Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110308

Approximation by Bivariate Bernstein-Durrmeyer Operators on a Triangle

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Received: 28 Feb. 2016, Revised: 19 Dec. 2016, Accepted: 21 Dec. 2016
Published online: 1 May 2017

Abstract: In the present paper, we obtain some approximation properties for the bivariate Bernstein-Durrmeyer operators on a triangle. We characterize the rate of convergence in terms of $K-$functional and the usual and second order modulus of continuity. We estimate the order of approximation by Voronovskaja type result and illustrate the convergence of these operators to a certain function through graphics using Mathematica algorithm. We also discuss the comparison of the convergence of the bivariate Bernstein-Durrmeyer operators and the bivariate Bernstein-Kantorovich operators to the function through illustrations using Mathematica. Lastly, we study the simultaneous approximation for first order partial derivatives and the shape preserving properties of these operators.

Keywords: Modulus of continuity, rate of convergence, simultaneous approximation, shape preserving properties

1 introduction

Let $\psi(x,y)$ be a continuous function in a closed region $R: 0\le x\le 1, 0\le y\le 1$. Kingsley [7] introduced the Bernstein polynomials for functions of two variables as

$$B_{m,n}(\psi;x,y)=\sum_{k=0}^{n}\sum_{l=0}^{m}\psi\left(\frac{k}{n},\frac{l}{m}\right)\lambda_{n,k}(x)\lambda_{m,l}(y),$$

where $\lambda_{r,i}(x)=\binom{r}{i}x^i(1-x)^{r-i}, x\in[0,1]$. He studied the simultaneous approximation for these operators. Butzer [3] also discussed the simultaneous approximation in a direct manner. In [8], Pop obtained the rate of convergence in terms of the modulus of continuity and established the Voronovskaja type asymptotic theorem for the operators $B_{m,n}(\psi;x,y)$.

Stancu [10] defined another bivariate Bernstein operators on the triangle $\triangle := S = \{(x,y) : x+y\le 1, 0\le x,y\le 1\}$ for functions $f: S\to\mathbb{R}$, as

$$M_n(f;x,y)=\sum_{k=0}^{n}\sum_{l=0}^{n-k}b_{n,k,l}(x,y)f\left(\frac{k}{n},\frac{l}{n}\right),\ (x,y)\in S$$

where $b_{n,k,l}(x,y)=\binom{n}{k}\binom{n-k}{l}x^ky^l(1-x-y)^{n-k-l}$. He derived the rate of convergence in terms of complete modulus of continuity for $M_n(f;x,y)$.
Pop and Fărcaş [9] discussed the convergence and approximation properties of the Bernstein-Kantorovich type operators defined as

$$\mathscr{U}_n(f;x,y)=(n+1)^2\sum_{k=0}^{n}\sum_{l=0}^{n-k}b_{n,k,l}(x,y)\times\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}}\int_{\frac{l}{n+1}}^{\frac{l+1}{n+1}}f(s,t)\,ds\,dt$$

and the associated GBS operators on the triangle $\triangle$. In [1], Acar and Aral studied the approximation properties of two dimensional Bernstein-Stancu-Chlodowsky operators on a triangular domain with mobile boundaries, and gave shape preserving properties and also obtained weighted approximation properties of these operators.
Derrienic [6] studied multivariate Bernstein polynomials defined for integral functions on a triangle and proved the convergence of theses operators and its derivative in L_p spaces. In 1992, Zhou [11] defined the two-dimensional Bernstein-Durrmeyer operators $\mathscr{V}_n : f\to\mathscr{V}_n(f;.,.)$ with $f\in C(S)$(the space of all continuous functions on S),

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endowed with the norm $\| f \| = \sup_{(x,y)\in S} |f(x,y)|$, as

$$\mathscr{V}_n(f;x,y) = (n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y) \times \int_0^1\int_0^{1-t} b_{n,k,l}(s,t)f(s,t)dsdt$$

and obtained the rate of convergence in terms of the $K-$functional and the smoothness of the functions in L_p spaces. Deo and Bhardwaj [5] also studied some direct theorems and established an inverse theorem for the operators $\mathscr{V}_n$ on S.
The aim of this paper is to study the approximation properties of bivariate Bernstein-Durrmeyer operators $\mathscr{V}_n$ on the triangle S. We obtain the rate of convergence by means of $K-$functional, usual and second order modulus of continuity and establish the asymptotic formula to find the order of approximation for the operators $\mathscr{V}_n$ in continuous function spaces. We demonstrate the convergence of the operators $\mathscr{V}_n$ to a certain function and the comparison of the convergence with the bivariate Benstein-Kantorovich operators to the function using Mathematica. We also study the simultaneous approximation for first order partial derivatives and shape preserving properties of these operators.

2 Preliminary results

Lemma 1. *For $e_{ij} = s^i t^j, (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0$, $\mathbb{N}^0 = \mathbb{N}\cup\{0\}$, we have*

$$(i)\mathscr{V}_n(e_{00};x,y) = 1;$$
$$(ii)\mathscr{V}_n(e_{10};x,y) = \frac{1+nx}{n+3};$$
$$(iii)\mathscr{V}_n(e_{01};x,y) = \frac{1+ny}{n+3};$$
$$(iv)\mathscr{V}_n(e_{20};x,y) = \frac{n(n-1)x^2+4nx+2}{(n+3)(n+4)};$$
$$(v)\mathscr{V}_n(e_{02};x,y) = \frac{n(n-1)y^2+4ny+2}{(n+3)(n+4)};$$
$$(vi)\mathscr{V}_n(e_{11};x,y) = \frac{n(n-1)xy+n(x+y)+1}{(n+3)(n+4)};$$
$$(vii)\mathscr{V}_n(e_{40};x,y) = \frac{1}{(n+3)(n+4)(n+5)(n+6)}\{n^4x^4 + n^3x^3(16-6x) + n^2x^2(72-48x+11x^2) + nx(96-72x+32x^2-6x^3)+24\};$$
$$(viii)\mathscr{V}_n(e_{04};x,y) = \frac{1}{(n+3)(n+4)(n+5)(n+6)}\{n^4y^4 + n^3y^3(16-6y) + n^2y^2(72-48y+11y^2) + ny(96-72y+32y^2-6y^3)+24\}.$$

The moments $(i)-(vi)$ are given in [5]. The proof of (vii) and $(viii)$ can be obtained by a simple computation. Hence the details are omitted.

Lemma 2.[5] *For $h_{ij} = (s-x)^i(t-y)^j, (i,j) \in \mathbb{N}^0 \times \mathbb{N}^0$, we have*

$$(i)\mathscr{V}_n(h_{00};x,y) = 1;$$
$$(ii)\mathscr{V}_n(h_{10};x,y) = \frac{1-3x}{n+3};$$
$$(iii)\mathscr{V}_n(h_{01};x,y) = \frac{1-3y}{n+3};$$
$$(iv)\mathscr{V}_n(h_{20};x,y) = \frac{2\{(6-n)x^2+(n-4)x+1\}}{(n+3)(n+4)};$$
$$(v)\mathscr{V}_n(h_{02};x,y) = \frac{2\{(6-n)y^2+(n-4)y+1\}}{(n+3)(n+4)}.$$

Lemma 3. *For the bivariate operators $\mathscr{V}_n(f;x,y)$, we have*

$$(i)\lim_{n\to\infty} n\mathscr{V}_n((s-x);x,y) = 1-3x;$$
$$(ii)\lim_{n\to\infty} n\mathscr{V}_n((t-y);x,y) = 1-3y;$$
$$(iii)\lim_{n\to\infty} n\mathscr{V}_n((s-x)^2;x,y) = 2x(1-x);$$
$$(iv)\lim_{n\to\infty} n\mathscr{V}_n((t-y)^2;x,y) = 2y(1-y);$$
$$(v)\lim_{n\to\infty} n\mathscr{V}_n((s-x)(t-y);x,y) = -2xy;$$
$$(vi)\lim_{n\to\infty} n^2\mathscr{V}_n((s-x)^4;x,y) = 12x^2(x-1)^2;$$
$$(vii)\lim_{n\to\infty} n^2\mathscr{V}_n((t-y)^4;x,y) = 12y^2(y-1)^2.$$

*Proof.*The proof of this lemma easily follows. Hence we omit the details.

Lemma 4. *For every $x \in [0,1]$ and $n \in \mathbb{N}$, we have*

$$\mathscr{V}_n((s-x)^2;x,y) + \left(\frac{1+nx}{n+3} - x\right)^2 < \frac{3}{n+3}\left(\phi^2(x) + \frac{1+9x^2}{n+3}\right),$$

where $\phi(x) = \sqrt{x(1-x)}$.

*Proof.*From Lemma 2, we have

$$\mathscr{V}_n((s-x)^2;x,y) + \left(\tfrac{1+nx}{n+3} - x\right)^2$$
$$< \frac{2\{(6-n)x^2+(n-4)x+1\}+(1-3x)^2}{(n+3)^2}$$
$$= \frac{(21-2n)x^2+(2n-14)x+3}{(n+3)^2}$$
$$= \frac{(2n-14)x(1-x)+7x^2+3}{(n+3)^2}$$
$$= \frac{1}{n+3}\left(2\phi^2(x) + \frac{27x^2-20x+3}{n+3}\right)$$
$$\leq \frac{3}{n+3}\left(\phi^2(x) + \frac{1+9x^2}{n+3}\right).$$

3 Main results

Basic convergence theorem

Theorem 1[12] *Let $\mathscr{V}_n : C(S) \to C(\mathbb{R})$, $n \in \mathbb{N}$, be linear positive operators. If*

$$\lim_{n\to\infty} \mathscr{V}_n(e_{ij}) = e_{ij},\ (i,j) \in \{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n\to\infty} \mathscr{V}_n(e_{20}+e_{02}) = e_{20}+e_{02}$$

uniformly in S, *then the sequence* $\mathscr{V}_n(f)$ *converges to* f *uniformly in* S, *for any* $f \in C(S)$.

Estimates of rate of convergence

For $f \in C(S)$, the complete modulus of continuity for the bivariate case is defined as follows:

$\omega(f;\delta_1,\delta_2)$

$= \sup\{|f(s,t)-f(x,y)| : |s-x| \le \delta_1, |t-y| \le \delta_2\}$,

where $\delta_1, \delta_2 > 0$. Taking into account that on triangle S, we have

$|f(s,t)-f(x,y)| \le \omega(|s-x|,|t-y|) \le \omega(f;\delta_1,\delta_2)$

whenever $|s-x| \le \delta_1, |t-y| \le \delta_2, \delta_1 > 0, \delta_2 > 0$ and

$\omega(f;\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1+\lambda_2)\omega(f;\delta_1,\delta_2), \lambda_1 > 0, \lambda_2 > 0.$

Further, the partial moduli of continuity with respect to x and y is defined as

$$\omega_1(f;\delta) = \sup\Big\{|f(x_1,y)-f(x_2,y)| : y \in [0,1] \text{ and } |x_1-x_2| \le \delta, \delta > 0\Big\},$$

$$\omega_2(f;\delta) = \sup\Big\{|f(x,y_1)-f(x,y_2)| : x \in [0,1] \text{ and } |y_1-y_2| \le \delta, \delta > 0\Big\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity. The details of the modulus of continuity for the bivariate case can be found in [2].

In what follows, $\delta_n(x) = \sqrt{\mathscr{V}_n((s-x)^2;x,y)}$, $\delta_n(y) = \sqrt{\mathscr{V}_n((t-y)^2;x,y)}$.

Theorem 2 *Let* f *be continuous on* S, *then we have*

$$|\mathscr{V}_n(f;x,y)-f(x,y)| \le 3\omega(f;\delta_n(x),\delta_n(y)).$$

*Proof.*Applying Lemma 2 and the Cauchy-Schwarz inequality, the proof of this theorem is straightforward. Hence the details are omitted.

Theorem 3 *Let* $f \in C(S)$. *Then, we have the following inequality*

$$|\mathscr{V}_n(f(s,t);x,y)-f(x,y)| \le 2(\omega_1(f;\delta_n(x))+\omega_2(f;\delta_n(y))).$$

*Proof.*The definition of partial moduli of continuity and using Cauchy-Schwarz inequality, proof of this theorem easily follows.

Local approximation

For $f \in C(S)$, let $C^2(S) = \{f \in C(S) : f^{(i,j)} \in C(S), 0 \le i+j \le 2\}$, where $f^{(i,j)}$ is (i,j)th-order partial derivative with respect to x,y of f, endowed with the norm

$$||f||_{C^2(S)} = ||f|| + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\| + \left\|\frac{\partial^i f}{\partial y^i}\right\|\right).$$

The Peetre's $K-$functional of the function $f \in C(S)$ is given by

$$\mathscr{K}(f;\delta) = \inf_{g\in C^2(S)}\{||f-g|| + \delta||g||_{C^2(S)}, \delta > 0\}.$$

It is also known that the following inequality

$$\mathscr{K}(f;\delta) \le M_1\{\overline{\omega_2}(f;\sqrt{\delta}) + \min(1,\delta)||f||\}, \quad (1)$$

holds for all $\delta > 0$ ([4], page 192). The constant M_1 is independent of δ and f and $\overline{\omega_2}(f;\sqrt{\delta})$ is the second order modulus of continuity.

Now, we find the order of approximation of the sequence $\mathscr{V}_n f;x,y)$ to the function $f(x,y) \in C(S)$ by Peetre's K-functional.

Theorem 4 *For the function* $f \in C(S)$, *the following inequality*

$$\begin{aligned}|\mathscr{V}_n(f;x,y)-f(x,y)| &< 4\mathscr{K}(f;J_n(x,y)) \\ &+\omega\left(f;\sqrt{\left(\frac{1-3x}{n+3}\right)^2+\left(\frac{1-3y}{n+3}\right)^2}\right) \\ &\le M\Big\{\overline{\omega_2}\left(f;\sqrt{J_n(x,y)}\right) \\ &+min\{1,J_n(x,y)\}||f||_{C^2(S)}\Big\} \\ &+\omega\left(f;\sqrt{\left(\frac{1-3x}{n+3}\right)^2+\left(\frac{1-3y}{n+3}\right)^2}\right),\end{aligned}$$

holds. The constant $M > 0$ *is independent of* f *and* $J_n(x,y)$, *where*

$$J_n(x,y) = \frac{1}{n+3}\left(\phi^2(x)+\frac{1+9x^2}{n+3}\right)+\frac{1}{n+3}\left(\phi^2(y)+\frac{1+9y^2}{n+3}\right)$$

and $\phi(x) = \sqrt{x(1-x)}$.

Proof. We define the auxiliary operators as follows:

$\overline{\mathscr{V}}_n(f;x,y)$

$$= \mathscr{V}_n(f;x,y) - f\left(\frac{1+nx}{n+3},\frac{1+ny}{n+3}\right) + f(x,y). \quad (2)$$

Then, from Lemma 2, we have $\overline{\mathscr{V}}_n(1;x,y) = 1$, $\overline{\mathscr{V}}_n((s-x);x,y) = 0$ and $\overline{\mathscr{V}}_n((t-y);x,y) = 0$.

Let $g \in C^2(S)$ and $(s,t) \in S$. Using the Taylor's theorem,

we have
$g(s,t)-g(x,y)$

$$= \frac{\partial g(x,y)}{\partial x}(s-x)+\int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha + \frac{\partial g(x,y)}{\partial y}(t-y)+\int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta. \quad (3)$$

Operating by $\overline{\mathscr{V}}_n$ on the equation (3), we get
$\overline{\mathscr{V}}_n(g;x,y)-g(x,y)$

$$= \overline{\mathscr{V}}_n\left(\int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) + \overline{\mathscr{V}}_n\left(\int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right)$$
$$= \mathscr{V}_n\left(\int_x^s (s-\alpha)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha;x,y\right) - \int_x^{\frac{1+nx}{n+3}}\left(\frac{1+nx}{n+3}-\alpha\right)\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}d\alpha + \mathscr{V}_n\left(\int_y^t (t-\beta)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta;x,y\right) - \int_y^{\frac{1+ny}{n+3}}\left(\frac{1+ny}{n+3}-\beta\right)\frac{\partial^2 g(x,\beta)}{\partial \beta^2}d\beta.$$

Hence,
$|\overline{\mathscr{V}}_n(g;x,y)-g(x,y)|$

$$\leq \mathscr{V}_n\left(\left|\int_x^s |s-\alpha|\left|\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}\right|d\alpha\right|;x,y\right) + \left|\int_x^{\frac{1+nx}{n+3}}\left|\frac{1+nx}{n+3}-\alpha\right|\left|\frac{\partial^2 g(\alpha,y)}{\partial \alpha^2}\right|d\alpha\right| + \mathscr{V}_n\left(\left|\int_y^t |t-\beta|\left|\frac{\partial^2 g(x,\beta)}{\partial \beta^2}\right|d\beta\right|;x,y\right) + \left|\int_y^{\frac{1+ny}{n+3}}\left|\frac{1+ny}{n+3}-\beta\right|\left|\frac{\partial^2 g(x,\beta)}{\partial \beta^2}\right|d\beta\right|$$
$$\leq \left\{\mathscr{V}_n((s-x)^2;x,y)+\left(\frac{1+nx}{n+3}-x\right)^2\right\}||g||_{C^2(S)} + \left\{\mathscr{V}_n((t-y)^2;x,y)+\left(\frac{1+ny}{n+3}-y\right)^2\right\}||g||_{C^2(S)}$$
$$< \left\{\frac{3}{n+3}\left(\phi^2(x)+\frac{1+9x^2}{n+3}\right) + \frac{3}{n+3}\left(\phi^2(y)+\frac{1+9y^2}{n+3}\right)\right\}||g||_{C^2(S)}.$$

Also,

$$\overline{\mathscr{V}}_n(f;x,y)| \leq |\mathscr{V}_n(f;x,y)| + \left|f\left(\frac{1+nx}{n+3},\frac{1+ny}{n+3}\right)\right| + |f(x,y)| \leq 3||f||_{C(S)}. \quad (4)$$

Now, for every $g\in C^2(S)$ and from equation (4), we get
$|\mathscr{V}_n(f;x,y)-f(x,y)|$

$$\leq |\overline{\mathscr{V}}_n(f-g;x,y)|+|\overline{\mathscr{V}}_n(g;x,y)-g(x,y)|+|g(x,y)-f(x,y)| + \left|f\left(\frac{1+nx}{n+3},\frac{1+ny}{n+3}\right)-f(x,y)\right|$$
$$< 4||f-g||_{C(S)} + \left\{\frac{3}{n+3}\left(\phi^2(x)+\frac{1+9x^2}{n+3}\right) + \frac{3}{n+3}\left(\phi^2(y)+\frac{1+9y^2}{n+3}\right)\right\}||g||_{C^2(S)} + \left|f\left(\frac{1+nx}{n+3},\frac{1+ny}{n+3}\right)-f(x,y)\right|$$
$$\leq \left(4||f-g||_{C(S)}+3J_n(x,y)||g||_{C^2(S)}\right) + \omega\left(f;\sqrt{\left(\frac{1-3x}{n+3}\right)^2+\left(\frac{1-3y}{n+3}\right)^2}\right).$$

Taking the infimum on the right hand side over all $g\in C^2(S)$ and using (1), we obtain

$$|\mathscr{V}_n(f;x,y)-f(x,y)| < 4\mathscr{K}(f;J_n(x,y)) + \omega\left(f;\sqrt{\left(\frac{1-3x}{n+3}\right)^2+\left(\frac{1-3y}{n+3}\right)^2}\right)$$
$$\leq M\left\{\overline{\omega_2}\left(f;\sqrt{J_n(x,y)}\right) + \min\{1,J_n(x,y)\}||f||_{C^2(S)}\right\} + \omega\left(f;\sqrt{\left(\frac{1-3x}{n+3}\right)^2+\left(\frac{1-3y}{n+3}\right)^2}\right),$$

where $M=4M_1$. Hence, the proof is completed.

Theorem 5 *Let $f\in C^1(S)$ and $(x,y)\in S$. Then, we have*
$|\mathscr{V}_n(f;x,y)-f(x,y)| \leq \| f'_x \| \delta_n(x) + \| f'_y \| \delta_n(y)$.

*Proof.*Let $(x,y)\in S$ be a fixed point. Then, we may write

$$f(s,t)-f(x,y)=\int_x^s f'_u(u,t)du+\int_y^t f'_v(x,v)dv.$$

Now, applying $\mathscr{V}_n(.;x,y)$ on both sides of the above equation,

$$|\mathscr{V}_n(f(s,t);x,y)-f(x,y)| \leq \mathscr{V}_n\left(\left|\int_x^s f'_u(u,t)du\right|;x,y\right) + \mathscr{V}_n\left(\left|\int_y^t f'_v(x,v)dv\right|;x,y\right).$$

By using the inequalities,

$$\left|\int_x^s f'_u(u,t)du\right| \leq \| f'_x \| \, |s-x|$$

and

$$\left|\int_y^t f'_v(x,v)dv\right| \leq \| f'_y \| \, |t-y|,$$

we get

$$|\mathscr{V}_n(f(s,t);x,y) - f(x,y)| \leq \| f'_x \| \mathscr{V}_n(|s-x|;x,y) + \| f'_y \| \mathscr{V}_n(|t-y|;x,y).$$

Now, by applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|\mathscr{V}_n(f(s,t);x,y) - f(x,y)| &\leq \| f'_x \| (\mathscr{V}_n((s-x)^2;x,y)^{1/2} \\ &\quad + \| f'_y \| (\mathscr{V}_n((t-y)^2;x,y)^{1/2} \\ &= \| f'_x \| \delta_n(x) + \| f'_y \| \delta_n(y).\end{aligned}$$

This completes the proof.

Voronovskaja type theorem

Theorem 6 *Let $f \in C^2(S)$. Then, we have* $\lim_{n\to\infty} n(\mathscr{V}_n(f;x,y) - f(x,y))$

$$\begin{aligned}&= f'_x(x,y)(1-3x) + f'_y(x,y)(1-3y) + f''_{xx}(x,y)x(1-x) \\ &\quad -2f''_{xy}(x,y)xy + f''_{yy}(x,y)y(1-y),\end{aligned}$$

uniformly in $(x,y) \in S$.

*Proof.*Let $(x,y) \in S$. By the Taylor's theorem, we have

$$\begin{aligned}f(s,t) &= f(x,y) + f'_x(x,y)(s-x) + f'_y(x,y)(t-y) \\ &\quad + \frac{1}{2}\{f''_{xx}(x,y)(s-x)^2 + 2f''_{xy}(x,y)(s-x)(t-y) \\ &\quad + f''_{yy}(x,y)(t-y)^2\} \\ &\quad + \eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\}, \qquad (5)\end{aligned}$$

where $\eta(s,t;x,y) \to 0$, as $(s,t) \to (x,y)$.
Operating $\mathscr{V}_n(.;x,y)$ on both sides of (5), we get

$$\begin{aligned}\mathscr{V}_n(f;x,y) &= f(x,y) + f'_x(x,y)\mathscr{V}_n((s-x);x,y) \\ &\quad + f'_y(x,y)\mathscr{V}_n((t-y);x,y) \\ &\quad + \frac{1}{2}\{f''_{xx}(x,y)\mathscr{V}_n((s-x)^2;x,y) \\ &\quad + 2f''_{xy}(x,y)\mathscr{V}_n((s-x)(t-y);x,y) \\ &\quad + f''_{yy}(x,y)\mathscr{V}_n((t-y)^2;x,y)\} \\ &\quad + \mathscr{V}_n(\eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\};x,y) (6)\end{aligned}$$

Now, by applying Cauchy-Schwarz inequality to the last term of (6), we have
$\mathscr{V}_n\left(\eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\};x,y\right)$

$$\begin{aligned}&\leq \{\mathscr{V}_n(\eta^2(s,t;x,y);x,y)\}^{1/2}\{\sqrt{\mathscr{V}_n((s-x)^4;x,y)} \\ &\quad + \sqrt{\mathscr{V}_n((t-y)^4;x,y)}\}.\end{aligned}$$

Since $\eta(.,.;x,y) \in C(S)$ and $\eta(s,t;x,y) \to 0$, as $(s,t) \to (x,y)$, applying Theorem 1

$$\lim_{n\to\infty} \mathscr{V}_n(\eta^2(s,t;x,y);x,y) = 0$$

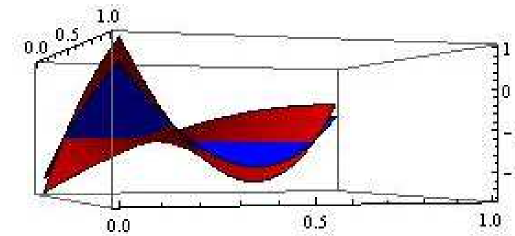

Fig. 1: The convergence of $\mathscr{V}_{20}(f;x,y)$ (blue) to $f(x,y)$ (red)

uniformly in $(x,y) \in S$. From (vii) and (viii) of Lemma 3, we have

$$\mathscr{V}_n((s-x)^4;x,y) = O\left(\frac{1}{n^2}\right),$$

and

$$\mathscr{V}_n((t-y)^4;x,y) = O\left(\frac{1}{n^2}\right), \text{ uniformly in } S.$$

Thus,

$$n\mathscr{V}_n\Big(\eta(s,t;x,y)\{(s-x)^2 + (t-y)^2\};x,y\Big) \to 0, \text{ as } n \to \infty. \text{ uniformly in } S. \qquad (7)$$

By using Lemma 3 and (7), we obtain
$\lim_{n\to\infty} n(\mathscr{V}_n(f;x,y) - f(x,y))$

$$\begin{aligned}&= f'_x(x,y)(1-3x) + f'_y(x,y)(1-3y) \\ &\quad + f''_{xx}(x,y)x(1-x) - 2f''_{xy}(x,y)xy \\ &\quad + f''_{yy}(x,y)y(1-y), \text{ uniformly in } S.\end{aligned}$$

Thus, the proof is completed.

Numerical Examples

Let us consider
$f : S \to \mathbb{R}, f(x,y) = x^2 - \sqrt{7}(1-x-y)^2 - 10xy$. The convergence of bivariate Bernstein-Durrmeyer operators $\mathscr{V}_n(f;x,y)$ to the function f is illustrated in Examples 1 and 2.

Example 1. For $n = 20, 50$ the convergence of the operators $\mathscr{V}_n(f;x,y)$ (blue) to the function (red) is demonstrated in figures 1 and 2 respectively. We notice that the error in the approximation of the function by the operators becomes smaller as n increases.

Example 2. For $n = 20, 50$ the comparison of the convergence of bivariate Bernstein-Durrmeyer operators $\mathscr{V}_n(f;x,y)$ (blue) and bivariate Bernstein-Kantorovich operators $\mathscr{U}_n(f;x,y)$ (green) to the function $f(x,y) = x^2 - \sqrt{7}(1-x-y)^2 - 10xy$ (red) is illustrated in figures 3 and 4 respectively. It is observed that the error in the approximation of f by the operators $\mathscr{V}_n$ is smaller than the operators $\mathscr{U}_n$.

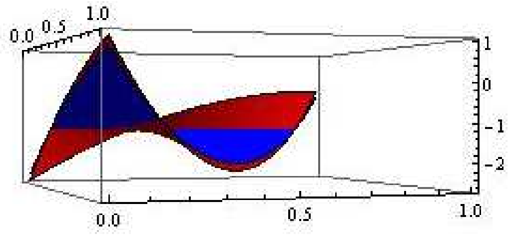

Fig. 2: The convergence of $\mathscr{V}_{50}(f;x,y)$ (blue) to $f(x,y)$ (red)

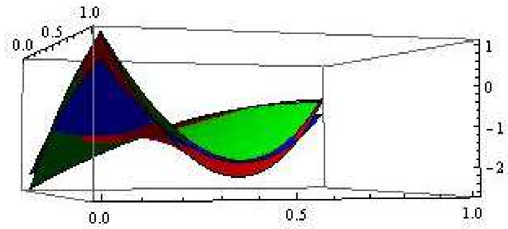

Fig. 3: The comparison of bivariate Bernstein-Kantorovich $\mathscr{U}_{20}(f;x,y)$ (green) and *bivariateBernstein − Durrmeyer* $\mathscr{V}_{20}(f;x,y)$ (blue) to $f(x,y)$ (red)

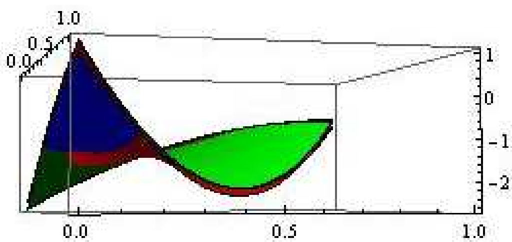

Fig. 4: The comparison of bivariate Bernstein-Kantorovich$\mathscr{U}_{50}(f;x,y)$ (green) and *bivariateBernstein − Durrmeyer*$\mathscr{V}_{50}(f;x,y)$ (blue) to $f(x,y)$ (red)

4 Simultaneous approximation

In this section we study the simultaneous approximation property of the operators $\mathscr{V}_n(\cdot;x,y)$.

Theorem 7 *Let $f \in C^1(S)$. Then for every $(x,y) \in S^\circ$ (the interior of S),*

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(f;\omega,y)\right)_{\omega=x} = \frac{\partial f}{\partial x}(x,y), \tag{8}$$

and

$$\lim_{n\to\infty}\left(\frac{\partial}{\partial v}\mathscr{V}_n(f;x,v)\right)_{v=y} = \frac{\partial f}{\partial y}(x,y). \tag{9}$$

*Proof.*We shall prove only (8) because the proof of (9) is similar.
By the Taylor formula for $f \in C^1(S)$, we have

$$f(s,t) = f(x,y) + f_x(x,y)(s-x) + f_y(x,y)(t-y) + \psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2} \text{ for } (s,t)\in S,$$

where $\psi(s,t;x,y) \equiv \psi(\cdot,\cdot) \in C(S)$ and $\psi(x,y) = 0$.
Operating $\mathscr{V}_n(\cdot;\cdot,y)$ to the above inequality and then by using Lemma 1, we get

$$\begin{aligned}
&\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(f(s,t);\omega,y)\right)_{\omega=x}\\
&= f(x,y)\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(1;\omega,y)\right)_{\omega=x}\\
&\quad + f_x(x,y)\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(s-x;\omega,y)\right)_{\omega=x}\\
&\quad + f_y(x,y)\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(t-y;\omega,y)\right)_{\omega=x}\\
&\quad + \left(\frac{\partial}{\partial\omega}\mathscr{V}_n(\psi(s,t;x,y)\right.\\
&\quad \left.\sqrt{(s-x)^2+(t-y)^2};\omega,y)\right)_{\omega=x}, \ for\ (s,t)\in S\\
&= f_x(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{1+n\omega}{n+3}\right)\right\}_{\omega=x}\\
&\quad + f_y(x,y)\left\{\frac{\partial}{\partial\omega}\left(\frac{1+ny}{n+3}\right)\right)\right\}_{\omega=x}\\
&\quad + \left(\frac{\partial}{\partial\omega}\mathscr{V}_n(\psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2};\omega,y)\right)_{\omega=x}\\
&= f_x(x,y)\left(\frac{n}{n+3}\right) + E, \text{ (say)}.
\end{aligned}$$

Hence, it is sufficient to prove that $E \to 0$, for every $(x,y) \in S^\circ$, as $n \to \infty$.

$$\begin{aligned}
E &= (n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k}\left(\frac{\partial}{\partial\omega}b_{n,k,l}(\omega,y)\right)_{\omega=x}\\
&\quad \times\int_0^1\int_0^{1-t}\psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2}ds\,dt\\
&= (n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k}\frac{\{(k-nx)(1-y)+x(l-ny)\}}{x(1-x-y)^2}\\
&\quad \times b_{n,k,l}(x,y)\int_0^1\int_0^{1-t}\psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2}ds\,dt\\
&= \frac{(n+1)(n+2)(1-y)}{x(1-x-y)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-k}(k-nx)b_{n,k,l}(x,y)\\
&\quad \times\int_0^1\int_0^{1-t}\psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2}ds\,dt\\
&\quad + \frac{(n+1)(n+2)}{(1-x-y)^2}\sum_{k=0}^{n}\sum_{l=0}^{n-k}(l-ny)b_{n,k,l}(x,y)\\
&\quad \times\int_0^1\int_0^{1-t}\psi(s,t;x,y)\sqrt{(s-x)^2+(t-y)^2}ds\,dt\\
&= E_1 + E_2, \text{ (say)}.
\end{aligned}$$

First, we estimate E_1. Applying Cauchy-Schwarz inequality, we have

$$E_1 \le \frac{(1-y)}{x(1-x-y)^2}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y)(k-nx)^2\right)^{1/2}$$
$$\times\left((n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y)\right.$$
$$\left.\times\int_0^1\int_0^{1-t}\psi^2(s,t;x,y)((s-x)^2+(t-y)^2)dsdt\right)^{1/2}$$
$$\le \frac{n(1-y)}{x(1-x-y)^2}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y)\left(\frac{k}{n}-x\right)^2\right)^{1/2}$$
$$\times\left(\mathscr{V}_n\left(\psi^2(s,t;x,y)\left((s-x)^2+(t-y)^2\right);x,y\right)\right)^{1/2}$$
$$= \frac{n(1-y)}{x(1-x-y)^2}\left(\sum_{k=0}^{n}\sum_{l=0}^{n-k} b_{n,k,l}(x,y)\left(\frac{k}{n}-x\right)^2\right)^{1/2}$$
$$\times\left\{\mathscr{V}_n\left(\psi^2(s,t;x,y)(s-x)^2;x,y\right)\right.$$
$$\left.+\mathscr{V}_n\left(\psi^2(s,t;x,y)(t-y)^2;x,y\right)\right\}^{1/2}$$
$$\le \frac{n(1-y)}{x(1-x-y)^2}\left(M_n((s-x)^2;x,y\right)^{1/2}$$
$$\times\left(\left\{\mathscr{V}_n(\psi^4(s,t;x,y);x,y)\right\}^{1/4}\right.$$
$$\left.\times\left\{(\mathscr{V}_n((s-x)^4;x,y))^{1/2}+(\mathscr{V}_n((t-y)^4;x,y))^{1/2}\right\}\right)^{1/2}.$$

By making use of ([5], Lemma (2.5)), for every $(x,y)\in S^\circ$, we have $M_n((s-x)^2;x,y)=O\left(\frac{1}{n}\right)$, as $n\to\infty$.

Thus, we get

$$|E_1| \le M(x,y)\{\mathscr{V}_n(\psi^4(s,t;x,y);x,y)\}^{1/4},$$
in view of Lemma 3 ((*vii*) and (*viii*)).

From Theorem 1, for every $(x,y)\in S^\circ$, we obtain

$$\lim_{n\to\infty}\mathscr{V}_n(\psi^4(s,t;x,y);x,y)=\psi^4(x,y)=0.$$

To estimate E_2, proceeding in a manner similar to the estimate of E_1, for every $(x,y)\in S^\circ$, we get $E_2\to 0$, as $n\to\infty$.
Combining the estimates of E_1 and E_2, it follows that for every $(x,y)\in S^\circ$, $E\to 0$, as $n\to\infty$. Hence the proof is completed.

Similarly, we can prove the following theorem:

Theorem 8 *Let $f\in C^3(S)$. Then for every $(x,y)\in S^\circ$, we have*

$$\lim_{n\to\infty} n\left\{\left(\frac{\partial}{\partial\omega}\mathscr{V}_n(f;\omega,y)\right)_{\omega=x}-\frac{\partial f}{\partial x}(x,y)\right\}$$
$$= -3f_x(x,y)+(2-5x)f_{xx}(x,y)+(1-5y)f_{xy}(x,y)$$
$$+x(1-x)f_{xxx}(x,y)-2xyf_{xxy}(x,y)+y(1-y)f_{xyy}(x,y)$$

and

$$\lim_{n\to\infty} n\left\{\left(\frac{\partial}{\partial v}\mathscr{V}_n(f;x,v)\right)_{v=y}-\frac{\partial f}{\partial y}(x,y)\right\}$$
$$= -3f_y(x,y)+(2-5y)f_{yy}(x,y)+(1-5x)f_{xy}(x,y)$$
$$+y(1-y)f_{xxx}(x,y)-2xyf_{xyy}(x,y)+x(1-x)f_{xxy}(x,y).$$

5 Shape preserving properties

In this section, we study convexity properties of the operators $\mathscr{V}_n$ by proving that the operators $\mathscr{V}_n$ is convex of order (i,j) if $f(x,y)$ is convex of order (i,j) for $0<i+j\le r$.
We first recall the usual definition of convexity for bivariate functions.
For $f\in C(S)$, $(x,y)\in S$ and $h\in\mathbb{R}$, $\triangle_h^{(i,j)}$ is defined by

$$\triangle_h^{(1,0)}f(x,y) = f(x+h,y)-f(x,y),$$
$$\triangle_h^{(0,1)}f(x,y) = f(x,y+h)-f(x,y),$$
$$\triangle_h^{(1,1)}f(x,y) = f(x+h,y+h)+f(x,y)-f(x+h,y)-f(x,y+h),$$
$$\triangle_h^{(2,0)}f(x,y) = f(x+2h,y)-2f(x+h,y)+f(x,y),$$
$$\triangle_h^{(0,2)}f(x,y) = f(x,y+2h)-2f(x,y+h)+f(x,y),$$
$$\triangle_h^{(r,0)}f(x,y) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x+(r-i)h,y),$$
$$\triangle_h^{(0,r)}f(x,y) = \sum_{i=0}^{r}(-1)^i\binom{r}{i}f(x,y+(r-i)h).$$

Definition 1. *$f(x,y)$ is convex of order (i,j), $i,j\in\mathbb{N}^0$, $0<i+j\le r$, if for $h\in\mathbb{R}$, $\triangle_h^{(i,j)}f\ge 0$.*

Remark. Let $i,j\in\mathbb{N}^0$, $0<i+j\le r$. If $f\in C^{i+j}(S)$ and for all $(x,y)\in S$, $\dfrac{\partial^{i+j}f}{\partial x^i\partial y^j}(x,y)\ge 0$, then $f(x,y)$ is convex of order (i,j).

Lemma 5. *For $r=0,1,2\cdots$, $\dfrac{\partial^r}{\partial x^r}\mathscr{V}_n(f;x,y)$ and $\dfrac{\partial^r}{\partial y^r}\mathscr{V}_n(f;x,y)$ can be put in the form*

$$(a)\frac{\partial^r}{\partial x^r}\mathscr{V}_n(f;x,y)=\frac{(n+2)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{n-k-r}b_{n-r,k,l}(x,y)$$
$$\times\int_0^1\int_0^{1-t}b_{n+r,k+r,l}(s,t)\frac{\partial^r}{\partial s^r}f(s,t)dsdt.$$
$$(b)\frac{\partial^r}{\partial y^r}\mathscr{V}_n(f;x,y)=\frac{(n+2)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{n-k-r}b_{n-r,k,l}(x,y)$$
$$\times\int_0^1\int_0^{1-t}b_{n+r,k,l+r}(s,t)\frac{\partial^r}{\partial t^r}f(s,t)dsdt.$$

Proof.(a) By Leibnitz theorem, we get

$$\frac{\partial^r}{\partial x^r}\mathscr{V}_n(f;x,y)$$

$$= (n+1)(n+2)\sum_{k=0}^{n}\sum_{l=0}^{n-k}\binom{n}{k}\binom{n-k}{l}y^l\sum_{j=0}^{r}\binom{r}{j} \times \frac{(-1)^{r-j}k!(n-k-l)!x^{k-j}(1-x-y)^{n-k-l-r+j}}{(k-j)!(n-k-l-r+j)!} \times \int_0^1\int_0^{1-t} b_{n,k,l}(s,t)f(s,t)dsdt$$

$$= (n+1)(n+2)\sum_{j=0}^{r}\sum_{k=j}^{n}\sum_{l=0}^{n-k-r+j}\binom{r}{j} \times \frac{(-1)^{r-j}n!x^{k-j}y^l(1-x-y)^{n-k-l-r+j}}{l!(k-j)!(n-k-l-r+j)!} \times \int_0^1\int_0^{1-t} b_{n,k,l}(s,t)f(s,t)dsdt$$

$$= \frac{(n+2)!}{(n-r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{n-k-r} b_{n-r,k,l}(x,y) \times \int_0^1\int_0^{1-t}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}b_{n,k+j,l}(s,t)f(s,t)dsdt.$$

Again by Leibnitz theorem, we have

$$\frac{\partial^r}{\partial x^r}b_{n+r,k+r,l}(x,y)$$

$$= \frac{\partial^r}{\partial x^r}\left(\binom{n+r}{k+r}\binom{n-k}{l}x^{k+r}y^l(1-x-y)^{n-k-l}\right)$$

$$= \frac{(n+r)!}{l!}\sum_{j=0}^{r}\binom{r}{j}\frac{(-1)^{r-j}y^lx^{k+r-j}(1-x-y)^{n-k-l-r+j}}{(k+r-j)!(n-k-l-r+j)!}$$

$$= \frac{(n+r)!}{n!}\sum_{j=0}^{r}\binom{r}{j}\binom{n}{k+r-j}\binom{n-k-r+j}{l} \times(-1)^{r-j}y^lx^{k+r-j}(1-x-y)^{n-k-l-r+j}$$

$$= \frac{(n+r)!}{n!}\sum_{j=0}^{r}\binom{r}{j}(-1)^j b_{n,k+j,l}(x,y).$$

Thus, we get

$$\frac{\partial^r}{\partial x^r}\mathscr{V}_n(f;x,y) = \frac{(n+2)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{n-k-r} b_{n-r,k,l}(x,y) \times\int_0^1\int_0^{1-t}(-1)^r\frac{\partial^r}{\partial s^r}b_{n+r,k+r,l}(s,t) \times f(s,t)dsdt. \tag{10}$$

Now,

$$\int_0^1\int_0^{1-t}\frac{\partial^r}{\partial s^r}b_{n+r,k+r,l}(s,t)f(s,t)dsdt = \int_0^1\int_0^{1-t}(-1)^r b_{n+r,k+r,l}(s,t)\frac{\partial^r}{\partial s^r}f(s,t)dsdt. \tag{11}$$

From (10) and (11), we obtain

$$\frac{\partial^r}{\partial x^r}\mathscr{V}_n(f;x,y) = \frac{(n+2)!n!}{(n-r)!(n+r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{n-k-r} b_{n-r,k,l}(x,y) \times\int_0^1\int_0^{1-t} b_{n+r,k+r,l}(s,t)\frac{\partial^r}{\partial s^r}f(s,t)dsdt.$$

The proof of (b) is similar to the proof of (a). Hence it is omitted.

Based on definition 1, Remark 5 and using Lemma 5, we give the following theorem:

Theorem 9 *Let $f \in C^{i+j}(S)$ such that $i,j \in \mathbb{N}^0$ and $0 < i+j \le r$. Then the following statement holds: If $f(x,y)$ is convex of order $(r,0)$ (resp. $(0,r)$), then $\mathscr{V}_n(f;x,y)$ is also convex of order $(r,0)$ (resp. $(0,r)$).*

Algorithm:

For the purpose of clarity we mention below the algorithm for one of the figures e.g. figure 4. The domain used in the graphics is $\{(x,y) : x+y \le 1, x,y \ge 0\}$.

$$Plot3D\Bigg[\Bigg\{x^2-\sqrt{7}(1-x-y)^2-10*x*y,$$
$$\frac{50*49*x^2+4*50*x+2}{(50+3)*(50+4)}-$$
$$\sqrt{7}*\frac{1}{(50+3)*(50+4)}*[50*(50-1)*(x+y)^2$$
$$-2*50*(50+1)*(x+y)+(50^2+3*50+2)]$$
$$-\frac{10*(50*(50-1)*x*y+50*(x+y)+1)}{(50+3)*(50+4)},$$
$$\frac{50^2*x^2+2*50*x-50*x^2+\frac{1}{3}}{(50+1)^2}$$
$$-\sqrt{7}*\Bigg(1+\frac{1}{(50+1)^2}*[50^2*x^2+50^2*y^2+2*50*(x+y)$$
$$-50*(x^2+y^2)+\frac{2}{3}]$$
$$-2*x-2*y+\frac{4*50*(50-1)*x*y+2*x+2*y+1}{2*(50+1)^2}\Bigg)$$
$$-\frac{10*(4*50*(50-1)*x*y+2*(x+y)+1)}{4*(50+1)^2}\Bigg\},$$
$$\{x,0,1\},\{y,0,1\},\ \text{PlotStyle}\to\{\text{Red, Blue, Green}\},$$
$$\text{RegionFunction}\to\text{Function}[\{x,y,z\},0\le x+y\le 1],$$
$$\text{Mesh}\to\text{None}\Bigg]$$

Conclusion: The rate of convergence of the bivariate Bernstein-Durrmeyer type operators introduced by Zhou [11] is obtained in terms of the $K-$functional and moduli of continuity. We estimate the order of approximation by Voronovskaja type result and illustrate the convergence of these operators to a certain function through graphics using Mathematica algorithm. We also discuss the comparison of the convergence of the bivariate Bernstein-Durrmeyer operators and the bivariate Bernstein-Kantorovich operators to the function through illustrations using Mathematica. Furthermore, we study the simultaneous approximation for first order partial derivatives and the shape preserving properties of these operators.

Acknowledgement

The authors are extremely grateful to the reviewers for their careful reading of the manuscript and making valuable suggestions and comments leading to a better presentation of the paper. The first and second authors are thankful to the "Council of Scientific and Industrial Research" (Grant code: 09/143(0836)/2013-EMR-1) India" and "University Grant Commission" India respectively for financial support to carry out the above research work.

References

[1] T. Acar and A. Aral, Approximation properties of two dimensional Bernstein-Stancu-Chlodowsky operators, Matematiche, **68** (2) (2013) 1531.
[2] G. A. Anastassiou and S. Gal, Approximation Theory. Moduli of Continuity and Global Smoothness Preservation, Birkhäuser, Boston (2000).
[3] P. L. Butzer, On two dimensional Bernstein polynomials, Canadian J. Math. **5** (1953) 107-113.
[4] P. L. Butzer and H. Berens, Semi-groups of operators and approximation, Springer, New York, xi+318 pp. (1967).
[5] N. Deo and N. Bhardwaj, Some approximation theorems for multivariate Bernstein operators, Southeast Asian Bulletin Math. **34** (2010) 1023-1034.
[6] M. M. Derriennic, On multivariate approximation by Bernstein-type polynomials, J. Approx. Theory **45** (2) (1985) 155-166.
[7] E. H. Kingsley, Bernstein polynomials for functions of two variables of class $C^{(k)}$, Proc. Amer. Math. Soc. **2** (1) (1951) 64-71.
[8] O. T. Pop, The generalization of Voronovskaja's theorem for a class of bivariate operators, Stud. Univ. Babes-Bolyai Math. **53** (2) (2008) 85-107.
[9] O. T. Pop and M. D. Fărcaş, About the Bivariate operators of Kantorovich type, Acta Math. Univ. Comenian. **78** (1) (2009) 43-52.
[10] D. D. Stancu, A method for obtaining polynomials of Bernstein type of two variables, Amer. Math. Monthly **70** (3) (1963) 260-264.
[11] D. X. Zhou, Inverse theorems for multidimensional Bernstein-Durrmeyer operators in L_p^*, J. Approx. Theory, **70** (1992) 68-93.
[12] C. Badea, I. Badea and H.H. Gonska, A test function theorem and approximation by pseudopolynomials, Bull. Austral. Math. Soc. **34** (1986) 53-64.

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