseteq T_n^{all}$.

Lemma 7.25. *Assume $\alpha', \beta' \in T[0]$. If*

$$\alpha = \vartheta_1 \vartheta_2^{n_1} \dots \vartheta_1 \vartheta_2^{n_p} \alpha' < \beta = \vartheta_1 \vartheta_2^{l_1} \dots \vartheta_1 \vartheta_2^{l_q} \beta'$$

with $n_i, l_i > 0$, then

$$\begin{aligned}&\Omega_1 \cdot (\omega^{n_1} + \dots + \omega^{n_p}) + \omega^{\omega^{n_1} + \dots + \omega^{n_p} + n_p} \cdot \alpha' + \omega^{\omega^{n_1} + \dots + \omega^{n_p}} \\ &+ \omega^{\omega^{n_1} + \dots + \omega^{n_{p-1}}} + \dots + \omega^{\omega^{n_1}} \\ &< \Omega_1 \cdot (\omega^{l_1} + \dots + \omega^{l_q}) + \omega^{\omega^{l_1} + \dots + \omega^{l_q} + l_q} \cdot \beta' + \omega^{\omega^{l_1} + \dots + \omega^{l_q}} \\ &+ \omega^{\omega^{l_1} + \dots + \omega^{l_{q-1}}} + \dots + \omega^{\omega^{l_1}}.\end{aligned}$$

Proof. Note that $f(\omega^{n_1} + \dots + \omega^{n_p}) = \omega^{n_1} + \dots + \omega^{n_p} + n_p$ and that $\omega^{n_1} + \dots + \omega^{n_p}$ is not necessarily in Cantor normal form. We prove by induction on $lh(\alpha) - lh(\alpha') + lh(\beta) - lh(\beta')$ that the assumption yields

$$\begin{aligned}&(\omega^{n_1} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1}) \\ <_{lex}\ &(\omega^{l_1} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1}).\end{aligned}$$

From this inequality, the lemma follows.

If $lh(\alpha) = lh(\alpha')$, then $p = 0$. If $q > 0$, then this is trivial, so we can assume that q is also 0. But then $\omega^{n_1} + \dots + \omega^{n_p} = \omega^{l_1} + \dots + \omega^{l_q} = 0$ and $\alpha' = \alpha < \beta = \beta'$. Now assume that $p > 0$. It is impossible that $q = 0$. $\alpha < \beta$ yields either $\vartheta_1\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_2}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ or $\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ and $\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$.

In the former case, the induction hypothesis yields

$$\begin{aligned}&(\omega^{n_1}+\dots+\omega^{n_p},\alpha',\omega^{n_1}+\dots+\omega^{n_{p-1}},\dots,\omega^{n_1})\\ <_{lex}\ &(\omega^{l_2}+\dots+\omega^{l_q},\beta',\omega^{l_2}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_2}).\end{aligned}$$

If $l_2 \le l_1$, then trivially

$$\begin{aligned}&(\omega^{l_2}+\dots+\omega^{l_q},\beta',\omega^{l_2}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_2},\omega^{l_1})\\ <_{lex}\ &(\omega^{l_1}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}+\omega^{l_2},\omega^{l_1}).\end{aligned}$$

If $l_2 > l_1$, then

$$\begin{aligned}&(\omega^{l_2}+\dots+\omega^{l_q},\beta',\omega^{l_2}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_2})\\ =\ &(\omega^{l_1}+\omega^{l_2}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\omega^{l_2}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}+\omega^{l_2})\\ <_{lex}\ &(\omega^{l_1}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}+\omega^{l_2},\omega^{l_1}).\end{aligned}$$

Assume that we are in the latter case. $\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ yields $n_1 < l_1$ or $n_1 = l_1$ and $\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_2}\dots\vartheta_1\vartheta_2^{l_q}\beta'$.

Suppose $n_1 < l_1$. The induction hypothesis on

$$\vartheta_1\vartheta_2^{n_2}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$$

implies

$$\begin{aligned}&(\omega^{n_2}+\dots+\omega^{n_p},\alpha',\omega^{n_2}+\dots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ <_{lex}\ &(\omega^{l_1}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}).\end{aligned}$$

Let

$$\begin{aligned}s &:= (\omega^{n_2}+\dots+\omega^{n_p},\alpha',\omega^{n_2}+\dots+\omega^{n_{p-1}},\dots,\omega^{n_2})\\ s' &:= (\omega^{l_1}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}).\end{aligned}$$

Note that $lh(s) = p$ and $lh(s') = q+1$. If $lh(s) < lh(s')$ and $s_i = s'_i$ for all $i < lh(s)$, then

$$\begin{aligned}&(\omega^{n_1}+\dots+\omega^{n_p},\alpha',\omega^{n_1}+\dots+\omega^{n_{p-1}},\dots,\omega^{n_1})\\ =\ &(\omega^{n_2}+\dots+\omega^{n_p},\alpha',\omega^{n_2}+\dots+\omega^{n_{p-1}},\dots,\omega^{n_2},\omega^{n_1})\\ <_{lex}\ &(\omega^{l_1}+\dots+\omega^{l_q},\beta',\omega^{l_1}+\dots+\omega^{l_{q-1}},\dots,\omega^{l_1}),\end{aligned}$$

where for the last inequality we need $n_1 < l_1$ if $p = q$. If there exists an index $j < \min\{lh(s), lh(s')\}$ such that $s_j < s'_j$ and $s_i = s'_i$ for all $i < j$, then

$$\begin{aligned}&(\omega^{n_1} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1})\\&(\omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_2} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_2}, \omega^{n_1})\\<_{lex}\ &(\omega^{l_1} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1}).\end{aligned}$$

Now assume $n_1 = l_1$. The induction hypothesis on $\vartheta_1\vartheta_2^{n_2} \dots \vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_2} \dots \vartheta_1\vartheta_2^{l_q}\beta'$ implies

$$\begin{aligned}&(\omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_2} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_2})\\<_{lex}\ &(\omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_2}).\end{aligned}$$

Let

$$\begin{aligned}s &:= (\omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_2} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_2})\\s' &:= (\omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_2} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_2}).\end{aligned}$$

Note that $lh(s) = p$ and $lh(s') = q$. If $lh(s) < lh(s')$ and $s_i = s'_i$ for all $i < lh(s)$, then one can easily prove

$$\begin{aligned}&(\omega^{n_1} + \omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1} + \omega^{n_2})\\<_{lex}\ &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}),\end{aligned}$$

hence

$$\begin{aligned}&(\omega^{n_1} + \omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1} + \omega^{n_2}, \omega^{n_1})\\<_{lex}\ &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2})\\<_{lex}\ &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}, \omega^{l_1}).\end{aligned}$$

If there exists an index $j < \min\{lh(s), lh(s')\}$ such that $s_j < s'_j$ and $s_i = s'_i$ for all $i < j$, then also

$$\begin{aligned}&(\omega^{n_1} + \omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1} + \omega^{n_2})\\<_{lex}\ &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}),\end{aligned}$$

hence

$$\begin{aligned}&(\omega^{n_1} + \omega^{n_2} + \dots + \omega^{n_p}, \alpha', \omega^{n_1} + \dots + \omega^{n_{p-1}}, \dots, \omega^{n_1} + \omega^{n_2}, \omega^{n_1})\\<_{lex}\ &(\omega^{l_1} + \omega^{l_2} + \dots + \omega^{l_q}, \beta', \omega^{l_1} + \dots + \omega^{l_{q-1}}, \dots, \omega^{l_1} + \omega^{l_2}, \omega^{l_1}).\end{aligned}$$

□

Define τ_0 as the mapping from $T'_3[0]$ to $T_0^{all} = E$ as follows: let $\tau_0 0 := 0$. If $\alpha = \vartheta_0\vartheta_1\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha'$ with $\alpha' \in T'_3[0]$ and $n_1,\dots,n_p,p > 0$, define $\tau_0\alpha$ as

$$\omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\dots+\omega^{n_p}}+\omega^{\omega^{n_1}+\dots+\omega^{n_{p-1}}}+\dots+\omega^{\omega^{n_1}}).$$

Lemma 7.26. *Assume $\alpha,\beta \in T'_3[0]$. If $\alpha < \beta$, then $\tau_0\alpha < \tau_0\beta$.*

Proof. We prove this by induction on the length of α and β. If $\alpha = 0$, then this is trivial. So we can assume that $0 < \alpha < \beta$. Hence,

$$\alpha = \vartheta_0\vartheta_1\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha'$$

and

$$\beta = \vartheta_0\vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$$

with $\alpha',\beta' \in T'_3[0]$ and $n_1,\dots,n_p,l_1,\dots,l_q,p,q > 0$.

We want to prove that

$$\begin{aligned}&\tau_0\alpha = \omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\dots+\omega^{n_p}}+\dots+\omega^{\omega^{n_1}})\\ &< \tau_0\beta = \omega^{\omega^{\omega^{l_1}+\dots+\omega^{l_q}}}\cdot(\omega^{\omega^{l_1}+\dots+\omega^{l_q}+l_q}\cdot\tau_0\beta'+\omega^{\omega^{l_1}+\dots+\omega^{l_q}}+\dots+\omega^{\omega^{l_1}}).\end{aligned}$$

$\alpha = \vartheta_0\vartheta_1\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \beta = \vartheta_0\vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ yields two cases: either $\alpha \leq k_0(\vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta') = \beta'$ or $\vartheta_1\vartheta_2^{n_1}\dots\vartheta_1\vartheta_2^{n_p}\alpha' < \vartheta_1\vartheta_2^{l_1}\dots\vartheta_1\vartheta_2^{l_q}\beta'$ and $\alpha' < \beta$. In the former case, the induction hypothesis yields $\tau_0\alpha \leq \tau_0\beta' < \tau_0\beta$.

So assume the latter case. Then the induction hypothesis yields $\tau_0\alpha' < \tau_0\beta$. Using Lemma 7.25, we know that

$$\begin{aligned}&\Omega_1\cdot(\omega^{n_1}+\dots+\omega^{n_p})+\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\dots+\omega^{n_p}}\\ &\quad+\omega^{\omega^{n_1}+\dots+\omega^{n_{p-1}}}+\dots+\omega^{\omega^{n_1}}\\ &<\Omega_1\cdot(\omega^{l_1}+\dots+\omega^{l_q})+\omega^{\omega^{l_1}+\dots+\omega^{l_q}+l_q}\cdot\tau_0\beta'+\omega^{\omega^{l_1}+\dots+\omega^{l_q}}\\ &\quad+\omega^{\omega^{l_1}+\dots+\omega^{l_{q-1}}}+\dots+\omega^{\omega^{l_1}}.\end{aligned}$$

If $\omega^{n_1}+\dots+\omega^{n_p} < \omega^{l_1}+\dots+\omega^{l_q}$, then

$$\omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha' < \omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\tau_0\beta = \tau_0\beta.$$

Therefore,

$$\begin{aligned}&\omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'+\omega^{\omega^{n_1}+\dots+\omega^{n_p}}+\dots+\omega^{\omega^{n_1}})\\ &<\omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot\omega^{\omega^{n_1}+\dots+\omega^{n_p}+n_p}\cdot\tau_0\alpha'\\ &\quad+\omega^{\omega^{\omega^{n_1}+\dots+\omega^{n_p}}}\cdot(\omega^{\omega^{n_1}+\dots+\omega^{n_p}}+\dots+\omega^{\omega^{n_1}})\\ &<\tau_0\beta,\end{aligned}$$

because $\omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}} \cdot (\omega^{\omega^{n_1}+\cdots+\omega^{n_p}} + \cdots + \omega^{\omega^{n_1}}) < \omega^{\omega^{\omega^{l_1}+\cdots+\omega^{l_q}}}$. We used the standard observation that $\xi < \rho + \omega^{\mu}$ and $\lambda < \mu$ imply $\xi + \omega^{\lambda} < \rho + \omega^{\mu}$.

Assume $\omega^{n_1} + \cdots + \omega^{n_p} = \omega^{l_1} + \cdots + \omega^{l_q}$ and $\tau_0\alpha' < \tau_0\beta'$. Then $\tau_0\alpha < \omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}} \cdot \omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p} \cdot (\tau_0\alpha'+1) \leq \omega^{\omega^{\omega^{n_1}+\cdots+\omega^{n_p}}} \cdot \omega^{\omega^{n_1}+\cdots+\omega^{n_p}+n_p} \cdot \tau_0\beta' \leq \tau_0\beta$.

Assume $\omega^{n_1} + \cdots + \omega^{n_p} = \omega^{l_1} + \cdots + \omega^{l_q}$, $\tau_0\alpha' = \tau_0\beta'$ and $\omega^{\omega^{n_1}+\cdots+\omega^{n_p}} + \omega^{\omega^{n_1}+\cdots+\omega^{n_{p-1}}} + \cdots + \omega^{\omega^{n_1}} < \omega^{\omega^{l_1}+\cdots+\omega^{l_q}} + \omega^{\omega^{l_1}+\cdots+\omega^{l_{q-1}}} + \cdots + \omega^{\omega^{l_1}}$. Then trivially, $\tau_0\alpha < \tau_0\beta$.

□

Proving the upper bound in general: $\vartheta_0 \ldots \vartheta_n\Omega_{n+1} \leq \omega_{n+2}$

We show that $otype(T'_{n+1}[0]) \leq \omega_{n+2}$. The previous section give us the idea of how to deal with this question, however the order-preserving embeddings in this subsection are slightly different than the ones proposed in the previous Subsection 7.3.2 for technical reasons. Fix a natural number n strictly bigger than 0.

Definition 7.27. *τ_m are functions from $T'_{n+1}[m]$ to T^{all}_m. We define $\tau_m\alpha$ for all m simultaneously by induction on the length of α. If $m \geq n+1$, then $T'_{n+1}[m] = T'_{n+1}$ and define $\tau_m\alpha = \alpha = \Omega_m 0 + \omega^0\alpha + 0$ for all α. Note that $\alpha \in T'_{n+1} \subseteq T[n] \subseteq T[m-1]$. Assume $m \leq n$. Define $\tau_m 0$ as 0. Define $\tau_m\vartheta_j\alpha$ as $\vartheta_j\alpha$ if $j < m$. Define $\tau_m\vartheta_m\alpha$ as $\Omega_m\omega^{\beta} + \omega^{\omega^{\beta}}(\omega^{f(\beta)} \cdot \tau_m k_m\alpha + \eta) + 1$ if $\tau_{m+1}\alpha = \Omega_{m+1}\beta + \omega^{f(\beta)}k_m\alpha + \eta$.*

First we prove that τ_m is well-defined.

Lemma 7.28. *For all $m > 0$ and $\alpha \in T'_{n+1}[m]$, there exist uniquely determined β and η with $\eta < \omega^{f(\beta)}$ such that $\tau_m\alpha = \Omega_m\beta + \omega^{f(\beta)}k_{m-1}\alpha + \eta$. Furthermore, η is either zero or a successor.*

Proof. We prove the first claim by induction on $lh(\alpha)$ and $n+1-m$. If $m \geq n+1$, then this is trivial by definition. Assume $0 < m \leq n$. From the induction hypothesis, we know that there exist β, η, β_1, η_1 such that $\tau_{m+1}\alpha = \Omega_{m+1}\beta + \omega^{f(\beta)}k_m\alpha + \eta$ with $\eta < \omega^{f(\beta)}$ and $\tau_m k_m\alpha = \Omega_m\beta_1 + \omega^{f(\beta_1)}k_{m-1}k_m\alpha + \eta_1$ with $\eta_1 < \omega^{f(\beta_1)}$. We want to prove that there exist β' and η' such that

$\tau_m\vartheta_m\alpha = \Omega_m\beta' + \omega^{f(\beta')}k_{m-1}\vartheta_m\alpha + \eta'$ with $\eta' < \omega^{f(\beta')}$. Using the definition,

$$\begin{aligned}
&\tau_m\vartheta_m\alpha \\
= \;& \Omega_m\omega^\beta + \omega^{\omega^\beta}(\omega^{f(\beta)}\cdot\tau_m k_m\alpha + \eta) + 1 \\
= \;& \Omega_m\omega^\beta + \omega^{\omega^\beta}(\omega^{f(\beta)}\cdot(\Omega_m\beta_1 + \omega^{f(\beta_1)}k_{m-1}k_m\alpha + \eta_1) + \eta) + 1 \\
= \;& \Omega_m(\omega^\beta + \beta_1) + \omega^{\omega^\beta}\omega^{f(\beta)}(\omega^{f(\beta_1)}k_{m-1}k_m\alpha + \eta_1) + \omega^{\omega^\beta}\eta + 1 \\
= \;& \Omega_m(\omega^\beta + \beta_1) + \omega^{\omega^\beta}\omega^{f(\beta)}\omega^{f(\beta_1)}k_{m-1}k_m\alpha + \omega^{\omega^\beta}\omega^{f(\beta)}\eta_1 + \omega^{\omega^\beta}\eta + 1 \\
= \;& \Omega_m(\omega^\beta + \beta_1) + \omega^{f(\omega^\beta+\beta_1)}k_{m-1}k_m\alpha + \omega^{\omega^\beta}\omega^{f(\beta)}\eta_1 + \omega^{\omega^\beta}\eta + 1 \\
= \;& \Omega_m(\omega^\beta + \beta_1) + \omega^{f(\omega^\beta+\beta_1)}k_{m-1}\vartheta_m\alpha + \omega^{\omega^\beta}\omega^{f(\beta)}\eta_1 + \omega^{\omega^\beta}\eta + 1.
\end{aligned}$$

Define β' as $\omega^\beta + \beta_1 > 0$ and η' as $\omega^{\omega^\beta}\omega^{f(\beta)}\eta_1 + \omega^{\omega^\beta}\eta + 1$. Note that $\omega^{\omega^\beta}\omega^{f(\beta)}\eta_1 < \omega^{\omega^\beta}\omega^{f(\beta)}\omega^{f(\beta_1)} = \omega^{f(\beta')}$, $\omega^{\omega^\beta}\eta < \omega^{\omega^\beta+f(\beta)} \leq \omega^{f(\beta')}$ and $1 < \omega^{f(\beta')}$, hence $\eta' < \omega^{f(\beta')}$.

That η is either zero or a successor for all m and α follows by construction.

□

The argument in the proof of Lemma 7.28 is crucially based on the property of f regarding non-normal forms. The lemma implies that τ_m is well-defined for all $m > 0$ and it does not make sense for $m = 0$ because we did not define $k_{-1}\alpha$. But, looking to the definition of τ_0, it is easy to see that τ_0 is also well-defined.

Note that one can easily prove $\tau_0\alpha \in T_0^{all}$ for all $\alpha \in T'_{n+1}[0]$. Furthermore, $\tau_0\alpha$ is also either zero or a successor ordinal. For all m and α, define $(\tau_m\alpha)^-$ as $\tau_m\alpha$, if η is zero, and as $\tau_m\alpha$ but with $\eta - 1$ instead of η, if η is a successor. Additionally, note that if $m > 0$ and $\tau_m\alpha = \Omega_m\beta + \omega^{f(\beta)}k_{m-1}\alpha + \eta$ we have $\beta > 0$ iff $\eta > 0$.

In the next theorem, we will again use the standard observation that $\xi < \rho + \omega^\mu$ and $\lambda < \mu$ imply $\xi + \omega^\lambda < \rho + \omega^\mu$.

Theorem 7.29. *For all natural m and $\alpha, \beta \in T'_{n+1}[m]$, if $\alpha < \beta$, then $\tau_m\alpha < \tau_m\beta$.*

Proof. We prove this theorem by induction on $lh\alpha + lh\beta$. If α and/or β are zero, this is trivial. So we can assume that $\alpha = \vartheta_i\alpha'$ and $\beta = \vartheta_j\beta'$. One can easily prove the statement if $i < j$, even if $j = m$. So we can assume that $i = j$. If $i = j < m$, then this is also easily proved. So suppose that

$i = j = m$. If $m > n$, then $\tau_m\alpha = \alpha < \beta = \tau_m\beta$, hence we are done. So we can also assume that $m \leq n$.

$\alpha = \vartheta_m\alpha' < \vartheta_m\beta'$ yields $\alpha \leq k_m\beta'$ or $\alpha' < \beta'$ and $k_m\alpha' < \beta$. In the former case, the induction hypothesis yields $\tau_m\alpha \leq \tau_m k_m\beta' < \tau_m\vartheta_m\beta' = \tau_m\beta$, where $\tau_m k_m\beta' < \tau_m\vartheta_m\beta'$ follows from the definition of $\tau_m\vartheta_m\beta'$. (One can also look at the proof of Lemma 7.28 for $m > 0$. The case $m = 0$ is straightforward.) So we only have to prove the assertion in the latter case, i.e. if $\alpha' < \beta'$ and $k_m\alpha' < \beta$. The induction hypothesis yields $\tau_{m+1}\alpha' < \tau_{m+1}\beta'$ and $\tau_m k_m\alpha' < \tau_m\beta$. Assume

$$\begin{aligned}\tau_{m+1}\alpha' =&\Omega_{m+1} \cdot \alpha_1 + \omega^{f(\alpha_1)} \cdot k_m\alpha' + \alpha_2,\\ \tau_{m+1}\beta' =&\Omega_{m+1} \cdot \beta_1 + \omega^{f(\beta_1)} \cdot k_m\beta' + \beta_2,\end{aligned}$$

where $\alpha_2 < \omega^{f(\alpha_1)}$, $\beta_2 < \omega^{f(\beta_1)}$. Then

$$\begin{aligned}\tau_m\alpha =&\Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} \cdot \tau_m k_m\alpha' + \alpha_2) + 1,\\ \tau_m\beta =&\Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m\beta' + \beta_2) + 1.\end{aligned}$$

The inequality $\tau_{m+1}\alpha' < \tau_{m+1}\beta'$ yields $\alpha_1 \leq \beta_1$. Assume first that $\alpha_1 = \beta_1$. Then $\tau_{m+1}\alpha' < \tau_{m+1}\beta'$ yields $k_m\alpha' \leq k_m\beta'$. If $k_m\alpha' = k_m\beta'$, then $\alpha_2 < \beta_2$ and $\tau_m\alpha < \tau_m\beta$. If $k_m\alpha' < k_m\beta'$ then the induction hypothesis yields $\tau_m k_m\alpha' < \tau_m k_m\beta'$ and $\omega^{f(\alpha_1)} \cdot \tau_m k_m\alpha' + \alpha_2 < \omega^{f(\alpha_1)} \cdot \tau_m k_m\beta' + \beta_2$, since $\alpha_2 < \omega^{f(\alpha_1)}$. We then find that $\tau_m\alpha < \tau_m\beta$. So we may assume that $\alpha_1 < \beta_1$.

Case 1: $k_m\alpha' < \vartheta_m 0$. Then $\tau_m k_m\alpha' = k_m\alpha'$. Hence,

$$\begin{aligned}\tau_m\alpha &= \Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} \cdot k_m\alpha' + \alpha_2) + 1\\ &< \Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m\beta' + \beta_2) + 1\\ &= \tau_m\beta\end{aligned}$$

follows in a straightforward way.

Case 2: $k_m\alpha' \geq \vartheta_m 0$. Using the definition, we then have $(\tau_m k_m\alpha')^- + 1 = \tau_m k_m\alpha'$. We show that

$$\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m\alpha')^- + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1 < (\tau_m\beta)^-$$

holds, hence

$$\begin{aligned}\tau_m\alpha &= \Omega_m \cdot \omega^{\alpha_1} + \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m\alpha')^- + \omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1\\ &< \Omega_m \cdot \omega^{\alpha_1} + (\tau_m\beta)^-\\ &= (\tau_m\beta)^-\\ &< \tau_m\beta.\end{aligned}$$

We know $\tau_m k_m \alpha' < \tau_m \beta$, hence

$$(\tau_m k_m \alpha')^- < (\tau_m \beta)^- = \Omega_m \cdot \omega^{\beta_1} + \omega^{\omega^{\beta_1}}(\omega^{f(\beta_1)} \cdot \tau_m k_m \beta' + \beta_2).$$

Therefore, $\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m k_m \alpha')^- < \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} \cdot (\tau_m \beta)^- = (\tau_m \beta)^-$ because $\omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} = \omega^{f(\omega^{\alpha_1})}$ and $f(\omega^{\alpha_1}) < \omega^{\alpha_1+1} \leq \omega^{\beta_1}$.

The last term in the normal form of $\omega^{\omega^{\beta_1}} \cdot \beta_2$ is bigger than $\omega^{\omega^{\beta_1}}$. Note that $\tau_{m+1}\beta' = \Omega_{m+1} \cdot \beta_1 + \omega^{f(\beta_1)} \cdot k_m \beta' + \beta_2$. The observation just before this theorem yields $\beta_2 > 0$ otherwise β_1 is zero, a contradiction (because $\beta_1 > \alpha_1$). So if

$$\omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1 < \omega^{\omega^{\beta_1}},$$

we can finish the proof by the standard observation $\xi < \rho + \omega^\mu$ and $\lambda < \mu$ imply $\xi + \omega^\lambda < \rho + \omega^\mu$.

Now,

$$\begin{aligned} &\omega^{\omega^{\alpha_1}}(\omega^{f(\alpha_1)} + \alpha_2) + 1 \\ &= \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} + \omega^{\omega^{\alpha_1}}\alpha_2 + 1 \\ &< \omega^{\omega^{\beta_1}} \end{aligned}$$

because $\omega^{\omega^{\alpha_1}}\alpha_2 < \omega^{\omega^{\alpha_1}}\omega^{f(\alpha_1)} = \omega^{f(\omega^{\alpha_1})}$ and $f(\omega^{\alpha_1}) < \omega^{\alpha_1+1} \leq \omega^{\beta_1}$.

□

Lemma 7.30. *For all $\alpha \in T'_{n+1}[m+1]$ we have that if $\tau_{m+1}\alpha = \Omega_{m+1}\beta + \omega^{f(\beta)}k_m\alpha + \eta$, then*

$$\begin{cases} \beta < \omega^0 = \omega_0 & \text{if } m \geq n, \\ \beta < \omega_{n-m} & \text{if } m < n. \end{cases}$$

Proof. We prove this by induction. If $m \geq n$, then $\tau_{m+1}\alpha = \Omega_{m+1}0 + \omega^0\alpha$, hence we are done. Assume $m < n$. If $\alpha = \vartheta_j\alpha'$ with $j < m+1$, then $\beta = 0 < \omega_{n-m}$. Assume $\alpha = \vartheta_{m+1}\alpha'$. Assume $\tau_{m+2}\alpha' = \Omega_{m+2}\beta' + \omega^{f(\beta')}k_{m+1}\alpha' + \eta'$ and $\tau_{m+1}k_{m+1}\alpha' = \Omega_{m+1}\beta_1 + \omega^{f(\beta_1)}k_m k_{m+1}\alpha' + \eta_1$. From the induction hypothesis, we know $\beta' < \omega_{n-m-1}$ and $\beta_1 < \omega_{n-m}$. Then

$$\begin{aligned} &\tau_{m+1}\alpha \\ &= \Omega_{m+1}\omega^{\beta'} + \omega^{\omega^{\beta'}}(\omega^{f(\beta')}(\Omega_{m+1}\beta_1 + \omega^{f(\beta_1)}k_m k_{m+1}\alpha' + \eta_1) + \eta') + 1 \\ &= \Omega_{m+1}\omega^{\beta'} + \omega^{\omega^{\beta'}}\omega^{f(\beta')}(\Omega_{m+1}\beta_1 + \omega^{f(\beta_1)}k_m\alpha' + \eta_1) + \omega^{\omega^{\beta'}}\eta' + 1 \\ &= \Omega_{m+1}(\omega^{\beta'} + \beta_1) + \omega^{\omega^{\beta'}}\omega^{f(\beta')}(\omega^{f(\beta_1)}k_m\alpha' + \eta_1) + \omega^{\omega^{\beta'}}\eta' + 1. \end{aligned}$$

Now, $\omega^{\beta'} + \beta_1 < \omega_{n-m}$. □

Lemma 7.31. *Let $n \geq 1$. For all $\alpha \in T'_{n+1}[0]$ we have that $\tau_0\alpha < \omega_{n+2}$.*

Proof. We prove this by induction on $lh(\alpha)$. If $\alpha = 0$, this is trivial. Assume $\alpha \in T'_{n+1}[0]$, meaning $\alpha = \vartheta_0\alpha'$ with $\alpha' \in T'_{n+1}[1]$. Assume $\tau_1\alpha' = \Omega_1\beta' + \omega^{f(\beta')}k_0\alpha' + \eta'$ with $\eta' < \omega^{f(\beta')}$. Using Lemma 7.30, we know that $\beta' < \omega_{n-0} = \omega_n$. Additionally, the induction hypothesis yields $\tau_0 k_0\alpha' < \omega_{n+2}$. Now,

$$\tau_0\vartheta_0\alpha' = \omega^{\omega^{\beta'}}(\omega^{f(\beta')}\tau_0 k_0\alpha' + \eta') + 1.$$

From the definition of f, one obtains that $f(\beta') \leq \beta' \cdot \omega$. Hence, $\omega^{f(\beta')}\tau_0 k_0\alpha' + \eta' < \omega^{f(\beta')}(\tau_0 k_0\alpha' + 1) < \omega_{n+2}$, so $\tau_0\vartheta_0\alpha' < \omega_{n+2}$. □

Corollary 7.32. *$otype(T'_{n+1}) \leq \omega_{n+2}$.*

Proof. By Theorem 7.29, τ_0 is an order preserving embedding from $T'_{n+1}[0]$ to $T_0^{all} = E$. Furthermore, from Lemma 7.31, we know $\tau_0\alpha < \omega_{n+2}$ for all $\alpha \in T'_{n+1}[0]$. Hence $otype(T'_{n+1}) \leq \omega_{n+2}$. □

Corollary 7.33. *$\vartheta_0\vartheta_1 \ldots \vartheta_n\Omega_{n+1} \leq \omega_{n+2}$.*

Proof. By Lemma 7.19, we know

$$\vartheta_0\vartheta_1 \ldots \vartheta_n\Omega_{n+1} = otype(T_{n+1}[0]) = otype(T'_{n+1}[0]),$$

hence the previous corollary yields $\vartheta_0\vartheta_1 \ldots \vartheta_n\Omega_{n+1} \leq \omega_{n+2}$. □

7.3.3 Binary ϑ-functions

We were not able to give a positive answer to the following question: *Does there exist a suitable choice of unary functions that realizes a maximal linear extension of $\overline{\mathbb{S}}_n^{wgap}$?*. However, if we allow binary functions, this is possible. For the sake of completeness, we show this here, although the proofs are due to Weiermann.

Let n be a fixed non-negative integer. We also use the notation T_n, however this is different than the previous one.

Definition 7.34. *Let T_n be the least set such that the following holds. On T_n, define S and K_i.*

1. $0 \in T_n$, $S0 := -1$, $K_i0 := \emptyset$,

2. *if* $\alpha, \beta \in T_n$, $S\alpha \leq i+1$ *and* $S\beta \leq i < n$, *then* $\overline{\theta}_i\alpha\beta \in T_n$, $S\overline{\theta}_i\alpha\beta := i$ *and*
$$K_j\overline{\theta}_i\alpha\beta := \begin{cases} K_j\alpha \cup K_j\beta & \text{if } j < i, \\ \{\overline{\theta}_i\alpha\beta\} & \text{otherwise.} \end{cases}$$

Note that all indices in T_n are strictly smaller than n.

Definition 7.35. *For* $\overline{\theta}_i\alpha\beta, \overline{\theta}_i\gamma\delta \in T_n$, *define* $\overline{\theta}_i\alpha\beta < \overline{\theta}_i\gamma\delta$ *iff either* $i < j$ *or* $i = j$ *and one of the following alternatives holds:*

- $\alpha < \gamma$ & $K_i\alpha \cup \{\beta\} < \overline{\theta}_j\gamma\delta$,
- $\alpha = \gamma$ & $\beta < \delta$,
- $\alpha > \gamma$ & $\overline{\theta}_i\alpha\beta \leq K_i\gamma \cup \{\delta\}$.

Let $0 < \overline{\theta}_i\alpha\beta$ *for all* $\overline{\theta}_i\alpha\beta \in T_n \backslash \{0\}$.

Here $\overline{\theta}_i\alpha\beta \leq K_i\gamma \cup \{\delta\}$ means that $\overline{\theta}_i\alpha\beta \leq \xi$ for every $\xi \in K_i\gamma \cup \{\delta\}$.

Lemma 7.36. *For* $\overline{\theta}_i\alpha\beta \in T_n$, *we have* $\beta < \overline{\theta}_i\alpha\beta$.

Proof. This can be proven by induction on $lh(\beta)$. □

Definition 7.37. *Define* $OT_n \subseteq T_n$ *as follows.*

1. $0 \in OT_n$,
2. *if* $\alpha, \beta \in OT_n$, $S\alpha \leq i+1$, $S\beta \leq i < n$ *and* $K_i\alpha = \emptyset$, *then* $\overline{\theta}_i\alpha\beta \in OT_n$

Note that $K_i\alpha = \emptyset$ yields that α does not contain any $\overline{\theta}_j$ for $j \leq i$.

Definition 7.38. *If* $K_0\alpha = \emptyset$, *let* α^- *be the result of replacing every occurence of* $\overline{\theta}_i$ *by* $\overline{\theta}_{i-1}$.

Lemma 7.39. *If* $\alpha < \beta$ & $K_0\alpha = K_0\beta = \emptyset$, *then* $\alpha^- < \beta^-$ *and* $(K_{i+1}\alpha)^- = K_i\alpha^-$.

Proof. This can be proven in a straightforward way by induction on $lh(\alpha) + lh(\beta)$. □

Therefore, if $\overline{\theta}_i\alpha\beta \in OT_n$, then α^- is defined and it is an element of OT_{n-1}. Additionally, if $i = 0$, then $S(\alpha^-), S(\beta) \leq 0$.

Definition 7.40. *Define* $OT_n[0]$ *as* $OT_n \cap \Omega_1$, *where* $\Omega_1 := \overline{\theta}_0 00$

Definition 7.41. *Define* $o_1 : OT_1[0] \to \omega$ *as follows. An arbitrary element of* OT_1 *is of the form* $\overline{\theta}_0(0, \overline{\theta}_0(0, \ldots \overline{\theta}_0(0,0)\ldots))$. *Define the image of this element under* o_1 *as* k *if* $\overline{\theta}_0(\cdot,\cdot)$ *occurs* k *many times. Define* $o_n : OT_n[0] \to \omega_{2n-1}$ *for* $n > 1$ *as follows.*

1. $o_n(0) := 0$,
2. $o_n(\overline{\theta}_0\alpha\beta) := \varphi_{o_{n-1}(\alpha^-)}o_n(\beta)$.

Note that $S(\alpha^-), S(\beta) \leq 0$ if $\overline{\theta}_0\alpha\beta \in OT_n[0]$.

Theorem 7.42. *For every* $n \geq 1$, o_n *is order-preserving and surjective.*

Proof. The surjectivity of o_n is easy to prove. We prove that o_n is order-preserving. If $n = 1$, this is trivial. Assume $n > 1$ and assume that o_{n-1} is order preserving. We will show that for all $\alpha, \beta \in OT_n[0]$, $\alpha < \beta$ yields $o_n(\alpha) < o_n(\beta)$. If α and/or β are equal to zero, this is trivial. Assume $0 < \alpha < \beta$. Let $\alpha = \overline{\theta}_0\alpha_1\alpha_2$ and $\beta = \overline{\theta}_0\beta_1\beta_2$. Then $\alpha < \beta$ iff one of the following cases holds:

1. $\alpha_1 < \beta_1$ and $\alpha_2 < \overline{\theta}_0\beta_1\beta_2$,
2. $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$,
3. $\alpha_1 > \beta_1$ and $\overline{\theta}_0\alpha_1\alpha_2 \leq \beta_2$.

Note that $\alpha_1 < \beta_1$ yields $\alpha_1^- < \beta_1^-$ by Lemma 7.39, hence $o_{n-1}(\alpha_1^-) < o_{n-1}(\beta_1^-)$. Furthermore, the induction hypothesis yields that the previous case i. is equivalent with the following case i. for all i.

1. $o_{n-1}\alpha_1^- < o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\overline{\theta}_0\beta_1\beta_2$,
2. $o_{n-1}\alpha_1^- = o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\beta_2$,
3. $o_{n-1}\alpha_1^- > o_{n-1}\beta_1^-$ and $o_n\overline{\theta}_0\alpha_1\alpha_2 \leq o_n\beta_2$.

Hence the above case i. is equivalent with the following case i.:

1. $o_{n-1}\alpha_1^- < o_{n-1}\beta_1^-$ and $o_n\alpha_2 < \varphi_{o_{n-1}\beta_1^-}o_n\beta_2$,
2. $o_{n-1}\alpha_1^- = o_{n-1}\beta_1^-$ and $o_n\alpha_2 < o_n\beta_2$,
3. $o_{n-1}\alpha_1^- > o_{n-1}\beta_1^-$ and $\varphi_{o_{n-1}\alpha_1^-}o_n\alpha_2 \leq o_n\beta_2$.

This is actually the definition of $\varphi_{o_{n-1}\alpha_1^-}o_n\alpha_2 < \varphi_{o_{n-1}\beta_1^-}o_n\beta_2$, so $o_n\overline{\theta}_0\alpha_1\alpha_2 < o_n\overline{\theta}_0\beta_1\beta_2$.

□

This yields the following corollary.

Corollary 7.43. *$otype(OT_n[0]) = \omega_{2n-1}$ if $n \geq 1$.*

This ordinal notation system corresponds to a maximal linear extension of $\overline{\mathbb{S}}^s_n[0] = \overline{\mathbb{S}}^w_n[0]$.

Definition 7.44. *Define f from $\overline{\mathbb{S}}^s_n$ to OT_n as follows. $f(\varepsilon) := 0$ if ε is the empty sequence. $f(ii_1 \dots i_k j\vec{s}) := \overline{\theta}_i(f(i_1 \dots i_k))(f(j\vec{s}))$ if $i < i_1, \dots, i_k$ and $j \leq i$. This yields that $f(i)$ is defined as $\overline{\theta}_i(0,0)$.*

Lemma 7.45. *OT_n is a linear extension of $\overline{\mathbb{S}}^s_n$.*

Proof. We prove by induction on the length of s and t that $s \leq^s_{gap} t$ yields $f(s) \leq f(t)$. If s and/or t are ε, then this is trivial. Assume not, then $s = ii_1 \dots i_k j\vec{s'}$ and $t = pp_1 \dots p_r q\vec{t'}$ with $i_1, \dots, i_k > i \geq j$ and $p_1, \dots, p_r > p \geq q$. If $i < p$, then $f(s) \leq f(t)$ is trivial. Furthermore, $s \leq^s_{gap} t$ yields that $i > p$ is impossible. Therefore we can assume that $i = p$. If the first i of s is mapped into $q\vec{t'}$ according to the inequality $s \leq^s_{gap} t$, then $i = q$ and $s \leq^s_{gap} q\vec{t'}$, hence $f(s) \leq f(qt')$. From Lemma 7.36, we know $f(q\vec{t'}) < f(t)$, hence we are done. Assume that the first i of s is mapped onto the first $i = p$ of t according to the $s \leq^s_{gap} t$ inequality. Then $j\vec{s'} \leq^s_{gap} q\vec{t'}$ and $i_1 \dots i_k \leq^s_{gap} p_1 \dots p_r$. The induction hypothesis yields $f(j\vec{s'}) \leq f(q\vec{t'})$ and $f(i_1 \dots i_k) \leq f(p_1 \dots p_r)$. If $f(i_1 \dots i_k) = f(p_1 \dots p_r)$, then $f(s) \leq f(t)$ follows from $f(j\vec{s'}) \leq f(q\vec{t'})$. If $f(i_1 \dots i_k) < f(p_1 \dots p_r)$, then $f(s) \leq f(t)$ follows from $f(j\vec{s'}) \leq f(q\vec{t'}) < f(t)$ and $K_i(f(i_1 \dots i_k)) = \emptyset$.

□

Corollary 7.46. *$OT_n[0]$ is a maximal linear extension of $\overline{\mathbb{S}}^w_n[0] = \overline{\mathbb{S}}^s_n[0]$.*

Proof. The previous lemma yields that $OT_n[0]$ is a linear extension of $\overline{\mathbb{S}}_n[0]$. We also know that $otype(OT_n[0]) = \omega_{2n-1} = o(\overline{\mathbb{S}}_n[0])$. □

In a sequel project, we intend to determine the relationship between other ordinal notation systems *without* addition with the systems studied here. More specifically, we intend to look at ordinal diagrams [79], Gordeev-style ordinal notation systems [34] and non-iterated ϑ-functions [14, 91].

7.4 From a reverse mathematical point of view

The subsection explores the reverse mathematical strength of a statement about the sequences with the gap-embeddability relation. Well-ordering principles, i.e.

$$\forall X(WO(X) \rightarrow WO(f(X))),$$

are interesting statements that are quite often strong enough to prove axioms of specific theories over a fixed base theory. For example if $f(X) = \omega^X$, then this well-ordering principle yields ACA_0 over RCA_0 (see Lemma 1.63). A similar observation is also valid for $\mathbb{wpo}$'s: a well-partial-ordering principle of the form $\forall X$('X is a $\mathbb{wpo}$' $\rightarrow$ '$f(X)$ is a $\mathbb{wpo}$') is normally strong enough to imply a specific theory in the context of reverse mathematics over a fixed base theory. For example, if $f(X)$ is the Higman order on X, then this statement also implies ACA_0 (again, see Lemma 1.63).

We are interested in the reverse mathematical strength of sequences with the gap-embeddability relation. We only have the true statement $\forall n$('$\mathbb{S}_n^{wgap}$ is a $\mathbb{wpo}$'), but this assertion is too weak to imply a theory in the context of reverse mathematics over RCA_0. So we have to generalize this statement to arbitrary $\mathbb{wpo}$'s. If one considers X^* with the gap-embeddability relation instead of $\mathbb{S}_n = \{0, \dots, n-1\}^*$ (with X a $\mathbb{wpo}$), the statement is not true in general. So we have to modify the statement $\forall n$('$\mathbb{S}_n^{wgap}$ is a $\mathbb{wpo}$') in a different way. The proposed form was stated by Keita Yokoyama in a private communication with the author. More precisely, Keita Yokoyama wondered what the reverse mathematical strength is of the statement

$$\forall n \forall Q \forall \eta \left[(WPO(Q) \wedge \eta : Q \rightarrow \{0, \dots, n\}) \rightarrow WPO(Q^*, \leq^{\eta, gap}) \right].$$

Here, the $\mathbb{wpo}$ $(Q^*, \leq^{\eta,gap})$ has the following ordering:

$$\begin{aligned}
& (q_1, \dots, q_k) \leq^{\eta,gap} (q'_1, \dots, q'_l) \\
\iff\ & \exists 1 \leq i_1 < \dots < i_k \leq l \text{ such that } q_j \leq_Q q'_{i_j} \text{ for every } j, \\
& (\eta(q_1), \dots, \eta(q_k)) = (\eta(q'_{i_1}), \dots, \eta(q'_{i_k})), \\
& \text{and for all } j = 1, \dots, k-1\text{: if } i_j < r < i_{j+1}, \text{ then } \eta(q'_r) \geq \eta(q'_{i_{j+1}}).
\end{aligned}$$

As you can see, this ordering uses a labeling function η. On Q^*, we can define the strong gap-embeddability relation $\leq^{\eta,sgap}$ in the same way, but we have to add the following line:

$$\text{if } r < i_1, \text{ then } \eta(q'_r) \geq \eta(q'_{i_1}).$$

Although, one can define a $\mathbb{wpo}$ in many different ways (see Lemma 1.49), we have to fix one definition because the different versions of the definition are not computably equivalent, meaning that they are not equivalent over the base theory RCA_0. For more information on this subject from a reverse mathematical point of view, we refer to [17, 30]. We use the usual definition (see Definition 1.46).

Before we answer Yokoyama's question, it is also worth to mention other results from the literature. For example, Gordeev [36] investigated a related question. He introduced a new gap-embeddability relation (called the symmetric gap-ordering) on the sequences X^* such that X^* with this ordering is now a $\mathbb{wpo}$. Therefore, he could investigate the reverse mathematical strength of the, now true, statement $\forall X$('X is a $\mathbb{wpo}$' $\to$ 'X^* with the symmetric gap-ordering is a $\mathbb{wpo}$'). This turned out to be the theory ATR_0. Also in this context, we should say that he also explored the symmetric gap-condition on trees. For more information, see [35, 36].

Let us go back to the question raised by Keita Yokoyama. Let Q be a partial ordering. Define on $(Q \times \{0, \dots, n\})^*$ the following ordering:

$$
\begin{aligned}
& ((q_1, n_1), \dots, (q_k, n_k)) \leq^{*,gap} ((q'_1, m_1), \dots, (q'_l, m_l)) \\
\iff & \exists 1 \leq i_1 < \dots < i_k \leq l \text{ such that } q_j \leq_Q q'_{i_j} \text{ for every } j, \\
& (n_1, \dots, n_k) = (m_{i_1}, \dots, m_{i_k}), \\
& \text{and for all } j = 1, \dots, k-1\text{: if } i_j < r < i_{j+1}, \text{ then } m_r \geq m_{i_{j+1}}.
\end{aligned}
$$

Define $\leq^{*,sgap}$ (strong gap-embeddability relation) in the same way, but add the following line:

$$\text{if } r < i_1, \text{ then } m_r \geq m_{i_1}.$$

Let us recall Definition 1.62.

Definition 7.47. *Let X be a linear order. Define ω^X as the subset of X^* such that $(x_0, \dots, x_{n-1}) \in \omega^X$ if $x_0 \geq_X \dots \geq_X x_{n-1}$. Define the ordering on ω^X as the lexicographic one: $(x_0, \dots, x_{n-1}) \leq_{\omega^X} (y_0, \dots, y_{m-1})$ if either $n \leq m$ and $x_i = y_i$ for all $i \leq n$, or there exists a $j < \min n, m$ such that $x_j <_X y_j$ and $x_i = y_i$ for all $i < j$.*

Definition 7.48. *Define for a linear order X, $\omega^{<0,X>}$ as X and $\omega^{<n+1,X>}$ as the linear order $\omega^{\omega^{<n,X>}}$. For an ordinal or well-order α, we sometimes write $\omega_n(\alpha)$ instead of $\omega^{<n,\alpha>}$.*

A tree in the reverse mathematical setting is defined as a subset T of $\mathbb{S} = \mathbb{N}^*$ such that any initial segment of a finite sequence in T also belongs to T. The empty sequence denotes the root of T. If X is a countable wpo, we encode X and $\leq_X$ as a subset of the natural numbers. Therefore, we can talk about the tree of the finite bad sequences $Bad(X)$ in reverse mathematics. More specifically, if X is a countable wpo, we can define $Bad(X)$ in RCA_0. A path in a tree is defined by a function $f : \mathbb{N} \to \mathbb{N}$ such that $\forall n(f[n] \in T)$, where $f[n]$ is (the code for) the finite sequence $(f(0), \ldots, f(n-1))$.

On $\mathbb{S}$, define the Kleene/Brouwer ordering as follows (see [77]): $\sigma \leq_{KB} \tau$ iff either τ is an initial segment of σ or

$$\exists j < \min(lh(\sigma), lh(\tau))(\sigma_j < \tau_j \text{ and } \forall i < j(\sigma_i = \tau_i)).$$

If $T \subseteq \mathbb{S}$ is a tree, define $KB(T)$ as the linear order $(T, \leq_{KB}\restriction (T \times T))$. The definitions of $\leq_{KB}$ and $KB(T)$ can be taken care of in RCA_0. Hence, if X is a countable wpo, we can define $KB(\mathrm{Bad}(X))$ in a decent way over RCA_0. A very useful lemma of the Kleene/Brouwer ordering states that for an arbitrary tree T, $KB(T)$ is well-ordered if and only if T is a well-founded tree (a tree is well-founded if it does not have an infinite path).

Lemma 7.49. *The following is provable in* ACA_0. *Let* $T \subseteq \mathbb{S}$ *be a tree. Then*

$$KB(T) \text{ is a well-order} \Leftrightarrow \forall f \exists n(f[n] \notin T).$$

Proof. See Lemma V.1.3 in [77]. □

Recall that if there exists a reification f from $\mathrm{Bad}(X)$ to a well-ordering, than one can prove that X is a wpo. If X is a wpo, one can naturally define a reification from $Bad(X)$ to the well-ordering $KB(\mathrm{Bad}(X))$ (if we can use the theory ACA_0).

Theorem 7.50. *The following are equivalent over* RCA_0*:*

1. ACA_0',
2. $\forall n \forall X(WO(X) \to WO(\omega^{<n,X>}))$,
3. $\forall n \forall Q \forall \eta\, [(WPO(Q) \wedge \eta : Q \to \{0, \ldots, n\}) \to WPO(Q^*, \leq^{\eta,gap})]$,
4. $\forall n \forall Q \forall \eta\, [(WPO(Q) \wedge \eta : Q \to \{0, \ldots, n\}) \to WPO(Q^*, \leq^{\eta,sgap})]$,
5. $\forall n \forall Q\, [WPO(Q) \to WPO((Q \times \{0, \ldots, n\})^*, \leq^{*,gap})]$.
6. $\forall n \forall Q\, [WPO(Q) \to WPO((Q \times \{0, \ldots, n\})^*, \leq^{*,sgap})]$.

Proof. It is well-known that 1. and 2. are equivalent. See [53] for a proof. The rest of the proof comes down to the determination of the maximal order types of the occurring wpo's, or at least determining upper and lower bounds for the maximal order types. We do not present sharp upper and lower bounds. The techniques are based on the article of Schütte and Simpson [72].

5. implies 3. and 6. implies 4.: is easy by looking to sequences of the form $((q_1, \eta(q_1)), \dots, (q_k, \eta(q_k)))$.

4. implies 3. and 6. implies 5.: trivial.

3. implies 2.: Fix a well-ordering X. We now recursively construct quasi-embeddings e_n from $\omega^{<n,X>}$ into the partial ordering $((\{0, \dots, n-1\} + X)^*, \leq^{\eta_n, gap})$ with $\eta_n(i) = i$ and $\eta_n(x) = n$ for $x \in X$. If we have such e_n's, we obtain that $\omega^{<n,X>}$ is a well-ordering using 3.

If $n = 0$, define $e_0(x)$ as (x) for every $x \in X = \omega^{<0,X>}$. It is trivial to see that e_0 is a quasi-embedding. Now, let $n = m+1$ and assume that we have a quasi-embedding e_m from $\omega^{<m,X>}$ into $((\{0, \dots, m-1\} + X)^*, \leq^{\eta_m, gap})$ with $\eta_m(i) = i$ and $\eta_m(x) = m$ for $x \in X$.

Take $\alpha < \omega^{<m+1,X>} = \omega^{\omega^{<m,X>}}$. Then $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_k}$ with $\omega^{<m,X>} > \alpha_1 \geq \dots \geq \alpha_k$. Therefore, one can assume that we have $e_m(\alpha_i) \in (\{0, \dots, m-1\} + X)^*$. Define $(e_m(\alpha_i))^+$ as the sequence in $(\{1, \dots, m\} + X)^*$, where an element j is replaced by $j+1$ for $j = 0, \dots, m-1$. Define $e_{m+1}(\alpha)$ as

$$\overline{0}(e_m(\alpha_1))^+\overline{0} \dots \overline{0}(e_m(\alpha_k))^+\overline{0},$$

which is an element of partial ordering $(\{0, \dots, m\} + X)^*$. We want to prove that e_{m+1} is a quasi-embedding from $\omega^{<m+1,X>}$ into $((\{0, \dots, m\} + X)^*, \leq^{\eta_{m+1}, gap})$ with $\eta_{m+1}(i) = i$ and $\eta_{m+1}(x) = m+1$ for $x \in X$. So take $\alpha = \omega^{\alpha_1} + \dots + \omega^{\alpha_k} < \omega^{<m+1,X>}$ and $\beta = \omega^{\beta_1} + \dots + \omega^{\beta_l} < \omega^{<m+1,X>}$ and assume that

$$\overline{0}(e_m(\alpha_1))^+\overline{0} \dots \overline{0}(e_m(\alpha_k))^+\overline{0} \leq^{\eta_{m+1}, gap} \overline{0}(e_m(\beta_1))^+\overline{0} \dots \overline{0}(e_m(\beta_l))^+\overline{0}.$$

$\eta_{m+1}(0) = 0$ and $\eta_{m+1}(a_i) > 0$ for all elements a_i in $(e_m(\alpha_j))^+$. Therefore, there exists $1 \leq i_1 < \dots < i_k \leq l$ such that $(e_m(\alpha_j))^+ \leq^{\eta_{m+1}, gap} (e_m(\beta_{i_j}))^+$ for every j. Hence $e_m(\alpha_j) \leq^{\eta_m, gap} e_m(\beta_{i_j})$, which implies $\alpha_j \leq \beta_{i_j}$ by the induction hypothesis. So there exist indices $1 \leq i_1 < \dots < i_k \leq l$ such that $\alpha_j \leq \beta_{i_j}$ for $j = 1, \dots, k$. Therefore $\alpha \leq \beta$.

2. implies 6.: Trivially, 2. implies $\forall X(WO(X) \rightarrow WO(\omega^X))$, hence we can make use of the theory ACA_0 (see Lemma 1.63). Assume that Q is a wpo. Therefore, the tree Bad(Q) is well-founded, hence the linear ordering

$KB(\mathrm{Bad}(Q))$ is a well-ordering (see Lemma 7.49). As mentioned before, there exists a natural primitive recursive reification from $\mathrm{Bad}(Q)$ to this well-ordering which we denote by $f : \mathrm{Bad}(Q) \to \alpha + 1$ with $\alpha = KB(\mathrm{Bad}(Q))$.

From Lemma 5.2 in [72], there exists a primitive recursive reification f_0 from $\mathrm{Bad}((Q \times \{0\})^*, \leq^{*,sgap})$ to the linear ordering $\omega_2(\alpha + 1)$.
Now, assume that we have a primitive recursive reification f_n from $\mathrm{Bad}((Q \times \{0, \dots, n\})^*, \leq^{*,sgap})$ to $\omega_{3n+2}(\alpha + 1)$. We can recursively define a reification f_{n+1} from $\mathrm{Bad}((Q \times \{0, \dots, n + 1\})^*, \leq^{*,sgap})$ to $\omega_{3n+5}(\alpha + 1)$ from f_n. The proof is more or less similar as Lemma 5.5 in [72] (see Theorem 7.8). We have to change the mapping h_{n+1} in that lemma as follows: h_{n+1} is now a mapping from

$$((Q \times \{0, \dots, n + 1\})^*, \leq^{*,sgap})$$

to

$$((Q \times \{0, \dots, n\})^*, \leq^{*,sgap}) \times [Q \times ((Q \times \{0, \dots, n\})^*, \leq^{*,sgap})]^*.$$

It maps the element $()$ to $\{(), ()\}$ and it maps

$$\begin{gathered}((q_1, n_1), \dots, (q_{i_1-1}, n_{i_1-1}), (q_{i_1}, 0), (q_{i_1+1}, n_{i_1+1}), \dots, \\ (q_{i_k-1}, n_{i_k-1}), (q_{i_k}, 0), (q_{i_k+1}, n_{i_k+1}), \dots, (q_m, n_m)),\end{gathered}$$

with $n_i > 0$ for every i to

$$\begin{gathered}\{((q_1, n_1 - 1), \dots, (q_{i_1-1}, n_{i_1-1} - 1)), [(q_{i_1}, ((q_{i_1+1}, n_{i_1+1} - 1), \dots)), \dots, \\ (q_{i_{k-1}}, (\dots, (q_{i_k-1}, n_{i_k-1} - 1))), (q_{i_k}, ((q_{i_k+1}, n_{i_k+1} - 1), \dots, (q_m, n_m - 1)))]\}.\end{gathered}$$

Then, let f_n be a reification from $\mathrm{Bad}((Q\times\{0, \dots, n\})^*, \leq^{*,sgap})$ to $\omega_{3n+2}(\alpha+1)$. Construct from f_n a reification g_n, similar as in Lemma 5.2 and Lemma 5.4 in [72], from

$$\mathrm{Bad}\left(((Q \times \{0, \dots, n\})^*, \leq^{*,sgap}) \times [Q \times ((Q \times \{0, \dots, n\})^*, \leq^{*,sgap})]^*\right)$$

to $\omega_{3n+2}(\alpha+1) \times \omega^{\omega^{\left(\omega^{\omega^{\alpha}} \times \omega_{3n+2}(\alpha+1)\right)+1}}$, which is strictly smaller than $\omega_{3n+5}(\alpha+1)$. Define $f_{n+1}(x_1, \dots, x_m)$ as $g_n(h_{n+1}(x_1), \dots, h_{n+1}(x_m))$. This procedure can be done all at once, meaning that one can prove that there exists a function f such that $f(\langle i, s\rangle) = f_i(s)$ because the construction never explicitly depends on n itself. Now pick an arbitrary natural number m. From 2. it follows that $\omega_{3m+2}(\alpha+1)$ is a well-ordering, hence $((Q\times\{0, \dots, m\})^*, \leq^{*,sgap})$ is a wpo. $\square$

We conclude that the reverse mathematical strength of the proposed assertion of Keita Yokoyama is the theory ACA_0', which lies between ACA_0 and ATR_0.

Appendix A

Nederlandstalige samenvatting

A.1 Inleiding

Ordinalen zijn in zekere zin veralgemeende natuurlijke getallen ontwikkeld door Cantor in de 19de eeuw. Ze laten toe te tellen tot in het transfiniete. Een getal wordt transfiniet genoemd indien het groter is dan alle eindige getallen. In het begin start men met $0, 1, 2, \ldots$ Men definieert de limiet van deze rij ook als een getal (genoteerd als ω) en men telt dan gewoon verder: $\omega, \omega + 1$, enzovoort. Zo creëert men ook $\omega^\omega, \omega^{\omega^\omega}, \ldots,$ net zoals exponentiatie op natuurlijke getallen. De limiet $\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}$ wordt, in wiskundige kringen, genoteerd als ε_0. Aan dit proces komt nooit een eind: men heeft ook $\varepsilon_0 + 1$ $\varepsilon_0 + 2$, enzovoort. De ordinaal getallen zorgen ervoor dat men verschillende soorten oneindigheden van elkaar kan onderscheiden.

Ordinaal notatiesystemen zijn ontwikkeld om (aftelbare) ordinalen op een effectieve manier voor te stellen met de gewone natuurlijke getallen. Er bestaat geen ordinaal notatiesysteem die *alle* aftelbare ordinaal getallen kan voorstellen, waardoor men eigenlijk voor elk ordinaal getal α een nieuw ordinaal notatiesysteem moet creëren. Uit Cantors theorie kan men bijvoorbeeld een ordinaal notatiesysteem voor ε_0 afleiden: elk ordinal getal strikt kleiner dan ε_0 kan voorgesteld worden door middel van het symbool 0 en de functies $\xi, \eta \mapsto \xi + \eta$ en $\xi \mapsto \omega^\xi$. In 1908 publiceerde Veblen een belangrijk artikel waarin hij functies op ordinaal getallen introduceerde gebaseerd op iteratie en afleiding (tegenwoordig bekend onder de naam *Veblen hiërarchie*). Dit levert een ordinaal notatiesysteem op voor Γ_0 als we ons beperken tot binaire functies. Veblen beschouwde ook functies met een groter aantal argumenten. Dit resulteerde in notatiesystemen voor het *kleine Veblen (ordinaal) getal* $\vartheta\Omega^\omega$ en het *grote Veblen (ordinaal) getal* $\vartheta\Omega^\Omega$.

In de geschiedenis zijn verschillende methodes van het creëren van ordinaal notatiesystemen ontwikkeld. Bachmanns vernieuwende methode gebruikte fundamentale rijen en overaftelbare kardinaalgetallen. Later werd dit verfijnd door gebruik te maken van *collapsing functies*. Een voorbeeld van dit soort functies zijn de ϑ_i- en ϑ-functies, die grote ordinalen kunnen beschrijven, zelfs tot het bewijstheoretisch ordinaalgetal van de theorie Π^1_1-CA_0. Collapsing functies zijn functies die overaftelbare ordinaal getallen afbeelden op aftelbare. Met andere woorden, ze leggen een verband tussen de aftelbare en overaftelbare wereld.

In de praktijk is een ordinaal notatiesysteem T een koppel $(T, <_T)$, waarbij T de kleinste verzameling van ordinaal getallen is en $<_T$ de natuurlijke ordening tussen twee ordinaal getallen, zodat

- $0 \in T$,
- Als $\alpha_1, \ldots, \alpha_n \in T$, dan $f(\alpha_1, \ldots, \alpha_n) \in T$, waarbij de symbolen f functies voorstellen die werken op de klasse van ordinaal getallen.

Het ordinaal notatiesysteem T stelt het ordinaal getal α voor, waarbij α het kleinste ordinaal getal is zodat $\alpha \notin T$. α wordt ook het *closure ordinaal (getal)* van T genoemd. Een belangrijk aspect van ordinaal notatiesystemen is de studie van geassocieerde *goede partiële ordeningen*. Deze thesis is geschreven in de lijn van dit soort onderzoek.

Een goede partiële ordening is een partiële ordening $(X, \leq_X)$ zodat voor elke oneindige rij $(x_i)_i$ van elementen in X er twee natuurlijke getallen i en j bestaan zodat $i < j$ en $x_i \leq_X x_j$. Deze karakteristieke eigenschap zorgt ervoor dat goede partiële ordeningen gezien kunnen worden als een veralgemening van goede ordeningen. Een van de meest bekende voorbeelden van een goede partiële ordening staat bekend als *Kruskals stelling*, dat zegt dat voor elke oneindige rij $(T_i)_i$ van eindige gewortelde bomen T_i, er twee natuurlijke getallen $i < j$ bestaan zodat T_i *inbedbaar* is in T_j. Deze stelling wordt onder meer gebruikt in de informatica. Ook *Higmans stelling*, dat zegt dat de verzameling van eindige rijen over een goede partiële ordening ook een goede partiële ordening is onder Higman zijn inbeddingsrelatie, is welbekend in de contreien van de wiskunde.

Eén van de belangrijkste voorbeelden van goede partiële ordeningen in de context van de *onbewijsbaarheidsleer* is die van H. Friedman over eindige gewortelde bomen met n labels met de *gap-inbeddingsrelatie*. Zijn goede partiële ordeningen, $\mathbb{T}_n^{wgap}$ en $\mathbb{T}_n^{sgap}$, kunnen worden gebruikt in een uitspraak die *onbewijsbaar* is in de sterkste theorie (Π^1_1-CA_0) van *reverse mathematics*. Er zijn nog tal van open problemen rond deze beruchte partiële ordening en één daarvan is het specifiek verband vinden met gekende ordinaal notatiesystemen.

A.2 De resultaten

Diana Schmidt toonde in haar Habilitationsthesis aan dat één van de betere manieren om closure ordinaal getallen van ordinaal notatiesystemen T te bestuderen, het overgaan was naar corresponderende goede partiële ordeningen. Specifieker bestudeerde ze *maximale ordeningstypes* van goede partiële ordeningen bestaande uit gewortelde gestructureerde en gelabelde bomen. De maximale ordeningstypes van deze goede partiële ordeningen zijn gelijk aan de closure ordinaal getallen van ordinaal notatiesystemen die geconstrueerd worden met monotone stijgende functies.

Later werd dit soort onderzoek verdergezet, bijvoorbeeld door Andreas Weiermann, de promotor van deze thesis. Hij breidde Diana Schmidts methode uit naar transfiniete argumenten. Specifieker bestudeerde hij een goede partiële ordening, die in deze thesis genoteerd zou worden als $\mathcal{T}(M^\diamond(\tau \times \cdot))$, en bewees hij dat deze een maximale ordeningstype $\vartheta\Omega^\tau$ heeft. Hierdoor kan deze goede partiële ordening resulteren in een ordinaal notatiesysteem voor dit ordinaal getal. In deze thesis tonen we aan dat als we het ordinaalgetal τ vervangen door eerdere gedefinieerde termen, dit een notatiesysteem voor het grote Veblen ordinaal getal oplevert. Specifieker tonen we de volgende stelling aan.

Stelling A.1. *Het maximale ordeningstype van* $\mathcal{T}(M^\diamond(\cdot \times \cdot))$ *is gelijk aan* $\vartheta\Omega^\Omega$*, het grote Veblen ordinaal getal.*

$M^\diamond$ stelt de multiset-constructor voor. Met andere woorden, de vermelde goede partiële ordening bestaat in zekere zin uit ongestructureerde bomen. Als we deze multiset-constructor vervangen door zijn geordende versie, dan bekomen we verrassend genoeg een groter ordinaal getal.

Stelling A.2. *Het maximale ordeningstype van* $\mathcal{T}((\cdot \times \cdot)^*)$ *is gelijk aan* $\vartheta\Omega^{\Omega^\omega}$.

Stellingen A.1 en A.2 worden behandeld in Hoofdstuk 3.

In een andere paper toonde Andreas Weiermann aan dat het Howard-Bachmann ordinaal getal een bovengrens is van closure ordinalen van notatiesystemen die gebruik maken van *essentieel monotone stijgende* functies. Sindsdien was het nog onbekend of we een corresponderende goede partiële ordening konden vinden met hetzelfde ordinaal getal. We tonen in Hoofdstuk 4 de volgende stelling aan.

Stelling A.3. *Het maximale ordeningstype van* $\mathcal{T}(\mathbb{B}(\cdot))$ *is gelijk aan* $\vartheta\varepsilon_{\Omega+1}$*, het Howard-Bachmann ordinaal getal.*

In Hoofdstuk 2 wordt aangetoond dat het maximale ordeningstype van de ongestructureerde bomen (zonder labels) gelijk is aan het maximale ordeningstype van de gestructureerde versie van Diana Schmidt.

In 2008 introduceerde Andreas Weiermann een vermoeden over het maximale ordeningstypes van structuren met de gap-inbeddingsrelatie. Specifieker claimde hij dat het maximale ordeningstype van zijn geïntroduceerde partiële ordeningen $\mathcal{T}(W)$ gelijk is aan $\vartheta(o(W(\Omega)))$. Dit zou kunnen resulteren in een classificatie van Friedmans bekende goede partiële ordeningen $\mathbb{T}_n^{wgap}$ en $\mathbb{T}_n^{sgap}$. In Hoofdstuk 5 tonen we aan dat Weiermanns vermoeden inderdaad de maximale ordeningstypes van $\mathbb{T}_2^{wgap}$ en $\mathbb{T}_2^{sgap}$ goed raadt.

Stelling A.4. *1. Het maximale ordeningstype van* $\mathbb{T}_2^{wgap}$ *is gelijk aan*

$$\vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega).$$

2. Het maximale ordeningstype van $\mathbb{T}_2^{sgap}$ *is gelijk aan*

$$\vartheta_0(\Omega_1^\omega + \vartheta_0(\vartheta_1(\Omega_2^\omega)^\omega)).$$

Voor het geval $n > 2$ is de exacte karakterisatie nog steeds een open vraag, maar we vermoeden dat de oplossing binnen handbereik is.

Hoofdstuk 6 behandelt onbewijsbaarheidsuitspraken over de goede partiële ordeningen $\mathcal{T}(W)$. We bestuderen het bewijs-theoretisch ordinaal getal van bepaalde theorieën die bestaan uit *lightface* Π_1^1*-comprehensie*. Met deze bewijs-theoretische ordinaal getallen kunnen we onbewijsbaarheidsuitspraken bewijzen, die gebruik maken van specifieke $\mathcal{T}(W)$'s, in concrete theorieën.

Tenslotte bestudeert Hoofdstuk 7 of Andreas Weiermanns vermoeden ook correct is als we werken met rijen over de verzameling van natuurlijke getallen $\{0, \ldots, n-1\}$ met de gap-inbeddingsrelatie in plaats van gewortelde bomen. Verrassend genoeg is dit niet het geval. Indien we strikt meer dan 2 labels hebben (met andere woorden $n > 2$), dan komt het corresponderende ordinaal notatiesysteem van de ϑ_i's niet meer overeen met het maximaal ordeningstype van de rijen met de gap-inbeddingsrelatie.

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