Helena Isabel dos Santos Ribeiro Ferreira **Natural and Complex Dynamical Systems**

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Universidade do Minho
Escola de Ciências

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Natural and Complex Dynamical Systems

Tese de Doutoramento em Ciências
Especialidade de Matemática

Trabalho realizado sob a orientação do
Professor Doutor Alberto Adrego Pinto
e do
Professor Doutor Rui Gonçalves

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DECLARAÇÃO

Nome: Helena Isabel dos Santos Ribeiro Ferreira

Endereço electrónico: helenaisafer@gmail.com Telefone: 917239215

Número do Bilhete de Identidade: 10273715

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Orientadores:

Professor Doutor Alberto Adrego Pinto

Professor Doutor Rui Gonçalves

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Abstract

In this PhD thesis, we apply several mathematical concepts to sciences, like, Finances, Hydrology, Energy and Psychology. We analyze real data of different areas and develop techniques of Dynamical Systems, Statistics and Game Theory to study the data. We, also, build mathematical theoretical models suitable to investigate the decisions/behavior of an individual by establishing an analogy to a psychological theory.

In chapter 1, we do an introduction mentioning the main scientific contributions presented in this thesis.

In chapter 2, we exploit ideas of nonlinear dynamics and statistical physics in a complex non-deterministic dynamical setting using the Ruelle-Takens embedding. We present some new insights on the quality of the prediction in the laminar regime and we exhibit the data collapse of the predicted relative first difference fluctuations to the universal BHP distribution. We observe that the nearest neighbor method of prediction acts as a filter that does not eliminate the randomness, but exhibits its universal character.

In chapter 3, we consider the α re-scaled I_p index positive returns $r(t)^\alpha$ and negative returns $(-r(t))^\alpha$ that we call, after normalization, the α pos-

itive fluctuations and α negative fluctuations. We use the Kolmogorov-Smirnov statistical test, as a method, to find the values of α that optimize the data collapse of the histogram of the α fluctuations with the truncated Bramwell-Holdsworth-Pinton (BHP) probability density function (pdf). Using the optimal $\alpha's$ we compute the analytical approximations of the pdf of the normalized positive and negative I_p index returns $r(t)$, with periodicity p. The main indices I_p that we study are the PSI-20 and the Dow Jones Industrial Average but we extend our analysis to world wide indices. The periodicity p varies from daily (d), weekly (w) and monthly (m) returns to intraday data (60 min, 30 min, 15 min and 5 min). We also compute the analytical approximations of the pdf of the normalized positive and negative spot daily prices or daily returns $r(t)$ of distinct energy sources ES and exchange rates ER. Since the BHP probability density function appears in several other dissimilar phenomena, our results reveal a universal feature of the stock market exchange.

In chapter 4, we construct a model, using Game Theory, for the Theory of Planned Behavior and we propose the Bayesian-Nash Equilibria as one of many possible mechanisms to transform human intentions into behavior decisions. We show that saturation can lead to the adoption of a variety of different behavior decisions, as opposed to no saturation, which leads to the adoption of a single consistent behavior decision. Furthermore, we use the new game theoretical model to understand the impact of the leaders and of their characteristics in the decision-making of other individuals or groups. We also apply the model to a students success example, describing Nash equilibria and "herding" effects, identifying a hysteresis in the process.

Resumo

Nesta tese, aplicamos diversos conceitos matemáticos em ciências, como, Finanças, Hidrologia, Energia e Psicologia. Analisamos dados reais de diferentes áreas e desenvolvemos técnicas de Sistemas Dinâmicos, Estatística e Teoria dos Jogos para estudar esses dados. Construímos, também, modelos matemáticos teóricos adequados para analisar decisões ou comportamentos de indivíduos, estabelecendo uma analogia com uma teoria da Psicologia.

No capítulo 1, efectuamos uma introdução na qual mencionamos as principais contribuições científicas apresentadas nesta tese.

No capítulo 2, exploramos ideias de dinâmica não-linear e física estatística, num contexto dinâmico complexo não determinístico, usando o método de reconstrução de Ruelle-Takens. Apresentamos novas percepções sobre a qualidade da previsão no regime laminar e exibimos a sobreposição do histograma da primeira diferença prevista das flutuações com o da distribuição universal BHP. Observamos que o método do vizinho mais próximo da previsão actua como um filtro que não elimina a aleatoriedade, mas evidencia a sua universalidade.

No capítulo 3, consideramos os retornos re-escalados positivos $r(t)^\alpha$ e os retornos re-escalados negativos $(-r(t))^\alpha$, do índice I_p, que chamamos, após

a normalização, flutuações α positivas e flutuações α negativas. Usamos o teste estatístico Kolmogorov-Smirnov, como um método para encontrar os valores de α que optimizam a sobreposição do histograma das flutuações α com o da função densidade de probabilidade Bramwell-Holdsworth-Pinton (BHP) truncada. Usando os valores óptimos de α calculamos uma aproximação analítica das funções densidade de probabilidade dos retornos $r(t)$ positivos e negativos normalizados do índice I_P, com periodicidade p. Os principais índices I_P que estudamos são o PSI-20 e o Dow Jones, alargando o nosso estudo a outros índices mundiais. A periodicidade p varia desde períodos de 5 minutos até períodos mensais (m). Estudamos, também, diferentes fontes de energia ES e taxas de câmbio ER. Dado que a função densidade de probabilidade BHP aparece em vários outros fenómenos diferentes, os nossos resultados revelam um carácter universal do mercado bolsista.

No capítulo 4, construímos um modelo, usando conceitos de Teoria de Jogos, para a Teoria do Comportamento Planeado e propomos o equilibrio Bayesian-Nash como um, dos muitos, mecanismos possíveis de transformar intenções em decisões comportamentais. Mostramos que a saturação pode levar à adopção de uma variedade de diferentes decisões comportamentais, em oposição à não-saturação que conduz à adopção de uma decisão comportamental consistente. Além disso, utilizamos este modelo de teoria de jogos para compreender o impacto dos líderes e das suas características na tomada de decisão de outros indivíduos ou grupos. Aplicamos, também, o modelo a um exemplo de sucesso de estudantes, descrevendo os equilibrios de Nash e efeitos de "rebanho", identificando uma histerese no processo.

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Chapter 1

Introduction

This PhD Thesis is the result of different research projects where several mathematical concepts of Dynamical Systems, Statistics and Game Theory are applied to sciences, like, Finances, Hydrology, Energy and Psychology.

The probability density function (pdf) of a global measure in a large class of highly correlated systems has been suggested to be of the same functional form. Here, we identify the analytical form of the pdf of one such measure, the magnetic order parameter in the low temperature phase (critical) of the 2D XY model using a quadratic (spi-wave) approximation. We present strong evidence that this pdf describes the fluctuations of global quantities in other correlated systems.

The application of dynamical systems methods (see [22, 55]) found a firm ground on the reconstruction theorem of Ruelle-Takens [66] and in the probabilistic justification due to Sauer, Yorke and Casdagli [64]. We pretend to develop some new insights between the quality of the prediction, the embedding dimension [43] and the number of nearest neighbors considered for river data and stock market data. In particular, our aim is to study the collapse of the data to the universal BHP distribution.

We also create mathematical theoretical models, using Game Theory concepts, suitable to investigate the decisions/behavior of an individual by establishing an analogy to a psychological theory.

Universality in nonlinear prediction of complex systems

A direct link between the real world and deterministic dynamical systems theory is given by the analysis of real systems time series in terms of nonlinear dynamics with noise (see [40, 42, 58]). Advances have been made to exploit ideas of dynamical systems theory in cases where the system is not necessarily deterministic, but it displays a structure not captured by classical stochastic methods. Here, the real system time series is the daily runoff of the river Paiva. The time series of the daily runoff reveals that the daily runoff is an intermittent dynamical system, this intermittent dynamical behavior is characterized by a laminar and an irregular phase. The laminar phase occurs in the absence of rainfall and the irregular phase occurs under the action of rain (see [41, 47, 60, 61]). Hence, the forcing of the dynamical system is not of a deterministic type (see [34, 35]). Our goal is to built a Markov process for the laminar regime that models the runoff stochastic process and that can be useful to do one step ahead prediction. To built the Markov process for the laminar regime, we start by introducing a modified version of the usual nonlinear prediction methods based in the Ruelle-Takens embedding (see [39, 61]). After a careful study of the reconstruction vectors sets taking into account the Ruelle-Takens embedding dimension, we find that a one-dimensional Ruelle-Takens embedding already gives good prediction results. We also study the improvement of the prediction results with the increase of the Ruelle-Takens embedding dimensions. Since the one-dimensional Ruelle-Takens embedding gives good predictability results, we built a first order Markov process for the laminar regime. Given the present day runoff value, we find the data collapse of the histogram of the predicted relative first difference fluctuations, that determine the transition probabilities of the first order Markov process, to the universal BHP pdf. Hence, we link the predicted relative first difference fluctuations in a prediction method for a natural and complex dynamical system, based in the Ruelle-Takens embedding, with the universal BHP pdf (see [32, 56]).

Universality in the Stock Market Exchange

Modeling the time series of stock prices is important in economics, finance and energy market, and it is essential in the management of large stock portfolios (see [6, 8, 9, 14, 16, 17, 23, 38, 45, 46, 48, 49, 50, 52, 59]). Here, we analyze specifically the well known Dow Jones Industrial Average (DJIA) index (see [30]) and compare it to the S&P 100 (see [25]), the NASDAQ Composite, the S&P 500 index, and the Russell 2000 index, that correspond to the most closely-watched benchmark indices in terms of stock market activity. We extend your study to European indices, in particular the Portuguese PSI20 (see [29, 56]) and world wide indices.

Let $Y(t)$ be the index I_p adjusted close value at day t. We define the I_p *index return* on day t by

$$r(t) = \frac{Y(t) - Y(t-1)}{Y(t-1)}$$

We define the α *re-scaled* I_p *index positive returns* $r(t)^\alpha$, for $r(t) > 0$, that we call, after normalization, the α *positive fluctuations*. We define the α *re-scaled* I_p *index negative returns* $(-r(t))^\alpha$, for $r(t) < 0$, that we call, after normalization, the α *negative fluctuations*.

We analyze, separately, the α positive and α negative fluctuations that can have different statistical and economic natures (see, for example, [6, 8, 42, 44, 49]). Our aim is to find the values of α that optimize the data collapse of the histogram of the α positive and α negative fluctuations to to the Bramwell-Holdsworth-Pinton (truncated BHP) probability density function (pdf) truncated to the support range of the data (see chapter 2 and Bramwell et al [11]). Our approach is to apply the Kolmogorov-Smirnov (K-S) statistic test as a method to find the values of α that optimize the data collapse. Using this data collapse we do a change of variable that allows us to compute the analytical approximations of the pdf of the normalized positive and negative I_p index returns

$$\begin{aligned}
f_{BHP,I_p,+}(x) &= A_1 x^{-(1-\alpha^+_{BHP})} f_{BHP}(B_1 x^{\alpha^+_{BHP}} - C_1) \\
f_{BHP,I_p,-}(x) &= A_2 x^{-(1-\alpha^-_{BHP})} f_{BHP}(B_2 x^{\alpha^-_{BHP}} - C_2)
\end{aligned}$$

in terms of the BHP pdf f_{BHP}. We exhibit the data collapse of the histogram of the positive and negative returns to our proposed theoretical pdf´s $f_{BHP,I_p,+}$ and $f_{BHP,I_p,-}$. We also extend our study to energy sources prices ES and exchange rates ER, obtaining similar results (see [27, 29, 30, 36]). Since the BHP probability density function appears in several other dissimilar phenomena (see, for example, [18, 24, 32, 33, 36, 57]), our results reveal a universal feature of the stock market exchange. Furthermore, these results lead to the construction of a new qualitative and quantitative econophysics model for the stock market based on the two-dimensional spin model (2dXY) at criticality (see [31]) and to a new stochastic differential equation model for the stock exchange market indices (see [53]) that provides a better understanding of several stock exchange crises (see [54]).

Modeling Human Decisions

The main goal in the study of Planned Behavior or Reasoned Action theory (see [1, 7]) is to understand and predict how individuals turn intentions into behaviors. We construct a general model for the Theory of Planned Behavior or Reasoned Action, inspired by the works of J. Cownley and M. Wooders [15], where specific characteristics of individuals, defined as taste type and crowding type are considered. The crowding type of an individual determines his influence on the welfare (utility) function of the other individuals. The taste type determines the characteristics of an individual specifying his welfare function taking in account the decisions of the other individuals. We start by constructing a model, that we call the platonic idealized psychological model, which consists of individuals with no uncertainties in their taste and crowding types and welfare function. Then we construct the general model, that we call the cave psychological model, which consists of individuals whose taste and crowding types follow the shadows of the taste and crowding types of the platonic idealized psychological model, according to a given probability distribution. Furthermore, in the cave psychological model, individuals know only the expected value of their welfare function. In both models, we present sufficient conditions for an individual or group to adopt a certain behavior decision according to both the Nash and the Bayesian-Nash Equilibria, i.e. the best strategic individual decision taking into account the collective response. We show how saturation, boredom and frustration can lead to different behavior decisions and how no saturation can lead to a single consistent behavior decision. Following the works of J. Driskel, E. Salas and R. Sternberg [20, 21, 65] on leadership, we use the new game theoretical model to understand the impact of the leaders in the decision-making of individuals or groups. We study how the characteristics of the leaders have an influence over the others' decisions (see [3, 5]). We apply the model to a students success example and we describe Nash equilibria and "herding" effects, identifying a hysteresis in the process.

Chapter 2

Universality in nonlinear prediction of complex systems

We exploit ideas of nonlinear dynamics and statistical physics in a complex non-deterministic dynamical setting using the Ruelle-Takens embedding. We present some new insights on the quality of the prediction in the laminar regime and we exhibit the data collapse of the predicted relative first difference fluctuations to the universal BHP distribution. We observe that the nearest neighbor method of prediction acts as a filter that does not eliminate the randomness, but exhibits its universal character.

2.1 Universality of the Bramwell-Hodsworth-Pinton distribution

The universal nonparametric BHP pdf was discovered by Bramwell, Holdsworth and Pinton [12]. The universal nonparametric BHP pdf is the pdf of the fluctuations of the total magnetization, in the strong coupling (low temperature) regime for a two-dimensional spin model (2dXY), using the spin wave approximation (see [11]). The magnetization distribution, that they found, is named, after them, the Bramwell-Holdsworth-Pinton (BHP) distribution. The *BHP probability density function (pdf)* is given by

$$p(\mu) = \int_{-\infty}^{\infty} \frac{dx}{2\pi} \sqrt{\frac{1}{2N^2} \sum_{k=1}^{N-1} \frac{1}{\lambda_k^2}} \quad e^{ix\mu\sqrt{\frac{1}{2N^2}\sum_{k=1}^{N-1}\frac{1}{\lambda_k^2}} - \sum_{k=1}^{N-1}\left[\frac{ix}{2N}\frac{1}{\lambda_k} - \frac{i}{2}\arctan\left(\frac{x}{N\lambda_k}\right)\right]}$$
$$.e^{-\sum_{k=1}^{N-1}\left[\frac{1}{4}\ln\left(1+\frac{x^2}{N^2\lambda_k^2}\right)\right]}, \tag{2.1}$$

where the $\{\lambda_k\}_{k=1}^{L}$ are the eigenvalues, as determined in [11], of the adjacency matrix. It follows, from the formula of the BHP pdf, that the asymptotic values for large deviations, below and above the mean, are exponential and double exponential, respectively (in this work, we use the approximation of the BHP pdf obtained by taking $L = 10$ and $N = L^2$ in equation (2.1)). As one can see, the BHP distribution does not have any parameter (except the mean that is normalized to 0 and the standard deviation that is normalized to 1). Furthermore, the BHP distribution is universal in the sense that appears in several physical phenomena (see [29, 63]). For instance, the universal nonparametric BHP distribution is a good model to explain the fluctuations of order parameters in theoretical examples such as the Sneppen model (see [12, 19]), auto-ignition fire models (see [62]), self-organized models, and percolation models (see [12]). The universal nonparametric BHP distribution is, also, an explanatory model for fluctuations of several phenomenon such as width power in steady state systems (see [12]), plasma density and electrostatic turbulent fluxes measured at the scrape-off layer of the Alcator C-Mod Tokamak (see [51]), the Wolf's sunspot numbers (see [26, 33]) and the stock exchange's indices, daily returns of stocks and commodities (see [25, 27, 29, 30, 36]). The universal BHP distribution also appears in river heights and flow (see [10, 19, 24, 32]). However, the approaches used to study the deseasonalised Danube height data (see [13]) and the Mississippi runoff fluctuations (see [19]) to find the universal distribution BHP in these dynamical systems do not give a similar result in river Paiva due to its different character (see [37]). In this chapter, we show the re-appearance of the BHP pdf as an approximation of the distribution of the predicted relative first difference of the river Paiva runoff.

2.2 Data and preliminary analysis

The most relevant data for this chapter consist of the time series of mean daily runoff of the river Paiva, measured at Fragas da Torre in the North of Portugal. The data is available for download in the *Instituto Nacional da Água* webpage[1]. The sample period runs from 1st of October of 1946 to

[1] http://www.inag.pt

30th of September of 2006 for a total of 21900 observations (see chronogram of Figure 2.1).

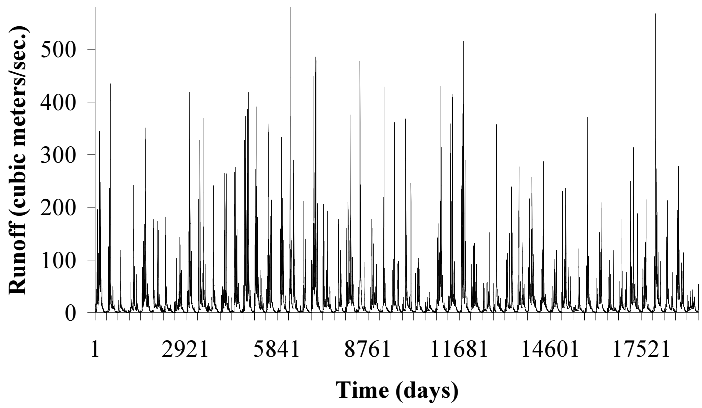


Figure 2.1: *Chronogram of the daily mean riverflow of Paiva measured at Fragas da Torre 1946-99*

The riverflow of Paiva is the closest to a natural flow one might expect. The river Paiva has a small basin of about $700Km^2$ and it is not a runoff intermittent river in the sense that at the referred location and in the 60 years of observation the surface stream never disappeared. The river Paiva is a mountain river with a rocky bed reacting very fast to rainfall. The river Paiva basin does not have regulators such as dams or glaciers. The water of river Paiva is used for public supply in the metropolitan area of Porto.

2.3 Nonlinear prediction

Following Ruelle-Takens [66], we consider the m-dimensional embedding set $R_m = \{\mathbf{X}_t = (X_{t-m+1}, \cdots, X_t), t = [m, \cdots, 21900]\}$ of the runoff data. Our goal is to predict the runoff value X_{t+1} during the laminar regime (absence of rain phase). Hence, we filter appropriately the reconstruction vectors $\mathbf{X}_t \in R_m$ by considering only those $\mathbf{X}_t$ that satisfy the following *δ-relative non-increasing rule* $X_{t+i-1} \geq X_{t+i}(1+\delta)$, for all $1 \leq i \leq m$. We call the reconstruction vectors satisfying the δ-relative non-increasing rule by *laminar reconstruction vectors* and we denote by LR_m the set

$$LR_m = \{\mathbf{X}_{i_j} : X_{i_j+i-1} \geq X_{t+i}(1+\delta),\ 1 \leq i \leq m\}$$

of all laminar reconstruction vectors. We define the *regime reconstruction vectors set* $D[x_0, x_1]$ by

$$D[x_0, x_1] = \{\mathbf{X}_t = (X_{t-m+1}, \cdots, X_t) \in LR_m : x_0 \leq X_t \leq x_1\}$$

We define the *real relative first difference* $b(t)$ of the runoff X_t, at time t, by

$$b(t) = \frac{\nabla X_t}{X_t},$$

where $\nabla X_t = X_{t+1} - X_t$. We define the *real relative first difference mean* $b^{\mu}[x_0, x_1]$, in the regime $D[x_0, x_1]$, by

$$b^{\mu}[x_0, x_1] = \frac{1}{\#D[x_0, x_1]} \sum_{\mathbf{X}_t \in D[x_0, x_1]} b(t).$$

Similarly, we define the *relative first difference standard deviation* $b^{\sigma}[x_0, x_1]$, in the regime $D[x_0, x_1]$, by

$$b^\sigma[x_0, x_1] = \sqrt{\frac{1}{\#\{D[x_0, x_1]\}} \sum_{\mathbf{X}_t \in D[x_0, x_1]} b^2(t) - (b^\mu[x_0, x_1])^2}.$$

We denote by $\|\mathbf{X}_t\|$ the maximum norm of the vector $\mathbf{X}_t = (X_{t-m+1}, \cdots, X_t)$. We define the *relative distance* $rd(\mathbf{X}_i, \mathbf{X}_j)$ of two reconstruction vectors $\mathbf{X}_i$ and $\mathbf{X}_j$ by

$$rd(X_i, X_j) = \frac{\|\mathbf{X}_i - \mathbf{X}_j\|}{\|\mathbf{X}_i\|}.$$

We define the *neighboring set* $N_m(x)$ of x by

$$N_m(x) = \{\mathbf{X}_r = (X_{r-m+1}, \cdots, X_r) \in LR_m : X_r \leq x\}.$$

In Figure 2.2, we show the curve of the cardinal $\#N_1(x)$ and $\#N_2(x)$ of the neighboring sets $N_1(x)$ and $N_2(x)$ with embedding dimensions 1 and 2, respectively.

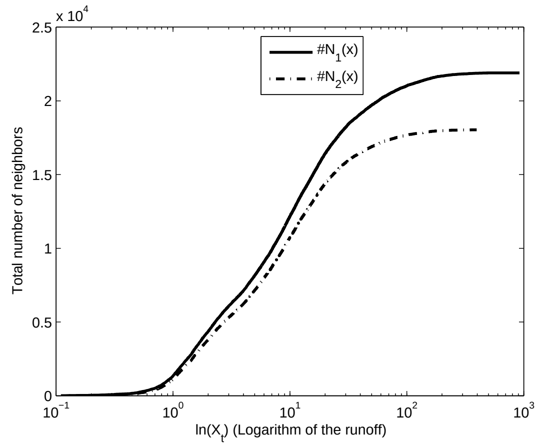


Figure 2.2: *Plot of* $\#N_1(x)$ *and* $\#N_2(x)$

For every runoff value X_t, we order the vectors

$$A_{1,m}(\mathbf{X}_t), \cdots, A_{\#N_m(x)-1,m}(\mathbf{X}_t)$$

in $N_m(X_t)$ by their distance to $\mathbf{X}_t$, i.e. $\|\mathbf{X}_t - A_{j,m}\| \leq \|\mathbf{X}_t - A_{j+1,m}\|$. We define the *reconstruction vectors set* $RV_{i,m}(t)$ of the runoff value X_t by

$$RV_{i,m}(t) = \{A_{j,m}(\mathbf{X}_t) \in N_m(X_t) : j \leq \#N_m(X_t)P_i\}$$

as the collection of the first $\#N_m(X_t)P_i$ ordered vectors in $N_m(X_t)$. In table 2.1, we exhibit the values of P_i, for $i \in \{1, \cdots, 9\}$, that we use in this work.

Table 2.1: Percentages P_i of the total number of neighbors

i	1	2	3	4	5	6	7	8	9
P_i	0.125	0.25	0.5	1.0	2.0	3.0	4.0	5.0	7.5

We define the *predicted relative first difference* $a_{i,m}(t)$ of the runoff at time t by

$$a_{i,m}(t) = \frac{1}{\#\{RV_{i,m}(t)\}} \sum_{\mathbf{X}_r \in RV_{i,m}(t)} \frac{\nabla X_r}{X_r}.$$

We define the *predicted relative first difference mean* $a^{\mu}_{i,m}[x_0, x_1]$, in the regime $D_m[x_0, x_1]$, by

$$a^{\mu}_{i,m}[x_0, x_1] = \frac{1}{\#\{a_{i,m}(t) : t \in D_m[x_0, x_1]\}} \sum_{\mathbf{X}_t \in D_m[x_0,x_1]} a_{i,m}(t) \tag{2.2}$$

We define the *predicted relative first difference standard deviation* $a^{\sigma}_{i,m}[x_0, x_1]$, in the regime $D_m[x_0, x_1]$, by

$$a^{\sigma}_{i,m}[x_0, x_1] = \sqrt{\frac{1}{\#\{D_m[x_0, x_1]\}} \sum_{\mathbf{X}_t \in D_m[x_0,x_1]} a^2_{i,m}(t) - (a^{\mu}_{i,m}[x_0, x_1])^2}. \quad (2.3)$$

We define the *predicted relative first difference fluctuations* $a^f_{i,m}(t)$, in the regime $D_m[x_0, x_1]$, by

$$a^f_{i,m}(t) = \frac{a_{i,m}(t) - a^{\mu}_{i,m}[x_0, x_1]}{a^{\sigma}_{i,m}[x_0, x_1]} \quad (2.4)$$

In Figures 2.3, 2.4 and 2.5, we observe the data collapse of the histogram of the predicted relative first difference fluctuations to the BHP pdf, for the regimes $D[0, 3]$, $D[3, 9]$ and $D[9, 30]$.

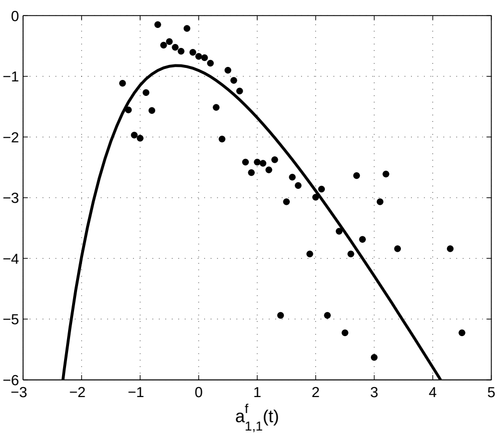


Figure 2.3: *Histogram of the fluctuations* $a^f_{1,1}(t)$ *with the BHP pdf on top, for the regime* $D[0, 3]$

The *error* $c_{i,m}(t)$ is the difference between the real relative first difference

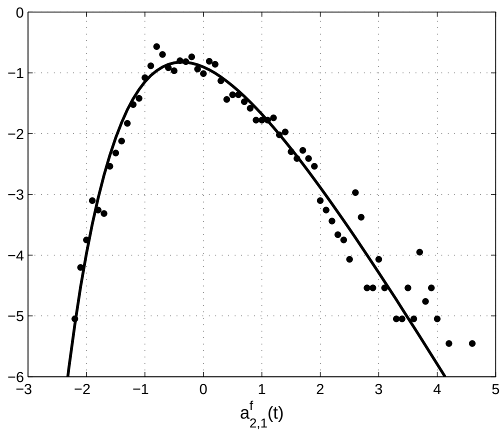


Figure 2.4: *Histogram of the fluctuations* $a^f_{2,1}(t)$ *with the BHP pdf on top, for the regime* $D[3,9]$

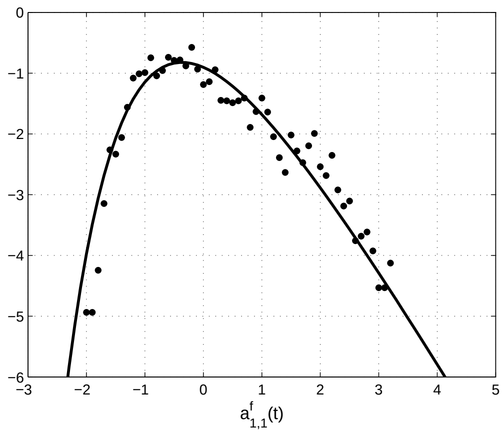


Figure 2.5: *Histogram of the fluctuations* $a^f_{1,1}(t)$ *with the BHP pdf on top, for the regime* $D[9,30]$

$b(t)$ and the predicted relative first difference $a_{i,m}(t)$, i.e.

$$c_{i,m}(t) = a_{i,m}(t) - b(t).$$

We define the *error mean* $c^{\mu}_{i,m}[x_0, x_1]$, in the regime $D_m[x_0, x_1]$, by

$$c^{\mu}_{i,m}[x_0, x_1] = \frac{1}{\#\{c_{i,m}(t) : t \in D_m[x_0, x_1]\}} \sum_{\mathbf{X}_t \in D_m[x_0, x_1]} c_{i,m}(t) \qquad (2.5)$$

We define the *error standard deviation* $c^{\sigma}_{i,m}[x_0, x_1]$, in the regime $D_m[x_0, x_1]$, by

$$c^{\sigma}_{i,m}[x_0, x_1] = \sqrt{\frac{1}{\#\{D_m[x_0, x_1]\}} \sum_{\mathbf{X}_t \in D_m[x_0, x_1]} c^2_{i,m}(t) - (c^{\mu}_{i,m}[x_0, x_1])^2}. \qquad (2.6)$$

In Figure 2.6 and 2.7, we present the error mean $c^{\mu}_{i,m}[a, b]$ and the error standard deviation $c^{\sigma}_{i,m}[a, b]$ for the regimes $[a, b] \in \{[0, 3], [3, 9], [9, 30]\}$. We observe that the values of the error mean $c^{\mu}_{i,m}[a, b]$ considered are close to 0. The values of the error standard deviation $c^{\sigma}_{i,m}[a, b]$ decrease, approximately, to half, when we increase the embedding dimension from 1 to 2. After that the changes are small except for the regime $D[9, 30]$. In the regime $D[9, 30]$ the lack of close neighbors starts to be visible for the values observed for m equal to 4 and 5.

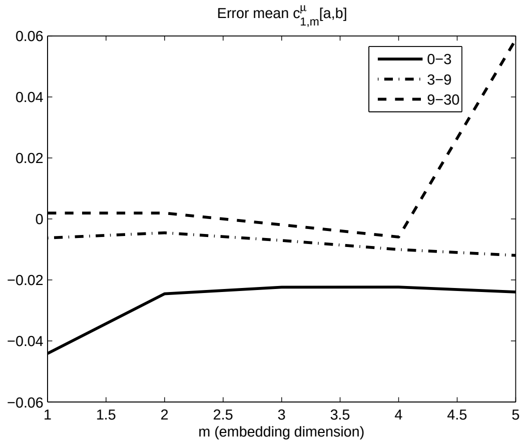


Figure 2.6: *Error mean* $c^{\mu}_{1,m}[a,b]$ *for* $m \in \{1,\cdots,5\}$ *and* $[a,b] \in \{[0,3],[3,9],[9,30]\}$

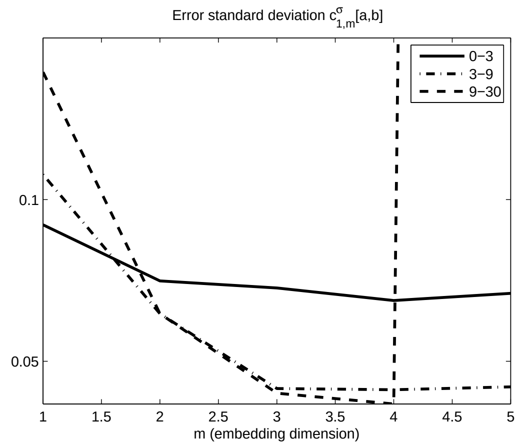


Figure 2.7: *Error standard deviation* $c^{\sigma}_{1,m}[a,b]$ *for* $m \in \{1,\cdots,5\}$ *and* $[a,b] \in \{[0,3],[3,9],[9,30]\}$

2.4 Conclusion

A dynamical analysis of the river Paiva data was performed using the Ruelle-Takens method of dynamical reconstruction. The river Paiva is an intermittent complex dynamical system. We studied the dependence of the nearest neighbors predictor of the relative first difference on the embedding dimension and on the relative average distance of the nearest neighbors with respect to the runoff value. The prediction results revealed that it is essential to know the current runoff to predict future values that lead us to reconstruct an approximation of the one-dimensional stochastics dynamics of the runoff. We noticed improvements in prediction when the former runoffs are used. The mean of the error between the real and predicted values, computed for different regimes and embedding dimensions, gives the correction of the runoff predictor. The standard deviation of the error between the real and the predicted values gives an insight for the best predictor. We observe the data collapse of the histograms of the predicted relative first difference fluctuations to the universal BHP pdf.

Chapter 3

Universality in the Stock Market Exchange

In this chapter, we consider the α re-scaled I_p index positive returns $r(t)^\alpha$ and negative returns $(-r(t))^\alpha$ that we call, after normalization, the α positive fluctuations and α negative fluctuations. We use the Kolmogorov-Smirnov statistical test, as a method, to find the values of α that optimize the data collapse of the histogram of the α fluctuations with the truncated Bramwell-Holdsworth-Pinton (BHP) probability density function. Using the optimal $\alpha's$ we compute the analytical approximations of the pdf of the normalized positive and negative I_p index returns $r(t)$, with periodicity p. The main indices I_p that we study are the PSI-20 and the Dow Jones Industrial Average but we extend our analysis to world wide indices. The periodicity p varies from daily (d), weekly (w) and monthly (m) returns to intraday data (60 min, 30 min, 15 min and 5 min). We also compute the analytical approximations of the pdf of the normalized positive and negative spot daily prices or daily returns $r(t)$ of distinct energy sources ES and of the pdf of the normalized positive and negative spot daily prices or daily returns $r(t)$ of distinct exchange rates ER. Since the BHP probability density function appears in several other dissimilar phenomena, our results reveal a universal feature of the stock market exchange.

3.1 Dow Jones Industrial Average

The Dow Jones Industrial Average, also referred to as the Industrial Average or the Dow Jones, is one of several stock market indices created by Wall Street Journal editor and Dow Jones & Company co-founder Charles Dow. It is an index that shows how 30 large, publicly-owned companies based in the United States have traded during a standard trading session in the stock market. In our analysis we investigate the time series of the *DJIA* index from October of 1928 to October of 2009, considering, respectively, daily, weekly and monthly returns as well as intraday values (see [30]).

3.1.1 DJIA index daily returns

Positive DJIA index daily returns

Let T^+ be the set of all days t with positive returns, i.e.

$$T^+ = \{t : r(t) > 0\}.$$

Let $n^+ = 10605$ be the cardinal of the set T^+. The α *re-scaled DJIA daily index positive returns* are the returns $r(t)^\alpha$ with $t \in T^+$. Since the total number of observed days is $n = 20404$, we obtain that $n^+/n = 0.52$. The *mean* μ_α^+ of the α re-scaled DJIA daily index positive returns is given by

$$\mu_\alpha^+ = \frac{1}{n^+} \sum_{t \in T^+} r(t)^\alpha \tag{3.1}$$

The *standard deviation* σ_α^+ of the α re-scaled DJIA daily index positive returns is given by

$$\sigma_\alpha^+ = \sqrt{\frac{1}{n^+} \sum_{t \in T^+} r(t)^{2\alpha} - (\mu_\alpha^+)^2} \tag{3.2}$$

We define the α *positive fluctuations* by

$$r_\alpha^+(t) = \frac{r(t)^\alpha - \mu_\alpha^+}{\sigma_\alpha^+} \tag{3.3}$$

for every $t \in T^+$. Hence, the α positive fluctuations are the normalized α re-scaled $DJIA$ daily index positive returns.

Let L_α^+ be the *smallest* α positive fluctuation, i.e.

$$L_\alpha^+ = \min_{t \in T^+} \{r_\alpha^+(t)\}.$$

Let R_α^+ be the *largest* α positive fluctuation, i.e.

$$R_\alpha^+ = \max_{t \in T^+} \{r_\alpha^+(t)\}.$$

We denote by $F_{\alpha,+}$ the *probability distribution of the α positive fluctuations.*

Let the *truncated BHP probability distribution* $F_{BHP,\alpha,+}$ be given by

$$F_{BHP,\alpha,+}(x) = \frac{F_{BHP}(x)}{F_{BHP}(R_\alpha^+) - F_{BHP}(L_\alpha^+)}$$

where F_{BHP} is the BHP probability distribution (see definition in Chapter 2). We apply the K-S statistic test to the null hypothesis claiming that the probability distributions $F_{\alpha,+}$ and $F_{BHP,\alpha,+}$ are equal. The Kolmogorov-Smirnov *P value* P_α^+ is plotted in Figure 3.1. Hence, we observe that $\alpha_{BHP}^+ = 0.45...$ is the point where the P value $P_{\alpha_{BHP}^+}^+ = 0.055...$ attains its maximum. We note that

$$\mu_{\alpha_{BHP}^+}^+ = 0.098..., \quad \sigma_{\alpha_{BHP}^+}^+ = 0.046..., \quad L_{\alpha_{BHP}^+}^+ = -1.964... \text{ and } R_{\alpha_{BHP}^+}^+ = 7.266...$$

It is well-known that the Kolmogorov-Smirnov P value P_α^+ decreases with the distance

$$D_{\alpha,+} = \|F_{\alpha,+} - F_{BHP,\alpha,+}\|$$

between $F_{\alpha,+}$ and $F_{BHP,\alpha,+}$. In Figure 3.2, we plot

$$D_{\alpha_{BHP}^+,+}(x) = \left|F_{\alpha_{BHP}^+,+}(x) - F_{BHP,\alpha_{BHP}^+,+}(x)\right|$$

and we observe that $D_{\alpha^+,+}(x)$ attains its maximum value 0.0130 for the α_{BHP}^+ positive fluctuations above the mean of the probability distribution. In Figures 3.3 and 3.4, we show the data collapse of the histogram $f_{\alpha_{BHP}^+,+}$ of the α_{BHP}^+ positive fluctuations to the truncated BHP pdf $f_{BHP,\alpha_{BHP}^+,+}$.

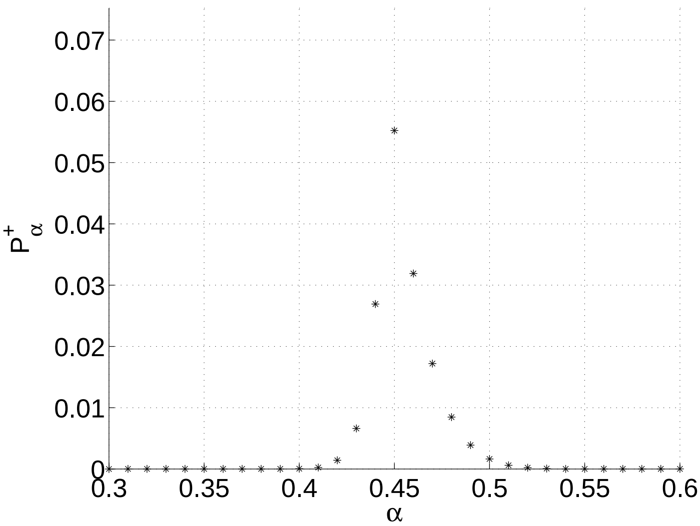


Figure 3.1: *The Kolmogorov-Smirnov P value* P^+_α *for values of* α *in the range* $[0.3, 0.6]$*, in DJIA.*

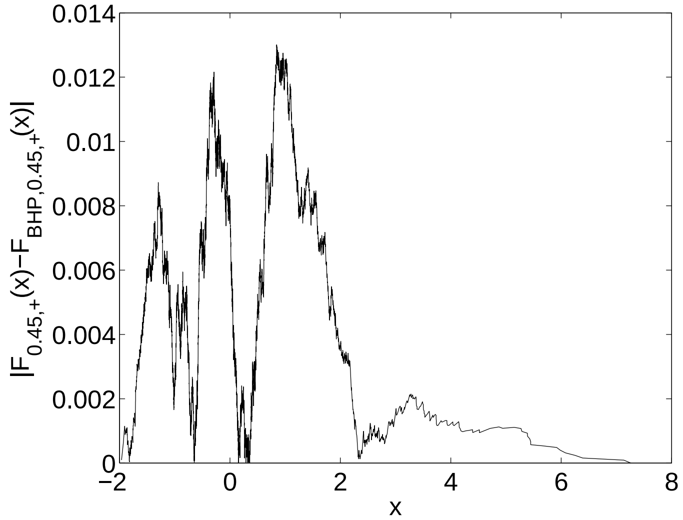


Figure 3.2: *The map* $D_{0.45,+}(x) = |F_{0.45,+}(x) - F_{BHP,0.45,+}(x)|$*, in DJIA.*

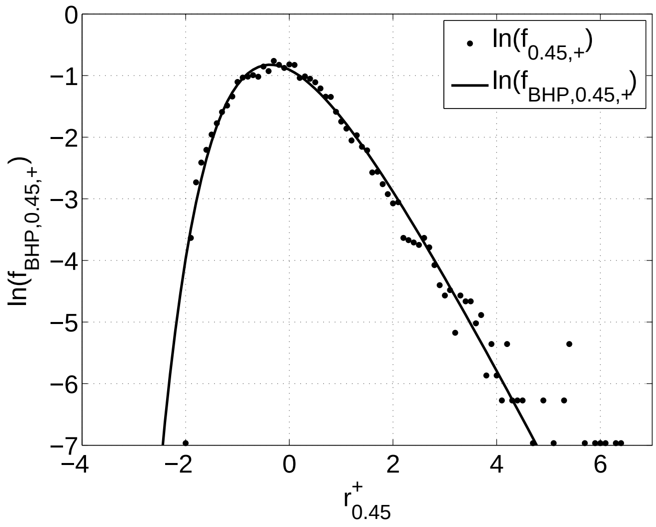


Figure 3.3: *The histogram of the α^{+}_{BHP} positive fluctuations with the truncated BHP pdf $f_{BHP,0.45,+}$ on top, in the semi-log scale, in DJIA*

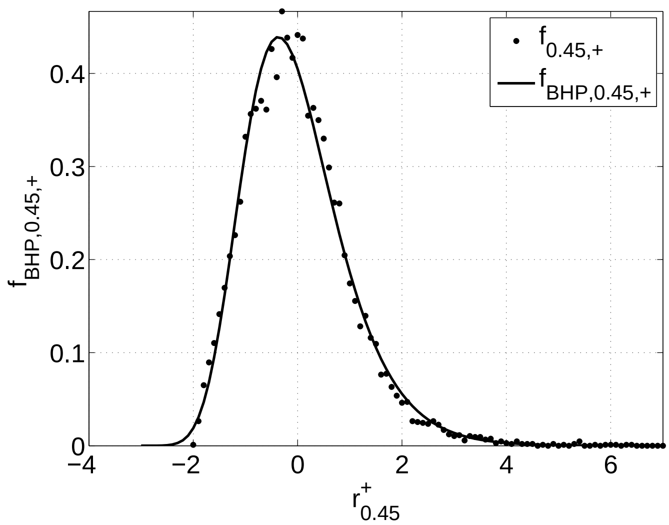


Figure 3.4: *The histogram of the α^{+}_{BHP} positive fluctuations with the truncated BHP pdf $f_{BHP,0.45,+}$ on top, in DJIA*

Theorem 3.1.1 (DJIA pdf daily index positive returns pdf $f_{DJIA_d,+}$) *Assume that the probability distribution of the α^+_{BHP} positive fluctuations $r^+_{\alpha^+_{BHP}}(t)$ is approximated by $F_{BHP,\alpha^+_{BHP},+}$, the pdf of the DJIA daily index positive returns $r(t)$ is approximated by*

$$f_{BHP,DJIA,+}(x) = \frac{\alpha^+_{BHP} x^{\alpha^+_{BHP}-1} f_{BHP}\left(\left(x^{\alpha^+_{BHP}} - \mu^+_{\alpha^+_{BHP}}\right)/\sigma^+_{\alpha^+_{BHP}}\right)}{\sigma^+_{\alpha^+_{BHP}}\left(F_{BHP}\left(R^+_{\alpha^+_{BHP}}\right) - F_{BHP}\left(L^+_{\alpha^+_{BHP}}\right)\right)}.$$

Hence, we get

$$f_{BHP,DJIA,+}(x) = 4.60x^{-0.55} f_{BHP}(21.86x^{0.45} - 2.14).$$

In Figures 3.5 and 3.6, we show the data collapse of the histogram of the positive returns to our proposed theoretical pdf $f_{BHP,DJIA,+}$.

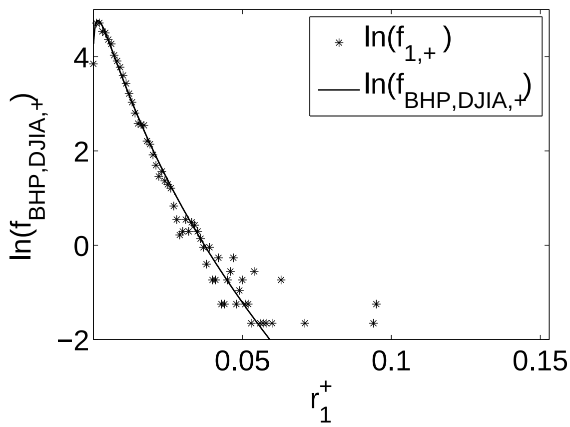


Figure 3.5: *The histogram of the fluctuations of the positive returns with the pdf $f_{BHP,DJIA,+}$ on top, in the semi-log scale, in DJIA*

Proof.

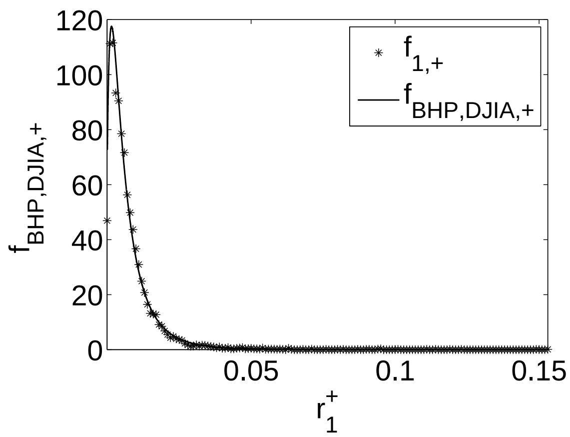


Figure 3.6: *The histogram of the fluctuations of the positive returns with the pdf $f_{BHP,DJIA,+}$ on top, in DJIA*

Let X be a positive random variable. Let μ_α and σ_α be, respectively, the mean and standard deviation of the random variable X^α, with $\alpha > 0$. Let $f_Y : [L, R] \to \mathbb{R}^+$ be the smooth pdf of the normalized random variable $Y = (X^\alpha - \mu_\alpha)/\sigma_\alpha$. We note that

$$\begin{aligned} P\left(X \leq x\right) &= P\left(\frac{X^\alpha - \mu_\alpha}{\sigma_\alpha} \leq \frac{x^\alpha - \mu_\alpha}{\sigma_\alpha}\right) \\ &= P\left(Y \leq \frac{x^\alpha - \mu_\alpha}{\sigma_\alpha}\right). \end{aligned}$$

Hence,

$$F_X(x) = F_Y\left(\frac{x^\alpha - \mu_\alpha}{\sigma_\alpha}\right).$$

Therefore,

$$\frac{dF_X(x)}{dx} = \frac{\alpha x^{\alpha-1} f_Y\left(\left(x^\alpha - \mu_\alpha\right)/\sigma_\alpha\right)}{\sigma_\alpha}.$$

■

Negative DJIA index daily returns

Let T^- be the set of all days t with negative returns, i.e.

$$T^- = \{t : r(t) < 0\}.$$

Let $n^- = 9713$ be the cardinal of the set T^-. Since the total number of observed days is $n = 20404$, we obtain that $n^-/n = 0.48$. The α *re-scaled DJIA daily index negative returns* are the returns $(-r(t))^\alpha$ with $t \in T^-$. We note that $-r(t)$ is positive. The *mean* μ_α^- of the α re-scaled DJIA daily index negative returns is given by

$$\mu_\alpha^- = \frac{1}{n^-} \sum_{t \in T^-} (-r(t))^\alpha \tag{3.4}$$

The *standard deviation* σ_α^- of the α re-scaled DJIA daily index negative returns is given by

$$\sigma_\alpha^- = \sqrt{\frac{1}{n^-} \sum_{t \in T^-} (-r(t))^{2\alpha} - (\mu_\alpha^-)^2} \tag{3.5}$$

We define the α *negative fluctuations* by

$$r_\alpha^-(t) = \frac{(-r(t))^\alpha - \mu_\alpha^-}{\sigma_\alpha^-} \tag{3.6}$$

for every $t \in T^-$. Hence, the α *negative fluctuations* are the normalized α re-scaled $DJIA$ daily index negative returns.

Let L_α^- be the *smallest* α negative fluctuation, i.e.

$$L_\alpha^- = \min_{t \in T^-}\{r_\alpha^-(t)\}.$$

Let R_α^- be the *largest* α negative fluctuation, i.e.

$$R_\alpha^- = \max_{t \in T^-}\{r_\alpha^-(t)\}.$$

We denote by $F_{\alpha,-}$ the *probability distribution of the* α *negative fluctuations.* Let the *truncated BHP probability distribution* $F_{BHP,\alpha,-}$ be given by

$$F_{BHP,\alpha,-}(x) = \frac{F_{BHP}(x)}{F_{BHP}(R_\alpha^-) - F_{BHP}(L_\alpha^-)}$$

where F_{BHP} is the BHP probability distribution (see definition in Chapter 2). We apply the K-S statistic test to the null hypothesis claiming that the probability distributions $F_{\alpha,-}$ and $F_{BHP,\alpha,-}$ are equal. The Kolmogorov-Smirnov *P value* P_α^- is plotted in Figure 3.7. Hence, we observe that that $\alpha_{BHP}^- = 0.46...$ is the point where the P value $P_{\alpha^- BHP}^- = 0.147...$ attains its maximum. We note that

$$\mu_{\alpha_{BHP}^-}^- = 0.093..., \quad \sigma_{\alpha_{BHP}^-}^- = 0.047..., \quad L_{\alpha_{BHP}^-}^- = -1.894... \text{ and } R_{\alpha_{BHP}^-}^- = 8.797...$$

It is well-known that the Kolmogorov-Smirnov P value P_α^- decreases with the distance

$$D_{\alpha,-} = \|F_{\alpha,-} - F_{BHP,\alpha,-}\|$$

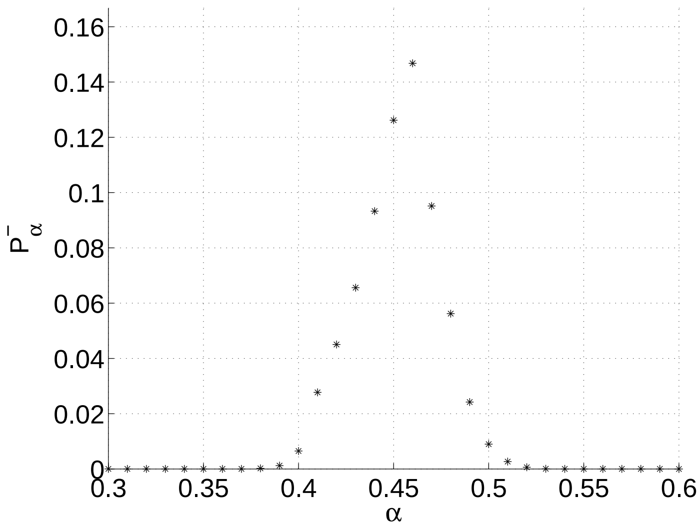


Figure 3.7: *The Kolmogorov-Smirnov P value* P_α^- *for values of* α *in the range* $[0.3, 0.6]$*, in DJIA.*

between $F_{\alpha,-}$ and $F_{BHP,\alpha,-}$. In Figure 3.8, we plot

$$D_{\alpha^-_{BHP},-}(x) = \left|F_{\alpha^-_{BHP},-}(x) - F_{BHP,\alpha^-_{BHP},-}(x)\right|$$

and we observe that $D_{\alpha^-_{BHP},-}(x)$ attains its maximum value 0.0116 for the α^-_{BHP} negative fluctuations above the mean of the probability distribution.

In Figure 3.9 and Figure 3.10, we show the data collapse of the histogram $f_{\alpha^-_{BHP},-}$ of the α^-_{BHP} negative fluctuations to the truncated BHP pdf $f_{BHP,\alpha^-_{BHP},-}$.

Theorem 3.1.2 (DJIA daily index negative returns pdf $f_{DJIA,-}$) *Assume that the probability distribution of the* α^-_{BHP} *negative fluctuations* $r^-_{\alpha^-_{BHP}}(t)$ *is approximated by* $F_{BHP,\alpha^-_{BHP},-}$*, the pdf of the DJIA daily index (symmetric)*

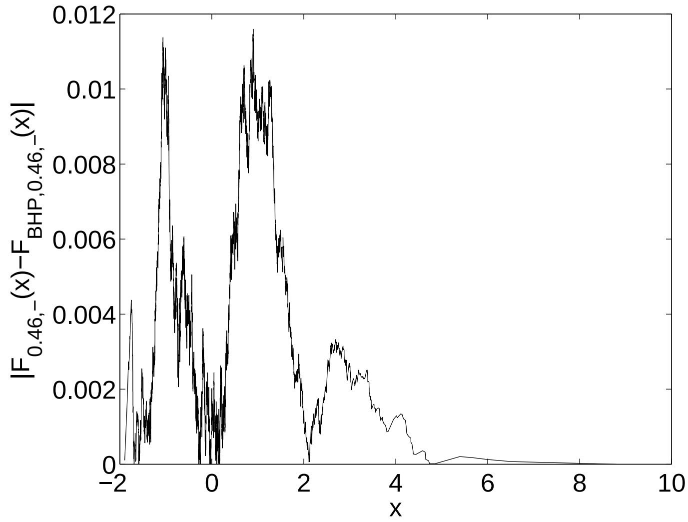


Figure 3.8: *The map* $D_{0.46,-}(x) = |F_{0.46,-}(x) - F_{BHP,0.46,-}(x)|$, *in DJIA.*

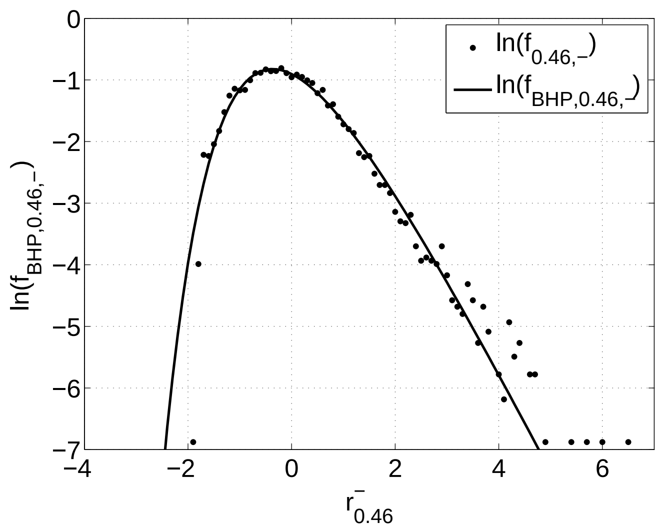


Figure 3.9: *The histogram of the* α^{+}_{BHP} *negative fluctuations with the truncated BHP pdf* $f_{BHP,0.46,-}$ *on top, in the semi-log scale, in DJIA.*

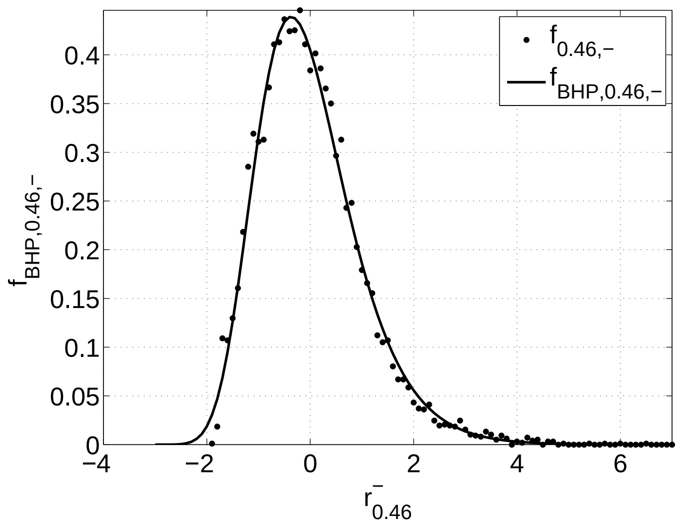


Figure 3.10: *The histogram of the α^+_{BHP} negative fluctuations with the truncated BHP pdf $f_{BHP,0.46,-}$ on top, in DJIA.*

negative returns $-r(t)$, with $T \in T^-$, is approximated by

$$f_{BHP,DJIA,-}(x) = \frac{\alpha^-_{BHP} x^{\alpha^-_{BHP} 1} f_{BHP}\left(\left(x^{\alpha^-_{BHP}} - \mu^-_{\alpha^-_{BHP}}\right)/\sigma^-_{\alpha^-_{BHP}}\right)}{\sigma^-_{\alpha^-_{BHP}}\left(F_{BHP}\left(R^-_{\alpha^-_{BHP}}\right) - F_{BHP}\left(L^-_{\alpha^-_{BHP}}\right)\right)}.$$

Hence, we get

$$f_{BHP,DJIA,-}(x) = 4.95x^{-0.54} f_{BHP}(21.37x^{0.46} - 1.99).$$

The proof of Theorem 3.1.2 follows similarly to the proof of Theorem 3.1.1. In Figures 3.11 and 3.12, we show the data collapse of the histogram of the negative returns to our proposed theoretical pdf $f_{BHP,DJIA,-}$.

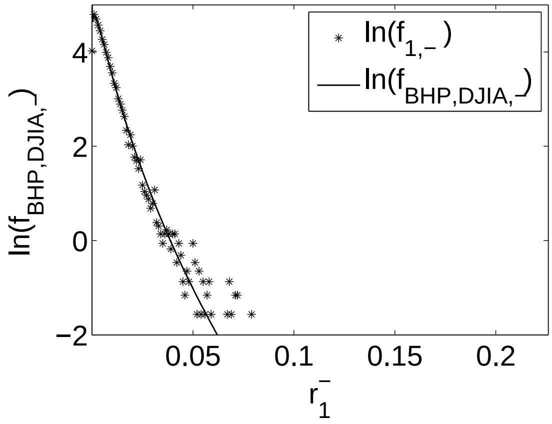


Figure 3.11: *The histogram of the negative returns with the pdf $f_{BHP,DJIA,-}$ on top, in the semi-log scale, in DJIA.*

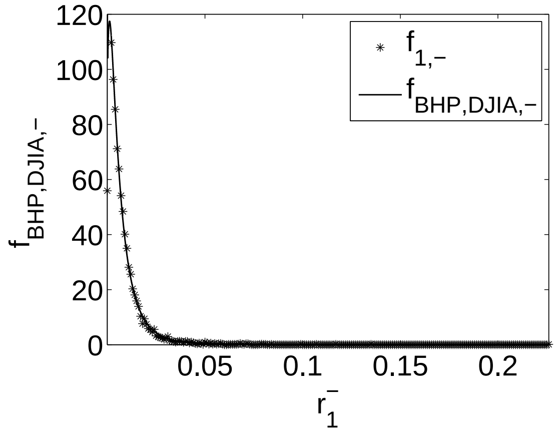


Figure 3.12: *The histogram of the negative returns with the pdf $f_{BHP,DJIA,-}$ on top, in DJIA.*

3.1.2 DJIA index daily returns through the decades

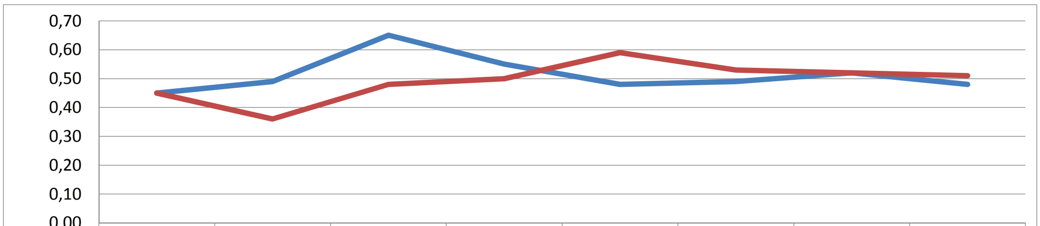
We divide the time series of our analysis in 8 distinct time series that correspond to 8 different decades and study each decade using the methodology described previously. The results are presented in Tables 3.1, 3.2 and in Figure 3.13, where we can observe that the values of α^+_{BHP} vary between 0.45 and 0.65 and the values of α^-_{BHP} vary between 0.36 and 0.59. Both $P^+_{\alpha^+_{BHP}}$ and $P^-_{\alpha^-_{BHP}}$ are higher that 0.01.

Table 3.1: $DJIA$ trough the decades-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^-_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
1930-1939	0.45	0.45	0.84	0.31	0.017	0.028
1940-1949	0.49	0.36	0.32	0.54	0.027	0.023
1950-1959	0.65	0.48	0.03	0.75	0.039	0.020
1960-1969	0.55	0.50	0.43	0.12	0.024	0.034
1970-1979	0.48	0.59	0.29	0.03	0.028	0.041
1980-1989	0.49	0.53	0.11	0.72	0.033	0.020
1990-1999	0.52	0.52	0.16	0.22	0.031	0.031
2000-2009	0.48	0.51	0.93	0.20	0.015	0.031

Table 3.2: $DJIA$ trough the decades-2

	$\mu^+_{\alpha^+_{BHP}}$	$\mu^-_{\alpha^-_{BHP}}$	$\sigma^+_{\alpha^+_{BHP}}$	$\sigma^-_{\alpha^-_{BHP}}$
1930-1939	0.13	0.13	0.06	0.06
1940-1949	0.07	0.13	0.03	0.05
1950-1959	0.03	0.07	0.02	0.03
1960-1969	0.05	0.06	0.02	0.03
1970-1979	0.08	0.05	0.04	0.03
1980-1989	0.08	0.07	0.04	0.04
1990-1999	0.07	0.06	0.03	0.03
2000-2009	0.09	0.08	0.05	0.04



	1930-1939	1940-1949	1950-1959	1960-1969	1970-1979	1980-1989	1990-1999	2000-2009
α+BHP	0,45	0,49	0,65	0,55	0,48	0,49	0,52	0,48
α-BHP	0,45	0,36	0,48	0,5	0,59	0,53	0,52	0,51

Figure 3.13: *Values of α^{+}_{BHP} and α^{-}_{BHP} for each decade, in DJIA.*

3.1.3 DJIA index weekly and monthly returns

We consider the same time series between 1928 and 2009 but with different periodicity, respectively, weekly and monthly returns. Using the same methodology, we obtain the results presented in Tables 3.3, 3.4 and in Figure 3.14.

Table 3.3: $DJIA$ weekly and monthly returns-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^-_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
weekly	0.56	0.47	0.55	0.59	0.016	0.018
monthly	0.37	0.65	0.94	0.90	0.026	0.024

Table 3.4: $DJIA$ weekly and monthly returns-2

	$\mu^+_{\alpha^+_{BHP}}$	$\mu^-_{\alpha^-_{BHP}}$	$\sigma^+_{\alpha^+_{BHP}}$	$\sigma^-_{\alpha^-_{BHP}}$
weekly	0.09	0.14	0.05	0.07
monthly	0.27	0.10	0.11	0.06

Hence, considering weekly returns, we get

$$f_{BHP,DJIA_w,+}(x) = 6.15x^{-0.44} f_{BHP}(20.34x^{0.56} - 1.85)$$

$$f_{BHP,DJIA_w,-}(x) = 3.44x^{-0.53} f_{BHP}(15.37x^{0.47} - 2.10)$$

Considering monthly returns, we get

$$f_{BHP,DJIA_m,+}(x) = 1.36x^{-0.63} f_{BHP}(8.80x^{0.37} - 2.40)$$

$$f_{BHP,DJIA_m,-}(x) = 6.21x^{-0.35} f_{BHP}(17.22x^{0.65} - 1.80).$$

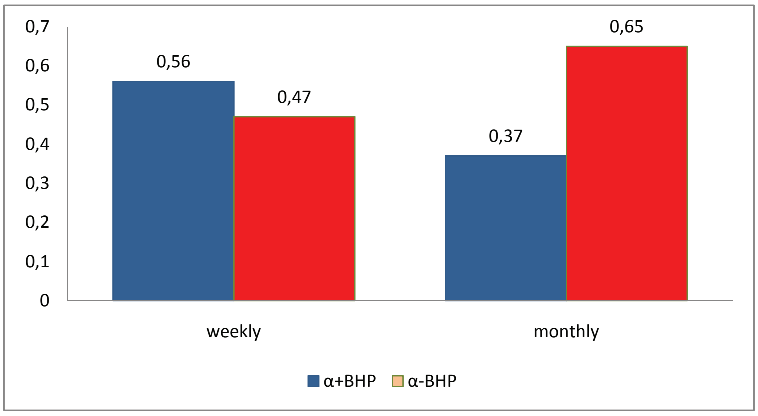


Figure 3.14: *Values of* α^+_{BHP} *and* α^-_{BHP} *for weekly and monthly returns, in DJIA.*

3.1.4 DJIA intraday returns

We analyze four time series of intraday closure values of $DJIA$ in periods of 60 minutes, 30 minutes, 15 minutes and 5 minutes. We compare the same number of observations (7000) in all series, studying in $f_{DJIA_{60m}}$ 292 days, in $f_{DJIA_{30m}}$ 146 days, in $f_{DJIA_{15m}}$ 73 days and in $f_{DJIA_{5m}}$ 24 days.

The values of the parameters are presented in Tables 3.5, 3.6 and in Figure 3.15, where we can observe that $P^+_{\alpha^+_{BHP}}$ and $P^-_{\alpha^-_{BHP}}$ are higher that 0.01 and α^+_{BHP} and α^-_{BHP} vary between 0.36 and 0.45.

Table 3.5: DJIA intraday returns-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^-_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
60 min	0.37	0.36	0.18	0.02	0.018	0.026
30 min	0.37	0.40	0.34	0.32	0.016	0.016
15 min	0.44	0.41	0.49	0.58	0.014	0.013
5 min	0.41	0.45	0.24	0.16	0.019	0.021

Table 3.6: DJIA intraday returns-2

	$\mu^+_{\alpha^+_{BHP}}$	$\mu^-_{\alpha^-_{BHP}}$	$\sigma^+_{\alpha^+_{BHP}}$	$\sigma^-_{\alpha^-_{BHP}}$
60 min	0.10	0.10	0.05	0.05
30 min	0.10	0.08	0.04	0.04
15 min	0.05	0.06	0.03	0.03
5 min	0.04	0.03	0.02	0.02

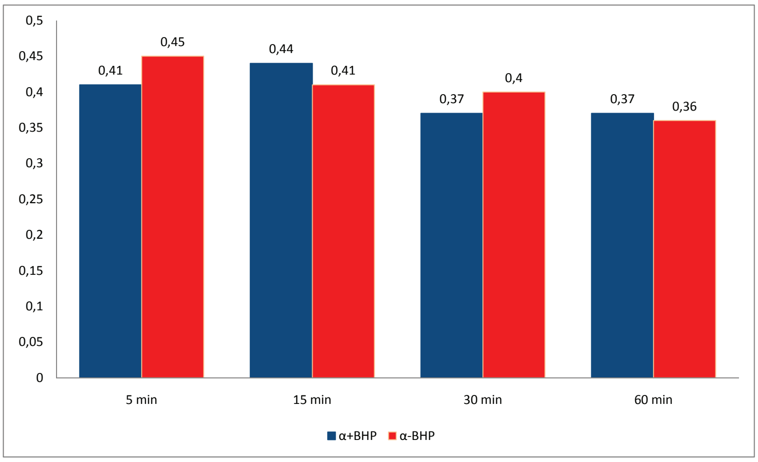


Figure 3.15: *Values of α^+_{BHP} and α^-_{BHP} for intraday returns, in DJIA.*

Considering 60 min returns, we get

$$f_{BHP,DJIA_{60min},+}(x) = 3.83x^{-0.63} f_{BHP}(21.22x^{0.37} - 2.05)$$

$$f_{BHP,DJIA_{60min},-}(x) = 3.43x^{-0.64} f_{BHP}(19.64x^{0.36} - 2.06).$$

Considering 30 min returns, we get

$$f_{BHP,DJIA_{30min},+}(x) = 3.78x^{-0.63} f_{BHP}(22.57x^{0.37} - 2.21)$$

$$f_{BHP,DJIA_{30min},-}(x) = 4.83x^{-0.60} f_{BHP}(24.64x^{0.40} - 2.04).$$

Considering 15 min returns, we get

$$f_{BHP,DJIA_{15min},+}(x) = 8.50x^{-0.56} f_{BHP}(38.95x^{0.44} - 2.02)$$

$$f_{BHP,DJIA_{15min},-}(x) = 6.45x^{-0.59} f_{BHP}(32.96x^{0.41} - 2.10).$$

Considering 5 min returns, we get

$$f_{BHP,DJIA_{5min},+}(x) = 9.79x^{-0.59} f_{BHP}(53.98x^{0.41} - 2.26)$$

$$f_{BHP,DJIA_{5min},-}(x) = 14.62x^{-0.55} f_{BHP}(64.32x^{0.45} - 1.98).$$

3.2 Dow Jones and other North American Indices

After making a particular study of the *DJIA* index, we extend our study to other North American relevant indices and analyze if the values of the α^+_{BHP} and α^-_{BHP} are similar.

We consider four time series, the *S&P 100* index from January of 1987 to September of 2008 (see [25]), the *Nasdaq* index from February of 1971 to January of 2010, the *Russell 2000* index from September of 1987 to January of 2010 and the *S&P 500* index from January of 1950 to January of 2010.

Considering daily returns, we obtain the results presented in Figures 3.16, 3.17, 3.18, 3.19 and 3.20.

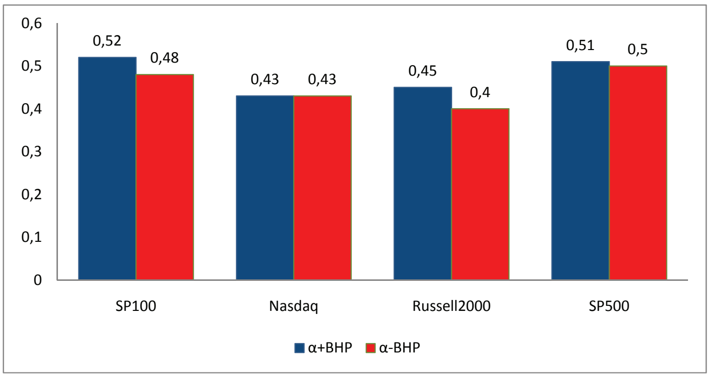


Figure 3.16: *Values of α^+BHP and α^-_{BHP} for each index daily returns*

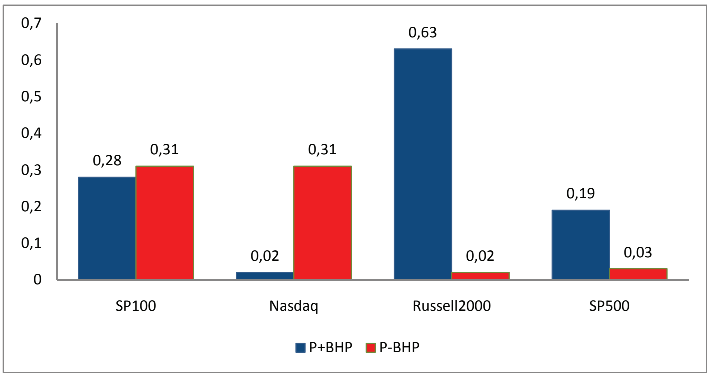


Figure 3.17: *Values of* $P^{+}_{\alpha^{+}_{BHP}}$ *and* $P^{-}_{\alpha^{-}_{BHP}}$ *for each index daily returns*

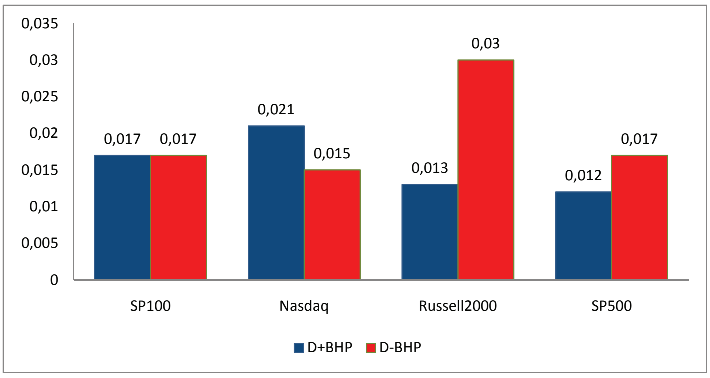


Figure 3.18: *Values of* $D^{+}_{\alpha^{+}_{BHP}}$ *and* $D^{-}_{\alpha^{-}_{BHP}}$ *for each index daily returns*

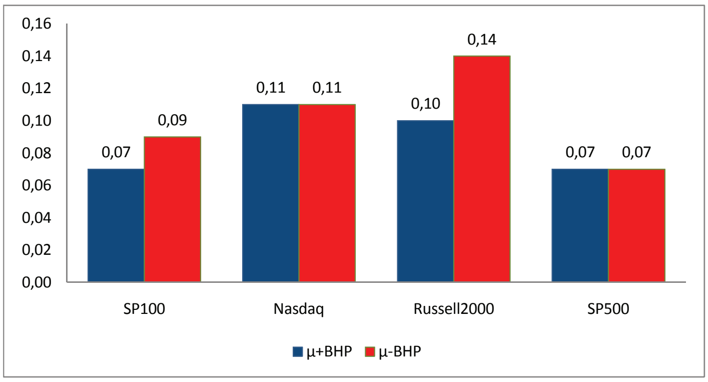


Figure 3.19: *Values of* $\mu^+_{\alpha^+_{BHP}}$ *and* $\mu^-_{\alpha^-_{BHP}}$ *for each index daily returns*

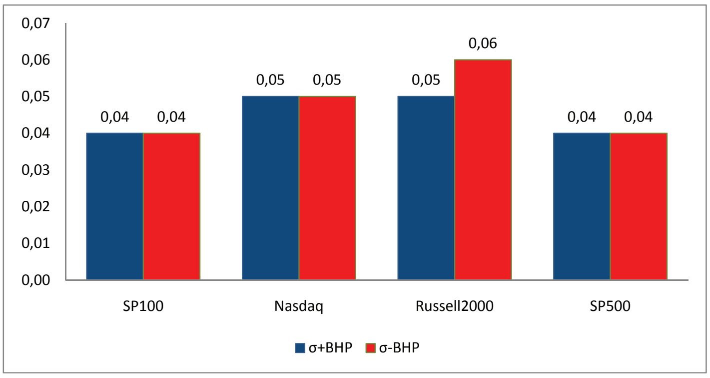


Figure 3.20: *Values of* $\sigma^+_{\alpha^+_{BHP}}$ *and* $\sigma^-_{\alpha^-_{BHP}}$ *for each index daily returns*

For weekly returns, we obtain the results presented in Figures 3.21, 3.22, 3.23, 3.24 and 3.25.

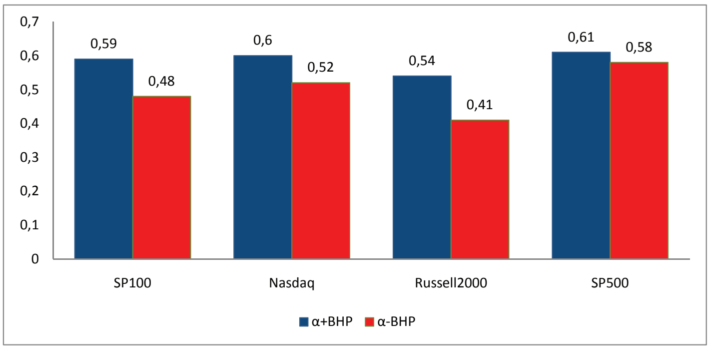


Figure 3.21: *Values of* α^{+}_{BHP} *and* α^{-}_{BHP} *for each index weekly returns*

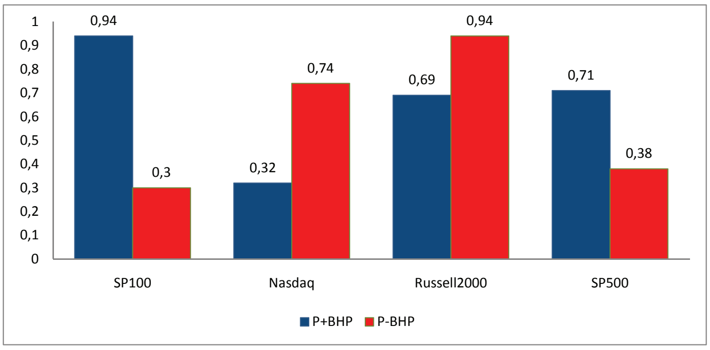


Figure 3.22: *Values of* $P^+_{\alpha^+_{BHP}}$ *and* $P^-_{\alpha^-_{BHP}}$ *for each index weekly returns*

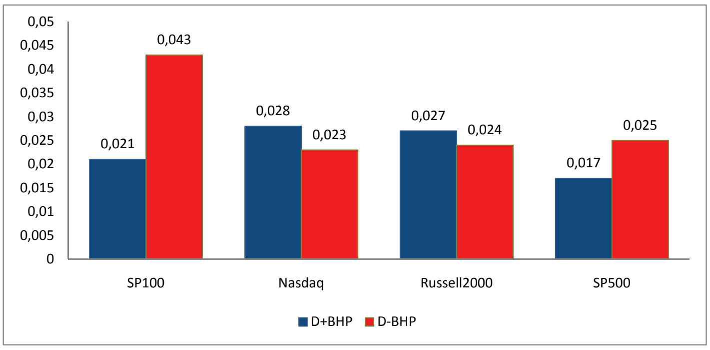


Figure 3.23: *Values of* $D^+_{\alpha^+_{BHP}}$ *and* $D^+_{\alpha^-_{BHP}}$ *for each index weekly returns*

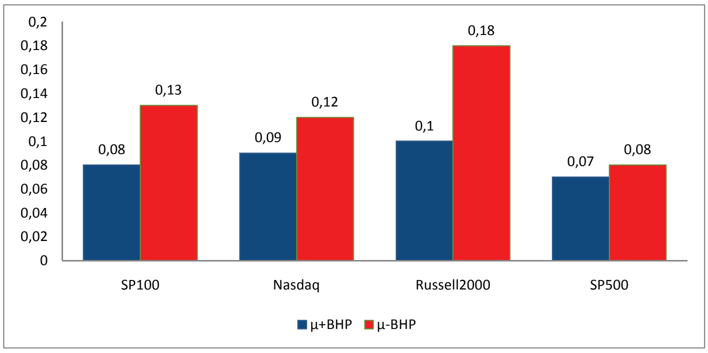


Figure 3.24: *Values of* $\mu^{+}_{\alpha^{+}_{BHP}}$ *and* $\mu^{-}_{\alpha^{-}_{BHP}}$ *for each index weekly returns*

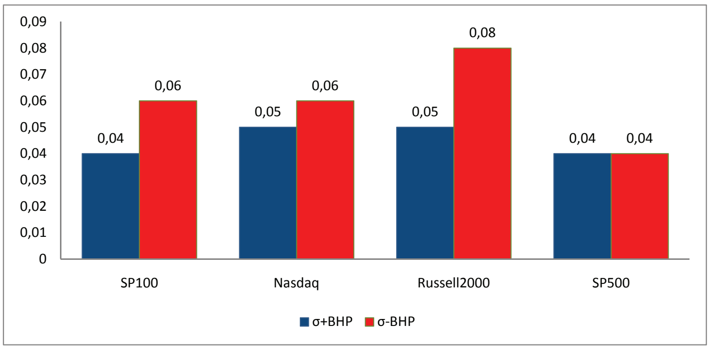


Figure 3.25: *Values of* $\sigma^{+}_{\alpha^{+}_{BHP}}$ *and* $\sigma^{-}_{\alpha^{-}_{BHP}}$ *for each index weekly returns*

Considering monthly returns, we obtain the results presented in Figures 3.26, 3.27, 3.28, 3.29 and 3.30.

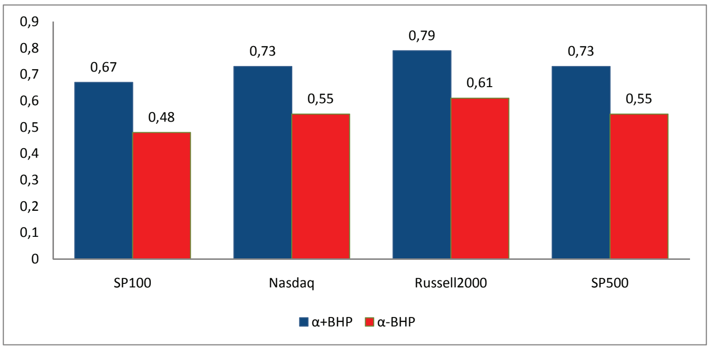


Figure 3.26: *Values of* α^+_{BHP} *and* α^-_{BHP} *for each index monthly returns*

In all the periodicities, especially in daily returns, we observe that the values of α^+_{BHP} and α^-_{BHP} are similar in the four indices, and close to 0.50.

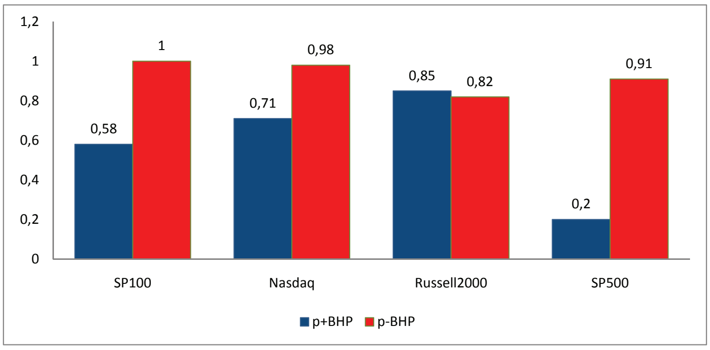


Figure 3.27: *Values of* $P^{+}_{\alpha^{+}_{BHP}}$ *and* $P^{-}_{\alpha^{-}_{BHP}}$ *for each index monthly returns*

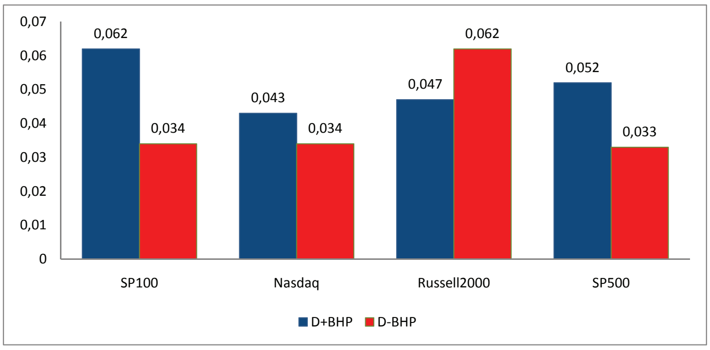


Figure 3.28: *Values of* $D^{+}_{\alpha^{+}_{BHP}}$ *and* $D^{-}_{\alpha^{-}_{BHP}}$ *for each index monthly returns*

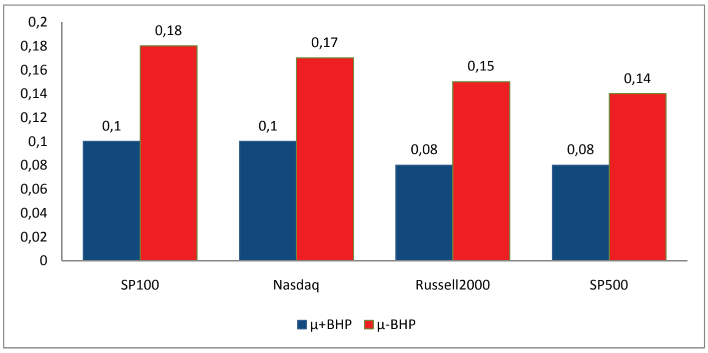


Figure 3.29: *Values of* $\mu^+_{\alpha^+_{BHP}}$ *and* $\mu^-_{\alpha^-_{BHP}}$ *for each index monthly returns*

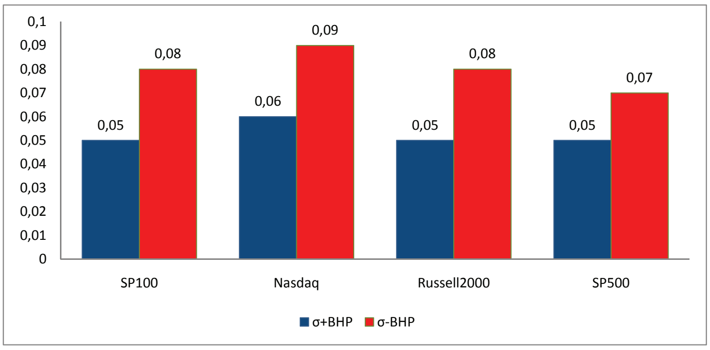


Figure 3.30: *Values of* $\sigma^+_{\alpha^+_{BHP}}$ *and* $\sigma^-_{\alpha^-_{BHP}}$ *for each index monthly returns*

3.3 European Indices

We use the same methodology to study European indices, in particular the Portuguese index, PSI-20 (see [29, 56]).

3.3.1 PSI-20

The PSI-20 (an acronym of Portuguese Stock Index) is a benchmark stock market index of companies that trade on Euronext Lisbon, the main stock exchange of Portugal. The index tracks the prices of the twenty listings with the largest market capitalization and share turnover in the PSI Geral, the general stock market of the Lisbon exchange. It is one of the main national indices of the pan-European stock exchange group Euronext alongside Brussels (BEL20), Paris (CAC 40) and Amsterdam (AEX).

For the positive PSI-20 index daily returns, the Kolmogorov-Smirnov P *value* P_{α}^{+} is plotted in Figure 3.31. We observe that $\alpha_{BHP}^{+} = 0.48...$ is the point where the P value $P_{\alpha_{BHP}^{+}}^{+} = 0.95...$ attains its maximum. In Figure 3.32, we plot

$$D_{\alpha_{BHP}^{+},+}(x) = \left| F_{\alpha_{BHP}^{+},+}(x) - F_{BHP,\alpha_{BHP}^{+},+}(x) \right|$$

and we observe that $D_{\alpha_{BHP}^{+},+}(x)$ attains its maximum value 0.0151 for the α^{+} positive fluctuations below the mean of the probability distribution. In Figures 3.33 and 3.34, we show the data collapse of the histogram $f_{\alpha_{BHP}^{+},+}$ of the α_{BHP}^{+} positive fluctuations to the truncated BHP pdf $f_{BHP,\alpha_{BHP}^{+},+}$.

The pdf of the PSI-20 daily index positive returns $r(t)$ is approximated

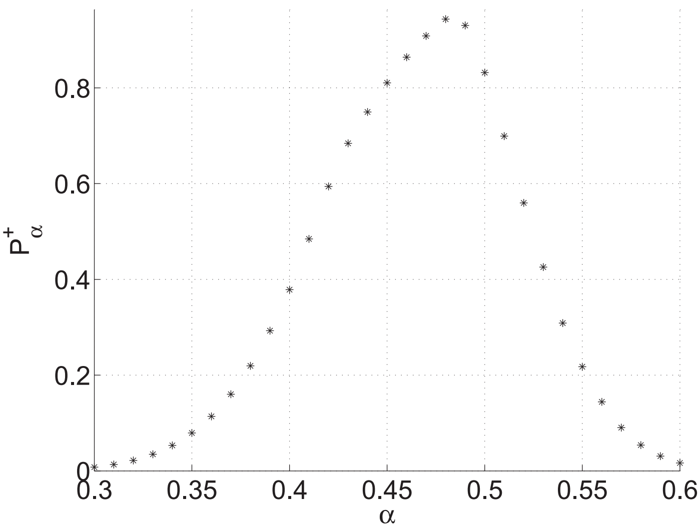


Figure 3.31: *The Kolmogorov-Smirnov P value P^+_α for values of α in the range $[0.3, 0.6]$, in PSI-20.*

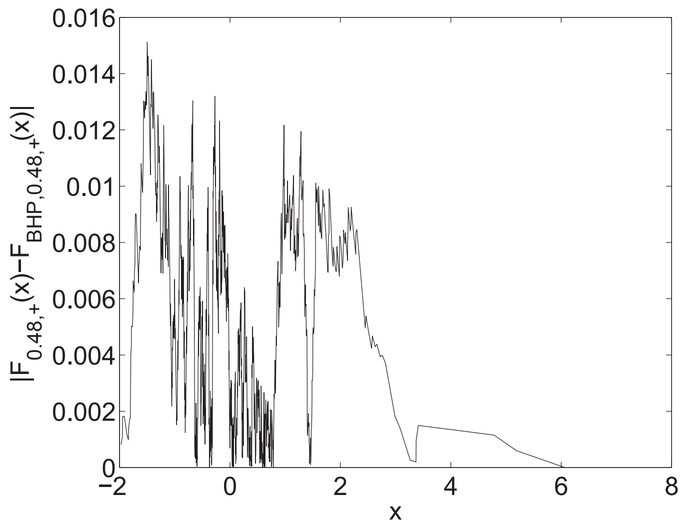


Figure 3.32: *The map $D_{0.48,+}(x) = |F_{0.48,+}(x) - F_{BHP,0.48,+}(x)|$, in PSI-20*

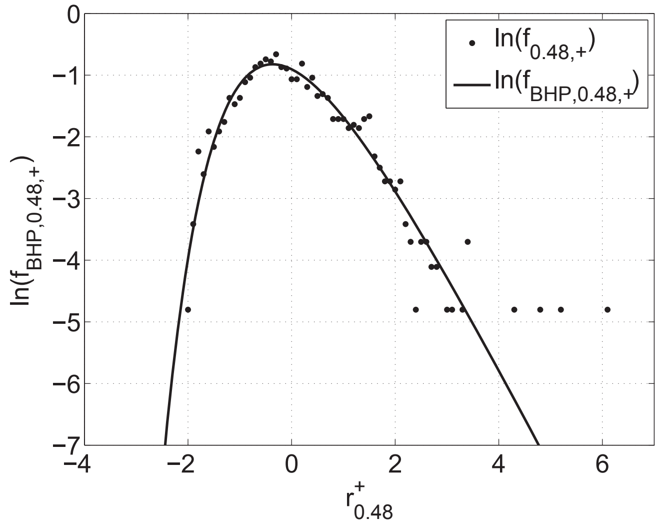


Figure 3.33: *The histogram of the α^+_{BHP} positive fluctuations with the truncated BHP pdf $f_{BHP,0.48,+}$ on top, in the semi-log scale, in PSI-20.*

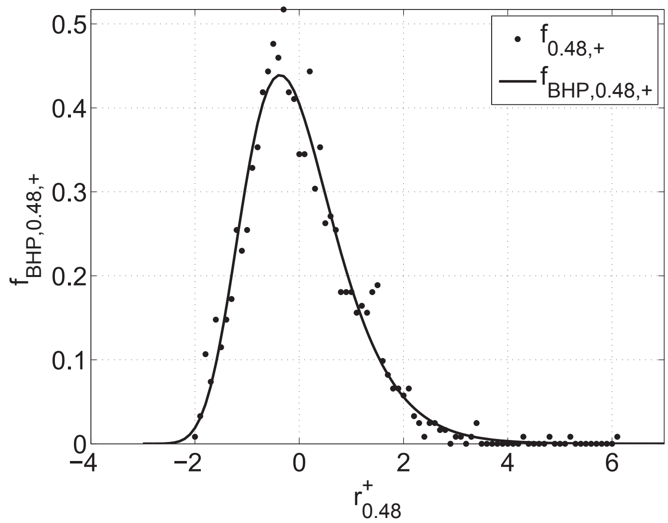


Figure 3.34: *The histogram of the α^+_{BHP} positive fluctuations with the truncated BHP pdf $f_{BHP,0.48,+}$ on top, in PSI-20.*

by

$$f_{BHP,PSI20,+}(x) = \frac{\alpha^+_{BHP} x^{\alpha^+_{BHP}-1} f_{BHP}\left(\left(x^{\alpha^+_{BHP}} - \mu^+_{\alpha^+_{BHP}}\right)/\sigma^+_{\alpha^+_{BHP}}\right)}{\sigma^+_{\alpha^+_{BHP}}\left(F_{BHP}\left(R^+_{\alpha^+_{BHP}}\right) - F_{BHP}\left(L^+_{\alpha^+_{BHP}}\right)\right)}.$$

Hence, we get

$$f_{BHP,PSI20,+}(x) = 5.71x^{-0.52} f_{BHP}(24.3x^{0.48} - 2.04).$$

For negative PSI-20 index daily returns, the Kolmogorov-Smirnov P *value* P^-_α is plotted in Figure 3.35. Hence, we observe that $\alpha^-_{BHP} = 0.46...$ is the point where the P value $P^-_{\alpha^-_{BHP}} = 0.77...$ attains its maximum. In Figure 3.36, we plot

$$D_{\alpha^-_{BHP},-}(x) = \left|F_{\alpha^-_{BHP},-}(x) - F_{BHP,\alpha^-_{BHP},-}(x)\right|$$

and we observe that $D_{\alpha^-_{BHP},-}(x)$ attains its maximum value 0.0202 for the α^-_{BHP} negative fluctuations below the mean of the probability distribution. In Figures 3.37 and 3.38, we show the data collapse of the histogram $f_{\alpha^-_{BHP},-}$ of the α^-_{BHP} negative fluctuations to the truncated BHP pdf $f_{BHP,\alpha^-_{BHP},-}$.

The pdf of the $PSI-20$ daily index (symmetric) negative returns $-r(t)$, with $T \in T^-$, is approximated by

$$f_{BHP,PSI20,-}(x) = \frac{\alpha^-_{BHP} x^{\alpha^-_{BHP}1} f_{BHP}\left(\left(x^{\alpha^-_{BHP}} - \mu^-_{\alpha^-_{BHP}}\right)/\sigma^-_{\alpha^-_{BHP}}\right)}{\sigma^-_{\alpha^-_{BHP}}\left(F_{BHP}\left(R^-_{\alpha^-_{BHP}}\right) - F_{BHP}\left(L^-_{\alpha^-_{BHP}}\right)\right)}.$$

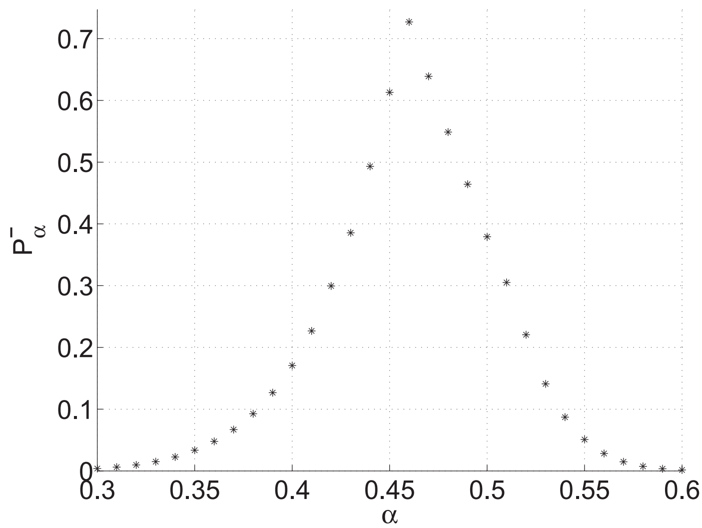


Figure 3.35: *The Kolmogorov-Smirnov P value* P_α^- *for values of* α *in the range* $[0.3, 0.6]$*, in PSI-20.*

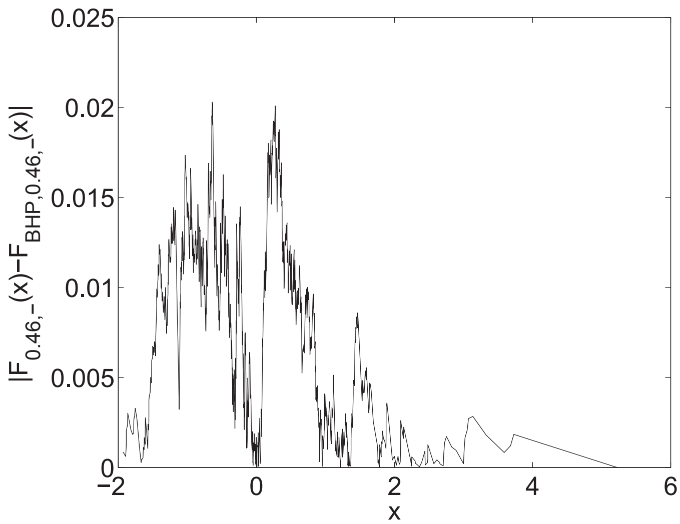


Figure 3.36: *The map* $D_{0.46,-}(x) = |F_{0.46,-}(x) - F_{BHP,0.46,-}(x)|$*, in PSI-20.*

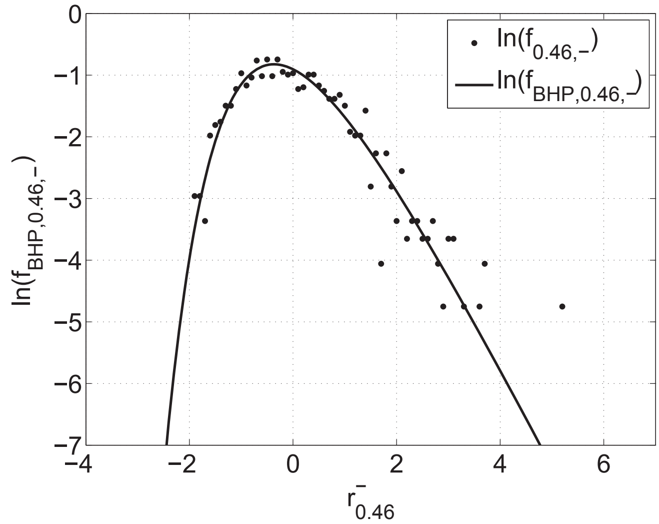


Figure 3.37: *The histogram of the α^-_{BHP} negative fluctuations with the truncated BHP pdf $f_{BHP,0.46,-}$ on top, in the semi-log scale, in PSI-20.*

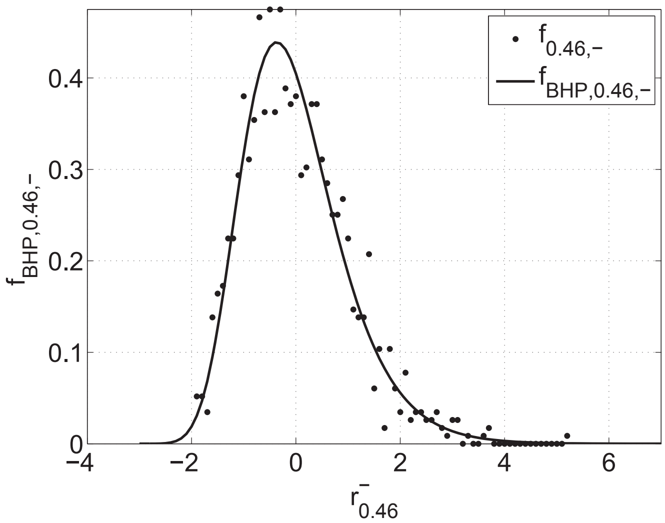


Figure 3.38: *The histogram of the α^-_{BHP} negative fluctuations with the truncated BHP pdf $f_{BHP,0.46,-}$ on top, in PSI-20.*

Hence, we get

$$f_{BHP,PSI20,-}(x) = 4.80x^{-0.54} f_{BHP}(21.0x^{0.46} - 2.0)$$

3.3.2 Other European Indices

We analyze the time series of the following European indices: from France, the *(FCHI)- CAC 40* index between March of 1990 and September of 2009; from Germany, the *(GDAXI)- DAX* index between November of 1990 and September of 2009; from Italy, the *(MIBTEL)- MIBTEL* index between January of 2000 and May of 2009; from Netherlands, the *(AEX)- AEX General* index between October of 1992 and September of 2009; from Norway, the *(OSEAX)- OSE All Share* index between February of 2001 and September of 2009; from Spain, the *(SMSI)-Madrid General* index between June of 2006 and September of 2009; from Sweden, the *(OMXSPI)- Stockholm General* index between January of 2001 and September of 2009; from Switzerland, the *(SSMI)- Swiss Market* index between November of 1990 and September of 2009; and from UK, the *(FTSE)- FTSE 100* index between April of 1984 and September of 2009. The results obtained for these indices are presented in Tables 3.7, 3.8 and in Figure 3.39.

In all the indices, we observe that the values of α^{+}_{BHP} and α^{-}_{BHP} are similar and vary between 0.40 and 0.57 and that $P^{+}_{\alpha^{+}_{BHP}}$ and $P^{-}_{\alpha^{-}_{BHP}}$ are higher that 0.01, which can indicate universality in European indices.

Table 3.7: European Indices-1

	α^{+}_{BHP}	α^{-}_{BHP}	$P^{+}_{\alpha^{+}_{BHP}}$	$P^{-}_{\alpha^{-}_{BHP}}$	$D^{+}_{\alpha^{+}_{BHP}}$	$D^{-}_{\alpha^{-}_{BHP}}$
FCHI	0.56	0.53	0.07	0.08	0.026	0.026
DAX	0.50	0.48	0.19	0.24	0.021	0.022
MIBTEL	0.47	0.43	0.83	0.15	0.018	0.034
AEX	0.46	0.43	0.59	0.31	0.016	0.021
OSEAX	0.57	0.47	0.79	0.94	0.019	0.017
SMCI	0.40	0.44	0.90	0.95	0.027	0.026
OMXSPI	0.52	0.50	0.44	0.15	0.026	0.035
SSMI	0.53	0.53	0.22	0.62	0.021	0.016
FTSE	0.55	0.55	0.19	0.14	0.019	0.021

Table 3.8: European Indices-2

	$\mu^{+}_{\alpha^{+}_{BHP}}$	$\mu^{-}_{\alpha^{-}_{BHP}}$	$\sigma^{+}_{\alpha^{+}_{BHP}}$	$\sigma^{-}_{\alpha^{-}_{BHP}}$
FCHI	0.07	0.08	0.04	0.04
DAX	0.09	0.10	0.05	0.05
MIBTEL	0.09	0.12	0.05	0.06
AEX	0.10	0.12	0.05	0.06
OSEAX	0.07	0.11	0.04	0.05
SMCI	0.15	0.13	0.06	0.06
OMXSPI	0.08	0.10	0.04	0.05
SSMI	0.07	0.07	0.04	0.04
FTSE	0.06	0.06	0.03	0.03

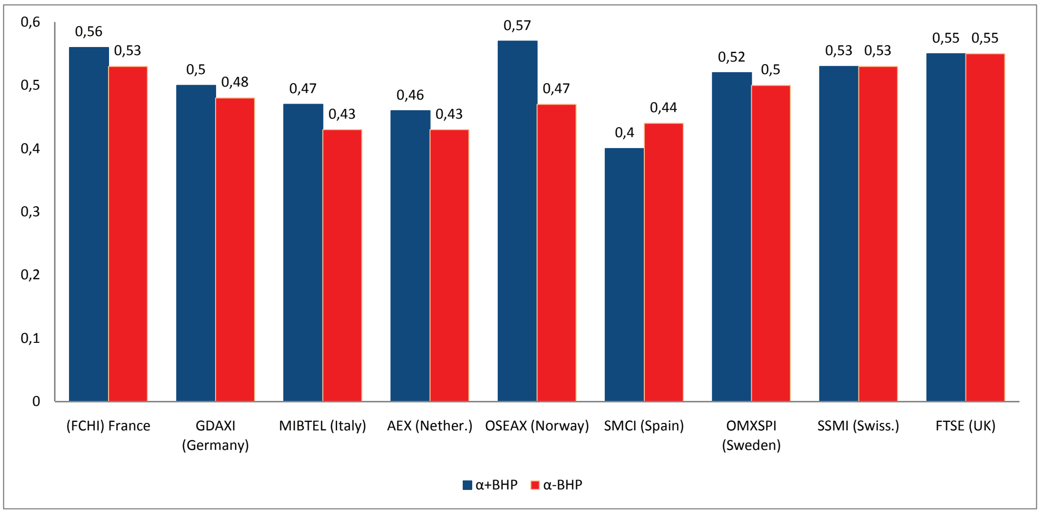


Figure 3.39: *Values of α^+_{BHP} and α^-_{BHP} in European indices.*

3.4 World Wide Indices

We analyze the time series of the following World wide Indices: from Argentina, the *(MERV)* index between October of 1996 and September of 2009; from Brazil, the *(BVSP)- Bovespa* index between April of 1993 and September of 2009; from Mexico, the *(MXX)- IPC* index between November of 1991 and September of 2009; from Japan, the *(N225)- Nikkei 225* index between January of 1984 and September of 2009; and from Hong-Kong, the *(HSI)- Hang Seng* index between January of 2000 and October of 2010. The results obtained for these indices are presented in Tables 3.9, 3.10 and in Figure 3.40.

Table 3.9: World Wide Indices-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^-_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
MERV	0.48	0.46	0.92	0.50	0.016	0.025
BVSP	0.51	0.50	0.29	0.15	0.021	0.027
MXX	0.54	0.53	0.68	0.19	0.015	0.023
N225	0.47	0.45	0.45	0.07	0.015	0.023
HSI	0.48	0.42	0.51	0.12	0.023	0.035

Table 3.10: World Wide Indices-2

	$\mu^+_{\alpha^+_{BHP}}$	$\mu^-_{\alpha^-_{BHP}}$	$\sigma^+_{\alpha^+_{BHP}}$	$\sigma^-_{\alpha^-_{BHP}}$
MERV	0.12	0.13	0.06	0.06
BVSP	0.12	0.12	0.06	0.06
MXX	0.08	0.08	0.04	0.04
N225	0.10	0.11	0.05	0.05
HSI	0.10	0.14	0.05	0.06

In all the indices, we observe that α^+_{BHP} and α^-_{BHP} are similar and vary between 0.42 and 0.54 and that $P^+_{\alpha^+_{BHP}}$ and $P^-_{\alpha^-_{BHP}}$ are higher that 0.01.

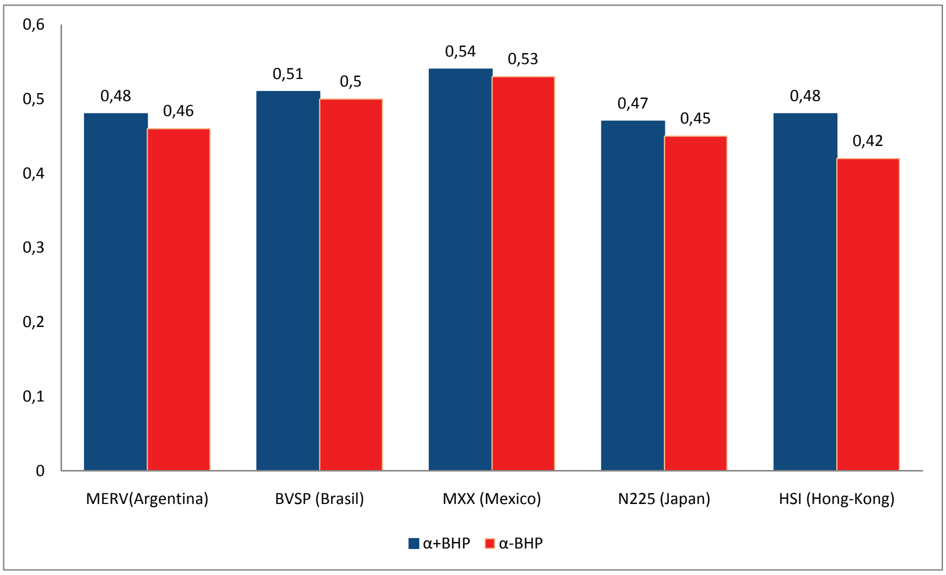


Figure 3.40: *Values of* α^+_{BHP} *and* α^-_{BHP} *in World wide indices.*

3.5 Energy Sources

We consider, in our analysis, several energy sources daily data series that correspond to periods of time between 6-14 years. We study energy sources data of two distinct groups: Non-renewable Sources and Renewable Sources (see [28]). In the first group, we consider oil and petroleum products, namely crude oil (from January 1986 to January of 2010), heating oil (from June 1986 to January of 2010), gasoline (from June 1986 to January of 2010) and propane (from May 1992 to January of 2010). In the second group, we analyze biofuels such as ethanol (from March 2005 to October of 2009) and biodiesel (from December of 2006 to October of 2009). We also study a product from which renewable energy is produced, respectively corn (from February 1998 to October of 2009)

The results obtained for the different energy sources are presented in Tables 3.11, 3.12, 3.13, 3.14 and in Figures 3.41 and 3.42, respectively.

Table 3.11: Non-renewable Energy Sources Prices-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^-_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
Crude Oil	0.52	0.51	0.26	0.43	0.018	0.016
Heating Oil	0.52	0.57	0.16	0.70	0.021	0.013
Propane	0.32	0.30	0.64	0.70	0.017	0.017
Gasoline	0.55	0.55	0.13	0.02	0.021	0.029

In the non-renewable energy sources prices, we observe that α^+_{BHP} and α^-_{BHP} vary between 0.30 and 0.57.

In the renewable energy sources prices, we observe that α^+_{BHP} and α^-_{BHP} vary between 0.20 and 0.50.

The results indicate universality in energy sources prices.

Table 3.12: Non-renewable Energy Sources Prices-2

	$\mu^{+}_{\alpha^{-}_{BHP}}$	$\mu^{-}_{\alpha^{-}_{BHP}}$	$\sigma^{+}_{\alpha^{+}_{BHP}}$	$\sigma^{-}_{\alpha^{-}_{BHP}}$
Crude Oil	0.11	0.12	0.06	0.06
Heating Oil	0.11	0.09	0.06	0.05
Propane	0.25	0.28	0.08	0.08
Gasoline	0.11	0.11	0.05	0.05

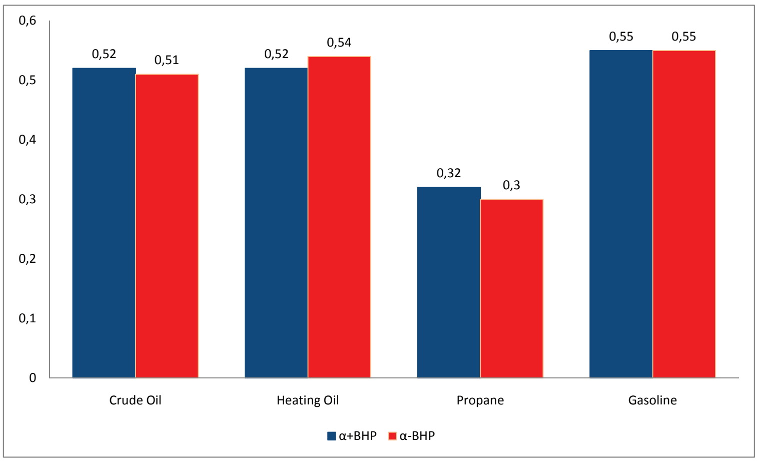


Figure 3.41: *Values of α^{+}_{BHP} and α^{-}_{BHP} in non-renewable energy sources prices.*

Table 3.13: Renewable Energy Sources and Products-1

	α^{+}_{BHP}	α^{-}_{BHP}	$P^{+}_{\alpha^{+}_{BHP}}$	$P^{-}_{\alpha^{+}_{BHP}}$	$D^{+}_{\alpha^{+}_{BHP}}$	$D^{-}_{\alpha^{-}_{BHP}}$
Ethanol	0.20	0.26	0.02	0.46	0.070	0.034
Biodiesel	0.30	0.50	0.46	0.94	0.064	0.035
Corn	0.41	0.41	0.56	0.25	0.021	0.027

Table 3.14: Renewable Energy Sources and Products-2

	$\mu^{+}_{\alpha^{+}_{BHP}}$	$\mu^{-}_{\alpha^{-}_{BHP}}$	$\sigma^{+}_{\alpha^{+}_{BHP}}$	$\sigma^{-}_{\alpha^{-}_{BHP}}$
Ethanol	0.51	0.40	0.14	0.12
Biodiesel	0.56	0.35	0.16	0.14
Corn	0.16	0.17	0.07	0.07

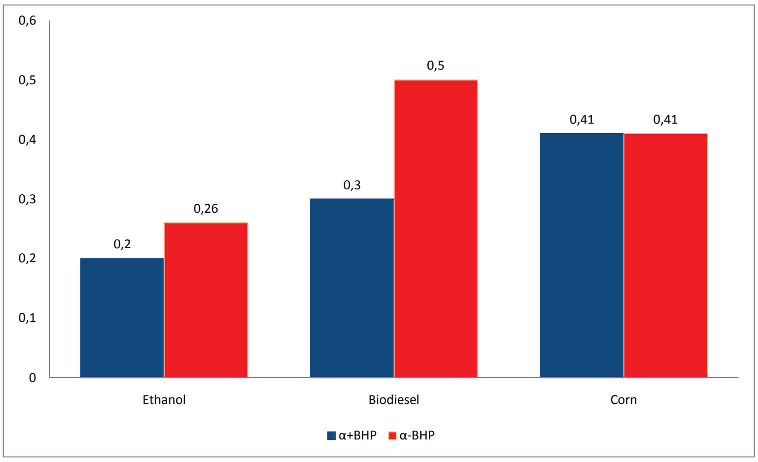


Figure 3.42: *Values of α^{+}_{BHP} and α^{-}_{BHP} in renewable energy sources prices.*

3.6 Exchange Rates

In finance, the exchange rates (also known as the foreign-exchange rate, forex rate or FX rate) between two currencies specify how much one currency is worth in terms of the other. It is the value of a foreign nation's currency in terms of the home nation's currency. The foreign exchange market is one of the largest markets in the world. We study the daily returns of the following exchange rates: *EURCHF* Euro vs. Swiss franc (from March 1979 to January 2010) and *EURJPY* Euro vs. Japanese yen (from March 1979 to January 2010).

The results obtained for the different exchange rates that we analyzed are presented in Tables 3.15, 3.16 and in Figure 3.43.

Table 3.15: Exchange Rates-1

	α^+_{BHP}	α^-_{BHP}	$P^+_{\alpha^+_{BHP}}$	$P^-_{\alpha^+_{BHP}}$	$D^+_{\alpha^+_{BHP}}$	$D^-_{\alpha^-_{BHP}}$
EUR-CHF	0.38	0.42	0.05	0.02	0.021	0.024
EUR-JPY	0.58	0.52	0.03	0.01	0.023	0.028

Table 3.16: Exchange Rates-2

	$\mu^+_{\alpha^+_{BHP}}$	$\mu^-_{\alpha^-_{BHP}}$	$\sigma^+_{\alpha^+_{BHP}}$	$\sigma^-_{\alpha^-_{BHP}}$
EUR-CHF	0.10	0.08	0.04	0.04
EUR-JPY	0.04	0.06	0.02	0.03

In the studied exchange rates, we observe that α^+_{BHP} and α^-_{BHP} vary between 0.38 and 0.58 and $P^+_{\alpha^+_{BHP}}$ and $P^-_{\alpha^-_{BHP}}$ are higher that 0.01, which can indicate universality in these data series.

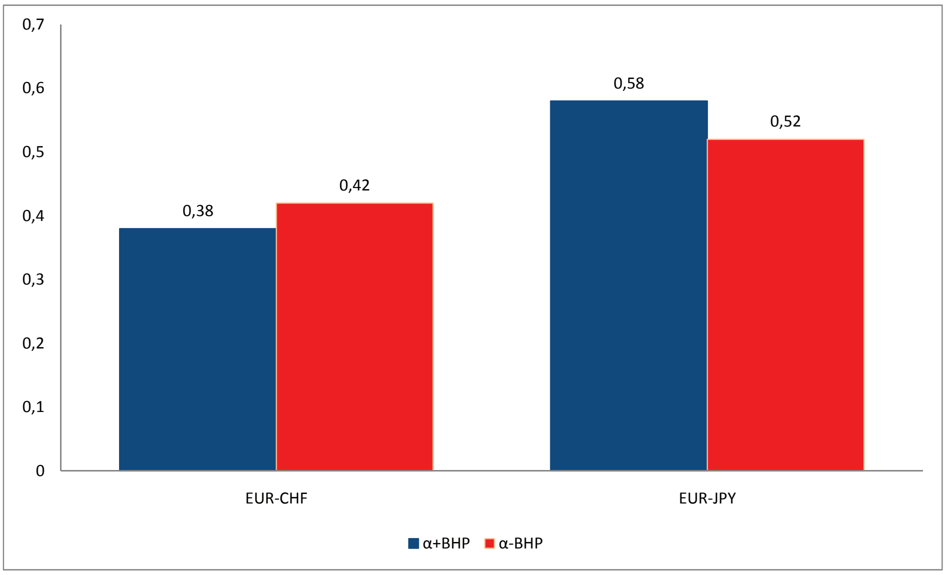


Figure 3.43: *Values of* α^{+}_{BHP} *and* α^{-}_{BHP} *in exchange rates.*

3.7 Conclusion

We used the Kolmogorov-Smirnov statistical test to compare the histogram of the α positive fluctuations and α negative fluctuations with the truncated Bramwell-Holdsworth-Pinton (BHP) probability density function. We found that the parameters α^{+}_{BHP} and α^{-}_{BHP} for the positive and negative fluctuations, respectively, vary mostly around 0.50. The fact that α^{+}_{BHP} is different from α^{-}_{BHP} can be due to leverage effects. We presented the data collapse of the corresponding fluctuations histograms to the BHP pdf. Furthermore, we computed the analytical approximations of the pdf of the normalized I_p index positive and negative returns in terms of the BHP pdf. We showed the data collapse of the histogram of the positive and negative returns to our proposed theoretical pdfs $f_{BHP,I_p,+}$ and $f_{BHP,I_p,-}$. We also computed the analytical approximations of the pdf of the normalized positive and negative spot daily prices or daily returns $r(t)$ of distinct energy sources ES and exchange rates ER.

Since the BHP probability density function appears in several other dissimilar phenomena (see, for example, [18, 24, 32, 33, 36, 57]), our results reveal a universal feature of the stock market exchange. Furthermore, these results lead to the construction of a new qualitative and quantitative econophysics model for the stock market based on the two-dimensional spin model (2dXY) at criticality (see [31]) and to a new stochastic differential equation model for the stock exchange market indices (see [53]) that provides a better understanding of several stock exchange crises (see [54]).

Chapter 4

Modeling Human Decisions

In this chapter, we construct a model, using Game Theory, for the Theory of Planned Behavior and we propose the Bayesian-Nash Equilibria as one of many possible mechanisms to transform human intentions into behavior decisions. We show that saturation, boredom and frustration can lead to the adoption of a variety of different behavior decisions, as opposed to no saturation, which leads to the adoption of a single consistent behavior decision. Furthermore, we use the new game theoretical model to understand the impact of the leaders in the decision-making of individuals or groups and we study how the characteristics of the leaders have an influence over the others' decisions. We also apply the model to a students success example, describing Nash equilibria and "herding" effects, identifying a hysteresis in the process.

4.1 Theory of Planned Behavior or Reasoned Action

The Theory of Planned Behavior or Reasoned Action is summarized in Figure 4.1 (see [1]), where we observe that external variables are divided in three categories: intrapersonal associated to individual actions; interpersonal associated to the interaction of the individual with others; and sociocultural associated to social values. These external variables influence, especially, the intermediate variables which are also subdivided in three major groups: social norms, attitude, and self-efficacy. The social norms can be the opinions, conceptions and judgments that others have about a certain behavior; attitudes are personal opinions in favor or against a specific behavior; and self-efficacy is the extent of ability to control a certain behavior. These external and intermediate variables lead to a consequent intention to adopt a certain behavior.

4.2 Platonic Idealized Psychological World

In the *platonic idealized psychological model*, inspired in Plato‘s world of thoughts or of the intelligible reality, the individuals have no uncertainties in their taste and crowding types and welfare function. We consider that the individuals are pure in the sense that the external and intermediate variables of the model are known, to all individuals.

In this model, the individuals will choose a certain behavior/group $g \in G$. Those choices will be done, taking in account their characteristics and

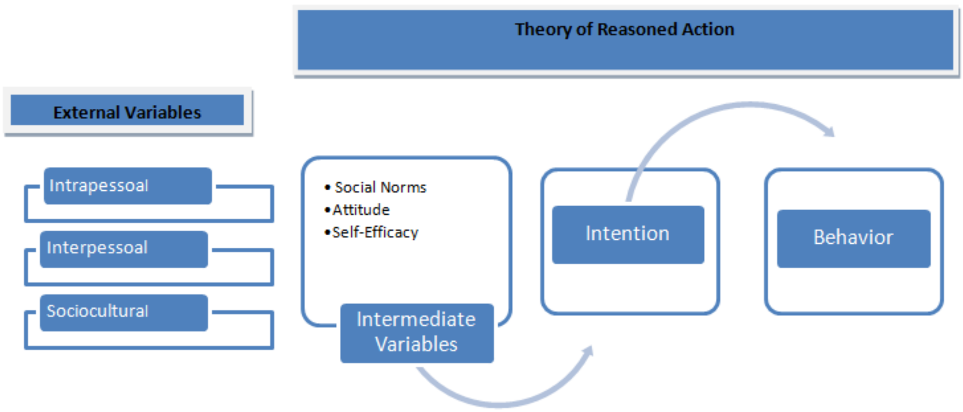


Figure 4.1: *Theory of Planned Behavior*

personal preferences (*taste type*) and the other individuals observable characteristics (*crowding club vector*). The goal is to present a decision mechanism for the individuals, taking in account their and the others types.

Let us consider a finite number S of individuals. For each individual $s \in S$, we distinguish two types of characteristics: *taste type* $\mathcal{T} : S \rightarrow \ T$ and *crowding type* $\mathcal{C} : S \rightarrow \ C$. We associate to each individual $s \in S$ one *taste type* $\mathcal{T}(s) = t \in T$ that describes the individual's inner characteristics, which are not always observable by the other individuals. We also associate to each individual $s \in S$ one *crowding type* $\mathcal{C}(s) = c \in C$ that describes the individual's characteristics observed by the others and that can influence the welfare of the others. In accordance with the Theory of Planned Behavior or Reasoned Action, we associate the intrapersonal external variables and the attitude and self efficacy intermediate variables with the taste type, and the interpersonal and sociocultural external variables and the social norms intermediate variable with the crowding type.

The individuals, with their own characteristics, can define a strategy $\mathcal{G} : S \to G$, i.e. each individual $s \in S$ chooses a behavior/group $\mathcal{G}(s)$. Each strategy $\mathcal{G}$ corresponds to an intention in the Theory of Planned Behavior (see [5]). Given a behavior/group strategy $\mathcal{G} : S \to G$, the *crowding vector* $m(\mathcal{G}) \in (\mathbb{N}^C)^G$ is the vector whose components $m_c^g = m_c^g(\mathcal{G})$ determine the number of individuals that choose behavior/group g with crowding type $c \in C$, i.e.

$$m_c^g = \#\left\{s \in S : \mathcal{G}\left(s\right) = g \wedge \mathcal{C}\left(s\right) = c\right\}.$$

We denote by $s_{t,c}$ the individual s with taste type t and crowding type c. We measure the level of welfare, or personal satisfaction, that an individual $s_{t,c}$ acquires by choosing a behavior/group $g \in G$ with *crowding vector* $m = m(\mathcal{G})$, using a utility function $u_{t,c} : G \times (\mathbb{N}^C)^G \to \mathbb{R}$ given by

$$u_{t,c}(g, m) = V_{t,c}^g + \sum_{c' \in \mathcal{C}} A_{t,c}^{g,c'} m_{c'}^g$$

where $V_{t,c}^g$ measures the satisfaction level that each individual $s_{t,c}$ has in choosing a behavior/group $g \in G$, and $A_{t,c}^{g,c'}$ evaluates the satisfaction that each individual $s_{t,c}$ has with the presence of an individual with crowding type c' that chooses the same behavior/group g.

The strategy $\mathcal{G}^* : S \to G$ is a *(pure) Nash Equilibrium* behavior/group, if given the choice options of all individuals, no individual feels motivated to change his behavior/group choice, i.e. his utility does not increase by changing his behavior/group decision (see A.A.Pinto [57]).

The platonic idealized psychological model gives rise to a dictionary between Game Theory and Theory of Planned Behavior that is summarized in Figure 4.2 (see Almeida [4, 5]).

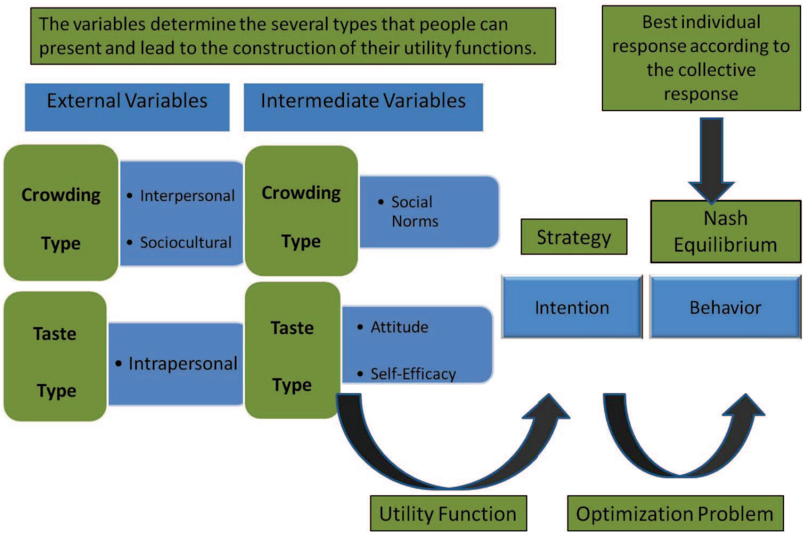


Figure 4.2: *Theory of Planned Behavior / Platonic idealized psychological world*

We denote by $S_{(t,c)}$ the group of all individuals $s_{t,c}$ with the same taste type $t \in T$ and the same crowding type $c \in C$. Let $n(t,c)$ correspond to the number of individuals in $S_{(t,c)}$.

Remark 1 *An interesting way to interpret $S_{(t,c)}$ is to consider that $n(t,c)$ is the number of times that a single individual $s_{t,c}$ has to take an action. In this case, $A^{g,c}_{t,c} > 0$ can be interpreted as the individual positive reward by repeating the same behavior/group choice $g \in G$, i.e. the individual $s_{t,c}$ does not feel a saturation effect by repeating the same choice. On the other hand, $A^{g,c}_{t,c} < 0$ can be interpreted as the individual negative reward by repeating the* *same behavior/group choice $g \in G$, i.e. the individual $s_{t,c}$ feels a saturation, boredom or frustration effect by repeating the same choice.*

4.2.1 Individuals that like to repeat the same behavior (no-saturation)

In this section, we consider the hypothesis that $A_{t,c}^{g,c} > 0$. We exploit situations where no-saturation can lead to the adoption of a single consistent behavior decision.

Lemma 4.2.1 *Let $\mathcal{G}^*$ be a Nash Equilibrium. Let $A_{t,c}^{g,c} > 0$, for every $g \in G$, $t \in T$ and $c \in C$. For every taste type $t \in T$ and every crowding type $c \in C$, all the individuals $s_{t,c} \in S_{(t,c)}$, with the same taste type and the same crowding type, choose the same behavior/group $\mathcal{G}^*(s_{t,c}) = \mathcal{G}^*(S_{(t,c)})$.*

Remark 2 *If Lemma 4.2.1 holds, considering that $n(t,c)$ represents the number of times that a same individual $s_{t,c}$ has to take an action, one concludes that the individual $s_{t,c}$ does not have a negative reward by repeating the same behavior/group choice and so has a single consistent behavior decision.*

Proof. Let us suppose that for a group strategy $\mathcal{G} : S \to G$ individuals with same taste type t and same crowding type c choose more than one behavior/group. Let us denote by g the behavior/group choice where these individuals $s_{t,c}$ attain the highest welfare (does not need to be unique). Then any individual $s_{t,c}$ that chooses another behavior/group g' by changing his choice to the behavior/group g will increase his welfare because $A_{t,c}^{g,c} > 0$. Hence $\mathcal{G} : S \to G$ is not a Nash Equibrium behavior/group. ■

4.2.2 Individuals that choose what they prefer

In this section, we exploit situations where individuals choose the behavior/group that they prefer, independently of the influence of the others.

We define the *worst neighbors* $WN_g(t,c)$ of the individual $s_{t,c}$ in choosing the behavior/group g by

$$WN_g(t,c) = V^g_{t,c} + \sum_{c' \in C, A^{g,c'}_{t,c} < 0} A^{g,c'}_{t,c} \sum_{t' \in T} n(t',c'),$$

where $V^g_{t,c}$ represents the valuation of the individual $s_{t,c}$ in choosing the behavior/group g, and $\sum_{c' \in C, A^{g,c'}_{t,c} < 0} A^{g,c'}_{t,c} \sum_{t' \in T} n(t',c')$ represents the worst neighbors that the individual $s_{t,c}$ can have for the same choice of behavior/group g.

We define the *best neighbors* $BN_g(t,c)$ of the individual $s_{t,c}$ in choosing the behavior/group g by

$$BN_g(t,c) = V^g_{t,c} + \sum_{c' \in C, A^{g,c'}_{t,c} > 0} A^{g,c'}_{t,c} \sum_{t' \in T} n(t',c'),$$

where $V^g_{t,c}$ represents the valuation of the individual $s_{t,c}$ in choosing the behavior/group g and $\sum_{c' \in C, A^{g,c'}_{t,c} > 0} A^{g,c'}_{t,c} \sum_{t' \in T} n(t',c')$ represents the best neighbors that the individual $s_{t,c}$ can have for the same choice of behavior/group g.

Let $g_W = \arg\max_{\{g \in G\}} WN_g(t,c)$ and $BN(t,c) = \max_{\{g \in G: g \neq g_W\}} BN_g(t,c)$.

Lemma 4.2.2 *If $WN_{g_W}(t,c) > BN(t,c)$ then $\mathcal{G}^*(S_{t,c}) = g_W$, for every Nash Equilibrium $\mathcal{G}^*$.*

We note that there is a value V_0 such that Lemma 4.2.2 holds, for all $V_{t,c}^{g_W} > V_0$.

Remark 3 *If Lemma 4.2.2 holds, considering that $n(t,c)$ represents the number of times that a same individual $s_{t,c}$ has to take an action, one concludes that the individual $s_{t,c}$ chooses the same behavior/group g_W, independently of the Nash Equilibrium behavior/group considered.*

Proof. Let us suppose, by contradiction, that $\mathcal{G}^*$ is a Nash equilibrium, such that, at least one individual $s_{t,c}$ chooses the behavior/group $g \in G \setminus \{g_w\}$. By construction of the best neighbors $BN(t,c)$, the utility function is bounded above by

$$u_{t,c}(g,m) \leq BN_g(t,c) \leq BN(t,c).$$

If the individual changes his behavior/group choice to g_W, then by construction of the worst neighbors $WN_{g_W}(t,c)$, the utility function is bounded below by

$$u_{t,c}(g_W,m) \geq WN_{g_W}(t,c).$$

Since $WN_{g_W}(t,c) > BN(t,c)$, we get

$$u_{t,c}(g_W,m) > u_{t,c}(g,m),$$

which is a contradiction. ■

4.2.3 Boredom and Frustration

In this section, we consider the hypothesis that $A^{g,c}_{t,c} < 0$. We exploit the situations where boredom and frustration can lead to the adoption of a variety of different behavior decisions.

We define the *worst lonely neighbors* $WNL_g(t,c)$ of the individual $s_{t,c}$ in choosing the behavior/group g by

$$WNL_g(t,c) = V^g_{t,c} + A^{g,c}_{t,c} + \sum_{c' \in C, c' \neq c, A^{g,c'}_{t,c} < 0} A^{g,c'}_{t,c} \sum_{t' \in T} n(t',c').$$

Let $g_1 = \arg\max_{\{g \in G\}} WNL_g(t,c)$ and $g_2 = \arg\max_{\{g \in G: g \neq g_1\}} WNL_g(t,c)$.

We define the *best lonely neighbors* $BNL(t,c)$ of the individual $s_{t,c}$ by

$$BNL(t,c) = \max_{\{g \in G: g_1 \neq g_w \neq g_2\}} BN_g(t,c)$$

Lemma 4.2.3 *Let $\mathcal{G}^*$ be a Nash Equilibrium, $A^{g_1,c}_{t,c} < 0$ and $A^{g_2,c}_{t,c} < 0$. If*

$$BNL(t,c) < WNLg_2(t,c)$$

and

$$A^{g_i,c}_{t,c} n(t,c) + BN_{g_i}(t,c) < WNL_{g_j}(t,c),$$

for every $i, j \in \{1,2\}$ with $i \neq j$, then $\mathcal{G}^*(S_{t,c}) \cap g_1 \neq \emptyset$ and $\mathcal{G}^*(S_{t,c}) \cap g_2 \neq \emptyset$.

Remark 4 *If Lemma 4.2.3 holds, considering that $n(t,c)$ represents the number of times that a same individual $s_{t,c}$ has to take an action, one* *concludes that the individual $s_{t,c}$ splits his decision, at least, between the behavior/groups g_1 and g_2, independently of the pure Nash Equilibrium behavior/group considered.*

Proof. Let us suppose, by contradiction that $\mathcal{G}^*$ is a Nash Equilibrium, such that either

a) $\mathcal{G}^*(S_{t,c}) \cap g_1 = \emptyset \wedge \mathcal{G}^*(S_{t,c}) \cap g_2 = \emptyset$; or

b) $\mathcal{G}^*(S_{t,c}) \cap g_1 \neq \emptyset \wedge \mathcal{G}^*(S_{t,c}) \cap g_2 = \emptyset$; or

c) $\mathcal{G}^*(S_{t,c}) \cap g_1 = \emptyset \wedge \mathcal{G}^*(S_{t,c}) \cap g_2 \neq \emptyset$

In case a), let $s_{t,c}$ be an individual that chooses a behavior/group $g \in G \setminus \{g_1, g_2\}$. By construction of the best lonely neighbors $BNL(t,c)$, the utility function is bounded above by

$$u_{t,c}(g,m) \leq BN_g(t,c) \leq BNL(t,c).$$

If the individual $s_{t,c}$ changes his behavior/group choice to g_2 (or g_1) then, by construction the worst lonely neighbors $WNLg_2(t,c)$, the utility function is bounded below by

$$u_{t,c}(g_2,m) \geq WNL_{g_2}(t,c).$$

Since $BNL(t,c) < WNLg_2(t,c)$ we get

$$u_{t,c}(g_2,m) > u_{t,c}(g,m),$$

which is a contradiction.

In case b), either i) all the individuals choose the behavior group g_1 or ii) there is at least one individual $s_{t,c}$ that chooses a behavior/group $g \in G \setminus \{g_1, g_2\}$. Case bii) does not occur and the proof follows similarly to the proof of case a). In case bi), by construction of $A_{t,c}^{g_1,c}n(t,c)$ and of the best neighbors $BN_{g_1}(t,c)$, the utility function is bounded above by

$$u_{t,c}(g_1, m) \leq A_{t,c}^{g_1,c}n(t,c) + BN_{g_1}(t,c).$$

If this individual changes his behavior/group choice to g_2, then by construction of the worst lonely neighbors $WNLg_2(t,c)$, the utility function is bounded below by

$$u_{t,c}(g_2, m) \geq WNL_{g_2}(t,c).$$

Since

$$A_{t,c}^{g_1,c}n(t,c) + BN_{g_1}(t,c) < WNL_{g_2}(t,c),$$

we get

$$u_{t,c}(g_2, m) > u_{t,c}(g_1, m),$$

which is a contradiction.

The proof of case c) follows similarly to the proof of case a).

■

4.3 Cave Psychological Model

Our *cave psychological world* is inspired in Plato´s concrete world, where all things are shadows of the intelligible reality in Plato´s world of thoughts. This world consists of individuals whose taste and crowding types follow the shadows of the idealized taste and crowding types according to a given probability distribution. Furthermore, the individuals know their welfare function just in expected value.

Let n be equal to the cardinality $\#S$ of S. Let us denote the individuals in S by $s_1, ..., s_n$. We represent by $\mathcal{E}$ the set of all external variables e and we represent by $\mathcal{I}$ the set of all intermediate variables i. The values of the external and intermediate variables determine the taste type t and the crowding type c of the individuals s_l, by a map $t_l \times c_l : \mathcal{E} \times \mathcal{I} \to \mathcal{T} \times \mathcal{C}$ given by

$$(e, i) \to (t_l(e, i), c_l(e, i)).$$

Let $V_l : G \times \mathcal{E} \times \mathcal{I} \to \mathbb{R}$ be a map, where $V_l(g, t_l(e_l, i_l), c_l(e_l, i_l))$ measures the satisfaction level that the individual s_l, with external and intermediate variables (e_l, i_l), has in choosing the behavior/group g. Let $f_l : (G \times \mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ be a map, where

$$f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))$$

measures the satisfaction level that the individual s_l has taking in account the crowding types c_k of the other individuals s_k and their behavior/group choices g_k.

The welfare $u_l : (G \times \mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ of the individual s_l is given by the utility function

$$u_l(g_1, ..., g_n, e_1, ..., e_n, i_1, ..., i_n) =$$
$$V_l(g_l, t_l(e_l, i_l), c_l(e_l, i_l)) + f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n)).$$

The *shadow* is the joint probability distribution P, of all the individuals in S, with support contained in $(\mathcal{E} \times \mathcal{I})^n$, and P_l is the marginal probability distribution, of the individual s_l, with support contained in $\mathcal{E}_l \times \mathcal{I}_l$. Let

$$E(V, l, g) = E_{P_l}\left[V_l(g, t_l(e_l, i_l), c_l(e_l, i_l))\right]$$

and

$$E(f, l, g_1, ..., g_n) = E_P\left[f_l(g_1, ...g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))\right].$$

The expected utility $E(u_l)$ of an individual $s_l \in S$ is given by

$$E(u_l) = E(V_l, l, g) + E(f, l, g_1, ..., g_n)$$

In this way, when the individuals pass from the intention to the behavior/group decision, the indetermination of their types is solved arbitrarily, for incidental reasons or reasons that the individuals are not able to predict except in probability. Another possible construction is presented in [3] where the individuals know their own taste and crowding types, but ignore the external and intermediate variables of the other individuals, except in probability.

The strategy $\mathcal{G}^* : S \to G$ is a (pure) *Bayesian-Nash Equilibrium*, if given the choice options of all individuals, no individual feels motivated to change his behavior/group choice, i.e. his expected utility does not increase by changing his behavior/group decision (see Pinto [57]).

The cave idealized psychological model gives rise to a dictionary between Game Theory and Theory of Planned Behavior that is summarized in Figure 4.3.

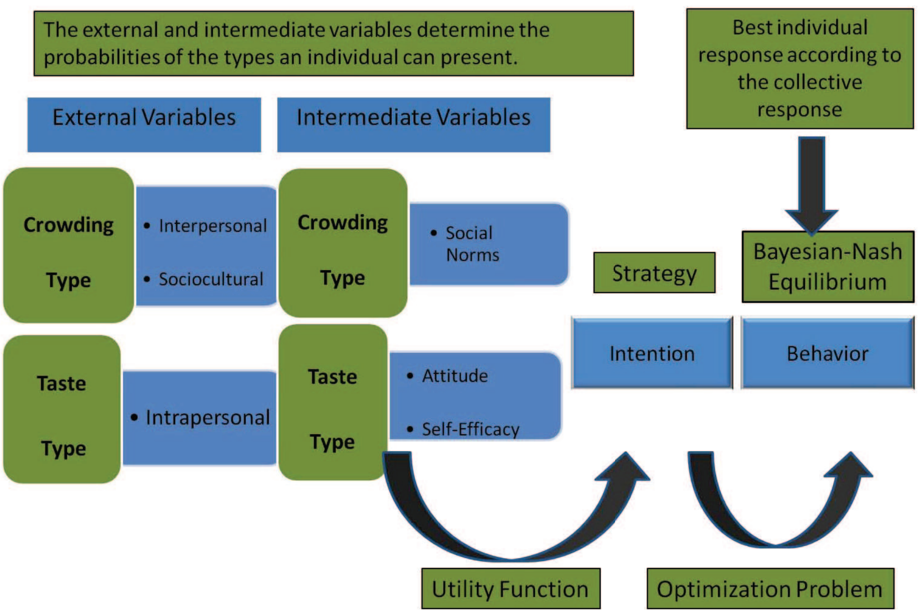


Figure 4.3: *Theory of Planned Behavior / Cave Psychological Model*

4.3.1 Individuals that like to repeat the same behavior (no-saturation)

In this section, we exploit situations where no-saturation can lead to the adoption of a single consistent behavior decision.

A *class* $F \subset S$ is *cohesive*, if for every $s_l, s_k \in F$ we have

$$E_P[u_k(g_1, ...\hat{g}_k, ..., g_n, e_1, ..., e_n, i_1, ..., i_n)] > E_P[u_l(g_1, ..., g_n, e_1, ..., e_n, i_1, ..., i_n)],$$

for every $(g_1, ..., g_n) \in G^n$ and $\hat{g}_k = g_l$.

Lemma 4.3.1 *Let $\mathcal{G}^*$ be a Bayesian-Nash Equilibrium. All the individuals of a cohesive class F choose the same behavior/group $\mathcal{G}^*(F) = \mathcal{G}^*(s_k)$, for all $s_k \in F$.*

The proof of Lemma 4.3.1 follows similarly to the proof of Lemma 4.2.1.

When all the marginal probabilities P_l are Dirac masses and P is the corresponding product measure, Lemma 4.2.1 is a sub-case of Lemma 4.3.1.

Remark 5 *An interesting way to interpret the class F is to consider that consists of a single individual $s_F \in S$ that has to take $\#F$ behavior/group decisions. If Lemma 4.3.1 holds, the individual s_F does not have a negative reward by repeating the same behavior/group choice and so has a single consistent behavior decision.*

4.3.2 Individuals that choose what they prefer

In this section, we exploit situations where individuals choose the behavior/group that they prefer, independently of the influence of the others.

Let $f_l^+ : G \times (\mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ be given by

$$f_l^+(g; e_1, ..., e_n, i_1, ..., i_n) =$$
$$\max_{\{(g_1,...,g_n)\in G^n : g_l = g\}} \{0, f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))\}.$$

Let $f_l^- : G \times (\mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ be given by

$$f_l^-(g; e_1, ..., e_n, i_1, ..., i_n) =$$
$$\min_{\{(g_1,...,g_n)\in G^n : g_l = g\}} \{0, f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))\}.$$

We define the *shadow worst neighbors* $SWN_g(l)$ of the individual s_l that chooses the behavior/group g by

$$SWN_g(l) = E(V, l, g) + E_P \left[f_l^-(g; e_1, ..., e_n, i_1, ..., i_n) \right]$$

We define the *shadow best neighbors* $SBN_g(l)$ of the individual s_l that chooses the behavior/group g by

$$SBN_g(l) = E(V, l, g) + E_P \left[f_l^+(g; e_1, ..., e_n, i_1, ..., i_n) \right]$$

Let $g_W(l) = \arg\max_{\{g\in G\}} SWN_g(l)$ and $SBN(l) = \max_{\{g\in G : g \neq g_W(l)\}} SBN_g(l)$.

Lemma 4.3.2 *If $SWN_{g_W(l)}(l) > SBN(l)$, then $\mathcal{G}^*(s_l) = g_W$ for every Bayesian-Nash Equilibrium $\mathcal{G}^*$.*

The proof follows similarly to the proof of Lemma 4.2.2.

The class $F \subset S$ is g_W *cohesive*, if, for every $s_l \in F$, $g_W(l) = g_W$ and $SWN_{g_W}(l) > SBN(l)$.

Corollary 4.3.1 *If a class F is g_W cohesive then $\mathcal{G}^*(F) = g_W$, for every Bayesian-Nash Equilibrium $\mathcal{G}^*$.*

When all the marginal probabilities P_l are Dirac masses and P is the corresponding product measure, Lemma 4.2.2 is a sub-case of Corollary 4.3.1.

Remark 6 *If Corollary 4.3.1 holds, considering that $\#F$ represents the number of times that a individual s_F has to take an action, one concludes that the individual s_F chooses the same behavior/group g_W, independently of the Nash Equilibrium behavior/group considered.*

4.3.3 Boredom and Frustration

In this section, we exploit the situations where boredom and frustration can lead to the adoption of a variety of different behavior decisions.

Let $F \subset S$ and $s_l \in F$. Let $h_l^- : G \times (\mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ be given by

$$h_l^-(g; e_1, ..., e_n, i_1, ..., i_n) =$$

$$\min_{\{(g_1,...,g_n)\in G^n : g_l = g \wedge g_k \neq g, \forall s_k \in F\setminus\{s_l\}\}} \{0, f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))\}.$$

We define the *worst family lonely neighbors* $WNF_g(l)$ of the individual s_l in choosing the behavior/group g by

$$WNF_g(l) = E(V, l, g) + E_P \left[h_l^-(g; e_1, ..., e_n, i_1, ..., i_n) \right].$$

Let $g_1(l) = \arg\max_{\{g \in G\}} WNF_g(l)$ and $g_2(l) = \arg\max_{\{g \in G: g \neq g_1(l)\}} WNF_g(l)$.

Let $h_l^+ : G \times (\mathcal{E} \times \mathcal{I})^n \to \mathbb{R}$ be given by

$$h_l^+(g; e_1, ..., e_n, i_1, ..., i_n) =$$
$$\max_{\{(g_1,...,g_n) \in G^n : g_k = g, \forall s_k \in F\}} \{0, f_l(g_1, ..., g_n, t_l(e_l, i_l); c_1(e_1, i_1), ..., c_n(e_n, i_n))\}.$$

We define the *best family lonely neighbors* $BNF_g(l)$ of the individual s_l in choosing the behavior/group g by

$$BNF_g(l) = E(V, l, g) + E_P \left[h_l^+(g; e_1, ..., e_n, i_1, ..., i_n) \right].$$

Let $SBN(l) = \max_{\{g \in G: g_1(l) \neq g \neq g_2(l)\}} SBN_g(l)$.

The class F is (g_1, g_2) *split*, if, for every $s_l \in F$, $g_1(l) = g_1$, $g_2(l) = g_2$, $SBN(l) < WNFg_2(l)$ and $BNF_{g_i}(l) < WNF_{g_j}(l)$, for every $i, j \in \{1, 2\}$ with $i \neq j$.

Lemma 4.3.3 *Let F be a (g_1, g_2) split class. For every Bayesian-Nash Equilibrium $\mathcal{G}^*$, $\mathcal{G}^*(F) \cap g_1 \neq \emptyset$ and $\mathcal{G}^*(F) \cap g_2 \neq \emptyset$.*

The proof of Lemma 4.3.3 follows similarly to the proof of Lemma 4.2.3.

When all the marginal probabilities P_l are Dirac masses and P is the corresponding product measure, Lemma 4.2.3 is a sub-case of Lemma 4.3.3

Remark 7 *If Lemma 4.3.3 holds, considering that $\#F$ represents the number of times that the individual s_F has to take an action, one concludes that the individual s_F splits his decision, at least, between the behavior/groups g_1 and g_2, independently of the pure Nash Equilibrium behavior/group considered.*

4.4 Leadership in a Game Theoretical Model

A leader is an individual who can influence others to choose a certain group/ behavior. We consider that the leader makes his group/behavior choice before the others, and therefore the others already know the leader's decision before taking their behavior/group decision. We study how the choice of the leader s_{t^l,c^l} can influence the followers s_{t^f,c^f} to choose the same behavior/group g as the leader, see [2, 3].

The leaders and the followers are characterized by the parameters (α, R, V, L) and we distinguish the following types:

- *Altruist and individualist leaders.* The leader s_{t^l,c^l} values $V > 0$ the behavior/group g and can donate a part $(1 - R)V$ to the followers. The parameter R determines the fraction $(1 - R)V$ of the good V donated from the leader to the followers. After the donation, the new valuation of the leader s_{t^l,c^l} for the group g is $V^g_{t^l,c^l} = RV$. The *altruist leader* is the one who distributes a valuation to the followers of the behavior/group g, i.e. $R < 1$ and the *individualist leader* is the one who gives a devaluation or debt to the followers of the behavior/group g, i.e. $R > 1$.

- *Consumption or wealth creation by the followers.* We define α as the parameter of the consumption or wealth creation on the valuation of the good distributed by the leader to the followers. Therefore, the new valuation of the followers s_{t^f,c^f} to choose the behavior/group g is given by
$$V^g_{t^f,c^f} = \bar{V}^g_{t^f,c^f} + \frac{\alpha(1-R)}{n(t^f,c^f)}V,$$
where $\bar{V}^g_{t^f,c^f}$ corresponds to the previous valuation of the followers to choose behavior/group g. There is *wealth creation by the followers* when $R < 1$ and $\alpha > 1$ or when $R > 1$ and $0 < \alpha < 1$. There is *wealth consumption by the followers* when $R < 1$ and $0 < \alpha < 1$ or when $R > 1$ and $\alpha > 1$.

- *Influent and persuasive leaders.* The influence or persuasiveness of the leaders s_{t^l,c^l} on the followers (t^f,c^f) is measured by the parameter L. We consider that
$$A^{g,c^l}_{t^f,c^f} = L$$
corresponds to the satisfaction that the followers have by choosing the same behavior/group as the leader. Alternatively, we consider that $A^{g,c^l}_{t^f,c^f} = 0$ and that the followers have a new valuation $V^{g'}_{t^f,c^f} =$

$V^{g'}_{t^f,c^f} - L$ when they choose the behavior/group $g' \in G \setminus \{g\}$ under the influence of the leader. If $L < 0$, the followers do not like to choose the same behavior/group as the leader, but if $L > 0$, the followers like to choose the same behavior/group as the leader.

We define the *leader worst neighbors* $LWN_g(t^f, c^f)$ of the individual s_{t^f,c^f} in choosing the behavior/group g by:

$$LWN_g(t^f, c^f) = \begin{cases} A^{g,c^f}_{t^f,c^f} + \sum\limits_{c' \in C, A^{g,c'}_{t^f,c^f} < 0} A^{g,c'}_{t^f,c^f} \sum\limits_{t' \in T} n(t', c') & \text{if} \quad A^{g,c^f}_{t^f,c^f} \geq 0 \\ \sum\limits_{c' \in C, A^{g,c'}_{t^f,c^f} < 0} A^{g,c'}_{t^f,c^f} \sum\limits_{t' \in T} n(t', c') & \text{if} \quad A^{g,c^f}_{t^f,c^f} < 0 \end{cases}$$

We define the *leader best neighbors* $LBN_g(t^f, c^f)$ of the individual s_{t^f,c^f} by:

$$LBN_g(t^f, c^f) = \begin{cases} \sum\limits_{c' \in C, A^{g,c'}_{t^f,c^f} > 0} A^{g,c'}_{t^f,c^f} \sum\limits_{t' \in T} n(t', c') & \text{if} \quad A^{g,c^f}_{t^f,c^f} \geq 0 \\ A^{g,c^f}_{t^f,c^f} + \sum\limits_{c' \in C, A^{g,c'}_{t^f,c^f} > 0} A^{g,c'}_{t^f,c^f} \sum\limits_{t' \in T} n(t', c') & \text{if} \quad A^{g,c^f}_{t^f,c^f} < 0 \end{cases}$$

Let

$$g_W = \arg\max_{\{g \in G\}} LWN_g(t^f, c^f)$$

and

$$LBN(t^f, c^f) = \max_{\{g \in G: g \neq g_W\}} LBN_g(t^f, c^f)$$

Lemma 4.4.1 *Let the leader s_{t^l,c^l} choose the behavior/group $g \in G$. If*

$$\frac{\alpha(1-R)}{n(t^f, c^f)} V + L > LBN(t^f, c^f) - LWN_{g_W}(t^f, c^f)$$

then $\mathcal{G}^*(s_{t^f,c^f}) = g_W$, for every Nash equilibrium $\mathcal{G}^*$.

Inequality above gives a sufficient condition, in the value of the donation $(1-R)V$, in the influence and persuasion L of the leader and, also, in the creation or consumption of wealth α by the followers, implying that the followers choose the same behavior/group as the leader.

Proof. Let us suppose, by contradiction, that $\mathcal{G}^*$ is a Nash equilibrium, such that, at least one follower s_{t^f,c^f} chooses the behavior/group $g \in G \setminus \{g_w\}$. By construction of the leader best neighbors $LBN(t^f, c^f)$, the utility function is bounded above by

$$u_{t^f,c^f}(g, m) \leq LBN(t^f, c^f)$$

If the follower changes his behavior/group choice to g_W, then by construction of the leaders worst neighbors $LWN_{g_w}(t^f, c^f)$, the utility function is bounded below by

$$u_{t^f,c^f}(g_w, m) \geq \frac{\alpha(1-R)}{n(t^f, c^f)} V + L + LWN_{g_w}(t^f, c^f).$$

Since $\frac{\alpha(1-R)}{n(t^f,c^f)}V + L > LBN(t^f, c^f) - LWN_{g_W}(t^f, c^f)$, we get

$$u_{t^f,c^f}(g_W, m) > u_{t^f,c^f}(g, m),$$

which is a contradiction. ■

4.5 Game Theory in an Educational Context

In this example each student chooses a behavior/group. We consider two different behavior/groups $g \in \{A, F\} = G$ that correspond to results that the students can have in the end of the academic year: A means that he will approve and F means that he will fail, with some probability.

The students have preferences, over different behavior/group and over the crowding profile of the other students in the same behavior/group, that are described by the taste type. We consider a student community with four taste types, $t \in \{t_{SW}, t_{SN}, t_{UW}, t_{UN}\} = T$, that can be defined by considering two possibilities of different learning skills and previously scientific knowledge obtained by the students, namely t_{SW}, t_{SN}, that correspond to students with skills for success (S) and t_{UW}, t_{UN} correspond to students without skills for success, that tend to be unsuccessful (U). The taste types can also be defined by considering two possibilities of different socializing behaviors, for instance, t_{SW}, t_{UW} (working-W), correspond to the students that like to be with students that work/study more than average and t_{SN}, t_{UN} (non-working-N), corresponds to the students that like to be with students that work/study less than average.

We consider that the choice of a behavior/group depends not only on the characteristics of each student and their behavior/group valuation but also on the characteristics of the other students that have chosen the same behavior/group. We refer these observable characteristics by crowding types. In this example we consider four illustrative crowding types related with the study frequency, namely $c \in \{C_V, C_F, C_O, C_R\} = C$. If a student has a crowding type C_V it means he studies very frequently, if a student has a crowding type C_F it means he studies frequently, if he has a C_O crowding type, he studies occasionally and with crowding type C_R he rarely studies.

Overall, we can now consider four possibilities of different socializing behaviors that we pass to describe: $T = \{t_{SW,C_V}, t_{SN,C_O}, t_{UW,C_F}, t_{UN,C_R}\}$.

Given the behavior/group $g \in G$, let us consider the crowding club vector:

$$m^g = \{m^g_{CV}, m^g_{CF}, m^g_{CO}, m^g_{CR}\}$$

where m^g_{CV} represents the number of students with crowding type C_V that choose the behavior/group g, m^g_{CF} represents the number of students with crowding type C_F that choose the behavior/group g, m^g_{CO} represents the number of students with crowding type C_O that choose the behavior/group g, m^g_{CR} represents the number of students with crowding type C_R that choose the behavior/group g.

Let us now introduce the payoff of the four taste types in the model.

Students of type t_{SW,C_V} are students with skills for success that study more than average. They also prefer to be with students that study very frequently.

$$u_{t_{SW,C_V}}(g, m^g) = V^g_{t_{SW}} + m^g_{CV} \quad , V^g_{t_{SW}} = \begin{cases} V_{t_{SW}} & \text{if} \quad g = A \\ 0 & \text{if} \quad g = F \end{cases}$$

Students of type t_{SN,C_O} are students with skills for success that study less than average. Furthermore, they prefer the company of students that study occasionally or rarely.

$$u_{t_{SN,C_O}}(g, m^g) = V^g_{t_{SN}} + m^g_{CO} + m^g_{CR} \quad , V^g_{t_{SN}} = \begin{cases} V_{t_{SN}} & \text{if} \quad g = A \\ 0 & \text{if} \quad g = F \end{cases}$$

Students of type t_{UW,C_F} are students without skills for success that study more than average. Furthermore, they prefer the company of students that study frequently or very frequently.

$$u_{t_{UW,C_F}}(g, m^g) = V^g_{t_{UW}} + m^g_{CF} + (1+\alpha)\, m^g_{CV} \quad , \alpha > 0,$$
$$V^g_{t_{UW}} = \begin{cases} V_{t_{UW}} & \text{if} \quad g = F \\ 0 & \text{if} \quad g = A \end{cases}$$

Students of type t_{UN,C_R} prefer to be with a group of students without skills that study less than average. Furthermore, they prefer the company of students that rarely study.

$$u_{t_{UN,C_R}}(g, m^g) = V^g_{t_{UN}} + m^g_{CR} \quad , V^g_{t_{UN}} = \begin{cases} V_{t_{UN}} & \text{if} \quad g = F \\ 0 & \text{if} \quad g = A \end{cases}$$

We can represent the utility function by

$$u_{t,c}(g, m^g) = V^g_{t,c} + \sum_{c' \in \mathcal{C}} A^{g,c'}_{t,c} m^g.$$

The information concerning the satisfaction function is summarized in two following tables: (A) a table of the behavior/group power $V^{g}_{t,c}$ in the welfare/payoff of each student taste type t and (B) a table of the crowding type influence $A^{g,c'}_{t,c}$ in the welfare/ payoff of each student taste type t. For this example the tables are described below:

A) The table of the behavior/group valuation in Figure 4.4 that shows the welfare/profit of a student depending on his taste type:

	A	F
tsw	Vtsw	0
tSN	VtSN	0
tUW	0	VtUW
tUN	0	VtUN

Figure 4.4: *Table of the behavior/group valuation*

B) The table of the crowding type influence, in Figure 4.5 that shows the positive or negative proportional effect of each crowding type for each student depending on his taste type:

	CV	CO	CF	CR
Tsw	1	0	0	0
TSN	0	1	0	1
TUW	1+α	0	1	0
TUN	0	0	0	1

Figure 4.5: *Table of the crowding type influence*

The distribution of all students is characterized in the table of Figure 4.6, where for each pair (t,c) we consider the corresponding number of students n(t,c) :

	TSW	TUW	TSN	TUN
CV	n(tSW,CV)	0	0	0
CF	0	n(tUW,CF)	0	0
CO	0	0	n(tSN,CO)	0
CR	0	0	0	n(tUN,CR)

Figure 4.6: *Table of the distribution of all students*

Given the utility function we can now determine the Nash equilibria. Studying all the cases we obtain four different equilibria, as we pass to describe.

We will assume from now on that:

$$\mathbf{H_1} : n(t_{UN}) \leq V_{UN}$$

and

$$\mathbf{H_2} : n(t_{SW}) \leq V_{SW}$$

and consider that $S_{(t,c)}$ represents the students $s \in S$ with taste type $t \in \mathcal{T}$ and crowding type $c \in \mathcal{C}$.

Lemma 7: Under hypothesis H_1 and H_2, for every Nash Equilibrium G^*, $G^*(s) = F$, for every $s \in S_{(t_{UN}, c_R)}$ and $G^*(s) = A$, for every $s \in S_{(t_{SW}, c_V)}$.

Hence, from the Nash Equilibrium point of view, we have some students with permanent choice of behavior/group, and we assume that the students with taste type t_{UN} and crowding type c_R always choose behavior/group F and students with taste type t_{SW} and crowding type c_V always choose behavior/group A, under H_1 and H_2.

Lemma 8 (Students of types t_{UW} and t_{SN} prefer what they like) : If $n(t_{UN}) < V_{t_{SN}} + n(t_{SN})$ and $(1 + \alpha)n(t_{SW}) < V_{t_{UW}} + n(t_{UW})$ then there is a Nash Equilibrium G^* such that $G^*(s) = A$ for every $s \in S_{(t_{SN}, c_O)}$ and $G^*(s) = F$ for every $s \in S_{(t_{UW}, c_F)}$.

In this case (Figure 4.7) the determinating factor for students of type t_{UW} is the valuation of the behavior/group and not the members of the behavior/group. For students of type t_{SN} the determinating factor is also the valuation of the behavior/group and not the members of the behavior/group. Then the students of type t_{UW} choosing behavior/group F and the students of type t_{SN} choosing behavior/group A form a Nash Equilibrium.

Lemma 9 (Students of type t_{SN} prefer who they like): If $n(t_{UN}) + n(t_{SN}) > V_{t_{SN}}$ and $(1 + \alpha)n(t_{SW}) \leq V_{t_{UW}} + n(t_{UW})$ then there is a Nash Equilibrium G^* such that $G^*(s) = F$ for every $s \in S_{(t_{SN}, c_O)}$ and $G^*(s) = F$

	A	F
(tSW,CV)	n(tSW,CV)	0
(tUW,CF)	0	n(tUW,CF)
(tSN,CO)	n(tSN,CO)	0
(tUN,CR)	0	n(tUN,CR)

Figure 4.7: *Students of types t_{UW} and t_{SN} prefer what they like*

for every $s \in S_{(t_{UW},c_F)}$.

In this case (Figure 4.8) the determinating factor for students of type t_{UW} is the valuation of the behavior/group and not the members of the behavior/group, but for students of type t_{SN} the determinating factor are the members of the behavior/group and not the valuation of the behavior/group. Then the students of type t_{UW} choosing behavior/group F and the students of type t_{SN} choosing behavior/group F form a Nash Equilibrium.

Lemma 10 (Students of type t_{UW} prefer who they like): If $n(t_{UN}) < V_{t_{SN}} + n(t_{SN})$ and $(1+\alpha)n(t_{SW}) + n(t_{UW}) \leq V_{t_{UW}}$ then there is a Nash Equilibrium G^* such that $G^*(s) = A$ for every $s \in S_{(t_{SN},c_O)}$ and $G^*(s) = A$ for every $s \in S_{(t_{UW},c_F)}$.

In this case (Figure 4.9) the determinating factor for students of type

	A	F
(tSW,CV)	n(tSW,CV)	0
(tUW,CF)	0	n(tUW,CF)
(tSN,CO)	0	n(tSN,CO)
(tUN,CR)	0	n(tUN,CR)

Figure 4.8: *Students of type t_{SN} prefer who they like*

t_{UW} are the members of the behavior/group and not the valuation of the behavior/group, but for students of type t_{SN} the determinating factor is the valuation of the behavior/group and not the members of the behavior/group. Then the students of type t_{UW} choosing behavior/group A and the students of type t_{SN} choosing behavior/group A form a Nash Equilibrium.

	A	F
(tSW,CV)	n(tSW,CV)	0
(tUW,CF)	n(tUW,CF)	0
(tSN,CO)	n(tSN,CO)	0
(tUN,CR)	0	n(tUN,CR)

Figure 4.9: *Students of type t_{UW} prefer who they like*

Lemma 11 (Students of types t_{UW} and t_{SN} prefer who they like) If $n(t_{UN}) + n(t_{SN}) > V_{t_{SN}}$ and $(1+\alpha)n(t_{SW}) + n(t_{UW}) \leq V_{t_{UW}}$ then there is a Nash Equilibrium G^* such that $G^*(s) = F$ for every $s \in S_{(t_{SN},c_O)}$ and $G^*(s) = A$ for every $s \in S_{(t_{UW},c_F)}$.

In this case (Figure 4.10) the determinating factor for students of type t_{UW} are the members of the behavior/group and not the valuation of the behavior/group, and the same happens for students of type t_{SN}. Then the students of type t_{UW} choosing behavior/group A and the students of type t_{SN} choosing behavior/group F form a Nash Equilibrium.

	A	F
(tSW,CV)	n(tSW,CV)	0
(tUW,CF)	n(tUW,CF)	0
(tSN,CO)	0	n(tSN,CO)
(tUN,CR)	0	n(tUN,CR)

Figure 4.10: *Students of types t_{UW} and t_{SN} prefer who they like*

Hence, we can have an "herding" effect in students of types t_{UW} and t_{SN} as we pass to explain:

- Herding Effect in students of type t_{UW}

Herding from F to A : Suppose that $(1+\alpha)n(t_{SW}) \leq n(t_{UW}) + V(t_{UW})$

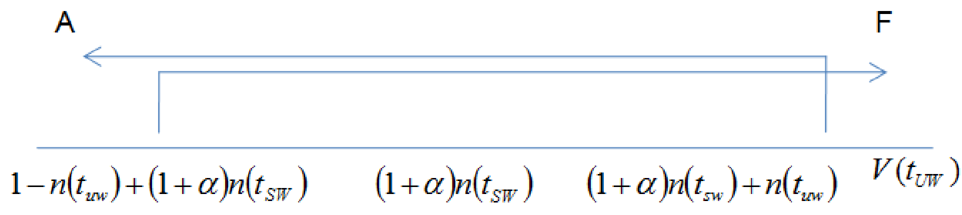


Figure 4.11: *Herding Effect in students of type t_{UW}*

and that $t_{UW} \subset F$, a small increase in $n(t_{SW})$ and $n(t_{UW})$ or decrease in valuation $V(t_{UW})$ can alter the above inequality to $>$ leading t_{UW} to change as a herd his choice from behavior/group F to behavior/group A.

Herding from A to F : Suppose that $(1+\alpha)n(t_{SW})+n(t_{UW}) > V(t_{UW})$ and that $t_{UW} \subset A$, a small decrease in $n(t_{SW})$ and $n(t_{UW})$ or increase in valuation $V(t_{UW})$ can alter de above inequality to $<$ (less than) leading t_{UW} to change as a herd his choice from A behavior/group to behavior/group F.

- Herding Effect in students of type t_{SN}

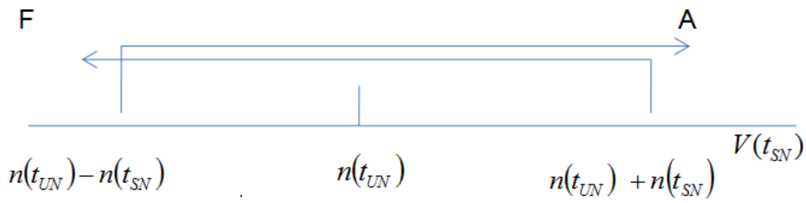


Figure 4.12: *Herding Effect in students of type t_{SN}*

Herding from F to A : Suppose that $n(t_{UN})+n(t_{SN}) > V(t_{SN})$ and that $t_{SN} \subset F$, a small decrease in $n(t_{SN})$ and $n(t_{UN})$ or increase in valuation $V(t_{SN})$ can alter the above inequality to $<$ leading t_{SN} to change as a herd his choice from behavior/group F to behavior/group A.

Herding from A to F : Suppose that $n(t_{UN}) < n(t_{SN}) + V(t_{SN})$ and that $t_{SN} \subset A$, a small increase in $n(t_{SN})$ and $n(t_{UN})$ or decrease in valuation $V(t_{SN})$ can alter the above inequality to $>$ leading t_{SN} to change as a herd his choice from behavior/group A to behavior/group F.

4.6 Conclusion

We constructed two game models for the theory of Planned Behavior or Reasoned Action. The first model, the platonic idealized psychological model, consists of individuals with no uncertainties in their taste and crowding types and welfare function. The second model, the cave psychological model, consists of individuals whose taste and crowding types follow the shadows of the taste and crowding types of the platonic idealized psychological model, according to a given probability distribution. Furthermore, the individuals know only the expected value of their welfare function. In both models, we presented sufficient conditions for an individual or group to adopt a certain behavior decision according to both the Nash and the Bayesian-Nash Equilibria. We demonstrated how saturation, boredom and frustration can lead to the adoption of a variety of different behavior decisions and how no saturation can lead to the adoption of a single consistent behavior decision. We studied how the characteristics of the leaders have influence over other individual's decisions. We presented a students success model, described Nash equilibria and "herding" effects, identifying a hysteresis in the process.

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