New Foundations for the Proof Theory of Bi-Intuitionistic and Provability Logics Mechanized in Coq

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The work in this dissertation is my own except where otherwise stated.

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Abstract

We contribute to the shift towards logic as a fully formalised science by formalising in the interactive theorem prover Coq both new and known results. Our contribution is threefold.

First, we define pen-and-paper general notions required in our dissertation: propositional and first-order syntax, axiomatic calculus, sequent calculus, Kripke semantics, soundness and completeness results. Furthermore, we instantiate these notions in modal logic, both in pen-and-paper and in Coq. Consequently, we constitute a readily available and usable Coq library containing foundational elements of the study of modal logic, which can thus serve as a basis for the formalisation of further works in non-classical logics.

Second, we exhibit and rectify a mistake that has gone unnoticed for almost fifty years in the foundations of propositional and first-order bi-intuitionistic logic. We show that both in the propositional and first-order case, what was conceived of as a unique bi-intuitionistic logic is in fact two distinct logics. We trace these confusions back to a fundamental problem in the axiomatic proof theory of propositional and first-order bi-intuitionistic logic: traditional Hilbert calculi are not designed to treat logics as consequence relations. They lead to an ambiguous notion of deduction from assumptions that can cause us to conflate distinct rules and thus distinct logics. More precisely, we show that the bi-intuitionistic rule (DN) is ambiguous in a traditional context and splits into two distinct rules in the context of generalized Hilbert calculi. As a consequence, we obtain two generalized Hilbert calculi in the propositional (wBI**H** and sBI**H**) and first-order (FOwBI**H** and FOsBI**H**) case, for bi-intuitionistic logic that differ only in the disambiguated version of the (DN) rule they incorporate. Unsurprisingly, these systems capture two distinct propositional logics (wBI**L** and sBI**L**) and first-order logics (FOwBI**L** and FOsBI**L**), which have been conflated in the literature.

Third, we obtain cut-elimination results for the sequent calculi GLS and GL4ip, which are sequent calculi for, respectively, the classical modal provability logic GL**L** and the intuitionistic modal provability logic iGL**L**. These results involve a novel proof technique for the admissibility of additive-cut called the "mhd proof technique", which relies on the fact that terminating backward proof-search allows us to attribute a "*m*aximal *h*eight of *d*erivations" to each sequent. So, for each of the calculi GLS and GL4ip, we define a backward proof-search procedure for it and develop a thorough termination argument using a local measure on sequents and a well-founded relations along which this measure decreases upwards in the proof-search. Then, we directly prove the admissibility of additive-cut for both calculi using the *mhd proof technique*. We use this number, the mhd of a sequent, as an induction measure in arguments involving local and syntactic transformations, allowing us to exhibit and hence extract from our Coq formalisation Haskell programs constituting cut-elimination procedures.

Contents

Introduction

Our proofs are unreliable. On the contrary to Brouwer [20, 3], we do not believe that this unreliability comes from the *principles* we, logicians, use in our proofs. Instead, we believe that it emerges from our *practice* of proving.

What is our practice of proving? To understand this, we need to explain what is the bread and butter of a nowadays logician. Abstractly, a logician produces formal definitions, states lemmas and theorems using these definitions, and builds proofs justifying the stated lemmas and theorems - all of these pertaining in some way or another to what is designated by "logic". Concretely, a logician most commonly does all of the above by writing these definitions, lemmas, theorems, and proofs (we call them *productions*) on a piece of paper, a whiteboard, or in LaTeX. We refer to this practice as the *pen-and-paper* practice. Thus, our practice of proving is the way we build proofs in the pen-and-paper practice.

Why do we trust productions of the pen-and-paper practice? We do so for two main reasons: education and control.

First, one does not become a logician in a day. Acquiring the skills required for this profession comes at the cost of several years spent at universities studying the common elements encountered in this scientific discipline, the core theorems of various aspects of logic, and the mathematical rigor inherent to the production of results in this field. Thus, at the end of their education students have wide and technical knowledge about the scientific discipline of logic, allowing them to design innovative and/or complex productions. So, we trust productions of the pen-and-paper practice notably because its practitioners are well-trained.

Second, any production of the pen-and-paper practice is reviewed before becoming an accepted item in the field. In a nutshell, the reviewing process is a control mechanism at the core of the scientific approach, which consists of the assessment of a journal article, a conference paper, or a dissertation, by one or several persons who are recognized as "experts" by the scientific community. These experts are meant to read the journal article, say, and amongst other things determine whether the productions are correct. The acceptance for publication of the journal article consequently implies that after a careful analysis the experts believe the productions of the pen-and-paper practice contained in the article, and thus label them as correct. This labeling carries the weight of the experts' status, thus generally leading the community to accept these productions.

Because of the education required to perform the pen-and-paper practice and the control enforced on any production of this practice, it seems that we have reasons to trust the productions thus obtained. So, why should we believe, as suggested above, that the pen-and-paper practice is unreliable? Because logicians and experts are humans after all, and hence fallible. More precisely, in face of this fallibility of humans, the combination of education and control cannot guarantee the reliability of the productions of the pen-and-paper practice. How so? Despite being well trained, a logician can: want their result to be true, and convince themselves of it ; be absent-minded and forget about technical aspects of a definition, say ; simply misunderstand ; have a bad day (being sad, tired, or hungover) thus leading them to be less careful in the design of their productions ; have deadlines to meet, inciting them to speed up their production process, potentially at the cost of quality ; deal with complex proofs with many more or less similar cases, thus tempting them to bundle many cases they believe to be very similar ; etc. Consequently, a logician can perfectly produce a mistake, contained in an article they submit to a journal or a conference. The control imposed by the reviewing process should prevent the publication of the article, right? In theory, yes. In reality, reviewers are also fallible, and very frequently cannot dedicate the required time to the evaluation of these productions. In fact, it is not uncommon for reviewers to not check the proofs of an article in detail, as this is a significantly time-consuming activity.

As a result mistakes in productions can make their way through the publication process. Once published, flawed productions are automatically disguised as items of scientific knowledge, allowing the creation of further flawed productions relying on them. This phenomenon can lead to at least two issues. First, entire scientific fields can be developed on shaky foundations, potentially causing their collapse once the mistakes are discovered. Second, mistakes can be present in proofs for statements that luckily happen to be holding. While this is less dramatic, it can be the source of great confusions, notably on the range of applicability of certain proof techniques. Both issues should obviously be avoided in any serious scientific approach.

All of the above is well-known, and so is a remedy to the unreliability of the pen-and-paper practice: the *formalised* practice. In the latter, we use interactive theorem provers (ITPs), which are systems providing an environment to write definitions, lemmas, theorems, and proofs in a formal language. The interaction between the ITP and the (human) user consists of the mechanical verification by the computer of the correctness of the elements written by the user, notably the steps of proofs. In particular, a proof of a statement is verified in an ITP only if the computer recognizes all the steps in the proof as applications of rules already agreed on, i.e. readily available in the ITP or declared by the user as an accepted rule. We refer to such proofs as *formalised* (sometimes called *formal proofs* [69]).

Why is the formalised practice more reliable than the pen-and-paper one? Because formalised proofs are verified by computers, which are far less fallible than we are when it comes to verification. Indeed, computers are not absent-minded, do not misunderstand, do not have bad days, have no deadlines to meet, and can deal with a great number of cases without forgetting one or confusing two. Because of this, formalised proofs are highly reliable: they are *necessarily* built using *only* rules and assumptions that are explicitly given. Thus, to determine with certainty whether the formalised proof of a statement constitutes a correct proof of this statement, it suffices to check, first, that the computer verifies the proof and, second, that we accept all the explicit rules and assumptions given in the ITP on which this proof relies.

While the formalised practice eliminates many issues encountered in the pen-and-paper practice, its use comes at a cost. First, the use of a specific ITP requires several months of learning. Formal languages used in ITPs can be alien to logicians, even in those ITPs which possess a language mimicking steps in pen-and-paper proofs. Their mastering is consequently a long process, which is best accomplished in a logician's education in university courses. Second, the reading of formalised proofs can be difficult. Often, formalised proofs are constituted by hundreds of lines of uncommented code, and can thus be very hard to identify with the pen-and-paper proofs. As a consequence, the content of some formalised proofs, e.g. the proof techniques used in them, can be intricate to extract. This intricacy can impede both the reviewing process and the teaching of proof techniques. Third, the multiplicity of ITPs, and their lack of compatibility, leads to philosophical problems about the universality of mathematics (see Dowek's recent paper [39]).

So, the formalised practice has greater reliability but is not perfect. Still, we believe that logic should shift to the formalised practice, thus transitioning to a fully formalised science, but with a twist. More precisely, we believe that formalised proofs and pen-and-paper proofs should always come together, to obtain the best of both worlds. With this combination, we can have clear communication of proofs and proof techniques on pen-and-paper, while having the reliability of formalised proofs. As a consequence, we concur with Avigad and Harrison's view [4, p.73]:

> *Formal verification is not supposed to replace human understanding or the development of powerful mathematical theories and concepts. Nor are formal proof scripts meant to replace ordinary mathematical exposition. Rather, they are intended to supplement the mathematics we do with precise formulations of our definitions and theorems and assurances that our theorems are correct.*

In what follows, we contribute to this shift towards a fully formalised science by both producing new results in logic which we formalised in the interactive theorem prover Coq, and formalising known results. Consequently, the work outlined below should be of interest to both the logic community and the ITP community. Our contribution is threefold. First, alongside the general notions we need in our dissertation, we constituted a library containing the formalisation of instances for modal logic of these general notions. This way, our library treats many foundational elements of the study of modal logic: propositional and first-order syntax, axiomatic calculus, sequent calculus, Kripke semantics, soundness and completeness results. Second, we justify the need to formalise proofs by exhibiting and rectifying a mistake that has gone unnoticed for almost fifty years in the foundations of bi-intuitionistic logic. More precisely, we show that both in the propositional and first-order case, what was conceived of as a unique bi-intuitionistic logic is in fact two distinct logics. Their treatment through axiomatic systems and Kripke semantics is explored and formalised, as well as their soundness and completeness proofs (only relative completeness proof in the first-order case). So, we claim to have produced the *only* coherent and correct account of bi-intuitionistic logic as two, rather than one, logics. Third, we obtain cut-elimination results for two provability logics GLL and iGLL using a novel, simpler proof technique. Our formalisation of these results provides us with Haskell programs that effectively eliminate cuts in the sequent calculi for these logics augmented with the cut rule.

Our dissertation is consequently divided into three parts. First, in Part I we introduce all the general notions and tools we use in this dissertation and instantiate them both on pen-and-paper and in Coq using the running example of modal logic. Second, in Part II we present all our results on bi-intuitionistic logics, propositional and first-order. Third, in Part III we exhibit our findings on sequent calculi for the provability logics GLL and iGLL.

Part I

Toolbox

In this part, we introduce all the general notions and tools we use in this dissertation. To make these notions more digestible, we instantiate them in the familiar setting of classical modal logic. Furthermore, we introduce the reader to key elements of our formalisation in Coq by exhibiting the formalisation of the aforementioned elements about classical modal logic.

Each chapter introducing a notion that we can instantiate in classical modal logic is thus divided into three parts:

1. general notions;
2. The classical modal logic example: on paper;
3. The classical modal logic example: in Coq.

Now, how should the reader use this chapter? Disclaimer: it is meant to be read neither as a preamble to this dissertation nor in a linear way. Instead, this chapter is built as a *toolbox*, which one goes to for a specific tool whenever one needs it. So, we specify at the beginning of each chapter which parts of this dissertation are required to understand the aforementioned chapter. Finally, note that the parts on the Coq formalisation can be altogether disregarded by the reader who is an exclusively pen-and-paper logician.

Chapter 1

Generalities and Notation

Any result that we formalised is accompanied by a clickable keyboard symbol "⌨" leading to the location of its formalised version. We flag non-formalised proofs by proof environments starting by "*Proof*✍.".

In proofs, we often use the notion of *strong induction*, which relies on what is called a *strong induction principle*. This type of principle assumes given a type X and a binary relation R on elements of type X. Then, the strong induction principle is presented as follows, where $\Rightarrow$ is the meta-level implication, $WellFounded$ is a predicate for the property of binary relations of being *well-founded* (there are no infinite descending R-chains of the shape $\dots x_2 R x_1 R x_0$), and $Prop$ is a predicate for a property on elements of type X.

$$WellFounded(R) \quad \Rightarrow \quad (\forall x, (\forall y, (xRy \Rightarrow Prop(y)) \Rightarrow Prop(x))) \quad \Rightarrow \quad (\forall x, Prop(x))$$

In essence, this principle tells us that we can prove that a property $Prop$ holds for all elements of type X (i.e. $(\forall x, Prop(x))$) under two conditions. The first condition consists in proving that the relation R is *well-founded* (i.e. $WellFounded(R)$). The second amounts to proving that for any x, if all the R-predecessors of x satisfy $Prop$, then x also satisfies $Prop$ (i.e. $(\forall x, (\forall y, (Rxy \Rightarrow Prop(y)) \Rightarrow Prop(x)))$).

In this dissertation we use the letters $\varphi, \psi, \chi, \dots$ for formulas and $\Gamma, \Delta, \Phi, \Psi, \dots$ for sets and multisets of formulas, which should be clear from context. We also make use of apostrophes and numbers to designate formulas $(\varphi', \varphi'', \dots, \varphi_0, \varphi_1, \dots)$ and sets of formulas $(\Gamma', \Gamma'', \dots, \Gamma_0, \Gamma_1, \dots)$.

Below is a definition of some general notations we use throughout this dissertation.

Definition 1.0.1. We define the following:

- $\mathbb{N}$ is the set of all natural numbers;
- S is the successor operator on natural numbers;
- For a set X the set $\mathsf{Pow}(X) = \{X' \mid X' \subseteq X\}$ is the power set of X;
- An arity function for a set X is a function $Ar_X : X \to \mathbb{N}$;
- For a pair (X, Y), we define $\mathsf{proj1}((X, Y)) = X$ and $\mathsf{proj2}((X, Y)) = Y$ to be respectively the first and second projection of (X, Y).

Chapter 2

Syntax

The study of formal logics essentially relies on *formal languages* and ultimately their interpretation. In this context, the definition of formal languages is often said to constitute the *syntax* of a logic. So, we proceed in this chapter to provide the syntax for the two types of logics we consider throughout this dissertation: propositional and first-order.

2.1 Propositional languages

We start by defining propositional languages and their constituents.

Definition 2.1.1. A *propositional language* $\mathbb{L}$ is a pair $(\mathbb{V}, \mathcal{C})$ where:

- $\mathbb{V}$ is a countable set $\{p, q, r \dots\}$ of propositional variables on which equality is decidable ;
- $\mathcal{C}$ is a pair (Con, Ar_{Con}) such that Con is a countable set of *connectives* on which equality is decidable, and $Ar_{Con} : Con \to \mathbb{N}$ is an arity function for connectives.

Note that our requirement on the decidability of equality on the set $\mathbb{V}$ is necessary if one wants equality on formulas to be decidable. This is often overlooked (or implicit) in the literature.

For each propositional language, we define its set of propositional formulas.

Definition 2.1.2 (Propositional Formulas)**.** Let $\mathbb{L} = (\mathbb{V}, \mathcal{C})$ be a propositional language where $\mathcal{C} = (Con, Ar_{Con})$. The set $Form_{\mathbb{L}}$ of *propositional formulas of* $\mathbb{L}$ is inductively defined by the following prefix grammar.

$$\varphi ::= p \in \mathbb{V} \mid \bullet(\varphi, ..., \varphi)$$

where $\bullet \in Con$ and $(\varphi, ..., \varphi)$ is a tuple of length $Ar_{Con}(\bullet)$.

As $Form_{\mathbb{L}}$ is inductively defined, we automatically get from this definition an *inductive principle*, which we use when we do "proofs by induction". More precisely, we have the following principle for any property *Prop* on formulas where & and ⇒ are respectively the meta-level conjunction and the meta-level implication.

$$(\forall p, p \in \mathbb{V} \Rightarrow Prop(p)) \Rightarrow$$

$$(\forall \varphi_1 \dots \varphi_n, (Prop(\varphi_1) \& \dots \& Prop(\varphi_n)) \Rightarrow Prop(\bullet(\varphi_1, ..., \varphi_n))) \Rightarrow$$

$$(\forall \varphi, \varphi \in Form_{\mathbb{L}} \Rightarrow Prop(\varphi))$$

Put simply, we can prove that a property *Prop* holds for all formulas (the last line), if we can prove that (1) the property holds for all propositional variables (the first line), and (2) that it holds for any complex formula $\bullet(\varphi_1, ..., \varphi_n)$ under the assumption that *Prop* holds for all of $\varphi_1, ..., \varphi_n$ (the second line). This very powerful principle is used to prove statements in various places in this dissertation.

A particularly useful notion is the one of subformula of a formula, defined as follows.

Definition 2.1.3 (Propositional Subformulas)**.** Let $\mathbb{L}$ be a propositional language, and $\varphi \in Form_{\mathbb{L}}$. We define $\mathrm{Subf}(\varphi)$ the *set of subformulas of* φ by recursion on the structure of φ:

$\varphi = p$: $\mathrm{Subf}(p) := \{p\}$;

$$\varphi = \bullet(\varphi_0, \dots, \varphi_n) \textbf{ : } \mathrm{Subf}(\bullet(\varphi_0, \dots, \varphi_n)) := \{\bullet(\varphi_0, \dots, \varphi_n)\} \cup \bigcup_{0 \leq i \leq n} \mathrm{Subf}(\varphi_i).$$

We abuse the notation to designate the set (resp. multiset) of subformulas of all formulas in the set (resp. multiset) Γ by $\mathrm{Subf}(\Gamma)$.

We define the notion of propositional variable substitution, which allows us to replace propositional variables in a formula with formulas

Definition 2.1.4. Let $\mathbb{L} = (\mathbb{V}, \mathcal{C})$ be a propositional language. A *propositional variable substitution* for $\mathbb{L}$ is a function $\sigma : \mathbb{V} \to Form_{\mathbb{L}}$.

Definition 2.1.5. Let $\mathbb{L} = (\mathbb{V}, \mathcal{C})$ be a propositional language and σ a propositional variable substitution for $\mathbb{L}$. We define φ^σ the *propositional variable substitution of* φ *through* σ by recursion on the structure of φ:

$\varphi = p$: $p^\sigma := \sigma(p)$;

$\varphi = \bullet(\varphi_0, \dots, \varphi_n) \textbf{ : } (\bullet(\varphi_0, ..., \varphi_n))^\sigma := \bullet(\varphi_0^\sigma, \dots, \varphi_n^\sigma)$.

We abuse the notation to designate the set (resp. multiset) of propositional variable substitution of formulas through σ of all formulas in the set (resp. multiset) Γ by Γ^σ.

On top of the inductive principle given by the inductive definition of formulas, we often use additional *measures* on formulas which allow us to prove statements using the decreasing of these measures along a well-founded order on them. In this dissertation, the measures on formulas we use are natural numbers attributed to formulas. Consequently, we can use the well-founded order $<$ on $\mathbb{N}$ to obtain an inductive principle useful to prove statements about formulas.

For example, we use the notion of size of a formula.

Definition 2.1.6. Let $\mathbb{L} = (\mathbb{V}, \mathcal{C})$ be a propositional language and $\varphi \in Form_{\mathbb{L}}$. For a formula φ, we define the *size*, $size(\varphi)$, by recursion on the structure of φ.

$\varphi = p$: $size(p) := 1$;

$\varphi = \bullet(\varphi_0, ..., \varphi_n)$: $size((\bullet(\varphi_0, ..., \varphi_n))) := 1 + size(\varphi_0) + \dots + size(\varphi_n)$.

Throughout this dissertation, we fix $\mathbb{V} = \{p, q, r \dots\}$ to be an infinite set of propositional variables. Furthermore, we require that equality is decidable on this set.

Hypothesis 2.1.1. For all $p, q \in \mathbb{V}$, we can decide whether $p = q$ or $p \neq q$.

2.1.1 The classical modal logic example: on paper

Now, let us consider a specific set of connectives for **C**lassical **M**odal logic $\mathcal{C}_{\mathbf{CM}} = (Con_{\mathbf{CM}}, Ar_{Con_{\mathbf{CM}}})$ where:

- $Con_{\mathbf{CM}} = \{\bot, \to, \Box\}$;
- $Ar_{Con_{\mathbf{CM}}}$ is such that:

$$Ar_{Con_{\mathbf{CM}}}(\bot) = 0 \qquad Ar_{Con_{\mathbf{CM}}}(\to) = 2 \qquad Ar_{Con_{\mathbf{CM}}}(\Box) = 1$$

We can thus define the propositional language $\mathbb{L}_{\mathbf{CM}} = (\mathbb{V}, \mathcal{C}_{\mathbf{CM}})$ and obtain its set of *modal* formulas $Form_{\mathbb{L}_{\mathbf{CM}}}$ through its prefix grammar:

$$\varphi ::= p \in \mathbb{V} \mid \bot \mid \to (\varphi, \varphi) \mid \Box(\varphi)$$

However, we usually prefer the use of *infix* grammars. In infix notation, the above grammar takes its common shape:

$$\varphi ::= p \in \mathbb{V} \mid \bot \mid \varphi \to \varphi \mid \Box\varphi$$

As an example, the expression $p \to \Box q$ is indeed an element of $Form_{\mathbb{L}_{\mathbf{CM}}}$ because $p, q \in \mathbb{V}$ are formulas, and thus $\Box q$ and in turn $p \to \Box q$ are formulas. Furthermore, we can compute $\mathrm{Subf}(p \to \Box q)$ to obtain the set $\{p \to \Box q, p, \Box q, q\}$.

2.1.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

To formalise our work, we need to encode the above usual pen-and-paper definitions.

We encode modal formulas as a type `MPropF` over some implicit parametric set `V` of propositional variables (⌨), over which we assume equality to be decidable. Note that `<>` stands for $\neq$ in Coq, while `+` stands here for the `Type` sum.

```
Parameter eq_dec_propvar :
        forall p q : V, {p = q} + {p <> q}.
```

Thus, we encode our set of propositional variables in the following way (⌨).

```
Inductive MPropF {V : Set} : Type :=
 | Var : V -> MPropF
 | Bot : MPropF
 | Imp : MPropF -> MPropF -> MPropF
 | Box : MPropF -> MPropF.
```

This inductive definition essentially captures the grammar given above: propositional variables constitute formulas with the constructor `Var` ; there is a constructor for each connective (`Bot` for $\bot$, `Imp` for $\to$, `Box` for $\Box$) ; the arity of each connective is respected (`Bot` takes no argument, `Imp` takes two, `Box` takes one). Note how each case is flagged by the vertical bar `|`. For example, the formula $p \to \Box q$ we listed previously becomes the Coq term `Imp (Var p) (Box (Var p))`.

We can add notations for connectives (⌨), which allow us to use them in an infix way. We add a notation for the implication symbol, where parentheses on the second line specify properties of the notation for Coq to read it.

```
Notation "φ --> ψ" := (Imp φ ψ)
       (at level 16, right associativity).
```

For convenience, we also introduce the following notation for propositional variables (⌨).

```
Notation "# p" := (Var p)
        (at level 1).
```

Once modal formulas are defined, we can encode the notion of subformulas (⌨).

```
Fixpoint subform (φ : MPropF) : Ensemble MPropF :=
  match φ with
   | Var p => Singleton _ (Var p)
   | Bot => Singleton _ Bot
   | Imp ψ χ => Union _ (Singleton _ (Imp ψ χ))
                       (Union _ (subform ψ) (subform χ))
   | Box ψ =>  Union _ (Singleton _ (Box ψ)) (subform ψ)
  end.
```

There is a crucial use of recursion on the structure of a formula φ in this definition. This can be detected by two elements. First, the expression `Fixpoint` explicitly expresses the fact that the definition we are about to give relies on the structure of some inductively defined objects, formulas in this case [145]. Second, this recursion shows in the case-by-case analysis over the structure of φ and its declaration via the expression `match` φ `with`.

The content of the definition itself should be rather clear: on input φ (i.e. a formula), `subform` outputs the set of subformulas of φ, which is an element of type `Ensemble MPropF` (`Ensemble` is French for "set"). For example, to the boxed formula `Box` ψ we attach the `Union` of the `Singleton` {`Box` ψ} and the set `subform` ψ of subformulas of ψ. It has to be noted that the constructors `Singleton` and `Union` require the type of entities it contains to be specified, but we can ask Coq to infer the type by using the dash `_`. If Coq cannot infer which type we meant, an error message will be displayed.

The above definition uses the mathematical notion of sets via `Ensemble`. In Coq, sets are identified with their characteristic function. So, a set of formulas Γ is nothing but a function from $Form_{\mathbb{L}}$ to *Prop*, the type of propositions in Coq. Thus, the elementhood of φ in Γ can be detected in Coq by the holding of the $\Gamma\ \varphi$. That being said, note that we can reproduce the notion above it using the notion of *list* instead of `Ensemble` (⌨).

```
Fixpoint subformlist (φ : MPropF) : list MPropF :=
  match φ with
   | Var p => (Var p) :: nil
   | Bot => Bot :: nil
   | Imp ψ χ => (Imp ψ χ) ::
                (subformlist ψ) ++ (subformlist χ)
   | Box ψ =>  (Box ψ) :: (subformlist ψ)
  end.
```

There exists a well-developed library for lists in Coq [146] which uses a *polymorphic* definition, i.e. one which is parametrized in a type of objects. For example, we define lists of *formulas* with the type `list MPropF` and lists of natural numbers with `list nat`. For any type of lists, the empty list is `nil` and the usual operations "append" and "cons" are respectively represented by `++` and `::`. However, Coq also allows us to write lists in infix notation using `;`. Thus the terms `φ1 :: φ2 :: φ3 :: nil` and `[φ1] ++ [φ2] ++ [φ3]` and `[φ1 ; φ2 ; φ3]` all encode the same list $\varphi1, \varphi2, \varphi3$.

We formalise the common notion of propositional variable substitution of φ through σ (⌨). We use such substitutions later on to define logics.

```
Fixpoint subst (σ : V -> MPropF) (φ : MPropF) : MPropF :=
  match φ with
   | Var p => (σ p)
   | Bot => Bot
   | Imp ψ χ => Imp (subst σ ψ) (subst σ χ)
   | Box ψ => Box (subst σ ψ)
  end.
```

Finally, we formalise the notion of size of formulas (⌨).

```
Fixpoint size (φ : MPropF) : nat :=
  match φ with
   | Var p => 1
   | Bot => 1
   | Imp ψ χ => 1 + (size ψ) + (size χ)
   | Box ψ => 1 + (size ψ)
  end.
```

2.2 First-order languages

First-order languages differ from propositional languages in two main ways. First, while the elementary building blocks of propositional languages are propositional variables, in the first-order case these blocks are *atoms*, or atomic formulas. More precisely, atoms are expressions of the shape $P(t_1, ..., t_n)$ where P is a predicate symbol with a certain arity, here n, and $t_1, ..., t_n$ are terms. Thus, to define a first-order language we need to characterize predicate symbols as well as terms. We call such characterizations *signatures.*

Definition 2.2.1. A signature $\mathcal{S}$ is a pair $(\mathcal{F}, \mathcal{P})$ where:

- $\mathcal{F}$ is a pair (Fun, Ar_{Fun}) of a countable set Fun of *function symbols* $f, g, h \dots$ on which equality is decidable, and Ar_{Fun} an arity function on Fun;
- $\mathcal{P}$ is a pair $(Pred, Ar_{Pred})$ of a countable set $Pred$ of *predicate symbols* $P, Q, R \dots$ on which equality is decidable, and Ar_{Pred} an arity function on $Pred.$

It may come as a surprise that we do not explicitly consider constant symbols in signatures. However, it has to be noted that we do not need to give a special place to constant symbols: they are nothing but function symbols $c \in Fun$ such that $Ar_{Fun}(c) = 0$.

With a signature, we already have characterized a set of predicate symbols. However, we have not reached the characterization of a set of terms yet. We proceed to define terms next.

Definition 2.2.2. Let $\mathcal{S} = (\mathcal{F}, \mathcal{P})$ be a signature. We define inductively the set of terms $Term_{\mathcal{S}}$ of $\mathcal{S}$ in the following way:

- if $n \in \mathbb{N}$ then $n \in Term_{\mathcal{S}}$;
- if $t_1, ..., t_n \in Term_{\mathcal{S}}$, $f \in Fun$ and $Ar_{Fun}(f) = n$, then $f(t_1, ..., t_n) \in Term_{\mathcal{S}}$.

In other words, we have the following grammar:

$$t ::= n \in \mathbb{N} \mid f(t, ..., t)$$

Usually one would expect variable symbols to appear as basic constituents of terms, as in the usual syntax with names. Here instead, we involve natural numbers as we use a syntax à la de Bruijn [34]. This decision to deviate from the usual syntax is motivated by formalisation issues: binders, hence quantifiers, are difficult to formalise. However, a well-developed approach in Coq uses de Bruijn indices [140]. We consequently decided to follow this approach in our formalisation, hence the need to adopt this syntax in this dissertation.

So, with the notion of signature, we now have a thorough definition of the building blocks of our first-order languages. However, there is a second way in which first-order languages differ from propositional ones: the use of variable binders called *quantifiers.*

Definition 2.2.3. A *set of quantifiers* $\mathcal{Q}$ is a countable set of *symbols*, called quantifiers, on which equality is decidable.

In this dissertation we only consider *unary* quantifiers, hence the absence of an arity function in $\mathcal{Q}$.

We now have all the material to define first-order languages.

Definition 2.2.4. A *first-order language* $\mathbb{L}$ is a triple $(\mathcal{S}, \mathcal{C}, \mathcal{Q})$ where $\mathcal{S}$ is a signature, $\mathcal{C}$ is a set of connectives and $\mathcal{Q}$ is a set of quantifiers.

On the contrary to propositional languages, where we use a fixed set $\mathbb{V}$ of propositional variables, here we do not work with a fixed signature. As a consequence, when a first-order language is based on a signature $\mathcal{S}$, we say that it is a *first-order language in* $\mathcal{S}$.

With the above definition of a language, we define the set of formulas of a first-order language $\mathbb{L}$ in $\mathcal{S}$.

Definition 2.2.5. Let $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ be a first-order language where $\mathcal{S} = (\mathcal{F}, \mathcal{P})$. We define inductively the set $Form_{\mathbb{L}}$ of formulas of $\mathbb{L}$ in the following way:

- if $P \in \mathcal{P}$ and $Ar_{\mathcal{P}}(P) = n$ and $t_1, \ldots, t_n \in Terms_{\mathcal{S}}$, then $P(t_1, \ldots, t_n) \in Form_{\mathbb{L}}$;
- if $\bullet \in Con$ and $Ar_{Con}(\bullet) = n$ and $\varphi_1, \ldots, \varphi_n \in Form_{\mathbb{L}}$, then $\bullet(\varphi_1, \ldots, \varphi_n) \in Form_{\mathbb{L}}$;
- if $\triangledown \in \mathcal{Q}$ and $\varphi \in Form_{\mathbb{L}}$, then $\triangledown\varphi \in Form_{\mathbb{L}}$.

In other words, we have the following grammar:

$$\varphi ::= P(t_1, \ldots, t_n) \mid \bullet(\varphi, ..., \varphi) \mid \triangledown\varphi$$

Formulas of the form $P(t_1, \ldots, t_n)$ are called *atoms* or *atomic formulas.*

Here are two examples of formulas, assuming that we have the binary connective $\rightarrow$ and two quantifiers $\forall$ and $\exists$.

$$\forall(P(0) \rightarrow \exists Q(0,1,7)) \quad \forall R(0,0) \rightarrow \exists\forall P(1)$$

The left expression is a formula because of the following. Q is a ternary predicate, and $0, 1$ and 7 are terms, hence $Q(0,1,7)$ is a formula. Consequently, we get that $\exists Q(0,1,7)$ is a formula. Note that in $\exists Q(0,1,7)$ the variable 0 is bound by the quantifier $\exists$, as between the occurrence of $\exists$ and the location where the second occurrence of 0 is we encounter no other quantifiers. Then, we have that $P(0)$ is a formula. Thus, $P(0) \rightarrow \exists Q(0,1,7)$ is a formula too. Finally, we obtain that $\forall(P(0) \rightarrow \exists Q(0,1,7))$ is a formula. In this resulting formula, we see that the outermost quantifier bounds both 0 in $P(0)$ and 1 in $Q(0,1,7)$. The first bounding of 0 can be explained as above: no quantifier is interleaved in the structure of the formula between $\forall$ and 0. The second bounding of 1 happens as there is one quantifier, i.e. $\exists$, interleaved between $\forall$ and the location of 1. Note that the formula $\forall(P(0) \rightarrow \exists Q(0,1,7))$ is thus not a “closed” formula, as 7 is a “free” variable here. The right expression can be shown to be a formula in a similar way. In this formula, the quantifier $\exists$ in $\exists\forall P(1)$ bounds 1, and the quantifier $\forall$ in $\forall R(0,0)$ bounds both occurrences of 0. We can now see that in the usual syntax these formulas can be represented as:

$$\forall x(P(x) \rightarrow \exists y Q(y,x,z)) \quad \forall x R(x,x) \rightarrow \exists y \forall z P(y)$$

Note how the above are just constituting *one among many* representations of the formulas in the syntax à la de Bruijn: we could use other bound variables. In the de Bruijn syntax, there is only one representation for these formulas hence we avoid having to “rename” variables in a formula as is usual in the usual notation.

We now turn to the definition of variable substitution. Given that in a syntax à la de Bruijn natural numbers play the role of variables in a named syntax, we define the notion of variable substitution over terms on natural numbers.

Definition 2.2.6. Let $\mathbb{L}$ be a first-order language. A *variable substitution* for $\mathbb{L}$ is a function $\tau : \mathbb{N} \rightarrow Terms_{\mathcal{S}}$.

The next definition provides the details on how to apply such variable substitution on a term t.

Definition 2.2.7. Let $\mathcal{S} = (\mathcal{F}, \mathcal{P})$ be a signature, $t \in Terms_{\mathcal{S}}$ a term and $\tau : \mathbb{N} \rightarrow Terms_{\mathcal{S}}$ a variable substitution. We define $t[\tau]$, the application of τ to t, recursively on the structure of t as follows:

$t = n$: then $n[\tau] := \tau(n)$;

$t = f(t_1, ..., t_n)$: then $f(t_1, ..., t_n)[\tau] := f(t_1[\tau], ..., t_n[\tau])$.

Before stating and proving some lemmas about variable substitution on terms, we list some commonly used operations and variable substitutions.

Definition 2.2.8. We define the following.

- id is the identity function such that $id(n) = n$;
- For a type X, x an element of type X and a function $f : nat \to X$, $x .: f$ is the function such that $(x .: f)(0) = x$ and $(x .: f)(S\ n) = f(n)$;
- $\uparrow$ is the function such that $\uparrow(n) = S\ n$;
- $g \circ f$ is composition of g and f, i.e. the function such that forall n we have $(g \circ f)(n) = g(f(n))$;
- $\lambda t.t[\uparrow]$ is the function which takes a term t as an input and outputs the term $t[\uparrow]$;
- $\mathsf{up}(\tau) = 0 .: (\lambda t.t[\uparrow] \circ\ \tau)$.

The first lemma shows how the substitution of the first variable for a term $[s .: id]$ cancels out with a prior use of the substitution $[\uparrow]$.

Lemma 2.2.1. For all first-order language $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ and terms $t, s \in Term_{\mathcal{S}}$, we have: $t[\uparrow][s .: id] = t$.

Proof. (⌨) Informally, we can simply notice that the application of the substitution $[\uparrow]$ transforms n into $S\ n$, making the substitution of the first variable by t ineffective. In addition to that, the substitution $[s .: id]$ has as effect of decreasing all but the first variable by 1, thus transforming $S\ n$ into n. Thus, the term t is unchanged. For a formal proof we refer to the formalisation. ■

The second lemma explains how two specific substitutions can be inverted. This comes in handy in many situations.

Lemma 2.2.2. For all first-order language $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$, $t \in Term_{\mathcal{S}}$ and variable substitution τ, we have:

$$t[\uparrow][\mathsf{up}(\tau)] = t[\tau][\uparrow]$$

Proof. We refer to the formalisation (⌨). ■

Naturally, the definition of variable substitution on terms ports to formulas. In the next definition we precisely define the notion of variable substitution on formulas.

Definition 2.2.9. Let $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ be a first-order language, $\varphi \in Form_{\mathbb{L}}$ a formula and τ a variable substitution. We define $\varphi[\tau]$, the application of τ to φ, recursively on the structure of φ as follows:

- $\varphi := P(t_1, \dots, t_n)$: then $P(t_1, \dots, t_n)[\tau] = P(t_1[\tau], \dots, t_n[\tau])$;
- $\varphi := \bullet(\psi_1, \dots, \psi_n)$: then $(\bullet(\psi_1, \dots, \psi_n))[\tau] = \bullet(\psi_1[\tau], \dots, \psi_n[\tau])$;
- $\varphi := \triangledown\psi$: then $(\triangledown\psi)[\tau] = \triangledown(\psi[\mathsf{up}(\tau)])$

We abuse the notation and write $\varphi[t]$ for $\varphi[t .: id]$.

The lemmas we considered for variable substitutions on terms also hold for formulas.

Lemma 2.2.3. For all first-order language $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$, $t \in Term_{\mathcal{S}}$ and $\varphi \in \mathbb{L}$, we have:

$$\varphi[\uparrow][s .: id] = \varphi$$

Proof. (⌨) The intuition is similar to the one of Lemma 2.2.1. ■

Lemma 2.2.4. For all first-order anguage $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$, $\varphi \in \mathbb{L}$ and variable substitution τ, we have:

$$\varphi[\uparrow][\mathsf{up}(\tau)] = \varphi[\tau][\uparrow]$$

Proof. We refer to the formalisation (⌨). ■

In this dissertation, we use an unusual notion of free variables in a formula. More precisely, we define the property of variables of being *unused* in a formula.

Definition 2.2.10. Let $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ be a first-order language. We define the property of a variable $n \in \mathbb{N}$ of being *unused in a term* $t \in Terms_{\mathcal{S}}$ recursively on the structure of t as follows:

$t = m$: then n is unused in m if $n \neq m$;

$t = f(t_1, ..., t_k)$: then n is unused in $f(t_1, ..., t_k)$ if n is unused in t_i, for all $1 \leq i \leq k$.

Then, we define the property of n of being *unused in a formula* $\varphi \in Form_{\mathbb{L}}$ recursively on the structure of φ.

$\varphi = P(t_1, \ldots, t_m)$: then n is unused in $P(t_1, \ldots, t_n)$ if n is unused in t_i for all $1 \leq i \leq m$;

$\varphi = \bullet(\psi_1, \ldots, \psi_m)$: then n is unused in $\bullet(\psi_1, \ldots, \psi_m)$ if n is unused in ψ_i for all $1 \leq i \leq m$;

$\varphi = \triangledown\psi$: then n is unused in $\triangledown\psi$ if $S\ n$ is unused in ψ, where S is the successor function.

In essence, if a variable n is unused in a formula φ then this variable has no "occurrence" in φ which is not bound by a quantifier. What has to be noted here is that we use the term "occurrence" in a loose way. In a syntax with names, the occurrences of a variable x are detected by the symbol x itself. Here, what we mean by "occurrences" when considering the statement "n is unused in φ" is all the variable occurrences that would be modified by a substitution of n on φ. For example, when considering whether 0 is unused in $P(1) \to \forall Q(0, 1)$, we need to check in $P(1)$ whether 0 is unused and in $Q(0, 1)$ whether 1 is unused, given that once we go through a quantifier a variable m becomes $S\ m$. This fact is dealt with in the last clause of Definition 2.2.10. So, in our example $P(1) \to \forall Q(0, 1)$ we get that 0 is not unused, as 1 is not unused in $Q(0, 1)$ hence 0 is not unused in $\forall Q(0, 1)$.

With this reading in mind, we can see that n being unused in φ implies the intuitive idea that n is not a free variable of φ. Thus, we write $n \notin FV(\varphi)$ if n is unused in φ. Also, we write $FV(\varphi) = \emptyset$ if for all $n \in \mathbb{N}$ we have $n \notin FV(\varphi)$. Finally, we port these definitions to sets of formulas in the obvious way.

As in the propositional case, we need to obtain a definition of substitution of basic constituents of the language. We proceed to define the notion of atom substitution.

Definition 2.2.11. Let $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ be a first-order language. An *atom substitution* for $\mathbb{L}$ is a function $\sigma : \forall P \in Pred, (t_0, \ldots, t_{Ar_{Pred}(P)}) \to Form_{\mathbb{L}}$.

Technically, the above definition describes a function taking a predicate P and a tuple of terms $(t_0, \ldots, t_{Ar_{Pred}(P)})$ to output a formula φ. So, in essence this functions takes an atomic formula $P(t_0, \ldots, t_{Ar_{Pred}(P)})$ and outputs a formula φ. We proceed to define the application of such functions on complex formulas.

Definition 2.2.12. Let $\mathbb{L}$ be a first-order language and σ an atom substitution for $\mathbb{L}$. We define φ^{σ} the *atom substitution of* φ *through* σ by recursion on the structure of φ:

$\varphi = P(t_1, \ldots, t_m)$: then $P(t_1, \ldots, t_n)^{\sigma} := \sigma(P, (t_1, \ldots, t_n))$;

$\varphi = \psi_0 \bullet \psi_1$: then $(\psi_0 \bullet \psi_1)^{\sigma} := \psi_0^{\sigma} \bullet \psi_1^{\sigma}$;

$\varphi = \triangledown\psi$: then $(\triangledown\psi)^{\sigma} := \triangledown(\psi^{\sigma})$.

While general, this definition is not satisfying. Indeed, it allows to substitute an atomic formula $P(t_1, \ldots, t_n)$ by whatever formula ψ in a formula φ, regardless of whether this substitution will create or delete variables captured by quantifiers inside of φ. As a concrete example, for an atom substitution σ such that $\sigma(P, 0) = \top$ and $\sigma(P, 1) = \bot$, we have that $(\forall P(0) \to P(1))^\sigma = \forall \top \to \bot$. The initial formula expresses the instantiation of universal quantifiers, which we want to be a principle holding in the logics considered here, while the substituted one is equivalent to $\bot$ in our setting. So, the notion of atom substitution of φ through σ defined above does not preserve important properties of the logic, such as "being a holding principle".

To avoid such issues, we restrict atom substitutions in the following way.

Definition 2.2.13. Let $\mathbb{L}$ be a first-order language and σ an atom substitution for $\mathbb{L}$. We say that σ is a *restricted* atom substitution if it satisfies the following property for all atomic formulas $P(t_1, \ldots, t_n)$ and variable substitutions τ.

$$(\sigma(P, (t_1, \ldots, t_n)))[\tau] \;=\; \sigma(P, (t_1[\tau], \ldots, t_n[\tau]))$$

Intuitively, this restriction on the interplay between atom substitutions and variable substitutions does the following: it makes sure that (1) the structure of formulas outputted by atom substitutions are only parametric in predicate symbols, and that (2) no new free variable is introduced.

If an atom substitution σ is such that the structure of formulas it outputs is parametric not only in predicate symbols but also in the terms alongside these symbols, then we could obtain two radically different formulas by applying σ to the same predicate symbol with different terms. For example, we can consider the atom substitution parametric in terms we exhibited above, specified by $\sigma(P, 0) = \top$ and $\sigma(P, 1) = \bot$. It is clear that σ fails the property above: we have that $(\sigma(P, 0))[\uparrow] = \top[\uparrow] = \top$, while $\sigma(P, 0[\uparrow]) = \sigma(P, 1) = \bot$, making the equality fail. Thus, this type of atom substitution is avoided by the restriction above, hence (1) is satisfied.

Similarly, we can show that (2) is enforced by this restriction. Consider σ an atom substitution such that $\sigma(P, 0) = Q(0, 1)$ and $\sigma(P, 1) = Q(1, 1)$. In this case, σ introduces a new free variable: 1. Now, we see that the equality above fails again: we have that $(\sigma(P, 0))[\uparrow] = Q(0, 1)[\uparrow] = Q(1, 2)$, while $\sigma(P, 0[\uparrow]) = \sigma(P, 1) = Q(1, 1)$. Consequently, we know that such atom substitutions are prohibited.

We consider the properties (1) and (2) as sufficient for a satisfying notion of atom substitution. Consequently, in the rest of this dissertation we only use restricted atom substitutions. A nice property of these restricted atom substitutions is that they commute with variable substitutions.

Lemma 2.2.5. For all languages $\mathbb{L}$, $\varphi \in \mathbb{L}$, variable substitution τ and restricted atom substitution σ, we have:

$$\varphi^\sigma[\tau] = (\varphi[\tau])^\sigma$$

Proof. (⌨) We reason by induction on the structure of φ:

- $\varphi = P(t_1, \ldots, t_n)$: on one hand we have that $P(t_1, \ldots, t_n)^\sigma[\tau] = (\sigma(P(t_1, \ldots, t_n)))[\tau]$, and on the other hand we have that $P(t_1, \ldots, t_n)[\tau]^\sigma = P(t_1[\tau], \ldots, t_n[\tau])^\sigma = \sigma(P(t_1[\tau], \ldots, t_n[\tau]))$. However, the fact that σ is a restricted atom substitution gives us that $(\sigma(P(t_1, \ldots, t_n)))[\tau] = \sigma(P(t_1[\tau], \ldots, t_n[\tau]))$.

- $\varphi = \bullet(\psi_1, \ldots, \psi_n)$: on one hand we have the following.

$$\bullet(\psi_1, \ldots, \psi_n)^\sigma[\tau] = (\bullet(\psi_1^\sigma, \ldots, \psi_n^\sigma))[\tau] = \bullet(\psi_1^\sigma[\tau], \ldots, \psi_n^\sigma[\tau])$$

  On the other hand, we have the following.

$$\bullet(\psi_1, \ldots, \psi_n)[\tau]^\sigma = \bullet(\psi_1[\tau], \ldots, \psi_n[\tau])^\sigma = \bullet(\psi_1[\tau]^\sigma, \ldots, \psi_n[\tau]^\sigma)$$

We can then link the two sequences of equalities using the induction hypothesis, giving us:

$$\bullet(\psi_1^\sigma[\tau], \ldots, \psi_n^\sigma[\tau]) = \bullet(\psi_1[\tau]^\sigma, \ldots, \psi_n[\tau]^\sigma)$$

- $\varphi = \nabla\psi$: on one hand we have the following.

$$\nabla\psi^\sigma[\tau] = (\nabla\psi^\sigma)[\tau] = \nabla(\psi^\sigma)[\mathsf{up}(\tau)]$$

On the other hand, we have the following.

$$(\nabla\psi)[\tau]^\sigma = \nabla\psi[\mathsf{up}(\tau)]^\sigma = \nabla(\psi[\mathsf{up}(\tau)])^\sigma$$

We can then link the two sequences of equalities using the induction hypothesis, giving us:

$$\nabla(\psi^\sigma)[\mathsf{up}(\tau)] = \nabla(\psi[\mathsf{up}(\tau)])^\sigma$$

■

Now that we presented all the ingredients to obtain a first-order language in general, we turn to the specific instance of first-order classical modal logic.

2.2.1 The classical modal logic example: on paper

For the classical modal logic, we consider the same connectives: $\mathcal{C}_{\mathbf{CM}}$. In addition to that, we consider the set of quantifiers $\mathcal{Q} = \{\forall, \exists\}$. As this is the only set of quantifiers we consider in this dissertation we fix the notation $\mathcal{Q}$ to refer to this set.

Given a signature $\mathcal{S}$, we can define the first-order language $\mathbb{L}_{\mathbf{CM}} = (\mathcal{S}, \mathcal{C}_{\mathbf{CM}}, \mathcal{Q})$ and obtain its set of modal formulas $Form_{\mathbb{L}_{\mathbf{CM}}}$ through its infix grammar:

$$\varphi ::= P(t_1, \ldots, t_n) \mid \bot \mid \varphi \to \varphi \mid \Box\varphi \mid \forall\varphi \mid \exists\varphi$$

As an example, the expression $\bot \to \Box((\forall P(0)) \to Q((f2), 3))$ is an element of $Form_{\mathbb{L}_{\mathbf{CM}}}$, provided that $f \in Fun$ is such that $Ar_{Fun}(f) = 1$, and $P, Q \in Pred$ are such that $Ar_{Pred}(P) = 1$ and $Ar_{Pred}(Q) = 2$. In this formula, the instance of 2 and 3 are both free variable instances. A representation of this formula in the usual syntax is as follows.

$$\bot \to \Box((\forall x P(x)) \to Q((fy), z))$$

2.2.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

In this subsection, we present our instantiation of the syntax for first-order logic presented in the Coq library called “undecidability” [48], following the work of Kathryn Stark[141]. This library provides general tools to obtain a representation à la de Bruijn of first-order syntaxes, as witnessed in the various works where it has been put to use [46, 48, 75, 79, 71, 82, 80].

The first element we defined to obtain a first-order language is a signature. We consequently follow Definition 2.2.1 and define sets of function and predicate symbols with their arity (⌨).

```
Class funcs_signature :=
        { syms : Type; ar_syms : syms -> nat }.

Class preds_signature :=
        { preds : Type; ar_preds : preds -> nat }.
```

A `funcs_signature` is an entity having a collection of `syms` (i.e. symbols) and an arity function `ar_syms`. In a similar way, we can see that `preds_signature` is a collection of predicate symbols with an arity function on them.

For the rest of this subsection we assume given a signature consisting of two elements: a `funcs_signature` called `Σ_funcs` and a `preds_signature` called `Σ_preds`. In Coq, we declare these as `Context`, i.e. variables on which depend the definitions which follow (⌨).

```
Context {Σ_funcs : funcs_signature}.
Context {Σ_preds : preds_signature}.
```

Then, we can turn to the formalisation of terms, defined in Definition 2.2.2 (⌨).

```
Inductive term : Type :=
  | var : nat -> term
  | func : forall (f : syms), vec term (ar_syms f) -> term.
```

To distinguish natural numbers in general and natural numbers we use in our syntax, we use the constructor `var`. Consequently, we know that `1` is of type `nat`, while `var 1` is of type `term`. The constructor `func` takes a function symbol `f` and a *vector* of arity of `f`, i.e. `vec term (ar_syms f)`, and outputs a term. In essence, vectors in Coq are determined by two things: a type of elements constituting the vector, here `term`, and a length, here `ar_syms f`. This is why the constructor `vec` needs the two arguments `term` and `ar_syms f`, thus informing us that we are dealing with a vector of terms of length equal to the arity of `f`. Consequently, we can see that in the formalisation we are presenting strings of terms like $(t_1, \ldots, t_n)$ are in fact vectors.

Now that we formalised terms in a very general way, we can define formulas in a not less general way. We obtain this generality by defining what a collection of logical operators and quantifiers is (⌨).

```
Class operators :=
        {unop : Type ; binop : Type ; quantop : Type}.
```

A collection of `operators` is thus constituted by a collection `unop` of *un*ary *op*erators, a collection `binop` of *bin*ary *op*erators, and a collection `quantop` of *quant*ifiers. We assume given such a collection of operators to give a general definition of formulas (⌨).

```
Context {ops : operators}.
```

Relying on this collection of operators, we can define formulas (⌨).

```
Inductive kform : Type :=
  | Bot : kform
  | atom : forall (P : preds), vec term (ar_preds P) -> kform
  | un : unop -> kform -> kform
  | bin : binop -> kform -> kform -> kform
  | quant : quantop -> kform -> kform.
```

Obviously, the constructor `Bot` is corresponding to nothing but $\bot$. The constructor `atom` needs some explaining. In essence, it requires a predicate symbol `P` and a vector of terms of length equal to the arity of `P`, i.e. `vec term (ar_preds P)`, to output a formula which we call an `atom`. Finally, the constructors `un`, `bin` and `quant` can be explained in a similar way, so we only focus on `un`. The latter requires an element of type `unop`, i.e. unary operator, as well as an element of type `kform`, i.e. a formula, to provide a formula. Note that there is no need for variables in the constructor `quant` as we are in a syntax à la de Bruijn.

As we can see, the definition above is very general given that it implicitly relies on a variable `ops`, a collection of operators. Now, if we want to focus on a specific collection of operators we can instantiate this variable and get automatically a definition of formulas for this collection. We proceed to do so for the collection of operators of $\mathbb{L}_{\mathbf{CM}}$ (⌨).

```
Inductive full_logic_un_sym : Type :=
  | Box : full_logic_un_sym.

Inductive full_logic_bin_sym : Type :=
  | Imp : full_logic_bin_sym.

Inductive full_logic_quant : Type :=
  | All : full_logic_quant
  | Ex : full_logic_quant.

Definition full_operators : operators :=
  {| unop := full_logic_un_sym ; binop := full_logic_bin_sym
   ; quantop := full_logic_quant |}.
```

The above first defines a collection `full_logic_un_sym` of unary operators constituted by `Box` only. Second, we get the collection `full_logic_bin_sym` of binary operators with `Imp` as sole element. Third, we define the collection `full_logic_quant` of quantifiers with the universal `All` and existential `Ex` quantifiers. With these three collections in hand we can finally define our collection of operators and quantifiers `full_operators`, which has `full_logic_un_sym` as unary operators, `full_logic_bin_sym` as binary operators and `full_logic_quant_sym` as quantifiers.

This collection of operators can thus be used to define a specific language and its formulas. Next, we introduce some notations for formulas in our language (⌨).

```
Notation "∀ φ" := (@quant _ _ full_operators All φ)
        (at level 50) : syn.
Notation "∃ φ" := (@quant _ _ full_operators Ex φ)
        (at level 50) : syn.
Notation "φ '-->' ψ" := (@bin _ _ full_operators Impl φ ψ)
        (at level 43, right associativity) : syn.
```

In the remaining of this subsection, we treat the notion of variable substitution in a first-order language. As we mentioned above, this notion is quite tricky to deal with in a syntax à la de Bruijn. But this is the beauty of the library we are using: the tedious work is already done. So, by simply giving a collection of operators as we did above, we get for free not only a language and its formulas but also a notion of variable substitution. We give an account of this notion below.

The easy part of the notion of variable substitution in a syntax 'a la de Bruijn is its application on terms. Indeed, it is simply applied without modifications (⌨).

```
Fixpoint subst_term (τ : nat -> term) (t : term) : term :=
  match t with
    | var n => τ n
    | func f v => func f (map (subst_term τ) v)
  end.
```

Quite obviously, Definition 2.2.7 is formalised by the above. The application of the variable substitution τ on the term `t` is defined recursively on the structure of the latter. If it is a variable `var n`, then we simply apply τ to `n`, giving us a term. If `t` is a term of the form `func f v`, then we simply push the substitution to the terms in the vector `v`: this is what `map (subst_term τ) v` does, as `map` allows to apply the function `subst_term τ` to all the terms in the vector `v`.

To obtain a notion of variable substitution for formulas, we need to define some operations on functions. First, we have the usual operation of composition on functions, as formalised below (⌨).

```
Definition funcomp {X Y Z} (g : Y -> Z) (f : X -> Y)  :=
  fun x => g (f x).
```

Given two functions `g : Y -> Z` and `f : X -> Y`, we can compose them in a new function which consists in taking an `x` and apply `f` to `x` and then `g` to `f x`. This function is expressed by `fun x => g (f x)`.

Second, we need a function which "pushes" all the outputs of a function on natural numbers by one on the inputs, and adds a new value to 0 (⌨).

```
Definition scons {X: Type} (x : X) (f : nat -> X) :=
  fun n => match n with
                  | 0 => x
                  | S n => f n
                end.
```

The function `scons` thus takes an element `x` of type `X` and a function `f : nat -> X`, and creates a new function from `nat` to `X`. This new function outputs `x` on input `0`, and `f n` on the input `S n`. In essence, if we order the outputs of the function `f`, i.e. `f 0`= x_0, `f 1`= x_1, `f 2`= x_2, using the inputs, then we get an infinite string $x_0, x_1, x_2, \ldots$. What `scons x f` does is adding `x` at the beginning of this list, and consequently pushing all the elements of the infinite list by one place, thus giving $x, x_0, x_1, x_2, \ldots$.

We can put together the two above operations on functions to obtain a function that is crucial to the definition of variable substitution (⌨).

```
Definition up (τ : nat -> term) : nat -> term :=
    scons (var 0) (funcomp (subst_term (funcomp var S)) τ).
```

So, we give a variable substitution τ to the function `up`. Now, let us decompose this function from `nat` to `term`. Its main operation is `scons`, so we must expect a `term` followed by a function of type `nat -> term`. The term which is given is `var 0`, i.e. the 0, so the outputs of the function which follows will all be pushed by one on input and `var 0` will be matched to `0`. The function is `funcomp (subst_term (funcomp var S)) τ`. This function is a composition of two functions: `subst_term (funcomp var S)` and τ. The latter, which will be the function first applied in the composition, is nothing but the variable substitution which was initially given. The former is the application of the variable substitution `funcomp var S`, which first changes n to $S\ n$ and then makes it a variable, to terms (`subst_term`).

In essence, the function `up` takes a variable substitution τ, fixes `var 0` to `0` and shifts by one the natural numbers occurring in the outputs of τ on the other inputs. For example, let us say that `τ 0 = f (g (var 4) (var 7))` and `τ 1 = g (f (var 2)) (var 0)`. Then, `(up τ) 0 = var 0`, as `scons` fixes the output `var 0` to the input `0`. When we consider the input `1` we need to consider the function `funcomp (subst_term (funcomp var S)) τ` applied to `0` because of `scons`. So, `(up τ) 1` is equal to `(funcomp (subst_term ( funcomp var S)) τ) 0`. To determine the value of the latter, we first need to apply τ to `0` because of the function composition `funcomp`, giving us `f (g (var 4) (var 7))` as assumed. Now, we need to apply the function `subst_term (funcomp var S)` to `f (g ( var 4) (var 7))`. Using the definition of `subst_term` above, we get that this application gives us: `f (g ((funcomp var S) 4) ((funcomp var S) 7))`. Thus, we are left with apply `(funcomp var S)` to `4` and `7`. Let us proceed: `(funcomp var S) 4` forces us to first apply `S` to `4`, giving us `5`, and then apply `var`, giving us `var 5`. Similarly, we get `(funcomp var S) 7 = var 8`. So, we finally get that `(up τ) 1 = f (g (var 5) (var 8))`. Note that this is nothing but `τ 0 = f (g (var 4) (var 7))` with all the natural numbers "upped" by one.

This complex function `up` allows us to get a notion of substitution which avoids both the modification of bound variables and the bounding of new variables. We can thus finally formalise the application of a variable substitution to a formula (⌨), as defined in Definition 2.2.9.

```
Fixpoint subst_kform (τ : nat -> term) (φ : kform) : kform :=
 match φ with
```

```
   | Bot => Bot
   | atom P v => atom P (map (subst_term τ) v)
   | un op ψ => un op (subst_kform τ φ)
   | bin op ψ χ => bin op (subst_kform τ ψ) (subst_kform τ χ)
   | quant op ψ => quant op (subst_kform (up τ) ψ)
 end.
```

The only interesting case is the case of quantifiers, i.e. the line where we consider that φ is of the form `quant op` ψ. There, we define the application of τ to `quant op` ψ as `quant op (subst_kform (up` τ`)` ψ`)`. This essentially relies on the application of the *modified* variable substitution `(up` τ`)` to ψ. Then the question is: how does `(up` τ`)` modify ψ? Before encountering any quantifiers in ψ, it essentially does two things. First, it makes sure that any occurrence of `var 0` is not modified. This is to make sure that the variable bounded by quantifiers are not modified. Second, it allows to substitute any variable `var (S n)` by the application of the *upped* version of τ. The effect of this is to avoid the introduction of new bound variables. For example, assume that the application of τ to a variable in ψ would have introduced a `var 0` where there was none, thus creating a new bound variable. The modification of τ to `(up` τ`)` ensures that this `var 0` will be modified to `var 1`, thus avoiding the bounding. All of these comments hold for the first layer between the quantifier mentioned here and the next quantifier in ψ, but the recursive definition of `subst_kform` makes sure that this property of avoiding both the modification of bound variables and the bounding of new variables is preserved.

Chapter 3

Logics

Now that we have languages, defining formulas, we want to define logics in a general way.

Here, we take a set-theoretic view of logic: they are specific sets of elements built using languages.

In the literature, logics are commonly taken to be *sets of formulas* in a language $\mathbb{L}$ and not *consequence relations*, i.e. relations of the type $(\mathsf{Pow}(Form_{\mathbb{L}}) \times (Form_{\mathbb{L}}))$ satisfying some properties we provide below. The former takes the notion of *theorem* as central, while the latter is founded on the notion of *deduction*.

It seems that the notion of consequence relation is more adequate to capture logics for two reasons. First, intuitively deductions are the primary object we study in logic, which establishes the correctness of arguments, but not theorems, which are isolated statements. Second, it is more general than the notion of set of formulas, as sets of formulas are consequence relations with the left component of their pair being the empty set $\emptyset$.

However, as mentioned above, the approach to logics as sets of formulas is more common in the literature. While this decision can be motivated in many ways, we believe that it relies implicitly on the holding of properties like compactness and the deduction-detachment theorem. By compactness, we mean the following.

Definition 3.0.1. Let $\mathbb{L}$ be a language (propositional or first-order) and $R \subseteq (\mathsf{Pow}(\mathbb{L}) \times \mathbb{L})$ be a relation. We say that R is *compact* if $(\Gamma, \varphi) \in R$ implies the existence of a finite $\Gamma_f \subseteq \Gamma$ such that $(\Gamma_f, \varphi) \in R$.

We say that pairs of the shape (Γ, φ) are *consecutions*. So, if a consequence relation CR is compact, we can reduce any consecution $(\Gamma, \varphi) \in CR$ with an infinite set Γ to a consecution $(\Gamma', \varphi) \in CR$ with a finite set Γ'.

Now, in addition to that suppose a CR satisfies a type of deduction-detachment theorem for some logical connective $\rightharpoonup$ in $\mathbb{L}$, as shown below. This type of theorem expresses the fact that the connective $\rightharpoonup$ is a proxy in $\mathbb{L}$ for the relation captured by CR.

$$(\Gamma \cup \{\varphi\}, \psi) \in CR \quad \text{iff} \quad (\Gamma, \varphi \rightharpoonup \psi) \in CR$$

Then, what we obtain is the following process. Take any pair $(\Gamma, \varphi) \in CR$. First, make its left component finite using compactness and obtain $(\Gamma', \varphi) \in CR$. Second, use repetitively the deduction-detachment theorem to empty the left-hand side of the pair and obtain: $(\emptyset\ , \gamma'_0 \rightharpoonup \gamma'_1 \rightharpoonup \dots \gamma'_n \rightharpoonup \varphi) \in CR$. Consequently, with these two properties, we can reduce any pair (Γ, φ) to a pair $(\emptyset, \psi)$, i.e. a theorem: logics as sets of theorems or logics as consequence relations are in this case indistinguishable.

However, in this dissertation, we consider logics for which such properties are not granted. Thus, we adopt the more general approach and define logics to be consequence relations [45, 84, 164], i.e. relations satisfying the following properties.

Definition 3.0.2. Let $\mathbb{L}$ be a language. A *logic* in $\mathbb{L}$ is a set $\mathbf{L} \subseteq \{(\Gamma, \varphi) \mid \Gamma \cup \{\varphi\} \subseteq Form_{\mathbb{L}}\}$ satisfying the following properties:

Identity: if $\varphi \in \Gamma$, then $(\Gamma, \varphi) \in \mathbf{L}$;

Monotonicity: if $(\Gamma, \varphi) \in \mathbf{L}$ and $\Gamma \subseteq \Gamma'$, then $(\Gamma', \varphi) \in \mathbf{L}$;

Compositionality: if $(\Gamma_1, \varphi) \in \mathbf{L}$ and $(\Gamma_2, \gamma) \in \mathbf{L}$ for all $\gamma \in \Gamma_1$, then $(\Gamma_2, \varphi) \in \mathbf{L}$;

Structurality: if $(\Gamma, \varphi) \in \mathbf{L}$, then $(\Gamma^\sigma, \varphi^\sigma) \in \mathbf{L}$ for propositional variable substitution (resp. atom substitution) σ if $\mathbb{L}$ is a propositional (resp. first-order) language.

In this dissertation, we refer to a logic as defined above by using the suffix **L**.

Chapter 4

Proof Systems

To capture logics, one can use the proof-theoretic path. There, we define proof systems, which are essentially constituted of rules manipulating judgements, and help define which judgements hold in this system through the notion of proof.

While there are numerous types of proof systems (axiomatic [72], natural deduction [112], proof nets [55], sequent [53], tableaux [57], display [78], etc.), with various structures being used to define proofs (graphs, lists, trees, upside-down trees, non-well-founded structures, etc.), in this dissertation we focus on proof systems using trees as crucial structures. Thus, we define some general notions for these proof systems. To attain this generality, we assume given a set of judgements $\mathcal{J}$, as well as notions of judgement schema and instances of the latter. As type of judgements, we have in mind entities of the shape φ, or $\Gamma \vdash \varphi$, or $\Gamma \Rightarrow \Delta$, or $\Gamma_0 \Rightarrow \Delta_0 \mid \Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_0 \Rightarrow \Delta_0$, or $\Gamma_0 \Rightarrow \Delta_0 / \Gamma_1 \Rightarrow \Delta_1$, with their usual notions of schema and instances of a schema.

Definition 4.0.1. Let $n \in \mathbb{N}$. A *n-ary rule* R is a pair $(\mathit{Prems}, \mathit{Concl})$, where $\mathit{Prems} = [\mathit{Prem}_1, \dots, \mathit{Prem}_n]$ is a list of judgements schemas and Concl is a judgement schema. A rule R is usually presented as follows:

$$\frac{\mathit{Prem}_1 \quad \dots \quad \mathit{Prem}_n}{\mathit{Concl}}\ (\mathrm{R})$$

Note that there can be conditions on the way the judgement schemas of R can be instantiated. A *rule instance* is obtained by uniformly instantiating every variable in the rule, i.e. in the judgement schemas contained in the rule, with a concrete object of the corresponding type. This is the standard definition from structural proof theory. The set of instantiations of a rule R is noted R^{i}.

A proof system P on $\mathcal{J}$ is a finite set $\{\mathsf{R}_0, \dots, \mathsf{R}_n\}$ of rules.

Definition 4.0.2. Let $\mathcal{J}' \subseteq \mathcal{J}$ be a set of judgements. A derivation $\mathfrak{d}$ of j from $\mathcal{J}'$ in P is a tree of judgements with root j, defined inductively as follows:

(Leaf): if $j \in \mathcal{J}'$, then j is a derivation from $\mathcal{J}'$ in P.

(Rule): if $\mathfrak{d}_1, \mathfrak{d}_2, \dots, \mathfrak{d}_k$ are derivations of respectively $j_1, j_2, \dots, j_k$ from $\mathcal{J}'$ in P, and $(\{j_1, j_2, \dots, j_k\}, j) \in \mathsf{R}^{\mathsf{i}}$ for some $\mathsf{R} \in \mathsf{P}$, then the following is a derivation from $\mathcal{J}'$ in P:

$$\frac{\mathfrak{d}_1 \quad \dots \quad \mathfrak{d}_k}{j}\ (\mathrm{R})$$

If there is a derivation $\mathfrak{d}$ of j from $\mathcal{J}'$ in P, we say that j is *derivable from* $\mathcal{J}'$ *in* P. Furthermore, we say that j is *derivable in* P if j is derivable from $\mathcal{J}$ in P. If $\mathfrak{d}$ is a derivation of j from $\mathcal{J}'$ in P, we say that j is the *root* of $\mathfrak{d}$.

The definition of derivation we use is non-standard as it parametrizes derivations by a set of judgements. This set of judgements describes which judgements are allowed to be leaves of the derivations. As a consequence, the usual notion of derivation, omitting altogether the set of allowed leaves, is captured in our setting by derivations from $\mathcal{J}$. Our definition can thus be conceived as more general and fine-grained.

With this in mind, the notion of proof is easily defined: it is a derivation in which no judgement is allowed as a leaf, i.e. the set of judgements allowed to be leaves is empty.

Definition 4.0.3. A *proof* $\mathfrak{p}$ of j in P is a derivation of j from $\emptyset$ in P. If there is a proof of j in P, we say that j is *provable in* P.

In what follows, it should be clear from context whether the word "proof" refers to the object defined in Definition 4.0.3, or to the meta-level notion.

When clear from context, we do not mention the proof system considered and simply talk about a "proof of j" and a "derivation of j from $\mathcal{J}'$".

In many places, we prove lemmas ensuring the existence of a derivation or proof from the existence of other derivations or proofs. For example, we could obtain a lemma stating that if a specific judgement j_0 is derivable from $\{j_2, j_3\}$ in P, then another specific judgement j_1 is derivable from $\{j_2, j_3\}$ in P too. As we often make use of such lemmas while proving other lemmas or theorems, we allow ourselves to insert these lemmas in derivations as rules. To distinguish the use of lemmas, which are meta-level operations on proofs and derivations, from the use of object-level rules, we use dashed lines for the former and continuous lines for the latter. As an example, consider the tree below, where $([j_2, j_3], j_0) \in \mathsf{R}^{\mathsf{i}}$.

$$\dfrac{\dfrac{j_2 \qquad j_3}{j_0}\,(\mathrm{R})}{j_1}\,\mathit{Lem.}$$

As such a construction is not strictly speaking a derivation, we call it a *semi-derivation*. Similarly, we talk about semi-proof when building trees establishing the provability of a sequent in which such dashed lines are present.

Next, we define the height of a derivation.

Definition 4.0.4. Let $\mathfrak{d}$ be a derivation of j from $\mathcal{J}'$. The *height* $h(\mathfrak{d})$ of $\mathfrak{d}$ is defined recursively on the structure of $\mathfrak{d}$.

- if $\mathfrak{d}$ is of the form j, then $h(\mathfrak{d}) = 1$;
- if $\mathfrak{d}$ is of the form
$$\dfrac{\mathfrak{d}_1 \quad \ldots \quad \mathfrak{d}_k}{j}\,(\mathrm{R})$$
then $h(\mathfrak{d}) = 1 + max(h(\mathfrak{d}_1), \ldots, h(\mathfrak{d}_n))$, where $max(h(\mathfrak{d}_1), \ldots, h(\mathfrak{d}_n))$ is the maximum of $h(\mathfrak{d}_1), \ldots, h(\mathfrak{d}_n)$.

Finally, we consider general properties of rules with respect to proof systems.

Definition 4.0.5. Let P be a proof system and R a rule. We say that R is:

- *admissible in* P if, for any $([j_1, \ldots, j_n], j) \in \mathsf{R}^{\mathsf{i}}$, j is provable in P whenever $j_1, \ldots, j_n$ are all provable in P;
- *invertible in* P if, for any $([j_1, \ldots, j_n], j) \in \mathsf{R}^{\mathsf{i}}$, $j_1, \ldots, j_n$ are all provable in P whenever j is provable in P;
- *left-invertible (resp. right-invertible) in* P if R is of the form
$$\dfrac{\mathit{Prem}_1 \quad \mathit{Prem}_2}{\mathit{Concl}}\,(\mathrm{R})$$
and for any $([j_1, j_2], j) \in \mathsf{R}^{\mathsf{i}}$, j_1 (resp. j_2) is provable in P whenever j is provable in P;

- *height-preserving admissible in* P if, for any $([j_1, \ldots, j_n], j) \in \mathsf{R}^{\mathsf{i}}$, if there are proofs $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $j_1, \ldots, j_n$ in P such that $h(\mathfrak{p}_i) = k$, then there is a proof $\mathfrak{p}$ of j in P such that $h(\mathfrak{p}) = k$;

- *height-preserving invertible in* P if, for any $([j_1, \ldots, j_n], j) \in \mathsf{R}^{\mathsf{i}}$, if there is a proof $\mathfrak{p}$ of j in P such that $h(\mathfrak{p}) = k$, then there are proofs $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $j_1, \ldots, j_n$ in P such that $h(\mathfrak{p}_i) = k$ for all $1 \leq i \leq n$;

- *height-preserving left-invertible (resp. right-invertible) in* P if R is of the form

$$\frac{Prem_1 \quad Prem_2}{Concl}\ (\mathrm{R})$$

  and for any $([j_1, j_2], j) \in \mathsf{R}^{\mathsf{i}}$, if there is a proof $\mathfrak{p}$ of j in P such that $h(\mathfrak{p}) = k$, then there is a proof of $\mathfrak{p}_1$ (resp. $\mathfrak{p}_2$) of j_1 (resp. j_2) in P such that $h(\mathfrak{p}_1) = k$ (resp. $h(\mathfrak{p}_2) = k$).

With these general notions in hand, we can focus on the two type of proof systems in this dissertation: Generalized Hilbert calculi and sequent calculi.

4.1 Generalized Hilbert calculi

The traditional approach to axiomatic systems, consisting in creating proofs manipulating formulas starting from axioms, aims at capturing logics as *sets of formulas*. Here, to capture logics as *consequence relations*, as defined in Chapter 3, we define calculi describing relations between a set of formulas and a formula: generalized Hilbert calculi. Note that these calculi are not an invention of ours, as witnessed in the literature [154, Section 2.4], but are rarely used. Given that these are targeted at logics, the judgements manipulated through rules of generalized Hilbert calculi are of the form $\Gamma \vdash \varphi$, read "Γ deduces φ", where Γ is a set of formulas while φ is a formula.

Definition 4.1.1. Let $\mathbb{L}$ be a language, propositional or first-order. We define the following.

- An *axiom* is a formula schema of $\mathbb{L}$.

- If $\mathcal{A}$ is a set of axioms, we define $\mathcal{A}^{\mathsf{i}}$ to be the set of instances of axioms of $\mathcal{A}$.

- The rule (Ax) is as follows, where $\varphi \in \mathcal{A}^{\mathsf{i}}$: $\quad \dfrac{}{\Gamma \vdash \varphi}\ (\mathrm{Ax})$

- The rule (El) is as follows, where $\varphi \in \Gamma$: $\quad \dfrac{}{\Gamma \vdash \varphi}\ (\mathrm{El})$

- A generalized Hilbert calculus in $\mathbb{L}$ is a set of rules $\mathbf{H} = (\mathrm{Ax}, \mathrm{El}, \mathcal{R})$ for a given set of axiom $\mathcal{A}$ and some set of rules $\mathcal{R}$.

In what follows we designate a generalized Hilbert calculus using the suffix **H**.

As a proof system, a generalized Hilbert calculus **H** defines a set of rules $\mathcal{R}$ divided into two parts: rules shared by all generalized Hilbert calculi, i.e. (Ax) and (El) defined below, and the ones particular to the calculus, i.e. $\mathcal{R}_{\mathbf{H}}$. However, as it is *characterized* by a set of axioms and a specific set of rules $\mathcal{R}_{\mathbf{H}}$, we identify **H** with its corresponding pair $(\mathcal{A}, \mathcal{R}_{\mathbf{H}})$.

To let a generalized Hilbert calculus define a binary relation we must say which judgement of the form $\Gamma \vdash \varphi$ follows from this calculus. To do so, we use the notion of proof from Definition 4.0.3. If there is a proof $\mathfrak{p}$ in **H** with $\Gamma \vdash \varphi$ as root, we write $\Gamma \vdash_{\mathbf{H}} \varphi$.

4.1.1 The classical modal logic example: on paper

In this subsection, we define two generalized Hilbert calculi, $\mathsf{wK}\mathbf{H}$ and $\mathsf{sK}\mathbf{H}$, on the propositional language $\mathbb{L}_{\mathbf{CM}}$ and prove some key properties of these calculi. The lemmas and theorems presented here are of importance throughout this dissertation and constitute a convenient occasion to present standard proof techniques we regularly use.

First, we consider $\mathcal{A}_K$ a set of axioms for propositional classical modal logic based on axioms for classical logic due to Tarski, Bernays, and Wajsberg. $\mathcal{A}_K$ is shared by both $\mathsf{wK}\mathbf{H}$ and $\mathsf{sK}\mathbf{H}$.

Definition 4.1.2. We define the set of axioms $\mathcal{A}_K$ below:

$$\begin{array}{llll} MA_1 & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi)) & MA_4 & \bot \to \varphi \\ MA_2 & \varphi \to (\psi \to \varphi) & MA_5 & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\ MA_3 & ((\varphi \to \psi) \to \varphi) \to \varphi & & \end{array}$$

Second, we define the rules (MP), (wNec) and (sNec), respectively called *modus ponens*, *weak necessitation* and *strong necessitation.*

Definition 4.1.3. We define the following rules:

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \to \psi}{\Gamma \vdash \psi}\ (\mathrm{MP}) \qquad \frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box\varphi}\ (\mathrm{wNec}) \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \Box\varphi}\ (\mathrm{sNec})$$

We can finally define the generalized Hilbert calculi $\mathsf{wK}\mathbf{H}$ and $\mathsf{sK}\mathbf{H}$.

Definition 4.1.4. We define the generalized Hilbert calculi below, alongside their set of rules.

$$\begin{array}{rcl@{\qquad}rcl} \mathsf{wK}\mathbf{H} & = & (\mathcal{A}_K, \mathcal{R}_{\mathsf{w}}) & \mathcal{R}_{\mathsf{w}} & = & \{(\mathrm{MP}), (\mathrm{wNec})\} \\ \mathsf{sK}\mathbf{H} & = & (\mathcal{A}_K, \mathcal{R}_{\mathsf{s}}) & \mathcal{R}_{\mathsf{s}} & = & \{(\mathrm{MP}), (\mathrm{sNec})\} \end{array}$$

We abbreviate $\Gamma \vdash_{\mathsf{wK}\mathbf{H}} \varphi$ by $\Gamma \vdash_{\mathsf{w}} \varphi$ and $\Gamma \vdash_{\mathsf{sK}\mathbf{H}} \varphi$ by $\Gamma \vdash_{\mathsf{s}} \varphi$.

As we mentioned above, we defined proof systems to capture logics. Here, we define the consequence relation $\mathsf{wK}\mathbf{L} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{w}} \varphi\}$ characterized by $\mathsf{wK}\mathbf{H}$, and the consequence relation $\mathsf{sK}\mathbf{L} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{s}} \varphi\}$ characterized by $\mathsf{sK}\mathbf{H}$.

In the remaining of this subsection, we proceed to show that $\mathsf{wK}\mathbf{L}$ and $\mathsf{sK}\mathbf{L}$ are logics interacting in a specific way and satisfying characteristic properties.

First, we show that they are logics.

Lemma 4.1.1. The following holds for $i \in \{\mathsf{w}, \mathsf{s}\}$.

Monotonicity: If $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash_i \varphi$ then $\Gamma' \vdash_i \varphi$.

Compositionality: If $\Gamma \vdash_i \varphi$ and $\Delta \vdash_i \gamma$ for all $\gamma \in \Gamma$, then $\Delta \vdash_i \varphi$.

Structurality: If $\Gamma \vdash_i \varphi$ and σ is a propositional variable substitution then $\Gamma^\sigma \vdash_i \varphi^\sigma$.

Proof. **Monotonicity**: (⌨) Assume $\Gamma \vdash_i \varphi$. Then there is a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that $\Gamma' \vdash_i \varphi$ with $\Gamma \subseteq \Gamma'$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$. In this case we have $\varphi \in \Gamma'$ as $\Gamma \subseteq \Gamma'$, hence $\Gamma' \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^{\mathsf{i}}_K$ and thus $\Gamma' \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
In the case of (iNec) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$ then we simply use the given premise to obtain $\Gamma' \vdash_{\mathsf{w}} \varphi$ as desired.

Compositionality: (⌨) Assume $\Gamma \vdash_i \varphi$ and that $\Delta \vdash_i \psi$ for every $\psi \in \Gamma$. Then we have a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Delta \vdash_i \varphi$.

If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$ and so we have $\Delta \vdash_i \varphi$ by assumption.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}_K^{\mathsf{i}}$ and thus $\Delta \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
In the case of (iNec) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$, we apply appropriately the rule, i.e. from $\emptyset \vdash_{\mathsf{w}} \varphi$ to $\Delta \vdash_{\mathsf{w}} \varphi$, to obtain the desired result.

Structurality: (⌨) Assume $\Gamma \vdash_i \varphi$. Then we have a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi^\sigma \in \Gamma^\sigma$, hence $\Gamma^\sigma \vdash \varphi^\sigma$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then we have that $\varphi^\sigma \in \mathcal{A}_K^{\mathsf{i}}$ (⌨), hence $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If it is (iNec) then for both values of i we apply the induction hypothesis on the premise and then the rule. ■

The above is interesting for two reasons. First, it establishes that wKL and sKL are logics, as defined in Chapter 3. Second, the proof technique we just exhibited, by induction on the structure of proofs, is one we often use throughout this dissertation when proving proof-theoretic statements.

In addition to being logics, we can show that wKL and sKL are *compact* logics.

Lemma 4.1.2. For $i \in \{\mathsf{w}, \mathsf{s}\}$, if $\Gamma \vdash_i \varphi$, then $\Gamma' \vdash_i \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

Proof. (⌨) Assume $\Gamma \vdash_i \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_i \varphi$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\{\varphi\} \subseteq \Gamma$ and $\varphi \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, $\emptyset \subseteq \Gamma$ and $\emptyset \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain finite $\Gamma', \Gamma'' \subseteq \Gamma$ such that $\Gamma' \vdash_i \psi$ and $\Gamma'' \vdash_i \psi \to \varphi$. Lemma 4.1.1 delivers $\Gamma' \cup \Gamma'' \vdash_i \psi$ and $\Gamma' \cup \Gamma'' \vdash_i \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma' \cup \Gamma'' \vdash_i \varphi$, where $\Gamma' \cup \Gamma'' \subseteq \Gamma$ is finite.
In the case of (iNec) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$, we apply the induction hypothesis on the premise and then the rule. If $i = w$, we apply appropriately the rule to obtain the desired result. ■

Second, we show how these two logics interact as consequence relations. They are extensionally connected as follows: wKL is a subset of sKL, but they share the same set of theorems.

Lemma 4.1.3. The following holds:

1. If $\Gamma \vdash_{\mathsf{w}} \varphi$ then $\Gamma \vdash_{\mathsf{s}} \varphi$;
2. $\emptyset \vdash_{\mathsf{w}} \varphi$ if and only if $\emptyset \vdash_{\mathsf{s}} \varphi$.

Proof. We prove each item separately.

1. (⌨) Assume $\Gamma \vdash_{\mathsf{w}} \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove $\Gamma \vdash_{\mathsf{s}} \varphi$ by induction on the structure of $\mathfrak{p}$.

   If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$ hence $\Gamma \vdash_{\mathsf{s}} \varphi$.

   If (Ax) is the last rule applied in $\mathfrak{p}$, then $\Gamma \vdash_{\mathsf{s}} \varphi$.

   If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\Gamma \vdash_{\mathsf{s}} \psi$ and $\Gamma \vdash_{\mathsf{s}} \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma \vdash_{\mathsf{s}} \varphi$.

If (wNec) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{s}} \psi$. Then, we use Lemma 4.1.1 to obtain $\Gamma \vdash_{\mathsf{s}} \psi$. By an application of (sNec) we obtain $\Gamma \vdash_{\mathsf{s}} \Box\psi$.

2. (⌨) From right to left, we simply use item 1 above. We are thus left with the direction from left to right. Assume $\emptyset \vdash_{\mathsf{s}} \varphi$, giving a proof $\mathfrak{p}$ with root $\emptyset \vdash \varphi$. We prove $\emptyset \vdash_{\mathsf{w}} \varphi$ by induction on the structure of $\mathfrak{p}$.

   If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \emptyset$ which is a contradiction.

   If (Ax) is the last rule applied in $\mathfrak{p}$, then $\emptyset \vdash_{\mathsf{w}} \varphi$.

   If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\emptyset \vdash_{\mathsf{w}} \psi$ and $\emptyset \vdash_{\mathsf{w}} \psi \to \varphi$. Thus (MP) can be applied to get $\emptyset \vdash_{\mathsf{w}} \varphi$.

   If (sNec) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{w}} \psi$. By an application of (wNec) we obtain $\emptyset \vdash_{\mathsf{w}} \Box\psi$.

■

The converse of item 1 above does not hold, as we shall see in Subsection 6.1.1. Thus, wKL is a *strict* subset of sKL, making them two different logics.

Third, we proceed to show that these logics satisfy characteristic properties which they do not share. To get to these results, we consider an intermediary lemma.

Lemma 4.1.4. For $i \in \{\mathsf{w}, \mathsf{s}\}$ we have:

1. $\Gamma \vdash_i \varphi \to \varphi$
2. $\Gamma \vdash_i (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

Proof. In each case, we only prove it for $i = \mathsf{w}$ and use item 1 of Lemma 4.1.3. For the first item (⌨), consider the following proof, where $\psi = (\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((((\varphi \to \varphi) \to \varphi) \to \varphi) \to (\varphi \to \varphi))$.

$$\dfrac{\dfrac{}{\Gamma \vdash ((\varphi \to \varphi) \to \varphi) \to \varphi}\,(\mathrm{Ax}) \qquad \dfrac{\dfrac{}{\Gamma \vdash \varphi \to ((\varphi \to \varphi) \to \varphi)}\,(\mathrm{Ax}) \qquad \dfrac{}{\Gamma \vdash \psi}\,(\mathrm{Ax})}{\Gamma \vdash (((\varphi \to \varphi) \to \varphi) \to \varphi) \to (\varphi \to \varphi)}\,(\mathrm{MP})}{\Gamma \vdash \varphi \to \varphi}\,(\mathrm{MP})$$

From left to right we have an instance of MA_3, MA_2 and MA_1.

For the second item, we required several lemmas. In a nutshell, the proof relies on the two following theorems:

$$\begin{array}{lll} (\mathrm{A}) & (\varphi \to (\varphi \to \psi)) \to (\varphi \to \psi) & (⌨) \\ (\mathrm{B}) & (\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi)) & (⌨) \end{array}$$

With these in hand, we can construct the following semi-proof (⌨).

$$\begin{array}{cl} \hline (\varphi \to \psi) \to ((\psi \to (\varphi \to \chi)) \to (\varphi \to (\varphi \to \chi))) & (\mathrm{Ax}) \\ \hdashline (\varphi \to \psi) \to ((\psi \to (\varphi \to \chi)) \to (\varphi \to \chi)) & \mathrm{Thm.(A)} \\ \hdashline (\psi \to (\varphi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) & \mathrm{Thm.(B)} \\ \hdashline (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi)) & \mathrm{Thm.(B)} \end{array}$$

From bottom to top, we first use (B) to swap φ and ψ in $\varphi \to (\psi \to \chi)$. Second, we reuse (B) to swap $\psi \to (\varphi \to \chi)$ and $\varphi \to \psi$. Third, we use (A) to replace $\varphi \to \psi$ by $\varphi \to (\varphi \to \psi)$. Finally, the formula obtained, i.e. $(\varphi \to \psi) \to ((\psi \to (\varphi \to \chi)) \to (\varphi \to (\varphi \to \chi)))$, is an instance of MA_1. ■

These lemmas allow us to prove that the logic wKL satisfies the *deduction-detachment theorem*, a key result in this dissertation.

Theorem 4.1.1 (Deduction-Detachment Theorem)**.** $\mathsf{w}\mathbf{KL}$ enjoys the deduction-detachment theorem:

$$\Gamma, \varphi \vdash_{\mathsf{w}} \psi \quad \text{iff} \quad \Gamma \vdash_{\mathsf{w}} \varphi \to \psi$$

Proof. (⇐) (⌨) Assume $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$. Then by monotonicity (Lemma 4.1.1) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi \to \psi$. Moreover we have that $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi$ as $\varphi \in \Gamma \cup \{\varphi\}$. So by (MP) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$.

(⇒) (⌨) Assume $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$, i.e. there is a proof $\mathfrak{p}$ of $\Gamma, \varphi \vdash \psi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (El) is the last rule applied in $\mathfrak{p}$, then $\psi \in \Gamma \cup \{\varphi\}$. If $\psi = \varphi$ then we clearly have $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ by Lemma 4.1.4. If $\psi \in \Gamma$ then we can deduce $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ from the fact that we have $\emptyset \vdash_{\mathsf{w}} \psi \to (\varphi \to \psi)$, by axiom MA_2.

If (Ax) is the last rule applied in $\mathfrak{p}$, then $\psi \in \mathcal{A}^I_K$ and with a similar reasoning we get $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (MP) is the last rule applied in $\mathfrak{p}$, then we proceed as follows. We use the induction hypothesis on the premises of the rule to obtain proofs of $\Gamma \vdash \varphi \to \chi$ and $\Gamma \vdash \varphi \to (\chi \to \psi)$. We note that $\Gamma \vdash_{\mathsf{w}} (\varphi \to \chi) \to ((\varphi \to (\chi \to \psi)) \to (\varphi \to \psi))$ by Lemma 4.1.4. Then, we use (MP) several times to arrive at the establishment of $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If the last rule is (wNec), we have a proof of $\emptyset \vdash \chi$, so we can apply (wNec) to obtain $\emptyset \vdash_{\mathsf{w}} \Box\chi$. Then we can use the axiom MA_4 to get $\emptyset \vdash_{\mathsf{w}} \Box\chi \to (\varphi \to \Box\chi)$ and to obtain $\emptyset \vdash_{\mathsf{w}} \varphi \to \Box\chi$. By monotonicity we obtain $\Gamma \vdash_{\mathsf{w}} \varphi \to \Box\chi$.

■

The deduction-detachment theorem, first proven by Herbrand [70] and given its name by Hilbert and Bernays [72], shows a profound connection between the object-level connective → and the meta-level symbol ⊢: the former gives an account in the object language of the relation captured by the former. This specific connection holds in $\mathsf{w}\mathbf{KL}$, as shown above, but is lost in $\mathsf{s}\mathbf{KL}$, as we shall in Subsection 6.1.1. However, a modified version of the deduction-detachment theorem holds for the latter. To prove it, we need the following lemma.

Lemma 4.1.5. For $i \in \{\mathsf{w}, \mathsf{s}\}$ we have:

1. For all $n, m \in \mathbb{N}$ such that $n \leq m$, we have the following.

$$\Gamma \vdash_i (\varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi)\dots)) \to (\varphi \to (\Box\varphi \to \dots (\Box^m\varphi \to \psi)\dots))$$

2. For all $n \in \mathbb{N}$, the following rule is admissible.

$$\frac{\Gamma \vdash_i \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi)\dots) \quad \Gamma \vdash_i \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to (\psi \to \chi)\dots))}{\Gamma \vdash_i \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \chi)\dots)}$$

3. $\Gamma \vdash_i \Box(\varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi)\dots)) \to (\Box\varphi \to (\Box\Box\varphi \to \dots (\Box^{n+1}\varphi \to \Box\psi)\dots))$.

Proof. Each proof goes by induction on natural numbers. For brevity we refer to their respective formalisation: (1) ⌨, (2) ⌨ and (3) ⌨. Note that 2 is easily obtained using (MP) twice with the theorem below, which we formalised.

$$\begin{gathered}(\varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi)\dots)) \to \\ (\varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to (\psi \to \chi)\dots))) \to \\ (\varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \chi)\dots))\end{gathered}$$

■

Intuitively, each item of the above lemma says the following. First, item 1 shows that if a formula ψ is implied by a formula φ and all the boxed versions of φ until $\Box^n\varphi$, then we can add more boxed versions of φ, e.g. until $\Box^m\varphi$, while still preserving the provability of the consecution. Second, item 1 pertains to a deep application of the rule (MP). Indeed, it allows us to go from ψ and $\psi \to \chi$ to χ, while preserving the layers of boxed versions of φ. Third, Item 3 simply shows how the axiom MA_5, showing that $\Box$ distributes over $\to$, can be generalized to nested implications.

With this lemma in hand, we can finally prove the modified deduction-detachment theorem for s**KH**.

Theorem 4.1.2 (Boxed Deduction-Detachment Theorem)**.**

$$\Gamma, \varphi \vdash_{\mathsf{s}} \psi \quad \text{iff} \quad \exists n \in \mathbb{N} \text{ s.t. } \Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi) \dots)$$

Proof. (⇒) (⌨) Assume that $\Gamma, \varphi \vdash_{\mathsf{s}} \psi$, i.e. that we have a proof $\mathfrak{p}$ of $\Gamma, \varphi \vdash \psi$. We reason by induction on the structure of $\mathfrak{p}$.

If the last rule applied is (Ax), then we get $\emptyset \vdash_{\mathsf{s}} \psi$. As we have that $\emptyset \vdash_{\mathsf{s}} \psi \to (\varphi \to \psi)$ by axiom MA_4, we obtain $\emptyset \vdash_{\mathsf{s}} \varphi \to \psi$ by (MP). By Lemma 4.1.1 we get $\Gamma \vdash_{\mathsf{s}} \varphi \to \psi$. So, it suffices to take $n = 0$.

If the last rule applied is (El) then $\Gamma \vdash_{\mathsf{s}} \varphi \to \psi$ as $\Gamma \vdash_{\mathsf{s}} \psi \to (\varphi \to \psi)$ and $\Gamma \vdash_{\mathsf{s}} \psi$.

If the last rule applied is (MP) then we have by induction hypothesis the two following statements, for some $\chi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ and $m, k \in \mathbb{N}$.

$$\begin{array}{l} \Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^k\varphi \to \chi) \dots) \\ \Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^m\varphi \to (\chi \to \psi)) \dots) \end{array}$$

Then, by using item 1 of Lemma 4.1.5 we obtain the following.

$$\begin{array}{l} \Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^{max(k,m)}\varphi \to \chi) \dots) \\ \Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^{max(k,m)}\varphi \to (\chi \to \psi)) \dots) \end{array}$$

Thus, we can use item 2 of Lemma 4.1.5 on these results to obtain the following.

$$\Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^{max(k,m)}\varphi \to \psi) \dots)$$

If the last rule applied is (sNec) then we get by induction hypothesis that $\Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^m\varphi \to \chi) \dots)$ for some $m \in \mathbb{N}$. We apply (sNec) on the latter to obtain $\Gamma \vdash_{\mathsf{s}} \Box(\varphi \to (\Box\varphi \to \dots (\Box^m\varphi \to \chi) \dots))$. Then, by item 3 of Lemma 4.1.5 we get $\Gamma \vdash_{\mathsf{s}} (\Box\varphi \to (\Box^2\varphi \to \dots (\Box^{m+1}\varphi \to \Box\chi) \dots))$. Via (MP) and Lemma 4.1.4, we easily obtain $\Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to (\Box^2\varphi \to \dots (\Box^{m+1}\varphi \to \Box\chi) \dots))$.

(⇐) (⌨) Assume that there is a $n \in \mathbb{N}$ such that $\Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi) \dots)$. Then by monotonicity (Lemma 4.1.1) we obtain $\Gamma, \varphi \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi) \dots)$. Moreover we have that $\Gamma, \varphi \vdash_{\mathsf{s}} \varphi$ as $\varphi \in \Gamma \cup \{\varphi\}$. Furthermore, we obtain $\Gamma, \varphi \vdash_{\mathsf{s}} \Box^m\varphi$ for all $m \in \mathbb{N}$ by repeated applications of (sNec) on $\Gamma, \varphi \vdash_{\mathsf{s}} \varphi$. So by repetitive applications of (MP) with the latter and $\Gamma \vdash_{\mathsf{s}} \varphi \to (\Box\varphi \to \dots (\Box^n\varphi \to \psi) \dots)$, we obtain $\Gamma, \varphi \vdash_{\mathsf{s}} \psi$.

■

This theorem "patches" the deduction-detachment theorem by strengthening φ by adding boxed versions of it. This patch gives an account of the strength of the rule (sNec), which allows to prove $\varphi \vdash_{\mathsf{s}} \Box^n\varphi$ for any $n \in \mathbb{N}$.

So, we proved that w**KL** and s**KL** are (compact) logics such that: (1) w**KL** $\subseteq$ s**KL**, (2) w**KL** and s**KL** share the same set of theorems, (3) w**KL** satisfies the deduction theorem, (4) s**KL** only satisfies a modified version of the deduction theorem.

4.1.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

To build our generalized Hilbert calculi **wKH** and **sKH**, we rely on the definition of modal formulas given in Subsection 2.1.2.

We define our five axioms individually (⌨).

```
Definition MA1 (φ ψ χ : MPropF) : MPropF :=
       (φ --> ψ) --> ((ψ --> χ) --> (φ --> χ)).

Definition MA2 (φ ψ : MPropF) : MPropF := φ --> (ψ --> φ).

Definition MA3 (φ ψ : MPropF) : MPropF :=
       ((φ --> ψ) --> φ) --> φ.

Definition MA4 (φ : MPropF) : MPropF := Bot --> φ.

Definition MA5 (φ ψ : MPropF) : MPropF :=
       Box (φ --> ψ) --> (Box φ --> Box ψ).
```

In each of these definitions, we give: a name to the entity we are defining, the parameters we are relying on with their type, the type of the entity, and finally the entity itself. For example, we give the name `MA1` to our first axiom. Then, we specify that the axiom `MA1` needs three elements φ ψ χ of type `MPropF`, i.e. formulas. The output of `MA1`, once given three elements of the adequate type `MPropF`, is also a formula (`MPropF`). Now, we specify the structure of `MA1` by explicitly giving the formula `(φ --> ψ) --> ((ψ --> χ) --> (φ --> χ))`, which relies on the parameters φ ψ χ. Note how in Coq we do not need a notion of axiom schema: an axiom is simply a function from formulas to a formula.

We can thus define the convenient notion of being an instance of one of the axioms defined above (⌨).

```
Inductive KAxioms (φ : MPropF) : Prop :=
 | MA1_I : (exists ψ χ γ, φ = (MA1 ψ χ γ)) -> KAxioms φ
 | MA2_I : (exists ψ χ, φ = (MA2 ψ χ)) -> KAxioms φ
 | MA3_I : (exists ψ χ, φ = (MA3 ψ χ)) -> KAxioms φ
 | MA4_I : (exists ψ, φ = (MA4 ψ)) -> KAxioms φ
 | MA5_I : (exists ψ χ, φ = (MA5 ψ χ)) -> KAxioms φ.
```

This definition tells us how we can claim for a formula φ that `KAxioms φ`, i.e. the property (`Prop`) of φ being an instance of an axiom, holds. It does so by cases: if φ is identical to an instantiation of *one of the axioms*, then we get `KAxioms φ`. So, we give each of our axioms a specific constructor. For example, `MA1` it is `MA1_I`, where `I` stands for "introduction". This constructor allows to obtain `KAxioms φ` under one condition: that we satisfy `(exists ψ χ γ, φ = (MA1 ψ χ γ))`, i.e. that we show the existence of three formulas ψ χ γ such that φ is equal to `(MA1 ψ χ γ)` the instantiation of `MA1` with these three formulas.

Thus, we can formalise the rule (Ax) which we name `AxRule` (⌨).

```
Inductive AxRule : rls ((Ensemble MPropF) * MPropF) :=
  | AxRule_I : forall Γ (φ : MPropF),
          (KAxioms φ) -> AxRule [] (Γ , φ).
```

Rules in a generalized Hilbert calculus are constituted by finitely many premises of the form $\Gamma \vdash \varphi$ and a conclusion of the same form. The type `rls ((Ensemble MPropF) * MPropF)` of `AxRule` gives an account of this: judgements of the shape $\Gamma \vdash \varphi$ are of the type `((Ensemble MPropF) * MPropF)`, and `rls` is a type constructor in `Prop` for pairs of a finite list of elements of a given type and an element of the same type. The constructor `AxRule_I` allows to claim that we have `AxRule [] (Γ , φ)`, i.e. a correct

(Ax) rule application for some Γ and φ, under the condition that we can prove that φ is an axiom instance, i.e. `KAxioms φ`. Note that the fact that the (Ax) rule has no premises is encapsulated in `AxRule` by the presence of the empty list `[]` as premises of the rule.

We define the remaining rules in a similar way (⌨).

```
Inductive IdRule : rls ((Ensemble MPropF) * MPropF) :=
  | IdRule_I : forall φ (Γ : Ensemble _),
                  (In _ Γ φ) -> IdRule [] (Γ , φ).

Inductive MPRule : rls ((Ensemble MPropF) * MPropF) :=
  | MPRule_I : forall φ ψ Γ,
          MPRule [(Γ , φ --> ψ) ; (Γ , φ)]
                   (Γ , ψ).

Inductive wNecRule : rls ((Ensemble MPropF) * MPropF) :=
  | wNecRule_I : forall (φ : MPropF) Γ,
          wNecRule [(Empty_set _ , φ)]
                   (Γ , Box φ).

Inductive sNecRule : rls ((Ensemble MPropF) * MPropF) :=
  | sNecRule_I : forall (φ : MPropF) Γ,
          sNecRule [(Γ , φ)]
                   (Γ , Box φ).
```

The `IdRule` corresponds to the rule (El), as it relies on a proof of `(In _ Γ φ)`, i.e. the fact that φ is an element of Γ. Note that in all other rules the list of premises is not empty, and is filled with the adequate premises corresponding to the rules. For example, the rule `wNecRule` takes as premises all the elements in the list `[(Empty_set _ , φ)]`, i.e. $\emptyset \vdash \varphi$, and as conclusion `(Γ , Box φ)`, i.e. $\Gamma \vdash \Box\varphi$.

We can finally define our calculi w**KH** and s**KH** (⌨) (⌨).

```
Inductive wKH_rules (s : ((Ensemble _) * MPropF)) : Prop :=
  | Id : IdRule [] s -> wKH_rules s
  | Ax : AxRule [] s -> wKH_rules s
  | MP : forall ps,
        (forall prem, List.In prem ps -> wKH_rules prem) ->
        MPRule ps s -> wKH_rules s
  | wNec : forall ps,
        (forall prem, List.In prem ps -> wKH_rules prem) ->
        wNecRule ps s -> wKH_rules s.
```

```
Inductive sKH_rules (s : ((Ensemble _) * MPropF)) : Prop :=
  | Ids : IdRule [] s -> sKH_rules s
  | Axs : AxRule [] s -> sKH_rules s
  | MPs : forall ps,
        (forall prem, List.In prem ps -> sKH_rules prem) ->
        MPRule ps s -> sKH_rules s
  | sNec : forall ps,
        (forall prem, List.In prem ps -> sKH_rules prem) ->
        sNecRule ps s -> sKH_rules s.
```

In essence, the above definition creates the notion of proof, relying on correct rule applications in the adequate calculus with the corresponding rules. For example, we can have a proof of `s` if we have a rule application `wNecRule ps s` and proofs for all the elements of the list of premises `ps`, i.e. `(forall prem, List.In prem ps -> wKH_rules prem)`. All the other rules can be read similarly. We define a more intuitive macro for provability in these calculi (⌨).

```
Definition wKH_prv s := wKH_rules s.
Definition sKH_prv s := sKH_rules s.
```

In the remaining of this subsection, we exhibit and comment on all the results we presented in the previous subsection about the logics wK**L** and sK**L**.

First, we showed via Lemma 4.1.1 and Lemma 4.1.2 that both were compact logics, as defined in Definition 3.0.2. For the sake of brevity, we only show the results for wK**L**. We start with monotonicity (⌨).

```
Theorem wKH_monot : forall s, (wKH_prv s) ->
        (forall Γ1, (Included _ (fst s) Γ1) ->
        (wKH_prv (Γ1, (snd s)))).
```

This theorem shows the holding of monotonicity for wK**L** once we realize that `s` is an expression of the form $\Gamma \vdash \varphi$. More precisely, it is a pair of the shape `pair Γ φ`, on which we can use the projection function `fst` and `snd` to obtain, respectively, the first and second element of the pair. Then, we read that this theorem has as assumptions `wKH_prv s` the provability of `s`, and the set-theoretic inclusion of `(fst s)`, i.e. the left component Γ of $\Gamma \vdash \varphi$, in another set `Γ1`. The conclusion of this theorem is the provability of the pair `(Γ1, (snd s))`, i.e. $\Gamma 1 \vdash \varphi$. It is now obvious that this is nothing but monotonicity.

Then, we turn to compositionality (⌨).

```
Theorem wKH_comp : forall s, (wKH_prv s) ->
     (forall Γ,  (forall φ, ((fst s) φ) -> wKH_prv (Γ, φ)) ->
     wKH_prv (Γ, (snd s))).
```

We first start with the provability of `s`. Then, in the second line, we assume that all the elements in the left component of `s`, i.e. `fst s`, are provable from a given set Γ. Finally, in the third line, we claim the provability of `(Γ, (snd s))`. This corresponds to compositionality.

Structurality can be captured in a similar way using a propositional variable substitution `f` (⌨).

```
Theorem wKH_struct : forall s, (wKH_prv s) ->
 (forall (f : V -> MPropF),
 (wKH_prv
  ((fun y => (exists φ, prod ((fst s) φ) (y = (subst f φ)))),
  (subst f (snd s))))).
```

Under the assumption that `s` is provable in wK**H**, we get the provability of the pair expressed on the two last lines. Say that `s` is of the shape $\Gamma \vdash \varphi$. In the last line, we have the right component of the pair: it is nothing but φ^f the propositional variable substitution of φ through `f`. On the line right before, we have the left component of the pair. In essence, it is Γ^f, which is using defining the characteristic function of the latter. The result of the application of the characteristic function `(fun y => (exists φ, prod ((fst s) φ) (y = (subst f φ))))` on input ψ holds only if we can show that there is φ such that $\varphi \in \Gamma$ and $\psi = \varphi^f$. This condition is expressed by the expression `(exists φ, prod ((fst s) φ) (y = (subst f φ)))`, which requires the exhibition of a witness and a proof of the conjunction of both statements we mentioned. It should appear that the theorem above corresponds to structurality.

We showed that wK**L** is compact in Lemma 4.1.2, which is formalised as follows (⌨).

```
Theorem wKH_finite : forall s, (wKH_prv s) ->
  (exists Γ, prod (Included _ Γ (fst s))
      (prod (wKH_prv (Γ, snd s))
      (exists l, (forall φ, ((Γ φ) -> List.In φ l) *
                            (List.In φ l -> (Γ φ)))))).
```

Under the assumption that `s` is provable, we show that there is set Γ such that: (1) `(Included _ Γ (fst s))` it is included in the left component of `s`, (2) `(wKH_prv (Γ, snd s))`, and (3) it is finite. The third point is obtained via the existence of a list `l`, which is finite by definition and contains the same formulas as Γ. This is what the two last lines express.

Second, we showed in Lemma 4.1.3 that wKL was a subset of sKL, while both share the same set of theorems. These results are respectively formalised as follows (⌨) (⌨).

```
Theorem sKH_extens_wKH : forall s,
    (wKH_prv s) -> (sKH_prv s).
```

```
Theorem wKH_sKH_same_thms : forall φ,
    wKH_prv (Empty_set _, φ) <-> sKH_prv (Empty_set _, φ).
```

Third, we proved in Theorem 4.1.1 that the deduction theorem holds for wKL, while only a modified version was holding for sKL as shown in Theorem 4.1.2. The next two theorems express this (⌨) (⌨).

```
Theorem wKH_Detachment_Deduction_Theorem : forall φ ψ Γ,
      wKH_prv (Union _ Γ (Singleton _ φ), ψ) <->
      wKH_prv (Γ, φ --> ψ).
```

```
Theorem sKH_Boxed_Detachment_Deduction_Theorem: forall φ ψ Γ,
      sKH_prv (Union _ Γ (Singleton _ φ), ψ) <->
      exists n, sKH_prv (Γ, (Imp_Box_power n φ ψ)).
```

While the first theorem is clear, we provide some details on the second. The only new element there is the use of the constructor `Imp_Box_power`. It captures the wrapping of the formula ψ under the implication of n boxed version of φ, as in the following formula.

$$\varphi \to (\Box\varphi \to \ldots (\Box^n\varphi \to \psi)\ldots)$$

More precisely, we formalised this notion in the following way (⌨).

```
Fixpoint Imp_Box_power (n : nat) (φ ψ : MPropF) : MPropF :=
  match n with
   | 0 => φ --> ψ
   | S m => φ --> (Imp_Box_power m (Box φ) ψ)
  end.
```

This definition is defined recursively on $n \in \mathbb{N}$. If n is 0, it simply outputs the implication $\varphi \to \psi$. If n is $S\ m$ for some m, then we first add φ as an antecedent in an implication, and then apply `Imp_Box_power` on m with now $\Box\varphi$ instead of φ. This has the effect of adding φ, decreasing n, and adding a box in front of φ for the next application.

We showed the formalisation of all the main results we exhibited in the previous subsection. We consequently close this section on generalized Hilbert calculi.

4.2 Sequent calculi

Sequent calculi are proof systems that were first invented by Gerhard Gentzen [53] in an attempt to obtain better-behaved proof systems to capture logics.

Indeed, finding a proof for a judgement $\Gamma \vdash \varphi$ in a generalized Hilbert calculus can be an extremely complicated matter. First, the presence of axioms leads to great difficulties when trying to find a proof, summarized by the following questions: "Which axiom needs to be used?" or "Which instantiations of the axiom?". Second, rules like (MP) are tricky to use. While they can reduce the task of finding a proof for $\Gamma \vdash \varphi$ to the finding of proofs for both $\Gamma \vdash \psi \to \varphi$ and $\Gamma \vdash \psi$, they require us to guess what the adequate formula ψ is with very few hints. Thus, generalized Hilbert calculi are not well-behaved when it comes to the finding of proofs.

Instead of the hard-to-guess proofs of generalized Hilbert calculi, sequent calculi allow a somewhat mechanical construction of proofs from the judgements we are trying to prove, as we shall see below. In a nutshell, sequent calculi rules are goal-oriented, that is targeted at deconstructing the elements of judgements they manipulate in a most deterministic way, thus avoiding the introduction of random elements like the formula ψ in the rule (MP).

The judgements manipulated through rules of sequent calculi are called *sequents* and are of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas of a given (propositional or first-order) language $\mathbb{L}$. For multisets Γ and Δ, the multiset sum $\Gamma \uplus \Delta$ is the multiset whose multiplicity (at each formula) is the sum of the multiplicities of Γ and Δ. We write Γ, Δ to mean $\Gamma \uplus \Delta$. For a formula φ, we write φ, Γ and Γ, φ to mean $\{\varphi\} \uplus \Gamma$.

Following Definition 4.0.1, a *sequent calculus* **S** consists of a finite set $\mathcal{R}$ of *sequent rule schemas*. If a rule schema has no premise sequents, then it is called an initial sequent. In what follows we designate a sequent calculus using the suffix **S**.

A sequent calculus **S** can capture a logic in the sense of Chapter 3 with the set $\{(\Gamma, \varphi) \mid \text{ there is a finite } \Gamma' \subseteq \Gamma \text{ s.t. } \Gamma' \Rightarrow \varphi \text{ is provable in } \mathbf{S}\}$. As a consequence, we can only capture compact logics with sequent calculi.

In proof theory, there are several properties of a sequent calculus we are interested in. We present them in the shape of rules in the following definition.

Definition 4.2.1. We say that a sequent calculus **S** admits:

- *exchange on the left* if the following rule is admissible.

$$\frac{\Gamma_0, \psi, \varphi, \Gamma_1 \Rightarrow \Delta}{\Gamma_0, \varphi, \psi, \Gamma_1 \Rightarrow \Delta}$$

- *exchange on the right* if the following rule is admissible.

$$\frac{\Gamma \Rightarrow \Delta_0, \varphi, \psi, \Delta_1}{\Gamma \Rightarrow \Delta_0, \psi, \varphi, \Delta_1}$$

- *weakening on the left* if the following rule is admissible.

$$\frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

- *weakening on the right* if the following rule is admissible.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

- *contraction on the left* if the following rule is admissible.

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$$

- *contraction on the right* if the following rule is admissible.

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

- *multiplicative cut* if the following rule is admissible, where φ is called the *cut-formula*.

$$\frac{\Gamma_0 \Rightarrow \Delta_0, \varphi \quad \varphi, \Gamma_1 \Rightarrow \Delta_1}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}$$

- *additive cut* if the following rule is admissible, where φ is called the *cut-formula*.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Let us comment on these properties, their use, and their importance.

When considering sequents $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite multisets, any sequent calculus **S** admits exchange on the left and right: in multisets, the order of elements is irrelevant. So, why do we mention these at all? Simply because in our formalisation in Coq we decided to encode sequents using *lists* of formulas, and not multisets, as we explain below. When considering sequents based on lists, the admissibility of exchange becomes relevant.

The rules of weakening convey the idea that the relation between Γ and Δ expressed in $\Gamma \Rightarrow \Delta$ is stable under the extension of either multiset by any formula. Note that this stability implies the stability under extension by finite multisets of formulas: if $\Gamma \Rightarrow \Delta$ is provable, then $\Gamma', \Gamma \Rightarrow \Delta, \Delta'$ is also provable.

The multiplicity of formulas in multisets is made unnecessary by the rules of contraction. Indeed, in the presence of contraction, it suffices to focus on sequents having no multiple occurrences of a formula in either Γ or Δ. As for weakening, the multiplicity of finite multisets is also made unnecessary by contraction: if $\Gamma', \Gamma', \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \Delta', \Delta'$) is provable, then $\Gamma', \Gamma \Rightarrow \Delta$ (resp. $\Gamma \Rightarrow \Delta, \Delta'$) is also provable.

While the additive and multiplicative versions of the cut rule are equivalent in the presence of weakening and contraction (left and right) [154, Section 3.2], in this dissertation we only make use of the additive version, which we name (cut) from now on. For a sequent calculus **S**, we write **S** + (cut) to designate the system **S** extended with the rule (cut).

As we mentioned above, sequent calculi are designed to make the task of determining the provability of a sequent easier. Consequently, we want to remove the cut rule from the collection of acceptable rules, as it permits to reduce the provability of a sequent $\Gamma \Rightarrow \Delta$ to the provability of the sequents $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ where φ can be *any formula*. Thus, this rule can introduce an arbitrary formula which can be extremely hard to guess when just looking at the sequent $\Gamma \Rightarrow \Delta$. However, we want our calculi to have the *strength* of the cut rule by admitting it as it pertains to the crucial compositionality property of logics shown in Definition 3.0.2.

On top of the admissibility of cut in a sequent calculus **S**, proof theorists are often interested in the *eliminability of cut*: the property that each proof $\mathfrak{p}$ of $\Gamma \Rightarrow \Delta$ in **S** + (cut) can be transformed into a proof $\mathfrak{p}'$ of $\Gamma \Rightarrow \Delta$ in **S**. Usually, if the admissibility of cut is obtained in a purely proof-theoretic way through *local transformations*, then a cut-elimination result is straightforwardly obtained. Indeed, it suffices to tackle the topmost cut in a proof $\mathfrak{p}$ in **S** + (cut), and follow the local transformations described in the cut-admissibility proof to eliminate this instance of cut.

With a sequent calculus **S** which admits cut but does not contain it, one can hope to define a general strategy to find proofs for sequents starting from the latter: this is called *backward proof-search procedure*. In backward proof-search, we start from a sequent $\Gamma \Rightarrow \Delta$ and mechanically apply restricted versions of the rules of **S** *backwards*. The way rules are applied is determined by the backward proof-search procedure, which constitutes an algorithm. Ideally, the design of this procedure should lead to two key properties.

First, a backward proof-search procedure for **S** should be *equivalent* to **S**: we can build a proof of a sequent following the backward proof-search procedure if and only if this sequent is provable in **S**. This ensures that the designed procedure allows to find a proof for a sequent if there is one in **S**, thus giving a complete account of provable sequents in **S**.

Second, the algorithm described by the strategy should terminate: given a sequent $\Gamma \Rightarrow \Delta$, the backward application of rules of **S** following the strategy comes to an end in finite time. If this is the case, then we say that backward proof-search terminates.

With these two properties, we obtain an amazing result: the decidability of provability of sequents in **S**. Indeed, we can decide whether there is a proof of a given sequent $\Gamma \Rightarrow \Delta$ following the backward proof-search procedure, as the latter is terminating, and thus tell whether $\Gamma \Rightarrow \Delta$ is provable in **S** because of the equivalence of the procedure.

In this dissertation, we identify terminating and equivalent backward proof-search procedures for sequent calculi with sequent calculi we prefix with PS for Proof-Search. For example, if we define such a procedure for a sequent calculus **S**, we use the name PSS for the calculus identified with the procedure. Our justification in taking this decision pertains to the fact that such a procedure captures a subset of the set of all derivations of **S**, i.e. those which are built using the restricted version of the rules of **S**. So, if we consider the system PSS, consisting of the restricted rules of **S**, we capture the same set of derivations as the one obtained through the backward proof-search procedure.

4.2.1 The classical modal logic example: on paper

In this subsection we first define the sequent calculus wKS, which captures the logic wKL. Second, we show that wKS admits the rule of exchange, weakening, and contraction. Third, also show that wKS itself embodies a terminating proof-search procedure, which would grant this system the prefix PS. We finally use the termination of wKS to show the admissibility of cut through in this calculus through local transformations, giving us a cut-elimination result.

The sequent calculus wKS

The sequent calculus wKS is given in Figure 4.1.

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\ (\mathrm{IdP}) \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\ (\bot\mathrm{L})$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L}) \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\mathrm{R})$$

$$\frac{\Gamma \Rightarrow \psi}{\Phi, \Box\Gamma \Rightarrow \Box\psi, \Delta}\ (\mathrm{wKR})$$

Figure 4.1: The sequent calculus wKS. Here, Φ does not contain any (top-level) boxed formulas.

In a rule instance of $(\to\mathrm{L})$ or $(\to\mathrm{R})$, the formula instantiating the featured $\varphi \to \psi$ is the *principal formula* of that instance. In (IdP), a propositional variable instantiating either featured occurrence of p is principal. In $(\bot\mathrm{L})$, the formula $\bot$ is principal. In (wKR), the formula instantiating the featured $\Box\psi$ is the principal formula.

Properties of wKS

Here, we show the (height-preserving) admissibility of the rules of weakening and contraction, as defined in Definition 4.2.1. To obtain these results we notably need to show that the rules $(\to\mathrm{R})$ and $(\to\mathrm{L})$ are (height-preserving) invertible as defined in Definition 4.0.5. All these results are useful in proving the admissibility of the additive cut rule.

First, we prove that the rules of weakening on the left and right are height-preserving admissible in wKS.

Lemma 4.2.1. The following holds.

(i) The rule of weakening on the right is height-preserving admissible in wKS.

(ii) The rule of weakening on the left is height-preserving admissible in wKS.

Proof. We prove each statement independently.

(i) (⌨) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\Gamma \Rightarrow \Delta, \varphi$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ is an instance of an initial sequent.

  (b) $r = (\text{wKR})$: then r is

$$\frac{\Gamma_0 \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0}\,(\text{wKR})$$

  Then we can apply (wKR) on the premise to obtain $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0, \varphi$. Note that the height of the proof is preserved.

  (c) $r = (\to\text{R})$: then r is

$$\frac{\chi, \Gamma \Rightarrow \Delta_0, \psi}{\Gamma \Rightarrow \Delta_0, \chi \to \psi}\,(\to\text{R})$$

  Then we can apply the induction hypothesis on the premise to get a proof of $\chi, \Gamma \Rightarrow \Delta_0, \varphi, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\to\text{R})$ to get a proof of $\Gamma \Rightarrow \Delta_0, \varphi, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

  (d) $r = (\to\text{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\,(\to\text{L})$$

  Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p}) - 1$ of $\Gamma_0 \Rightarrow \Delta, \varphi, \chi$ and $\psi, \Gamma_0 \Rightarrow \Delta, \varphi$. We can consequently apply the rule $(\to\text{L})$ to obtain a proof of $\chi \to \psi, \Gamma_0 \Rightarrow \Delta, \varphi$ of height less than or equal to $h(\mathfrak{p}) - 1$.

(ii) (⌨) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\varphi, \Gamma \Rightarrow \Delta$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\varphi, \Gamma \Rightarrow \Delta$ is an instance of an initial sequent.

  (b) $r = (\text{wKR})$: then r is

$$\frac{\Gamma_0 \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0}\,(\text{wKR})$$

  Either φ is boxed, i.e. $\varphi = \Box\chi$ for some χ, and then we can apply the induction hypothesis to obtain a proof of $\chi, \Gamma_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the (wKR) rule to obtain a proof of $\Phi, \Box\chi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0$ of height less than or equal to $h(\mathfrak{p})$. Or φ is not boxed and then we can apply (wKR) on the premise to obtain a proof of $\varphi, \Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0$ of height $h(\mathfrak{p})$.

  (c) $r = (\to\text{R})$: then r is

$$\frac{\chi, \Gamma \Rightarrow \Delta_0, \psi}{\Gamma \Rightarrow \Delta_0, \chi \to \psi}\,(\to\text{R})$$

  Then we can apply the induction hypothesis on the premise to get a proof of $\chi, \Gamma, \varphi \Rightarrow \Delta_0, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\to\text{R})$ to get a proof of $\Gamma, \varphi \Rightarrow \Delta_0, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

  (d) $r = (\to\text{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\,(\to\text{L})$$

Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p})-1$ of $\Gamma_0, \varphi \Rightarrow \Delta, \chi$ and $\psi, \Gamma_0, \varphi \Rightarrow \Delta$. We can consequently apply the rule to obtain a proof of $\chi \to \psi, \Gamma_0, \varphi \Rightarrow \Delta$ of height less than or equal to $h(\mathfrak{p})$.

■

Second, we prove that the implication rules (→R) and (→L) are height-preserving invertible as defined in Definition 4.0.5.

Lemma 4.2.2. The following holds.

(i) The rule (→L) is height-preserving invertible in wKS.

(ii) The rule (→R) is height-preserving invertible in wKS.

Proof. (⌨) We prove the two statements, which we respectively call (i) and (ii), simultaneously by strong induction on the height of the given proof.

(i) Let $\mathfrak{p}$ be a proof of $\varphi \to \psi, \Gamma \Rightarrow \Delta$. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ are instances of an initial sequent.

(b) r is (wKR): then r is

$$\frac{\Gamma_0 \Rightarrow \chi}{\varphi \to \psi, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0}\ (\mathrm{wKR})$$

Then we can apply (wKR) on the premise to obtain a proof of $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0, \Psi, \varphi$ of height $h(\mathfrak{p})$. Now, to get a proof for the other desired sequent $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0$, we need to distinguish two cases. If ψ is boxed, i.e. there is a δ such that $\psi = \Box\delta$, then we apply Lemma 4.2.1(ii) on $\Gamma_0 \Rightarrow \chi$ to get a proof of $\delta, \Gamma_0 \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})-1$. We can thus apply (wKR) on the latter to obtain a proof of $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0$ of height less than or equal to $h(\mathfrak{p})$. If ψ is not boxed, then we can simply apply (wKR) on $\Gamma_0 \Rightarrow \chi$ to obtain a proof of $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0$ of height less than or equal to $h(\mathfrak{p})$.

(c) r is (→R):

$$\frac{\chi, \varphi \to \psi, \Gamma \Rightarrow \Delta, \delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta, \chi \to \delta}\ (\to\mathrm{R})$$

Applying the induction hypothesis to the proof of $\chi, \varphi \to \psi, \Gamma \Rightarrow \Delta, \delta$ we obtain a proof of height less than or equal to $h(\mathfrak{p})-1$ for

$$\chi, \Gamma \Rightarrow \Delta, \delta, \varphi \tag{4.1}$$

$$\psi, \chi, \Gamma \Rightarrow \Delta, \delta \tag{4.2}$$

Applying (→R) to the proof of (4.2) yields a proof of $\psi, \Gamma \Rightarrow \Delta, \chi \to \delta$ of height less than or equal to $h(\mathfrak{p})$. Also, we can apply (→R) to the proof of (4.1) to obtain a proof of $\Gamma \Rightarrow \Delta, \chi \to \delta, \varphi$ of height less than or equal to $h(\mathfrak{p})$. We have obtained proofs for the desired sequents.

(d) Now suppose that r is (→L). If $\varphi \to \psi$ is principal in r then the premises are $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ so we are done. If $\varphi \to \psi$ is not principal in r then we have the following.

$$\frac{\varphi \to \psi, \Gamma \Rightarrow \Delta, \chi \qquad \delta, \varphi \to \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \chi \to \delta, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L})$$

Applying the induction hypothesis to the proofs of the premises we can obtain proofs of height less than or equal to $h(\mathfrak{p}) - 1$ of

$$\Gamma \Rightarrow \Delta, \chi, \varphi \tag{4.3}$$

$$\psi, \Gamma \Rightarrow \Delta, \chi \tag{4.4}$$

$$\delta, \Gamma \Rightarrow \Delta, \varphi \tag{4.5}$$

$$\psi, \delta, \Gamma \Rightarrow \Delta \tag{4.6}$$

From the proofs of (4.3) and (4.5) we get a proof of height less than or equal to $h(\mathfrak{p})$ for one of the desired sequents.

$$\dfrac{\Gamma \Rightarrow \Delta, \chi, \varphi \qquad \delta, \Gamma \Rightarrow \Delta, \varphi}{\chi \to \delta, \Gamma \Rightarrow \Delta, \varphi}\ (\to\mathrm{L})$$

Also, from the proofs of (4.4) and (4.6):

$$\dfrac{\psi, \Gamma \Rightarrow \Delta, \chi \qquad \psi, \delta, \Gamma \Rightarrow \Delta}{\psi, \chi \to \delta, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L})$$

So we have obtained a proof of height less than or equal to $h(\mathfrak{p})$ for the desired sequents.

(ii) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta, \varphi \to \psi$. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma, \varphi \Rightarrow \Delta, \psi$ is an instance of an initial sequent.

(b) r is (wKR): then r is

$$\dfrac{\Gamma_0 \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0, \varphi \to \psi}\ (\mathrm{wKR})$$

We need to distinguish two cases. If φ is boxed, i.e. there is a δ such that $\varphi = \Box\delta$, then we apply Lemma 4.2.1(i) on the proof of $\Gamma_0 \Rightarrow \chi$ to get a proof of $\delta, \Gamma_0 \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply (wKR) on the latter to obtain a proof of $\Phi, \varphi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0, \psi$ of height less than or equal to $h(\mathfrak{p})$. If φ is not boxed, then we can simply apply (wKR) on $\Gamma_0 \Rightarrow \chi$ to obtain a proof of $\Phi, \varphi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0, \psi$ of height $h(\mathfrak{p})$.

(c) Suppose that r is ($\to$R). If $\varphi \to \psi$ is principal in r then the premise is $\varphi, \Gamma \Rightarrow \Delta, \psi$ so we are done. If $\varphi \to \psi$ is not principal in r then we have the following.

$$\dfrac{\chi, \Gamma \Rightarrow \Delta, \varphi \to \psi, \delta}{\Gamma \Rightarrow \Delta, \varphi \to \psi, \chi \to \delta}\ (\to\mathrm{R})$$

Applying the induction hypothesis to the proof of $\chi, \Gamma \Rightarrow \Delta, \varphi \to \psi, \delta$ we can obtain a proof of $\varphi, \chi, \Gamma \Rightarrow \Delta, \delta, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Then we can apply ($\to$R) to get a proof of $\varphi, \Gamma \Rightarrow \Delta, \psi, \chi \to \delta$ of height less than or equal to $h(\mathfrak{p})$, so we have obtained a proof for the desired sequent.

(d) Finally, suppose that r is ($\to$L).

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \to \psi, \chi \qquad \delta, \Gamma \Rightarrow \Delta, \varphi \to \psi}{\chi \to \delta, \Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\mathrm{L})$$

Applying the induction hypothesis to the proofs of the premises we can obtain proofs of height less than or equal to $h(\mathfrak{p}) - 1$ for

$$\varphi, \Gamma \Rightarrow \Delta, \chi, \psi \tag{4.7}$$

$$\varphi, \delta, \Gamma \Rightarrow \Delta, \psi \tag{4.8}$$

Then proceed:

$$\dfrac{\varphi, \Gamma \Rightarrow \Delta, \chi, \psi \qquad \varphi, \delta, \Gamma \Rightarrow \Delta, \psi}{\varphi, \chi \to \delta, \Gamma \Rightarrow \Delta, \psi}\ (\to\mathrm{L})$$

So we have obtained proofs of height less than or equal to $h(\mathfrak{p})$ for the desired sequent.

$\blacksquare$

Third, we proceed to show that the left and right rules of contraction are height-preserving admissible.

Lemma 4.2.3 (Height-preserving admissibility of contraction)**.** For all Γ, Δ, φ and ψ:

(i) The rule of contraction on the right is height-preserving admissible in wKS.

(ii) The rule of contraction on the left is height-preserving admissible in wKS.

Proof. (⌨) We prove (i) and (ii) simultaneously by strong induction on the height of the given proof.

(i) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta, \varphi, \varphi$ of height n. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: either $p \in \Gamma \cap (\Delta \cup \{\varphi, \varphi\})$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ is an instance of an initial sequent.

  (b) $r = (\mathrm{wKR})$: then r is

$$\frac{\Gamma_0 \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0}\ (\mathrm{wKR})$$

  Either is not principal, i.e. $\varphi \in \Delta_0$, and then we can apply (wKR) on the premise to obtain a proof of height $h(\mathfrak{p})$ of $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_1$ where $\Delta_1, \varphi = \Delta_0$. Or φ is principal, i.e. $\varphi = \Box\psi$, and we consequently have that the second $\Box\psi$ appearing in the conclusion is an element of Δ_0 and then we can apply the rule on the proof of the premise using Δ_1 instead of Δ_0.

  (c) $r = (\to\mathrm{R})$: then r is

$$\frac{\chi, \Gamma_0 \Rightarrow \Delta_0, \psi}{\Gamma_0 \Rightarrow \Delta_0, \chi \to \psi}\ (\to\mathrm{R})$$

  Either $\varphi = \chi \to \psi$, and then we use Lemma 4.2.2(ii) on the premise of the form $\chi, \Gamma_0 \Rightarrow \Delta_1, \chi \to \psi, \psi$ to obtain a proof $\chi, \chi, \Gamma_0 \Rightarrow \Delta_1, \psi, \psi$ of height less than or equal to $h(\mathfrak{p} - 1)$. Then we can apply the induction hypothesis to contract on both sides and obtain a proof of $\chi, \Gamma_0 \Rightarrow \Delta_1, \psi$ of height less than or equal to $h(\mathfrak{p} - 1)$. It consequently suffices to apply $(\to\mathrm{R})$ on the latter to get a proof of $\Gamma_0 \Rightarrow \Delta_1, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$. Or $\varphi \neq \chi \to \psi$, and then $\{\varphi, \varphi\} \subseteq \Delta_0$. Thus we can apply the induction hypothesis on the premise to get a proof of height less than or equal to $h(\mathfrak{p} - 1)$ of $\chi, \Gamma \Rightarrow \Delta_1, \psi$ where $\Delta_1, \varphi = \Delta_0$. We can thus apply the rule $(\to\mathrm{R})$ to get a proof of $\Gamma \Rightarrow \Delta_1, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

  (d) $r = (\to\mathrm{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\mathrm{L})$$

  Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of $\Gamma_0 \Rightarrow \Delta_0, \chi$ and $\psi, \Gamma_0 \Rightarrow \Delta_0$ where $\Delta_0, \varphi = \Delta$. We can consequently apply the rule to obtain a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_0 \Rightarrow \Delta_0$.

(ii) Let $\mathfrak{p}$ be a proof of $\varphi, \varphi, \Gamma \Rightarrow \Delta$ of height n. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: either $p \in (\{\varphi, \varphi\} \cup \Gamma) \cap \Delta$ or $\bot \in \{\varphi, \varphi\} \cup \Gamma$. In all cases $\varphi, \Gamma \Rightarrow \Delta$ is an instance of an initial sequent.

(b) $r = (\text{wKR})$: then r is

$$\frac{\Gamma_0 \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0}\ (\text{wKR})$$

Either φ is boxed, i.e. there is a χ such that $\varphi = \Box\chi$, and consequently the premise is of the form $\chi, \chi, \Gamma_1 \Rightarrow \psi$, where $\Gamma_1, \chi, \chi = \Gamma_0$. Then we can apply the induction hypothesis to obtain a proof of height less than or equal to $h(\mathfrak{p}-1)$ of $\chi, \Gamma_1 \Rightarrow \psi$. We can then apply the (wKR) rule to obtain a proof of height less than or equal to $h(\mathfrak{p})$ of $\Phi, \Box\chi, \Gamma_1 \Rightarrow \Box\psi, \Delta_0$. Or φ is not boxed and then we can apply (wKR) on the premise to obtain a proof of of height $h(\mathfrak{p})$ of $\Phi_0, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta_0$ where $\Phi_0, \varphi = \Phi$.

(c) $r = (\to\text{R})$: then r is

$$\frac{\chi, \Gamma_0 \Rightarrow \Delta_0, \psi}{\Gamma_0 \Rightarrow \Delta_0, \chi \to \psi}\ (\to\text{R})$$

Then $\{\varphi, \varphi\} \subseteq \Gamma_0$ and we can apply the induction hypothesis on the premise to get a proof of height less than or equal to $h(\mathfrak{p} - 1)$ of $\chi, \Gamma_1 \Rightarrow \Delta_0, \psi$ where $\Gamma_1, \varphi = \Gamma_0$. We can thus apply the rule $(\to\text{R})$ to get a proof of height less than or equal to $h(\mathfrak{p})$ of $\Gamma_1 \Rightarrow \Delta_0, \chi \to \psi$.

(d) $r = (\to\text{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\text{L})$$

Either $\varphi = \chi \to \psi$, and then we use Lemma 4.2.2(i) on both premises to get proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of:

$$\Gamma_1 \Rightarrow \Delta, \chi, \chi \tag{4.9}$$
$$\psi, \Gamma_1 \Rightarrow \Delta, \chi \tag{4.10}$$
$$\psi, \Gamma_1 \Rightarrow \Delta, \chi \tag{4.11}$$
$$\psi, \psi, \Gamma_1 \Rightarrow \Delta \tag{4.12}$$

where $\Gamma_1, \chi \to \psi = \Gamma_0$. Then we can apply the induction hypothesis on the proofs of (12.9) and (12.12) to obtain proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of $\Gamma_1 \Rightarrow \Delta, \chi$ and, respectively, $\psi, \Gamma_1 \Rightarrow \Delta$. It consequently suffices to apply $(\to\text{L})$ on these sequents get a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_1 \Rightarrow \Delta$. Or $\varphi \neq \chi \to \psi$, and then $\{\varphi, \varphi\} \subseteq \Delta_0$. Thus we can apply the induction hypothesis on the premise to get proofs of height less than or equal to $h(\mathfrak{p}-1)$ of $\chi, \Gamma_1 \Rightarrow \Delta$ and $\Gamma_1 \Rightarrow \Delta, \psi$, where $\Gamma_1, \varphi = \Gamma_0$. We can thus apply the rule $(\to\text{L})$ to get a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_1 \Rightarrow \Delta$.

■

Termination of wKS

Our next task is to show that the naive proof-search procedure on wKS, which consists in applying backwards any rule in any order, terminates.

To show termination we need to define a measure on sequents.

Definition 4.2.2. We define:

1. the *number of occurrences of* "$\to$" *imp*$(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$.

2. the *number of occurrences of* "$\Box$" *box*$(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$.

3. the *measure* $\Theta(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$ as

$$\Theta(\Gamma \Rightarrow \Delta) := \mathit{imp}(\Gamma \Rightarrow \Delta) + \mathit{box}(\Gamma \Rightarrow \Delta)$$

This measure is quite simple: it counts the number of → and □ in a sequent $\Gamma \Rightarrow \Delta$, and adds the two numbers. In a nutshell, it is the number of non-nullary operators in a sequent.

We can show that this measure decreases on the usual $<$ ordering on natural numbers, which is well-known to be well-founded, upwards through the rules of wKS. This is the essence of the next lemma.

Lemma 4.2.4. Let s_0 and $s_1, ..., s_n$ be sequents. If there is an instance of a rule r of wKS of the following form, then $\Theta(s_i) < \Theta(s_0)$ for $1 \leq i \leq n$.

$$\frac{s_1 \quad \ldots \quad s_n}{s_0}\ r$$

Proof ✍. We reason by case analysis on r:

1. If r is (IdP) or (IdB) or (⊥L), then we are done as there is no premise.

2. If r is (→R), then it must have the following form.

   $$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\mathrm{R})$$

   Then we get that $\mathit{box}(\Gamma, \varphi \Rightarrow \Delta, \psi) = \mathit{box}(\Gamma \Rightarrow \Delta, \varphi \to \psi)$ but also $\mathit{imp}(\Gamma, \varphi \Rightarrow \Delta, \psi) = \mathit{imp}(\Gamma \Rightarrow \Delta, \varphi \to \psi) - 1$ hence $\Theta(\Gamma, \varphi \Rightarrow \Delta, \psi) < \Theta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$.

3. If r is (→L), then it must have the following form.

   $$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L})$$

   We can easily establish that $\Theta(\Gamma \Rightarrow \Delta, \varphi) < \Theta(\varphi \to \psi, \Gamma \Rightarrow \Delta)$ as one implication symbol is deleted while the number of occurrences of boxes stays the same. We can prove that $\Theta(\psi, \Gamma \Rightarrow \Delta) < \Theta(\varphi \to \psi, \Gamma \Rightarrow \Delta)$ in a similar way.

4. If r is (wKR) then it must have the following form.

   $$\frac{\Gamma \Rightarrow \psi}{\Phi, \Box\Gamma \Rightarrow \Box\psi, \Delta}\ (\mathrm{wKR})$$

   We can straightforwardly show that $\Theta(\Gamma \Rightarrow \psi) < \Theta(\Box\Gamma \Rightarrow \Box\psi)$, as at least one box symbol is deleted, i.e. the one in front of ψ. Consequently, we get that $\Theta(\Gamma \Rightarrow \psi) < \Theta(\Phi, \Box\Gamma \Rightarrow \Box\psi, \Delta)$ as the addition of more formulas to a sequent cannot make the measure decrease.

■

Next, we prove that each sequent has a (finite) list of premises through rules of wKS.

Lemma 4.2.5. For all sequent s there is a list $\mathit{Prems}(s)$ such that for all s', s' is a premise of the conclusion s for an instance of a rule r in wKS iff s' is in $\mathit{Prems}(s)$.

Proof. (⌨) Intuitively, this lemma is rather straightforward to prove: it suffices to check which formula occurrences in a sequent can be principal in a rule, apply the rule backwards, and add the premises thus obtained to the list. As there are finitely many formula occurrences and finitely many finitary rules, we get a finite list of premises. However, in Coq, we need to give more fleshed-out details.

First, in Coq, we *effectively* construct such a list. More precisely, we prove that each rule can be applied in only finitely many ways on a sequent and that we can list all of the premises thus obtained. Once this is done for each rule, we can gather these lists of premises to obtain our list of all premises.

Second, as mentioned previously, while we use sequents built on multisets here, in our formalisation they are built on *lists*. So, we need to pay attention not only to the formula occurrence which is principal in the rule, but also to the location where residues of the principal formula, also called *side formulas*, are placed in the premise. For example, we can apply (→R) backwards on $r \Rightarrow p \to q$ in two ways, giving the premise $r, p \Rightarrow q$ or the premise $p, r \Rightarrow q$. While these sequents are indistinguishable from the multiset perspective, as order is irrelevant in multisets, they are distinct from the list point of view. Our formalisation gives an account of these details.

Still, the idea of the proof in Coq follows the intuition initially given. ■

Conjointly, the two previous lemmas imply the existence of a derivation in wKS of maximum height for all sequent. We formally prove this theorem.

Theorem 4.2.1. Every sequent s has a wKS derivation of maximum height.

Proof. (⌨) We reason by strong induction on the ordered pair $\Theta(s)$. As the applicability of the rules of wKS is decidable, we distinguish two cases:

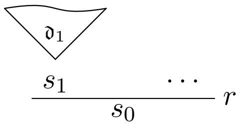
(I) No wKS rule is applicable to s. Then the derivation of maximum height sought after is simply the derivation constituted of s solely, which is the only derivation for s.

(II) Some wKS rule is applicable to s. Either only initial rules are applicable, in which case the derivation of maximum height sought after is simply the derivation of height 1 constituted of the application of the applicable initial rule to s. Or, some other rules than the initial rules are applicable. Then by Lemma 4.2.5 there is a list $Prems(s)$ of all sequents s_0 such that there is an application of a wKS rule r with s as conclusion of r and s_0 as premise of r. By Lemma 4.2.4 we know that every element s_0 in the list $Prems(s)$ is such that $\Theta(s_0) < \Theta(s)$. Consequently, the induction hypothesis allows us to consider the derivation of maximum height of all the sequents in $Prems(s)$. As $Prems(s)$ is finite, there must be an element s_{max} of $Prems(s)$ such that its derivation of maximum height is higher than the derivation of maximum height of all sequents in $Prems(s)$ or of the same height. It thus suffices to pick that s_{max}, use its derivation of maximum height, and apply the appropriate rule to obtain s as a conclusion: this is by choice the derivation of maximum height of s. ■

As the previous lemma implies the existence of a derivation $\mathfrak{d}$ of maximum height in wKS, we are entitled to let $\text{mhd}(s)$ denote the height of $\mathfrak{d}$. We crucially use this measure later on in our cut-admissibility. In fact, the sole property of mhd we use in our proof of cut-admissibility is the following.

Lemma 4.2.6. If r is a rule instance from wKS with conclusion s_0 and s_1 as one of the premises, then $\text{mhd}(s_1) < \text{mhd}(s_0)$.

Proof. (⌨) Suppose that $\text{mhd}(s_1) \geq \text{mhd}(s_0)$. Let $\mathfrak{d}_0$ and $\mathfrak{d}_1$ be the derivations of, respectively, s_0 and s_1 witnessing Theorem 4.2.1. Then the following $\mathfrak{d}_2$ is derivation of s_0 of height $\text{mhd}(s_1) + 1$.



Because of the maximality of $\mathfrak{d}_0$, we get that the height of $\mathfrak{d}_0$ is greater than the height of $\mathfrak{d}_2$, i.e. $\text{mhd}(s_1)+1 \leq \text{mhd}(s_0)$. As our initial assumption implies that $\text{mhd}(s_1)+1 > \text{mhd}(s_0)$, we reached a contradiction. ■

Note that the existence of a derivation of maximum height for each sequent in wKS shows that in the backward application of rules of wKS on a sequent, i.e. the carrying of the naive proof-search procedure, a halting point has to be encountered. As a consequence, the naive proof-search procedure is *terminating.*

Next, we use the notion of mhd, obtained through the termination of wKS, to establish cut-elimination for wKS.

Cut-Elimination for wKS

To prove cut-elimination we first prove through local proof transformations that additive cut, as defined in Definition 4.2.1, is admissible.

Theorem 4.2.2. The additive cut rule is admissible in wKS.

Proof. (⌨) Let $\mathfrak{p}_1$ (with last rule r_1) and $\mathfrak{p}_2$ (with last rule r_2) be proofs in wKS of $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ respectively, as shown below.

$$\dfrac{\mathfrak{p}_1}{\Gamma \Rightarrow \Delta, \varphi}\, r_1 \qquad\qquad \dfrac{\mathfrak{p}_2}{\varphi, \Gamma \Rightarrow \Delta}\, r_2$$

It suffices to show that there is a proof in wKS of $\Gamma \Rightarrow \Delta$. We reason by strong primary induction (PI) on the size of the cut-formula φ, giving the primary inductive hypothesis (PIH), and strong secondary induction (SI) on mhd(s) of the conclusion of a cut, giving the secondary inductive hypothesis (SIH).

There are five cases to consider for r_1: one for each rule in wKS. We separate them by using Roman numerals. The SIH is invoked in all of the following cases: (III-a), (III-b-1), (III-b-2) and (IV).

(I) $\mathbf{r_1}$ =(IdP) : If φ is not principal in r_1, then the latter must have the following form.

$$\dfrac{}{\Gamma_0, p \Rightarrow \Delta_0, p, \varphi}\,(\text{IdP})$$

where $\Gamma_0, p = \Gamma$ and $\Delta_0, p = \Delta$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, p \Rightarrow \Delta_0, p$, and is an instance of an initial sequent. So we are done.

If φ principal in r_1, i.e. $\varphi = p$, then $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, p \Rightarrow \Delta$. Thus, the conclusion of r_2 is of the form $\Gamma_0, p, p \Rightarrow \Delta$. We can consequently apply Lemma 4.2.3 (ii) to obtain a proof of $\Gamma_0, p \Rightarrow \Delta$.

(II) $\mathbf{r_1}$ =(⊥L): Then r_1 must have the following form.

$$\dfrac{}{\Gamma_0, \bot \Rightarrow \Delta, \varphi}\,(\bot\text{L})$$

where $\Gamma_0, \bot = \Gamma$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, \bot \Rightarrow \Delta$, and is an instance of an initial sequent. So we are done.

(III) $\mathbf{r_1}$ =(→R) : We distinguish two cases.
(III-a) If φ is not principal in r_1, then the latter must have the following form.

$$\dfrac{\Gamma, \psi \Rightarrow \Delta_0, \chi, \varphi}{\Gamma \Rightarrow \Delta_0, \psi \to \chi, \varphi}\,(\to\text{R})$$

where $\Delta_0, \psi \to \chi = \Delta$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\Gamma \Rightarrow \Delta_0, \psi \to \chi$ and $\varphi, \Gamma \Rightarrow \Delta_0, \psi \to \chi$. We can apply Lemma 4.2.2 (ii) on the proof of the latter to get a proof of $\varphi, \Gamma, \psi \Rightarrow \Delta_0, \chi$. Thus proceed as follows.

$$\dfrac{\dfrac{\Gamma, \psi \Rightarrow \Delta_0, \chi, \varphi \qquad \varphi, \Gamma, \psi \Rightarrow \Delta_0, \chi}{\Gamma, \psi \Rightarrow \Delta_0, \chi}\,\text{SIH}}{\Gamma \Rightarrow \Delta_0, \psi \to \chi}\,(\to\text{R})$$

Note that the use of SIH is justified here since the last rule in this proof is an instance of (→R) in wKS and hence $\text{mhd}(\Gamma, \psi \Rightarrow \Delta_0, \chi) < \text{mhd}(\Gamma \Rightarrow \Delta_0, \psi \to \chi)$ by Lemma 4.2.6.
(III-b) If φ principal in r_1, i.e. $\varphi = \psi \to \chi$, then r_1 must have the following form.

$$\dfrac{\psi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \psi \to \chi}\ (\to\text{R})$$

The conclusion of r_2 must be of the form $\psi \to \chi, \Gamma \Rightarrow \Delta$. In that case, we distinguish two further cases. In the first case, $\psi \to \chi$ is principal in r_2. Consequently, the latter must have the following form.

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \chi, \Gamma \Rightarrow \Delta}{\psi \to \chi, \Gamma \Rightarrow \Delta}\ (\to\text{L})$$

Proceed as follows.

$$\begin{array}{c} \Gamma \Rightarrow \Delta, \psi \qquad \begin{array}{c} \psi, \Gamma \Rightarrow \Delta, \chi \qquad \begin{array}{c} \chi, \Gamma \Rightarrow \Delta \\ \hdashline \chi, \psi, \Gamma \Rightarrow \Delta \end{array}\ \text{Lem.4.2.1 (ii)} \\ \hdashline \psi, \Gamma \Rightarrow \Delta \end{array}\ \text{PIH} \\ \hdashline \Gamma \Rightarrow \Delta \end{array}\ \text{PIH}$$

In the second case, $\psi \to \chi$ is not principal in r_2. In the cases where r_2 is one of (IdP) and (⊥L) proceed respectively as in **(I)** and **(II)** when the cut-formula is not principal in the rule considered. We are left with the cases where r_2 is one of (→R), (→L) and (wKR).
(III-b-1) If r_2 is (→R) then it must have the following form.

$$\dfrac{\psi \to \chi, \delta, \Gamma \Rightarrow \Delta_0, \gamma}{\psi \to \chi, \Gamma \Rightarrow \Delta_0, \delta \to \gamma}\ (\to\text{R})$$

where $\Delta_0, \delta \to \gamma = \Delta$. In that case, note that the provable sequent $\Gamma \Rightarrow \Delta, \psi \to \chi$ is of the form $\Gamma \Rightarrow \Delta_0, \delta \to \gamma, \psi \to \chi$. We can use Lemma 4.2.2 (ii) on the proof of the latter to get a proof of $\delta, \Gamma \Rightarrow \Delta_0, \gamma, \psi \to \chi$. Proceed as follows.

$$\dfrac{\begin{array}{c} \delta, \Gamma \Rightarrow \Delta_0, \gamma, \psi \to \chi \qquad \psi \to \chi, \delta, \Gamma \Rightarrow \Delta_0, \gamma \\ \hdashline \delta, \Gamma \Rightarrow \Delta_0, \gamma \end{array}\ \text{SIH}}{\Gamma \Rightarrow \Delta_0, \delta \to \gamma}\ (\to\text{R})$$

Note that the use of SIH is justified here as the last rule in this proof is effectively an instance of (→R) in wKS, hence $\text{mhd}(\Gamma, D \Rightarrow \Delta_0, \gamma) < \text{mhd}(\Gamma \Rightarrow \Delta_0, \delta \to \gamma)$ by Lemma 4.2.6.
(III-b-2) If r_2 is (→L) then it must have the following form.

$$\dfrac{\psi \to \chi, \Gamma_0 \Rightarrow \Delta, \delta \qquad \psi \to \chi, \gamma, \Gamma_0 \Rightarrow \Delta}{\psi \to \chi, \delta \to \gamma, \Gamma_0 \Rightarrow \Delta}\ (\to\text{L})$$

where $\Gamma_0, \delta \to \gamma = \Gamma$. In that case, note that the provable sequent $\Gamma \Rightarrow \Delta, \psi \to \chi$ is of the form $\Gamma_0, \delta \to \gamma \Rightarrow \Delta, \psi \to \chi$. We can use Lemma 4.2.2 (i) on the proof of the latter to get proofs of both $\Gamma_0 \Rightarrow \Delta, \delta, \psi \to \chi$ and $\Gamma_0, \gamma \Rightarrow \Delta, \psi \to \chi$. Thus proceed as follows.

$$\dfrac{\begin{array}{c} \Gamma_0 \Rightarrow \Delta, \delta, \psi \to \chi \qquad \psi \to \chi, \Gamma_0 \Rightarrow \Delta, \delta \\ \hdashline \Gamma_0 \Rightarrow \Delta, \delta \end{array}\ \text{SIH} \qquad \begin{array}{c} \Gamma_0, \gamma \Rightarrow \Delta, \psi \to \chi \qquad \psi \to \chi, \gamma, \Gamma_0 \Rightarrow \Delta \\ \hdashline \Gamma_0, \gamma \Rightarrow \Delta \end{array}\ \text{SIH}}{\Gamma_0, \delta \to \gamma \Rightarrow \Delta}\ (\to\text{L})$$

Note that both uses of SIH are justified here as the last rule in this proof is effectively an instance of (→L) in wKS, hence $\text{mhd}(\Gamma_0 \Rightarrow \Delta, \delta) < \text{mhd}(\Gamma_0, \delta \to \gamma \Rightarrow \Delta)$ and $\text{mhd}(\Gamma_0, \gamma \Rightarrow \Delta) < \text{mhd}(\Gamma_0, \delta \to \gamma \Rightarrow \Delta)$ by Lemma 4.2.6.
(III-b-3) If r_2 is (wKR) then it must have the following form.

$$\dfrac{\Gamma_0, \Rightarrow \delta}{\Phi, \psi \to \chi, \Box\Gamma_0 \Rightarrow \Box\delta, \Delta_0}\ (\text{wKR})$$

where $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\delta, \Delta_0 = \Delta$. In that case, note that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\delta, \Delta_0$. To obtain a proof of the latter, we apply the rule (wKR) on the premise of r_2 without weakening $\psi \to \chi$:

$$\dfrac{\Gamma_0 \Rightarrow \delta}{\Phi, \Box\Gamma_0 \Rightarrow \Box\delta, \Delta_0}\ (\mathrm{wKR})$$

(IV) $\mathbf{r_1}$ =(→L): Then r_1 must have the following form.

$$\dfrac{\Gamma_0 \Rightarrow \Delta, \psi, \varphi \qquad \chi, \Gamma_0 \Rightarrow \Delta, \varphi}{\psi \to \chi, \Gamma_0 \Rightarrow \Delta, \varphi}\ (\to\mathrm{L})$$

where $\psi \to \chi, \Gamma_0 = \Gamma$. Thus, we have that the sequents $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\psi \to \chi, \Gamma_0 \Rightarrow \Delta$ and $\varphi, \psi \to \chi, \Gamma_0 \Rightarrow \Delta$. It thus suffices to apply Lemma 4.2.2 (i) on the proof of the latter to obtain proofs of both $\varphi, \Gamma_0 \Rightarrow \Delta, \psi$ and $\varphi, \chi, \Gamma_0 \Rightarrow \Delta$, and then proceed as follows.

$$\dfrac{\dfrac{\Gamma_0 \Rightarrow \Delta, \psi, \varphi \qquad \varphi, \Gamma_0 \Rightarrow \Delta, \psi}{\Gamma_0 \Rightarrow \Delta, \psi}\ \mathrm{SIH} \qquad \dfrac{\chi, \Gamma_0 \Rightarrow \Delta, \varphi \qquad \varphi, \chi, \Gamma_0 \Rightarrow \Delta}{\chi, \Gamma_0 \Rightarrow \Delta}\ \mathrm{SIH}}{\psi \to \chi, \Gamma_0 \Rightarrow \Delta}\ (\to\mathrm{L})$$

Note that both uses of SIH are justified here as the last rule in this proof is effectively an instance of (→L) in wKS, hence $\mathrm{mhd}(\Gamma_0 \Rightarrow \Delta, \psi) < \mathrm{mhd}(\psi \to \chi, \Gamma_0 \Rightarrow \Delta)$ and $\mathrm{mhd}(\chi, \Gamma_0 \Rightarrow \Delta) < \mathrm{mhd}(\psi \to \chi, \Gamma_0 \Rightarrow \Delta)$ by Lemma 4.2.6.

(V) $\mathbf{r_1}$ =(wKR): Then we distinguish two cases.
(V-a) φ is the principal formula in r_1:

$$\dfrac{\Gamma_0 \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Delta}\ (\mathrm{wKR})$$

where $\varphi = \Box\psi$ and $\Phi, \Box\Gamma_0 = \Gamma$. Thus, we have that the sequents $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\Phi, \Box\Gamma_0 \Rightarrow \Delta$ and $\Box\psi, \Phi, \Box\Gamma_0 \Rightarrow \Delta$. We now consider r_2. If r_2 is one of (IdP), (⊥L), (→R) and (→L) then respectively proceed as in (I), (II), (III) and (IV) when the cut-formula is not principal in the rules considered by using SIH. We are consequently left to consider the case when r_2 is (wKR). Then r_2 is of the following form:

$$\dfrac{\psi, \Gamma_0 \Rightarrow \chi}{\Phi, \Box\psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0}\ (\mathrm{wKR})$$

where $\Box\chi, \Delta_0 = \Delta$. Then, proceed as follows.

$$\dfrac{\dfrac{\dfrac{\Gamma_0 \Rightarrow \chi}{\Gamma_0 \Rightarrow \chi, \psi}\ \text{Lem.4.2.1} \qquad \psi, \Gamma_0 \Rightarrow \chi}{\Gamma_0 \Rightarrow \chi}\ \mathrm{PIH}}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_1}\ (\mathrm{wKR})$$

(V-b) φ is not the principal formula in r_1:

$$\dfrac{\Gamma_0 \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \varphi, \Delta_0}\ (\mathrm{wKR})$$

where $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\chi, \Delta_0 = \Delta$. In that case, note that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0$. To obtain a proof of the latter, we apply the rule (wKR) on the premise of r_1 without weakening φ:

$$\dfrac{\Gamma_0 \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Delta_0}\ (\mathrm{wKR})$$

■

The previous operates local transformations on proofs. Thus, we can obtain from this proof a cut-elimination procedure by eliminating topmost cuts first. So, the above proof-theoretically establishes that cuts are eliminable in the calculus wKS extended with (cut), as we show next.

Theorem 4.2.3. The additive cut rule is eliminable from wKS + (cut).

Proof. (⌨) Let $\mathfrak{p}$ be the proof in wKS + (cut) of the sequent s. We prove the statement by induction on the structure of $\mathfrak{p}$. If the last rule applied is a rule in wKS, then it suffices to apply the induction hypothesis on the premises and then the rule. If the last rule applied is (cut), then we use the induction hypothesis on both premises and then Theorem 4.2.2. ■

Thus, in this subsection, we managed to prove cut-elimination for wKS by relying on the mhd measure we obtained through the proof of termination of the naive backward proof-search procedure on wKS. Next, we show how we formalised these results in Coq.

4.2.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

Our formalisation of sequent calculi heavily relies on tools developed by Jeremy E. Dawson and Rajeev Goré [27, 29, 30, 31, 32, 151]. The central notion from their work we need to exhibit is `derrec`. In a nutshell, `derrec` is a function ultimately returning the type of finite trees where nodes are labeled and built according to a specified set of rules, and allowed leaves are explicitly given. We give more details by analyzing the following definition, which is the most complex of this dissertation (⌨).

```
Inductive derrec X (rules : list X -> X -> Type)
  (prems : X -> Type) : X -> Type :=
  | dpI : forall concl ,
    prems concl -> derrec rules prems concl
  | derI : forall ps concl ,
    rules ps concl -> dersrec rules prems ps ->
    derrec rules prems concl
  with dersrec X (rules : list X -> X -> Type)
  (prems : X -> Type) : list X -> Type :=
    | dlNil : dersrec rules prems []
    | dlCons : forall seq seqs ,
      derrec rules prems seq -> dersrec rules prems seqs ->
      dersrec rules prems (seq :: seqs).
```

So, `derrec` is a function taking a type `X`, rules `rules` manipulating elements of `X`, a function `prems` characterizing allowed leaves, and outputs a function of type `X -> Type`. We explain the role of each of these elements. First, the type `X` dictates the type of elements we can label the nodes our finite trees with. For example, if we use `nat`, we can only create finite trees with natural numbers as nodes. Second, `rules` describes the way we can create nodes with children. The type of `rules` is `list X -> X -> Type`, so we can view an element of type `rules` as a usual rule on elements of `X`:

$$\frac{x_0 \quad \dots \quad x_n}{x_{n+1}}$$

With this in mind, we can see that `rules` describes how we can build nodes in the finite trees we are considering. Third, `prems` dictates which elements of `X` can appear in the finite trees as leaves. Given the type `X -> Type` of `prems`, we can roughly think of it as a *characteristic function*: for a given `x` of type `X`, `prems x` outputs an “accepted” or “rejected” value. Thus, to build a leaf labeled with `x` we need the value of `prems x` to be “accepted”.

Consequently, we have a way to build finite trees with nodes labeled by elements of `X` and built according to `rules`, and with leaves labeled by elements described by `prems`. However, it has to be noted that the expression `derrec X rules prems` is of type `X -> Type`. Why is that? The core idea is that for an element `x` of type `X`, the expression `derrec X rules prems x` is the type of all finite trees we described *with root* `x`. It is now rather clear how this general type can be used in the context of sequent calculi.

Now that we have a high-level understanding of what `derrec` is, let us turn to the details of the inductive definition. An element of this definition is striking: we are inductively defining `derrec` by relying on an inductive definition of `dersrec`, which also relies on `derrec`. This complex definition is targeted at defining what are called `mutually defined inductive types` [148]. We explain the constructors of each type individually.

The first constructor of `derrec` is `dpI`, which allows us to construct leaves. Indeed, under the assumption that the element `concl` is an accepted leaf, i.e. `prems concl`, we get that `derrec rules prems concl` (the type `X` is implicit). This means that if the element `concl` is an accepted leaf, then `concl` standing on its own constitutes a finite tree of type `derrec rules prems concl`. This corresponds to the clause (Leaf) of Definition 4.0.2. The second constructor of `derrec` is `derI`, which allows to build trees with complex nodes. Given a lists of elements `ps` and an element `concl`, we can create a tree with a complex node if we have that `ps` and `concl` constitute a rule instance, i.e. `rules ps concl`, and that we are in possession of an appropriate tree for all of the elements in `ps`, i.e. `dersrec rules prems ps`. Under these two conditions, we get a tree with root `concl`, i.e. `derrec rules prems concl`, which clearly corresponds to the clause (Rule) of Definition 4.0.2.

We are thus left to show that `dersrec` formalises the idea that we possess a tree for all the elements of a given list. `dersrec` requires the same inputs as `derrec`, unsurprisingly. The constructor `dlNil` tells us that nothing is required to possess finite trees for all elements of a given list *if this list is empty*. This is why we directly get `dersrec rules prems []`. The second constructor `dlCons` allows us to get the desired trees for a list `seq :: seqs` by requiring that we have a finite tree for `seq`, i.e. `derrec rules prems seq`, as well as trees for all of the elements of `seqs`, i.e. `dersrec rules prems seqs`.

The fact that this definition captures mutually defined inductive types should now appear clearly, as well as the content of each of these types. Quite crucially, to obtain an inhabitant of type `derrec rules prems concl`, one needs to *effectively build* the derivation of that type, appearing as a term in Coq. This is why we are talking about a *deep* embedding of finite trees [31].

Now that we have a very general way of capturing some types of finite trees, we can define the *height* of such trees. The following formalises a slightly modified version of Definition 4.0.4 (⌨).

```
Fixpoint derrec_height X rules prems concl
 (der : @derrec X rules prems concl) :=
 match der with
   | dpI _ _ _ _ => 0
   | derI _ _ ds => S (dersrec_height ds)
 end
 with dersrec_height X rules prems concls
 (ders : @dersrec X rules prems concls) :=
 match ders with
   | dlNil _ _ => 0
   | dlCons d ds => max (derrec_height d) (dersrec_height ds)
 end.
```

We define the height `derrec_height` of a finite tree `der` of type `derrec X rules prems concl` recursively on the structure of this tree. If it is a leaf built using `dpI`, then the height of `der` is `0`. If `der` is built with last step `derI`, i.e. through a complex node, then its height is the height of its list of premises `dersrec_height ds` to which we add one. `dersrec_height` also needs to be defined, and essentially gives `0` to the derivations of the empty list, and takes the maximum out of `derrec_height d` and `dersrec_height ds` for a list of elements `d :: ds`. This definition is nothing but Definition 4.0.4 modified in the initial cases: here we decided to give height 0 to leaves, while there we gave them 1. This difference does not technically matter.

In what follows, we consider finite trees which are derivations manipulating sequents. So, the type we implicitly feed `derrec` with is the type of sequents. Thus, we proceed to define sequents: they are *pairs of lists of formulas* (⌨).

```
Definition Seq := prod (list MPropF) (list MPropF).
```

The type `prod`, for *prod*uct, takes a type `A` and a type `B`, and outputs the type of pairs `(a, b)` with `a` an element of type `A` and `b` an element of type `B`. Consequently, we see that `prod (list MPropF) (list MPropF)` is the type of all pairs of lists of modal formulas. For example, `(Γ0++Γ1 , Δ0++Δ1)` corresponds to $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$. Note that `([φ1 ; φ2 ; φ3] , [ψ1 ; ψ2])` is different from `([φ2 ; φ1 ; φ3] , [ψ1 ; ψ2])` since the order of the elements is crucial, so our lists do not capture multisets (yet).

The sequent calculus wKS

The way we formalise the calculus wKS is rather simple: it is nothing but the collection of its rules. So, the effort lies in defining the rules from Figure 4.1. We start with general remarks on the type of our rules and then present each of them individually.

The type of rules in this setting is `rlsT Seq`, where `rlsT` is the `Type` counterpart of `rls` we presented in the rules of `wKH_rules`. So, `rlsT Seq` is the type in `Type` for pairs of a finite list of sequents and a sequent. We use `Type` here instead of `Prop` as the former contains computational content, on the contrary to the latter. By formalising in `Type` we can thus extract programs from the results we obtain in our formalisation, notably a cut-elimination program. In hindsight, our decision of using `Prop` instead of `Type` for `wKH_rules` was motivated by one thing: in our work on this calculus, we had to use classical logic as our meta-logic, which forces one to give up on the hope to get computationally meaningful results. Here, we do not require classical logic, so we can preserve that hope and use `Type`.

The rule (IdP) of identity on propositional variables is formalised as follows (⌨).

```
Inductive IdPRule : rlsT Seq :=
  | IdPRule_I : forall (P : V) (Γ0 Γ1 Δ0 Δ1 : list MPropF),
        IdPRule [] (Γ0 ++ # P :: Γ1 , Δ0 ++ # P :: Δ1).
```

This rule requires a propositional variable `P` as well as four lists of formulas `Γ0`, `Γ1`, `Δ0` and `Δ1`. This rule is then applicable in the presence of no premises, as shows the empty list `[]`, on a sequent which has a propositional variable `# P` on each of its side, i.e. of the shape `(Γ0 ++ # P :: Γ1 , Δ0 ++ # P :: Δ1)`. Our use of four lists as context is crucial, as it allows us to prove that exchange rules are admissible. Indeed, if the sequent `( Γ0 ++ φ :: # P :: Γ1 , Δ0 ++ # P :: Δ1)` is provable using the rule `IdPRule`, then we can show that the sequent `(Γ0 ++ # P :: φ :: Γ1 , Δ0 ++ # P :: Δ1)` also is using the same rule: it suffices to change the instantiation of the variables for the context around the propositional variable in the rule. More precisely, while in the first sequent the context on the left of the propositional variable in the antecedent is `Γ0 ++ [φ]` and the right context was `Γ1`, in the second second the left is `Γ0` and the right is `φ :: Γ1`, making the rule applicable. Note that if our rule was given in generality on sequents of the shape `(# P :: Γ , Δ0 ++ # P :: Δ1)`, then while `(# P :: [φ] , [# P])` is provable the sequent `(φ :: [# P] , [# P])` is not, as the rule requires the propositional variable on the leftmost position in the list. The admissibility of exchange is thus a feature we obtain through this type of tailoring of our rules.

With these explanations in mind, the formalisation of the rule (⊥L) should be self-explanatory (⌨).

```
Inductive BotLRule : rlsT Seq :=
  | BotLRule_I : forall Γ0 Γ1 Δ,
          BotLRule [] (Γ0 ++ Bot :: Γ1 , Δ).
```

We are thus left with rules with premises: (→R), (→L) and (wKR). We start with the implication rules (⌨).

```
Inductive ImpRRule : rlsT Seq :=
  | ImpRRule_I : forall A B Γ0 Γ1 Δ0 Δ1,
          ImpRRule [(Γ0 ++ A :: Γ1 , Δ0 ++ B :: Δ1)]
                    (Γ0 ++ Γ1 , Δ0 ++ (A --> B) :: Δ1).

Inductive ImpLRule : rlsT Seq :=
  | ImpLRule_I : forall A B Γ0 Γ1 Δ0 Δ1,
          ImpLRule [(Γ0 ++ Γ1 , Δ0 ++ A :: Δ1) ;
                    (Γ0 ++ B :: Γ1 , Δ0 ++ Δ1)]
                    (Γ0 ++ (A --> B) :: Γ1 , Δ0 ++ Δ1).
```

Their reading should be straightforward. As in the previous rules, we tailored these two rules to make sure that exchange is admissible. However, the main difference here is the presence of a list of premises: `[(Γ0 ++ A :: Γ1 , Δ0 ++ B :: Δ1)]` for `ImpRRule` and `[(Γ0 ++ Γ1 , Δ0 ++ A :: Δ1) ; (Γ0 ++ B :: Γ1 , Δ0 ++ Δ1)]` for `ImpLRule`. The premises of the rules (→R) and (→L) are easily recognized.

We finally turn to the more complex (wKR) (⌨).

```
Inductive wKRRule : rlsT Seq :=
  | wKRRule_I : forall A BΓ Γ0 Δ0 Δ1,
          (is_Boxed_list BΓ) ->
          (nobox_gen_ext BΓ Γ0) ->
           wKRRule [(unboxed_list BΓ , [A])]
             (Γ0 , Δ0 ++ Box A :: Δ1).
```

This rule requires a single premise of the shape `(unboxed_list BΓ , [A])` and a conclusion of the shape `(Γ0 , Δ0 ++ Box A :: Δ1)`. While we can recognize rather easily the right-hand sides of the sequents present in the pen-and-paper rule (wKR), the left-hand sides look rather mysterious. In the rule (wKR), the premise of the rule has Γ as left-hand side, and the conclusion has $\Phi, \Box\Gamma$ as left-hand side, where Φ contains no top-level boxed formula. How do we get that `unboxed_list BΓ` corresponds to Γ and `Γ0` corresponds to $\Phi, \Box\Gamma$. Let us first note that we have the assumption `is_Boxed_list BΓ`, ensuring us that `BΓ` is a *list of boxed formulas*. So, by unboxing all the elements of `BΓ` using the function `unboxed_list`, we get a set of formula corresponding to Γ in the rule (wKR). Now that we see that `BΓ` corresponds to $\Box\Gamma$, and `unboxed_list BΓ` corresponds to Γ, how do we add Φ in the left-hand side of the conclusion sequent in our formalisation? To answer this question, we need to define some general relation between lists (⌨).

```
Inductive univ_gen_ext (X : Type) (P : X -> Type) : relationT
    (list X) :=
    univ_gen_ext_nil : univ_gen_ext P [] []
  | univ_gen_ext_cons : forall (x : X) (l le : list X),
           univ_gen_ext P l le ->
           univ_gen_ext P (x :: l) (x :: le)
  | univ_gen_ext_extra : forall (x : X) (l le : list X),
           P x ->
           univ_gen_ext P l le ->
           univ_gen_ext P l (x :: le).
```

With `univ_gen_ext`, we have a relation in `Type` (i.e. `relationT`) on lists of elements of a given type `X` (i.e. `list X`). In essence, the holding of `univ_gen_ext P l0 l1` tells us that the list `l1` of elements of type `X` is generated from `l0` by interleaving elements of type `X` satisfying the property `P`, at the end, or between elements of `l0`. In more details, we can see that the constructor `univ_gen_ext_nil` allows us to say that the empty list satisfies `univ_gen_ext` with respect to itself for any property `P`. The constructor `univ_gen_ext_cons` allows us to add *the same element* in front of two lists which satisfy `univ_gen_ext`. Finally, under the assumption that two lists satisfy `univ_gen_ext`, the constructor `univ_gen_ext_extra` allows us to add an *extra* element to the second list but not to the first, under the condition that this element satisfies the property `P`. So, the first constructor is the starting point, the second constructor makes sure that the list `l0` is "preserved" in `l1`, while the third constructor allows interleaving elements satisfying `P` in `l0`.

Now, we can instantiate `X` with `MPropF` and `P` with the property of not being a boxed formula (⌨).

```
Definition nobox_gen_ext :=
        univ_gen_ext (fun x => (is_boxedT x) -> False).
```

`nobox_gen_ext` is an instance of `univ_gen_ext` with the property `fun x => (is_boxedT x) -> False`, which holds only if `(is_boxedT x) -> False` does, i.e. `x` is not a boxed formula. Consequently, `nobox_gen_ext l0 l1` tells us that the list `l1` is generated from `l0` by interleaving non-boxed formulas in front, at the end, or between elements of `l0`. For example, we have `nobox_gen_ext [P0 ; P1] [(P0 --> P1) ; P0 ; P3 ; P4 ; P1 ; (P1 --> Bot) ; Bot]` because we added `(P0 --> P1)` in front of the initial list, φ3 and φ4 in between the elements φ0 and φ1, and finally `(P1 --> Bot)` and `Bot` at the end of the initial list. Note that none of the formulas we added is a boxed formula.

With these elements in hand, we can explain the correspondence between `Γ0` and Φ, Γ via the assumption `nobox_gen_ext BΓ Γ0`: `Γ0` is nothing but `BΓ` (i.e. the formulas of $\Box\Gamma$) with some additional non-boxed formulas interleaved (i.e. the formulas of Φ).

Why did we bother using such a complex structure, instead of simply separating the boxed and non-boxed formulas? The whole point of our use of `nobox_gen_ext` is to allow *any* interleaving, leading to the admissibility of exchange. Indeed, if we followed the pen-and-paper rule and separated boxed and non-boxed formulas, an application of exchange moving a boxed formula from Γ in the non-boxed component Φ would lead to a potentially unprovable sequent.

As we formalised all our rules, we can finally define the collection of rules of our calculus wKS, which we use to instantiate the variable `rules` in `derrec` (⌨).

```
Inductive wKS_rules : rlsT Seq :=
  | IdP : forall ps c, IdPRule ps c -> wKS_rules ps c
  | BotL : forall ps c, BotLRule ps c -> wKS_rules ps c
  | ImpR : forall ps c, ImpRRule ps c -> wKS_rules ps c
  | ImpL : forall ps c, ImpLRule ps c -> wKS_rules ps c
  | wKR : forall ps c, wKRRule ps c -> wKS_rules ps c.
```

With `derrec`, we can use this collection of rules to define provability and derivability from a set of assumptions. We create a macro `wKS_prv` for provability, but also `wKS_drv` for derivability from the set of all assumptions (⌨).

```
Definition wKS_prv s := derrec wKS_rules (fun _ => False) s.
Definition wKS_drv s := derrec wKS_rules (fun _ => True) s.
```

In `wKS_prv`, the set of allowed leaves is described but the function `(fun _ => False)`. As a consequence, any derivation of type `wKS_prv s` cannot have sequents as leaves: any attempt to create such a leaf will output `False` because of the given function. We can thus see that `wKS_prv` expresses *provability* as defined in Definition 4.0.3. Similarly, we can see that `wKS_drv` expresses derivability defined in Definition 4.0.2, i.e. from the set of all possible sequents: the function `(fun _ => True)` allows us to use any sequent we would like as leaf.

With our calculus and these macros defined, we can turn to the formalisation of our results about wKS.

Properties of wKS

An early result we can obtain is the decidability of the applicability of rules from wKS. This result was used in the proof of Lemma 4.2.1 (⌨).

```
Lemma dec_wKS_rules : forall (concl :Seq) ,
      ((existsT2 prems, wKS_rules prems concl) -> False) +
      (existsT2 prems, wKS_rules prems concl).
```

For any sequent `concl`, we can determine whether there exists a an application of a rule in wKS with list of premises `prems`, i.e. `existsT2 prems, wKS_rules prems concl`, or not, i.e. `(existsT2 prems, wKS_rules prems concl) -> False`. Note that in this context the symbol `+` denotes type sum. We can prove this result by showing that for each rule we can determine whether this rule is applicable on `concl`. While tedious, it is rather straightforward.

Next, we consider the exchange rules. As our formalisation used sequents based on lists and not multisets, we had to make sure that exchange was admissible so that our list-sequents would indeed mimic multiset-sequents (⌨).

```
Inductive list_exch_L : relationT Seq :=
  | list_exch_LI Γ0 Γ1 Γ2 Γ3 Γ4 Δ : list_exch_L
      (Γ0 ++ Γ1 ++ Γ2 ++ Γ3 ++ Γ4, Δ)
      (Γ0 ++ Γ3 ++ Γ2 ++ Γ1 ++ Γ4, Δ).

Inductive list_exch_R : relationT Seq :=
  | list_exch_RI Γ Δ0 Δ1 Δ2 Δ3 Δ4 : list_exch_R
        (Γ, Δ0 ++ Δ1 ++ Δ2 ++ Δ3 ++ Δ4)
        (Γ, Δ0 ++ Δ3 ++ Δ2 ++ Δ1 ++ Δ4).
```

In the definition of `list_exch_L`, it should appear clearly that the second sequent is nothing but the first sequent with two lists of formulas on the left-hand side, i.e. `Γ1` and lstinlineΓ3, swapped. Note that we decided to consider a version of exchange that moves lists of formulas and not only formulas as it is more convenient to manipulate. The usual notion of exchange, swapping simply two formulas, is an instance of our formalisation. The property `list_exch_R` can be understood similarly.

So, we can formalise the result of admissibility of exchange rules as follows (⌨) (⌨).

```
Theorem wKS_adm_list_exch_L : forall s,
        (wKS_prv s) ->
        (forall se, list_exch_L s se ->
        wKS_prv se).
```

```
Theorem wKS_adm_list_exch_R : forall s,
        (wKS_prv s) ->
        (forall se, list_exch_R s se ->
        wKS_prv se).
```

We only analyze `wKS_adm_list_exch_L` as `wKS_adm_list_exch_R` can be explained in a similar way. This states that if a sequent `s` is provable in wKS, i.e. `wKS_prv s`, then any sequent `se` which is an exchanged version of `s`, i.e. `list_exch_L s se`, is also provable: `wKS_prv se`.

As we explained above, the admissibility of exchange is not an accident, nor is it hard-wired as an explicit rule in Coq. That is, our encoding of the multiset-based rules shown in Figure **??** is designed to entail exchange. The list-encoding requires a very pedantic analysis of the position of the occurrence of the principal formula, notably, in rule instances. While this is a major disadvantage of our approach, this general framework allows us to formalise sequent calculus for logics that require the presence of lists, like the Lambek calculus [87].

Let us now turn to the formalisation of weakening rules. Here again, we need properties similar to `list_exch_L` and `list_exch_R` (⌨).

```
Inductive wkn_L (φ : MPropF) : relationT Seq :=
  | wkn_LI Γ0 Γ1 Δ  : wkn_L φ
        (Γ0 ++ Γ1, Δ) (Γ0 ++ φ :: Γ1, Δ).

Inductive wkn_R (φ : MPropF) : relationT Seq :=
  | wkn_RI Γ  Δ0 Δ1 : wkn_R φ
        (Γ , Δ0 ++ Δ1) (Γ , Δ0 ++ φ :: Δ1).
```

These properties allow to express the fact that the second sequent is nothing but the first sequent weakened, on the left or the right, with one formula φ. The height-preserving admissibility of weakening rules is thus expressed as follows (⌨) (⌨).

```
Theorem wKS_hpadm_wkn_L : forall s (D0 : wKS_prv s),
          (forall sw φ, ((wkn_L φ s sw) ->
          existsT2 (D1 : wKS_prv sw),
          derrec_height D1 <= derrec_height D0)).
```

```
Theorem wKS_hpadm_wkn_R : forall s (D0 : wKS_prv s),
          (forall sw φ, ((wkn_R φ s sw) ->
          existsT2 (D1 : wKS_prv sw),
          derrec_height D1 <= derrec_height D0)).
```

We focus on `wKS_hpadm_wkn_L`. Given a sequent `s` and a proof `D0` of `s`, this theorem states that for any sequent `sw` which is a weakening on the left of the sequent `s`, i.e. `wkn_L φ s sw`, there exists a proof `D1` of `sw` which has its height lesser or equal to the height of `D0`, i.e. `derrec_height D1 <= derrec_height D0`. Note that to be able to talk about the height of the proof of `s`, we had to give it a name `D0`. To do so, we switched the assumption `wKS_prv s` under the quantifier `forall`, allowing us to name the given derivation.

The invertibility of the implication rules (→L) and (→R) is next. Their reading should be straightforward by now, so we do not comment on them (⌨) (⌨).

```
Theorem ImpL_inv : forall concl prem1 prem2,
        (wKS_prv concl) ->
        (ImpLRule [prem1;prem2] concl) ->
          (wKS_prv prem1) *
          (wKS_prv prem2).
```

```
Theorem ImpR_inv : forall concl prem, (wKS_prv concl) ->
        (ImpRRule [prem] concl) ->
        (wKS_prv prem).
```

Finally, we exhibit our formalisation about the contraction rules. Similarly to exchange and weakening, we define properties on sequent showing that one is the contracted version of the other (⌨).

```
Inductive ctr_L (φ : MPropF) : relationT Seq :=
  | ctr_LI Γ0 Γ1 Γ2 Δ : ctr_L φ
        (Γ0 ++ φ :: Γ1 ++ φ :: Γ2, Δ)
        (Γ0 ++ φ :: Γ1 ++ Γ2, Δ).

Inductive ctr_R (φ : MPropF) : relationT Seq :=
  | ctr_RI Γ Δ0 Δ1 Δ2 : ctr_R φ
        (Γ, Δ0 ++ φ :: Δ1 ++ φ :: Δ2)
        (Γ, Δ0 ++ φ :: Δ1 ++ Δ2).
```

The second sequent is a contracted version of the first, where the contracted formula is φ. The design of these properties allows to contract two formulas occurrences into one *wherever* these are standing on one side of the sequent. The height-preserving admissibility of these rules can then straightforwardly be expressed as follows, where `(ctr_L φ s sc ) + (ctr_R φ s sc)` implies that `sc` is a contracted version of `s` on the right *or* on the left (⌨).

```
Theorem wKS_hpadm_ctr_LR : forall s (D0 : wKS_prv s),
          (forall sc φ, ((ctr_L φ s sc) + (ctr_R φ s sc)) ->
          existsT2 (D1 : wKS_prv sc),
          derrec_height D1 <= derrec_height D0).
```

Next, we give the formalisation of our results about the termination of wKS.

Termination of wKS

To show the termination of the naive backward proof-search strategy on wKS, we defined in Definition 4.2.2 a measure Θ on sequents. We proceed to formalise this measure.

First, we need to give an account of the number of occurrences of the symbol → in a sequent, which constitutes one part of the measure. To count the number of such occurrences in a sequent, we need to be able to do so in a formula (⌨).

```
Fixpoint n_imp_subformF (φ : MPropF) : nat :=
match φ with
 | # P => 0
 | Bot => 0
 | ψ --> χ => 1 + (n_imp_subformF ψ) + (n_imp_subformF χ)
 | Box ψ => (n_imp_subformF ψ)
end.
```

Quite obviously, on input φ the function `n_imp_subformF` returns the number of occurrences of → in φ, a `nat`. It does so by adding 1 every time an implication symbol is encountered in the structure of the formula. Thus, we can extend this counting to lists of formulas (⌨).

```
Fixpoint n_imp_subformLF (l : list MPropF) : nat :=
match l with
  | [] => 0
  | h :: t => (n_imp_subformF h) + (n_imp_subformLF t)
end.
```

The function `n_imp_subformLF` crucially relies on the function `n_imp_subformF` in the inductive case, by counting the number of implication symbols in the formula `h`. We can now define the measure *imp* from Definition 4.2.2 (⌨).

```
Definition n_imp_subformS (s : Seq) : nat :=
    (n_imp_subformLF (fst s)) + (n_imp_subformLF (snd s)).
```

Given a sequent `s`, which is a pair of lists of formulas, `n_imp_subformS` adds up the number of occurrences of → in the left component of `s`, i.e. `n_imp_subformLF (fst s)`, with the number of such occurrences in the right component, i.e. `n_imp_subformLF (snd s)`. This is capturing the desired measure.

We can do a similar trick for counting the number of boxes in a sequent, i.e. the measure *box* from Definition 4.2.2 (⌨) (⌨) (⌨).

```
Fixpoint n_box_subformF (φ : MPropF) : nat :=
match φ with
 | # P => 0
 | Bot => 0
```

```
 | ψ --> χ => (n_box_subformF ψ) + (n_box_subformF χ)
 | Box ψ => 1 + (n_box_subformF ψ)
end.

Fixpoint n_box_subformLF (l : list MPropF) : nat :=
match l with
  | [] => 0
  | h :: t => (n_box_subformF h) + (n_box_subformLF t)
end.

Definition n_box_subformS (s : Seq) : nat :=
    (n_box_subformLF (fst s)) + (n_box_subformLF (snd s)).
```

With `n_imp_subformS` and `n_box_subformS`, we can easily define the measure Θ from Definition 4.2.2, which simply adds the two previously defined measures (⌨).

```
Definition term_meas (s : Seq) : nat :=
  (n_imp_subformS s) + (n_box_subformS s).
```

To show that naive backward proof-search terminates using this measure, we need to show that *whichever* rule we decide to apply backwards on a sequent makes the measure decrease. So, we need to consider all possible premises of a sequent, through any rule we have. The next lemma expresses that the set of such premises is finite, as it can be represented in a list. Consequently, it is a formalisation of Lemma 4.2.5 (⌨).

```
Lemma finite_premises_of_S : forall (s : Seq),
  existsT2 listprems, (forall prems,
    ((wKS_rules prems s) -> (InT prems listprems)) *
    ((InT prems listprems) -> (wKS_rules prems s))).
```

Any sequent is thus ensured to have a list `listprems` of *lists of sequents* (recall that our rules take list of premises as input) such that the following are equivalent: (1) a list of sequents `prems` with `s` constitute a rule application in wKS, i.e. `wKS_rules prems s` ; (2) `prems` is in `listprems`, i.e. `InT prem prems`. Note that we used `InT` here instead of `In`, i.e. the computationally-loaded `Type` version of the latter. In a similar way, we used `existsT2`, which is a `Type` version of `exists`. Thus, the above lemma allows us to effectively compute this list of all (list of) premises. We use that fact to define the list of all premises of a sequent (⌨).

```
Definition list_of_premises (s : Seq) : list Seq :=
         flatten_list (proj1_sigT2 (finite_premises_of_S s)).
```

The lemma `finite_premises_of_S`, when applied on `s`, is a statement of existence through `existsT2`. When holding, such a statement gives us two things: a witness, and a proof that this witness satisfies the property expressed after `existsT2`. So, behind the quantifier `existsT2` lies a pair, on which we can apply projections. Thus, the expression `proj1_sigT2 (finite_premises_of_S s)` is nothing but the first projection of the lemma `finite_premises_of_S` applied to `s`. In other words, it is the witness, the list of all lists of premises of `s`. However, for convenience, we would like to obtain a list of sequents, instead of having a list of lists of sequents. So, we simply flatten this list by using `flatten_list`. We are now able to manipulate for any sequent its list of all premises.

In our pen-and-paper proof, we proved as a separate statement (Lemma 4.2.4) the decreasing of the measure Θ in any rule application. However, in the formalisation, we embedded this proof in the proof of the existence of a derivation of maximal height for sequents. Thus, we turn to the formalisation of Theorem 4.2.1. For convenience, we first define the property of being a derivation of maximal height for a sequent (⌨).

```
Definition is_mhd (s: Seq) (D0 : wKS_drv s): Prop :=
```

```
      forall (D1 : wKS_drv s), derrec_height D1 <=
   derrec_height D0.
```

We have that `is_mhd s D0`, for `D0` a proof of `s`, holds if for any proof `D1` of `s` we have that `derrec_height D1 <= derrec_height D0`. This clearly expresses the intended property. We can now prove that there exists such a derivation for any sequent (⌨).

```
Theorem wKS_termin : forall s,
        existsT2 (DMax : wKS_drv s), (is_mhd s DMax).
```

Here again, we make use of the existential quantifier `existsT2`, thus giving us a witness which we can extract. We can consequently extract this witness, and check its height, thus giving us for a sequent `s` the height of the derivation of maximal height for `s`. This is the measure mhd, which we formalise next (⌨).

```
Definition mhd (s: Seq) : nat :=
        derrec_height (proj1_sigT2 (@wKS_termin s)).
```

As clearly expressed, `mhd s` is a natural number. We can then show that this measure decreases on the usual order on natural numbers $<$ upwards in rule applications of the calculus wKS (⌨).

```
Theorem RA_mhd_decreases : forall prems concl,
  (wKS_rules prems concl) ->
  (forall prem, (In prem prems) -> (mhd prem) < (mhd concl)).
```

We proceeded to show how we formalised the measure mhd and exhibited the formalisation of the only property we use about this measure, expressed in Lemma 4.2.6. We can finally turn to the formalisation of our results on the elimination of additive cuts.

Cut-Elimination for wKS

To reach cut-elimination we first proved cut-admissibility. The latter is formalised as follows (⌨).

```
Theorem wKS_cut_adm : forall φ Γ0 Γ1 Δ0 Δ1,
      (wKS_prv (Γ0 ++ Γ1, Δ0 ++ φ :: Δ1)) ->
      (wKS_prv (Γ0 ++ φ :: Γ1, Δ0 ++ Δ1)) ->
      (wKS_prv (Γ0 ++ Γ1, Δ0 ++ Δ1)).
```

The two premises of the additive cut rule are recognized in the second and third lines. Note however that in the above code, the cut rule does not exist as such: it has not been defined. To obtain a result of cut-*elimination*, we need to be able to talk about the calculus wKS with cut. Indeed, cut-elimination is a result showing that we can take a proof built with cut as a rule, and transform it into a proof without cut. So, we need to formalise this rule (⌨).

```
Inductive CutRule : rlsT Seq :=
 | CutRule_I : forall φ Γ0 Γ1 Δ0 Δ1,
    CutRule
   [(Γ0 ++ Γ1, Δ0 ++ φ :: Δ1) ; (Γ0 ++ φ :: Γ1, Δ0 ++ Δ1)]
                    (Γ0 ++ Γ1, Δ0 ++ Δ1).
```

The layout of the above is quite suggestive: we easily recognize the two premises of the additive cut rule with its conclusion. With this rule defined, we can thus define the set of rules for calculus wKS + (cut) (⌨).

```
Inductive wKS_cut_rules : rlsT  Seq :=
  | wKS_woc : forall ps c, wKS_rules ps c ->
                 wKS_cut_rules ps c
  | wKS_cut : forall ps c, CutRule ps c ->
                 wKS_cut_rules ps c.
```

The first constructor `wKS_woc` allows to use the already defined set of rules of wKS *w*ith*o*ut *c*ut. In addition to that, the second constructor `wKS_cut` allows us to use the cut rule. For convenience, we define a macro for provability in this calculus containing the cut rule (⌨).

```
Definition wKS_cut_prv s :=
        derrec wKS_cut_rules (fun _ => False) s.
```

Finally, we use this macro to formalise Theorem 4.2.3: if we have a proof of a sequent in wKS + (cut), then we have a proof of this sequent in wKS (⌨).

```
Theorem wKS_cut_elimination : forall s,
        (wKS_cut_prv s) -> (wKS_prv s).
```

This theorem only relies on expressions in `Type`: the sequent `s`, the provability of `s` in the cut-containing calculus `wKS_cut_prv s`, and the provability of `s` in the cut-free calculus `wKS_prv s`. Given its shape, and its proof crucially using the admissibility of cut (also in `Type`), we can extract a Haskell automatically using one of Coq's features (⌨). This code is certainly far from optimal, but constitutes an automatically generated program extracted from a formally verified proof.

Chapter 5

Kripke Semantics

Another way to capture a logic is through what is called a *semantics*. While we capture logics through the notion of *proof* in proof systems, by essentially relying on rules conferring meaning to logical symbols, with semantics we grasp consequence relations via the notion of *truth* in mathematical structures we call *models*.

The types of mathematical structures used in a semantics are legion: truth tables, topological spaces [144], subset spaces [28], neighborhood frames [98, 131], algebras [12], categories [88], Kripke frames [8, 73, 85]. All of these are not equivalent, as some allow to capture particular logics while others cannot.

We restrict our attention to Kripke frames, which constitute the core elements of what is canonically called *Kripke semantics* [8, 73, 85]. While this type of semantics is named after Saul Kripke, Arnould Bayart and Jaako Hintikka also deserve credit as they independently came up with a similar idea [8, 73]. Kripke semantics are philosophically grounded on the idea of possible worlds, i.e. alternative worlds to ours. In a nutshell, these worlds allow the evaluation of modal statements: "X is necessary" if X holds in all possible worlds, while "X is possible" if X holds in one possible world.

From a mathematical point of view, Kripke semantics are based on Kripke frames. In essence, the latter are directed multigraphs: a collection of vertices connected by edges of several types.

Definition 5.0.1. A *Kripke frame* $\mathcal{F}$ *is a pair* $(W, \{R_i\}_{i\in\mathbb{N}})$ *where* W *is a non-empty set and* $R_i \subseteq W \times W$ *for all* $i \in \mathbb{N}$. *We call elements of* W *points and elements of* $\{R_i\}_{i\in\mathbb{N}}$ *accessibility relations.*

Crucially, we can impose various restrictions on frames and thus obtain classes of frames satisfying these restrictions. More precisely, we can restrict frames in two ways. First, we can impose restrictions on the set of points W of a frame. For example, we can restrict the cardinality of W, ask that equality is decidable on it, or demand that $W = \mathbb{N}$. Second, we can restrict accessibility relations in various ways. Commonly, an accessibility relation R is required to satisfy some properties like reflexivity (i.e. $\forall x.xRx$), symmetry (i.e. $\forall x.\forall y.(xRy \to yRx)$), transitivity (i.e. $\forall x.\forall y.\forall z.(xRy \to yRz \to xRz)$) or totality (i.e. $\forall x.\forall y.(xRy \vee yRx)$).

Throughout this dissertation, we focus on Kripke frames with at most two accessibility relations.

In the remaining of this chapter, we present propositional and first-order variants of such semantics.

5.1 Propositional

Here we define Kripke semantics for propositional languages in a general way.

As mentioned above, semantics rely on models. So, we define Kripke models using Kripke frames defined in Definition 5.0.1.

Definition 5.1.1. Let $\mathbb{L} = (\mathbb{V}, \mathcal{C})$ be a propositional language. A *Kripke model* $\mathcal{M}$ for $\mathbb{L}$ is a tuple $(W, \{R_i\}_{i\in\mathbb{N}}, I)$, where $(W, \{R_i\}_{i\in\mathbb{N}})$ is a Kripke frame and $I : \mathbb{V} \to \mathsf{Pow}(W)$ is an *interpretation function* or *valuation*.

In a similar way that we restricted Kripke frames, we can here obtain specific classes of models by imposing restrictions on interpretation functions. The sole restriction on such functions we consider in this dissertation is called *persistence* and pertains to the preservation of the interpretation of propositional variables through accessibility relations. More precisely, the interpretation function I is persistent on the accessibility relation R in the model $\mathcal{M} = (W, R, I)$ if the following is satisfied: for every $w, v \in W$ with Rwv and $p \in \mathbb{V}$, if $w \in I(p)$ then $v \in I(p)$.

Then, to obtain a semantics for a propositional language $\mathbb{L} = (\mathbb{V}, \mathcal{C})$, we need to define a *forcing relation* $\mathcal{M}, w \Vdash \varphi$ where $\mathcal{M} = (W, R, I)$ is a model, $w \in W$ and $\varphi \in Form_{\mathbb{L}}$. Canonically, this relation is defined recursively on the structure of φ with one clause for each connective in $\mathcal{C}$, determining its semantic behavior, and as base case at least the following.

$$\mathcal{M}, w \Vdash p \quad \text{iff} \quad w \in I(p)$$

The forcing relation expresses the notion of *truth*, crucial to the semantic approach to logics. More precisely, the holding of $\mathcal{M}, w \Vdash \varphi$ expresses the fact that φ is true in the point w in the model $\mathcal{M}$.

Next, we introduce some notations related to the forcing relation.

Definition 5.1.2. Given $\Gamma \cup \{\varphi\} \subseteq Form_{\mathbb{L}}$ and a model $\mathcal{M} = (W, R, I)$ for $\mathbb{L}$, we define the following.

- $\mathcal{M}, w \nVdash \varphi$ if it is not the case that $\mathcal{M}, w \Vdash \varphi$.
- $\mathcal{M} \Vdash \varphi$ if for every point $w \in W$, $\mathcal{M}, w \Vdash \varphi$.
- $\mathcal{M}, w \Vdash \Gamma$ for a given $w \in W$ if for every $\gamma \in \Gamma$ we have $\mathcal{M}, w \Vdash \gamma$. In this case, we say w is a Γ-point.
- $\mathcal{M} \Vdash \Gamma$ if for every $\gamma \in \Gamma$, $\mathcal{M}, w \Vdash \gamma$. In this case, we say that $\mathcal{M}$ is a Γ-model.

Kripke semantics are traditionally used to define sets of formulas: the formulas *valid* in a given class of frames or models. More formally, we say that φ is valid, and write $\models \varphi$, if we have $\mathcal{M} \Vdash \varphi$ for all models $\mathcal{M}$ obeying the appropriate restrictions of the class considered. We can thus define the set of validities of a semantics with the set $\{\varphi \in Form_{\mathbb{L}} \mid \ \models \varphi\}$.

Traditionally, this set is intended to capture a logic *understood as a set of formulas.* However, as explained in Chapter 3, in this dissertation we consider logics to be *consequence relations.* So, we develop notions of semantic consequence relations using Kripke semantics.

Definition 5.1.3. The $\underline{\text{l}}$ocal and $\underline{\text{g}}$lobal consequence relations are as below:

$$\begin{array}{lll} \Gamma \models_l \varphi & \text{iff} & \forall \mathcal{M}. \forall w. (\mathcal{M}, w \Vdash \Gamma \ \Rightarrow \ \mathcal{M}, w \Vdash \varphi) \\ \Gamma \models_g \varphi & \text{iff} & \forall \mathcal{M}. (\mathcal{M} \Vdash \Gamma \ \Rightarrow \ \mathcal{M} \Vdash \varphi). \end{array}$$

With these notions in hand, we can then also capture logics understood as consequence relations with the sets $\{(\Gamma, \varphi) \in (\mathsf{Pow}(Form_{\mathbb{L}}) \times Form_{\mathbb{L}}) \mid \Gamma \models_l \varphi\}$ and $\{(\Gamma, \varphi) \in (\mathsf{Pow}(Form_{\mathbb{L}}) \times Form_{\mathbb{L}}) \mid \Gamma \models_g \varphi\}$ on given classes of frames or models.

We close our general presentation of Kripke semantics for propositional languages by noting that in some instances, we extend the local and global consequence relations to the following more general ones.

$$\begin{array}{lll} \Gamma \models_l \Delta & \text{iff} & \forall \mathcal{M}. \forall w. (\mathcal{M}, w \Vdash \Gamma \ \Rightarrow \ \exists \delta \in \Delta. \mathcal{M}, w \Vdash \delta) \\ \Gamma \models_g \Delta & \text{iff} & \forall \mathcal{M}. (\mathcal{M} \Vdash \Gamma \ \Rightarrow \ \forall w \in W. \exists \delta \in \Delta. \mathcal{M}, w \Vdash \delta). \end{array}$$

5.1.1 The classical modal logic example: on paper

The forcing relation for propositional classical modal logics is usually defined as follows.

Definition 5.1.4. Given a Kripke model $\mathcal{M} = (W, R, I)$, we define the forcing relation as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash p & \text{iff} & w \in I(p) \\ \mathcal{M}, w \Vdash \bot & & \text{never} \\ \mathcal{M}, w \Vdash \varphi \to \psi & \text{iff} & \text{if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \text{for all } v \text{ such that } wRv \text{ we have } \mathcal{M}, v \Vdash \varphi \end{array}$$

Later on we consider the class of all models for classical modal logic. In particular, we will show that the local (resp. global) consequence relation on that class, i.e. the set $\{(\Gamma, \varphi) \mid \Gamma \models_l \varphi\}$ (resp. $\{(\Gamma, \varphi) \mid \Gamma \models_g \varphi\}$), is identical to the logic $\mathsf{wK}\mathbf{L}$ (resp. $\mathsf{sK}\mathbf{L}$).

A central notion in the model theory of Kripke semantics is the notion of bisimulation [11, Sect.2.2]. While it relies on the language at stake, it relates models with similar structures, using a specific notion of "similarity". We define below the canonical notion of bisimulation when considering the language $\mathbb{L}_{\mathbf{CM}}$.

Definition 5.1.5. Let $\mathcal{M}_1 = (W_1, R_1, I_1)$ and $\mathcal{M}_2 = (W_2, R_2, I_2)$ be models. A *bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a relation $B \subseteq W_1 \times W_2$ such that for all $w_1 \in W_1$ and $w_2 \in W_2$, if $w_1 B w_2$ we have:

(B_1) for all $p \in \mathbb{V}$, $w_1 \in I_1(p)$ iff $w_2 \in I_2(p)$;

(B_2) for all $v_2 \in W_2$, if $w_2 R_2 v_2$ then there exists $v_1 \in W_1$ such that $w_1 R_1 v_1$ and $v_1 B v_2$;

(B_3) for all $v_1 \in W_1$, if $w_1 R_1 v_1$ then there exists $v_2 \in W_2$ such that $w_2 R_2 v_2$ and $v_1 B v_2$.

If there is a bisimulation B such that $w_1 B w_2$, we say that w_1 and w_2 are *bisimilar* and note $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$.

This notion is usually a sought-after one, as it often entails *logical equivalence* for points in models: two bisimilar points force the same formulas. This property, which will be of crucial use later on, is obtained with the above definition.

Proposition 5.1.1. Let $\mathcal{M}_1 = (W_1, R_1, I_1)$ and $\mathcal{M}_2 = (W_2, R_2, I_2)$ be models, $w_1 \in W_1$ and $w_2 \in W_2$. If $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$, then for all $\varphi \in Form_{\mathbb{L}_{\mathbf{CM}}}$ we have $\mathcal{M}_1, w_1 \Vdash \varphi$ iff $\mathcal{M}_2, w_2 \Vdash \varphi$.

Proof. (⌨) We reason by induction on the structure of φ.

- $\varphi := p$: As $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$ we notably have $w_1 \in I_1(p)$ iff $w_2 \in I_2(p)$ by item (B_1) of Definition 5.1.5. So, we get that $\mathcal{M}_1, w_1 \Vdash p$ iff $\mathcal{M}_2, w_2 \Vdash p$.

- $\varphi := \bot$: We have $\mathcal{M}_1, w_1 \nVdash \bot$ and $\mathcal{M}_2, w_2 \nVdash \bot$ by definition. So we trivially get the desired result.

- $\varphi := \chi \to \psi$: ($\Rightarrow$) Assume $\mathcal{M}_1, w_1 \Vdash \chi \to \psi$. We need to show $\mathcal{M}_2, w_2 \Vdash \chi \to \psi$. Assume that $\mathcal{M}_2, w_2 \Vdash \chi$. We need to show $\mathcal{M}_2, w_2 \Vdash \psi$. As $\mathcal{M}_1, w_1 \Vdash \chi \to \psi$ we get that if $\mathcal{M}_1, w_1 \Vdash \chi$ then $\mathcal{M}_1, w_1 \Vdash \psi$. But we know that $\mathcal{M}_2, w_2 \Vdash \chi$, so by $\mathcal{M}_1, w_1 \underline{\leftrightarrow} \mathcal{M}_2, w_2$ and induction hypothesis we get $\mathcal{M}_1, w_1 \Vdash \chi$. Thus, we get $\mathcal{M}_1, w_1 \Vdash \psi$. It then suffices to use the induction hypothesis again to obtain $\mathcal{M}_2, w_2 \Vdash \psi$. Consequently, we proved that $\mathcal{M}_2, w_2 \Vdash \chi \to \psi$.

  ($\Leftarrow$) We proceed similarly to ($\Rightarrow$).

- $\varphi := \Box\psi$: ($\Rightarrow$) Assume $\mathcal{M}_1, w_1 \Vdash \Box\psi$. We need to show $\mathcal{M}_2, w_2 \Vdash \Box\psi$. Let $v_2 \in W_2$ such that $w_2 R_2 v_2$. By $w_1 B w_2$ and item (B_2) of Definition 5.1.5 we get that there is a $v_1 \in W_1$ such that $w_1 R_1 v_1$ and $v_1 B w_2$. As $w_1 R_1 v_1$ and $\mathcal{M}_1, w_1 \Vdash \Box\psi$ we get $\mathcal{M}_1, v_1 \Vdash \psi$. By induction hypothesis and $v_1 B v_2$, we get $\mathcal{M}_2, w_2 \Vdash \psi$. As v_2 is arbitrary, we get $\mathcal{M}_2, w_2 \Vdash \Box\psi$.

  ($\Leftarrow$) We proceed similarly to ($\Rightarrow$), using the item (B_3) of Definition 5.1.5 instead.

■

5.1.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

In the above, to obtain a Kripke semantics for the language $\mathbb{L}_{\mathbf{CM}}$ we first defined models. We follow this first step by formalising models as follows (⌨).

```
Class kmodel :=
  {
    nodes : Type ;
    reachable : nodes -> nodes -> Prop ;
    val : nodes -> V -> Prop ;
  }.
```

A `kmodel` is an entity having a collection of `nodes` (or worlds), a relation `reachable` on them, and a valuation function `val`. Then, it appears that a model is of the shape (W, R, I), as expected.

We can turn to the definition of forcing (⌨).

```
Fixpoint wforces (M : kmodel) w (φ : MPropF) : Prop :=
      match φ with
      | # p => val w p
      | Bot => False
      | ψ --> χ => (wforces M w ψ) -> (wforces M w χ)
      | Box ψ => forall v, reachable w v -> wforces M v ψ
      end.
```

As we did in Definition 5.1.4, we define the notion of `wforces` (for *world*-force) relying on the presence of a model `M`, a world `w` and a formula φ. The definition goes by recursion on the structure of φ, and follows Definition 5.1.4. Indeed, we have that `# p` is forced in `w` if `val w p` holds: the valuation describes `w` as a `p`-world. In addition to that, the semantics of the operator $\Box$ is faithfully formalised as to force `Box` ψ in `w` we rely on the forcing of ψ by all successor of `w`, using `reachable w v` for all `v`.

We can also define a notion of forcing *for all points in a model* rather straightforwardly (⌨).

```
Definition mforces M φ : Prop := forall w , wforces M w φ.
```

The definition above is helpful to give a compact definition of the global semantic consequence relation. We formalise the latter and its local version in the definition below (⌨).

```
Definition loc_conseq Γ φ := forall M w,
  (forall ψ, (In _ Γ ψ) -> wforces M w ψ) ->
  (wforces M w φ).

Definition glob_conseq Γ φ := forall M,
  (forall ψ, (In _ Γ ψ) -> mforces M ψ) ->
  (mforces M φ).
```

We easily recognize both relations from Definition 5.1.3. First, `loc_conseq` is localized in `w` by requiring that all formulas in Γ are forced in `w` `(forall ψ, (In _ Γ ψ) -> wforces M w ψ)` and leading to `wforces M w φ`. Second, `glob_conseq` is globalized by requiring that all formulas in Γ are forced in all worlds of `M` `(forall ψ, (In _ Γ ψ) -> mforces M ψ)` and leading to `mforces M φ`.

We are thus left with the formalisation of bisimulation between models (⌨).

```
Definition bisimulation (M0 M1 : kmodel)
           (B : (@nodes M0) -> (@nodes M1) -> Prop) : Prop :=
  forall (w0 : @nodes M0) (w1 : @nodes M1),
 (B w0 w1) ->
 (*B1*) ((forall p, (@val M0) w0 p <-> (@val M1) w1 p)
 /\
 (*B2*) (forall v1, ((@reachable M1) w1 v1) ->
         (exists v0, ((@reachable M0) w0 v0) /\ (B v0 v1)))
 /\
 (*B3*) (forall v0, ((@reachable M0) w0 v0) ->
         (exists v1, ((@reachable M1) w1 v1) /\ (B v0 v1)))).
```

Following the formalisation of Definition 5.1.5, a relation `B` is a bisimulation between `M0` and `M1` if for any world `w0` of `M0` and `w1` of `M1` in the relation `B` (i.e. `B w0 w1`), the following holds. First, we need to satisfy the condition `(*B1*)` (note that `(*` and `*)` allow to define an environment for comments). More precisely, the valuation function `@val M0` of `M0` and `@val M1` of `M1` must coincide in `w0` and `w1`. We also need to satisfy the condition `(*B2*)`, which we add using the conjunction operation `/\` of Coq. This condition requires that for any `v1`, if we have that it is a successor of `w1` in `M1` (i.e. `(@reachable M1) w1 v1`), then there must be a `v0` successor of `w0` (i.e. `(@reachable M0) w0 v0`) such that `v0` and `v1` are in the relation `B` (i.e. `B v0 v1`). Finally, in a similar way, we can see that the condition `(*B3*)` is faithfully capturing the condition B_3 of Definition 5.1.5.

With the notion of bisimulation formalised, we can get a formalisation of Lemma 5.1.1 (⌨).

```
Lemma bisimulation_imp_modal_equiv : forall M0 M1 B,
  (bisimulation M0 M1 B) ->
    (forall φ w0 w1, (B w0 w1) ->
      (wforces M0 w0 φ <-> wforces M1 w1 φ)).
```

Clearly, this lemma simply says that if `B` is a bisimulation between the models `M0` and `M1` (i.e. `bisimulation M0 M1 B`), then any two bisimilar points `w0` and `w1` (i.e. `B w0 w1`) force the same formulas (i.e. `wforces M0 w0 φ <-> wforces M1 w1 φ`).

5.2 First-order

First-order languages contain elements other than propositional variables: predicate and function symbols, as well as variables. So, we need to tailor models in such a way that we can interpret these elements.

Definition 5.2.1. A constant domain FO Kripke model $\mathcal{M}$ is a tuple of the shape $(W, \{R_i\}_{i \in \mathbb{N}}, D, I_{fun}, I_{pred})$ where:

- $(W, \{R_i\}_{i \in \mathbb{N}})$ is a Kripke frame as in Definition 5.0.1;
- D is a non-empty set called the *domain*;
- I_{fun} is a function interpreting each function symbol $f \in Fun$ of arity $n = Ar_{Fun}(f)$ by a function $I_{fun}(f) : D^n \to D$;

- I_{pred} is a function interpreting, in each $w \in W$, each predicate symbol $P \in Pred$ of arity $n = Ar_{Pred}(P)$ by a set $I_{pred}(w, P) \subseteq D^n$.

An assignment α on D is a function $\alpha : \mathbb{N} \to D$. When clear from context, we simply say that α is an assignment.

Models can interpret all the syntactic elements mentioned above: predicate and function symbols using the function I_{pred} and I_{fun}, and variables using an assignment α. Note that on top of the restrictions on W and the accessibility relations mentioned in the propositional case, here we can decide to impose restrictions on I_{fun} or I_{pred} and hence characterize classes of models.

Anyone familiar with the literature of Kripke semantics for first-order languages notices the strong decision we made in our definition: the domain D is *constant*. Alternatively, we could have decided to allow each point w of W to have its own domain D_w. However, while less general, the definition above is sufficient for this dissertation as we only consider first-order logics semantically captured by constant domain semantics.

Next, we show how we interpret terms in a FO Kripke model using the interpretation function I_{fun} and the assignment α.

Definition 5.2.2. Let $\mathcal{S}$ be a signature, $\mathcal{M} = (W, R, D, I_{fun}, I_{pred})$ a FO Kripke model, α an assignment on D and $t \in Term_{\mathcal{S}}$. We define the interpretation of t in $\mathcal{M}$ given α, noted $\overline{\alpha}(t)$, inductively as following:

$$\begin{array}{rcl} \overline{\alpha}(x) & = & \alpha(x) \\ \overline{\alpha}(f(t_1, ..., t_n)) & = & I_{fun}(f)(\overline{\alpha}(t_1), ..., \overline{\alpha}(t_n)) \end{array}$$

We obtain a semantics for a first-order language $\mathbb{L} = (\mathcal{S}, \mathcal{C}, \mathcal{Q})$ by defining a forcing relation $\mathcal{M}, w, \alpha \Vdash \varphi$ where $\mathcal{M} = (W, R, D, I_{fun}, I_{pred})$ is a model, $w \in W$, α is an assignment on D and $\varphi \in Form_{\mathbb{L}}$. This relation is defined recursively on the structure of φ with one clause for each connective in $\mathcal{C}$ and each quantifier in $\mathcal{Q}$, and as base case at least the following.

$$\mathcal{M}, w, \alpha \Vdash P(t_1, \ldots, t_n) \quad \text{iff} \quad (\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I_{pred}(w, P)$$

With the forcing relation in hand, we can define the following notations.

Definition 5.2.3. Given $\Gamma \cup \{\varphi\} \subseteq Form_{\mathbb{L}}$, a model $\mathcal{M} = (W, R, D, I_{fun}, I_{pred})$ for $\mathbb{L}$ and an assignment α on D, we define the following.

- $\mathcal{M}, \alpha \Vdash \varphi$ if for every point $w \in W$, $\mathcal{M}, w, \alpha \Vdash \varphi$.
- $\mathcal{M}, w, \alpha \Vdash \Gamma$ for a given $w \in W$ if every $\gamma \in \Gamma$ we have $\mathcal{M}, w, \alpha \Vdash \gamma$. In this case, we say w is a Γ-point.
- $\mathcal{M}, \alpha \Vdash \Gamma$ if for every $\gamma \in \Gamma$, $\mathcal{M}, w, \alpha \Vdash \gamma$. In this case, we say that $\mathcal{M}$ is a Γ-model.

With the forcing relation and the notations above, we can port the notions of semantic consequence relation to the first-order case.

Definition 5.2.4. The $\underline{\text{l}}$ocal and $\underline{\text{g}}$lobal consequence relations are as below:

$$\begin{array}{lll} \Gamma \models_l \varphi & \text{iff} & \forall \mathcal{M}. \forall \alpha. \forall w. (\mathcal{M}, w, \alpha \Vdash \Gamma \;\; \Rightarrow \;\; \mathcal{M}, w, \alpha \Vdash \varphi) \\ \Gamma \models_g \varphi & \text{iff} & \forall \mathcal{M}. \forall \alpha. (\mathcal{M} \Vdash \Gamma \;\; \Rightarrow \;\; \mathcal{M}, \alpha \Vdash \varphi). \end{array}$$

We can then also capture logics understood as consequence relations with the following two sets, using the appropriate class of models.

$$\begin{array}{c} \{(\Gamma, \varphi) \in (\mathsf{Pow}(Form_{\mathbb{L}}) \times Form_{\mathbb{L}}) \;\mid\; \Gamma \models_l \varphi\} \\ \{(\Gamma, \varphi) \in (\mathsf{Pow}(Form_{\mathbb{L}}) \times Form_{\mathbb{L}}) \;\mid\; \Gamma \models_g \varphi\} \end{array}$$

These notions can be extended to the following more general ones.

$$\begin{array}{lll} \Gamma \models_l \Delta & \text{iff} & \forall \mathcal{M}. \forall \alpha. \forall w. (\mathcal{M}, w, \alpha \Vdash \Gamma \;\; \Rightarrow \;\; \exists \delta \in \Delta. \mathcal{M}, w, \alpha \Vdash \delta) \\ \Gamma \models_g \Delta & \text{iff} & \forall \mathcal{M}. \forall \alpha. (\mathcal{M}, \alpha \Vdash \Gamma \;\; \Rightarrow \;\; \forall w \in W. \exists \delta \in \Delta. \mathcal{M}, w, \alpha \Vdash \delta). \end{array}$$

5.2.1 The classical modal logic example: on paper

The forcing relation for constant domain first-order classical modal logics is usually defined as follows, where $(d .: \alpha)$ is the function defined in Definition 2.2.8.

Definition 5.2.5. Given a constant domain FO Kripke model $\mathcal{M} = (W, R, D, I_{fun}, I_{pred})$, we define the forcing relation as follows:

$$\begin{array}{lll}
\mathcal{M}, w, \alpha \Vdash P(t_1, \dots, t_n) & \text{iff} & (\overline{\alpha}(t_1), \dots, \overline{\alpha}(t_n)) \in I_{pred}(w, P) \\
\mathcal{M}, w, \alpha \Vdash \bot & & \text{never} \\
\mathcal{M}, w, \alpha \Vdash \varphi \to \psi & \text{iff} & \text{if } \mathcal{M}, w, \alpha \Vdash \varphi \text{ then } M, w, \alpha \Vdash \psi \\
\mathcal{M}, w, \alpha \Vdash \Box\varphi & \text{iff} & \text{for all } v \text{ such that } wRv \text{ we have } \mathcal{M}, v, \alpha \Vdash \varphi \\
\mathcal{M}, w, \alpha \Vdash \forall\varphi & \text{iff} & \text{for all } d \in D \text{ we have } M, w, (d .: \alpha) \Vdash \varphi \\
\mathcal{M}, w, \alpha \Vdash \exists\varphi & \text{iff} & \text{there exists } d \in D \text{ such that } M, w, (d .: \alpha) \Vdash \varphi
\end{array}$$

The only additions to Definition 5.1.4, apart from the different nature of models, concern the treatment of quantifiers. To treat them, we modify the assignment function α to $(d .: \alpha)$, which ensures that the occurrences bound by $\exists$ or $\forall$ in $\exists\varphi$ or $\forall\varphi$ are interpreted by d in the model.

It is well-known that the class of models for the first-order constant domain classical modal logic **CDKL** is the class of *all* constant domain models of the shape $\mathcal{M} = (W, R, D, I_{fun}, I_{pred})$ with no restriction [51]. However, to the best of our knowledge this result has not been formalised in an interactive theorem prover.

5.2.2 The classical modal logic example: in Coq

All the elements presented in this subsection can be found here: ⌨.

The main difference between propositional and first-order Kripke semantics lies in the presence of *domains*. As we could see above, a lot of our definitions are parametric in a given domain D. So, in most definitions below, we rely on a domain `D` given in the context. To declare this domain, we write the following (⌨).

```
Variable D : Type.
```

Then, we proceed to define what interpretation functions for function symbols are (⌨).

```
Class interp_fun :=
  {
  i_func : forall f : syms, vec D (ar_syms f) -> D
  }.
```

They satisfy a single property: given a function symbol `f`, and a vector of elements of `D` of arity corresponding to `f`, i.e. `vec D (ar_syms f)`, we get an element of `D`. This captures the idea expressed in the clause for I_{fun} given in Definition 5.2.1.

To interpret terms in a model, we need both the type of functions described above and assignments. So, we formalise what it is to be an assignment function on a domain `D` as follows (⌨).

```
Definition assign := nat -> D.
```

We are now able to formalize the interpretation of terms given an interpretation function for function symbols and an assignment, which we noted $\widehat{\alpha}$ above but call `eval` here (⌨).

```
Fixpoint eval (α : assign) (t : term) : D :=
  match t with
   | var s => α s
   | func f v => i_func f (Vector.map (eval α) v)
  end.
```

It follows very closely Definition 5.2.2 as it goes by recursion on the structure of the term `t`. If `t` is a natural number in the syntax (i.e. `var s`), then we simply apply the assignment α to `s`. If `t` is a function symbol `f` with a vector of terms `v` of the correct arity, then we use the interpretation function `i_interp` to interpret `f` as a function, and give to this function the vector of interpreted terms from `v` (i.e. `Vector.map (eval α) v`). Indeed, the function `(eval α)` is the interpretation function of terms $\widehat{\alpha}$ defined in Definition 5.2.2 and the function `Vector.map` allows the application of a function to all elements of a vector.

Then, we are ready to define first-order Kripke models by following Definition 5.2.1 and enhancing `kmodel` into `FOkmodel` (⌨).

```
Class FOkmodel :=
  {
    nodes : Type ;
    reachable : nodes -> nodes -> Prop ;
    k_interp : interp D ;
    k_P : nodes -> forall P : preds, Vector.t D (ar_preds P)
          -> Prop
  }.
```

As in the propositional case, we have an underlying Kripke frame: a collection of `nodes` (or worlds) and a relation `reachable` on them. Then, we have a domain `D` which is not explicitly mentioned in the definition but is a required `Variable` in the context. So, to define a `FOkmodel` we need to explicitly provide a domain. With this domain in hand, we require an interpretation function for function symbols `k_interp` on this domain, pertaining to I_{fun} of Definition 5.2.1. We finally get an interpretation function for predicate symbols with `k_P`, as it takes a node and sends a predicate symbol `P` together with a vector of the correct arity (i.e. `Vector.t domain (ar_preds P)`) to `Prop`. Then, it appears that a model is of the shape $(W, R, D, I_{fun}, I_{pred})$, as expected.

We can turn to the definition of forcing (⌨).

```
Fixpoint wforces w (α : assign D) (φ : kform) : Prop :=
 match φ with
  | atom P v => k_P w P (Vector.map (@eval _ D k_interp α) v)
  | Bot => False
  | bin Imp ψ χ => wforces w α ψ -> wforces w α χ
  | un Box ψ => forall v, reachable w v -> wforces v α ψ
  | quant All ψ => forall j, wforces w (j .: α) ψ
  | quant Ex ψ => exists j, wforces w (j .: α) ψ
 end.
```

For this definition, we require a `FOkmodel` on a domain `D` which are both implicit here, a node `w` in this model, an assignment α on `D`, and a formula φ. Then, we define `wforces` by recursion on the structure of φ, following Definition 5.2.5. The cases for the connectives $\bot$, $\Box$ and $\to$ are similar to the propositional encoding, so we focus on the forcing of atoms and quantified formulas.

For a predicate symbol `P` and a vector of terms `v` of the adequate arity, we have that `atom P v` is forced in `w` under the assignment α if we have `k_P w (Vector.map (@eval _ D k_interp α) v)`. The latter expression is obtained if we have that the result of the interpretation function for predicate symbols `k_P` holds, when given the node `w`, the predicate symbol `P` and the vector of interpretations of terms in `v` given α (i.e. `Vector.map (@eval _ D k_interp alpha) v`). Indeed, the function `(@eval _ D k_interp alpha)` is the interpretation function of terms $\widehat{\alpha}$ defined in Definition 5.2.2. Thus, we can clearly see that our formalisation of Definition 5.2.5: $\mathcal{M}, w, \alpha \Vdash P(t_1, \dots, t_n)$ if and only if $(\overline{\alpha}(t_1), \dots, \overline{\alpha}(t_n)) \in I_{pred}(w, P)$.

We now explain the quantifiers. As their forcing only differs in the use of the meta-level quantifiers `forall` and `exists`, we can restrict our attention to one: `All`. The encoding clearly follows the clause for ∀ in Definition 5.2.5 as it requires that for any element of the domain `j` we have `wforces w (j .: α) ψ`, i.e. ψ is forced in `w` with the modified assignment `(j .: α)`.

For convenience, we can define a notion of forcing *for all points in a model* rather straightforwardly (⌨).

```
Definition mforces α φ : Prop := forall w, wforces w α φ.
```

The definition above is helpful to give a compact definition of the global semantic consequence relation. We formalise the latter and its local version in the definition below (⌨).

```
Definition loc_conseq Γ φ :=
  forall D (M : FOkmodel D) w α,
     (forall ψ, In _ Γ ψ -> wforces D M w α ψ) ->
     (wforces D M w α φ).

Definition glob_conseq Γ φ :=
    forall D (M : FOkmodel D) α,
    (forall ψ, In _ Γ ψ -> mforces D M α ψ) ->
    (mforces D M α φ).
```

We easily recognize both relations from Definition 5.2.4.

Chapter 6

Soundness and Completeness

In the two previous chapters, we exhibited ways of capturing a logic. Each way allows to obtain results about logics with more or less ease. Consequently, we would quite naturally want to be able to capture a logic through a proof system and through a semantics to get the best of both worlds. The results traditionally called *soundness* and *completeness* allow us to do exactly this, by putting in correspondence a proof system and a semantics.

Definition 6.0.1. Let $\mathbb{L}$ be a language, P a proof system, S a semantics for $\mathbb{L}$ and R a semantic consequence relation built from S. We name $\mathbf{L}_\mathsf{P}$ the set of provable consecutions, i.e. $\{(\Gamma, \varphi) \mid (\Gamma, \varphi) \text{ is provable in } \mathsf{P}\}$, and $\mathbf{L}_\mathsf{S}$ the set $\{(\Gamma, \varphi) \mid (\Gamma, \varphi) \in \mathsf{R}\}$. We say that:

- P is *sound* with respect to S if $\mathbf{L}_\mathsf{P} \subseteq \mathbf{L}_\mathsf{S}$;
- P is *complete* with respect to S if $\mathbf{L}_\mathsf{S} \subseteq \mathbf{L}_\mathsf{P}$.

6.1 The classical modal logic example: on paper

Here we connect the generalized Hilbert calculi defined in Definition 4.1.4 with the semantic consequence relations defined in Definition 5.1.3 using the forcing relation given in Definition 5.1.4 on the class of all Kripke frames. More precisely, we aim to prove the following theorem.

Theorem 6.1.1. The following holds:

$$\begin{array}{lccc} (1) & \Gamma \vdash_\mathsf{w} \varphi & \text{iff} & \Gamma \models_l \varphi \\ (2) & \Gamma \vdash_\mathsf{s} \varphi & \text{iff} & \Gamma \models_g \varphi \end{array}$$

We break it down into two parts: soundness (from left to right) and completeness (from right to right).

6.1.1 Soundness

First, we prove soundness of each generalized Hilbert calculus with respect to its corresponding semantic consequence relation. In essence, the proof of this statement boils down to a proof of validity for each of the axioms, combined with a proof of preservation of consecutions in rules. So, we separately prove that all axiom instances are valid.

Lemma 6.1.1. For all φ, if $\varphi \in \mathcal{A}^{\mathsf{i}}_{K}$ then $\models \varphi$.

Proof. (⌨) Let $\varphi \in \mathcal{A}^{\mathsf{i}}_{K}$. We show the statement for each axiom φ can be an instance of. In what follows let $\mathcal{M} = (W, R, I)$ be a model and $w \in W$.

(MA_1) We show that $\mathcal{M}, w \Vdash (\gamma \to \psi) \to ((\psi \to \chi) \to (\gamma \to \chi))$. Assume that $\mathcal{M}, w \Vdash \gamma \to \psi$. We need to show that $\mathcal{M}, w \Vdash (\psi \to \chi) \to (\gamma \to \chi)$. Consequently, assume $\mathcal{M}, w \Vdash \psi \to \chi$. We then need to show that $\mathcal{M}, w \Vdash \gamma \to \chi$. Assume $\mathcal{M}, w \Vdash \gamma$.

We need to show that $\mathcal{M}, w \Vdash \chi$. As $\mathcal{M}, w \Vdash \gamma \to \psi$ and $\mathcal{M}, w \Vdash \gamma$, we get that $\mathcal{M}, w \Vdash \psi$. Moreover we have $\mathcal{M}, w \Vdash \psi \to \chi$, so $\mathcal{M}, w \Vdash \chi$. As $\mathcal{M}$ and w are arbitrary, we get that $\models (\gamma \to \psi) \to ((\psi \to \chi) \to (\gamma \to \chi))$.

(MA_2) We show that $\mathcal{M}, w \Vdash \psi \to (\chi \to \psi)$. Assume $\mathcal{M}, w \Vdash \psi$. We need to show that $\mathcal{M}, w \Vdash \chi \to \psi$. So, assume $\mathcal{M}, w \Vdash \chi$. Then, we need to show that $\mathcal{M}, w \Vdash \psi$, which is one of our assumptions. So, we are done. As $\mathcal{M}$ and w are arbitrary, we get that $\models \psi \to (\chi \to \psi)$.

(MA_3) We show that $\mathcal{M}, w \Vdash ((\psi \to \chi) \to \psi) \to \psi$. Assume $\mathcal{M}, w \Vdash (\psi \to \chi) \to \psi$. We need to show that $\mathcal{M}, w \Vdash \psi$. We reason classically using the law of excluded middle: we have that $\mathcal{M}, w \Vdash \psi$ or $\mathcal{M}, w \nVdash \psi$. In the first case, we are done. In the second case, we have $\mathcal{M}, w \nVdash \psi$. Consequently, we get that $\mathcal{M}, w \Vdash \psi \to \chi$ by semantic definition of $\to$. However, as we have $\mathcal{M}, w \Vdash (\psi \to \chi) \to \psi$, we need to have $\mathcal{M}, w \Vdash \psi$, hence a contradiction. So, in both cases, we obtained $\mathcal{M}, w \Vdash \psi$. As $\mathcal{M}$ and w are arbitrary, we get that $\models ((\psi \to \chi) \to \psi) \to \psi$.

(MA_4) We show that $\mathcal{M}, w \Vdash \bot \to \psi$. Assume that $\mathcal{M}, w \Vdash \bot$. We need to show that $\mathcal{M}, w \Vdash \psi$. Note that $\mathcal{M}, w \Vdash \bot$ cannot happen by definition, so we have a contradiction. As a consequence, we trivially get that $\mathcal{M}, w \Vdash \psi$. As $\mathcal{M}$ and w are arbitrary, we get that $\models \bot \to \psi$.

(MA_5) We show that $\mathcal{M}, w \Vdash \Box(\psi \to \chi) \to (\Box\psi \to \Box\chi)$. Assume $\mathcal{M}, w \Vdash \Box(\psi \to \chi)$. We need to show $\mathcal{M}, w \Vdash \Box\psi \to \Box\chi$. Assume $\mathcal{M}, w \Vdash \Box\psi$. Thus, we need to show $\mathcal{M}, w \Vdash \Box\chi$. So, let $v \in W$ such that wRv. Our goal is to show $\mathcal{M}, v \Vdash \chi$. As we have $\mathcal{M}, w \Vdash \Box(\psi \to \chi)$ and wRv, we get that $\mathcal{M}, v \Vdash \psi \to \chi$. Furthermore, as $\mathcal{M}, w \Vdash \Box\psi$ and wRv, we get $\mathcal{M}, w \Vdash \psi$. Thus, as we have $\mathcal{M}, v \Vdash \psi \to \chi$ and $\mathcal{M}, v \Vdash \psi$, we reach our goal: $\mathcal{M}, v \Vdash \chi$. So, we are done. As $\mathcal{M}$ and w are arbitrary, we get that $\models \Box(\psi \to \chi) \to (\Box\psi \to \Box\chi)$.

■

With this lemma in hand, we can turn to soundness results.

Theorem 6.1.2. The following holds:

$$\begin{array}{lccc} (1) & \Gamma \vdash_{\mathsf{w}} \varphi & \text{implies} & \Gamma \models_l \varphi \\ (2) & \Gamma \vdash_{\mathsf{s}} \varphi & \text{implies} & \Gamma \models_g \varphi \end{array}$$

Proof. We prove each statement independently.

(1) (⌨) Assume $\Gamma \vdash_{\mathsf{w}} \varphi$. We prove that $\Gamma \models_l \varphi$ by induction on the structure of the proof of $\Gamma \vdash_{\mathsf{w}} \Delta$.

(El) Then, we have that $\varphi \in \Gamma$. Consequently, if $\mathcal{M}, w \Vdash \Gamma$ then we automatically get $\mathcal{M}, w \Vdash \varphi$ for all $\mathcal{M}$ and w. So, $\Gamma \models_l \varphi$.

(Ax) Then, we have that $\varphi \in \mathcal{A}^{\mathsf{i}}_K$. By Lemma 6.1.1 we get $\models \varphi$. Thus, we easily get that $\Gamma \models_l \varphi$.

(MP) By induction hypothesis applied on the premises $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, we get that $\Gamma \models_l \varphi$ and $\Gamma \models_l \varphi \to \psi$ hold. We need to show $\Gamma \models_l \psi$. Let $\mathcal{M}$ be a model and $w \in W$. Assume that $\mathcal{M}, w \Vdash \Gamma$. We need to show that $\mathcal{M}, w \Vdash \psi$. As $\mathcal{M}, w \Vdash \Gamma$ and $\Gamma \models_l \varphi$ and $\Gamma \models_l \varphi \to \psi$, we directly obtain $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \varphi \to \psi$. Consequently, we obtain $\mathcal{M}, w \Vdash \psi$. As $\mathcal{M}$ and w are arbitrary, we get $\Gamma \models_l \psi$.

(wNec) By induction hypothesis applied on the premise $\emptyset \vdash \varphi$ we get that $\emptyset \models_l \varphi$ holds. We show that $\Gamma \models_l \Box\varphi$. Let $\mathcal{M}$ be a model and $w \in W$. Assume that $\mathcal{M}, w \Vdash \Gamma$. We need to show that $\mathcal{M}, w \Vdash \Box\varphi$. Thus we need to show that for every v such that wRv we have $\mathcal{M}, v \Vdash \varphi$. As $\models_l \varphi$ we get that $\mathcal{M}, v \Vdash \varphi$. As v is arbitrary, we have that $\mathcal{M}, w \Vdash \Box\varphi$. As $\mathcal{M}$ and w are arbitrary, we get $\Gamma \models_l \Box\varphi$.

(2) (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$. We prove that $\Gamma \models_l \Delta$ by induction on the structure of the proof of $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$.

(El) Then, we have that $\varphi \in \Gamma$. Consequently, if $\mathcal{M} \Vdash \Gamma$ then we automatically get $\mathcal{M} \Vdash \varphi$ for all $\mathcal{M}$. So, $\Gamma \models_g \varphi$.

(Ax) Then, we have that $\varphi \in \mathcal{A}^{\mathsf{i}}_{K}$. By Lemma 6.1.1 we get $\models \varphi$. Thus, we easily get that $\Gamma \models_g \varphi$.

(MP) By induction hypothesis applied on the premises $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \to \psi$, we get that $\Gamma \models_g \varphi$ and $\Gamma \models_g \varphi \to \psi$ hold. We need to show $\Gamma \models_g \psi$. Let $\mathcal{M}$ be a model. Assume that $\mathcal{M} \Vdash \Gamma$. We need to show that $\mathcal{M} \Vdash \psi$. Let $w \in W$. We need to show that $\mathcal{M} \Vdash \psi$. As $\mathcal{M} \Vdash \Gamma$ and $\Gamma \models_g \varphi$ and $\Gamma \models_g \varphi \to \psi$, we directly obtain $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \varphi \to \psi$. Consequently, we obtain $\mathcal{M}, w \Vdash \psi$. As w is arbitrary, we get $\mathcal{M} \Vdash \psi$. As $\mathcal{M}$ is arbitrary, we get $\Gamma \models_g \psi$.

(sNec) By induction hypothesis applied on the premise $\Gamma \vdash \varphi$ we get that $\Gamma \models_g \varphi$ holds. We show that $\Gamma \models_g \Box\varphi$. Let $\mathcal{M}$ be a model. Assume that $\mathcal{M} \Vdash \Gamma$. We need to show that $\mathcal{M} \Vdash \Box\varphi$. Let $w \in W$. We need to show $\mathcal{M}, w \Vdash \Box\varphi$. Thus we need to show that for every v such that wRv we have $\mathcal{M}, v \Vdash \varphi$. As $\Gamma \models_g \varphi$ and $\mathcal{M} \Vdash \Gamma$, we get that $\mathcal{M}, v \Vdash \varphi$. As v is arbitrary, we have that $\mathcal{M}, w \Vdash \Box\varphi$. As w is arbitrary, we get $\mathcal{M} \Vdash \Box\varphi$. As $\mathcal{M}$ is arbitrary, we get $\Gamma \models_g \Box\varphi$.

■

These soundness results allow us to show two differences between the logics $\mathsf{w}\mathsf{K}\mathbf{L}$ and $\mathsf{s}\mathsf{K}\mathbf{L}$ pointed out in Subsection 4.1.1.

Lemma 6.1.2. The following holds.

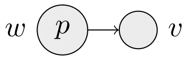
1. $\mathsf{sK}\mathbf{L} \not\subseteq \mathsf{wK}\mathbf{L}$;
2. The detachment-deduction theorem fails for $\mathsf{sK}\mathbf{L}$.

Proof. For both statements, we rely on the holding of $p \vdash_{\mathsf{s}} \Box p$, as proven below.

$$\dfrac{\dfrac{}{p \vdash p}\ \text{(El)}}{p \vdash \Box p}\ \text{(sNec)}$$

With this result in hand, we can prove each statement independently.

1. (⌨) While we have $(p, \Box p) \in \mathsf{sK}\mathbf{L}$, as shown above, we show that $(p, \Box p) \notin \mathsf{wK}\mathbf{L}$ using soundness. It thus suffices to show that $p \not\models_l \Box p$. Consider the following model $\mathcal{M}$.



We clearly have that $\mathcal{M}, w \Vdash p$, but also $\mathcal{M}, w \nVdash \Box p$ as $\mathcal{M}, w \nVdash p$ and wRv. Consequently, we have that $p \not\models_l \Box p$, hence $(p, \Box p) \notin \mathsf{wK}\mathbf{L}$ using soundness.

2. (⌨) We show that while we have $p \vdash_{\mathsf{s}} \Box p$, we do not have $\emptyset \vdash_{\mathsf{s}} p \to \Box p$. To show this, we use soundness and prove $\emptyset \not\models_g p \to \Box p$. In fact, the model $\mathcal{M}$ above suffices: we have that $\mathcal{M}, w \nVdash p \to \Box p$ as $\mathcal{M}, w \Vdash p$ and $\mathcal{M}, w \nVdash \Box p$, hence $\emptyset \not\models_g p \to \Box p$.

■

In the next subsection, we turn to the other half of Theorem 6.1.1: completeness results.

6.1.2 Completeness

Our goal is to prove the following theorem.

Theorem 6.1.3. The following holds:

$$\begin{array}{llll} (1) & \Gamma \models_l \varphi & \text{implies} & \Gamma \vdash_{\mathsf{w}} \varphi \\ (2) & \Gamma \models_g \varphi & \text{implies} & \Gamma \vdash_{\mathsf{s}} \varphi \end{array}$$

The strategy we adopt here consists of first proving (1) and second (2) using (1) through a transformation of models establishing bisimulations. We consequently divide our work into two parts.

Completeness of wKH

Here we focus on the logic **wKL** and show that it is complete with respect to the local semantic consequence relation. We first establish a Lindenbaum lemma, ensuring the existence for any set Γ such that $\Gamma \not\vdash_{\mathsf{w}} \varphi$ of a Lindenbaum extension Γ' such that it is a maximal consistent set and $\Gamma' \not\vdash_{\mathsf{w}} \varphi$. Second, we use this type of set, i.e. maximal consistent, to construct a canonical model which allows us to prove completeness.

To obtain a Lindenbaum lemma, showing the existence of a maximal consistent set, we need to proceed as follows. First, we need an encoding of formulas by natural numbers. Second, we define a decoding function, which is some type of inverse of the encoding. These two functions are then used in a third moment, where we define a selection function, picking formulas, which is involved in the step-by-step construction of the Lindenbaum extension of a set Γ. Fourth, the Lindenbaum extension, i.e. a maximal consistent set Γ' such that $\Gamma \subseteq \Gamma'$, is defined using the selection function. Finally, we prove the Lindenbaum lemma: the Lindenbaum extension is a maximal consistent set such that $\Gamma' \not\vdash_{\mathsf{w}} \varphi$.

First, we assume given an encoding as defined below.

Definition 6.1.1. A function $f : Form_{\mathbb{L}_{\mathbf{CM}}} \to \mathbb{N}$ is an *encoding* of $Form_{\mathbb{L}_{\mathbf{CM}}}$ if f is injective, i.e. if $f(\varphi) = f(\psi)$ then $\varphi = \psi$ for $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{CM}}}$.

Hypothesis 6.1.1. There is an encoding of $Form_{\mathbb{L}_{\mathbf{CM}}}$.

We fix this encoding and call it *encode*0. We then define the encoding we use in our Lindenbaum extension.

Definition 6.1.2. We define $encode : \varphi \mapsto S\ (encode0(\varphi))$, where S is the successor operator on natural numbers.

Note that *encode* is also injective. Its specificity in comparison to *encode*0 is that it has the additional property of making 0 the encoding of no formula, which is useful in our Lindenbaum construction as the stage 0 of the latter then corresponds to no formula.

Second, we need some type of inverse of *encode*: a decoding function. To define such a function, we define the option type $Opt(X)$ parametric in the type X, which represents the encapsulation of an optional value from X.

Definition 6.1.3. Let X be a type. We inductively define $Opt(X)$ the option type of X as follows.

- $None \in Opt(X)$;
- If $x \in X$ then $Some(x) \in Opt(X)$.

With this type defined, we can define what a decoding function is.

Definition 6.1.4. Given an encoding function f, a function $g : \mathbb{N} \to Opt(Form_{\mathbb{L}_{\mathbf{CM}}})$ is a *decoding* function for f if g satisfies the following:

1. for all $\varphi \in \mathit{Form}_{\mathbb{L}_{\mathbf{CM}}}$ we have $g(f(\varphi)) = \mathit{Some}(\varphi)$;
2. for all $n \in \mathbb{N}$, if there is no $\varphi \in \mathit{Form}_{\mathbb{L}_{\mathbf{CM}}}$ such that $f(\varphi) = n$, then $g(n) = \mathit{None}$.

Lemma 6.1.3. There is a decoding function for *encode*.

Proof. (⌨) We define the relation $\mathit{encode}^{-1} \subseteq \mathbb{N} \times \mathit{Opt}(\mathit{Form}_{\mathbb{L}_{\mathbf{CM}}})$ such that:

$$(n, o) \in \mathit{encode}^{-1} \text{ iff } \begin{cases} o = \mathit{None} \text{ and there is no } \varphi \text{ such that } \mathit{encode}(n) = \varphi \\ o = \mathit{Some}(\varphi) \text{ for some } \varphi \text{ and } \mathit{encode}(\varphi) = n \end{cases}$$

First, let us prove that this is a function, i.e. that for every $n \in \mathbb{N}$ there is a unique $o \in \mathit{Opt}(\mathit{Form}_{\mathbb{L}_{\mathbf{CM}}})$ such that $(n, o) \in \mathit{encode}^{-1}$. Let $n \in \mathbb{N}$. If there is no φ such that $\mathit{encode}(\varphi) = n$, then by the above definition we have that $(n, \mathit{None}) \in \mathit{encode}^{-1}$. Note that it is unique, as if another $o \in \mathit{Opt}(\mathit{Form}_{\mathbb{L}_{\mathbf{CM}}})$ we have $(n, o) \in \mathit{encode}^{-1}$, we need o to be of the shape $\mathit{Some}(\varphi)$. The latter can only happen if there is a φ such that $\mathit{encode}(n) = \varphi$, a contradiction. If there is a φ such that $\mathit{encode}(\varphi) = n$, then by definition we get that $(n, \mathit{Some}(\varphi)) \in \mathit{encode}^{-1}$. Assume for a contradiction that there is another $o \in \mathit{Opt}(\mathit{Form}_{\mathbb{L}_{\mathbf{CM}}})$ such that $(n, o) \in \mathit{encode}^{-1}$. Then, o is either of the form None or of the form $\mathit{Some}(\psi)$ for $\psi \neq \varphi$. If the former, then we can argue as above for a contradiction. If the latter, we would then have by definition that $\mathit{encode}(\varphi) = \mathit{encode}(\psi) = n$. However, as *encode* is injective, we have that $\varphi = \psi$, a contradiction. So, we obtain uniqueness.

Second, we proceed to prove the above properties.

- We have that $\mathit{encode}^{-1}(\mathit{encode}(\varphi)) = \mathit{encode}^{-1}(n)$ for some n. Now, by definition of encode^{-1}, we get that $\mathit{encode}^{-1}(n) = \varphi$ as $\mathit{encode}(\varphi) = n$.
- Let $n \in \mathbb{N}$ and assume that there is no φ such that $\mathit{encode}(\varphi) = n$. Then, by definition of encode^{-1}, we get that $\mathit{encode}^{-1}(n) = \mathit{None}$.

■

Using the previous lemma, we fix a decoding function of *encode* and call it *decode*.

Third, with these functions in hand, we can define a selection function crucial to our Lindenbaum extension.

Definition 6.1.5. We define the selection function sel which takes as inputs a set of formulas, a formula, and a natural number, and outputs a set of formulas (note that there is a priority order from bottom to top).

$$\mathsf{sel}(\Gamma, \varphi, n) = \begin{cases} \Gamma & \text{if } \mathit{decode}(n) = \mathit{None} \\ \Gamma \cup \{\psi\} & \text{if } \mathit{decode}(n) = \mathit{Some}(\psi) \text{ and } \Gamma, \psi \nvdash_{\mathsf{w}} \varphi \\ \Gamma \cup \{\psi \to \bot\} & \text{if } \mathit{decode}(n) = \mathit{Some}(\psi) \text{ and } \Gamma, \psi \vdash_{\mathsf{w}} \varphi \end{cases}$$

The above definition is simple. If n is the encoding of no formula, then sel outputs the set Γ it was given (case 1). If n is the encoding of some formula ψ, and the addition of ψ to Γ does not entail φ, i.e. $\Gamma, \psi \nvdash_{\mathsf{w}} \varphi$, then sel outputs $\Gamma \cup \{\psi\}$ (case 2). If n is the encoding of some formula ψ, and the addition of ψ to Γ does entail φ, i.e. $\Gamma, \psi \vdash_{\mathsf{w}} \varphi$, then sel outputs $\Gamma \cup \{\psi \to \bot\}$ (case 3).

The function sel is built with the intention of not entailing φ while adding to Γ either a formula ψ or its *negation* $\psi \to \psi$.

To obtain this selection property for all formulas, we need to go through all natural numbers. Our fourth point deals with this through the next function.

Definition 6.1.6. We define the function Lindf, which takes as inputs a set of formulas, a formula, and a natural number, and outputs a set of formulas, inductively on natural numbers.

- $\mathsf{Lindf}(\Gamma, \varphi, 0) = \Gamma$
- $\mathsf{Lindf}(\Gamma, \varphi, (S\ m)) = \mathsf{sel}(\,\mathsf{Lindf}(\Gamma, \varphi, m), \varphi\,,\ (S\ m)\,)$

We define the *Lindenbaum extension* of set Γ for a formula φ as follows:

$$\mathsf{Lind}(\Gamma, \varphi) = \bigcup_{n=0}^{\infty}(\mathsf{Lindf}(\Gamma, \varphi, n))$$

The function Lindf takes a set Γ, a formula φ and a natural number n, and proceeds to extend step-by-step from 0 to n the set Γ using at each step the function sel. With this function in hand, it is rather straightforward to obtain the Lindenbaum extension of a set Γ for a formula φ: we use the union of $\mathsf{Lindf}(\Gamma, \varphi, n)$ for all $n \in \mathbb{N}$, i.e. $\bigcup_{n:=0}^{\infty}(\mathsf{Lindf}(\Gamma, \varphi, n))$.

Then, we proceed to establish that the Lindenbaum extension of a set Γ for a formula φ is such that $\Gamma \nvdash_{\mathsf{w}} \varphi$ is a maximal consistent set, as defined below.

Definition 6.1.7. Let $\Gamma \subseteq Form_{\mathbb{L}_{\mathbf{CM}}}$ We say that Γ is

- *consistent* if $\Gamma \nvdash_{\mathsf{w}} \bot$;
- *maximal* if for every $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{CM}}}$, $\varphi \in \Gamma$ or $(\varphi \to \bot) \in \Gamma$;
- *closed under deducibility* if for every $\varphi \in Form_{\mathbb{L}_{\mathbf{CM}}}$, if $\Gamma \vdash_{\mathsf{w}} \varphi$ then $\varphi \in \Gamma$.

Next, we turn to the Lindenbaum lemma.

Lemma 6.1.4 (Lindenbaum Lemma)**.** If $\Gamma \nvdash_{\mathsf{w}} \varphi$ then there exist Γ' such that:

1. $\Gamma \subseteq \Gamma'$;
2. $\Gamma' \nvdash_{\mathsf{w}} \varphi$;
3. Γ' is maximal;
4. is consistent.

Proof. We claim that $\mathsf{Lind}(\Gamma, \varphi)$ is such a set. For convenience, let $\Gamma' = \mathsf{Lind}(\Gamma, \varphi)$.

First, we prove an intermediary result showing that each step of the construction $\mathsf{Lindf}(\Gamma, \varphi, n)$ preserves the unprovability of the judgement $\mathsf{Lindf}(\Gamma, \varphi, n) \vdash \varphi$ (⌨). So, we show that $\mathsf{Lindf}(\Gamma, \varphi, n) \nvdash_{\mathsf{w}} \varphi$ by induction on n. If $n := 0$, then we have that $\mathsf{Lindf}(\Gamma, \varphi, 0) \nvdash_{\mathsf{w}} \Delta$ by assumption as $\mathsf{Lindf}(\Gamma, \varphi, 0) = \Gamma$. If $n := S\ m$, then we need to show that $\mathsf{Lindf}(\Gamma, \varphi, (S\ m)) \nvdash_{\mathsf{w}} \Delta$. We make a first case distinction on *decode*$(S\ m)$.

1. If *decode*$(S\ m) = None$, then we have the following.

   $$\mathsf{Lindf}(\Gamma, \varphi, (S\ m)) = \mathsf{sel}(\,\mathsf{Lindf}(\Gamma, \varphi, m)\,,\ \varphi\,,\ (S\ m)\,) = \mathsf{Lindf}(\Gamma, \varphi, m)$$

   However, we get by induction hypothesis that $\mathsf{Lindf}(\Gamma, \varphi, m) \nvdash_{\mathsf{w}} \varphi$ so we are done.

2. If *decode*$(S\ m) = Some(\varphi)$, then we need to make a case distinction on the structure of ψ. There, depending on whether $\mathsf{Lindf}(\Gamma, \varphi, m) \vdash_{\mathsf{w}} \psi$, we have two cases.

   (a) If $\mathsf{Lindf}(\Gamma, \varphi, m), \psi \nvdash_{\mathsf{w}} \varphi$, then we have the following.

   $$\mathsf{sel}(\,\mathsf{Lindf}(\Gamma, \varphi, m)\,,\ \varphi\,,\ (S\ m)\,) = \mathsf{Lindf}(\Gamma, \varphi, m) \cup \{\psi\}$$

   But note that by assumption we get $\mathsf{Lindf}(\Gamma, \varphi, m), \psi \nvdash_{\mathsf{w}} \varphi$, that is we have $\mathsf{Lindf}(\Gamma, (S\ m)) \nvdash_{\mathsf{w}} \varphi$. So, we are done.

(b) If $\mathsf{Lindf}(\Gamma, \varphi, m), \psi \vdash_{\mathsf{w}} \varphi$, then we have the following.

$$\mathsf{sel}(\,\mathsf{Lindf}(\Gamma, \varphi, m)\,,\, \varphi\,,\, (S\ m)\,) = \mathsf{Lindf}(\Gamma, \varphi, m) \cup \{\psi \to \bot\}$$

Assume for a contradiction that $\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) \vdash_{\mathsf{w}} \varphi$, which is equal to the following.

$$\mathsf{Lindf}(\Gamma, \varphi, m), \psi \to \bot \vdash_{\mathsf{w}} \varphi$$

Using Theorem 4.1.1 on $\mathsf{Lindf}(\Gamma, \varphi, m), \psi \vdash_{\mathsf{w}} \varphi$, we get $\mathsf{Lindf}(\Gamma, \varphi, m) \vdash_{\mathsf{w}} \psi \to \varphi$. Furthermore, we use Theorem 4.1.1 on $\mathsf{Lindf}(\Gamma, \varphi, m), \psi \to \bot \vdash_{\mathsf{w}} \varphi$, we get the following.

$$\mathsf{Lindf}(\Gamma, \varphi, m) \vdash_{\mathsf{w}} (\psi \to \bot) \to \varphi$$

Now, we have a theorem pertaining to a reasoning by cases using the law of the excluded-middle (⌨). This theorem has the following shape.

$$((\psi \to \bot) \to \varphi) \to (\psi \to \varphi) \to \varphi$$

So, we can use the rule (MP) twice with the two previously established judgements to obtain a proof of $\mathsf{Lindf}(\Gamma, \varphi, m) \vdash_{\mathsf{w}} \varphi$. But this contradicts our induction hypothesis, so we completed the proof of this case.

In all possible cases we concluded that $\mathsf{Lindf}(\Gamma, \varphi, m) \nvdash_{\mathsf{w}} \varphi$, so we are done.

With this result in hand, we proceed to prove each of the items.

1. (⌨) We prove that $\Gamma \subseteq \Gamma'$. This is easily obtained as $\mathsf{Lind}(\Gamma, \varphi)$ successively extends Γ with ψ or $\psi \to \bot$ for all ψ.

2. (⌨) We need to show $\Gamma' \nvdash_{\mathsf{w}} \varphi$. Assume for a contradiction that $\Gamma' \vdash_{\mathsf{w}} \varphi$. However, from this assumption we can that that there is a m such that $\mathsf{Lindf}(\Gamma, \varphi, m) \vdash_{\mathsf{w}} \varphi$, as ultimately we can finitarize Γ'. But this is in contradiction with our preliminary result above. So, $\Gamma' \nvdash_{\mathsf{w}} \varphi$.

3. (⌨) We need to show that Γ' is maximal. Let $\psi \in Form_{\mathbb{L}_{\mathbf{CM}}}$. It is straightforward to see that at stage $m = \mathit{encode}(\psi)$, the function $\mathsf{Lindf}(\Gamma, \varphi, m)$ either adds ψ or adds $\psi \to \bot$ to the set we are building. Consequently, it must be the case that in the resulting set Γ' we have $\psi \in \Gamma'$ or $(\psi \to \bot) \in \Gamma'$.

4. (⌨) We need to show $\Gamma' \nvdash_{\mathsf{w}} \bot$. Assume for a contradiction that $\Gamma' \vdash_{\mathsf{w}} \bot$. Then, using MA_4 and (MP) we can straightforwardly show that $\Gamma' \vdash_{\mathsf{w}} \varphi$. But this is in contradiction with item 2.

So, Γ' satisfies all of the above items. ■

We just showed that for every Γ such that $\Gamma \nvdash_{\mathsf{w}} \varphi$ there is a maximal consistent set Γ' which an extension of Γ for φ such that $\Gamma' \nvdash_{\mathsf{w}} \varphi$. Next, we use these maximal consistent sets in a canonical model.

Definition 6.1.8. The canonical model $\mathcal{M}^c = (W^c, R^c, I^c)$ is defined in the following way:

1. $W^c = \{\Gamma \mid \Gamma \text{ is a maximal consistent set}\}$;

2. $\Gamma_1 R^c \Gamma_2$ iff for all $\Box\varphi \in \Gamma_1$ we have $\varphi \in \Gamma_2$;

3. $I^c(p) = \{\Gamma \in W^c \mid p \in \Gamma\}$.

We use this canonical model as a witness: we intend to show that if $\Gamma \nvdash_{\mathsf{w}} \varphi$, then we have that there is a point in $\mathcal{M}^c$ which is a Γ-point but does not force φ, hence $\Gamma \not\models_l \varphi$. As usual a canonical model proof technique, we prove the crucial Truth Lemma. To do so, we first require the next lemma on maximal consistent sets.

Lemma 6.1.5. Let $\Gamma \subseteq Form_{\mathbb{L}_{\mathbf{CM}}}$. If Γ is a maximal consistent set, then Γ is closed under deducibility.

Proof. (⌨) Assume Γ is a maximal consistent set. Let $\varphi \in Form_{\mathbb{L}_{\mathbf{CM}}}$. Assume that $\Gamma \vdash_{\mathsf{w}} \varphi$. As Γ is maximal, we have that $\varphi \in \Gamma$ or $(\varphi \to \bot) \in \Gamma$. If the former, we are done. If the latter, we obtain a contradiction as $(\varphi \to \bot) \in \Gamma$ and $\Gamma \vdash_{\mathsf{w}} \varphi$ leads to $\Gamma \vdash_{\mathsf{w}} \bot$. In both cases, we get $\varphi \in \Gamma$. ■

Then, we need the following lemma, ensuring that the canonical model is properly built: if a point Γ is such that all its successors contain φ, then $\Box\varphi$ must be in Γ.

Lemma 6.1.6 (Existence Lemma)**.** For every $\Gamma \in W^c$ and φ:

$$[\ \varphi \in \Gamma' \text{ for all } \Gamma' \in W^c \text{ such that } \Gamma R^c \Gamma'\ ] \quad \text{implies} \quad \Box\varphi \in \Gamma.$$

Proof. (⌨) Let $\Gamma \in W^c$ and φ, and assume that $\varphi \in \Gamma'$ for all $\Gamma' \in W^c$ such that $\Gamma R^c \Gamma'$. We proceed to show that $\Box\varphi \in \Gamma$. As $\Gamma \in W^c$, we have that Γ is maximal. Consequently, we have that $\Box\varphi \in \Gamma$ or $(\Box\varphi \to \bot) \in \Gamma$. In the former case, we are done. We proceed to show that the latter leads to a contradiction. Let us divide Γ into two sets Γ_0, of non-boxed formulas, and $\Box\Gamma_1$ of boxed formulas.

First, we show that $\Gamma_1, \varphi \to \bot \nvdash_{\mathsf{w}} \bot$. Assume for a contradiction that $\Gamma_1, \varphi \to \bot \vdash_{\mathsf{w}} \bot$. Then, we can use Theorem 4.1.1 to obtain $\Gamma_1 \vdash_{\mathsf{w}} (\varphi \to \bot) \to \bot$. However, we can show that $((\varphi \to \bot) \to \bot) \to \varphi$ is a theorem of $\mathsf{w}\mathbf{KH}$ (⌨), so we easily get $\Gamma_1 \vdash_{\mathsf{w}} \varphi$. Thus, we can apply the following admissible (⌨) rule to obtain $\Box\Gamma_1 \vdash_{\mathsf{w}} \Box\varphi$.

$$\frac{\Gamma' \vdash \psi}{\Box\Gamma' \vdash \Box\psi}$$

Thus, using Theorem 4.1.1 we obtain $\Gamma \vdash_{\mathsf{w}} \Box\varphi$. However, as $(\Box\varphi \to \bot) \in \Gamma$, we also have that $\Gamma \vdash_{\mathsf{w}} \Box\varphi \to \bot$. Thus, we straightforwardly obtain $\Gamma \vdash_{\mathsf{w}} \bot$, using MA_4 and (MP). But this is a contradiction, as Γ is consistent. So, we have that $\Gamma_1, \varphi \to \bot \nvdash_{\mathsf{w}} \bot$.

Second, we use Lemma 6.1.4 with $\Gamma_1, \varphi \to \bot \nvdash_{\mathsf{w}} \bot$ to obtain a maximal consistent set Γ' such that $\Gamma_1 \cup \{\varphi \to \bot\} \subseteq \Gamma'$. Moreover, we can show that $\Gamma R^c \Gamma'$. Indeed, we have that $\varphi \in \Gamma'$ for any φ such that $\Box\varphi \in \Gamma$: this holds because Γ' is an extension of Γ_1, while $\Box\Gamma_1$ is the set of all boxed formulas of Γ.

Finally, as $\Gamma R^c \Gamma'$ and our initial assumption that $\varphi \in \Gamma''$ for all $\Gamma'' \in W^c$ such that $\Gamma R^c \Gamma''$, we obtain $\varphi \in \Gamma'$. Now, we can easily obtain $\Gamma' \vdash_{\mathsf{w}} \bot$: $\varphi \to \bot \in \Gamma'$ as $\Gamma_1 \cup \{\varphi \to \bot\} \subseteq \Gamma'$, and $\varphi \in \Gamma'$ as we just argued. But this is a contradiction, as Γ' is consistent. So, we trivially get that $\Box\varphi \in \Gamma$.

In both cases, we obtained that $\Box\varphi \in \Gamma$, so we are done. ■

We then turn to the Truth Lemma, which relates forcing and elementhood in the canonical model: $\mathcal{M}^c, \Gamma \Vdash \varphi$ if and only if $\varphi \in \Gamma$. Note that our use of *maximal* consistent set entails that we have equivalently that $\mathcal{M}^c, \Gamma \nVdash \varphi$ if and only if $\varphi \notin \Gamma$. In essence, the Truth Lemma shows that a maximal consistent set $\Gamma \in W^c$ is a full description of which formula ψ is forced in the point Γ, if $\psi \in \Gamma$, and which formula is not, if $\psi \notin \Gamma$.

Lemma 6.1.7 (Truth Lemma)**.** For every $\Gamma \in W^c$:

$$\psi \in \Gamma \quad \text{iff} \quad \mathcal{M}^c, \Gamma \Vdash \psi.$$

Proof. (⌨) By induction on ψ.

- $\psi := p$: by definition $p \in \Gamma$ iff $\Gamma \in I^c(p)$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash p$.
- $\psi := \bot$: we have that $\bot \notin \Gamma$ as otherwise we get $\Gamma \nvdash_{\mathsf{w}} \bot$ which is a contradiction. In addition to that, we have $\mathcal{M}^c, \Gamma \nVdash \bot$ by definition. So, we have $\bot \in \Gamma$ iff $\mathcal{M}^c, \Gamma \Vdash \bot$ trivially.

- $\psi := \psi_1 \to \psi_2$: ($\Rightarrow$) Assume $\psi_1 \to \psi_2 \in \Gamma$. We need to show that $\mathcal{M}^c, \Gamma \Vdash \psi_1 \to \psi_2$. Assume $\mathcal{M}^c, \Gamma \Vdash \psi_1$. We need to show that $\mathcal{M}^c, \Gamma \Vdash \psi_2$. By the induction hypothesis, it is sufficient to show that $\psi_2 \in \Gamma$. From $\mathcal{M}^c, \Gamma \Vdash \psi_1$ we obtain $\psi_1 \in \Gamma$ by induction hypothesis. Also, we have $\psi_1 \to \psi_2 \in \Gamma$. Thus, it is straightforward to prove that $\Gamma \vdash_{\mathsf{w}} \psi_2$ using (MP). Using Lemma 6.1.5 we get $\psi_2 \in \Gamma$. The induction hypothesis gives us $\mathcal{M}^c, \Gamma \Vdash \psi_2$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, \Gamma \Vdash \psi_1 \to \psi_2$. As Γ is maximal, we have that $\psi_1 \in \Gamma$ or $(\psi_1 \to \bot) \in \Gamma$. In the first case, we can obtain $\mathcal{M}^c, \Gamma \Vdash \psi_1$ using the induction hypothesis. Consequently, as $\mathcal{M}^c, \Gamma \Vdash \psi_1 \to \psi_2$ we get $\mathcal{M}^c, \Gamma \Vdash \psi_2$. Using the induction hypothesis again, we obtain $\psi_2 \in \Gamma$. Then, it is straightforward to use $\psi_2 \to (\psi_1 \to \psi_2)$, an instance of MA_2, to prove that $\Gamma \vdash_{\mathsf{w}} \psi_1 \to \psi_2$. We get $\psi_1 \to \psi_2 \in \Gamma$ using Lemma 6.1.5. In the second case, where we have $(\psi_1 \to \bot) \in \Gamma$, we can easily obtain that $\Gamma, \psi_1 \vdash_{\mathsf{w}} \bot$ using (MP) and (El). Consequently, we can prove $\Gamma, \psi_1 \vdash_{\mathsf{w}} \psi_2$ using MA_4 and (MP). Thus, it suffices to apply Theorem 4.1.1 to get $\Gamma \vdash_{\mathsf{w}} \psi_1 \to \psi_2$. By Lemma 6.1.5, we obtain $\psi_1 \to \psi_2 \in \Gamma$. In both cases we reach $\psi_1 \to \psi_2 \in \Gamma$.

- $\psi := \Box\psi_1$: ($\Rightarrow$) Assume $\Box\psi_1 \in \Gamma$. We need to show $\mathcal{M}^c, \Gamma \Vdash \Box\psi_1$. Let $\Gamma' \in W^c$ such that $\Gamma R^c \Gamma'$. We need to show $\mathcal{M}^c, \Gamma' \Vdash \psi_1$. By the induction hypothesis, it is sufficient to show $\psi_1 \in \Gamma'$. Now, as $\Gamma R^c \Gamma'$ and by definition of R^c, we get that $\psi_1 \in \Gamma'$ as $\Box\psi_1 \in \Gamma$. So, we are done.

  ($\Leftarrow$) Assume $\mathcal{M}^c, \Gamma \Vdash \Box\psi_1$. Then, for all $\Gamma' \in W^c$ such that $\Gamma R^c \Gamma'$ we have $\mathcal{M}^c, \Gamma' \Vdash \psi_1$. Consequently, using the induction hypothesis we get that all such Γ' are such that $\psi_1 \Gamma'$. Then, we can use Lemma 6.1.6 to obtain $\Box\psi_1 \in \Gamma$. So, we are done.

■

We are now ready to prove that **wBIL** is complete with respect to the local semantic consequence relation.

Theorem 6.1.4. The following holds:

$$\Gamma \models_l \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathsf{w}} \varphi$$

Proof. (⌨) We reason contrapositively by showing that $\Gamma \nvdash_{\mathsf{w}} \varphi$ implies $\Gamma \not\models_l \varphi$, which is classically equivalent to our goal. Assume $\Gamma \nvdash_{\mathsf{w}} \Delta$. Lemma 6.1.4 gives us a maximal consistent set Γ' such that $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash_{\mathsf{w}} \varphi$. So, by Lemma 6.1.7 we obtain that in the canonical model of Definition 6.1.8 the following holds: $\mathcal{M}^c, \Gamma' \nVdash \varphi$, while $\mathcal{M}^c, [\Gamma \mid '] \Vdash \Gamma$. Consequently, we have that $\Gamma \not\models_l \varphi$. ■

Completeness of sKH

Now, we show that **sKL** is complete with respect to the global semantic consequence relation. As mentioned earlier, we rely on the completeness of **wBIL** with respect to the local semantic consequence shown just above. More precisely, we show that if a pair $\Gamma \vdash \varphi$ is unprovable in **sBIH**, then through Theorem 6.1.4 we extract the existence of a model $\mathcal{M}$ having a point w forcing $\Box^\omega\Gamma$ (which we define below) and not forcing φ. But this is not enough because $\mathcal{M}$ is not necessarily a Γ-model. Therefore, we make use of this model $\mathcal{M}$ by restricting it to another model containing w which is a Γ-model and where w still does not force φ, making use of a bisimulation between the two models. Thus, the latter model witnesses $\Gamma \not\models_g \varphi$.

First, we define what the set of formula $\Box^\omega\Gamma$ is.

Definition 6.1.9. We define:

1. for $n \in \mathbb{N}$, let $\Box^0\varphi := \varphi$ and let $\Box^{(n+1)}\varphi := \Box\Box^n\varphi$;
2. $\Box^n\Gamma = \{\Box^n\gamma \mid \gamma \in \Gamma\}$;

3. $\Box^\omega\Gamma = \bigcup\limits_{n\in\mathbb{N}} \Box^n\Gamma$.

Second, we show how to restrict a model $\mathcal{M}$ from a point w. Often in the literature, this restricted model is called the *point generated submodel* [11, p.56].

Definition 6.1.10. Let $\mathcal{M} = (W, R, I)$ be a model and $w \in W$. The *restriction of* $\mathcal{M}$ *in* w is the model $\mathcal{M}^w = (W^w, R^w, I^w)$, where:

- $W^w = \{v \in W \mid \text{ there is a chain } wR\ldots Rv\}$;
- $R^w \;=\; R \;\cap\; (W^w \times W^w)$;
- $I^w(p) \;=\; I(p) \cap W^w$.

As suggested above, we show that the point w in the initial model is bisimilar with itself in the restriction of the model.

Lemma 6.1.8. For all model $\mathcal{M} = (W, R, I)$ and $w \in W$, we have that $\mathcal{M}, w \leftrightharpoons \mathcal{M}^w, w$.

Proof. (⌨) We need to exhibit a bisimulation $B \subseteq (W \times W^w)$. Consider the relation $B = \{(v, v) \mid v \in W^w\}$. Note that $B \subseteq (W \times W^w)$, as required. We are left to show that B satisfies the conditions of the Definition 5.1.5. Let $v \in W^w$.

(B_1) Let $p \in \mathbb{V}$. We have $v \in I(p)$ iff $v \in I^w(p)$ by definition of the restriction.

(B_2) Let $u \in W^w$ such that vR^wu. As $u \in W^w$, we get that $u \in W$. Finally, as vR^wu we get that vRu by the definition of the restriction, so we are done.

(B_3) Let $u \in W$ such that vRu. As $v \in W^w$, we get that there is a chain $wR\ldots Rv$. So, there is a chain $wR\ldots RvRu$ as vRu. Consequently, we get that $u \in W^w$. Finally, as vRu we get that vR^wu, so we are done.

■

However, Lemma 5.1.1 informs us that bisimulations entail logical equivalence. So, we easily obtain the following corollary showing that the two models force the same formulas in w.

Corollary 6.1.1. For all model $\mathcal{M} = (W, R, I)$, $w \in W$ and $\varphi \in \mathbb{L}_{\mathbf{CM}}$:

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{M}^w, w \Vdash \varphi$$

Proof. (⌨) As we have that $\mathcal{M}, w \leftrightharpoons \mathcal{M}^w, w$ by Lemma 6.1.8, we know by Proposition 5.1.1 that $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^w, w \Vdash \varphi$. ■

Let us recall that our intention is to obtain a model forcing Γ *everywhere* out of a model forcing $\Box^\omega\Gamma$ in *one point.* Given a model $\mathcal{M}$ and a point w such that $\mathcal{M}, w \Vdash \Box^\omega\Gamma$, we can obtain the promised model: it is the restriction of $\mathcal{M}$ in w. We can sustain our claim by showing that if $\mathcal{M}, w \Vdash \Box^\omega\Gamma$, then for all $v \in W^w$ we have $\mathcal{M}^w, v \Vdash \Gamma$. This is the essence of the next lemma.

Lemma 6.1.9. Let $n \in \mathbb{N}$ be a natural number, $\mathcal{M} = (W, R, I)$ a model, $w, v \in W$ points, $\varphi \in Form_{\mathbb{L}_{\mathbf{CM}}}$ a formula and $wR\ldots Rv$ a chain in $\mathcal{M}$. If $\mathcal{M}, w \Vdash \Box^n\varphi$, then $\mathcal{M}, v \Vdash \varphi$.

Proof. (⌨) We prove the statement by induction on n.

- $n = 0$: then $v = w$ as the chain $wR\ldots Rv$ is of length 0. Consequently $\mathcal{M}, w \Vdash \varphi$ as $\mathcal{M}, w \Vdash \Box^0\varphi$ by assumption.

- $n = S\ m$: then there is u such that $wR \dots RuRv$ where the length of the chain $wR \dots Ru$ is of length m. Given that $\mathcal{M}, w \Vdash \Box^{Sm}\varphi$ by assumption, we easily obtain that $\mathcal{M}, w \Vdash \Box^m(\Box\varphi)$ as $\Box^{Sm}\varphi = \Box^m(\Box\varphi)$. Thus, we can use the induction hypothesis to obtain that $\mathcal{M}, u \Vdash \Box\varphi$. Consequently, as we have uRv we get by the definition of the semantics that $\mathcal{M}, v \Vdash \varphi$.

■

More formally, we now exploit these ideas to show the completeness of $\mathsf{s}\mathbf{KL}$ with respect to the global semantic consequence.

Theorem 6.1.5. The following holds:

$$\Gamma \models_g \varphi \quad \text{implies} \quad \Gamma \vdash_{\mathsf{s}} \varphi$$

Proof. (⌨) We reason contrapositively by showing that $\Gamma \nvdash_{\mathsf{s}} \varphi$ implies $\Gamma \not\models_g \varphi$, which is classically equivalent to our goal. Assume $\Gamma \nvdash_{\mathsf{s}} \varphi$. We show that $\Gamma \not\models_g \varphi$. Note that for all $\gamma \in \Box^\omega\Gamma$ we have $\Gamma \nvdash_{\mathsf{s}} \gamma$, as the rule (sNec) is applicable at will with any context in $\mathsf{s}\mathbf{KH}$. Consequently, we have that $\Box^\omega\Gamma \nvdash_{\mathsf{s}} \varphi$, else we would get $\Gamma \vdash_{\mathsf{s}} \varphi$ using Lemma 4.1.1. Thus, we get $\Box^\omega\Gamma \nvdash_{\mathsf{w}} \varphi$ by item 1 of Lemma 4.1.3. By Theorem 6.1.4 we know that $\Box^\omega\Gamma \not\models_l \Delta$. So, there is a model $\mathcal{M} = (W, R, I)$ and $w \in W$ such that $\mathcal{M}, w \Vdash \Box^\omega\Gamma$ and $\mathcal{M}, w \nVdash \varphi$. Now we consider the restriction of $\mathcal{M}$ in w, i.e. the model $\mathcal{M}^w = (W^w, R^w, I^w)$. As shown in Corollary 6.1.1 we have that $\mathcal{M}^w, w \Vdash \Box^\omega\Gamma$ and $\mathcal{M}^w, w \nVdash \varphi$. If we prove that $\mathcal{M}^w \Vdash \Gamma$, then we are done as we would then have exhibited a Γ-model $\mathcal{M}^w$ which has one point w that not a φ-point, hence $\Gamma \not\models_g \varphi$. We thus proceed to show that $\mathcal{M}^w \Vdash \Gamma$. Let $v \in W^w$ and $\gamma \in \Gamma$. We need to show $\mathcal{M}^w, v \Vdash \gamma$. As $v \in W^w$ we know that there is a chain $wR \dots Rv$. We straightforwardly obtain that $\mathcal{M}^w, w \Vdash \Box^n\gamma$ as $\mathcal{M}^w, w \Vdash \Box^\omega\Gamma$. We finally obtain that $\mathcal{M}^w, v \Vdash \gamma$ using Lemma 6.1.9 using $\mathcal{M}^w, w \Vdash \Box^n\gamma$ and the existence of the chain $wR \dots Rv$. As v and γ are arbitrary we get $\mathcal{M}^w \Vdash \Gamma$. ■

6.2 The classical modal logic example: in Coq

In this section, we provide the formalisation of the soundness and completeness results given above.

6.2.1 Soundness

All the elements presented in this subsection can be found here: ⌨.

We start with soundness. To obtain soundness of both generalized Hilbert calculi with respect to their semantic consequence relations, we first proved that any instance of an axiom in $\mathcal{A}_K$ is valid. This result is straightforwardly formalised as follows (⌨).

```
Lemma Ax_valid : forall φ,
        KAxioms φ -> (forall M w, wforces M w φ).
```

Using this result, we could obtain soundness results linking $\mathsf{w}\mathbf{KH}$ and $\mathsf{s}\mathbf{KH}$ with respectively the local and the global semantic consequence relations. These two results are formalised below (⌨) (⌨).

```
Theorem wSoundness : forall Γ φ,
        wKH_prv (Γ, φ) -> (loc_conseq Γ φ).
```

```
Theorem sSoundness : forall Γ φ,
        wKH_prv (Γ, φ) -> (loc_conseq Γ φ).
```

The two consequences we could draw from the soundness results above, presented in Lemma 6.1.2, are formalised as follows (⌨) (⌨).

```
Lemma Consequences_Soundness1 :
        exists s, (sKH_prv s) /\ ((wKH_prv s) -> False).
```

```
Lemma Consequences_Soundness2 :
    (sKH_prv (Singleton _ # p , Box # p)) /\
    ((sKH_prv (Empty_set _, (# p) --> (Box # p))) -> False).
```

6.2.2 Completeness

All the elements presented in this subsection can be found here: ⌨.

Below, we first show the formalisation of the results leading to the completeness of **wKH** and then turn to **sKH**.

Completeness of wKH

Let us recall that the structure of the argument for the completeness of **wKH** goes through the following elements: (1) a Lindenbaum lemma, (2) a canonical model construction, (3) an existence lemma, (4) a truth lemma, (5) a final proof of completeness. We show how we formalise each of these steps below.

First, we need to attain the Lindenbaum lemma, i.e. Lemma 6.1.4. The proof of this lemma relies on encoding and decoding functions for the set of propositional formulas. We assume given an encoding function `encode0`, with its injectivity as hypothesis (⌨).

```
Parameter encode0 : MPropF -> nat.

Hypothesis encode0_inj : forall φ ψ n,
        (encode0 φ = encode0 ψ) -> φ = ψ.
```

With this encoding function in hand, we can proceed as in Definition 6.1.2 and define the encoding function we effectively use (⌨).

```
Definition encode : MPropF -> nat := fun x => S (encode0 x).
```

Thus, `encode` is a function making sure that no formula is encoded as 0, as it takes the initial encoding of `encode0` and adds one using the successor function on natural numbers `S`.

We formalises Lemma 6.1.3, showing the existence of a decoding function for `encode`, as follows (⌨).

```
Lemma decoding_inh :
  {g : nat -> option (MPropF) |
    (forall A, g (encode A) = Some A) /\
    (forall n,
      (((exists A, encode A = n) -> False) -> g n = None)) }.
```

This lemma claims that the type described on lines 2 to 4 is *inhabited*: there is an element of that type. This element is a function `g` of type `nat -> option (MPropF)`, which we intend to be a decoding function for `encode`. To do so, the function is required to satisfy the conjunction of line 3 with the property expressed on lines 4 and 5. The first conjunct requires that the composition of `g` and `encode` on `A` gives back the option `Some A`. Thus, we can retrieve the encoded formula using the function `g`, as required by item 1 of Definition 6.1.4. The second conjunct corresponds to the clause expressed by item 2 of the same definition. Clearly, this second property requires that if a natural number `n` is the encoding of no formula, i.e. `(exists A, encode A = n) -> False`, then on the output corresponding to `n` via the function `g` is `None`. Put together, these two properties give us a decoding function for `encode`.

Then, we can obtain a decoding function for `encode` by extracting the witness of the above lemma using `proj1_sig`, which is a decoding function (⌨).

```
Definition decoding : nat -> option (MPropF) :=
        proj1_sig decoding_inh.
```

Next, we proceed to the definition in two steps of our selection function sel given in Definition 6.1.5. First, we define how to choose a formula (⌨).

```
Definition choice_form Γ φ ψ : Ensemble MPropF :=
fun x => (In _ Γ x) \/
(((wKH_prv (Union _ Γ (Singleton _ ψ), φ) -> False) -> x = ψ)
/\
(wKH_prv (Union _ Γ (Singleton _ ψ), φ) -> x = (ψ --> Bot))).
```

The function `choice_form`, when given a formula Γ and two formulas φ and ψ outputs a set of formulas. As explained above, in Coq an element of type `Ensemble MPropF` is a function from `MPropF` to `Prop` which can be thought of as the characteristic function of the set. Thus, the set above has as elements the formulas which satisfy the property described after `fun x =>`. We can thus see that `x` can be an element if it is an element of Γ, i.e. `In _ Γ x`, or (`\/`) if it satisfies the conjunction given in the last three lines. This conjunction is a case distinction on the holding of `wKH_prv (Union _ Γ (Singleton _ ψ), φ)`. If it does not, then we add `x` to the set we are constructing if it is equal to ψ, as expressed by the first conjunct. If it does, then we add `x` to the set if it is equal to $\psi - - > Bot$, as expressed by the second conjunct. As we are using classical logic as meta-logic, we can see that it is either the case that `(wKH_prv (Union _ Γ (Singleton _ ψ), φ))` or `( wKH_prv (Union _ Γ (Singleton _ ψ), φ) --> False)` holds. Consequently, we can infer that `choice_form` adds either ψ or `ψ --> Bot` to the set we are building. This looks already very similar to the two last cases of Definition 6.1.5.

Second, we formalise the selection function sel which takes as input a natural number (⌨).

```
Definition choice_code Γ φ n : Ensemble MPropF :=
  match decoding n with
    | None => Γ
    | Some ψ => choice_form Γ φ ψ
  end.
```

This function is such that if it is given a natural number not corresponding to an encoding, then it outputs the set of formulas Γ it was given. This corresponds to the first case of Definition 6.1.5. If the natural number `n` does correspond to an encoding, then the outputted set of formulas is nothing but `choice_form Γ φ ψ`, which extends Γ with ψ or `ψ --> Bot`. We see that we thus formalised the two last cases of Definition 6.1.5.

The selection function above allows us to formalise the function Lindf from Definition 6.1.6, which helps us build step-by-step our Lindenbaum extension (⌨). It strikingly resembles the pen-and-paper version.

```
Fixpoint Lindf Γ φ n : Ensemble MPropF :=
  match n with
    | 0 => Γ
    | S m => choice_code (Lindf Γ φ m) φ (S m)
  end.
```

We can thus formalise the Lindenbaum extension of a set Γ for a formula φ from Definition 6.1.6 (⌨).

```
Definition Lind Γ φ : Ensemble MPropF :=
        fun x => (exists n, In _ (Lindf Γ φ n) x).
```

Given its two arguments, it creates the set of all formulas which is an element of `Lindf Γ φ n`, for some `n`. This set corresponds to $\bigcup\limits_{n=0}^{\infty}(\mathsf{Lindf}(\Gamma,\varphi,n))$.

While we embedded the proof of various properties in the Lindenbaum lemma (Lemma 6.1.4), we decided in the formalisation to separate them into different statements. So, we started by showing that the Lindenbaum extension of a set Γ for a formula φ is an extension of Γ (⌨).

```
Lemma Lind_extens : forall Γ φ x,
        In _ Γ x -> In _ (Lind Γ φ) x.
```

Then, we formalised the fact that if the pair`(Γ, φ)` is unprovable, then so is the pair `(Lindf Γ φ n, φ)` for any `n` (⌨).

```
Lemma Unprv_Lindf : forall n Γ φ,
        (wKH_prv (Γ, φ) -> False) ->
        (wKH_prv (Lindf Γ φ n, φ) -> False).
```

The latter results leads to the unprovability of the pair `(Lind Γ φ, φ)` with the Lindenbaum extension (⌨).

```
Lemma Unprv_Lind : forall Γ φ,
        (wKH_prv (Γ, φ) -> False) ->
        (wKH_prv (Lind Γ φ , φ) -> False).
```

In addition to that, we could formalise the maximality of the Lindenbaum extension, expressed in the two last lines (⌨).

```
Lemma maximality_Lind_extens : forall Γ φ,
        (wKH_prv (Γ, φ) -> False) ->
        (forall ψ,
   (In _ (Lind Γ φ) ψ) \/ (In _ (Lind Γ φ) (ψ --> Bot))).
```

And the last result before our Lindenbaum lemma is the consistency of the Lindenbaum extension (⌨).

```
Lemma Consist_Lind : forall Γ φ,
        (wKH_prv (Γ, φ) -> False) ->
        (wKH_prv (Lind Γ φ, Bot) -> False).
```

Our formalisation of the Lindenbaum lemma is as follows. In essence, it shows that if the pair `(Γ, φ)` is unprovable, then there is a `Γm` which extends Γ (third line), is maximal (fourth line), and makes the pair `(Γm, φ)` unprovable (⌨).

```
Lemma Lindenbaum_lemma : forall Γ φ,
    (wKH_prv (Γ, φ) -> False) ->
    (exists Γm, Included _ Γ Γm /\
               (forall ψ, (Γm ψ) \/ (Γm (ψ --> Bot))) /\
               (wKH_prv (Γm, φ) -> False)).
```

Second, we formalise our canonical model construction as in Definition 6.1.8. To do so, we define the property of being a maximal consistent set (⌨).

```
Definition is_mcs Γ : Prop :=
        ((forall φ, (Γ φ) \/ (Γ (φ --> Bot))) /\
        (wKH_prv (Γ, Bot) -> False)).
```

As we can see from the definition, we have that `is_mcs Γ` if Γ is maximal, as shown in the second line, and consistent, as per the third line. With this property in hand, we can create the type of all elements satisfying this property (⌨).

```
Definition Canon_worlds : Type :=
        {x : Ensemble MPropF | is_mcs x}.
```

An element of `Canon_worlds` is a pair: a first element `x` which is of type `Ensemble MPropF`, and a second element which is a proof of `is_mcs x`, i.e. that `x` is a maximal consistent set. We use this type of elements as nodes in the canonical model we build. To obtain the canonical relation, we use the following relation between elements of `Canon_worlds` (⌨).

```
Definition Canon_rel (cw0 cw1 : Canon_worlds) : Prop :=
  forall φ, (In _ (proj1_sig cw0) (Box φ)) ->
            (In _ (proj1_sig cw1) φ).
```

As `cw0` and `cw1` are canonical worlds, we have that they are pairs. Consequently, `proj1_sig cw0` and `proj1_sig cw1` refer to the sets of formulas which are their respective first elements. Thus, `(In _ (proj1_sig cw0) (Box φ))` expresses the fact that the formula $\Box\varphi$ is an element of the set of formulas underlying `cw0`. With this reading in mind, we can see that the above formalises the canonical relation R^c given in Definition 6.1.8.

The canonical valuation can be understood in a similar way (⌨).

```
Definition Canon_val (cw : Canon_worlds) (q : V) : Prop :=
        In _ (proj1_sig cw) (# q).
```

Thus, we have all the components to define our canonical model `CM`. It is nothing but an `Instance` of a `kmodel` as defined in Subsection 5.1.2 (⌨).

```
Instance CM : kmodel :=
      {|
        nodes := Canon_worlds ;
        reachable := Canon_rel ;
        val := Canon_val
      |}.
```

Third, we turn to the formalisation of the existence lemma, i.e. Lemma 6.1.6 (⌨).

```
Lemma existence_lemma : forall φ (cw : Canon_worlds),
  (forall cw0, Canon_rel cw cw0 -> In _ (proj1_sig cw0) φ) ->
        (In _ (proj1_sig cw) (Box φ)).
```

The assumption of Lemma 6.1.6, given below is expressed by the second line.

$$[\ \varphi \in \Gamma' \text{ for all } \Gamma' \in W^c \text{ such that } \Gamma R^c \Gamma'\ ]$$

Its conclusion $\Box\varphi \in \Gamma$ appears on the second line.

Fourth, the Truth lemma (Lemma 6.1.7) is straightforwardly formalised as follows (⌨).

```
Lemma truth_lemma : forall φ (cw : Canon_worlds),
        (wforces CM cw φ) <-> (In _ (proj1_sig cw) φ).
```

Fifth, and finally, we can turn to the formalisation of our completeness result for wKH. Without using an explicit classical step in our meta-logic, we could prove a result of "counter" completeness: the contrapositive of the completeness result (⌨).

```
Theorem wCounterCompleteness : forall Γ φ,
    (wKH_prv (Γ, φ) -> False) -> ((loc_conseq Γ φ) -> False).
```

However, by relying on the former result and the use of classical logic, we can finally obtain in a non-constructive way our completeness result for wKH (⌨).

```
Theorem wCompleteness : forall Γ φ,
        (loc_conseq Γ φ) -> wKH_prv (Γ, φ).
```

Completeness of sKH

Below, we give the formalisation of the definitions and results leading to the completeness of sKH with respect to the global semantic consequence relation. To do so, we show how to restrict models, and then provide the formalisation of the completeness results.

To start, we formalise elements of Definition 6.1.9. First, we define $\Box^n\varphi$ (⌨).

```
Fixpoint Box_power n (φ : MPropF) : MPropF :=
  match n with
   | 0 => φ
   | S m => Box (Box_power m φ)
  end.
```

While in Definition 6.1.9 we defined $\Box^\omega\Gamma$ to be the union of all $\Box^n\Gamma$ for all n, in the formalisation we preferred to use an equivalent alternative. More precisely, we defined $\Box^\omega\Gamma$ as the *closure of* Γ *under* $\Box$ (⌨).

```
Inductive Box_clos_set Γ : Ensemble MPropF :=
  | InitClo : forall φ, In _ Γ φ -> Box_clos_set Γ φ
  | IndClo : forall φ,  Box_clos_set Γ φ ->
                        Box_clos_set Γ (Box φ).
```

The clause `InitClo` forces all elements of Γ to be in `Box_clos_set` Γ, while the clause `IndClo` requires that any element φ in `Box_clos_set` Γ must be such that `Box` φ is also in `Box_clos_set` Γ. So, we obviously get that `Box_clos_set` Γ is Γ closed under $\Box$, which is nothing but $\Box^\omega\Gamma$.

While the latter comes in handy in the proof f completeness, we now focus on the definition of restricted models. To define these models in a general way, we need the notion of chain for a given relation R (⌨).

```
Fixpoint n_reachable {W : Type} (R : W -> W -> Prop) (n: nat)
    (w v : W) : Prop :=
  match n with
    | 0 => w = v
    | S m => exists u, (n_reachable R m w u) /\ (R u v)
  end.
```

Clearly, we have that `n_reachable R n w v` if there is a chain of `n` `R`-steps from `w` to `v`. The case where `n` is 0 ensures us that `w` is reachable from itself through no `R`-step, while the inductive case allows us to build effective `R`-steps.

With this notion, we can define the property of being reachable for some n, which will help us define the restriction on models we impose (⌨).

```
Definition is_reachable {W : Type} R w v : Prop :=
        exists n, @n_reachable W R n w v.
```

Now, we can define the type of elements which are reachable from `w` through `R`, given a relation `R` and an element `w`. Note that all of the definitions below are parametrized in an element `w`, the point from where we generate the restricted model (⌨).

```
Definition restr_worlds {W : Type} R w : Type :=
        { x : W | is_reachable R w x}.
```

The above type will constitute the type of nodes for our restricted models. We proceed to define the restricted relation and valuation (⌨).

```
Definition restr_rel {W : Type} R w
        (rw0 rw1 : @restr_worlds W R w) : Prop :=
                R (proj1_sig rw0) (proj1_sig rw1).
```

Similarly to the canonical model construction, when we create a type of restricted nodes, we are enforcing elements of this type to be pairs of an element `v` and a proof that `v` satisfies the property we used, here being reachable. So, `R (proj1_sig rw0) ( proj1_sig rw1)` expresses the fact that the elements underlying `rw0` and `rw1`, i.e. their first element, are in the relation `R`. This implies that the restricted relation is simply the relation `R` restricted to the new set of nodes.

A restricted valuation can be obtained from a valuation `val` in a similar way (⌨).

```
Definition restr_val {W : Type} R (val : W -> V -> Prop) w
        (rw : @restr_worlds W R w) (p : V) : Prop :=
                val (proj1_sig rw) p.
```

We now have enough material to define the restriction of a model `M` in `w`, as in Definition 6.1.10 (⌨).

```
Definition restr_model (M : kmodel) (w : nodes) : kmodel :=
      {| nodes := (restr_worlds (@reachable M) w) ;
        reachable := (restr_rel (@reachable M) w) ;
        val := (restr_val (@reachable M) (@val M) w) |}.
```

As `M` is a `kmodel`, it has implicitly a collection of `nodes`, a relation `reachable` and a valuation `val`. However, to define the restricted model we need to make these notions explicitly appear, as they need to be arguments in the functions we defined above. To do so, we make use of `@`, which forces the implicit arguments to be made explicit. So, by writing `@reachable M` and `@val M`, we specify that the relation and valuation we are mentioning are indeed those of `M`. Having this in mind, we can see that the above formalises Definition 6.1.10.

Finally, we can turn to the formalisation of our completeness result for s**KH**, which we obtained via a proof involving restricted models. As in the case for w**KH**, we needed to use a non-constructive step to obtain completeness from counter-completeness (⌨) (⌨).

```
Theorem sCounterCompleteness : forall Γ φ,
    (sKH_prv (Γ, φ) -> False) ->
    ((glob_conseq Γ φ) -> False).
```

```
Theorem sCompleteness : forall Γ φ,
    (glob_conseq Γ φ) ->
    sKH_prv (Γ, φ).
```

Part II

Bi-Intuitionistic Logics

Chapter 7

Forewords on Bi-Intuitionistic Logics

Bi-intuitionistic logics are extensions of intuitionistic logics, both in propositional and first-order cases. More precisely, an extra binary connective $\prec$ is added to the intuitionistic language. Throughout this dissertation we call this connective the *exclusion*, while it can be called *subtraction* or *co-implication* elsewhere. The interest in studying this connective lies in its major peculiarity: it is the *dual* of the intuitionistic arrow $\rightarrow$, similarly that the conjunction $\wedge$ and disjunction $\vee$ are duals of each other. This duality is given an account of in the intended meaning of the exclusion: while the intended meaning of $\rightarrow$ is the notion of *implication*, it is the *refutation* or *exclusion* which is aimed at with the connective $\prec$. Consequently, the formula $\varphi \prec \psi$ is read "φ excludes ψ" or "φ refutes ψ".

The invention of bi-intuitionistic logic is commonly attributed to Cecylia Rauszer. She published a series of interdependent articles [115, 116, 117, 118, 119, 120, 121], from 1974 to 1977 culminating in her Ph.D. thesis in 1980 [122], characterizing this logic under various aspects. She first introduced propositional bi-intuitionistic logic in 1974 via an axiomatic calculus [116], which she showed to have a corresponding algebraic semantics on what she then called "semi-Boolean algebras" but are now also commonly known as "bi-Heyting algebras" [125]. Then, she provided a sequent calculus for propositional bi-intuitionistic logic and claimed a cut-admissibility result [115]. In 1976 Rauszer turned to a Kripke semantics for first-order bi-intuitionistic logic: she published a strong completeness result for the Hilbert system for first-order bi-intuitionistic logic with respect to a Kripke semantics [117]. She pursued the study of the first-order variant by defining an algebraic semantics for it [118], and returned to completeness results with respect to the Kripke semantics [119]. She then obtained a version of interpolation for first-order bi-intuitionistic logic [120], but also noted that many mistakes were present in a former article of hers [117]. Finally, she published a last article [121] where classic model-theoretic results are obtained. All of the results mentioned here were gathered in her Ph.D. thesis [122], published in 1980.

For historical purposes, it has to be noted that Klemke, without naming it, introduced in 1971 a sequent calculus for first-order bi-intuitionistic logic [81] similar to Rauszer's [115]. This fact was pointed out to us by Grigory Olkhovikhov in May 2022 in private communication, right after he discovered it. However, we believe that Rauszer deserves credit for the invention of this logic for three reasons. First, Rauszer's work on bi-intuitionistic logic is independent of Klemke's as far as we can tell. Second, the breadth of Klemke's contribution to bi-intuitionistic logic is in no way comparable to Rauszer's, as shown above. Third, Klemke's introduction of a sequent calculus is simply intended as a mere tool to obtain a completeness proof for constant domain first-order intuitionistic logic [66]. Rauszer's work on the contrary is primarily focused on the development of bi-intuitionistic logic *per se*.

After Rauszer's work, bi-intuitionistic logic received a great deal of attention as witnesses an abundant literature with multiple flourishing lines of research [6, 7, 21, 24, 36, 52, 58, 59, 60, 83, 94, 101, 106, 107, 127, 142, 152]. Alas, reviewing this literature on bi-intuitionistic logics can be quite confusing, because, in many places, the status of theorems is unclear if not puzzling. An account of this confusion can be given by three elements.

First, as is well-known, the usual deduction theorem is: $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \varphi \to \psi$. However the "deduction theorem" is claimed by Rauszer under the following various forms in chronological order: (1) $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \neg{\sim} \dots \neg{\sim}\varphi \to \psi$ [116]; (2) the usual version above [117]; an explicit retraction of (2) and replacement by (3) $\Gamma, \varphi \vdash \psi$ iff $\Gamma \vdash \neg{\sim}\varphi \to \psi$ [120]; (4) a return to (1) without retracting (3) [122]. Crolard [25] claims that yet another form of the deduction theorem fails to hold.

Second, the Pinto-Uustalu counterexample [106] not only breaks the admissibility of cut in Rauszer's sequent calculus [115] for bi-intuitionistic logic but also casts doubts on Crolard's work on a formulas-as-types interpretation for this logic. More precisely, Crolard made the following claim: *"as a by-product of the previous properties [proved by Crolard], we obtain a new proof of this result [Rauszer's cut admissibility]"* [26, p.3].

Third, Rauszer's [119] strong completeness of bi-intuitionistic logic with respect to rooted canonical models contradicts Crolard's [25] result that it is not complete for this class.

In this part of the dissertation, we trace these confusions back to a fundamental problem in the axiomatic proof theory of propositional and first-order bi-intuitionistic logic: traditional Hilbert calculi are not designed to treat logics as consequence relations. They lead to an ambiguous notion of deduction from assumptions that can cause us to conflate distinct logics. For example, modal logic as a consequence relation splits into a *strong* and a *weak* version, as shown in Subsection 4.1.1, depending on how the *necessitation rule* is interpreted. Conflating these logics leads to great confusion, notably regarding the deduction theorem [68]. A similar phenomenon, as yet undetected, occurs in propositional and first-order bi-intuitionistic logic where traditional Hilbert calculi cannot adequately separate two interpretations of a bi-intuitionistic rule called (DN) (an analog of the necessitation rule from modal logic).

We intend to provide reliable foundations for propositional and first-order bi-intuitionistic logics. So, to avoid the presence of any mistake in our foundations, which could go undetected for another fifty years, we decided to formalise our results in Coq. We notably formalised technical and detailed completeness proofs via canonical model constructions, leading to two positive outcomes. First, we contributed to the acceptance of these techniques as formally sound. Second, our use of classical logic as meta-logic in Coq for the completion of these completeness proofs shows the potential non-constructivity of such proof techniques and paves the way to their "constructivization" in an approach à la reverse mathematics.

We naturally divide this part of the dissertation on bi-intuitionistic logics into two chapters. In Chapter 8 we focus on *propositional* bi-intuitionistic logic, and tackle the *first-order* variant in Chapter 9.

Chapter 8

Propositional Bi-Intuitionistic Logics

For this chapter the following sections of the Toolbox I are required: Section 2.1, Chapter 3, the introduction of Chapter 4, Section 4.1, Section 5.1 and Chapter 6.

The results of this chapter are all adaptations or extensions of the article "Bi-Intuitionistic Logics: A New Instance of an Old Problem" [64], written jointly with Rajeev Goré.

Their formalisation can be found here: https://github.com/ianshil/PhD_thesis/tree/main/Prop_Bi_Int.

8.1 Propositional confusions

In this chapter we treat propositional bi-intuitionistic logic using generalized Hilbert calculi, to view it from a perspective of consequence relation rather than just theoremhood. By our use of such calculi we explain and fix the fundamental problem of Rauszer's axiomatic proof theory of propositional bi-intuitionistic logic.

More precisely, we show that the bi-intuitionistic rule (DN), similarly to the modal necessitation rule (Nec), is ambiguous in a traditional context but splits into two distinct rules. As a consequence, we obtain two generalized Hilbert calculi, wBIH and sBIH, for bi-intuitionistic logic that differ only in the disambiguated version of the (DN) rule they incorporate. Unsurprisingly, these systems capture two distinct logics wBIL and sBIL, which have been conflated in parts of the literature.

Finally, the logics wBIL and sBIL are shown, respectively, to be sound and complete with respect to the Kripkean *local* and *global* semantic consequence relations for bi-intuitionistic logic, mimicking similar results in modal logic via a canonical model construction while using techniques of Sano and Stell [127].

Section 8.2 contains the problems caused by traditional Hilbert calculi in modal logics, and how generalized Hilbert calculi solve them. Rauszer's traditional Hilbert calculus, as well as the syntax for propositional bi-intuitionistic logic, is in Section 8.3. Section 8.4 contains the two generalized Hilbert calculi obtained from Rauszer's axiomatization. In Section 8.5, we show they define two extensionally distinct logics. Section 8.6 contains significant theorems distinguishing these logics. Section 8.7 defines a Kripke semantics for the bi-intuitionistic language. We prove that wBIL corresponds to the local semantic consequence relation in Section 8.8, and that sBIL corresponds to the global semantic consequence relation in Section 8.9. In Section 8.10, we use these results to prove pending claims from Sections 8.4 and 8.6. In Section 8.11, we use the distinctions between our two bi-intuitionistic logics to expose the flaws in Rauszer's results.

8.2 Theorems and consequences in classical modal logic

To explain how a rule can be ambiguous in a traditional Hilbert calculus, we use the case of the modal necessitation rule (Nec). As shown in Subsection 4.1.1, generalized Hilbert calculi clearly demarcate the existence of two modal logics based on the usual axiomatization $\mathcal{A}_{\mathsf{KL}}$ of the basic modal logic KL. More precisely, these calculi differ in the disambiguated version of the necessitation rule used. This rule, often written as in the middle where logics are defined as sets of theorems, can be interpreted either as a <u>w</u>eak or <u>s</u>trong rule as shown at left and right, once we take logics to be consequence relations.

$$\frac{\emptyset \vdash \varphi}{\Gamma \vdash \Box\varphi}\ \text{(wNec)} \qquad \frac{\varphi}{\Box\varphi}\ \text{(Nec)} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \Box\varphi}\ \text{(sNec)}$$

As presented in Section 4.1, these rules respectively define the calculi wKH and sKH. The *distinct* logics wKL and sKL they define correspond to the extensionally different *local* and *global* Kripkean semantic consequence relations [84], as covered in Chapter 6.

The most obvious example of their difference, as consequence relations, is that we have $p \vdash_{\mathsf{sKH}} \Box p$ but $p \nvdash_{\mathsf{wKH}} \Box p$. Then, the long-standing debate [68] about the modal deduction theorem is resolved immediately via two simple facts: (1) $p \vdash_{\mathsf{sKH}} \Box p$ but $\nvdash_{\mathsf{sKH}} p \to \Box p$; (2) $p \vdash_{\mathsf{wKH}} \Box p$ iff $\vdash_{\mathsf{wKH}} p \to \Box p$.

Not only does this example show that the two rules added to the same axiomatization do not capture the same logics, as consequence relations, but it also gives sufficient tools to show that these logics differ in their meta-properties. In fact, this partially explains the failure of the deduction theorem in sKL, while it is proven to hold for wKL.

That is, traditional Hilbert calculi allow us to easily confuse the logics wKL and sKL. To capture both of them in a traditional Hilbert setting, one has to provide debatable modifications to the notion of deduction. In fact, to capture sKL one defines the notion of deduction from assumptions as follows [22]:

Definition 8.2.1. A deduction of φ from assumptions Γ is a list l of formulas ending with φ such that each formula in l is an instance of an axiom of $\mathcal{A}_{\mathsf{KL}}$, a member of Γ, or follows via (MP) or (Nec) from formulas appearing earlier in l.

While this definition is natural and unproblematic, the notion of deduction from assumptions has to be bent to capture wKL:

Definition 8.2.2. A deduction of φ from assumptions Γ is a list l of formulas ending with φ, and such that every formula in l is an instance of an axiom, a member of Γ, follows from formulas appearing before it l by (MP) or *follows from a deducible formula by* (Nec).

First, this definition relies on the notion of deducibility which really should just be a special case of deduction from assumptions. Second, as it involves the deducibility of a formula in the application of (Nec), to determine if a list of formulas is a deduction from assumptions or not it is not sufficient to check the list of formulas itself. In other words, the notion defined here is not local because the application of (Nec) is conditioned on the existence of another deduction . These features bring a lot of confusion on the nature of deductions from assumptions.

A common way to avoid these contortions is to define the notion of deduction from assumptions from the notion of deduction [11, 110]:

Definition 8.2.3. A deduction of φ from assumptions Γ is a deduction of the formula $(\gamma_0 \wedge ... \wedge \gamma_n) \to \varphi$ for some $n \in \mathbb{N}$ and $\gamma_i \in \Gamma$ for $0 \leq i \leq n$.

Here, some other criticisms can be given. Mainly, it is the striking lack of generality of this definition that we address. More precisely, this definition is not general as there are four types of logics that it cannot capture. First, logics without conjunction, such as implicational ticket entailment [32], cannot be captured. Indeed, the definition above requires the presence of the conjunction, thereby excluding any logic without conjunction. Second, the same remark can be made of logics devoid of implication, such as positive modal logics [40, 35, 37] and geometric logics [1, 9]. Third, logics that are not compact are ruled out: it is in their nature to be unable, in some circumstances, to reduce an infinite set of assumptions to a finite one, while this is forced here by the presence of $\gamma_0, ..., \gamma_n$. Finally, no logic for which the deduction theorem fails can be characterized via this definition, as this theorem is built in here.

Generalized Hilbert calculi avoid these issues while easily capturing the logic wKL by interpreting the necessitation rule as (wNec). Of course, all of this is well-known for modal logic. We next use generalized Hilbert calculi to show that bi-intuitionistic logic is the victim of a similar confusion.

8.3 Rauszer's Hilbert calculus for bi-intuitionistic logic

Before showing how bi-intuitionistic logic is captured via generalized Hilbert calculi, we recall Rauszer's traditional Hilbert calculus RBIH from 1974 [119].

As mentioned above, RBIH is expressed in the language of intuitionistic logic extended with a binary operator $\mathbin{-\!\!<}$. More formally, as in Section 2.1 we first define a set of connectives.

Definition 8.3.1. We define the **B**i-**I**ntuitionistic set of connectives $\mathcal{C}_{\mathbf{BI}}$ to be the pair $(Con_{\mathbf{BI}}, Ar_{Con_{\mathbf{BI}}})$ where:

- $Con_{\mathbf{BI}} = \{\top, \bot, \wedge, \vee, \rightarrow, \mathbin{-\!\!<}\}$;
- $Ar_{Con_{\mathbf{BI}}}$ is such that:

$$\begin{aligned} Ar_{Con_{\mathbf{BI}}}(\bot) = Ar_{Con_{\mathbf{BI}}}(\top) &= 0 \\ Ar_{Con_{\mathbf{BI}}}(\wedge) = Ar_{Con_{\mathbf{BI}}}(\vee) = Ar_{Con_{\mathbf{BI}}}(\rightarrow) = Ar_{Con_{\mathbf{BI}}}(\mathbin{-\!\!<}) &= 2 \end{aligned}$$

Then, we define the bi-intuitionistic propositional language we work with.

Definition 8.3.2. We define the propositional language $\mathbb{L}_{\mathbf{BI}} = (\mathbb{V}, \mathcal{C}_{\mathbf{BI}})$ and obtain its set of *bi-intuitionistic* formulas $Form_{\mathbb{L}_{\mathbf{BI}}}$ through its grammar:

$$\varphi ::= p \in \mathbb{V} \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \mathbin{-\!\!<} \varphi$$

We also define two unary operators called negation and weak negation:

$$\neg\varphi := (\varphi \rightarrow \bot) \qquad\qquad {\sim}\varphi := (\top \mathbin{-\!\!<} \varphi)$$

As explained in the introduction of this part of the dissertation, the added binary operator $\mathbin{-\!\!<}$ is meant to be the dual of $\rightarrow$. Also, the formula ${\sim}\varphi := \top \mathbin{-\!\!<} \varphi$, defined dually to $\neg\varphi := \varphi \rightarrow \bot$, is usually called "weak negation".

We can then define Rauszer's traditional Hilbert calculus RBIH [116].

Definition 8.3.3. RBIH consists of the axioms $\mathcal{A}_{RBI}$ and rules $\mathcal{R}_{RBI}$ below:

RA_1 $(\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi))$
RA_2 $\varphi \rightarrow (\varphi \vee \psi)$
RA_3 $\psi \rightarrow (\varphi \vee \psi)$
RA_4 $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$
RA_5 $(\varphi \wedge \psi) \rightarrow \varphi$
RA_6 $(\varphi \wedge \psi) \rightarrow \psi$
RA_7 $(\varphi \rightarrow \psi) \rightarrow ((\varphi \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \wedge \chi)))$
RA_8 $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \wedge \psi) \rightarrow \chi)$
RA_9 $((\varphi \wedge \psi) \rightarrow \chi) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$
RA_{10} $\varphi \rightarrow (\psi \vee (\varphi \mathbin{-\!\!<} \psi))$
RA_{11} $(\varphi \rightarrow \psi) \rightarrow (\neg\psi \rightarrow \neg\varphi)$
RA_{12} $(\varphi \mathbin{-\!\!<} \psi) \rightarrow {\sim}(\varphi \rightarrow \psi)$
RA_{13} $((\varphi \mathbin{-\!\!<} \psi) \mathbin{-\!\!<} \chi) \rightarrow (\varphi \mathbin{-\!\!<} (\psi \vee \chi))$
RA_{14} $\neg(\varphi \mathbin{-\!\!<} \psi) \rightarrow (\varphi \rightarrow \psi)$
RA_{15} $(\varphi \rightarrow (\psi \mathbin{-\!\!<} \psi)) \rightarrow \neg\varphi$
RA_{16} $\neg\varphi \rightarrow (\varphi \rightarrow (\psi \mathbin{-\!\!<} \psi))$
RA_{17} $((\psi \rightarrow \psi) \mathbin{-\!\!<} \varphi) \rightarrow {\sim}\varphi$
RA_{18} ${\sim}\varphi \rightarrow ((\psi \rightarrow \psi) \mathbin{-\!\!<} \varphi)$

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad\qquad \frac{\varphi}{\neg{\sim}\varphi}\ (\mathrm{DN})$$

For technical reasons, we work with the following axiomatization. Instead of working with primitive negations and defined $\top$ and $\bot$, as Rauszer did, we decided to do the opposite: take the latter as primitive and make $\neg$ and $\sim$ defined connectives. The resulting axiomatization is more traditional and more economic.

Definition 8.3.4. We define the set of axioms $\mathcal{A}_{BI}$ below:

A_1	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	A_9	$((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
A_2	$\varphi \to (\varphi \vee \psi)$	A_{10}	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
A_3	$\psi \to (\varphi \vee \psi)$	A_{11}	$\varphi \to (\psi \vee (\varphi \mathrel{-\!\!<} \psi))$
A_4	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$	A_{12}	$(\varphi \mathrel{-\!\!<} \psi) \to {\sim}(\varphi \to \psi)$
A_5	$(\varphi \wedge \psi) \to \varphi$	A_{13}	$((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi) \to (\varphi \mathrel{-\!\!<} (\psi \vee \chi))$
A_6	$(\varphi \wedge \psi) \to \psi$	A_{14}	$\neg(\varphi \mathrel{-\!\!<} \psi) \to (\varphi \to \psi)$
A_7	$(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$	A_{15}	$\varphi \to \top$
A_8	$(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$	A_{16}	$\bot \to \varphi$

Note that the set $\mathcal{A}_I = \{A_n \mid 1 \leq n \leq 10 \text{ or } 15 \leq n \leq 16\}$, constituted by the axioms A_1 to A_{10} plus A_{15} and A_{16}, is a set of axioms for intuitionistic logic.

Next, we show that the *Double Negation* rule (DN) above can be interpreted in the context of generalized Hilbert calculi in two main ways, giving different logics.

8.4 Bi-intuitionistic logic as a consequence relation

As in the modal case, the traditional Hilbert calculus hides a distinction in the shape of rules. To be more precise, it overlooks the multiple interpretations of (DN) that are clearly expressed in a generalized Hilbert calculus:

$$\frac{\emptyset \vdash \varphi}{\Gamma \vdash \neg{\sim}\varphi}\ (\mathrm{wDN}) \qquad\qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \neg{\sim}\varphi}\ (\mathrm{sDN})$$

As we shall see, not only are these rules formally different, but they also have a significantly different strength, implying a difference in the consequence relations they define and hence a difference in their logics. To see the difference between the two logics, erroneously identified in Rauszer's work, that emerge from the set of axioms $\mathcal{A}_{BI}$, we define the following generalized Hilbert calculi.

Definition 8.4.1. We define the generalized Hilbert calculi below, alongside their set of rules.

$$\begin{array}{rclcrcl} \mathsf{wBI}\mathbf{H} & = & (\mathcal{A}_{BI}, \mathcal{R}_{\mathsf{w}}) & \qquad & \mathcal{R}_{\mathsf{w}} & = & \{(\mathrm{MP}), (\mathrm{wDN})\} \\ \mathsf{sBI}\mathbf{H} & = & (\mathcal{A}_{BI}, \mathcal{R}_{\mathsf{s}}) & \qquad & \mathcal{R}_{\mathsf{s}} & = & \{(\mathrm{MP}), (\mathrm{sDN})\} \end{array}$$

We abbreviate $\Gamma \vdash_{\mathsf{wBI}\mathbf{H}} \varphi$ by $\Gamma \vdash_{\mathsf{w}} \varphi$ and let $\mathsf{wBI}\mathbf{L} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{w}} \varphi\}$ be the consequence relation characterized by $\mathsf{wBI}\mathbf{H}$. Similarly we abbreviate $\Gamma \vdash_{\mathsf{sBI}\mathbf{H}} \varphi$ by $\Gamma \vdash_{\mathsf{s}} \varphi$, and define $\mathsf{sBI}\mathbf{L} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{s}} \varphi\}$.

As there is no guarantee that generalized Hilbert calculi define logics, to assert that $\mathsf{sBI}\mathbf{L}$ and $\mathsf{wBI}\mathbf{L}$ are logics we must show they satisfy Definition 3.0.2. The single rule proof of $\Gamma \vdash \varphi$ via (El) shows that **Identity** is satisfied both in $\mathsf{sBI}\mathbf{L}$ and $\mathsf{wBI}\mathbf{L}$. The other properties need to be proved.

Lemma 8.4.1. The following holds for $i \in \{\mathsf{w}, \mathsf{s}\}$.

Monotonicity: If $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash_i \varphi$ then $\Gamma' \vdash_i \varphi$.

Compositionality: If $\Gamma_1 \vdash_i \varphi$ and $\Gamma_2 \vdash_i \psi$ for all $\psi \in \Gamma_1$, then $\Gamma_2 \vdash_i \varphi$.

Structurality: If $\Gamma \vdash_i \varphi$ and σ is a propositional variable substitution then $\Gamma^\sigma \vdash_i \varphi^\sigma$.

Proof. **Monotonicity**: (⌨) Assume $\Gamma \vdash_i \varphi$. Then there is a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that $\Gamma' \vdash_i \varphi$ if $\Gamma \subseteq \Gamma'$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$. In this case we have $\varphi \in \Gamma'$ as $\Gamma \subseteq \Gamma'$, hence $\Gamma' \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^I_{BI}$ and thus $\Gamma' \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$ then we simply use the given premise to obtain $\Gamma' \vdash_{\mathsf{w}} \varphi$ as desired.

Compositionality: (⌨) Assume $\Gamma_1 \vdash_i \varphi$ and that $\Gamma_2 \vdash_i \psi$ for every $\psi \in \Gamma_1$. Then we have a proof $\mathfrak{p}$ of $\Gamma_1 \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma_2 \vdash_i \varphi$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma_1$ and so we have $\Gamma_2 \vdash_i \varphi$ by assumption.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^I_{BI}$ and thus $\Gamma_2 \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$, we apply appropriately the rule, i.e. from $\emptyset \vdash_{\mathsf{w}} \varphi$ to $\Gamma_2 \vdash_{\mathsf{w}} \varphi$, to obtain the desired result.

Structurality: (⌨) Assume $\Gamma \vdash_i \varphi$. Then we have a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi^\sigma \in \Gamma^\sigma$, hence $\Gamma^\sigma \vdash \varphi^\sigma$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi^\sigma \in \mathcal{A}^I_{BI}$ (⌨), hence $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If it is (iDN) then for both values of i we apply the induction hypothesis on the premise and then the rule. ■

We are now in position to claim that sBIL and wBIL are both logics. Furthermore, we can add that they are compact logics:

Lemma 8.4.2. For $i \in \{s, w\}$, if $\Gamma \vdash_i \varphi$, then $\Gamma' \vdash_i \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

Proof. (⌨) Assume $\Gamma \vdash_i \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_i \varphi$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\{\varphi\} \subseteq \Gamma$ and $\varphi \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, $\emptyset \subseteq \Gamma$ and $\emptyset \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain finite $\Gamma', \Gamma'' \subseteq \Gamma$ such that $\Gamma' \vdash_i \psi$ and $\Gamma'' \vdash_i \psi \to \varphi$. Lemma 8.4.1 delivers $\Gamma' \cup \Gamma'' \vdash_i \psi$ and $\Gamma' \cup \Gamma'' \vdash_i \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma' \cup \Gamma'' \vdash_i \varphi$, where $\Gamma' \cup \Gamma'' \subseteq \Gamma$ is finite.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$, we apply the induction hypothesis on the premise and then the rule. If $i = w$, we apply appropriately the rule to obtain the desired result. ■

That sBIL and wBIL are (compact) logics is all well and good, but we require further work to show that they are *different* logics, as explained next.

8.5 Extensional interactions

To prove our claim that sBIL and wBIL are two different logics that were erroneously conflated in the literature we first show they differ on an extensional level, i.e. they differ as sets.

Claim 8.5.1. For $p \in \mathbb{V}$, $p \vdash_{\mathsf{s}} \neg{\sim}p$ and $p \nvdash_{\mathsf{w}} \neg{\sim}p$.

While it is clear that $p \vdash_{\mathsf{s}} \neg{\sim}p$ holds because (sDN) can be applied on $p \vdash p$ (⌨), we need a semantic argument, that we provide later, to prove that $p \nvdash_{\mathsf{w}} \neg{\sim}p$. By accepting this result for now, we can see that the two consequence relations sBIL and wBIL are extensionally different. However, the two consequence relations are closely related. In fact, sBIL is an extension of wBIL: this justifies our use of the terms "weak" and "strong" to qualify our calculi.

Theorem 8.5.1. If $\Gamma \vdash_{\mathsf{w}} \varphi$ then $\Gamma \vdash_{\mathsf{s}} \varphi$.

Proof. (⌨) Assume $\Gamma \vdash_{\mathsf{w}} \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove $\Gamma \vdash_{\mathsf{s}} \varphi$ by induction on the structure of $\mathfrak{p}$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$ hence $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\Gamma \vdash_{\mathsf{s}} \psi$ and $\Gamma \vdash_{\mathsf{s}} \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (wDN) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{s}} \psi$. Then, we use Lemma 8.4.1 to obtain $\Gamma \vdash_{\mathsf{s}} \psi$. By an application of (sDN) we obtain $\Gamma \vdash_{\mathsf{s}} \neg{\sim}\psi$. ■

Moreover, they coincide on their sets of *theorems* (derivable from $\emptyset$):

Theorem 8.5.2. $\emptyset \vdash_{\mathsf{s}} \varphi$ if and only if $\emptyset \vdash_{\mathsf{w}} \varphi$.

Proof. (⌨) From right to left, we simply use Theorem 8.5.1. We are thus left with the direction from left to right. (⌨) Assume $\emptyset \vdash_{\mathsf{s}} \varphi$, giving a proof $\mathfrak{p}$ with root $\emptyset \vdash \varphi$. We prove $\emptyset \vdash_{\mathsf{w}} \varphi$ by induction on the structure of $\mathfrak{p}$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \emptyset$ which is a contradiction.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\emptyset \vdash_{\mathsf{w}} \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\emptyset \vdash_{\mathsf{w}} \psi$ and $\emptyset \vdash_{\mathsf{w}} \psi \to \varphi$. Thus (MP) can be applied to get $\emptyset \vdash_{\mathsf{w}} \varphi$.
If (sDN) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{w}} \psi$. By an application of (wDN) we obtain $\emptyset \vdash_{\mathsf{w}} \neg{\sim}\psi$. ■

Traditionally, Theorem 8.5.2 is an argument against our distinction between sBIL and wBIL as it identifies the two logics on their sets of theorems. However, as mentioned previously, they are different consequence relations. Given Claim 8.5.1, sBIL and wBIL are thus different logics.

8.6 Deduction and dual-deduction theorems

In this section, we proceed to show that sBIL and wBIL are distinct on a meta-level by proving that both the deduction-detachment theorem and its dual hold for wBIL, while none hold for sBIL. In addition to that, the rule (DMP), dual to the rule (MP), can safely be added to the former logic but not to the latter.

To get to these results, we first need to prove preliminary results about our generalized Hilbert calculi.

Lemma 8.6.1. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

1. $\Gamma \vdash_i \varphi \to (\psi \to \varphi)$
2. $\Gamma \vdash_i \varphi \to \varphi$
3. $\Gamma \vdash_i (\varphi \wedge \psi) \to (\psi \wedge \varphi)$
4. $\Gamma \vdash_i (\varphi \wedge \neg\varphi) \to \bot$
5. If $\Gamma \vdash_i \varphi \to \psi$ then $\Gamma \vdash_i (\varphi \vee \chi) \to (\psi \vee \chi)$
6. $\Gamma \vdash_i \varphi \to \psi$ then $\Gamma \vdash_i (\chi \vee \varphi) \to (\chi \vee \psi)$
7. $\Gamma \vdash_i (\varphi \vee \psi) \to (\psi \vee \varphi)$
8. $\Gamma \vdash_i \top$
9. $\Gamma \vdash_i \neg{\sim}\varphi \to \varphi$
10. $\Gamma \vdash_i \bot \to \varphi$
11. If $\Gamma \vdash_i \varphi \vee \bot$ then $\Gamma \vdash_i \varphi$

Proof. Apart from the item 9, all other items are commonly accepted but rarely explicitly proven in the literature. As their proofs are tedious and uninteresting we do not display them here. Instead, for most of them we refer to their formalisation. For three items we need separate proofs for each case: (5) ⌨ and ⌨, (6) ⌨ and ⌨, (11) ⌨ and ⌨. For the remaining items, we can restrict our attention to w thanks to Theorem 8.5.1: (1) ⌨, (2) ⌨, (3) ⌨, (4) ⌨, (7) ⌨, (8) ⌨, (10) ⌨. To give the reader an idea of the tediousness of these proofs, we display a semi-proof of item 9 (⌨):

$$\dfrac{\dfrac{}{\Gamma \vdash \top}\,\textit{Item 8} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \neg(\top \mathrel{-\!\!<} \varphi) \to (\top \to \varphi)}\,(\mathrm{Ax})}{\Gamma \vdash \neg{\sim}\varphi \to (\top \to \varphi)}\,\textit{Thm.}}{\Gamma \vdash (\neg{\sim}\varphi \wedge \top) \to \varphi}\,\textit{Thm.}}{\Gamma \vdash (\top \wedge \neg{\sim}\varphi) \to \varphi}\,\textit{Thm.}}{\Gamma \vdash \top \to (\neg{\sim}\varphi \to \varphi)}\,\textit{Thm.}}{\Gamma \vdash \neg{\sim}\varphi \to \varphi}\,(\mathrm{MP})$$

The rule (Ax) on top of the proof involves the axiom $A_{14} : \neg(\varphi' \mathrel{-\!\!<} \psi') \to (\varphi' \to \psi')$, where φ' is instantiated by $\top$ and ψ' by φ. All the instances of *Thm.* refer to instantiations of theorems which are themselves tedious to prove. Without these theorems, the size of the proof would be significantly bigger. ■

Second, to express the deduction-detachment theorem and its dual we use notions from Sano and Stell [127]. These can be interpreted as an extension of the notion of logic as a consequence relation of the form (Γ, φ) to the more general form (Γ, Δ).

Definition 8.6.1. Let $i \in \{\mathsf{w}, \mathsf{s}\}$ and $\bigvee \Delta$ be the disjunction of all the members of Δ. We define the following:

1. $\bigvee \Delta := \bot$ if $\Delta = \emptyset$;
2. $\vdash_i [\Gamma \mid \Delta]$ if $\Gamma \vdash_i \bigvee \Delta'$ for some finite $\Delta' \subseteq \Delta$;
3. $\nvdash_i [\Gamma \mid \Delta]$ if it is not the case that $\vdash_i [\Gamma \mid \Delta]$;
4. $[\Gamma \mid \Delta]$ is complete if $\Gamma \cup \Delta = Form_{\mathbb{L}_{\mathbf{BI}}}$.

Pairs of the form $[\Gamma \mid \Delta]$ bring a symmetry, witnessed by the presence of potentially infinite sets of formulas on both sides of the vertical bar, which is not present in expressions such as $\Gamma \vdash \varphi$. Conceptually, this symmetry and the presence of a non-orientated separation symbol $\mid$ suggests a bidirectional reading of a pair $[\Gamma \mid \Delta]$. From left to right such a pair should be read as a *deduction*, while from right to left it should be read as a *refutation*. This interpretation helps us understand the duality between $\to$ and $\mathrel{-\!\!<}$.

We require a preliminary result using pairs. This result is central to the two logics.

Proposition 8.6.1. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

$$\vdash_i [\emptyset \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi] \quad \text{iff} \quad \vdash_i [\emptyset \mid \varphi \to (\psi \vee \chi)]$$

Proof. (⌨) (⌨) We treat each direction separately.

($\Rightarrow$) Assume $\vdash_i [\emptyset \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi]$, i.e. $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to \chi$. From it we can easily obtain $\emptyset \vdash_i ((\varphi \mathrel{-\!\!<} \psi) \vee \psi) \to (\chi \vee \psi)$. But as we have $\emptyset \vdash_i \varphi \to ((\varphi \mathrel{-\!\!<} \psi) \vee \psi)$ we get $\emptyset \vdash_i \varphi \to (\chi \vee \psi)$, hence $\vdash_i [\emptyset \mid \varphi \to (\chi \vee \psi)]$.

($\Leftarrow$) Assume $\vdash_i [\emptyset \mid \varphi \to (\psi \vee \chi)]$, i.e. $\emptyset \vdash_i \varphi \to (\psi \vee \chi)$. First we have, as an instance of the axiom A_{11}, $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to (\chi \vee ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi))$. But we have that $\emptyset \vdash_i ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi) \leftrightarrow (\varphi \mathrel{-\!\!<} (\psi \vee \chi))$, so we obtain that $\emptyset \vdash_i (\chi \vee ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi)) \to (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi)))$. Thus $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi)))$. However we have that $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} (\psi \vee \chi)) \to \sim(\varphi \to (\psi \vee \chi))$ by axiom A_{12}. And as we have $\emptyset \vdash_i \varphi \to (\psi \vee \chi)$ by (iDN) we obtain $\emptyset \vdash_i \neg\sim(\varphi \to (\psi \vee \chi))$. Consequently we can obtain that $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} (\psi \vee \chi)) \to \bot$, hence $\emptyset \vdash_i (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi))) \to \chi$. This finally implies $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to \chi$, hence $\vdash_i [\emptyset \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi]$.

■

We have not given Rauszer's [116] algebraic semantics for **RBIL**, but Proposition 8.6.1 is an object language analog of the dual residuation property below:

$$\begin{array}{c} a \leq b \vee c \\ \hline\hline a \mathrel{-\!\!<} b \leq c \end{array}$$

Note that as **sBIL** allows contexts to be manipulated more liberally thanks to the rule (sDN), we can prove for this logic the above statement in a stronger form:

Proposition 8.6.2. $\vdash_{\mathsf{s}} [\Gamma \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi]$ iff $\vdash_{\mathsf{s}} [\Gamma \mid \varphi \to (\psi \vee \chi)]$

Proof. (⌨) We treat each direction separately.

($\Rightarrow$) Assume $\vdash_{\mathsf{s}} [\Gamma \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi]$, i.e. $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} \psi) \to \chi$. From it we can easily obtain $\Gamma \vdash_{\mathsf{s}} ((\varphi \mathrel{-\!\!<} \psi) \vee \psi) \to (\chi \vee \psi)$. But as we have $\Gamma \vdash_{\mathsf{s}} \varphi \to ((\varphi \mathrel{-\!\!<} \psi) \vee \psi)$ we get $\Gamma \vdash_{\mathsf{s}} \varphi \to (\chi \vee \psi)$, hence $\vdash_{\mathsf{s}} [\Gamma \mid \varphi \to (\chi \vee \psi)]$.

($\Leftarrow$) Assume $\vdash_{\mathsf{s}} [\Gamma \mid \varphi \to (\psi \vee \chi)]$, i.e. $\Gamma \vdash_{\mathsf{s}} \varphi \to (\psi \vee \chi)$. First we have, as an instance of the axiom A_{11}, $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} \psi) \to (\chi \vee ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi))$. But we have that $\Gamma \vdash_{\mathsf{s}} ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi) \leftrightarrow (\varphi \mathrel{-\!\!<} (\psi \vee \chi))$, so we obtain that $\Gamma \vdash_{\mathsf{s}} (\chi \vee ((\varphi \mathrel{-\!\!<} \psi) \mathrel{-\!\!<} \chi)) \to (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi)))$. Thus $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} \psi) \to (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi)))$. However we have that $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} (\psi \vee \chi)) \to \sim(\varphi \to (\psi \vee \chi))$ by axiom A_{12}. And as we have $\Gamma \vdash_{\mathsf{s}} \varphi \to (\psi \vee \chi)$ by (sDN) we obtain $\Gamma \vdash_{\mathsf{s}} \neg\sim(\varphi \to (\psi \vee \chi))$. Consequently we can obtain that $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} (\psi \vee \chi)) \to \bot$, hence $\Gamma \vdash_{\mathsf{s}} (\chi \vee (\varphi \mathrel{-\!\!<} (\psi \vee \chi))) \to \chi$. This finally implies $\Gamma \vdash_{\mathsf{s}} (\varphi \mathrel{-\!\!<} \psi) \to \chi$, hence $\vdash_{\mathsf{s}} [\Gamma \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi]$.

■

With Proposition 8.6.1 in hand we can prove interesting results:

Lemma 8.6.2. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

1. $\Gamma \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to \varphi$;
2. $\Gamma \vdash_i \varphi \vee \sim\varphi$;
3. if $\emptyset \vdash_i \varphi \to \psi$, then $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \chi) \to (\psi \mathrel{-\!\!<} \chi)$;
4. if $\emptyset \vdash_i \varphi \to \psi$, then $\emptyset \vdash_i (\chi \mathrel{-\!\!<} \psi) \to (\chi \mathrel{-\!\!<} \varphi)$;
5. $\Gamma \vdash_i \neg(\varphi \mathrel{-\!\!<} \psi) \to (\sim\psi \to \sim\varphi)$;

6. $\Gamma \vdash_i \neg{\sim}(\varphi \to \psi) \to (\neg{\sim}\varphi \to \neg{\sim}\psi)$.

Proof. We treat each statement individually, and restrict ourselves to w using Theorem 8.5.1 when possible:

1. (⌨) Consider the following semi-proof:

$$\dfrac{\dfrac{}{\Gamma \vdash \varphi \to (\psi \lor \varphi)}\,(\mathrm{Ax})}{\Gamma \vdash (\varphi \prec \psi) \to \varphi}\,\textit{Prop.}8.6.1$$

2. (⌨) Consider the following semi-proof:

$$\dfrac{\dfrac{}{\Gamma \vdash \top}\,\textit{Lem.}8.6.1 \qquad \dfrac{\dfrac{}{\Gamma \vdash (\top \prec \varphi) \to (\top \prec \varphi)}\,\textit{Lem.}8.6.1}{\Gamma \vdash \top \to (\varphi \lor (\top \prec \varphi))}\,\textit{Prop.}8.6.1}{\Gamma \vdash \varphi \lor (\top \prec \varphi)}\,(\mathrm{MP})$$

3. (⌨) (⌨) Given that $\emptyset \vdash_i \psi \to (\chi \lor (\psi \prec \chi))$ and $\emptyset \vdash_i \varphi \to \psi$, we get the following: $\emptyset \vdash_i \varphi \to (\chi \lor (\psi \prec \chi))$. By Proposition 8.6.1 (or Proposition 8.6.2), we consequently get $\emptyset \vdash_i (\varphi \prec \chi) \to (\psi \prec \chi)$.

4. (⌨) (⌨) Assume $\emptyset \vdash_i \varphi \to \psi$. Note that we have $\emptyset \vdash_i \chi \to (\varphi \lor (\chi \prec \varphi))$ as the formula is an instance of A_{11}. Given that $\emptyset \vdash_i \varphi \to \psi$, we can straightforwardly obtain $\emptyset \vdash_i (\varphi \lor (\chi \prec \varphi)) \to (\psi \lor (\chi \prec \varphi))$. It then suffices to use $\emptyset \vdash_i \chi \to (\varphi \lor (\chi \prec \varphi))$ and $\emptyset \vdash_i (\varphi \lor (\chi \prec \varphi)) \to (\psi \lor (\chi \prec \varphi))$ to obtain $\emptyset \vdash_i \chi \to (\psi \lor (\chi \prec \varphi))$. By Proposition 8.6.1 we finally get $\emptyset \vdash_i (\chi \prec \psi) \to (\chi \prec \varphi)$.

5. (⌨) We have $\emptyset \vdash_i \varphi \to (\psi \lor (\varphi \prec \psi))$. We can use item 4 to obtain the following.

$$\emptyset \vdash_i (\top \prec (\psi \lor (\varphi \prec \psi))) \to {\sim}\varphi$$

Next we prove that $\emptyset \vdash_i ({\sim}\psi \land \neg(\varphi \prec \psi)) \to {\sim}(\psi \lor (\varphi \prec \psi))$ to obtain that $\emptyset \vdash_i ({\sim}\psi \land \neg(\varphi \prec \psi)) \to {\sim}\varphi$, and hence $\emptyset \vdash_i \neg(\varphi \prec \psi) \to ({\sim}\psi \to {\sim}\varphi)$. First, $\emptyset \vdash_i \top \to ((\psi \lor (\varphi \prec \psi)) \lor (\top \prec (\psi \lor (\varphi \prec \psi))))$ is an instance of the axiom A_{11}. Then by associativity of disjunction, we obtain the following.

$$\emptyset \vdash_i \top \to (\psi \lor ((\varphi \prec \psi) \lor (\top \prec (\psi \lor (\varphi \prec \psi)))))$$

By Proposition 8.6.1 we get $\emptyset \vdash_i {\sim}\psi \to ((\varphi \prec \psi) \lor (\top \prec (\psi \lor (\varphi \prec \psi))))$. Consequently we easily obtain $\emptyset \vdash_i ({\sim}\psi \land \neg(\varphi \prec \psi)) \to (\top \prec (\psi \lor (\varphi \prec \psi)))$, i.e. $\emptyset \vdash_i ({\sim}\psi \land \neg(\varphi \prec \psi)) \to {\sim}(\psi \lor (\varphi \prec \psi))$. Monotonicity finally gives us the following.

$$\Gamma \vdash_i ({\sim}\psi \land \neg(\varphi \prec \psi)) \to {\sim}(\psi \lor (\varphi \prec \psi))$$

6. (⌨) First note that $\emptyset \vdash_i (\varphi \prec \psi) \to {\sim}(\varphi \to \psi)$ as it is an instance of the axiom A_{12}. By using A_{10} and (MP) we obtain $\emptyset \vdash_i \neg{\sim}(\varphi \to \psi) \to \neg(\varphi \prec \psi)$. Thus, using item 5 just above, we can obtain $\emptyset \vdash_i \neg{\sim}(\varphi \to \psi) \to ({\sim}\psi \to {\sim}\varphi)$. We can instantiate A_{10} again to obtain $\emptyset \vdash_i ({\sim}\psi \to {\sim}\varphi) \to (\neg{\sim}\varphi \to \neg{\sim}\psi)$, and use this fact with the previous result to get $\emptyset \vdash_i \neg{\sim}(\varphi \to \psi) \to (\neg{\sim}\varphi \to \neg{\sim}\psi)$. Monotonicity finally gives us $\Gamma \vdash_i \neg{\sim}(\varphi \to \psi) \to (\neg{\sim}\varphi \to \neg{\sim}\psi)$.

■

The above theorems are of importance as they show interesting behaviors of the exclusion arrow. Notably, with the second item of Lemma 8.6.2 we exhibit a form of the law of excluded middle with the weak negation ∼. As a consequence, neither wBIL nor sBIL can satisfy the disjunction property, important to many constructivists.

Also, item 6 above and item 9 of Lemma 8.6.1 inform us that $\neg{\sim}$ is at least behaving like a **T** modality, i.e. validating the axiom **T**: $\Box\varphi \to \varphi$. This fact can certainly be explained in the light of the tense interpretation of bi-intuitionistic logic, where $\varphi \prec \psi$ is interpreted as $\blacklozenge(\varphi \wedge \neg\psi)$ and the intuitionistic $\to$ as $\Box(\varphi \to \psi)$. Thus, we obtain that $\neg{\sim}\varphi$ is interpreted as $\Box\blacksquare\varphi$. As it is known that the interpretation of bi-intuitionistic in tense logic is $\mathsf{S4}_t$ [107, 93, 137, 165] , we naturally get that $\Box\blacksquare$ is a **T** modality, explaining the properties of $\neg{\sim}$.

The *deduction theorem* (from left to right) is the first theorem to separate the two logics. Note that the direction from right to left is called the *detachment theorem*.

Theorem 8.6.1 (Deduction-Detachment Theorem)**.** $\mathsf{wBI}\mathbf{L}$ enjoys the deduction-detachment theorem:

$$\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi] \quad \text{iff} \qquad \vdash_{\mathsf{w}} [\Gamma \mid \varphi \to \psi]$$

Proof. ($\Leftarrow$) (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma \mid \varphi \to \psi]$, i.e. $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$. Then by monotonicity (Lemma 8.4.1) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi \to \psi$. Moreover we have that $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi$ as $\varphi \in \Gamma \cup \{\varphi\}$. So by (MP) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$, hence $\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi]$.

($\Rightarrow$) (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi]$, i.e. $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (El) is the last rule applied in $\mathfrak{p}$, then $\psi \in \Gamma \cup \{\varphi\}$. If $\psi = \varphi$ then we clearly have $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ by Lemma 8.6.1. If $\psi \in \Gamma$ then we can deduce $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ from the fact that we have $\emptyset \vdash_{\mathsf{w}} \psi \to (\varphi \to \psi)$, proven in Lemma 8.6.1.

If (Ax) is the last rule applied in $\mathfrak{p}$, then $\psi \in \mathcal{A}^{I}_{BI}$ and with a similar reasoning we get $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (MP) is the last rule applied in $\mathfrak{p}$, then we proceed as follows. We use the induction hypothesis on the premises of the rule to obtain proofs of $\Gamma \vdash \varphi \to \chi$ and $\Gamma \vdash \varphi \to (\chi \to \psi)$. We also note that $\emptyset \vdash_{\mathsf{w}} (\varphi \to \chi) \to ((\varphi \to (\chi \to \psi)) \to (\varphi \to \psi))$. Then, we use (MP) several times to arrive at the establishment of $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If the last rule is (wDN), we have a proof of $\emptyset \vdash \chi$, so we can apply (wDN) to obtain $\emptyset \vdash_{\mathsf{w}} \neg{\sim}\chi$. Then we can use the fact that $\emptyset \vdash_{\mathsf{w}} \neg{\sim}\chi \to (\varphi \to \neg{\sim}\chi)$ via Lemma 8.6.1 to obtain $\emptyset \vdash_{\mathsf{w}} \varphi \to \neg{\sim}\chi$. By monotonicity we obtain $\Gamma \vdash_{\mathsf{w}} \varphi \to \neg{\sim}\chi$.

■

Next, we give a counter-example for the deduction theorem for $\mathsf{sBI}\mathbf{L}$.

Fact 8.6.1. We have that $\vdash_{\mathsf{s}} [p \mid \neg{\sim}p]$ but $\not\vdash_{\mathsf{s}} [\emptyset \mid p \to \neg{\sim}p]$.

Proof. Obviously we have $p \vdash_{\mathsf{s}} p$. So we can apply the rule (sDN) to obtain $p \vdash_{\mathsf{s}} \neg{\sim}p$, hence $\vdash_{\mathsf{s}} [p \mid \neg{\sim}p]$. We leave the following claim pending until Section 8.10:

Claim 8.6.1. We have that $\not\vdash_{\mathsf{s}} [\emptyset \mid p \to \neg{\sim}p]$.

■

This situation is very similar to the modal case: it is well-known that $\mathsf{wK}\mathbf{L}$ satisfies the deduction theorem while $\mathsf{sK}\mathbf{L}$ does not [84, Sect.3.1], as shown in Subsection 4.1.1, Subsection 5.1.1 and Section 6.1. However, a variant of this theorem does hold for $\mathsf{sK}\mathbf{L}$: $\Gamma, \varphi \vdash_{\mathsf{sK}\mathbf{H}} \psi$ *iff there exists a* $n \in \mathbb{N}$ *such that* $\Gamma \vdash_{\mathsf{sK}\mathbf{H}} (\varphi \wedge \Box\varphi \wedge ... \wedge \Box^n\varphi) \to \psi$ [22, p.85]. A similar variant of the deduction theorem holds for $\mathsf{sBI}\mathbf{L}$, but we first need some notation to express it.

Definition 8.6.2. We define:

1. for $n \in \mathbb{N}$, let $(\neg{\sim})^0\varphi := \varphi$ and let $(\neg{\sim})^{(n+1)}\varphi := \neg{\sim}(\neg{\sim})^n\varphi$;
2. $(\neg{\sim})^n\Gamma = \{(\neg{\sim})^n\gamma \mid \gamma \in \Gamma\}$;

3. $(\neg{\sim})^\omega\Gamma = \bigcup\limits_{n\in\mathbb{N}} (\neg{\sim})^n\Gamma$.

The variant of the deduction theorem below uses the pattern $\neg{\sim}$ as the modal variant uses $\Box$. But it suffices to replace φ by just $(\neg{\sim})^n\varphi$, without the conjunction of all $(\neg{\sim})^i\varphi$ for $i \leq n$, as $\neg{\sim}$ is a $\mathbf{T}$ modality satisfying $\neg{\sim}\varphi \to \varphi$ (Lemma 8.6.1).

Theorem 8.6.2 (Double-Negated Deduction-Detachment Theorem)**.**

$$\vdash_{\mathsf{s}} [\Gamma, \varphi \mid \psi] \quad \text{iff} \quad \exists n \in \mathbb{N} \text{ s.t. } \vdash_{\mathsf{s}} [\Gamma \mid (\neg{\sim})^n\varphi \to \psi]$$

Proof. ($\Rightarrow$) (⌨) Assume that $\vdash_{\mathsf{s}} [\Gamma, \varphi \mid \psi]$, i.e. that we have a proof $\mathfrak{p}$ of $\Gamma, \varphi \vdash \psi$. We reason by induction on the structure of $\mathfrak{p}$.

If the last rule applied is (Ax), then we get $\emptyset \vdash_{\mathsf{s}} \psi$, and as we have that $\emptyset \vdash_{\mathsf{s}} \psi \to ((\neg{\sim})^n\varphi \to \psi)$ for any $n \in \mathbb{N}$ we obtain by (MP): $\emptyset \vdash_{\mathsf{s}} (\neg{\sim})^n\varphi \to \psi$. By Lemma 8.4.1 we get $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^n\varphi \to \psi$. So, it suffices to take $n = 0$.

If the last rule applied is (El) then $\Gamma \vdash_{\mathsf{s}} \varphi \to \psi$ as $\Gamma \vdash_{\mathsf{s}} \psi \to (\varphi \to \psi)$ and $\Gamma \vdash_{\mathsf{s}} \psi$.

If the last rule applied is (MP) then we have by induction hypothesis $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^k\varphi \to \chi$ and $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^m\varphi \to (\chi \to \psi)$ for some $\chi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ and $m, k \in \mathbb{N}$. As we have that $\emptyset \vdash_{\mathsf{s}} (\lambda_1 \to \lambda_2) \to ((\lambda_1 \to (\lambda_2 \to \lambda_3)) \to (\lambda_1 \to \lambda_3))$ and $\emptyset \vdash_{\mathsf{s}} \neg{\sim}\lambda \to \lambda$ (Lemma 8.6.1) we obtain that $\Gamma \vdash_{\mathsf{s}} \neg{\sim}^n\varphi \to \chi$ for $n = max(m, k)$.

If the last rule applied is (sDN) then we get by induction hypothesis that $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^n\varphi \to \chi$. We apply (sDN) on $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^n\varphi \to \chi$ to obtain $\Gamma \vdash_{\mathsf{s}} \neg{\sim}((\neg{\sim})^n\varphi \to \chi)$. Then, we get $\Gamma \vdash_{\mathsf{s}} (\neg{\sim})^{n+1}\varphi \to \neg{\sim}\chi$ by using item 6 of Lemma 8.6.2 and (MP). So, we have $\vdash_{\mathsf{s}} [\Gamma \mid (\neg{\sim})^{n+1}\varphi \to \neg{\sim}\chi]$.

($\Leftarrow$) (⌨) Straightforward use of the rules (sDN) and (MP) with Lemma 8.4.1. ■

Another theorem distinguishing the two logics is the dual of the deduction-detachment theorem.

Theorem 8.6.3 (Dual Deduction-Detachment Theorem)**.** The following holds:

$$\vdash_{\mathsf{w}} [\varphi \mid \psi, \Delta] \qquad \text{iff} \qquad \vdash_{\mathsf{w}} [\varphi \prec \psi \mid \Delta].$$

Proof. (⌨) (⌨) Assume that $\vdash_{\mathsf{w}} [\varphi \mid \psi, \Delta]$. By definition we get $\varphi \vdash_{\mathsf{w}} \psi \vee \bigvee \Delta'$ where $\Delta' \subseteq \Delta$ is finite. Using Theorem 8.6.1 we get $\emptyset \vdash_{\mathsf{w}} \varphi \to (\psi \vee \bigvee \Delta')$. We obtain $\emptyset \vdash_{\mathsf{w}} (\varphi \prec \psi) \to \bigvee \Delta'$ by Proposition 8.6.1. By Theorem 8.6.1 again, we obtain $\varphi \prec \psi \vdash_{\mathsf{w}} \bigvee \Delta'$. By definition we get $\vdash_{\mathsf{w}} [\varphi \prec \psi \mid \Delta]$. Note that all the steps used here are based on equivalences. ■

Before demonstrating that $\mathsf{sBI}\mathbf{L}$ fails the dual deduction theorem, we remark on the previous theorem. Pairs $[\Gamma \mid \Delta]$ express the duality between $\to$ and $\prec$ on the syntactic level in $\mathsf{wBI}\mathbf{L}$ by showing that $\prec$ plays the same role as $\to$ on the left-hand side of $|$: it internalizes in the object language the relation expressed by our pairs. Just as $\to$ internalizes the deduction relation of expressions such as $\Gamma \vdash \varphi$, dually $\prec$ internalizes the *refutation* relation of expressions such as $\Delta \dashv \varphi$, read "Δ refutes φ" and formalized here as $[\varphi \mid \Delta]$. This interpretation, already explored by Goré and Postniece [59], relies on the aforementioned reading of our pairs, from right to left, to express refutations.

The following witnesses the failure of the dual detachment theorem for $\mathsf{sBI}\mathbf{L}$.

Fact 8.6.2. $\vdash_{\mathsf{s}} [p \prec q \mid \neg{\sim}{\sim}q]$ while $\nvdash_{\mathsf{s}} [p \mid q, \neg{\sim}{\sim}q]$.

Proof. First, let us prove that $\vdash_{\mathsf{s}} [p \prec q \mid \neg{\sim}{\sim}q]$. By definition, we need to show that $p \prec q \vdash_{\mathsf{s}} \neg{\sim}{\sim}q$. We have that $\emptyset \vdash_{\mathsf{w}} q \vee {\sim}q$, hence $\emptyset \vdash_{\mathsf{w}} p \to (q \vee {\sim}q)$. By Proposition 8.6.1 we obtain $\emptyset \vdash_{\mathsf{w}} (p \prec q) \to {\sim}q$. In turn, by Theorem 8.6.2 we get $p \prec q \vdash_{\mathsf{w}} {\sim}q$. Then, by Theorem 8.5.1 we get that $p \prec q \vdash_{\mathsf{s}} {\sim}q$. Finally, we can apply the rule (sDN) to obtain $p \prec q \vdash_{\mathsf{s}} \neg{\sim}{\sim}q$, hence $\vdash_{\mathsf{s}} [p \prec q \mid \neg{\sim}{\sim}q]$. We leave the following claim pending until Section 8.10:

Claim 8.6.2. $\nvdash_{\mathsf{s}} [p \mid q, \neg{\sim}{\sim}q]$.

■

While a variant of the deduction-detachment theorem exists for sBIL, the form or the existence of a variant of the dual deduction-detachment theorem is still a mystery to us. For the interested reader: while the deduction-detachment theorem fails for sBIL because of the rule (sDN), the dual deduction-detachment theorem fails for this logic because of the rule (MP). Thus, it appears that if a variant of the dual deduction theorem exists for sBIL, then it must use a "patch" inspired by the structure of (MP), as done in the double-negated deduction theorem with (sDN).

Another interesting feature pertaining to the duality between $\to$ and $\mathrel{-\!\!<}$ distinguishes the two logics. Given this duality, one would expect that a dual version of the *modus ponens* rule (MP) holds for $\mathrel{-\!\!<}$. In fact, the Dual of Modus Ponens rule, presented below, can be added to the calculus wBIH without changing the set of judgements it derives.

$$\frac{[\psi \mid \Delta] \quad [\varphi \mathrel{-\!\!<} \psi \mid \Delta]}{[\varphi \mid \Delta]}\ (\text{DMP})$$

Lemma 8.6.3. If $\vdash_{\mathsf{w}} [\varphi \mathrel{-\!\!<} \psi \mid \Delta]$ and $\vdash_{\mathsf{w}} [\psi \mid \Delta]$ then $\vdash_{\mathsf{w}} [\varphi \mid \Delta]$.

Proof. (⌨) Assume $\vdash_{\mathsf{w}} [\varphi \mathrel{-\!\!<} \psi \mid \Delta]$ and $\vdash_{\mathsf{w}} [\psi \mid \Delta]$. Then, we have proofs of $\varphi \mathrel{-\!\!<} \psi \vdash \bigvee \Delta'$ and $\psi \vdash \bigvee \Delta''$. We can easily obtain that $\varphi \mathrel{-\!\!<} \psi \vdash \bigvee \Delta'''$ and $\psi \vdash \bigvee \Delta'''$, where $\bigvee \Delta''' = \bigvee \Delta' \vee \bigvee \Delta''$. Thus, we can use Theorem 8.6.1 to obtain proofs of $\emptyset \vdash (\varphi \mathrel{-\!\!<} \psi) \to \bigvee \Delta'''$ and $\emptyset \vdash \psi \to \bigvee \Delta'''$. By applying Proposition 8.6.1 on $\emptyset \vdash (\varphi \mathrel{-\!\!<} \psi) \to \bigvee \Delta'''$ we get a proof of $\emptyset \vdash \varphi \to (\psi \vee \bigvee \Delta''')$. Given the latter and $\emptyset \vdash \psi \to \bigvee \Delta'''$ we can conclude that there is a proof of $\emptyset \vdash \varphi \to \bigvee \Delta'''$. So, we have that $\vdash_{\mathsf{w}} [\varphi \mid \Delta]$. ■

However, as already suggested by the failure of the dual deduction-detachment theorem for sBIL, in the latter the duality between $\to$ and $\mathrel{-\!\!<}$ seems to be broken. This assessment is further backed by the fact that (DMP) does not hold in sBIL.

Fact 8.6.3. $\vdash_{\mathsf{s}} [p \mathrel{-\!\!<} q \mid q, \neg{\sim}{\sim}q]$ and $\vdash_{\mathsf{s}} [q \mid q, \neg{\sim}{\sim}q]$, while $\nvdash_{\mathsf{s}} [p \mid q, \neg{\sim}{\sim}q]$.

Proof. The proof of $\vdash_{\mathsf{s}} [p \mathrel{-\!\!<} q \mid q, \neg{\sim}{\sim}q]$ can be found in the proof of Fact 8.6.2. We obtain $\vdash_{\mathsf{s}} [q \mid q, \neg{\sim}{\sim}q]$ straightforwardly via the use of (El). We show $\nvdash_{\mathsf{s}} [p \mid q, \neg{\sim}{\sim}q]$ using Claim 8.6.2. ■

On top of the extensional difference between wBIL and sBIL, the deduction- and dual deduction theorems with the rule (DMP) expose their meta-difference. But both differences rely on claims that are still pending. The next section builds on Rauszer's Kripke semantics to resolve these claims.

8.7 Kripke semantics

The logics wBIL and sBIL, proof-theoretically characterized via the generalized Hilbert calculi wBIH and sBIH, can be captured model-theoretically in a Kripke semantics using the *local* and *global* notions of semantic consequence defined in Section 5.1. In this section, we define such a semantics and provide useful results about its use.

First, we need to define a Kripke semantics for the propositional bi-intuitionistic language $\mathbb{L}_{\mathbf{BI}}$ [122]. In this context, we use the symbol $\leq$ for the generic accessibility relation R on models.

Definition 8.7.1. A Kripke model $\mathcal{M}$ is a tuple $(W, \leq, I)$, where $(W, \leq)$ is a poset, i.e. $\leq$ is reflexive and transitive, and $I : \mathbb{V} \to \mathsf{Pow}(W)$ is an interpretation function obeying persistence: for every $v, w \in W$ with $w \leq v$ and $p \in \mathbb{V}$, if $w \in I(p)$ then $v \in I(p)$.

The forcing relation of bi-intuitionistic logic extends the forcing relation of intuitionistic to $\mathrel{-\!\!<}$. We define it in its entirety below.

Definition 8.7.2. Given a Kripke model $\mathcal{M} = (W, \leq, I)$, we define the forcing relation between a point $w \in W$ and a formula as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in I(p) \\
\mathcal{M}, w \Vdash \top & & \text{always} \\
\mathcal{M}, w \Vdash \bot & & \text{never} \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \text{ s.t. } w \leq v, \text{ if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, v \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \mathrel{-\!\!<} \psi & \text{iff} & \text{there exists a } v \text{ s.t. } v \leq w,\ \mathcal{M}, v \Vdash \varphi \text{ and } \mathcal{M}, v \nVdash \psi
\end{array}$$

Note that the semantic interpretation of the exclusion arrow is peculiar, as it goes *backwards* on the relation $\leq$.

The main feature of the Kripke semantics for intuitionistic logic is arguably persistence. This property, which we use in various places, is preserved here:

Lemma 8.7.1 (Persistence)**.** Let $\mathcal{M} = (W, \leq, I)$ be a Kripke model and $w \in W$. For all $v \in W$ such that $w \leq v$, we have that if $\mathcal{M}, w \Vdash \varphi$ then $\mathcal{M}, v \Vdash \varphi$.

Proof. (⌨) Let $w \in W$. We reason by induction on φ:

- $\varphi := p$: Let $v \in W$ and assume that $w \leq v$. We have $\mathcal{M}, w \Vdash p$ implies $w \in I(p)$, which implies $v \in I(p)$ by the persistence condition on models, which in turn implies $\mathcal{M}, v \Vdash p$. We thus get the desired result for all v.

- $\varphi := \top$: Let $v \in W$ and assume that $w \leq v$. We have $\mathcal{M}, w \Vdash \top$ and $\mathcal{M}, v \Vdash \top$ by definition. So we trivially get the desired result for all v.

- $\varphi := \bot$: Let $v \in W$ and assume that $w \leq v$. We have $\mathcal{M}, w \nVdash \bot$ and $\mathcal{M}, v \nVdash \bot$ by definition. So we trivially get the desired result for all v.

- $\varphi := \chi \wedge \psi$: Let $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w \Vdash \chi \wedge \psi$ then $\mathcal{M}, w \Vdash \chi$ and $\mathcal{M}, w \Vdash \psi$. By induction hypothesis $\mathcal{M}, v \Vdash \chi$ and $\mathcal{M}, v \Vdash \psi$. Then $\mathcal{M}, v \Vdash \chi \wedge \psi$. We thus get the desired result for all v.

- $\varphi := \chi \vee \psi$: Let $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w \Vdash \chi \vee \psi$ then $\mathcal{M}, w \Vdash \chi$ or $\mathcal{M}, w \Vdash \psi$. By induction hypothesis $\mathcal{M}, v \Vdash \chi$ or $\mathcal{M}, v \Vdash \psi$. Then $\mathcal{M}, v \Vdash \chi \vee \psi$. We thus get the desired result for all v.

- $\varphi := \chi \to \psi$: Let $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w \Vdash \chi \to \psi$ then for every $u \in W$ s.t. $w \leq u$, if $\mathcal{M}, u \Vdash \chi$ then $\mathcal{M}, u \Vdash \psi$. Note that for every $u \in W$, if $v \leq u$ then $w \leq u$ by transitivity. Consequently we have that for every $u \in W$ s.t. $v \leq u$, if $\mathcal{M}, u \Vdash \chi$ then $\mathcal{M}, u \Vdash \psi$. We get $\mathcal{M}, v \Vdash \chi \to \psi$. We thus get the desired result for all v.

- $\varphi := \chi \mathrel{-\!\!<} \psi$: Let $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w \Vdash \chi \mathrel{-\!\!<} \psi$ then there is a $u \leq w$ s.t. $\mathcal{M}, u \Vdash \chi$ and $\mathcal{M}, u \nVdash \psi$. By transitivity we have $u \leq v$, so there is a $u \leq v$ s.t. $\mathcal{M}, u \Vdash \chi$ and $\mathcal{M}, u \nVdash \psi$. Thus $\mathcal{M}, v \Vdash \chi \mathrel{-\!\!<} \psi$. We thus get the desired result for all v.

■

As explained in Subsection 5.1.1, a central notion in the model theory of Kripke semantics is the notion of bisimulation [11, Sect.2.2]. It relates models with similar structures, using a specific notion of "similarity". Here, we present the notion of bisimulation developed by de Groot and Pattinson [36, Def.4.1]. Note that an alternative notion of bisimulation was presented by Badia [6].

Definition 8.7.3. Let $\mathcal{M}_1 = (W_1, \leq_1, I_1)$ and $\mathcal{M}_2 = (W_2, \leq_2, I_2)$ be models. A *bisimulation* between $\mathcal{M}_1$ and $\mathcal{M}_2$ is a relation $B \subseteq W_1 \times W_2$ such that for all $w_1 \in W_1$ and $w_2 \in W_2$, if $w_1 B w_2$ we have:

(B_1) for all $p \in \mathbb{V}$, $w_1 \in I_1(p)$ iff $w_2 \in I_2(p)$;

(B_2) for all $v_2 \in W_2$, if $w_2 \leq v_2$ then there exists $v_1 \in W_1$ such that $w_1 \leq v_1$ and $v_1 B v_2$;

(B_3) for all $v_1 \in W_1$, if $w_1 \leq v_1$ then there exists $v_2 \in W_2$ such that $w_2 \leq v_2$ and $v_1 B v_2$;

(B_4) for all $v_2 \in W_2$, if $v_2 \leq w_2$ then there exists $v_1 \in W_1$ such that $v_1 \leq w_1$ and $v_1 B v_2$;

(B_5) for all $v_1 \in W_1$, if $v_1 \leq w_1$ then there exists $v_2 \in W_2$ such that $v_2 \leq w_2$ and $v_1 B v_2$.

If there is a bisimulation B such that $w_1 B w_2$, we say that w_1 and w_2 are *bisimilar* and note $\mathcal{M}_1, w_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, w_2$.

This notion is usually a sought-after one, as it often entails *logical equivalence* for points in models: two bisimilar points force the same formulas. This property, which will be of crucial use later on, is obtained with the above definition [36, Prop.4.4]

Proposition 8.7.1. Let $\mathcal{M}_1 = (W_1, \leq_1, I_1)$ and $\mathcal{M}_2 = (W_2, \leq_2, I_2)$ be models, $w_1 \in W_1$ and $w_2 \in W_2$. If $\mathcal{M}_1, w_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, w_2$, then for all $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ we have $\mathcal{M}_1, w_1 \Vdash \varphi$ iff $\mathcal{M}_2, w_2 \Vdash \varphi$.

Proof. (⌨) We reason by induction on the structure of φ.

- $\varphi := p$: As $\mathcal{M}_1, w_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, w_2$ we notably have $w_1 \in I_1(p)$ iff $w_2 \in I_2(p)$ by item (B_1) of Definition 8.7.3. So, we get that $\mathcal{M}_1, w_1 \Vdash p$ iff $\mathcal{M}_2, w_2 \Vdash p$.

- $\varphi := \top$: We have $\mathcal{M}_1, w_1 \Vdash \top$ and $\mathcal{M}_2, w_2 \Vdash \top$ by definition. So we trivially get the desired result.

- $\varphi := \bot$: We have $\mathcal{M}_1, w_1 \nVdash \bot$ and $\mathcal{M}_2, w_2 \nVdash \bot$ by definition. So we trivially get the desired result.

- $\varphi := \chi \wedge \psi$: We have that $\mathcal{M}_1, w_1 \Vdash \chi \wedge \psi$ iff $[\mathcal{M}_1, w_1 \Vdash \chi$ and $\mathcal{M}_1, w_1 \Vdash \psi]$. By induction hypothesis the latter holds iff $[\mathcal{M}_2, w_2 \Vdash \chi$ and $\mathcal{M}_2, w_2 \Vdash \psi]$. We finally reach $\mathcal{M}_2, w_2 \Vdash \chi \wedge \psi$ through a chain of equivalences. We thus get the desired result.

- $\varphi := \chi \vee \psi$: We have that $\mathcal{M}_1, w_1 \Vdash \chi \vee \psi$ iff $[\mathcal{M}_1, w_1 \Vdash \chi$ or $\mathcal{M}_1, w_1 \Vdash \psi]$. By induction hypothesis the latter holds iff $[\mathcal{M}_2, w_2 \Vdash \chi$ or $\mathcal{M}_2, w_2 \Vdash \psi]$. We finally reach $\mathcal{M}_2, w_2 \Vdash \chi \vee \psi$ through a chain of equivalences. We thus get the desired result.

- $\varphi := \chi \to \psi$: ($\Rightarrow$) Assume $\mathcal{M}_1, w_1 \Vdash \chi \to \psi$. We need to show $\mathcal{M}_2, w_2 \Vdash \chi \to \psi$. Let $v_2 \in W_2$ such that $w_2 \leq v_2$, and assume $\mathcal{M}_2, v_2 \Vdash \chi$. We need to show $\mathcal{M}_2, v_2 \Vdash \psi$. Given that $\mathcal{M}_1, w_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, w_2$ and $w_2 \leq v_2$, we get that there is $v_1 \in W_1$ such that $w_1 \leq v_1$ and $v_1 B v_2$ by item (B_2) of Definition 8.7.3. As $\mathcal{M}_1, w_1 \Vdash \chi \to \psi$ we get that if $\mathcal{M}_1, v_1 \Vdash \chi$ then $\mathcal{M}_1, v_1 \Vdash \psi$. But we know that $\mathcal{M}_2, v_2 \Vdash \chi$, so by $\mathcal{M}_1, v_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, v_2$ and induction hypothesis we get $\mathcal{M}_1, v_1 \Vdash \chi$. Thus, we get $\mathcal{M}_1, v_1 \Vdash \psi$. It then suffices to use the induction hypothesis again to obtain $\mathcal{M}_2, v_2 \Vdash \psi$. Consequently, we proved that $\mathcal{M}_2, w_2 \Vdash \chi \to \psi$.

  ($\Leftarrow$) We proceed similarly to ($\Rightarrow$), using the item (B_3) of Definition 8.7.3 instead.

- $\varphi := \chi \mathrel{-\!\!<} \psi$: ($\Rightarrow$) Assume $\mathcal{M}_1, w_1 \Vdash \chi \mathrel{-\!\!<} \psi$. We need to show $\mathcal{M}_2, w_2 \Vdash \chi \mathrel{-\!\!<} \psi$. As $\mathcal{M}_1, w_1 \Vdash \chi \mathrel{-\!\!<} \psi$ there is a $v_1 \in W_1$ such that $v_1 \leq w_1$ and $\mathcal{M}_1, v_1 \Vdash \chi$ and $\mathcal{M}_1, v_1 \nVdash \psi$. Given that $\mathcal{M}_1, w_1 \mathrel{\underline{\leftrightarrow}} \mathcal{M}_2, w_2$ and $v_1 \leq w_1$, we get that there is

$v_2 \in W_2$ such that $v_2 \leq w_2$ and $v_1 B v_2$ by item (B_5) of Definition 8.7.3. But we know that $\mathcal{M}_1, v_1 \Vdash \chi$ and $\mathcal{M}_1, v_1 \nVdash \psi$, so by $\mathcal{M}_1, v_1 \leftrightharpoons \mathcal{M}_2, v_2$ and induction hypothesis we get $\mathcal{M}_2, v_2 \Vdash \chi$ and $\mathcal{M}_2, v_2 \nVdash \psi$. Thus, we get $\mathcal{M}_2, w_2 \Vdash \chi \prec \psi$.

$(\Leftarrow)$ We proceed similarly to $(\Rightarrow)$, using the item (B_4) of Definition 8.7.3 instead.

$\blacksquare$

Finally, let us recall the definition of the local and global semantic consequence relations given in Section 5.1.

Definition 8.7.4. The local and global consequence relations are as below:

$$\begin{array}{lll} \Gamma \models_l \Delta & \text{iff} & \forall \mathcal{M}. \forall w. \, (\mathcal{M}, w \Vdash \Gamma \, \Rightarrow \, \exists \delta \in \Delta. \, \mathcal{M}, w \Vdash \delta) \\ \Gamma \models_g \Delta & \text{iff} & \forall \mathcal{M}. \, (\mathcal{M} \Vdash \Gamma \, \Rightarrow \, \forall w \in W. \, \exists \delta \in \Delta. \, \mathcal{M}, w \Vdash \delta) \end{array}$$

While the two notions are not generally equivalent in modal logic, as shown in Section 5.1, they are equivalent on the intuitionistic language as opposed to the bi-intuitionistic language.

Lemma 8.7.2. For $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_\mathbf{I}}$:

$$\Gamma \models_l \Delta \quad \text{iff} \quad \Gamma \models_g \Delta$$

Proof ✍. We prove each direction separately.

$(\Rightarrow)$ Assume $\Gamma \models_l \Delta$. Let $\mathcal{M}$ be a model. Assume that $\mathcal{M} \Vdash \Gamma$. We need to show that for all $w \in W$, there is a $\delta \in \Delta$ such that $\mathcal{M}, w \Vdash \delta$. Let $w \in W$. Given that $\mathcal{M} \Vdash \Gamma$, we have in particular $\mathcal{M}, w \Vdash \Gamma$. Consequently, as we have $\Gamma \models_l \Delta$, we obtain that there is a $\delta \in \Delta$ such that $\mathcal{M}, w \Vdash \delta$. As w is arbitrary, we get $\forall w \in W. \exists \delta \in \Delta. \mathcal{M}, w \Vdash \delta$. So, $\Gamma \models_g \Delta$.

$(\Leftarrow)$ Assume $\Gamma \models_g \Delta$. Let $\mathcal{M}$ be a model and $w \in W$. Assume that $\mathcal{M}, w \Vdash \Gamma$. We need to show that there is a $\delta \in \Delta$ such that $\mathcal{M}, w \Vdash \delta$. Consider the model $\mathcal{M}' = (W', \leq', I')$ such that $W' = \{v \in W \mid w \leq v\}$, and $\leq'$ and I' are respective restrictions of $\leq$ and I to W'. We have that w and w' are bisimilar if we restrict our notion of bisimulation in Definition 8.7.3 to the clauses (B_1) to (B_3), which is the usual notion of bisimulation in the intuitionistic context [22, p.54][103, p.8]. So, we have $\mathcal{M}, w \leftrightharpoons \mathcal{M}', w$. In addition to that, in the intuitionistic context, the notion of bisimulation also implies logical equivalence between bisimilar points as expressed in Proposition 8.7.1 [22, p.54]. So, we obtain that $\mathcal{M}, w \Vdash \Gamma$ implies $\mathcal{M}', w \Vdash \Gamma$. Then, the intuitionistic version of Lemma 8.7.1 implies that for all $v \in W'$ such that $w \leq v$, we have $\mathcal{M}', w \Vdash \Gamma$. However, for all $v \in W'$ we have $w \leq v$, by definition of W'. Consequently, we have $\mathcal{M}', v \Vdash \Gamma$ for all $v \in W'$, hence $\mathcal{M}' \Vdash \Gamma$. We use $\Gamma \models_g \Delta$ on the latter to obtain $\forall v \in W. \exists \delta \in \Delta. \mathcal{M}', v \Vdash \delta$. In particular, we get that for w there is $\delta \in \Delta$ such that $\mathcal{M}', w \Vdash \delta$. It then suffices to use the fact that bisimulation implies logical equivalence [22, p.54] once again to obtain $\mathcal{M}, w \Vdash \delta$. So, $\Gamma \models_l \Delta$.

$\blacksquare$

Let us comment on this proof. It is easy to see that the local implies the global in full generality. The converse holds for intuitionistic logic for two reasons. First, persistence plays an important role: if a formula is true at a point then it is true at all the successors (the upcone) of that point. Second and more crucially, in an intuitionistic Kripke model, a point in a model is bisimilar with itself in its upcone.

Nonetheless, in the semantics for bi-intuitionistic logics just defined it is not the case that $\Gamma \models_g \Delta$ implies $\Gamma \models_l \Delta$, similarly to modal logic. This can easily be shown by the fact that $p \models_g \neg{\sim}p$ while $p \not\models_l \neg{\sim}p$. We suspect that the equivalence between the two notions in the intuitionistic setting may have led experts in the field to overlook the difference between them in the bi-intuitionistic case, notably causing the confusions presented in this chapter.

We still need to prove the claims we left pending, to substantiate our claims that wBIL and sBIL are different logics, both extensionally and on a meta-level. These differences are shown by proving that the local semantic consequence corresponds to wBIL and the global semantic consequence corresponds to sBIL.

8.8 Weak is local

In this section, we focus on the logic wBIL and show that it corresponds to the local semantic consequence relation. We first establish a Lindenbaum lemma, ensuring the existence for any unprovable pair of a Lindenbaum extension that is both complete and unprovable. Second, we use this type of pair, i.e. complete and unprovable, to construct a canonical model which allows us to prove completeness.

8.8.1 A Lindenbaum lemma

To obtain a Lindenbaum lemma, showing the existence of a complete and unprovable extension for any unprovable pair, we need to proceed as follows. First, we need an encoding of formulas by natural numbers. Second, we define a decoding function, which is some type of inverse of the encoding. These two functions are then used in a third moment, where we define a selection function, picking formulas, which is involved in the step-by-step construction of the Lindenbaum extension of a pair $[\Gamma \mid \Delta]$. Fourth, the Lindenbaum extension, i.e. a pair $[\Gamma' \mid \Delta']$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, is defined using the selection function. Finally, we prove the Lindenbaum lemma: the Lindenbaum extension is complete and unprovable.

First, we assume given an encoding (Definition 6.1.1) as defined below.

Hypothesis 8.8.1. There is an encoding of $Form_{\mathbb{L}_{\mathbf{BI}}}$.

We fix this encoding and call it *encode*0. We then define the encoding we use in our Lindenbaum extension.

Definition 8.8.1. We define $encode : \varphi \mapsto S\ (encode0(\varphi))$, where S is the successor operator on natural numbers.

As in the modal case, *encode* is injective and makes 0 the encoding of no formula. Second, we need a decoding function, as defined in Definition 6.1.4.

Lemma 8.8.1. There is a decoding function for *encode*.

Proof. (⌨) Identical to the proof of Lemma 6.1.3. ■

Using the previous lemma, we fix a decoding function of *encode* and call it *decode*.

Third, with these functions in hand, we can define a selection function crucial to our Lindenbaum extension.

Definition 8.8.2. We define the selection function sel which takes as inputs a pair of sets of formulas and a natural number, and outputs a set of formulas (note that there is a priority order from bottom to top).

$$\mathsf{sel}([\Gamma \mid \Delta], n) = \begin{cases} \Gamma & \text{if } decode(n) = None \\ \Gamma & \text{if } decode(n) = Some(\varphi \vee \psi) \text{ and } \Gamma \nvdash_{\mathsf{w}} \varphi \vee \psi \\ \Gamma \cup \{\varphi\} & \text{if } decode(n) = Some(\varphi \vee \psi) \text{ and } \Gamma \vdash_{\mathsf{w}} \varphi \vee \psi \text{ and} \\ & \nvdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta] \\ \Gamma \cup \{\psi\} & \text{if } decode(n) = Some(\varphi \vee \psi) \text{ and } \Gamma \vdash_{\mathsf{w}} \varphi \vee \psi \text{ and} \\ & \vdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta] \\ \Gamma & \text{if } decode(n) = Some(\varphi) \end{cases}$$

While the above definition may look complicated, it is in essence rather simple. If n is the encoding of no formula, then sel outputs the left-hand side Γ of the pair it was given (case 1). If n is the encoding of some formula χ, then we need to look at the structure of χ (cases 2-5).

If $\chi := \varphi \vee \psi$, then we need to verify whether $\varphi \vee \psi$ is derived from Γ (cases 2-4). If $\Gamma \nvdash_{\mathsf{w}} \varphi \vee \psi$, sel outputs the left-hand side of the pair it was given (case 2). If $\Gamma \vdash_{\mathsf{w}} \varphi \vee \psi$, we check whether adding φ to the left of the pair makes the pair unprovable (cases 3-4). If we have $\nvdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta]$, then sel outputs $\Gamma \cup \{\varphi\}$ (case 3). If we have $\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta]$, then sel outputs $\Gamma \cup \{\psi\}$ (case 4).

Finally, if χ is any other formula, then sel outputs the left-hand side of the pair it was given (case 5).

The function sel is built to make sure that if a disjunction $\varphi \vee \psi$ is deduced by the left-hand side of a pair $[\Gamma \mid \Delta]$ and its encoding is the natural number under consideration, then one of the disjuncts is *selected* and added to the left-hand side of the pair without making the resulting pair provable.

To obtain this selection property for all disjunctions, we need to go through all natural numbers. Our fourth point deals with this through the next function.

Definition 8.8.3. We define the function Lindf, which takes as inputs a pair of sets of formulas and a natural number, and outputs a set of formulas, inductively on natural numbers.

- $\mathsf{Lindf}([\Gamma \mid \Delta], 0) = \Gamma$
- $\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\ (S\ m)\,)$

We define the *Lindenbaum extension* of a pair $[\Gamma \mid \Delta]$ as follows: $\mathsf{Lind}([\Gamma \mid \Delta]) = (X, Y)$ where:

$$X = \bigcup_{n=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n))$$

$$Y = \{\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}} \mid \bigcup_{n=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n)) \nvdash_{\mathsf{w}} \varphi\}$$

Here again, behind heavy notations lie simple ideas. The function Lindf takes a pair $[\Gamma \mid \Delta]$ and a natural number n, and proceeds to extend step-by-step from 0 to n the set Γ using at each step the function sel. With this function in hand, it is rather straightforward to obtain the Lindenbaum extension of a pair $[\Gamma \mid \Delta]$. First, we use the union of $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ for all $n \in \mathbb{N}$, i.e. $\bigcup_{n:=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n))$, to define X, i.e. the left-hand side of the extension. Second, we define Y the right-hand side of the extension, to be the set of formulas not deduced by the left-hand side, i.e. the following set.

$$\{\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}} \mid \bigcup_{n:=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n)) \nvdash_{\mathsf{w}} \varphi\}$$

In what is remaining of this subsection, we proceed to establish our fifth point: we show that the Lindenbaum extension of an unprovable pair $[\Gamma \mid \Delta]$ is complete, as in Definition 8.6.1, and unprovable.

Definition 8.8.4. Let $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_{\mathbf{BI}}}$ We say that $[\Gamma \mid \Delta]$ is

- *prime* if for every $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\Gamma \vdash_{\mathsf{w}} \varphi \vee \psi$ then $\varphi \in \Gamma$ or $\psi \in \Gamma$;
- *closed under deducibility* if for every $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\Gamma \vdash_{\mathsf{w}} \varphi$ then $\varphi \in \Gamma$.

Before turning to the Lindenbaum lemma, note that our notion of primeness is non-standard in two ways. To witness this, let us recall that traditionally a set of formulas Γ is prime if and only if $\varphi \vee \psi \in \Gamma$ implies $\varphi \in \Gamma$ or $\psi \in \Gamma$. First, we bend this definition by saying that a *pair* is prime, and not a *set*. In fact, our notion of primeness on a pair entails the traditional notion of primeness on the set of formulas on the left of the pair. Second, we require that in our definition the disjunction is *deducible* from the left component of the pair, and not just an element of it.

Lemma 8.8.2 (Lindenbaum Lemma)**.** If $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$ then there exist $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ such that:

1. $[\Gamma' \mid \Delta']$ is prime;
2. $[\Gamma' \mid \Delta']$ is closed under deducibility;
3. $[\Gamma' \mid \Delta']$ is complete;
4. $\nvdash_{\mathsf{w}} [\Gamma' \mid \bot]$;
5. $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$.

Proof. We claim that $\mathsf{Lind}([\Gamma \mid \Delta])$ is such a pair. For convenience, let Γ' and Δ' be such that $\mathsf{Lind}([\Gamma \mid \Delta]) = [\Gamma' \mid \Delta']$. Note that $\Delta' = \{\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}} \mid \Gamma' \nvdash_{\mathsf{w}} \varphi\}$ by definition. First, we prove that each step of the construction $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ preserves the unprovability of the pair $[\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \Delta]$ (⌨). So, we show that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \Delta]$ by induction on n. If $n := 0$, then we have that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], 0) \mid \Delta]$ by assumption as $\mathsf{Lindf}([\Gamma \mid \Delta], 0) = \Gamma$. If $n := S\ m$, then we need to show that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) \mid \Delta]$. We make a first case distinction on $decode(S\ m)$.

1. If $decode(S\ m) = None$, then we have the following chain of equalities.

   $$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\ (S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m)$$

   However, we get by induction hypothesis that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m) \mid \Delta]$ so we are done.

2. If $decode(S\ m) = Some(\varphi)$, then we need to make a case distinction on the structure of φ.

   (a) If $\varphi := \psi \vee \chi$, then note that we have the following equality.

   $$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\ (S\ m)\,)$$

   There, depending on whether the disjunction is deduced from $\mathsf{Lindf}([\Gamma \mid \Delta], m)$, we have two cases.

   i. If $\mathsf{Lindf}([\Gamma \mid \Delta], m) \nvdash_{\mathsf{w}} \psi \vee \chi$, then an application of the induction hypothesis suffices to reach our goal, as we have the following equality.

   $$\mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\ (S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m)$$

   ii. If $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \psi \vee \chi$, then we need to verify whether the following holds or not.

   $$\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \mid \Delta]$$

   A. If $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \mid \Delta]$, then we have the following equality.

   $$\mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\ (S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m) \cup \psi$$

   But note that by assumption we get $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \mid \Delta]$, that is $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) \mid \Delta]$. So, we are done.

B. If $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \mid \Delta]$, then we have the following equality.

$$\mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\,(S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m) \cup \chi$$

Assume for a contradiction that $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) \mid \Delta]$, that is $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \chi \mid \Delta]$. Consequently, by definition there is a finite $\Delta' \subseteq \Delta$ such that $\mathsf{Lindf}([\Gamma \mid \Delta], m), \chi \vdash_{\mathsf{w}} \bigvee \Delta'$. As we have $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \mid \Delta]$ we also obtain a finite $\Delta'' \subseteq \Delta$ such that $\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \vdash_{\mathsf{w}} \bigvee \Delta''$. From $\mathsf{Lindf}([\Gamma \mid \Delta], m) \cup \chi \vdash_{\mathsf{w}} \bigvee \Delta'$ and $\mathsf{Lindf}([\Gamma \mid \Delta], m), \psi \vdash_{\mathsf{w}} \bigvee \Delta''$, we straightforwardly obtain the two following statements.

$$\mathsf{Lindf}([\Gamma \mid \Delta], m) \cup \chi \vdash_{\mathsf{w}} \bigvee \Delta' \vee \bigvee \Delta''$$
$$\mathsf{Lindf}([\Gamma \mid \Delta], m) \cup \psi \vdash_{\mathsf{w}} \bigvee \Delta' \vee \bigvee \Delta''$$

Using Theorem 8.6.1 we get $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \chi \to (\bigvee \Delta' \vee \bigvee \Delta'')$ and $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \psi \to (\bigvee \Delta' \vee \bigvee \Delta'')$. Note that the following formula is an instance of the axiom A_4:

$$(\psi \to (\bigvee \Delta' \vee \bigvee \Delta'')) \to (\chi \to (\bigvee \Delta' \vee \bigvee \Delta'')) \to (\psi \vee \chi \to (\bigvee \Delta' \vee \bigvee \Delta''))$$

So, we easily obtain that $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \bigvee \Delta' \vee \bigvee \Delta''$ by using $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \psi \vee \chi$ and the above. But this is in contradiction with our induction hypothesis $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m) \mid \Delta]$: we found a finite subset $\Delta' \cup \Delta'' \subseteq \Delta$ such that $\mathsf{Lindf}([\Gamma \mid \Delta], m) \vdash_{\mathsf{w}} \bigvee(\Delta' \cup \Delta'')$. So, we get that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], m), \chi \mid \Delta]$.

(b) If φ is not a disjunction, then an application of the induction hypothesis suffices to reach our goal, as we have the following chain of equalities.

$$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,[\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\ \mid \Delta]\,,\,(S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m)$$

In all possible cases we concluded that $\nvdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) \mid \Delta]$, so we are done.

With this result in hand, we proceed to prove each of the items.

1. (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \psi]$. So, we can prove that there must be a number n such that $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \varphi \vee \psi]$. The reason being that Γ' is the collection of all such $\mathsf{Lindf}([\Gamma \mid \Delta], m)$, and so in the proof $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \psi]$ we can replace Γ' for a big enough $\mathsf{Lindf}([\Gamma \mid \Delta], m)$ in all the instances of the application of the rule (El). Now, given that $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \varphi \vee \psi]$, we have $\vdash_{\mathsf{w}} [\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \varphi \vee \psi(\vee\bot)^k]$ for any $k \in \mathbb{N}$ and where $(\vee\bot)^k$ designate the pattern $\vee\bot$ repeated k times. The latter implies that even if at stage n we have already treated the case of $m = \mathit{encode}\varphi \vee \psi$ (where we may not have picked any of φ or ψ because $\mathsf{Lindf}([\Gamma \mid \Delta], m) \nvdash_{\mathsf{w}} \varphi \vee \psi$), there will always be a disjunction $\varphi \vee \psi(\vee\bot)^k$ which will be encountered at a later stage j. When this does happen, the disjunction $\varphi \vee \psi(\vee\bot)^k$ will be deduced from the corresponding $\mathsf{Lindf}([\Gamma \mid \Delta], j)$ as the latter is a superset of $\mathsf{Lindf}([\Gamma \mid \Delta], n)$. So, given that $\bot$ cannot be selected by sel at stage j, either φ or ψ will be added to the set. As a consequence, we obtain that $\varphi \in \Gamma'$ or $\psi \in \Gamma'$.

2. (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi]$. We need to show $\varphi \in \Gamma'$. We easily obtain $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \varphi]$ using our assumption and A_2. Item 1 proved above gives us that $\varphi \in \Gamma'$ or $\varphi \in \Gamma'$, hence $\varphi \in \Gamma'$.

3. (⌨) We need to show that $[\Gamma' \mid \Delta']$ is complete. Let $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. We use the law of excluded middle on the derivability of φ from Γ'. If $\Gamma' \vdash_{\mathsf{w}} \varphi$, then by item 2 we obtain $\varphi \in \Gamma'$. If $\Gamma' \nvdash_{\mathsf{w}} \varphi$, then by definition of Δ' we get that $\varphi \in \Delta'$. So, $[\Gamma' \mid \Delta']$ is complete.

4. (⌨) We need to show $\nvdash_{\mathsf{w}} [\Gamma' \mid \bot]$. Assume for a contradiction that $\vdash_{\mathsf{w}} [\Gamma' \mid \bot]$. Item 2 allows us to infer that $\bot \in \Gamma'$. However, we have that $\Gamma' = \bigcup_{n:=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n))$. So, we must have that $\bot \in \mathsf{Lindf}([\Gamma \mid \Delta], n)$ for some n. The latter contradicts the established fact that the pair $[\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \delta]$ is unprovable.

5. (⌨) We need to show $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. Assume for a contradiction that $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. Then, by definition there must be a finite $\Delta'' \subseteq \Delta'$ such that $\Gamma' \vdash_{\mathsf{w}} \bigvee \Delta''$. Using item 1, we get that there must be a $\delta \in \Delta''$ such that $\delta \in \Gamma'$. However, as $\delta \in \Delta'' \subseteq \Delta'$, we obtain by definition of $\Delta' = \{\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}} \mid \Gamma' \nvdash_{\mathsf{w}} \varphi\}$ that $\Gamma' \nvdash_{\mathsf{w}} \delta$, a contradiction. So, $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$.

(⌨) We prove $\Delta \subseteq \Delta'$. Take $\delta \in \Delta$ and assume $\delta \notin \Delta'$. Then, we get $\delta \in \Gamma'$ by item 3. As we have that $\Gamma' = \bigcup_{n:=0}^{\infty} (\mathsf{Lindf}([\Gamma \mid \Delta], n))$, we get that $\delta \in \mathsf{Lindf}([\Gamma \mid \Delta], n)$ for some n. However, this implies that the pair $[\mathsf{Lindf}([\Gamma \mid \Delta], n) \mid \delta]$ is provable, which contradicts our point above.

So, $[\Gamma' \mid \Delta']$ is such $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$, and satisfies all of the above items. ■

8.8.2 A canonical model construction

We just showed that for every unprovable pair $[\Gamma \mid \Delta]$ there is an extension $[\Gamma' \mid \Delta']$ of this pair which is complete and unprovable. We use this type of pair to define the points of a canonical model, taking inspiration from Sano and Stell [127].

Definition 8.8.5. The canonical model $\mathcal{M}^c = (W^c, \leq^c, I^c)$ is defined in the following way:

1. $W^c = \{[\Gamma \mid \Delta] \mid [\Gamma \mid \Delta]$ is complete and $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]\}$;
2. $[\Gamma_1 \mid \Delta_1] \leq^c [\Gamma_2 \mid \Delta_2]$ iff $\Gamma_1 \subseteq \Gamma_2$;
3. $I^c(p) = \{[\Gamma \mid \Delta] \in W^c \mid p \in \Gamma\}$.

We use this canonical model as a witness: we intend to show that if $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$, then we have that there is a point in $\mathcal{M}^c$ which is a Γ-point but forces no formula in Δ, hence $\Gamma \not\models_l \Delta$. As usual a canonical model proof technique, we prove the crucial Truth Lemma. To do so, we require the next lemma.

Lemma 8.8.3. Let $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_{\mathbf{BI}}}$. If $[\Gamma \mid \Delta]$ is complete and $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$, then $[\Gamma \mid \Delta]$ is prime and closed under deducibility.

Proof. Assume $[\Gamma \mid \Delta]$ is complete and $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$.

(⌨) We first prove that it is closed under deducibility. Let $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. Assume that $\Gamma \vdash_{\mathsf{w}} \varphi$. As $[\Gamma \mid \Delta]$ is complete, we have that $\varphi \in \Gamma$ or $\varphi \in \Delta$. If the former, we are done. If the latter, we obtain a contradiction as $\varphi \in \Delta$ and $\Gamma \vdash_{\mathsf{w}} \varphi$ leads to $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$. In both cases, we get $\varphi \in \Gamma$.

(⌨) Second, we prove that $[\Gamma \mid \Delta]$ is prime. Let $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. Assume that $\Gamma \vdash_{\mathsf{w}} \varphi \vee \psi$. As $[\Gamma \mid \Delta]$ is complete, we have that $\varphi \in \Gamma$ or $\varphi \in \Delta$, and $\psi \in \Gamma$ or $\psi \in \Delta$. If one of the formulas is in Γ, then we are done. Consequently, it suffices to consider the case where $\varphi \in \Delta$ and $\psi \in \Delta$. If this is the case, note that there is a finite set $\{\varphi, \psi\} \subseteq \Delta$ such that $\Gamma \vdash_{\mathsf{w}} \bigvee\{\varphi, \psi\}$, i.e. $\Gamma \vdash_{\mathsf{w}} \varphi \vee \psi$. So, we have $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$, in contradiction with our initial assumption. ■

We then turn to the Truth Lemma, which relates forcing and elementhood in the canonical model: $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \varphi$ if and only if $\varphi \in \Gamma$. Note that our use of *complete* pairs entails that we have equivalently that $\mathcal{M}^c, [\Gamma \mid \Delta] \nVdash \varphi$ if and only if $\varphi \in \Delta$. In essence, the Truth Lemma shows that a pair $[\Gamma \mid \Delta] \in W^c$ is a full description of which formula ψ is forced in $[\Gamma \mid \Delta]$, if $\psi \in \Gamma$, and which formula is not, if $\psi \in \Delta$.

Lemma 8.8.4 (Truth Lemma)**.** For every $[\Gamma \mid \Delta] \in W^c$:

$$\psi \in \Gamma \quad \text{iff} \quad \mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi.$$

Proof. (⌨) By induction on ψ.

- $\psi := p$: by definition $p \in \Gamma$ iff $\Gamma \in I^c(p)$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash p$.

- $\psi := \top$: we have that $\top \in \Gamma$ by Lemma 8.8.3 and $\Gamma \vdash_{\mathsf{w}} \top$. In addition to that, we have $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \top$ by definition. So, we have $\top \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \top$ trivially.

- $\psi := \bot$: we have that $\bot \notin \Gamma$ as otherwise we get $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$ which is a contradiction. In addition to that, we have $\mathcal{M}^c, [\Gamma \mid \Delta] \nVdash \bot$ by definition. So, we have $\bot \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \bot$ trivially.

- $\psi := \psi_1 \wedge \psi_2$: $\psi_1 \wedge \psi_2 \in \Gamma$ iff $\psi_1 \in \Gamma$ and $\psi_2 \in \Gamma$ by Lemma 8.8.3 and $\Gamma \vdash_{\mathsf{w}} \psi_i$ for $i \in \{1, 2\}$. By induction hypothesis this holds if and only if $\mathcal{M}^c, \Gamma \vdash_{\Vdash} \Delta\psi_1$ and $\mathcal{M}^c, \Gamma \vdash_{\Vdash} \Delta\psi_2$. Then $\psi_1 \wedge \psi_2 \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \wedge \psi_2$.

- $\psi := \psi_1 \vee \psi_2$: $\psi_1 \vee \psi_2 \in \Gamma$ iff $[\psi_1 \in \Gamma$ or $\psi_2 \in \Gamma]$ by Lemma 8.8.3. By induction hypothesis this holds if and only if $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1$ or $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_2$. Then $\psi_1 \vee \psi_2 \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \vee \psi_2$.

- $\psi := \psi_1 \to \psi_2$: ($\Rightarrow$) Assume $\psi_1 \to \psi_2 \in \Gamma$. We need to show that $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \to \psi_2$. Let $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$, and assume $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$. We need to show that $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_2$. By the induction hypothesis, it is sufficient to show that $\psi_2 \in \Gamma'$. From $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ we obtain $\psi_1 \in \Gamma'$ by induction hypothesis. Also, as $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$, we have $\psi_1 \to \psi_2 \in \Gamma \subseteq \Gamma'$. It is straightforward to prove that $\Gamma' \vdash_{\mathsf{w}} \psi_2$ using (MP). Using Lemma 8.8.3 we get $\psi_2 \in \Gamma'$. The induction hypothesis gives us $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_2$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \to \psi_2$. Assume for reductio that $\psi_1 \to \psi_2 \notin \Gamma$. Then by Theorem 8.6.1 we obtain $\nvdash_{\mathsf{w}} [\Gamma, \psi_1 \mid \psi_2]$. By Lemma 8.8.2 there is a $\Gamma' \supseteq \Gamma \cup \{\psi_1\}$ and $\Delta' \supseteq \{\psi_2\}$ such that $[\Gamma' \mid \Delta']$ is complete and $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. Note that we have $\vdash_{\mathsf{w}} [\Gamma' \mid \psi_1]$, hence $\psi_1 \in \Gamma'$ by Lemma 8.8.3. By induction hypothesis we get $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \psi_2$. But as $\Gamma' \supseteq \Gamma$ and $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \to \psi_2$ we reached a contradiction.

- $\psi := \psi_1 \mathrel{-\!\!<} \psi_2$: ($\Rightarrow$) Assume $\psi_1 \mathrel{-\!\!<} \psi_2 \in \Gamma$. We claim that $\nvdash_{\mathsf{w}} [\psi_1 \mid \psi_2, \Delta]$. Suppose it is not the case. Then by definition there is a finite $\Delta_f \subseteq \Delta$ such that $\psi_1 \vdash_{\mathsf{w}} \psi_2 \vee \bigvee \Delta_f$, hence $\vdash_{\mathsf{w}} [\psi_1 \mid \psi_2 \vee \bigvee \Delta_f]$. By Theorem 8.6.1 we thus obtain $\vdash_{\mathsf{w}} [\emptyset \mid \psi_1 \to (\psi_2 \vee \bigvee \Delta_f)]$. And then by Proposition 8.6.1 we obtain that $\vdash_{\mathsf{w}} [\emptyset \mid (\psi_1 \mathrel{-\!\!<} \psi_2) \to \bigvee \Delta_f]$. But as $\psi_1 \mathrel{-\!\!<} \psi_2 \in \Gamma$ and we get that $\bigvee \Delta_f \in \Gamma$ using Lemma 8.8.3, which leads to an obvious contradiction. So $\nvdash_{\mathsf{w}} [\psi_1 \mid \psi_2, \Delta]$. Thus by Lemma 8.8.2 there are $\Gamma' \supseteq \{\psi_1\}$ and $\Delta' \supseteq \Delta \cup \{\psi_2\}$ such that $[\Gamma' \mid \Delta']$ is complete and $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. Note that $\psi_1 \in \Gamma'$ and $\psi_2 \notin \Gamma'$, hence $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \psi_2$ by induction hypothesis. But we have that $\Delta \subseteq \Delta'$, which implies by completeness that $\Gamma' \subseteq \Gamma$. So $[\Gamma' \mid \Delta'] \leq^c [\Gamma \mid \Delta]$. Consequently $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \mathrel{-\!\!<} \psi_2$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta] \Vdash \psi_1 \mathrel{-\!\!<} \psi_2$. Then, there is $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$, $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \psi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \psi_2$. By induction hypothesis, we obtain $\psi_1 \in \Gamma'$ and $\psi_2 \notin \Gamma'$. Now, note that $\Gamma' \vdash_{\mathsf{w}} \psi_1 \to (\psi_2 \vee (\psi_1 \mathrel{-\!\!<} \psi_2))$ as the formula on the right is an instance of A_{11}. Consequently, we get $\Gamma' \vdash_{\mathsf{w}} \psi_2 \vee (\psi_1 \mathrel{-\!\!<} \psi_2)$ as $\psi_1 \in \Gamma'$. Using Lemma 8.8.3 we obtain that $\psi_2 \in \Gamma'$ or $\psi_1 \mathrel{-\!\!<} \psi_2 \in \Gamma'$. As the former is impossible, we have $\psi_1 \mathrel{-\!\!<} \psi_2 \in \Gamma'$. Finally, note that $\Gamma' \subseteq \Gamma$ as $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$. So, we obtain $\psi_1 \mathrel{-\!\!<} \psi_2 \in \Gamma$.

■

We are now ready to prove that wBIL corresponds to the local semantic consequence relation.

Theorem 8.8.1. The following holds:

$$\vdash_{\mathsf{w}} [\Gamma \mid \Delta] \quad \text{iff} \quad \Gamma \models_l \Delta$$

Proof. We prove each direction separately.

($\Rightarrow$) (⌨) Here we prove *soundness*. Assume $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$. We prove that $\Gamma \models_l \Delta$ by induction on the structure of the proof of $\vdash_{\mathsf{w}} [\Gamma \mid \Delta]$. The case where the last rule applied is (El) is trivial. So, we need to prove the case where the last rule applied is (Ax), (MP) or (wDN). First, we proceed to prove that all axioms are valid, which suffices to deal with the (Ax). Second, we treat the cases (MP) and (wDN) by showing that these rules preserve semantic consecutions. In what follows let $\mathcal{M} = (W, \leq, I)$ be a model and $w \in W$.

(A_1) We show that $\mathcal{M}, w \Vdash (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$. Let $w_1 \geq w$ and assume $\mathcal{M}, w_1 \Vdash \varphi \to \psi$. We need to show that $\mathcal{M}, w_1 \Vdash (\psi \to \chi) \to (\varphi \to \chi)$. Let $w_2 \geq w_1$ and assume $\mathcal{M}, w_2 \Vdash \psi \to \chi$. We need to show that $\mathcal{M}, w_2 \Vdash \varphi \to \chi$. Let $w_3 \geq w_2$ and assume $\mathcal{M}, w_3 \Vdash \varphi$. We need to show that $\mathcal{M}, w_3 \Vdash \chi$. As $w_1 \leq w_3$ by transitivity of $\leq$ and $\mathcal{M}, w_1 \Vdash \varphi \to \psi$, we get that $\mathcal{M}, w_3 \Vdash \psi$. Moreover $w_2 \leq w_3$ and $\mathcal{M}, w_2 \Vdash \psi \to \chi$, so $\mathcal{M}, w_3 \Vdash \chi$.

(A_2) We show that $\mathcal{M}, w \Vdash \varphi \to (\varphi \vee \psi)$. Let $w_1 \geq w$ and assume $\mathcal{M}, w_1 \Vdash \varphi$. Then by definition $\mathcal{M}, w_1 \Vdash \varphi \vee \psi$.

(A_3) We show that $\mathcal{M}, w \Vdash \psi \to (\varphi \vee \psi)$ as in (A_2).

(A_4) We show that $\mathcal{M}, w \Vdash (\varphi \to \psi) \to ((\chi \to \psi) \to ((\varphi \vee \chi) \to \psi))$. Let $w_1 \geq w$ and assume that $\mathcal{M}, w_1 \Vdash \varphi \to \psi$. We need to show that $\mathcal{M}, w_1 \Vdash (\chi \to \psi) \to ((\varphi \vee \chi) \to \psi)$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \chi \to \psi$. We need to show that $\mathcal{M}, w_2 \Vdash (\varphi \vee \chi) \to \psi$. Let $w_3 \leq w_2$ s.t. $\mathcal{M}, w_3 \Vdash \varphi \vee \chi$. By definition $\mathcal{M}, w_3 \Vdash \varphi$ or $\mathcal{M}, w_3 \Vdash \chi$. Assume $\mathcal{M}, w_3 \Vdash \varphi$. As $w_1 \leq w_3$ by transitivity and $\mathcal{M}, w_1 \Vdash \varphi \to \psi$, we get $\mathcal{M}, w_3 \Vdash \psi$. The case where $\mathcal{M}, w_3 \Vdash \chi$ is similar. So $\mathcal{M}, w_3 \Vdash \psi$.

(A_5) We show that $\mathcal{M}, w \Vdash (\varphi \wedge \psi) \to \varphi$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \varphi \wedge \psi$. By definition we get that $\mathcal{M}, w_1 \Vdash \varphi$.

(A_6) We show that $\mathcal{M}, w \Vdash (\varphi \wedge \psi) \to \psi$ as in (A_5).

(A_7) We show that $\mathcal{M}, w \Vdash (\chi \to \varphi) \to ((\chi \to \psi) \to (\chi \to (\varphi \wedge \psi)))$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \chi \to \varphi$. We need to show that $\mathcal{M}, w_1 \Vdash (\chi \to \psi) \to (\chi \to (\varphi \wedge \psi))$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \chi \to \psi$. We need to show $\mathcal{M}, w_2 \Vdash \chi \to (\varphi \wedge \psi)$. Let $w_3 \geq w_2$ s.t. $\mathcal{M}, w_3 \Vdash \chi$. We need to show that $\mathcal{M}, w_3 \Vdash \varphi \wedge \psi$, i.e. $\mathcal{M}, w_3 \Vdash \varphi$ and $\mathcal{M}, w_3 \Vdash \psi$. First, as $w_2 \geq w_3$ and $\mathcal{M}, w_2 \Vdash \chi \to \psi$ we get that $\mathcal{M}, w_3 \Vdash \psi$. Second, as $w_1 \leq w_3$ and $\mathcal{M}, w_1 \Vdash \chi \to \varphi$ we get that $\mathcal{M}, w_3 \Vdash \varphi$.

(A_8) We show that $\mathcal{M}, w \Vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \varphi \to (\psi \to \chi)$. We need to show that $\mathcal{M}, w_1 \Vdash (\varphi \wedge \psi) \to \chi$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \varphi \wedge \psi$ i.e. $\mathcal{M}, w_2 \Vdash \varphi$ and $\mathcal{M}, w_2 \Vdash \psi$. We need to show that $\mathcal{M}, w_2 \Vdash \chi$. As $w_1 \leq w_2$ and $\mathcal{M}, w_1 \Vdash \varphi \to (\psi \to \chi)$ we get that $\mathcal{M}, w_2 \Vdash \psi \to \chi$. By reflexivity and the fact that $\mathcal{M}, w_2 \Vdash \psi$ we get $\mathcal{M}, w_2 \Vdash \chi$.

(A_9) We show $\mathcal{M}, w \Vdash ((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash (\varphi \wedge \psi) \to \chi$. We need to show that $\mathcal{M}, w_1 \Vdash \varphi \to (\psi \to \chi)$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \varphi$. We need to show that $\mathcal{M}, w_2 \Vdash \psi \to \chi$. Let $w_3 \geq w_2$ s.t. $\mathcal{M}, w_3 \Vdash \psi$. We need to show that $\mathcal{M}, w_3 \Vdash \chi$. As $w_2 \leq w_3$ and $\mathcal{M}, w_2 \Vdash \varphi$ we get by Lemma 9.7.1 that $\mathcal{M}, w_3 \Vdash \varphi$, thus $\mathcal{M}, w_3 \Vdash \varphi \wedge \psi$. As $w_1 \leq w_3$ and $\mathcal{M}, w_1 \Vdash (\varphi \wedge \psi) \to \chi$ we get $\mathcal{M}, w_3 \Vdash \chi$.

(A_{10}) We show that $\mathcal{M}, w \Vdash (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \varphi \to \psi$. We need to show $\mathcal{M}, w_1 \Vdash \neg\psi \to \neg\varphi$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \neg\psi$. We need to show that $\mathcal{M}, w_2 \Vdash \neg\varphi$. Let $w_3 \geq w_2$. Let us assume for contradiction that $\mathcal{M}, w_3 \Vdash \varphi$. Then as $w_1 \leq w_3$ and $\mathcal{M}, w_1 \Vdash \varphi \to \psi$ we get $\mathcal{M}, w_3 \Vdash \psi$. But this is a contradiction as $\mathcal{M}, w_2 \Vdash \neg\psi$. So $\mathcal{M}, w_3 \nVdash \varphi$.

(A_{11}) We show $\mathcal{M}, w \Vdash \varphi \to (\psi \vee (\varphi \mathrel{-\!\!\!<} \psi))$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \varphi$. We need to show that $\mathcal{M}, w_1 \Vdash \psi \vee (\varphi \mathrel{-\!\!\!<} \psi)$, i.e. $\mathcal{M}, w_1 \Vdash \psi$ or $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} \psi$. If $\mathcal{M}, w_1 \Vdash \psi$ then we're done. If $\mathcal{M}, w_1 \nVdash \psi$ then by reflexivity of $\leq$ we get $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} \psi$. In every case we get $\mathcal{M}, w_1 \Vdash \psi \vee (\varphi \mathrel{-\!\!\!<} \psi)$. Note that we used the law of excluded middle in this situation, making our proof non-constructive.

(A_{12}) We show $\mathcal{M}, w \Vdash (\varphi \mathrel{-\!\!\!<} \psi) \to \sim(\varphi \to \psi)$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} \psi$. We need to show that $\mathcal{M}, w_1 \Vdash \sim(\varphi \to \psi)$. In other words we need to show that there is a $w_2 \leq w_1$ s.t. $\mathcal{M}, w_2 \nVdash \varphi \to \psi$. But as $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} \psi$ there is a $w_3 \leq w_1$ s.t. $\mathcal{M}, w_3 \Vdash \varphi$ and $\mathcal{M}, w_1 \nVdash \psi$. Thus $\mathcal{M}, w_3 \nVdash \varphi \to \psi$ by reflexivity of $\leq$.

(A_{13}) We show that $\mathcal{M}, w \Vdash ((\varphi \mathrel{-\!\!\!<} \psi) \mathrel{-\!\!\!<} \chi) \to (\varphi \mathrel{-\!\!\!<} (\psi \vee \chi))$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash (\varphi \mathrel{-\!\!\!<} \psi) \mathrel{-\!\!\!<} \chi$. We need to show that $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} (\psi \vee \chi)$. As $\mathcal{M}, w_1 \Vdash (\varphi \mathrel{-\!\!\!<} \psi) \mathrel{-\!\!\!<} \chi$ there is a $w_2 \leq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \varphi \mathrel{-\!\!\!<} \psi$ and $\mathcal{M}, w_2 \nVdash \chi$. As $\mathcal{M}, w_2 \Vdash \varphi \mathrel{-\!\!\!<} \psi$ there is a $w_3 \leq w_2$ s.t. $\mathcal{M}, w_3 \Vdash \varphi$ and $\mathcal{M}, w_3 \nVdash \psi$. And as $\mathcal{M}, w_2 \nVdash \chi$ and $w_3 \leq w_2$ we get by Lemma 8.7.1 that $\mathcal{M}, w_3 \nVdash \chi$. So we get that $\mathcal{M}, w_3 \nVdash \psi \vee \chi$. And as $w_3 \leq w_1$, $\mathcal{M}, w_3 \Vdash \varphi$ and $\mathcal{M}, w_3 \nVdash \psi \vee \chi$ we get that $\mathcal{M}, w_1 \Vdash \varphi \mathrel{-\!\!\!<} (\psi \vee \chi)$.

(A_{14}) We show that $\mathcal{M}, w \Vdash \neg(\varphi \mathrel{-\!\!\!<} \psi) \to (\varphi \to \psi)$. Let $w_1 \geq w$ s.t. $\mathcal{M}, w_1 \Vdash \neg(\varphi \mathrel{-\!\!\!<} \psi)$. We need to show $\mathcal{M}, w_1 \Vdash \varphi \to \psi$. Let $w_2 \geq w_1$ s.t. $\mathcal{M}, w_2 \Vdash \varphi$. We need to show that $\mathcal{M}, w_2 \Vdash \psi$. Assume for *reductio* that $\mathcal{M}, w_2 \nVdash \psi$. Then $\mathcal{M}, w_2 \Vdash \varphi \mathrel{-\!\!\!<} \psi$, but this is a contradiction as $w_1 \leq w_2$ and $\mathcal{M}, w_1 \Vdash \neg(\varphi \mathrel{-\!\!\!<} \psi)$. So $\mathcal{M}, w_2 \Vdash \psi$.

(A_{15}) We show that $\mathcal{M}, w \Vdash \varphi \to \top$. Let $v \geq w$ s.t. $\mathcal{M}, v \Vdash \varphi$. We need to show that $\mathcal{M}, v \Vdash \top$, but this holds by definition.

(A_{16}) We show $\mathcal{M}, w \Vdash \bot \to \varphi$. Let $v \geq w$ s.t. $\mathcal{M}, v \Vdash \bot$. We need to show that $\mathcal{M}, v \Vdash \varphi$. Given that $\mathcal{M}, v \Vdash \bot$ is a contradiction by definition, we infer $\mathcal{M}, v \Vdash \varphi$.

(MP) Assume that $\Gamma \models_l \varphi$ and $\Gamma \models_l \varphi \to \psi$ hold. We show that $\Gamma \models_l \psi$. Let $\mathcal{M}$ be a model and $w \in W$. Assume that $\mathcal{M}, w \Vdash \Gamma$. We need to show that $\mathcal{M}, w \Vdash \psi$. As $\mathcal{M}, w \Vdash \Gamma$ and $\Gamma \models_l \varphi$ and $\Gamma \models_l \varphi \to \psi$, we directly obtain $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \varphi \to \psi$. By reflexivity of $\leq$ we consequently obtain $\mathcal{M}, w \Vdash \psi$.

(wDN) Assume that $\models_l \varphi$. We show that $\Gamma \models_l \neg\sim\varphi$. Let $\mathcal{M}$ be a model and $w \in W$. Assume that $\mathcal{M}, w \Vdash \Gamma$. We need to show that $\mathcal{M}, w \Vdash \neg\sim\varphi$. Thus we need to show that for every $w_1 \geq w$ we have $\mathcal{M}, w_1 \nVdash \sim\varphi$. Let $w_1 \geq w$ and $w_2 \leq w_1$. As $\models_l \varphi$ we get that $\mathcal{M}, w_2 \Vdash \varphi$. As w_2 is arbitrary, we have that for all $w_2 \leq w_1$, we have $\mathcal{M}, w_2 \Vdash \varphi$. Thus $\mathcal{M}, w_1 \nVdash \sim\varphi$. As w_1 is arbitrary we reached our goal.

($\Leftarrow$) (⌨) Here we prove *completeness*. Assume $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$. Lemma 8.8.2 gives us a complete pair $[\Gamma' \mid \Delta']$ such that $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$, where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Moreover there is no $\delta \in \Delta$ such that $\delta \in \Gamma'$, so by Lemma 8.8.4 we obtain that in the canonical model of Definition 8.8.5 the following holds: $\mathcal{M}^c, [\Gamma' \mid \Delta'] \nVdash \delta$ for every $\delta \in \Delta$, while $\mathcal{M}^c, [\Gamma' \mid \Delta'] \Vdash \Gamma$. Consequently, we have that $\Gamma \not\models_l \Delta$.

■

8.9 Strong is global

In this section we show that sBIL corresponds to the global semantic consequence relation, by notably following the argument exhibited in Subsection 6.1.2. We rely on the correspondence between wBIL and the local semantic consequence shown just above. More precisely, we show that if a pair $[\Gamma \mid \Delta]$ is unprovable in sBIH, then through Theorem 8.8.1 we extract the existence of a model $\mathcal{M}$ having a point w forcing $(\neg\sim)^{\omega}\Gamma$ and forcing no formula in Δ. But this is not enough because $\mathcal{M}$ is not necessarily a Γ-model. Therefore, we make use of this model $\mathcal{M}$ by restricting it to another model containing w which is a Γ-model and where w still forces no formula in Δ, making use of a bisimulation between the two models. Thus, the latter model witnesses $\Gamma \not\models_g \Delta$.

First, we show how to restrict a model $\mathcal{M}$ from a point w.

Definition 8.9.1. Let $\mathcal{M} = (W, \leq, I)$ be a model and $w \in W$. The *restriction of $\mathcal{M}$ in w* is the model $\mathcal{M}^w = (W^w, \leq^w, I^w)$, where:

- $W^w = \{v \in W \mid$ there is a chain $wR_1 ... R_n v$, where $R_j \in \{\leq, \geq\}$ for $j \in \mathbb{N}\}$;
- $\leq^w \;=\; \leq \;\cap\; (W^w \times W^w)$;
- $I^w(p) \;=\; I(p) \cap W^w$.

Note that on the contrary to Definition 6.1.10, here we consider chains constituted by steps not only going forward on the relation of accessibility $\leq$, but also *backward*. As suggested at the beginning of this section, we show that the point w in the initial model is bisimilar with itself in the restriction of the model.

Lemma 8.9.1. For all model $\mathcal{M} = (W, \leq, I)$ and $w \in W$, we have that $\mathcal{M}, w \leftrightharpoons \mathcal{M}^w, w$.

Proof. (⌨) We need to exhibit a bisimulation $B \subseteq (W \times W^w)$. Consider the relation $B = \{(v, v) \mid v \in W^w\}$. Note that $B \subseteq (W \times W^w)$ as required. We are left to show that B satisfies the conditions of the Definition 8.7.3. Let $v \in W^w$.

(B_1) Let $p \in \mathbb{V}$. We have $v \in I(p)$ iff $v \in I^w(p)$ by definition of the restriction.

(B_2) Let $u \in W^w$ such that $v \leq^w u$. As $u \in W^w$, we get that $u \in W$. Finally, as $v \leq^w u$ we get that $v \leq u$ by the definition of the restriction, so we are done.

(B_3) Let $u \in W$ such that $v \leq u$. As $v \in W^w$, we get that there is a chain $wR_1 ... R_n v$. So, there is a chain $wR_1 ... R_n v \leq u$ as $v \leq u$. Consequently, we get that $u \in W^w$. Finally, as $v \leq u$ we get that $v \leq^w u$, so we are done.

(B_4) Let $u \in W^w$ such that $u \leq^w v$. As $u \in W^w$, we get that $u \in W$. Finally, as $u \leq^w v$ we get that $u \leq v$ by the definition of the restriction, so we are done.

(B_5) Let $u \in W$ such that $u \leq v$. As $v \in W^w$, we get that there is a chain $wR_1 ... R_n v$. So, there is a chain $wR_1 ... R_n v \geq u$ as $v \geq u$. Consequently, we get that $u \in W^w$. Finally, as $v \geq u$ we get that $v \geq^w u$, so we are done.

■

However, Lemma 8.7.1 informs us that bisimulations entail logical equivalence. So, we easily obtain the following corollary showing that the two models force the same formulas in w.

Corollary 8.9.1. For all model $\mathcal{M} = (W, \leq, I)$, $w \in W$ and $\varphi \in \mathbb{L}_{\mathbf{BI}}$:

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{M}^w, w \Vdash \varphi$$

Proof. (⌨) As we have that $\mathcal{M}, w \leftrightharpoons \mathcal{M}^w, w$ by Lemma 8.9.1, we know by Proposition 8.7.1 that $\mathcal{M}, w \Vdash \varphi$ iff $\mathcal{M}^w, w \Vdash \varphi$. ■

Let us recall that our intention is to obtain a model forcing Γ *everywhere* out of a model forcing $(\neg\sim)^\omega\Gamma$ in *one point*. Given a model $\mathcal{M}$ and a point w such that $\mathcal{M}, w \Vdash (\neg\sim)^\omega\Gamma$, we can obtain the promised model: it is the restriction of $\mathcal{M}$ in w. We can sustain our claim by showing that if $\mathcal{M}, w \Vdash (\neg\sim)^\omega\Gamma$, then for all $v \in W^w$ we have $\mathcal{M}^w, v \Vdash \Gamma$. This is the essence of the next lemma.

Lemma 8.9.2. Let $n \in \mathbb{N}$ be a natural number, $\mathcal{M} = (W, \leq, I)$ a model, $w, v \in W$ points, $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ a formula and $wR_1...R_nv$ a chain in $\mathcal{M}$ such that $R_j \in \{\leq, \geq\}$ for every $j \in \{1, ..., n\}$. If $\mathcal{M}, w \Vdash (\neg\sim)^n\varphi$, then $\mathcal{M}, v \Vdash \varphi$.

Proof. (⌨) We prove the statement by induction on n.

- $n = 0$: then $v = w$ as the chain $wR_1...R_nv$ is of length 0. Consequently $\mathcal{M}, w \Vdash \varphi$ as $\mathcal{M}, w \Vdash (\neg\sim)^0\varphi$ by assumption.
- $n = S\ m$: then there is u such that $uR_{Sm}v$. Given that $\mathcal{M}, w \Vdash (\neg\sim)^{Sm}\varphi$ by assumption, we easily obtain that $\mathcal{M}, w \Vdash (\neg\sim)^m(\neg\sim\varphi)$ as $(\neg\sim)^{Sm}\varphi = (\neg\sim)^m(\neg\sim\varphi)$. Thus, we can use the induction hypothesis to obtain that $\mathcal{M}, u \Vdash \neg\sim\varphi$. If $R_{Sm} =\leq$, then we have that $u \leq v$. Consequently, we get by the definition of the semantics that $\mathcal{M}, v \nVdash \sim\varphi$. As $v \leq v$, we get that $\mathcal{M}, v \Vdash \varphi$. If $R_{Sm} =\geq$, then we have $u \geq v$. As $u \leq u$, we have that $\mathcal{M}, u \nVdash \sim\varphi$. So, we get $\mathcal{M}, v \Vdash \varphi$ as $v \leq u$. In both cases, we obtained $\mathcal{M}, v \Vdash \varphi$.

■

It appears clearly in this proof that we use the fact that $\neg\sim$ semantically corresponds to a *zig-zag*: if $\mathcal{M}, w \Vdash \neg\sim\varphi$ then $\mathcal{M}, v \Vdash \varphi$ for any point v such that $w \leq u \geq v$ for some u. So, if a point v is accessible from w by n zig-zags $\leq\geq$, it is sufficient to have $\mathcal{M}, w \Vdash (\neg\sim)^n\varphi$ to get $\mathcal{M}, v \Vdash \varphi$. Thus, by closing Γ under $\neg\sim$ and obtaining $(\neg\sim)^\omega\Gamma$, we make sure that a point accessible by *any* number of zig-zag is forcing Γ. But the restriction of a model is *exactly* based on these points. So, using the bisimulation between $\mathcal{M}$ and $\mathcal{M}^w$, we can conclude that $\mathcal{M}^w$ is a Γ-model if the initial point w is a $(\neg\sim)^\omega\Gamma$-point.

More formally, we now exploit these ideas to show that sBIL corresponds to the global semantic consequence.

Theorem 8.9.1. The following holds:

$$\vdash_{\mathsf{s}} [\Gamma \mid \Delta] \quad \text{iff} \quad \Gamma \models_g \Delta.$$

Proof. We prove each direction separately.

($\Rightarrow$) (⌨) Here we prove *soundness*. The proofs for the axioms are identical to the ones above in Theorem 8.9.1 ($\Rightarrow$). We are left to prove that the rules are sound:

(MP) Assume that $\Gamma \models_g \varphi$ and $\Gamma \models_g \varphi \to \psi$ hold. We show that $\Gamma \models_g \psi$. Let $\mathcal{M}$ be a model. Assume that for all $w \in W$, we have $\mathcal{M}, w \Vdash \Gamma$. We need to show that for all $w \in W$, we have $\mathcal{M}, w \Vdash \psi$. Given that $\Gamma \models_g \varphi$ and $\Gamma \models_g \varphi \to \psi$, we directly obtain that for all $w \in W$, we have $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \varphi \to \psi$. Let $w \in W$. We can use our previous results to get $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \varphi \to \psi$. By reflexivity of $\leq$ we consequently obtain $\mathcal{M}, w \Vdash \psi$. As w is arbitrary we get that for all $w \in W$, we have $\mathcal{M} \Vdash \psi$.

(sDN) Assume that $\Gamma \models_g \varphi$. We show that $\Gamma \models_g \neg\sim\varphi$. Let $\mathcal{M}$ be a model. Assume that $\mathcal{M} \Vdash \Gamma$. We need to show that $\mathcal{M} \Vdash \neg\sim\varphi$. As $\mathcal{M} \Vdash \Gamma$ and $\Gamma \models_g \varphi$, we get $\mathcal{M} \Vdash \varphi$. Let $w \in W$. We need to show that $\mathcal{M}, w \Vdash \neg\sim\varphi$. Let $v \in W$ such that $w \leq v$. We need to show that $\mathcal{M}, v \nVdash \sim\varphi$, i.e. there is no $u \leq v$ such that $\mathcal{M}, u \nVdash \varphi$. Let $u \leq v$. As $\mathcal{M} \Vdash \varphi$ we get $\mathcal{M}, u \Vdash \varphi$. As u is arbitrary we get $\mathcal{M}, v \nVdash \sim\varphi$. As v is also arbitrary we get $\mathcal{M}, w \Vdash \neg\sim\varphi$. Finally, as w is arbitrary we get $\mathcal{M} \Vdash \neg\sim\varphi$.

($\Leftarrow$) (⌨) Here we prove *completeness*. Assume $\nvdash_{\mathsf{s}} [\Gamma \mid \Delta]$. We show that $\Gamma \not\models_g \Delta$. Note that $\vdash_{\mathsf{s}} [\Gamma \mid (\neg\sim)^\omega\Gamma]$ as the rule (sDN) is applicable at will with any context in sBIH. Consequently, we have that $\nvdash_{\mathsf{s}} [(\neg\sim)^\omega\Gamma \mid \Delta]$, else we would get $\vdash_{\mathsf{s}} [\Gamma \mid \Delta]$ by transitivity of deducibility which contradicts our assumption. Thus, we get $\nvdash_{\mathsf{w}} [(\neg\sim)^\omega\Gamma \mid \Delta]$ by Theorem 8.5.1. By Theorem 8.8.1 we know that $(\neg\sim)^\omega\Gamma \not\models_l \Delta$. So, there is a model $\mathcal{M} = (W, \leq, I)$ and $w \in W$ such that $\mathcal{M}, w \Vdash (\neg\sim)^\omega\Gamma$ and for all $\delta \in \Delta$ we have $\mathcal{M}, w \nVdash \delta$. Now we consider the restriction of $\mathcal{M}$ in w, i.e. the model $\mathcal{M}^w = (W^w, \leq^w, I^w)$. As shown in Corollary 8.9.1 we have that $\mathcal{M}^w, w \Vdash (\neg\sim)^\omega\Gamma$ and for all $\delta \in \Delta$ we have $\mathcal{M}^w, w \nVdash \delta$. If we prove that $\mathcal{M}^w \Vdash \Gamma$, then we are done as we would then have exhibited a Γ-model $\mathcal{M}^w$ which has one point w that is a δ-point for no $\delta \in \Delta$, hence $\Gamma \not\models_g \Delta$. We thus proceed to show that $\mathcal{M}^w \Vdash \Gamma$. Let $v \in W^w$ and $\gamma \in \Gamma$. We need to show $\mathcal{M}^w, v \Vdash \gamma$. As $v \in W^w$ we know that there is a chain $wR_1 \ldots R_n v$. We straightforwardly obtain that $\mathcal{M}^w, w \Vdash (\neg\sim)^n\gamma$ as $\mathcal{M}^w, w \Vdash (\neg\sim)^\omega\Gamma$. We finally obtain that $\mathcal{M}^w, v \Vdash \gamma$ using Lemma 8.9.2 using $\mathcal{M}^w, w \Vdash (\neg\sim)^n\gamma$ and the existence of the chain $wR_1 \ldots R_n v$. As v and γ are arbitrary we get $\mathcal{M}^w \Vdash \Gamma$.

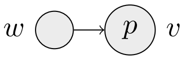
■

8.10 A semantic look back

We use Theorem 8.8.1 and Theorem 8.9.1, respectively stating that the logics sBIL and wBIL are respectively sound and complete with respect to the global and local consequence relations, to fill in the gaps of Sections 8.5 and 8.6 by proving the claims left pending there.

First, we can show the extensional difference of the two logics by proving Claim 8.5.1, which claims that $\mathsf{sBIL} \not\subseteq \mathsf{wBIL}$:

Proof of Claim 8.5.1. (⌨) On the one hand we obviously have that $\vdash_{\mathsf{s}} [p \mid p]$ hence $\vdash_{\mathsf{s}} [p \mid \neg\sim p]$ by (sDN). On the other hand, we have that $p \not\models_l \neg\sim p$ as shown by the following model $\mathcal{M}_0$ where reflexive arrows are not depicted:



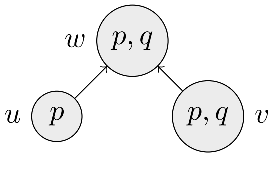
We have $\mathcal{M}_0, v \Vdash p$. We also have $\mathcal{M}_0, v \nVdash \neg\sim p$ as $\mathcal{M}_0, v \Vdash \sim p$ because $\mathcal{M}_0, w \nVdash p$ and $w \leq v$. By Theorem 8.8.1 we obtain $\nvdash_{\mathsf{w}} [p \mid \neg\sim p]$. ■

Second, we resolve Claim 8.6.1 that sBIL fails the deduction theorem.

Proof of Claim 8.6.1. (⌨) We need to prove that $\nvdash_{\mathsf{s}} [\emptyset \mid p \to \neg\sim p]$. Consider the model $\mathcal{M}_0$ above. We have that $\mathcal{M}_0, v \nVdash p \to \neg\sim p$, hence $\not\models_g p \to \neg\sim p$. By Theorem 8.9.1 we obtain $\nvdash_{\mathsf{s}} [\emptyset \mid p \to \neg\sim p]$. Since applying (sDN) to $p \vdash_{\mathsf{s}} p$ gives $\vdash_{\mathsf{s}} [p \mid \neg\sim p]$, the deduction theorem does not hold for sBIL. ■

Lastly, we prove Claim 8.6.2, showing that the rule (DMP) is unsound for sBIL, and that the latter fails the dual deduction theorem.

Proof of Claim 8.6.2. (⌨) We need to prove that $\nvdash_{\mathsf{s}} [p \mid q \vee \neg\sim\sim q]$. Consider the following model $\mathcal{M}_1$:



First, we have that $\mathcal{M}_1 \Vdash p$. We also have $\mathcal{M}_1, u \nVdash q$ by definition of the valuation. But we also have $\mathcal{M}_1, u \nVdash \neg{\sim}{\sim}q$. In fact $\mathcal{M}_1, w \Vdash {\sim}{\sim}q$ as $v \leq w$ and $\mathcal{M}_1, v \nVdash {\sim}q$: its only predecessor is itself, and it forces q. Consequently we have $p \not\models_g q \vee \neg{\sim}{\sim}q$, which by Theorem 8.9.1 gives $\nvdash_{\mathsf{s}} [p \mid q \vee \neg{\sim}{\sim}q]$. ■

Finally, with the soundness and completeness results in hand, we can show that both wBIL and sBIL are conservative extensions of intuitionistic logic.

Theorem 8.10.1. For $\Gamma \cup \Delta \subseteq \mathit{Form}_{\mathbb{L}_\mathbf{I}}$ and $i \in \{\mathsf{w}, \mathsf{s}\}$: $\vdash_{\mathsf{IH}} [\Gamma \mid \Delta]$ iff $\vdash_i [\Gamma \mid \Delta]$

Proof ✍. ($\Rightarrow$) Assume $\vdash_{\mathsf{IH}} [\Gamma \mid \Delta]$. As all axioms and rules of IH are axioms and rules of wBIH and sBIH, we get that the proof of $[\Gamma \mid \Delta]$ in IH is also a proof in both wBIH and sBIH. So, we get $\vdash_i [\Gamma \mid \Delta]$ for $i \in \{\mathsf{w}, \mathsf{s}\}$.
($\Leftarrow$) Assume $\vdash_i [\Gamma \mid \Delta]$ for $i \in \{\mathsf{w}, \mathsf{s}\}$. By Theorem 8.5.1 in both cases we get $\vdash_{\mathsf{s}} [\Gamma \mid \Delta]$. By Theorem 8.9.1, we get that $\Gamma \models_g \Delta$. Given that $\Gamma \cup \Delta \subseteq \mathit{Form}_{\mathbb{L}_\mathbf{I}}$, we have that $\Gamma \models_g \Delta$ holds in the Kripke semantics for intuitionistic logic IL. However, by Lemma 8.7.2 we get that $\Gamma \models_l \Delta$ in the Kripke semantics for IL. By completeness of the IL with respect to this semantics [86] we finally get $\vdash_{\mathsf{IH}} [\Gamma \mid \Delta]$. ■

8.11 Why Rauszer's proofs are erroneous

The existence of sBIL and wBIL justifies our use of the plural bi-intuitionistic logics. We now trace the effect of this bifurcation on Rauszer's works.

As far as we know, neither the existence of wBIL and sBIL, nor the distinction between them has been highlighted in the literature. While it was certainly not noted in Rauszer's works, it has to be acknowledged that Hiroakira Ono may have suspected something [122, p.7]. Our bifurcation is not important if we only focus on one of the logics and use properties only belonging to it. However if one confuses them by using properties of these logics that are not shared by both of them, then troubles arrive. Unfortunately, such a confusion is made in some of Rauszer's works. As a consequence, various important theorems are asserted with erroneous proofs. The most important of them is the theorem of strong completeness with respect to the Kripke semantics [119]. More precisely there are two flawed proofs for this theorem.

The first one [119, Lemma 2.3] is flawed because it ignores restrictions on the use of a lemma proved by Gabbay [49, p.135]:

Lemma 8.11.1. For any ψ without $\vee$ and $\exists$ we have:

$$[\psi]_{(0,\ldots,\varphi)} = T \qquad \text{iff} \qquad \vdash \varphi \to \psi$$

This lemma puts in correspondence truth in a model ($[\psi]_{(0,\ldots,\varphi)} = T$) with provability ($\vdash \varphi \to \psi$). As a consequence, it expresses a completeness result, notably when restricted to the propositional case. However, one has to note the restriction on ψ: it must be expressed in the intuitionistic language without the disjunction. Consequently, ψ cannot contain a disjunction $\vee$, an exclusion $-\!\!<$ or $\sim$. This restriction is ignored in Rauszer's proof when she combines the above lemma with a dual one of her own:

Lemma 8.11.2. For any ψ without $\wedge$, $\forall$, $\to$, $\neg$ we have:

$$[\psi]_{(0,\ldots,\varphi)} = F \qquad \text{iff} \qquad \dashv \psi -\!\!< \varphi$$

These lemmas cannot be used together as they both impose language restrictions that are incompatible with each other. This was not detected by Rauszer, who used them both in her proof of Lemma 2.3 [119]. Note however that this mistake is extraneous to the confusion between the logics sBIL and wBIL.

The second one [119, Theorem 3.5], is a standard completeness proof, involving the construction of a canonical model. However, in this proof, some intermediate lemmas are proved using features that are distinct for these logics. For example, the fact that $\vdash_{\mathsf{s}} [\varphi \mid \neg{\sim}\varphi]$ holds is used in the proof of Lemma 3.1 [119], where it is erroneously claimed that a prime filter A is such that if $a \in A$ then $\neg{\sim}a \in A$ and hence ${\sim}a \notin A$. In addition, in the proof of point (3) of Lemma 3.3 [119] the deduction theorem is used implicitly as it relies on a proof provided by Thomason [150] which uses it. Thus, while the proof of Lemma 3.1 [119] suggests the logic used is sBIL, the proof of Lemma 3.3 indicates that it must be wBIL. Thus the proof of completeness given there, which relies on these two lemmas, is a proof for none of the logics discussed here.

Another strong completeness proof [117, Theorem 3.6] suffers from the same confusion because it ultimately relies on the aforementioned completeness proofs [119]. Interestingly, some elements of this paper [117] were corrected [120], but the corrections do not suffice to fix the issue. More precisely, one side of the deduction theorem is changed from $\Gamma \vdash \varphi \to \psi$ to $\Gamma \vdash \neg{\sim}\varphi \to \psi$ [120], but this version also fails for sBIL and, in any case, the proofs [119] are not modified to handle the change.

In a nutshell, as the proofs of strong completeness for bi-intuitionistic logic given in Rauszer's Ph.D. thesis [122] are taken from the articles mentioned above, we are left with no actual trace in Rauszer's papers of a correct proof of strong completeness of bi-intuitionistic logic with respect to the Kripke semantics defined. To the best of our knowledge such a proof has only been provided by Sano and Stell [127], but for a different axiomatization. So, our proofs are the first to ensure that Rauszer's axiomatization is strongly complete for the appropriate Kripke semantics in a non-ambiguous way: sBIL (wBIL) is strongly complete for global (local) semantic consequence in Kripke semantics.

Providing such a proof is necessary to set the record straight for Rauszer's axiomatization. Furthermore, when compared with the initial proofs, our proofs are useful for avoiding false conclusions hinted at by the former. Most importantly, two proofs of strong completeness [119] involve the construction of a *rooted* canonical model where by "rooted" we understand the following.

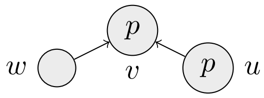
Definition 8.11.1. Let $\mathcal{F} = (W, \leq)$ be a Kripke frame. We say that $\mathcal{F}$ is rooted if there is a $w \in W$ such that for every $v \in W$ we have $w \leq^* v$ (but since $\leq$ is reflexive and transitive we can replace $\leq^*$ with $\leq$).

The use of rooted models immediately implies that bi-intuitionistic logic is sound and complete with respect to the class of *rooted* Kripke frames. However, we show that this result fails for both sBIL and wBIL! Specifically, ${\sim}p \vee \neg{\sim}p$ is valid on rooted frames but not valid on the full class of frames.

Lemma 8.11.3. Let $\mathcal{F} = (W, \leq)$ be a rooted Kripke frame. For any interpretation function I, we have that $(W, \leq, I) \Vdash {\sim}p \vee \neg{\sim}p$.

Proof. (⌨) Let r be the root of $\mathcal{F}$ and I an interpretation function, $\mathcal{M} = (W, \leq, I)$ and $w \in W$. As $\mathcal{F}$ is rooted we have that $r \leq w$. If $r \in I(p)$ then persistence and rootedness give $\mathcal{M}, v \Vdash p$ for every $v \in W$. Thus, for all points $v \in W$ such that $w \leq v$, we have that $\mathcal{M}, v \nVdash {\sim}p$ as this requires the existence of a predecessor not forcing p. Consequently, we obtain that $\mathcal{M}, w \Vdash \neg{\sim}p$. If $r \notin I(p)$ then we get $\mathcal{M}, w \Vdash {\sim}p$. In each case we obtain $\mathcal{M}, w \Vdash {\sim}p \vee \neg{\sim}p$. ■

Thus the formula ${\sim}p \vee \neg{\sim}p$ is valid on the class of rooted Kripke frames. Now we show that there is a Kripke model $\mathcal{N}$ such that $\mathcal{N} \nVdash {\sim}p \vee \neg{\sim}p$ (⌨). Consider the following model where reflexive arrows are omitted:



We have that $\mathcal{N}, u \nVdash {\sim}p$ as the only predecessor of u is itself and $\mathcal{N}, u \Vdash p$. Moreover we have that $\mathcal{N}, w \nVdash p$, hence $\mathcal{N}, v \Vdash {\sim}p$ which in turn implies $\mathcal{N}, u \nVdash \neg{\sim}p$. Consequently $\mathcal{N}, u \nVdash {\sim}p \vee \neg{\sim}p$.

It can be argued that Crolard [25, p.168] proved that **wBIL** is not complete for the class of rooted frames. But he does not make the distinction between the two logics presented here, nor pinpoint the flaws in Rauszer's proof.

Theorem 8.8.1 and Theorem 8.9.1 allow us to claim that $\nvdash_i [\emptyset \mid {\sim}p \vee \neg{\sim}p]$ for $i \in \{s, w\}$ as $\not\models_j {\sim}p \vee \neg{\sim}p$ for $j \in \{g, l\}$. From this, we conclude that neither **sBIL** nor **wBIL** is complete, with their corresponding semantic consequence relations, for the class of rooted frames: the formula ${\sim}p \vee \neg{\sim}p$ is a counterexample to such a claim.

However, we suspect that **wBIL** and **sBIL** are respectively sound and complete with respect to the local and global semantic consequence relations on another class of frame: the class of *finite* frames containing a point w from which all other points are accessible via combinations of $\leq$ and $\geq$ steps. This should follow in part quite straightforwardly from the work of Postniece [111, Corollary 3.4.19], which would then give us the finite model property for both logics.

8.12 Conclusion

As already mentioned in Section 4.1, generalized Hilbert calculi effectively provide the tools to clarify the status of rules in axiomatic systems. The distinction between the two logics **sBIL** and **wBIL** can easily be tracked to the obvious difference between the rules (wDN) and (sDN) in the calculi defining them. Effectively, as in the modal case exhibited in Section 8.2, different syntactic consequence relations stem from the traditional Hilbert calculus for bi-intuitionistic logic, formalized as generalized Hilbert calculi. The logics **wBIL** and **sBIL** are distinguishable on an extensional level in a similar way to **wKL** and **sKL**. The similarity with modal logic goes even further as the famous deduction theorem is not a property common to both **sBIL** and **wBIL**. As we have shown, the deduction theorem can be modified to hold in **sBIL**, and the dual deduction theorem holds in **wBIL**, but we have not yet found a modification of the dual deduction theorem for **sBIL**. So, on top of allowing one to clearly detect which logic satisfies the deduction theorem or its dual, generalized Hilbert calculi also prevent the confusions that existed in both the modal [68] and bi-intuitionistic case.

As we have shown, the logics **wBIL** and **sBIL**, respectively, have a local and global semantic counterpart on the class of Kripke frames. Although quite common, this phenomenon finally clarifies the relation between the two logics. It also helps rectify the status of some properties of **sBIL** and **wBIL**, such as the fact that they are not strongly complete with respect to the class of rooted frames.

Finally, the difference between the two logics allows us to look at the proof theory of bi-intuitionistic logic from a different angle. We conjecture that the various calculi which have been designed to capture bi-intuitionistic logic [52, 59, 60, 106, 152] are in fact sound and strongly complete for **wBIL**. One exception is Sano and Stell's axiomatization [127]: when considered in a generalized Hilbert calculus context, it also suffers from the same phenomenon as Rauszer's axiomatization. More precisely, their rule (Mon-≺) can be interpreted in the same ways as (DN) giving two rules (sMon-≺) and (wMon-≺), as shown below.

$$\frac{\emptyset \vdash \varphi \to \psi}{\Gamma \vdash (\varphi \prec \chi) \to (\psi \prec \chi)}\ \text{(wMon-}\prec\text{)} \qquad \frac{\varphi \to \psi}{(\varphi \prec \chi) \to (\psi \prec \chi)}\ \text{(Mon-}\prec\text{)} \qquad \frac{\Gamma \vdash \varphi \to \psi}{\Gamma \vdash (\varphi \prec \chi) \to (\psi \prec \chi)}\ \text{(sMon-}\prec\text{)}$$

We strongly believe that the generalized Hilbert calculus involving the rule (wMon-≺) corresponds to **wBIL**, while generalized Hilbert calculus involving the rule (sMon-≺) corresponds to **sBIL**. Thus, Sano and Stell's axiomatization could be targeted to either logic once ported to the framework of generalized Hilbert calculi.

There are several directions for further work on propositional bi-intuitionistic logics. First, the diversity of interpretations of the (MP) rule should be investigated. While we made a case of the multiplicity of interpretations (which we have not exhausted) of the rules (DN) and (Nec), we did not question the shape of the rule (MP). We could modify one of the generalized Hilbert calculi defined above to use a modified version of (MP) where the premisses would be $\emptyset \vdash \varphi$ and $\emptyset \vdash \varphi \to \psi$. This system would define a logic, but a weird one where $p, p \to q \vdash q$ would not be guaranteed to hold. A second direction leads to the algebraic treatment of $\mathsf{wBI}\mathbf{L}$ and $\mathsf{sBI}\mathbf{L}$ as consequence relations [45]. Third, the use of pairs $[\Gamma \mid \Delta]$ suggests a general treatment of logics that would capture both *deducibility* and *refutability* calculi in one shot, in a similar spirit to Gore and Postniece's calculus which combines both notions as first-class citizens [59]. Finding if such a general framework exists would require further investigations.

Chapter 9

First-Order Bi-Intuitionistic Logics

For this chapter the following sections of the Toolbox I are required: Section 2.2, Section 3, the introduction of Section 4, Section 4.1, Section 5.2 and Section 6.

The results of this chapter are unpublished but were mentioned in an extended abstract [136] and presented at the AAL 2022 Conference (https://sites.google.com/view/aalogic/aal-conference-2022).

Their formalisation can be found here: https://github.com/ianshil/PhD_thesis/tree/main/FO_Bi_Int.

9.1 First-order confusions

The previous chapter established that there is a worryingly weak point in the foundations of propositional bi-intuitionistic logic: two versions of the rule (DN) are conflated, leading to the identification of two significantly different logics, i.e. **wBIL** and **sBIL**. We fixed this issue using generalized Hilbert calculi, with which both logics can be faithfully captured unambiguously. To make sure that the foundations we offer are flawless we formalised all our results in Coq.

This is it for the propositional case. Quite naturally, one may wonder whether the same weak point is present in the foundations of *first-order* bi-intuitionistic logic, where the universal and existential quantifiers are added to the language. Unfortunately, Rauszer's work on first-order bi-intuitionistic is simply founded on the propositional case: the same type of traditional Hilbert calculus is used. Consequently, the conflation of (wDN) and (sDN) into (DN) must also occur here.

This casts an important shadow on Rauszer's work on first-order bi-intuitionistic logic, leading to the querying of the results developed on the later [7, 25, 92]. So, we need to reconsider the foundations she laid in a formalised context to determine if the same splitting phenomenon happens in the first-order case, leading to similar mistakes in proofs as the ones exhibited in the previous chapter. Notably, we need to reconsider the axiomatic system developed for first-order bi-intuitionistic logic [117], as well as the first-order Kripke semantics shown to correspond to it [119].

On top of allowing us to provide solid foundations to first-order bi-intuitionistic logic, this formal investigation allows us to confirm in a trustworthy way some peculiarity of this logic. In fact, it is claimed that first-order bi-intuitionistic logic is not a conservative extension of first-order intuitionistic logic [122, p.56][92], on the contrary to the propositional case. More precisely, the *constant domain axiom* CD, displayed below, is a theorem of first-order bi-intuitionistic logic while it is not in the intuitionistic counterpart [50]. Note that in this axiom x is crucially not appearing free in φ.

$$\forall x(\varphi \vee \psi(x)) \to (\varphi \vee \forall x \psi(x))$$

On the semantic side, this axiom characterizes the *constant domain property* on the traditional Kripke semantics for the intuitionistic language [66, 50, 102]. While the holding of this axiom was only proven in 1980 by Rauszer [122], she already noticed in 1976 that constant domains were required to semantically capture this logic [117]. The first-order Kripke semantics she developed used the same frames as the ones of intuitionistic logic, but had the specificity of having the same domain across all points, similarly to the semantics for CDL, i.e. first-order intuitionistic logic extended with the axiom CD. So, in this light, it should not come as a surprise that first-order bi-intuitionistic logic is a conservative extension of CDL [122, p.57][25].

However, all of the above heavily relies on the traditional axiomatic system and its soundness and completeness proofs with respect to the class of first-order Kripke models with constant domains given by Rauszer [119]. If Rauszer's first-order bi-intuitionistic logic was to split into two logics in a similar way to the propositional case, then these proofs, which notably involve canonical model constructions, suffer from similar problems.

So, in this chapter, we proceed to provide solid foundations to first-order bi-intuitionistic logic by formalising in Coq generalized Hilbert calculi and a first-order Kripke semantics for them and attempt at connecting them. As such, our work is part of an extremely active line of research consisting in formalising traditional results for first-order logics [33, 47, 67, 75, 79].

Section 9.2 contains the two generalized Hilbert calculi obtained from Rauszer's axiomatization of first-order bi-intuitionistic logic. In Section 9.3, we show they define two logics FOwBIL and FOsBIL. In Section 9.4, we show that these logics are extensionally distinct. Section 9.5 contains significant theorems distinguishing these logics. Using the results obtained at that point, we formally prove in Section 9.6 that the axiom CD is a theorem of both logics. Section 9.7 defines a Kripke semantics using constant domains for the first-order bi-intuitionistic language. We prove that FOwBIL is sound with respect to the local semantic consequence relation, and that FOsBIL is sound with respect to the global semantic consequence relation in Section 9.8. We use these results to prove claims we left pending in Section 9.4 and Section 9.5, establishing for good the differences between FOwBIL and FOsBIL. In Section 9.9, we investigate the various possibilities of completeness for our logics. In fact, we show that if FOwBIL is complete with respect to the local semantics consequence relation, then FOsBIL is complete with respect to the global semantics consequence relation. Section 9.10 gives the reasons for our failure to find a proof of completeness for FOwBIL, boiling down to frictions between the Lindenbaum lemma and existence lemmas. In Section 9.11, we use the distinctions between our two bi-intuitionistic logics, as well as the various scenarios we explored, to expose the flaws in Rauszer's results.

9.2 Generalized Hilbert calculi for FOwBIL and FOsBIL

In this section, we define two generalized Hilbert calculi for first-order bi-intuitionistic logic. So, we need to define languages à la de Bruijn for the latter. Here, we reuse $\mathcal{C}_{\mathbf{BI}}$ the set of connectives from Definition 8.3.1, as well as $\mathcal{Q} = \{\forall, \exists\}$ the set of quantifiers fixed in Section 2.2.

Definition 9.2.1. Given a signature $\mathcal{S}$, we define the first-order language $\mathbb{L}_{\mathbf{BI}} = (\mathcal{S}, \mathcal{C}_{\mathbf{BI}}, \mathcal{Q})$ and obtain its set of *bi-intuitionistic* formulas $Form_{\mathbb{L}_{\mathbf{BI}}}$ through its grammar:

$$\varphi ::= P(t_1, \ldots, t_n) \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \prec \varphi \mid \forall \varphi \mid \exists \varphi$$

where $t_1, \ldots, t_n \in Term_{\mathcal{S}}$, $P \in Pred$ and $Ar_{Pred}(P) = n$. As in the propositional case, we also define two unary operators:

$$\neg\varphi := (\varphi \rightarrow \bot) \qquad \sim\varphi := (\top \prec \varphi)$$

To define generalized Hilbert Calculi for FOwBIL and FOsBIL we can reuse the calculi defined in the propositional case (see Definition 8.3.4 and Definition 8.4.1) and extend them with axioms and rules dealing with quantifiers. Note, however, that the nature of the axioms and rules which we extract here from the propositional calculi becomes different, as they are now instantiated using *first-order* formulas.

Definition 9.2.2. We define the set of axioms $\mathcal{A}_{FOBI}$ as the set of axioms $\mathcal{A}_{BI}$ extended with the axioms A_{17}, A_{18} and A_{19} below. We also define the rules (Gen) and (EC).

$$A_{17} \quad \forall(\psi[\uparrow] \to \varphi) \to (\psi \to \forall\varphi) \qquad A_{18} \quad \forall\varphi \to \varphi[t .: id] \qquad A_{19} \quad \varphi[t .: id] \to \exists\varphi$$

$$\frac{\Gamma[\uparrow] \vdash \varphi}{\Gamma \vdash \forall\varphi}\ (\text{Gen}) \qquad \frac{\Gamma[\uparrow] \vdash \varphi \to \psi[\uparrow]}{\Gamma \vdash \exists\varphi \to \psi}\ (\text{EC})$$

The name of the rule (Gen) stands for *Gen*eralization, while the name of the rule (EC) stands for for *E*xistential *C*onditionalization. Below, you can find these rules presented more familiarly, using a more common syntax. Note that in this presentation, it is required that x appears free neither in Γ in (Gen) nor in Γ or ψ in the rule (EC).

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x\varphi}\ (\text{Gen}) \qquad \frac{\Gamma \vdash \varphi \to \psi}{\Gamma \vdash (\exists x\varphi) \to \psi}\ (\text{EC})$$

We need to answer a rather natural question here: how do our rules in the de Bruijn syntax capture the meaning of these common rules? In essence, our rules capture the condition about x's non-freeness using the variable substitution $\uparrow$. Indeed, in (Gen) à la de Bruijn, by shifting from Γ in the conclusion to $\Gamma[\uparrow]$ in the premise, we change any free variable n in Γ into its successor $S\ n$. As a consequence, any variable made free by the change from $\forall\varphi$ to φ is mechanically not present in $\Gamma[\uparrow]$, thus creating a "fresh" variable. For example, consider judgement $P(0) \vdash \forall P(0)$. If we apply upwards the rule (Gen), we obtain the judgement $P(0)[\uparrow] \vdash P(0)$, which is nothing but the judgement $P(1) \vdash P(0)$. Thus, we made sure that the variable 0 we made free in the change from $\forall P(0)$ to $P(0)$ is not present in the context in the premise (while it was in the conclusion!). This explains the use of the substitution $\uparrow$ in our rules and axioms.

With these axioms and rules in hand, we can define our generalized Hilbert calculi.

Definition 9.2.3. We define the generalized Hilbert calculi below, alongside their set of rules.

$$\begin{array}{rcl} \mathsf{FOwBIH} & = & (\mathcal{A}_{FOBI}, \mathcal{R}_{wFO}) \\ \mathsf{FOsBIH} & = & (\mathcal{A}_{FOBI}, \mathcal{R}_{sFO}) \end{array} \qquad \begin{array}{rcl} \mathcal{R}_{wFO} & = & \{(\text{MP}), (\text{wDN}), (\text{Gen}), (\text{EC})\} \\ \mathcal{R}_{sFO} & = & \{(\text{MP}), (\text{sDN}), (\text{Gen}), (\text{EC})\} \end{array}$$

We abbreviate $\Gamma \vdash_{\mathsf{FOwBIH}} \varphi$ by $\Gamma \vdash_{\mathsf{w}} \varphi$ and let $\mathsf{FOwBIL} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{w}} \varphi\}$ be the consequence relation characterized by FOwBIH. Similarly we abbreviate $\Gamma \vdash_{\mathsf{FOsBIH}} \varphi$ by $\Gamma \vdash_{\mathsf{s}} \varphi$, and define $\mathsf{FOsBIL} = \{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{s}} \varphi\}$.

Some may argue that our use of the same abbreviations $\vdash_{\mathsf{w}}$ and $\vdash_{\mathsf{s}}$ for the propositional and first-order case is confusing. However, not only do we believe that the separation between the two chapters is clear enough, but we also want to give an account of the fact that the same phenomenon occurs in both cases: a split between a *weak* and a *strong* logic.

We note that, similarly to the propositional case, the set $\mathcal{A}_{FOI} = \{A_n \mid 1 \leq n \leq 10 \text{ or } 15 \leq n \leq 19\}$ together with the set of rules $\mathcal{R}_{FOI} = \{(\text{MP}), (\text{Gen}), (\text{EC})\}$ constitute a generalized Hilbert calculus for first-order intuitionistic logic. Also, if we add CDto $\mathcal{A}_{FOI}$, then we get a calculus for CDL, the constant domains first-order intuitionistic logic.

Before turning to the next section, let us comment on the absence of a variable substitution rule, as shown below, in our calculi.

$$\frac{\Gamma \vdash \varphi}{\Gamma[\tau] \vdash \varphi[\tau]}$$

It can come as a surprise that such a rule is not present in our calculi, while widely used in the literature. In fact, we can show that it can be safely added to our calculi without changing the set of provable consecutions. We need a preliminary lemma to show that axiomhood is preserved under variable substitution.

Lemma 9.2.1. For all $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ and variable substitution τ, if $\varphi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ then $\varphi[\tau] \in \mathcal{A}^{\mathsf{i}}_{FOBI}$.

Proof. (⌨) For instances of axioms A_1 to A_{16}, it suffices to realize that the variable substitution via τ is pushed on the subformulas of φ in $\varphi[\tau]$. So, $\varphi[\tau]$ is an instance of the same axiom. We are thus left with the axioms A_{17}, A_{18} and A_{19}.

For axiom A_{17}, we consider a formula of the shape $\forall(\chi[\uparrow] \to \psi) \to (\chi \to \forall\psi)$. By definition, we obtain that $(\forall(\chi[\uparrow] \to \varphi) \to (\chi \to \forall\varphi))[\tau] = \forall(\chi[\uparrow][\mathsf{up}(\tau)] \to \varphi[\mathsf{up}(\tau)]) \to (\chi[\tau] \to \forall\varphi[\mathsf{up}(\tau)])$. By Lemma 2.2.4 we get that $\chi[\uparrow][\mathsf{up}(\tau)] = \chi[\tau][\uparrow]$. So, we clearly obtain that $\forall(\chi[\tau][\uparrow] \to \varphi[\mathsf{up}(\tau)]) \to (\chi[\tau] \to \forall\varphi[\mathsf{up}(\tau)])$ is an instance of A_{17}.

For axiom A_{18}, we consider a formula of the shape $\forall\psi \to \psi[t .: id]$. By definition, we obtain that $(\forall\psi \to \psi[t .: id])[\tau] = \forall\psi[\mathsf{up}(\tau)] \to \psi[t .: id][\tau]$. We claim that $\psi[t .: id][\tau] = \psi[\mathsf{up}(\tau)][t[\tau] .: id]$, which allows us to show that $\forall\psi[\mathsf{up}(\tau)] \to \psi[\mathsf{up}(\tau)][t[\tau] .: id]$ is an instance of A_{18}. Our claim $\psi[t .: id][\tau] = \psi[\mathsf{up}(\tau)][t[\tau] .: id]$ holds as any term in r appearing in ψ is such that $r[t .: id][\tau] = r[\mathsf{up}(\tau)][t[\tau] .: id]$. Why? The substitution $[t .: id]$ replaces 0 by t, and replaces $S\ m$ by m. So, the combination of substitutions $[t .: id]$ and $[\tau]$ replaces 0 by $t[\tau]$, and replaces $S\ m$ by $m[\tau] = \tau(m)$. But this is exactly what the combination of substitutions $[\mathsf{up}(\tau)]$ and $[t[\tau] .: id]$ on r does: it replaces 0 by itself and then $t[\tau]$, and replaces $S\ m$ by $\tau(m)[\uparrow]$ and then by $\tau(m)[\uparrow][t[\tau] .: id]$ which is nothing but $\tau(m)$ by Lemma 2.2.1.

For axiom A_{19}, we consider a formula of the shape $\psi[t .: id] \to \exists\psi$. By definition, we obtain that $(\psi[t .: id] \to \exists\psi)[\tau] = \psi[t .: id][\tau] \to \exists\psi[\mathsf{up}(\tau)]$. As in the previous case, we use the fact that $\psi[t .: id][\tau] = \psi[\mathsf{up}(\tau)][t[\tau] .: id]$ and obtain that $\psi[t .: id][\tau] \to \exists\psi[\mathsf{up}(\tau)]$ is an instance of A_{19}. ■

Then, we can show that the substitution is implicitly already contained in our calculi.

Lemma 9.2.2. For $i \in \{\mathsf{w}, \mathsf{s}\}$, if $\Gamma \vdash_i \varphi$, then for any variable substitution τ we have $\Gamma[\tau] \vdash_i \varphi[\tau]$.

Proof. (⌨) Assume $\Gamma \vdash_i \varphi$. Then there is a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that $\Gamma[\tau] \vdash_i \varphi[\tau]$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$. In this case we have $\varphi[\tau] \in \Gamma[\tau]$, hence $\Gamma[\tau] \vdash_i \varphi[\tau]$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$. In this case we have $\varphi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ and then $\varphi[\tau] \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ by Lemma 9.2.1. Thus, we obtain $\Gamma[\tau] \vdash_i \varphi[\tau]$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If (Gen) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi$ where $\varphi = \forall\psi$. Note that $(\forall\psi)[\tau] = \forall\psi[0 .: \tau]$. We can thus apply the induction hypothesis on the proof of $\Gamma[\uparrow] \vdash \psi$ with the variable substitution $(0; \tau)$ to obtain a proof of $\Gamma[\uparrow][0 .: \tau] \vdash \psi[0 .: \tau]$. Thus, using Lemma 2.2.4 we obtain that the latter is a proof of $\Gamma[\tau][\uparrow] \vdash \psi[0 .: \tau]$. We can apply (Gen) to obtain a proof of $\Gamma[\tau] \vdash \forall(\psi[0 .: \tau])$ i.e. of $\Gamma[\tau] \vdash \forall\psi[\tau]$ by definition.
If (EC) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi \to \chi[\uparrow]$ where $\varphi = (\exists\psi) \to \chi[\uparrow]$. Note that $((\exists\psi) \to \chi)[\tau] = (\exists\psi[0 .: \tau]) \to \chi[\tau]$. We can thus apply the

induction hypothesis on the proof of $\Gamma[\uparrow] \vdash \psi \to \chi[\uparrow]$ with the variable substitution $(0;\tau)$ to obtain a proof of $\Gamma[\uparrow][0 .: \tau] \vdash \psi[0 .: \tau] \to \chi[\uparrow][0 .: \tau]$. Thus, using Lemma 2.2.4 we obtain that the latter is a proof of $\Gamma[\tau][\uparrow] \vdash \psi[0 .: \tau] \to \chi[\tau][\uparrow]$. Thus, we can apply (EC) to obtain a proof of $\Gamma[\tau] \vdash (\exists(\psi[0 .: \tau])) \to \chi[\tau]$ i.e. of $\Gamma[\tau] \vdash ((\exists\psi) \to \chi)[\tau]$ by definition.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$ then we simply use the given premise to obtain $\Gamma[\tau] \vdash_i \varphi[\tau]$ as desired. ■

In the next section, we show that the calculi **FOwBIH** and **FOsBIH** capture logics.

9.3 Consequence relations

As in the propositional case, we show that our calculi capture logics, i.e. consequence relations satisfying the properties defined in Chapter 3.

Lemma 9.3.1. The following holds for $i \in \{\mathsf{w}, \mathsf{s}\}$, where σ is an atom substitution as in Definition 2.2.12 and Definition 2.2.13.

Monotonicity: if $\Gamma \subseteq \Gamma'$ and $\Gamma \vdash_i \varphi$ then $\Gamma' \vdash_i \varphi$.

Compositionality: if $\Gamma_1 \vdash_i \varphi$ and $\Gamma_2 \vdash_i \psi$ for all $\psi \in \Gamma_1$, then $\Gamma_2 \vdash_i \varphi$

Structurality: if $\Gamma \vdash_i \varphi$ then $\Gamma^\sigma \vdash_i \varphi^\sigma$.

Proof. **Monotonicity**: (⌨) Assume $\Gamma \vdash_i \varphi$. Then there is a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that $\Gamma' \vdash_i \varphi$ with $\Gamma \subseteq \Gamma'$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$. In this case we have $\varphi \in \Gamma'$ as $\Gamma \subseteq \Gamma'$, hence $\Gamma' \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ and thus $\Gamma' \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If (Gen) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi$ where $\varphi = \forall\psi$. We straightforwardly obtain that $\Gamma[\uparrow] \subseteq \Gamma'[\uparrow]$. Thus, we can apply the induction hypothesis to obtain a proof of $\Gamma'[\uparrow] \vdash \psi$. It then suffices to apply (Gen) to obtain $\Gamma' \vdash \forall\psi$.
If (EC) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi \to \chi[\uparrow]$ where $\varphi = (\exists\psi) \to \chi[\uparrow]$. Here again we have that $\Gamma[\uparrow] \subseteq \Gamma'[\uparrow]$, so we can apply the induction hypothesis to obtain a proof of $\Gamma'[\uparrow] \vdash \psi \to \chi[\uparrow]$. An application of (EC) leads us to a proof of $\Gamma' \vdash (\exists\psi) \to \chi$.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$ then we simply use the given premise to obtain $\Gamma' \vdash_{\mathsf{w}} \varphi$ as desired.

Compositionality: (⌨) Assume $\Gamma_1 \vdash_i \varphi$ and that $\Gamma_2 \vdash_i \psi$ for every $\psi \in \Gamma_1$. Then we have a proof $\mathfrak{p}$ of $\Gamma_1 \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma_2 \vdash_i \varphi$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma_1$ and so we have $\Gamma_2 \vdash_i \varphi$ by assumption.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ and thus $\Gamma_2 \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If (Gen) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma_1[\uparrow] \vdash \chi$ where $\varphi = \forall\chi$. Using Lemma 9.2.2 and the assumptions, we get that $\Gamma_2[\uparrow] \vdash_i \psi[\uparrow]$ for every $\psi[\uparrow] \in \Gamma_1[\uparrow]$. Thus, we can apply the induction hypothesis to obtain that $\Gamma_2[\uparrow] \vdash_i \chi$. It then suffices to apply (Gen) to obtain $\Gamma_2 \vdash \forall\psi$.
If (EC) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma_1[\uparrow] \vdash \chi \to \delta[\uparrow]$ where $\varphi = (\exists\chi) \to \delta[\uparrow]$. Using Lemma 9.2.2 and the assumptions, we get that $\Gamma_2[\uparrow] \vdash_i \psi[\uparrow]$ for every $\psi[\uparrow] \in \Gamma_1[\uparrow]$. Thus, we can apply the induction hypothesis to obtain that $\Gamma_2[\uparrow] \vdash \chi \to \delta[\uparrow]$. It then suffices to apply (EC) to obtain $\Gamma_2 \vdash (\exists\chi) \to \delta$.

In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$ then we can simply use the induction hypothesis on the premises and then apply the rule. If $i = w$, we apply appropriately the rule, i.e. from $\emptyset \vdash_{\mathsf{w}} \varphi$ to $\Gamma_2 \vdash_{\mathsf{w}} \varphi$, to obtain the desired result.

Structurality: (⌨) Assume $\Gamma \vdash_i \varphi$. Then we have a proof $\mathfrak{p}$ of $\Gamma \vdash \varphi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi^\sigma \in \Gamma^\sigma$, hence $\Gamma^\sigma \vdash \varphi^\sigma$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\varphi^\sigma \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ (⌨) , hence $\Gamma^\sigma \vdash_i \varphi^\sigma$.
If (MP) is the last rule applied in $\mathfrak{p}$, then we simply apply the induction hypothesis on the premises and apply the rule to obtain our result.
If (Gen) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi$ where $\varphi = \forall\psi$. Thus, we can apply the induction hypothesis to obtain a proof of $(\Gamma[\uparrow])^\sigma \vdash \psi^\sigma$. However, as σ is restricted as in Definition 2.2.13, we have that $(\Gamma[\uparrow])^\sigma = \Gamma^\sigma[\uparrow]$ by Lemma 2.2.5. So, we can apply (Gen) on $\Gamma^\sigma[\uparrow] \vdash \psi^\sigma$ to obtain a proof of $\Gamma^\sigma \vdash \forall\psi^\sigma$.
If (EC) is the last rule applied in $\mathfrak{p}$, then we have a proof of $\Gamma[\uparrow] \vdash \psi \to \chi[\uparrow]$ where $\varphi = (\exists\psi) \to \chi[\uparrow]$. Thus, we can apply the induction hypothesis to obtain a proof of $(\Gamma[\uparrow])^\sigma \vdash \psi^\sigma \to (\chi[\uparrow])^\sigma$. However, as σ is restricted as in Definition 2.2.13, we both have that $(\Gamma[\uparrow])^\sigma = \Gamma^\sigma[\uparrow]$ and $(\chi[\uparrow])^\sigma = \chi^\sigma[\uparrow]$ by Lemma 2.2.5. So, we can apply (EC) on $\Gamma^\sigma[\uparrow] \vdash \psi^\sigma \to \chi^\sigma[\uparrow]$ to obtain a proof of $\Gamma^\sigma \vdash \exists\psi^\sigma \to \chi^\sigma$.
If it is (iDN) then for both values of i we apply the induction hypothesis on the premise and then the rule. ■

In the first-order case as well, we can prove that these logics are finitary.

Lemma 9.3.2. For $i \in \{\mathsf{w}, \mathsf{s}\}$, if $\Gamma \vdash_i \varphi$, then $\Gamma' \vdash_i \varphi$ for some finite $\Gamma' \subseteq \Gamma$.

Proof. (⌨) Assume $\Gamma \vdash_i \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove by induction on the structure of $\mathfrak{p}$ that there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_i \varphi$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\{\varphi\} \subseteq \Gamma$ and $\varphi \vdash_i \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, $\emptyset \subseteq \Gamma$ and $\emptyset \vdash_i \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain finite $\Gamma', \Gamma'' \subseteq \Gamma$ such that $\Gamma' \vdash_i \psi$ and $\Gamma'' \vdash_i \psi \to \varphi$. Lemma 9.3.1 delivers $\Gamma' \cup \Gamma'' \vdash_i \psi$ and $\Gamma' \cup \Gamma'' \vdash_i \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma' \cup \Gamma'' \vdash_i \varphi$, where $\Gamma' \cup \Gamma'' \subseteq \Gamma$ is finite.
If the last rule is (Gen) or (EC) then apply the induction hypothesis on the premise and then the rule.
In the case of (iDN) we have to distinguish between the case where $i = s$ and $i = w$. If $i = s$, we apply the induction hypothesis on the premise and then the rule. If $i = w$, we apply appropriately the rule to obtain the desired result. ■

Now that we have formally established that our calculi define finitary logics, we proceed to show that these logics are different. We do so in an identical way to the propositional case.

9.4 Extensional interactions

In this section, we show that despite their tight connections, FOwBIL and FOsBIL are extensionally different.

First, here again we justify our use of the terms "weak" and "strong" to qualify our calculi by showing that FOwBIL is a subset of FOsBIL.

Theorem 9.4.1. If $\Gamma \vdash_{\mathsf{w}} \varphi$ then $\Gamma \vdash_{\mathsf{s}} \varphi$.

Proof. (⌨) Assume $\Gamma \vdash_{\mathsf{w}} \varphi$, giving a proof $\mathfrak{p}$ with root $\Gamma \vdash \varphi$. We prove $\Gamma \vdash_{\mathsf{s}} \varphi$ by induction on the structure of $\mathfrak{p}$.

If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \Gamma$ hence $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\Gamma \vdash_{\mathsf{s}} \psi$ and $\Gamma \vdash_{\mathsf{s}} \psi \to \varphi$. Thus (MP) can be applied to get $\Gamma \vdash_{\mathsf{s}} \varphi$.
If (wDN) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{s}} \psi$. Then, we use Lemma 8.4.1 to obtain $\Gamma \vdash_{\mathsf{s}} \psi$. By an application of (sDN) we obtain $\Gamma \vdash_{\mathsf{s}} \neg{\sim}\psi$.
If (Gen) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\Gamma[\uparrow] \vdash_{\mathsf{s}} \psi$. Thus (Gen) can be applied to get $\Gamma \vdash_{\mathsf{s}} \forall\psi$.
If (EC) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\Gamma[\uparrow] \vdash_{\mathsf{s}} \psi \to \chi[\uparrow]$. Thus (EC) can be applied to get $\Gamma[\uparrow] \vdash_{\mathsf{s}} \exists\psi \to \chi$. ■

Second, we prove that they are indistinguishable on their set

Theorem 9.4.2. $\emptyset \vdash_{\mathsf{s}} \varphi$ if and only if $\emptyset \vdash_{\mathsf{w}} \varphi$.

Proof. (⌨) From right to left we can use Theorem 9.4.1. We are thus left with the direction from left to right (⌨). Assume $\emptyset \vdash_{\mathsf{s}} \varphi$, giving a proof $\mathfrak{p}$ with root $\emptyset \vdash \varphi$. We prove $\emptyset \vdash_{\mathsf{w}} \varphi$ by induction on the structure of $\mathfrak{p}$.
If (El) is the last rule applied in $\mathfrak{p}$, then $\varphi \in \emptyset$ which is a contradiction.
If (Ax) is the last rule applied in $\mathfrak{p}$, then $\emptyset \vdash_{\mathsf{w}} \varphi$.
If (MP) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premises to obtain $\emptyset \vdash_{\mathsf{w}} \psi$ and $\emptyset \vdash_{\mathsf{w}} \psi \to \varphi$. Thus (MP) can be applied to get $\emptyset \vdash_{\mathsf{w}} \varphi$.
If (sDN) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{w}} \psi$. By an application of (wDN) we obtain $\emptyset \vdash_{\mathsf{w}} \neg{\sim}\psi$.
If (Gen) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{w}} \psi$. Thus (Gen) can be applied to get $\emptyset \vdash_{\mathsf{w}} \forall\psi$.
If (EC) is the last rule applied in $\mathfrak{p}$, then apply the induction hypothesis on the premise to obtain $\emptyset \vdash_{\mathsf{w}} \psi \to \chi[\uparrow]$. Thus (EC) can be applied to get $\emptyset \vdash_{\mathsf{w}} \exists\psi \to \chi$. ■

Finally, as in the propositional case, we have that FOwBIL and FOsBIL are two different logics. The following claim extensionally distinguishes them. However, as the second item relies on a soundness result proved later on, we leave this claim pending.

Claim 9.4.1. For $P(0) \in Form_{\mathbb{L}_{\mathbf{BI}}}$, $P(0) \vdash_{\mathsf{s}} \neg{\sim}P(0)$ and $P(0) \nvdash_{\mathsf{w}} \neg{\sim}P(0)$.

Proof. The proof of the first statement is straightforward (⌨):

$$\dfrac{\dfrac{}{P(0) \vdash P(0)}\,(\mathrm{El})}{P(0) \vdash \neg{\sim}P(0)}\,(\mathrm{sDN})$$

The second statement follows from Theorem 9.8.1 (1). ■

9.5 Deduction and dual-deduction theorems

In this section, we proceed to show that FOsBIL and FOwBIL are distinct on a meta-level by proving that both the deduction-detachment theorem and its dual hold for FOwBIL, while none hold for FOsBIL. In addition to that, the rule (DMP), dual to the rule (MP), can safely be added to the former logic but not to the latter.

Note that the structure of this section is identical to the propositional Section 8.6. Most proofs are identical to their propositional counterparts, but some require some adaptations.

To get to the results mentioned above, we first need to prove preliminary results about our generalized Hilbert calculi.

Lemma 9.5.1. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

1. $\Gamma \vdash_i \varphi \to (\psi \to \varphi)$
2. $\Gamma \vdash_i \varphi \to \varphi$
3. $\Gamma \vdash_i (\varphi \wedge \psi) \to (\psi \wedge \varphi)$
4. $\Gamma \vdash_i (\varphi \wedge \neg\varphi) \to \bot$
5. if $\Gamma \vdash_i \varphi \to \psi$ then $\Gamma \vdash_i (\varphi \vee \chi) \to (\psi \vee \chi)$
6. $\Gamma \vdash_i \varphi \to \psi$ then $\Gamma \vdash_i (\chi \vee \varphi) \to (\chi \vee \psi)$
7. $\Gamma \vdash_i (\varphi \vee \psi) \to (\psi \vee \varphi)$
8. $\Gamma \vdash_i \top$
9. $\Gamma \vdash_i \neg{\sim}\varphi \to \varphi$
10. $\Gamma \vdash_i \bot \to \varphi$
11. if $\Gamma \vdash_i \varphi \vee \bot$ then $\Gamma \vdash_i \varphi$

Proof. The proofs are identical to those in Lemma 8.6.1: (1) ⌨, (2) ⌨, (3) ⌨, (4) ⌨, (5) ⌨ and ⌨, (6) ⌨ and ⌨, (7) ⌨, (8) ⌨, (9) ⌨, (10) ⌨, (11) ⌨ and ⌨. ■

Here again, we define pairs $[\Gamma \mid \Delta]$ to express the deduction-detachment theorem and its dual in a way that makes obvious their duality. To define these pairs, we simply port the propositional Definition 8.6.1 to the first-order case.

Before turning to the meta-level duality between the implication and the exclusion arrows, we show part of their object-level duality.

Proposition 9.5.1. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

$$\vdash_i [\emptyset \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi] \quad \text{iff} \quad \vdash_i [\emptyset \mid \varphi \to (\psi \vee \chi)]$$

Proof. Identical to the proof of the propositional Proposition 8.6.1 (⌨). ■

Note that as FOsBIL allows contexts to be manipulated more liberally thanks to the rule (sDN), we can prove for this logic the same statement in a stronger form.

Proposition 9.5.2.

$$\vdash_{\mathsf{s}} [\Gamma \mid (\varphi \mathrel{-\!\!<} \psi) \to \chi] \quad \text{iff} \quad \vdash_{\mathsf{s}} [\Gamma \mid \varphi \to (\psi \vee \chi)]$$

Proof. Identical to the proof of the propositional Proposition 8.6.2 (⌨). ■

However, the strength of Proposition 9.5.1 already suffices to prove interesting results, notably showing that neither logic satisfies the disjunction property, and that $\neg{\sim}$ behaves like a **T** modality here too.

Lemma 9.5.2. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

1. $\Gamma \vdash_i (\varphi \mathrel{-\!\!<} \psi) \to \varphi$;
2. $\Gamma \vdash_i \varphi \vee {\sim}\varphi$;
3. if $\emptyset \vdash_i \varphi \to \psi$, then $\emptyset \vdash_i (\varphi \mathrel{-\!\!<} \chi) \to (\psi \mathrel{-\!\!<} \chi)$;
4. if $\emptyset \vdash_i \varphi \to \psi$, then $\emptyset \vdash_i (\chi \mathrel{-\!\!<} \psi) \to (\chi \mathrel{-\!\!<} \varphi)$;
5. $\Gamma \vdash_i \neg(\varphi \mathrel{-\!\!<} \psi) \to ({\sim}\psi \to {\sim}\varphi)$;
6. $\Gamma \vdash_i \neg{\sim}(\varphi \to \psi) \to (\neg{\sim}\varphi \to \neg{\sim}\psi)$.

Proof. The proofs here are identical to the proofs of Lemma 8.6.2: (1) ⌨, (2) ⌨, (3) ⌨, (4) ⌨, (5) ⌨, (6) ⌨. ■

In addition to these results, Proposition 9.5.1 allows us to prove the *deduction-detachment theorem* for FOwBIL.

Theorem 9.5.1 (Deduction-Detachment Theorem)**.** FOwBIL enjoys the deduction-detachment theorem:

$$\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi] \quad \text{iff} \quad \vdash_{\mathsf{w}} [\Gamma \mid \varphi \to \psi]$$

Proof. (⇐) (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma \mid \varphi \to \psi]$, i.e. $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$. Then by monotonicity (Lemma 9.3.1) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi \to \psi$. Moreover we have that $\Gamma, \varphi \vdash_{\mathsf{w}} \varphi$ as $\varphi \in \Gamma \cup \{\varphi\}$. So by (MP) we obtain $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$, hence $\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi]$.

(⇒) (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma, \varphi \mid \psi]$, i.e. $\Gamma, \varphi \vdash_{\mathsf{w}} \psi$ giving a proof $\mathfrak{p}$ of $\Gamma, \varphi \vdash \psi$. We show by induction on the structure of $\mathfrak{p}$ that $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (El) is the last rule applied in $\mathfrak{p}$, then $\psi \in \Gamma \cup \{\varphi\}$. If $\psi = \varphi$ then we clearly have $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ by Lemma 9.5.1. If $\psi \in \Gamma$ then we can deduce $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$ from the fact that we have $\emptyset \vdash_{\mathsf{w}} \psi \to (\varphi \to \psi)$, proven in Lemma 9.5.1.

If (Ax) is the last rule applied in $\mathfrak{p}$, then $\psi \in \mathcal{A}^{\mathsf{i}}_{FOBI}$ and with a similar reasoning we get $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If (MP) is the last rule applied in $\mathfrak{p}$, then we proceed as follows. We use the induction hypothesis on the premises of the rule to obtain proofs of $\Gamma \vdash \varphi \to \chi$ and $\Gamma \vdash \varphi \to (\chi \to \psi)$. We also note that $\emptyset \vdash_{\mathsf{w}} (\varphi \to \chi) \to ((\varphi \to (\chi \to \psi)) \to (\varphi \to \psi))$. Then, we use (MP) several times to arrive at the establishment of $\Gamma \vdash_{\mathsf{w}} \varphi \to \psi$.

If the last rule is (Gen), then we apply the induction hypothesis on the premise to obtain $\Gamma[\uparrow] \vdash_{\mathsf{w}} \varphi[\uparrow] \to \psi$. We can apply (Gen) on the previous proof to obtain a proof of $\Gamma \vdash \forall(\varphi[\uparrow] \to \chi)$. Moreover, we have from A_{17} that $\Gamma \vdash_{\mathsf{w}} \forall(\varphi[\uparrow] \to \chi) \to (\varphi \to \forall\chi)$. So, we can use (MP) to obtain a proof of $\Gamma \vdash_{\mathsf{w}} \varphi \to \forall\chi$.

If the last rule is (EC), then we apply the induction hypothesis on the premise to obtain $\Gamma[\uparrow] \vdash_{\mathsf{w}} \varphi[\uparrow] \to (\chi \to \gamma[\uparrow])$. We get $\Gamma[\uparrow] \vdash_{\mathsf{w}} (\varphi[\uparrow] \wedge \chi) \to \gamma[\uparrow]$ as we have that $\Gamma[\uparrow] \vdash_{\mathsf{w}} (\varphi[\uparrow] \to (\chi \to \gamma[\uparrow])) \to ((\varphi[\uparrow] \wedge \chi) \to \gamma[\uparrow])$ from axiom A_8. Then, we can obtain $\Gamma[\uparrow] \vdash_{\mathsf{w}} (\chi \wedge \varphi[\uparrow]) \to \gamma[\uparrow]$, and using A_9 we get $\Gamma[\uparrow] \vdash_{\mathsf{w}} \chi \to (\varphi[\uparrow] \to \gamma[\uparrow])$. Thus we can apply (EC) and get $\Gamma \vdash_{\mathsf{w}} (\exists\chi) \to (\varphi \to \gamma)$. We straightforwardly obtain $\Gamma \vdash_{\mathsf{w}} \varphi \to ((\exists\chi) \to \gamma)$.

If the last rule is (wDN), we have a proof of $\emptyset \vdash \chi$, so we can apply (wDN) to obtain $\emptyset \vdash_{\mathsf{w}} \neg\!\sim\!\psi$. Then we can use the fact that $\emptyset \vdash_{\mathsf{w}} \neg\!\sim\!\chi \to (\varphi \to \neg\!\sim\!\chi)$ via Lemma 9.5.1 to obtain $\emptyset \vdash_{\mathsf{w}} \varphi \to \neg\!\sim\!\chi$. By monotonicity we obtain $\Gamma \vdash_{\mathsf{w}} \varphi \to \neg\!\sim\!\chi$.

■

However, as in the propositional case, the deduction theorem (from left to right) in the above form does not hold for FOsBIL. This failure is witnessed through the following claim. As this claim is proved using a soundness result, we leave it as pending.

Claim 9.5.1. We have that $\vdash_{\mathsf{s}} [P(0) \mid \neg\!\sim P(0)]$ but $\nvdash_{\mathsf{s}} [\emptyset \mid P(0) \to \neg\!\sim P(0)]$.

Proof. A proof of $\vdash_{\mathsf{s}} [P(0) \mid \neg\!\sim P(0)]$ is given in Claim 9.4.1. The second statement $\nvdash_{\mathsf{s}} [\emptyset \mid P(0) \to \neg\!\sim P(0)]$ follows from Theorem 9.8.1 (2). ■

Still, as in the propositional case, the same variant of the deduction theorem holds for FOsBIL. This variant is expressed using expressions like $(\neg\!\sim)^n\varphi$ and $(\neg\!\sim)^\omega\Gamma$, define in an identical way to the propositional Definition 8.6.2.

Theorem 9.5.2 (Double-Negated Deduction Theorem)**.**

$$\vdash_{\mathsf{s}} [\Gamma, \varphi \mid \psi] \quad \text{iff} \quad \exists n \in \mathbb{N} \text{ s.t. } \vdash_{\mathsf{s}} [\Gamma \mid (\neg\!\sim)^n\varphi \to \psi]$$

Proof. We prove each direction separately.

($\Rightarrow$) (⌨) Assume that $\vdash_{\mathsf{s}} [\Gamma, \varphi \mid \psi]$, i.e. that we have a proof $\mathfrak{p}$ of $\Gamma, \varphi \vdash \psi$. We reason by induction on the structure of $\mathfrak{p}$.

If the last rule applied is (Ax), then we get $\emptyset \vdash_{\mathsf{s}} \psi$, and as we have that $\emptyset \vdash_{\mathsf{s}} \psi \to ((\neg\sim)^n\varphi \to \psi)$ for any $n \in \mathbb{N}$ we obtain by (MP): $\emptyset \vdash_{\mathsf{s}} (\neg\sim)^n\varphi \to \psi$. By Lemma 9.3.1 we get $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^n\varphi \to \psi$. So, it suffices to take $n = 0$.

If the last rule applied is (El) then $\Gamma \vdash_{\mathsf{s}} \varphi \to \psi$ as $\Gamma \vdash_{\mathsf{s}} \psi \to (\varphi \to \psi)$ and $\Gamma \vdash_{\mathsf{s}} \psi$.

If the last rule applied is (MP) then we have by induction hypothesis $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^k\varphi \to \chi$ and $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^m\varphi \to (\chi \to \psi)$ for some $\chi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ and $m, k \in \mathbb{N}$. As we have that $\emptyset \vdash_{\mathsf{s}} (\lambda_1 \to \lambda_2) \to ((\lambda_1 \to (\lambda_2 \to \lambda_3)) \to (\lambda_1 \to \lambda_3))$ and $\emptyset \vdash_{\mathsf{s}} \neg\sim\lambda \to \lambda$ we obtain that $\Gamma \vdash_{\mathsf{s}} \neg\sim^n\varphi \to \chi$ for $n = max(m, k)$.

If the last rule applied is (Gen) then we apply the induction hypothesis on the premise to obtain $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid \neg\sim^n\varphi[\uparrow] \to \chi]$. Thus, we can apply (Gen) and get $\vdash_{\mathsf{s}} [\Gamma \mid \forall(\neg\sim^n\varphi[\uparrow] \to \chi)]$. Moreover, we know from A_{17} that the following holds.

$$\vdash_{\mathsf{s}} [\Gamma \mid \forall(\neg\sim^n\varphi[\uparrow] \to \chi) \to (\neg\sim^n\varphi \to \forall\chi)]$$

So, an instance of (MP) leads to $\vdash_{\mathsf{s}} [\Gamma \mid \neg\sim^n\varphi \to \forall\chi]$.

If the last rule applied is (EC) then apply the induction hypothesis on the premise to obtain $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid \varphi[\uparrow] \to (\chi \to \gamma[\uparrow])]$. Then, we get $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid (\varphi[\uparrow] \wedge \chi) \to \gamma[\uparrow]]$ as we have that $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid (\varphi[\uparrow] \to (\chi \to \gamma[\uparrow])) \to ((\varphi[\uparrow] \wedge \chi) \to \gamma[\uparrow])]$ from A_8. In turn, we get $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid (\chi \wedge \varphi[\uparrow]) \to \gamma[\uparrow]]$ and then $\vdash_{\mathsf{s}} [\Gamma[\uparrow] \mid \chi \to (\varphi[\uparrow] \to \gamma[\uparrow])]$. Thus we can apply (EC) and get $\vdash_{\mathsf{s}} [\Gamma \mid (\exists\chi) \to (\varphi \to \gamma)]$. We can then easily obtain $\vdash_{\mathsf{s}} [\Gamma \mid \varphi \to ((\exists\chi) \to \gamma)]$.

If the last rule applied is (sDN) then we get by induction hypothesis that $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^n\varphi \to \chi$. We apply (sDN) on $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^n\varphi \to \chi$ to obtain $\Gamma \vdash_{\mathsf{s}} \neg\sim((\neg\sim)^n\varphi \to \chi)$. Then, we get $\Gamma \vdash_{\mathsf{s}} (\neg\sim)^{n+1}\varphi \to \neg\sim\chi$ by using item 6 of Lemma 9.5.2 and (MP). So, we have $\vdash_{\mathsf{s}} [\Gamma \mid (\neg\sim)^{n+1}\varphi \to \neg\sim\chi]$.

($\Leftarrow$) (⌨) Straightforward use of the rules (sDN) and (MP) with Lemma 9.3.1.

■

In addition to that, the dual of the deduction-detachment theorem distinguishes the two logics in the first-order case too.

Theorem 9.5.3 (Dual Deduction-Detachment Theorem)**.** The following holds:

$$\vdash_{\mathsf{w}} [\varphi \mid \psi, \Delta] \qquad \text{iff} \qquad \vdash_{\mathsf{w}} [\varphi \prec \psi \mid \Delta].$$

Proof. (⌨) (⌨) Assume that $\vdash_{\mathsf{w}} [\varphi \mid \psi, \Delta]$. By definition we get $\varphi \vdash_{\mathsf{w}} \psi \vee \bigvee \Delta'$ where $\Delta' \subseteq \Delta$ is finite. Using Theorem 9.5.1 we get $\emptyset \vdash_{\mathsf{w}} \varphi \to (\psi \vee \bigvee \Delta')$. We obtain $\emptyset \vdash_{\mathsf{w}} (\varphi \prec \psi) \to \bigvee \Delta'$ by Proposition 9.5.1. By Theorem 9.5.1 again, we obtain $\varphi \prec \psi \vdash_{\mathsf{w}} \bigvee \Delta'$. By definition we get $\vdash_{\mathsf{w}} [\varphi \prec \psi \mid \Delta]$. Note that all the steps used here are based on equivalences. ■

This theorem does not port to FOsBIL, showing that the duality between $\to$ and $\prec$ breaks in it. We formulate the following claim, relying on soundness, proved later on.

Claim 9.5.2. $\vdash_{\mathsf{s}} [P(0) \prec Q(0) \mid \neg\sim\sim Q(0)]$ while $\nvdash_{\mathsf{s}} [P(0) \mid Q(0), \neg\sim\sim Q(0)]$.

Proof. To show $\nvdash_{\mathsf{s}} [P(0) \mid Q(0), \neg\sim\sim Q(0)]$ we rely on Theorem 9.8.1. In addition to that, we provide a semi-proof for $\vdash_{\mathsf{s}} [P(0) \prec Q(0) \mid \neg\sim\sim Q(0)]$ (⌨).

$$\begin{array}{cl}
\hline
\emptyset \vdash_{\mathsf{w}} P(0) \to \top & (\mathrm{Ax}) \\
\hdashline
\emptyset \vdash_{\mathsf{w}} (P(0) \prec Q(0)) \to (\top \prec Q(0)) & Lem.9.5.2 \\
\hdashline
P(0) \prec Q(0) \vdash_{\mathsf{w}} \sim Q(0) & Thm.9.5.1 \\
\hdashline
P(0) \prec Q(0) \vdash_{\mathsf{s}} \sim Q(0) & Thm.9.4.1 \\
\hline
P(0) \prec Q(0) \vdash_{\mathsf{s}} \neg\sim\sim Q(0) & (\mathrm{DN})
\end{array}$$

■

A final property of importance that distinguishes the two logics pertains to the Dual of Modus Ponens rule. The latter can here too be added to the weak calculus FOwBIH in the first-order case, without changing the set of expressions it derives.

$$\frac{[\psi \mid \Delta] \quad [\varphi \mathrel{-\!\!<} \psi \mid \Delta]}{[\varphi \mid \Delta]} \ (\mathrm{DMP})$$

Lemma 9.5.3. If $\vdash_{\mathsf{w}} [\varphi \mathrel{-\!\!<} \psi \mid \Delta]$ and $\vdash_{\mathsf{w}} [\psi \mid \Delta]$ then $\vdash_{\mathsf{w}} [\varphi \mid \Delta]$.

Proof. (⌨) Assume $\vdash_{\mathsf{w}} [\varphi \mathrel{-\!\!<} \psi \mid \Delta]$ and $\vdash_{\mathsf{w}} [\psi \mid \Delta]$. Then, we have proofs of $\varphi \mathrel{-\!\!<} \psi \vdash \bigvee \Delta'$ and $\psi \vdash \bigvee \Delta''$. We can easily obtain that $\varphi \mathrel{-\!\!<} \psi \vdash \bigvee \Delta'''$ and $\psi \vdash \bigvee \Delta'''$, where $\bigvee \Delta''' = \bigvee \Delta' \vee \bigvee \Delta''$. Thus, we can use Theorem 9.5.1 to obtain proofs of $\emptyset \vdash (\varphi \mathrel{-\!\!<} \psi) \to \bigvee \Delta'''$ and $\emptyset \vdash \psi \to \bigvee \Delta'''$. By applying Proposition 9.5.1 on $\emptyset \vdash (\varphi \mathrel{-\!\!<} \psi) \to \bigvee \Delta'''$ we get a proof of $\emptyset \vdash \varphi \to (\psi \vee \bigvee \Delta''')$. Given the latter and $\emptyset \vdash \psi \to \bigvee \Delta'''$ we can conclude that there is a proof of $\emptyset \vdash \varphi \to \bigvee \Delta'''$. So, we have that $\vdash_{\mathsf{w}} [\varphi \mid \Delta]$. ■

But this does not hold for FOsBIL. This fact is another evidence of the lack of duality between $\mathrel{-\!\!<}$ and $\to$ in this logic.

Claim 9.5.3. $\vdash_{\mathsf{s}} [P(0) \mathrel{-\!\!<} Q(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$ and $\vdash_{\mathsf{s}} [Q(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$, while $\nvdash_{\mathsf{s}} [P(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$.

Proof. The proof of $\vdash_{\mathsf{s}} [P(0) \mathrel{-\!\!<} Q(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$ can be found in the proof of Claim 9.5.2. We obtain $\vdash_{\mathsf{s}} [P(0) \mathrel{-\!\!<} Q(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$ straightforwardly via the use of (El). As in Claim 9.5.2, we show $\nvdash_{\mathsf{s}} [P(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$ using Theorem 9.8.1. ■

As we announced at the beginning of the section, the same differences are present in the first-order case: all of the deduction-detachment, its dual, and the safety of the addition of the rule (DMP) hold in FOwBIL; none does in FOsBIL. In the next section, we make use of the proof-theoretical results we obtained so far to provide a formally verified proof of an important theorem: the constant domain axiom.

9.6 A theorem: constant domains axiom

The results we obtained this far are sufficient to attain a milestone in first-order bi-intuitionistic logic: a formally verified proof of the constant domain axiom.

Lemma 9.6.1. For $i \in \{\mathsf{w}, \mathsf{s}\}$:

$$\vdash_i [\Gamma \mid \forall(\varphi[\uparrow] \vee \psi) \to (\varphi \vee \forall\psi)]$$

Proof. (⌨) Given Theorem 9.4.1, we can restrict our attention to FOwBIH and prove that $\vdash_{\mathsf{w}} [\Gamma \mid \forall(\varphi[\uparrow] \vee \psi) \to (\varphi \vee \forall\psi)]$. By Lemma 9.3.1 and the fact that $\emptyset \subset \Gamma$, it suffices to prove $\vdash_{\mathsf{w}} [\emptyset \mid \forall(\varphi[\uparrow] \vee \psi) \to (\varphi \vee \forall\psi)]$. By Theorem 9.5.1 the latter is equivalent to $\vdash_{\mathsf{w}} [\forall(\varphi[\uparrow] \vee \psi) \mid \varphi \vee \forall\psi]$. In turn, this is equivalent to $\vdash_{\mathsf{w}} [\forall(\varphi[\uparrow] \vee \psi) \mathrel{-\!\!<} \varphi \mid \forall\psi]$ using Theorem 9.5.3. We provide the following semi-proof for the latter:

$$\begin{array}{cl}
\hline \emptyset \vdash \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow] \vee \psi) & (\mathrm{Ax}) \\
\hdashline \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \vdash \varphi[\uparrow] \vee \psi & Thm. 9.5.1 \\
\hdashline \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \mathrel{-\!\!<} \varphi[\uparrow] \vdash \psi & Thm. 9.5.3 \\
\hline \forall(\varphi[\uparrow] \vee \psi) \mathrel{-\!\!<} \varphi \vdash \forall\psi & (\mathrm{Gen})
\end{array}$$

To show that $\forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow] \vee \psi)$ is an instance of A_{18}, consider the following chain of equalities.

$$\begin{aligned}\forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow] \vee \psi) &= \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow][\uparrow][0 \mathbin{.:} id] \vee \psi[\uparrow][0 \mathbin{.:} id]) \\ &= \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow][\mathsf{up}(\uparrow)][0 \mathbin{.:} id] \vee \psi[\mathsf{up}(\uparrow)][0 \mathbin{.:} id]) \\ &= \forall(\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)]) \to (\varphi[\uparrow][\mathsf{up}(\uparrow)] \vee \psi[\mathsf{up}(\uparrow)])[0 \mathbin{.:} id]\end{aligned}$$

The first line is justified by Lemma 2.2.3, essentially saying that the combination of substitutions $[\uparrow][0 \mathbin{.:} id]$ does not modify the formula. The second line involves Lemma 2.2.4 showing notably that $\varphi[\uparrow][\uparrow] = \varphi[\uparrow][\mathsf{up}(\uparrow)]$. Finally, the third line follows by definition of substitution. The formula thus obtained is obviously an instance of A_{18} where we instantiated the bound variables by the term 0. ■

First, let us note that the above formula, in a syntax à la de Bruijn, does capture the meaning of the constant domain axiom in the syntax with names. Indeed, we make sure that no variable in φ is bound by the leftmost universal quantifier by lifting all its free variables through $[\uparrow]$.

Second, we comment on the proof of this theorem. This proof is extremely insightful as it shows that we obtain the constant domain axiom straightforwardly through properties of *propositional* connectives: the deduction-detachment theorem and its dual. In essence, the idea of the proof is to use these theorems to isolate $\forall\psi$ to apply (Gen). Note that this isolation is possible only thanks to the dual detachment-deduction theorem: it allows to extract φ from the disjunction $\varphi \vee \forall\psi$, and adds it on the left-hand side of $\vdash$. This mechanism is not available in first-order intuitionistic logic. So, while this formula is not a theorem in first-order intuitionistic logic, the addition of the exclusion arrow as characterized here adds this mechanism to our toolbox and thus entails this axiom.

With this last theorem, we close our proof-theoretic study of FOwBIL and FOsBIL. Next, we introduce a semantics for first-order bi-intuitionistic languages based on constant domains models. With it, we will resolve the claims we left pending this far, which distinguish the two logics.

9.7 Constant domains Kripke semantics

The Kripke semantics for first-order bi-intuitionistic languages we present here is a modified extension of the usual one for first-order intuitionistic languages. First, it is modified as instead of allowing *increasing* domains we require *constant* domains. Second, it is an extension as it caters for the exclusion $\mathrel{-\!\!<}$ as in the propositional case.

Throughout this section we consider given a signature $\mathcal{S}$ and the bi-intuitionistic language $\mathbb{L}_{\mathbf{BI}}$ in $\mathcal{S}$. Here again, the accessibility relation on first-order frames is represented by the symbol $\leq$. For conciseness, and as it is clear given the context, we say "model" instead of "first-order model".

As we use the same frames and models as defined in Section 5.2 of our Toolbox, we are only required to provide the interpretation of connectives and quantifiers.

Definition 9.7.1. Given a model $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ and an assignment α for $\mathcal{M}$, we define a forcing relation between a point $w \in W$ and a formula in $Form_{\mathbb{L}_{\mathbf{BI}}}$ as follows:

$$\begin{array}{lll}
\mathcal{M}, w, \alpha \Vdash P(t_1, \ldots, t_n) & \text{iff} & (\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I_{pred}(w, P) \\
\mathcal{M}, w, \alpha \nVdash \bot & & \\
\mathcal{M}, w, \alpha \Vdash \top & & \\
\mathcal{M}, w, \alpha \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w, \alpha \Vdash \varphi \text{ and } \mathcal{M}, w, \alpha \Vdash \psi \\
\mathcal{M}, w, \alpha \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, w, \alpha \Vdash \varphi \text{ or } \mathcal{M}, w, \alpha \Vdash \psi \\
\mathcal{M}, w, \alpha \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \text{ s.t. } w \leq v,\ \mathcal{M}, v, \alpha \Vdash \varphi \text{ implies } \mathcal{M}, v, \alpha \Vdash \psi \\
\mathcal{M}, w, \alpha \Vdash \varphi \mathrel{-\!\!<} \psi & \text{iff} & \text{there exists a } v \text{ s.t. } v \leq w,\ \mathcal{M}, v, \alpha \Vdash \varphi \text{ and } \mathcal{M}, v, \alpha \nVdash \psi \\
\mathcal{M}, w, \alpha \Vdash \forall\varphi & \text{iff} & \text{for all } d \in D,\ \mathcal{M}, w, (d \mathbin{.:} \alpha) \Vdash \varphi \\
\mathcal{M}, w, \alpha \Vdash \exists\varphi & \text{iff} & \text{there is } d \in D \text{ s.t. } \mathcal{M}, w, (d \mathbin{.:} \alpha) \Vdash \varphi
\end{array}$$

The reader familiar with the usual first-order Kripke semantics for first-order intuitionistic languages will certainly notice that the clause for the universal quantifier is *local.* Indeed, given that we work with a semantics based on constant domains models, we do not need to check in successors of the point w to determine whether $\forall\varphi$ is forced in w. This is simply because, on the contrary to the intuitionistic case, at a point in the model no element of the domain cannot yet be witnessed "here", but will be in further points up the relation accessibility. As the domain is constant, any individual that is ever to exist is present anywhere in the model. More formally, the above clause and the following are equivalent on constant domains models.

$$\mathcal{M}, w, \alpha \Vdash \forall\varphi \quad \text{iff} \quad \text{for all } v \in W \text{ s.t. } w \leq v \text{ and } d \in D,\ \mathcal{M}, v, (d .: \alpha) \Vdash \varphi$$

This equivalence can be shown using persistence, the main feature of the Kripke semantics for intuitionistic logic.

Lemma 9.7.1 (Persistence). Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, $w \in W$ and $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. For all $v \in W$ such that $w \leq v$ we have that for all assignment α for $\mathcal{M}$, if $\mathcal{M}, w, \alpha \Vdash \varphi$ then $\mathcal{M}, v, \alpha \Vdash \varphi$.

Proof. (⌨) Let $w \in W$. We reason by induction on φ:

- $\varphi := P(t_1, \ldots, t_n)$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. We have $\mathcal{M}, w, \alpha \Vdash P(t_1, \ldots, t_n)$ implies $(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I_{pred}(w, P)$, which implies $(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I_{pred}(v, P)$ by the persistence condition on models, which in turn implies $\mathcal{M}, v, \alpha \Vdash P(t_1, \ldots, t_n)$. As α and v are arbitrary we get that for all α, if $\mathcal{M}, w, \alpha \Vdash P(t_1, \ldots, t_n)$ then $\mathcal{M}, v, \alpha \Vdash P(t_1, \ldots, t_n)$;

- $\varphi := \chi \wedge \psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \chi \wedge \psi$ then $\mathcal{M}, w, \alpha \Vdash \chi$ and $\mathcal{M}, w, \alpha \Vdash \psi$. By induction hypothesis $\mathcal{M}, v, \alpha \Vdash \chi$ and $\mathcal{M}, v, \alpha \Vdash \psi$. Then $\mathcal{M}, v, \alpha \Vdash \chi \wedge \psi$. We thus get the desired result for all α and v.

- $\varphi := \chi \vee \psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \chi \vee \psi$ then $\mathcal{M}, w, \alpha \Vdash \chi$ or $\mathcal{M}, w, \alpha \Vdash \psi$. By induction hypothesis $\mathcal{M}, v, \alpha \Vdash \chi$ or $\mathcal{M}, v, \alpha \Vdash \psi$. Then $\mathcal{M}, v, \alpha \Vdash \chi \vee \psi$. We thus get the desired result for all α and v.

- $\varphi := \chi \rightarrow \psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \chi \rightarrow \psi$ then for every $u \in W$ s.t. $w \leq u$, if $\mathcal{M}, u, \alpha \Vdash \chi$ then $\mathcal{M}, u, \alpha \Vdash \psi$. Note that for every $u \in W$, if $v \leq u$ then $w \leq u$ by transitivity. Consequently we have that for every $u \in W$ s.t. $v \leq u$, if $\mathcal{M}, u, \alpha \Vdash \chi$ then $\mathcal{M}, u, \alpha \Vdash \psi$. We get $\mathcal{M}, v, \alpha \Vdash \chi \rightarrow \psi$. We thus get the desired result for all α and v.

- $\varphi := \chi \prec \psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \chi \prec \psi$ then there is a $u \leq w$ s.t. $\mathcal{M}, u, \alpha \Vdash \chi$ and $\mathcal{M}, u, \alpha \nVdash \psi$. By transitivity we have $u \leq v$, so there is a $u \leq v$ s.t. $\mathcal{M}, u, \alpha \Vdash \chi$ and $\mathcal{M}, u, \alpha \nVdash \psi$. Thus $\mathcal{M}, v, \alpha \Vdash \chi \prec \psi$. We thus get the desired result for all α and v.

- $\varphi := \exists\psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \exists\psi$ then there is a $d \in D$ s.t. $\mathcal{M}, w, (d .: \alpha) \Vdash \psi$. By induction hypothesis we get that for all α', if $\mathcal{M}, w, \alpha' \Vdash \psi$ then $\mathcal{M}, v, \alpha' \Vdash \psi$. In particular, this holds for $(d .: \alpha)$ hence $\mathcal{M}, v, (d .: \alpha) \Vdash \psi$. Thus $\mathcal{M}, v, \alpha \Vdash \exists\psi$. We thus get the desired result for all α and v.

- $\varphi := \forall\psi$: Let α be an assignment, $v \in W$ and assume that $w \leq v$. $\mathcal{M}, w, \alpha \Vdash \forall\psi$ then for all $d \in D$, $\mathcal{M}, w, (d .: \alpha) \Vdash \psi$. By induction hypothesis we get $\mathcal{M}, v, (d .: \alpha) \Vdash \psi$. As d was arbitrary, we get that $\mathcal{M}, v, \alpha \Vdash \forall\psi$. We thus get the desired result for all α and v.

■

We turn to the proof of the equivalence of the two clauses.

Lemma 9.7.2. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, $w \in W$ and $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. The following statements are equivalent.

1. For all $d \in D$, we have $\mathcal{M}, w, (d .: \alpha) \Vdash \varphi$.
2. For all $v \in W$ such that $w \leq v$ and $d \in D$, we have $\mathcal{M}, v, (d .: \alpha). \Vdash \varphi$

Proof ✍. The direction from 2 to 1 holds trivially by instantiation of v by w. The direction from 1 to 2 relies on persistence as shown below. Assume that for all $d \in D$, $\mathcal{M}, w, (d .: \alpha) \Vdash \varphi$. We need to show that for all $v \in W$ s.t. $w \leq v$ and $d \in D$, $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$. Let $v \in W$ and $d \in D$ and assume that $w \leq v$. We have $\mathcal{M}, w, (d .: \alpha) \Vdash \varphi$ by assumption, and consequently we get $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$ ■

It can already be noticed that the notations required so far easily become heavy. Indeed, the bureaucracy inherent to the treatment of first-order logics shows itself in the manipulation of the first-order Kripke semantics, and adds up to the complexity of syntactic notations. In particular, the interactions between substitutions and assignments can give birth to extremely thorny situations. To help us in dealing with the latter, we prove some useful lemmas.

The first one relates to a property in Coq: in it, we do not have functional extensionality as an axiom [105, Chap. Logic]. While the latter can safely be added [147], in general we cannot consider identical two functions which are identical on all inputs. For example, having that two assignments α_1 and α_2 agree on all inputs is not sufficient to substitute one for the other in a given context. This is the reason why we need to provide a proof for the next lemma.

Lemma 9.7.3. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, α_1 and α_2 assignments and $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. If for all $n \in \mathbb{N}$ we have $\alpha_1(n) = \alpha_2(n)$, then:

$$\mathcal{M}, w, \alpha_1 \Vdash \varphi \quad \text{iff} \quad \mathcal{M}, w, \alpha_2 \Vdash \varphi$$

Proof. (⌨) We reason by induction on φ.

- $\varphi := P(t_1, \ldots, t_n)$: We have the following equivalence.

  $$\mathcal{M}, w, \alpha_1 \Vdash P(t_1, \ldots, t_n) \quad \text{iff} \quad (\overline{\alpha_1}(t_1), \ldots, \overline{\alpha_1}(t_n)) \in I_{pred}(w, P)$$

  However, we can prove for all $t \in Term_{\mathcal{S}}$ we have $\overline{\alpha_1}(t) = \overline{\alpha_2}(t)$ if the assignments agree on all inputs (⌨). So, we have the following equivalence.

  $$(\overline{\alpha_1}(t_1), \ldots, \overline{\alpha_1}(t_n)) \in I_{pred}(w, P) \quad \text{iff} \quad (\overline{\alpha_2}(t_1), \ldots, \overline{\alpha_2}(t_n)) \in I_{pred}(w, P)$$

  Consequently, we get $\mathcal{M}, w, \alpha_2 \Vdash P(t_1, \ldots, t_n)$.

- $\varphi := \top$: We have that $\top$ is always forced, so we are done.

- $\varphi := \bot$: We have that $\bot$ is never forced, so we are done.

- $\varphi := \chi \wedge \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \chi \wedge \psi$ iff $[\mathcal{M}, w, \alpha_1 \Vdash \chi$ and $\mathcal{M}, w, \alpha_1 \Vdash \psi]$. By the induction hypothesis, we get that the latter holds iff $[\mathcal{M}, w, \alpha_2 \Vdash \chi$ and $\mathcal{M}, w, \alpha_2 \Vdash \psi]$. Thus $\mathcal{M}, w, \alpha_1 \Vdash \chi \wedge \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \chi \wedge \psi$.

- $\varphi := \chi \vee \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \chi \vee \psi$ iff $[\mathcal{M}, w, \alpha_1 \Vdash \chi$ or $\mathcal{M}, w, \alpha_1 \Vdash \psi]$. By the induction hypothesis, we get that the latter holds iff $[\mathcal{M}, w, \alpha_2 \Vdash \chi$ or $\mathcal{M}, w, \alpha_2 \Vdash \psi]$. Thus $\mathcal{M}, w, \alpha_1 \Vdash \chi \vee \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \chi \vee \psi$.

- $\varphi := \chi \to \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \chi \to \psi$ iff for every $v \in W$ s.t. $w \leq v$, if $\mathcal{M}, v, \alpha_1 \Vdash \chi$ then $\mathcal{M}, v, \alpha_1 \Vdash \psi$. By induction hypothesis this holds if and only if for every $v \in W$ s.t. $w \leq v$, if $\mathcal{M}, v, \alpha_2 \Vdash \chi$ then $\mathcal{M}, v, \alpha_2 \Vdash \psi$. We consequently get $\mathcal{M}, w, \alpha_2 \Vdash \chi \to \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \chi \to \psi$.

- $\varphi := \chi \mathrel{-\!\!<} \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \chi \mathrel{-\!\!<} \psi$ iff there is a $v \leq w$ s.t. $\mathcal{M}, v, \alpha_1 \Vdash \chi$ and $\mathcal{M}, v, \alpha_1 \nVdash \psi$. By induction hypothesis this holds if and only if there is a $v \leq w$ s.t. $\mathcal{M}, v, \alpha_2 \Vdash \chi$ and $\mathcal{M}, v, \alpha_2 \nVdash \psi$. Thus $\mathcal{M}, w, \alpha_2 \Vdash \chi \mathrel{-\!\!<} \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \chi \mathrel{-\!\!<} \psi$.

- $\varphi := \forall \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \forall \psi$ iff for all $d \in D$ we have $\mathcal{M}, w, (d .\!: \alpha_1) \Vdash \psi$. We use the induction hypothesis to obtain that this holds if and only if for all $d \in D$ we have $\mathcal{M}, w, (d .\!: \alpha_2) \Vdash \psi$. To do so, we need to prove that the assignment $(d .\!: \alpha_1)$ and $(d .\!: \alpha_2)$ agree on all inputs. This is rather straightforward, as they both output d on 0 and output $\alpha_1(m) = \alpha_2(m)$ on $S\ m$. So, we finally get that $\mathcal{M}, w, \alpha_1 \Vdash \forall \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \forall \psi$.

- $\varphi := \exists \psi$: $\mathcal{M}, w, \alpha_1 \Vdash \exists \psi$ iff there is a $d \in D$ s.t. $\mathcal{M}, w, (d .\!: \alpha_1) \Vdash \psi$. We use the induction hypothesis to obtain that this holds if and only if there is a $d \in D$ s.t. $\mathcal{M}, w, (d .\!: \alpha_2) \Vdash \psi$. Here again, we have that the assignment $(d .\!: \alpha_1)$ and $(d .\!: \alpha_2)$ agree on all inputs. So, we finally get that $\mathcal{M}, w, \alpha_1 \Vdash \exists \psi$ iff $\mathcal{M}, w, \alpha_2 \Vdash \exists \psi$.

■

The second bureaucratic lemma we prove shows that if we apply an assignment to a term obtained through a substitution, we can equivalently modify the assignment to take into account the substitution, and then apply it to the initial term.

Lemma 9.7.4. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, $w \in W$ and τ a variable substitution. For all term t, $\overline{\alpha}(t[\tau]) = \overline{\overline{\alpha} \circ \tau}(t)$.

Proof. (⌨) We reason by induction on the structure of t:

- $t := x$: Then $x[\tau] = \tau(x)$, hence $\overline{\alpha}(x[\tau]) = \overline{\alpha}(\tau(x)) = (\overline{\alpha} \circ \tau)(x) = \overline{\overline{\alpha} \circ \tau}(x)$.

- $t := f(t_1, \ldots, t_n)$: as $f(t_1, \ldots, t_n)[\tau] = f(t_1[\tau], \ldots, t_n[\tau])$, by definition we get that $\overline{\alpha}(f(t_1, \ldots, t_n)[\tau]) = \overline{\alpha}(f(t_1[\tau], \ldots, t_n[\tau])) = I_{fun}(f)(\overline{\alpha}(t_1[\tau]), \ldots, \overline{\alpha}(t_n[\tau]))$. By induction hypothesis we have that for all $1 \leq i \leq n$, $\overline{\alpha}(t_i[\tau]) = \overline{\overline{\alpha} \circ \tau}(t_i)$. So we obtain that $\overline{\alpha}(f(t_1, \ldots, t_n)[\tau]) = \overline{\overline{\alpha} \circ \tau}(f(t_1, \ldots, t_n))$ as the following chain of equalities holds.

$$I_{fun}(f)(\overline{\alpha}(t_1[\tau]), \ldots, \overline{\alpha}(t_n[\tau])) = I_{fun}(f)(\overline{\overline{\alpha} \circ \tau}(t_1), \ldots, \overline{\overline{\alpha} \circ \tau}(t_n)) = \overline{\overline{\alpha} \circ \tau}(f(t_1, \ldots, t_n))$$

■

The third technical lemma essentially ports to formulas the idea expressed in the previous one, leading to an equivalence in the forcing relation.

Lemma 9.7.5. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model and $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$. For all variable substitution τ, $w \in W$ and all assignment α, we have that:

$$\mathcal{M}, w, \alpha \Vdash \varphi[\tau] \quad \text{iff} \quad \mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \varphi$$

Proof. (⌨) We prove our claim by induction on φ:

- $\varphi := P(t_1, \ldots, t_n)$: We have that $\mathcal{M}, w, \alpha \Vdash P(t_1, \ldots, t_n)[\tau]$ if and only if $\mathcal{M}, w, \alpha \Vdash P(t_1[\tau], \ldots, t_n[\tau])$ by definition. Equivalently, we have $(\overline{\alpha}(t_1[\tau]), \ldots, \overline{\alpha}(t_n[\tau])) \in I_{pred}(w, P)$. By Lemma 9.7.4 we get that $\overline{\alpha}(t_1[\tau]) = \overline{\overline{\alpha} \circ \tau}(t_1), \ldots, \overline{\alpha}(t_n[\tau]) = \overline{\overline{\alpha} \circ \tau}(t_n)$. Consequently, we have equivalently that $(\overline{\overline{\alpha} \circ \tau}(t_1), \ldots, \overline{\overline{\alpha} \circ \tau}(t_n)) \in I_{pred}(w, P)$. We easily get the last statement is equivalent to $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash P(t_1, \ldots, t_n)$.

- $\varphi := \top$: We have that $\top[\tau] = \top$, and the latter is always forced, so we are done.

- $\varphi := \bot$: We have that $\bot[\tau] = \bot$, and the latter is never forced, so we are done.

- $\varphi := \chi \wedge \psi$: $\mathcal{M}, w, \alpha \Vdash (\chi \wedge \psi)[\tau]$ iff $\mathcal{M}, w, \alpha \Vdash \chi[\tau]$ and $\mathcal{M}, w, \alpha \Vdash \psi[\tau]$. By induction hypothesis $\mathcal{M}, w, \alpha \Vdash \chi[\tau]$ and $\mathcal{M}, w, \alpha \Vdash \psi[\tau]$ iff $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi$ and $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \psi$. Thus $\mathcal{M}, w, \alpha \Vdash (\chi \wedge \psi)[\tau]$ iff $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi \wedge \psi$.

- $\varphi := \chi \vee \psi$: $\mathcal{M}, w, \alpha \Vdash (\chi \vee \psi)[\tau]$ iff $\mathcal{M}, w, \alpha \Vdash \chi[\tau]$ or $\mathcal{M}, w, \alpha \Vdash \psi[\tau]$. By induction hypothesis $[\mathcal{M}, w, \alpha \Vdash \chi[\tau]$ or $\mathcal{M}, w, \alpha \Vdash \psi[\tau]]$ iff $[\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi$ or $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \psi]$. Thus $\mathcal{M}, w, \alpha \Vdash (\chi \vee \psi)[\tau]$ iff $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi \vee \psi$.

- $\varphi := \chi \to \psi$: $\mathcal{M}, w, \alpha \Vdash (\chi \to \psi)[\tau]$ iff for every $v \in W$ s.t. $w \leq v$, if $\mathcal{M}, v, \alpha \Vdash \chi[\tau]$ then $\mathcal{M}, v, \alpha \Vdash \psi[\tau]$. By induction hypothesis this holds if and only if for every $v \in W$ s.t. $w \leq v$, if $\mathcal{M}, v, (\overline{\alpha} \circ \tau) \Vdash \chi$ then $\mathcal{M}, v, (\overline{\alpha} \circ \tau) \Vdash \psi$. We consequently get $\mathcal{M}, w, \alpha \Vdash (\chi \to \psi)[\tau]$ iff $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi \to \psi$.

- $\varphi := \chi \mathrel{-\!\!<} \psi$: $\mathcal{M}, w, \alpha \Vdash (\chi \mathrel{-\!\!<} \psi)[\tau]$ iff there is a $v \leq w$ s.t. $\mathcal{M}, v, \alpha \Vdash \chi[\tau]$ and $\mathcal{M}, v, \alpha \nVdash \psi[\tau]$. By induction hypothesis this holds if and only if there is a $v \leq w$ s.t. $\mathcal{M}, v, (\overline{\alpha} \circ \tau) \Vdash \chi$ and $\mathcal{M}, v, (\overline{\alpha} \circ \tau) \nVdash \psi$. Thus $\mathcal{M}, w, \alpha \Vdash (\chi \mathrel{-\!\!<} \psi)[\tau]$ iff $\mathcal{M}, w, (\overline{\alpha} \circ \tau) \Vdash \chi \mathrel{-\!\!<} \psi$.

- $\varphi := \exists \psi$: $\mathcal{M}, w, \alpha \Vdash \exists \psi[\tau]$ iff there is a $d \in D$ s.t. $\mathcal{M}, w, (d .: \alpha) \Vdash \psi[\mathsf{up}(\tau)]$. By induction hypothesis this holds if and only if there is a $d \in D$ s.t. $\mathcal{M}, w, (\overline{(d .: \alpha)} \circ \mathsf{up}(\tau)) \Vdash \psi$.

  We claim that for all $n \in \mathbb{N}$, $(\overline{(d .: \alpha)} \circ \mathsf{up}(\tau))(n) = (d .: (\overline{\alpha} \circ \tau))(n)$. We prove this by analyzing n. If $n := 0$, then we have the following chain of equalities.

  $$(\overline{(d .: \alpha)} \circ \mathsf{up}(\tau))(0) = \overline{(d .: \alpha)}(\mathsf{up}(\tau)(0)) = \overline{(d .: \alpha)}(0) = d = (d .: (\overline{\alpha} \circ \tau))(0)$$

  If $n := Sm$, then we have the following chain of equalities.

  $$(\overline{(d .: \alpha)} \circ \mathsf{up}(\tau))(Sm) = \overline{(d .: \alpha)}(\mathsf{up}(\tau)(Sm)) = \overline{(d .: \alpha)}((\lambda t.t[\uparrow] \circ \tau)(m)) = \overline{(d .: \alpha)}(\tau(m)[\uparrow])$$

  The last element of this chain is equal to $(\overline{\overline{(d .: \alpha)} \circ \uparrow})(\tau(m))$ by Lemma 9.7.4. We can see that $\overline{(d .: \alpha)} \circ \uparrow = \alpha$, as the left function takes n, applies $\uparrow$ to give $S\ n$, and then applies $\overline{(d .: \alpha)}$ which gives $\alpha(n)$ by definition. So, we get $(\overline{\overline{(d .: \alpha)} \circ \uparrow})(\tau(m)) = \overline{\alpha}(\tau(m))$. We finally obtain $\overline{\alpha}(\tau(m)) = (\overline{\alpha} \circ \tau)(m) = (d .: (\overline{\alpha} \circ \tau))(Sm)$.

  So, using Lemma 9.7.3, we get that there is a $d \in D$ s.t. $\mathcal{M}, w, (\overline{(d .: \alpha)} \circ \mathsf{up}(\tau)) \Vdash \psi$, if and only if there is a $d \in D$ s.t. $\mathcal{M}, w, (d .: (\overline{\alpha} \circ \tau)) \Vdash \psi$, hence $\mathcal{M}, w, \overline{\alpha} \circ \tau \Vdash \exists \psi$. Thus $\mathcal{M}, w, \alpha \Vdash \exists \psi[\tau]$ iff $\mathcal{M}, w, \overline{\alpha} \circ \tau \Vdash \exists \psi$.

- $\varphi := \forall \psi$: $\mathcal{M}, w, \alpha \Vdash \forall \psi[\tau]$ iff for all $d \in D$ s.t. $\mathcal{M}, w, (d .: \alpha) \Vdash \psi[\mathsf{up}(\tau)]$. By induction hypothesis this holds if and only if for all $d \in D$ s.t. $\mathcal{M}, w, (\overline{(d .: \alpha)} \circ \mathsf{up}(\tau)) \Vdash \psi$. As above, we can prove that for all $n \in \mathbb{N}$, $(\overline{(d .: \alpha)} \circ \mathsf{up}(\tau))(n) = (d .: (\overline{\alpha} \circ \tau))(n)$. So, using Lemma 9.7.3, we have that for all $d \in D$ s.t. $\mathcal{M}, w, (\overline{(d .: \alpha)} \circ \mathsf{up}(\tau)) \Vdash \psi$, if and only for all $d \in D$ s.t. $\mathcal{M}, w, (d .: (\overline{\alpha} \circ \tau)) \Vdash \psi$, hence $\mathcal{M}, w, \overline{\alpha} \circ \tau \Vdash \forall \psi$. Thus $\mathcal{M}, w, \alpha \Vdash \forall \psi[\tau]$ iff $\mathcal{M}, w, \overline{\alpha} \circ \tau \Vdash \forall \psi$.

■

We are done with bureaucratic lemmas. Before turning to a more model-theoretic part of our study of the semantics, let us comment on the de Bruijn approach to first-order syntax. This approach implies a significant amount of manipulation of functions: to show their equality on all inputs, to compose them, and to modify them. So, many lemmas need to be proven about these manipulations in Coq. But this is where the existence of such a well-developed library as the one we use makes our life easier.

Now, let us turn to notions relating models. While there are notions of bisimulation which imply logical equivalence in first-order Kripke semantics [109], we show instead that a much weaker relation, but sufficient for our purpose, entails logical equivalence: between a model $\mathcal{M}$ and its restriction in w, i.e. $\mathcal{M}^w$ as in Definition 8.9.1. We precisely define this notion in the first-order case.

Definition 9.7.2. Let $\mathcal{M} = (W, \leq, I_{fun}, I_{pred})$ be a model and $w \in W$. We define the restriction of $\mathcal{M}$ in w as the model $\mathcal{M}^w = (W^w, \leq^w, D, I_{fun}, I^w_{pred})$, where:

- $W^w = \{v \in W \mid \text{ there is a chain } wR_1 ... R_n v, \text{ where } R_j \in \{\leq, \geq\} \text{ for } j \in \mathbb{N}\}$;
- $\leq^w \;=\; \leq \;\cap\; (W^w \times W^w)$;
- $I^w_{pred}(v, P) \;=\; I_{pred}(v, P)$.

As promised, we prove that a model and its restriction are logically equivalent in all points of $\mathcal{M}^w$.

Proposition 9.7.1. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$, $w \in W$ and $\mathcal{M}^w = (W^w, \leq^w, D, I^w_{fun}, I^w_{pred})$ the restriction of $\mathcal{M}$ in w. For all assignment on D α, $v \in W^w$ and $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ we have $\mathcal{M}, v, \alpha \Vdash \varphi$ iff $\mathcal{M}^w, v, \alpha \Vdash \varphi$.

Proof. (⌨) We reason by induction on the structure of φ.

- $\varphi := P(t_1, \ldots, t_n)$: We have $\mathcal{M}, v, \alpha \Vdash P(t_1, \ldots, t_n)$ iff $(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I_{pred}(v, P)$. However, we have that $I_{pred}(v, P) = I^w_{pred}(v, P)$ by definition, so the latter holds iff $(\overline{\alpha}(t_1), \ldots, \overline{\alpha}(t_n)) \in I^w_{pred}(v, P)$, i.e. $\mathcal{M}^w, v, \alpha \Vdash P(t_1, \ldots, t_n)$.

- $\varphi := \top$: We have $\mathcal{M}, v, \alpha \Vdash \top$ and $\mathcal{M}^w, v, \alpha \Vdash \top$ by definition. So we trivially get the desired result.

- $\varphi := \bot$: We have $\mathcal{M}, v, \alpha \nVdash \bot$ and $\mathcal{M}^w, v, \alpha \nVdash \bot$ by definition. So we trivially get the desired result.

- $\varphi := \chi \wedge \psi$: We have that $\mathcal{M}, v, \alpha \Vdash \chi \wedge \psi$ iff $[\mathcal{M}, v, \alpha \Vdash \chi$ and $\mathcal{M}, v, \alpha \Vdash \psi]$. By the induction hypothesis, the latter holds iff $[\mathcal{M}^w, v, \alpha \Vdash \chi$ and $\mathcal{M}^w, v, \alpha \Vdash \psi]$. We finally reach $\mathcal{M}^w, v, \alpha \Vdash \chi \wedge \psi$ through a chain of equivalences. We thus get the desired result.

- $\varphi := \chi \vee \psi$: We have that $\mathcal{M}, v, \alpha \Vdash \chi \vee \psi$ iff $[\mathcal{M}, v, \alpha \Vdash \chi$ or $\mathcal{M}, v, \alpha \Vdash \psi]$. By induction hypothesis the latter holds iff $[\mathcal{M}^w, v, \alpha \Vdash \chi$ or $\mathcal{M}^w, v, \alpha \Vdash \psi]$. We finally reach $\mathcal{M}^w, v, \alpha \Vdash \chi \vee \psi$ through a chain of equivalences. We thus get the desired result.

- $\varphi := \chi \rightarrow \psi$: $(\Rightarrow)$ Assume $\mathcal{M}, v, \alpha \Vdash \chi \rightarrow \psi$. We need to show $\mathcal{M}^w, v, \alpha \Vdash \chi \rightarrow \psi$. Let $u \in W^w$ such that $v \leq u$, and assume $\mathcal{M}^w, u, \alpha \Vdash \chi$. We need to show $\mathcal{M}^w, u, \alpha \Vdash \psi$. As $\mathcal{M}, v, \alpha \Vdash \chi \rightarrow \psi$ we get that if $\mathcal{M}, u, \alpha \Vdash \chi$ then $\mathcal{M}, u\alpha \Vdash \psi$, as $u \in W$ by definition. But we know that $\mathcal{M}^w, u, \alpha \Vdash \chi$, so by induction hypothesis we get $\mathcal{M}, u, \alpha \Vdash \chi$. Thus, we get $\mathcal{M}, u, \alpha \Vdash \psi$. It then suffices to use the induction hypothesis again to obtain $\mathcal{M}^w, u, \alpha \Vdash \psi$. Consequently, we proved that $\mathcal{M}^w, v, \alpha \Vdash \chi \rightarrow \psi$.

  $(\Leftarrow)$ Assume $\mathcal{M}^w, v, \alpha \Vdash \chi \rightarrow \psi$. We need to show $\mathcal{M}, v, \alpha \Vdash \chi \rightarrow \psi$. Let $u \in W$ such that $v \leq u$, and assume $\mathcal{M}, u, \alpha \Vdash \chi$. We need to show $\mathcal{M}, u, \alpha \Vdash \psi$. Note that $u \in W^w$ as there is a chain from w to u going through v, as the latter is in W^w. So, as $\mathcal{M}^w, v, \alpha \Vdash \chi \rightarrow \psi$ we get that if $\mathcal{M}^w, u, \alpha \Vdash \chi$ then $\mathcal{M}^w, u\alpha \Vdash \psi$, as $v \leq^w u$ by definition. But we know that $\mathcal{M}, u, \alpha \Vdash \chi$, so by induction hypothesis we get $\mathcal{M}^w, u, \alpha \Vdash \chi$. Thus, we get $\mathcal{M}^w, u, \alpha \Vdash \psi$. It then suffices to use the induction hypothesis again to obtain $\mathcal{M}, u, \alpha \Vdash \psi$. Consequently, we proved that $\mathcal{M}, v, \alpha \Vdash \chi \rightarrow \psi$.

- $\varphi := \chi \mathrel{-\!\!<} \psi$: ($\Rightarrow$) Assume $\mathcal{M}, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$. We need to show $\mathcal{M}^w, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$. As $\mathcal{M}, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$ there is a $u \in W$ such that $u \leq v$ and $\mathcal{M}, u, \alpha \Vdash \chi$ and $\mathcal{M}, u, \alpha \nVdash \psi$. Note that $u \in W^w$, as there is a chain from w to u going through v, as $v \in W^w$. But we know that $\mathcal{M}, u, \alpha \Vdash \chi$ and $\mathcal{M}, u, \alpha \nVdash \psi$, so by induction hypothesis we get $\mathcal{M}^w, u, \alpha \Vdash \chi$ and $\mathcal{M}^w, u, \alpha \nVdash \psi$. Thus, we get $\mathcal{M}^w, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$.

  ($\Leftarrow$) Assume $\mathcal{M}^w, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$. We need to show $\mathcal{M}, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$. As $\mathcal{M}^w, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$ there is a $u \in W^w$ such that $u \leq^w v$ and $\mathcal{M}^w, u, \alpha \Vdash \chi$ and $\mathcal{M}^w, u, \alpha \nVdash \psi$. Note that $u \in W$ by definition. But we know that $\mathcal{M}^w, u, \alpha \Vdash \chi$ and $\mathcal{M}^w, u, \alpha \nVdash \psi$, so by induction hypothesis we get $\mathcal{M}, u, \alpha \Vdash \chi$ and $\mathcal{M}, u, \alpha \nVdash \psi$. Thus, we get $\mathcal{M}, v, \alpha \Vdash \chi \mathrel{-\!\!<} \psi$ as $u \leq v$ by definition.

- $\varphi := \forall\psi$: We have that $\mathcal{M}, v, \alpha \Vdash \forall\psi$ iff [for all $d \in D$, $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$]. By induction hypothesis the latter holds iff [for all $d \in D$, $\mathcal{M}^w, v, (d .: \alpha) \Vdash \varphi$]. We finally reach $\mathcal{M}^w, v, \alpha \Vdash \forall\psi$ through a chain of equivalences. We thus get the desired result.

- $\varphi := \exists\psi$: We have that $\mathcal{M}, v, \alpha \Vdash \exists\psi$ iff [there exists a $d \in D$ such that $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$]. By induction hypothesis the latter holds iff [fthere exists a $d \in D$ such that $\mathcal{M}^w, v, (d .: \alpha) \Vdash \varphi$]. We finally reach $\mathcal{M}^w, v, \alpha \Vdash \exists\psi$ through a chain of equivalences. We thus get the desired result.

■

While this is a quite weak relation between models, it suffices for our purpose thanks to the previous lemma and the next one, relating $\leq\geq$ zig-zags and double negations $\neg\sim$.

Lemma 9.7.6. Let $n \in \mathbb{N}$ be a natural number, $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ a model, $w, v \in W$ points, α an assignment, $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$ a formula and $wR_1...R_nv$ a chain in $\mathcal{M}$ such that $R_j \in \{\leq, \geq\}$ for every $j \in \{1, ..., n\}$. If $\mathcal{M}, w, \alpha \Vdash (\neg\sim)^n\varphi$, then $\mathcal{M}, v, \alpha \Vdash \varphi$.

Proof. (⌨) Identical to the propositional proof of Lemma 8.9.2. ■

We conclude this section on the semantics by recalling the definition of the local and global semantic consequence relations given in Section 5.2.

Definition 9.7.3. The $\underline{l}$ocal and $\underline{g}$lobal consequence relations are as below.

$$\begin{array}{lll} \Gamma \models_l \Delta & \text{iff} & \forall\mathcal{M}.\forall\alpha.\forall w.\,(\mathcal{M}, w, \alpha \Vdash \Gamma \;\Rightarrow\; \exists\delta \in \Delta.\,\mathcal{M}, w, \alpha \Vdash \delta) \\ \Gamma \models_g \Delta & \text{iff} & \forall\mathcal{M}.\forall\alpha.\,((\forall w.\mathcal{M}, \alpha, w \Vdash \Gamma) \;\Rightarrow\; (\forall w.\,\exists\delta \in \Delta.\,\mathcal{M}, w, \alpha \Vdash \delta)) \end{array}$$

As in the propositional case, we can prove that these notions are equivalent in the usual interpretation of the intuitionistic language (restricted to the class of constant domains models), on the contrary to the modal ones shown in Section 5.2.

Lemma 9.7.7. For $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_{\mathbf{I}}}$:

$$\Gamma \models_l \Delta \quad \text{iff} \quad \Gamma \models_g \Delta$$

Proof ✍. We prove each direction separately.

($\Rightarrow$) Assume $\Gamma \models_l \Delta$. Let $\mathcal{M}$ be a model and α an assignment. Assume that $\mathcal{M}, \alpha \Vdash \Gamma$. We need to show that for all $w \in W$, there is a $\delta \in \Delta$ such that $\mathcal{M}, w, \alpha \Vdash \delta$. Let $w \in W$. Given that $\mathcal{M}, \alpha \Vdash \Gamma$, we have in particular $\mathcal{M}, w, \alpha \Vdash \Gamma$. Consequently, as we have $\Gamma \models_l \Delta$, we obtain that there is a $\delta \in \Delta$ such that $\mathcal{M}, w, \alpha \Vdash \delta$. As w is arbitrary, we get $\forall w \in W.\exists\delta \in \Delta.\mathcal{M}, w, \alpha \Vdash \delta$. So, $\Gamma \models_g \Delta$.

($\Leftarrow$) Assume $\Gamma \models_g \Delta$. Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, α an assignment and $w \in W$. Assume that $\mathcal{M}, w, \alpha \Vdash \Gamma$. We need to show that there is a $\delta \in \Delta$ such that $\mathcal{M}, w, \alpha \Vdash \delta$. Consider the model $\mathcal{M}' = (W', \leq', D, I_{fun}, I'_{pred})$ such that $W' = \{v \in W \mid w \leq v\}$, and $\leq'$, I'_{fun} and I'_{pred} are respective restrictions of $\leq$, I_{fun} and I_{pred} to W'. In essence, this model is a restriction of $\mathcal{M}$ in w where we keep points which are successors of w, i.e. the upcone of w in the initial model. We can show that for such a restriction an equivalent of Proposition 9.7.1 holds, on the intuitionistic language this time. So, we obtain that $\mathcal{M}, w, \alpha \Vdash \Gamma$ implies $\mathcal{M}', w, \alpha \Vdash \Gamma$. Then, the intuitionistic version of Lemma 9.7.1 implies that for all $v \in W'$ such that $w \leq v$, we have $\mathcal{M}', w, \alpha \Vdash \Gamma$. However, for all $v \in W'$ we have $w \leq v$, by definition of W'. Consequently, we have $\mathcal{M}', v, \alpha \Vdash \Gamma$ for all $v \in W'$, hence $\mathcal{M}', \alpha \Vdash \Gamma$. We use $\Gamma \models_g \Delta$ on the latter to obtain $\forall v \in W. \exists \delta \in \Delta. \mathcal{M}', v, \alpha \Vdash \delta$. In particular, we get that for w there is $\delta \in \Delta$ such that $\mathcal{M}', w, \alpha \Vdash \delta$. It then suffices to use the intuitionistic version of Proposition 9.7.1 once again to obtain $\mathcal{M}, w, \alpha \Vdash \delta$. So, $\Gamma \models_l \Delta$.

■

9.8 Soundness

In this section, we show a similar configuration as in the propositional case: FOwBIL is sound for the local semantic consequence relation just defined, while FOwBIL is sound for the global one. With these results in hand, we can resolve all the claims we left pending this far.

Theorem 9.8.1 (Soundness)**.** The following holds:

$$\begin{array}{llll} (1) & \vdash_{\mathsf{w}} [\Gamma \mid \Delta] & \text{entails} & \Gamma \models_l \Delta \\ (2) & \vdash_{\mathsf{s}} [\Gamma \mid \Delta] & \text{entails} & \Gamma \models_g \Delta. \end{array}$$

Proof. We refer to the propositional case (Theorem 8.8.1 and Theorem 8.9.1) for A_1 to A_{16}, (MP), (wDN) and (sDN). Indeed, as these axioms and rules do not involve quantifiers, their proofs can be adapted from the propositional case by simply adding a mention of an inert assignment α in the forcing relation.

(1) (⌨) Let $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ be a model, $w \in W$ and α an assignment for D.

(A_{17}) We show that $\mathcal{M}, w, \alpha \Vdash \forall(\psi[\uparrow] \to \varphi) \to (\psi \to \forall\varphi)$. Let $v \geq w$ s.t. $\mathcal{M}, v, \alpha \Vdash \forall(\psi[\uparrow] \to \varphi)$. We need to show that $\mathcal{M}, v, \alpha \Vdash \psi \to \forall\varphi$. Let $u \geq v$ s.t. $\mathcal{M}, u, \alpha \Vdash \psi$. We need to show that $\mathcal{M}, u, \alpha \Vdash \forall\varphi$. Let $d \in D$. We need to show that $\mathcal{M}, u, (d .: \alpha) \Vdash \varphi$. As $\mathcal{M}, v, \alpha \Vdash \forall(\psi[\uparrow] \to \varphi)$ and $v \leq u$, we get by Lemma 9.7.1 that $\mathcal{M}, u, \alpha \Vdash \forall(\psi[\uparrow] \to \varphi)$. We notably obtain $\mathcal{M}, u, (d .: \alpha) \Vdash \psi[\uparrow] \to \varphi$. Now, we claim that $\mathcal{M}, u, (d .: \alpha) \Vdash \psi[\uparrow]$, which helps us show $\mathcal{M}, u, (d .: \alpha) \Vdash \varphi$ for an arbitrary $d \in D$, giving us $\mathcal{M}, u, \alpha \Vdash \forall\varphi$. To show our claim, we notice that by Lemma 9.7.5 it is equivalent to $\mathcal{M}, u, (\overline{(d .: \alpha)} \circ \uparrow) \Vdash \psi$. However, by noticing that $(\overline{(d .: \alpha)} \circ \uparrow)$ and α are the same functions, the latter is then identical to $\mathcal{M}, u, \alpha \Vdash \psi$ which is an assumption.

(A_{18}) We show that $\mathcal{M}, w, \alpha \Vdash \forall\varphi \to \varphi[t .: id]$. Let $v \geq w$ s.t. $\mathcal{M}, v, \alpha \Vdash \forall\varphi$. We need to show that $\mathcal{M}, v, \alpha \Vdash \varphi[t .: id]$. To show this, we first use Lemma 9.7.5 to focus on the equivalent statement $\mathcal{M}, v, (\overline{\alpha} \circ (t .: id)) \Vdash \varphi$. Second, we note that as $\mathcal{M}, v, \alpha \Vdash \forall\varphi$ we get $\mathcal{M}, v, (\overline{\alpha}(t) .: \alpha) \Vdash \varphi$. Third, we give a proof of $\mathcal{M}, v, (\overline{\alpha} \circ (t .: id)) \Vdash \varphi$ using Lemma 9.7.3 and $\mathcal{M}, v, (\overline{\alpha}(t) .: \alpha) \Vdash \varphi$. We are thus left to prove that for all $n \in \mathbb{N}$ we have $(\overline{\alpha} \circ (t .: id))(n) = (\overline{\alpha}(t) .: \alpha)(n)$. We prove this by case distinction on n. First, we have that $(\overline{\alpha} \circ (t .: id))(0) = \overline{\alpha}((t .: id)(0)) = \overline{\alpha}(t) = (\overline{\alpha}(t) .: \alpha)(0)$. Second, we have that $(\overline{\alpha} \circ (t .: id))(Sm) = \overline{\alpha}((t .: id)(Sm)) = \overline{\alpha}(id(m)) = \overline{\alpha}(m) = \alpha(m) = (\overline{\alpha}(t) .: \alpha)(Sm)$. So, we are done.

(A_{19}) We show that $\mathcal{M}, w, \alpha \Vdash \varphi[t .: id] \to \exists\varphi$. Let $v \geq w$ s.t. $\mathcal{M}, v, \alpha \Vdash \varphi[t .: id]$. We need to show that $\mathcal{M}, v, \alpha \Vdash \exists\varphi$. As $\mathcal{M}, v, \alpha \Vdash \varphi[t .: id]$ we get $\mathcal{M}, v, (\overline{\alpha} \circ (t .: id)) \Vdash \varphi$ by Lemma 9.7.5. We claim that $\mathcal{M}, v, (\overline{\alpha}(t) .: \alpha) \Vdash \varphi$, which gives us $\mathcal{M}, v, \alpha \Vdash \exists\varphi$. To show this, we use Lemma 9.7.3 and $\mathcal{M}, v, (\overline{\alpha} \circ (t .: id)) \Vdash \varphi$. We are thus left to prove that $(\overline{\alpha} \circ (t .: id))$ and $(\overline{\alpha}(t) .: \alpha)$ are the same function, which is done as above.

(Gen) Assume that $\Gamma[\uparrow] \models_l \varphi$. We show that $\Gamma \models_l \forall\varphi$. Let $\mathcal{M}$ be a model, $w \in W$ and α be an assignment. Assume that $\mathcal{M}, w, \alpha \Vdash \Gamma$. We need to show that $\mathcal{M}, w, \alpha \Vdash \forall\varphi$. Let $d \in D$. We need to show that $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$. As $w \leq v$ and $\mathcal{M}, w, \alpha \Vdash \Gamma$ we get $\mathcal{M}, v, \alpha \Vdash \Gamma$ by Lemma 9.7.1. We claim that $\mathcal{M}, v, (d .: \alpha) \Vdash \Gamma[\uparrow]$, which in combination with $\Gamma[\uparrow] \models_l \varphi$ gives us our goal: $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$. To show $\mathcal{M}, v, (d .: \alpha) \Vdash \Gamma[\uparrow]$ we use Lemma 9.7.5 and prove the equivalent statement $\mathcal{M}, v, (\overline{(d .: \alpha)} \circ \uparrow) \Vdash \Gamma$. Now, note that for all $n \in \mathbb{N}$ we have $(\overline{(d .: \alpha)} \circ \uparrow)(n) = \overline{(d .: \alpha)}(\uparrow (n)) = \overline{(d .: \alpha)}(Sn) = (d .: \alpha)(Sn) = \alpha(n)$. In other words, $(\overline{(d .: \alpha)} \circ \uparrow)$ and α are the same functions. So, our assumption $\mathcal{M}, v, \alpha \Vdash \Gamma$ gives us our claim $\mathcal{M}, v, (d .: \alpha) \Vdash \Gamma[\uparrow]$.

(EC) Assume that $\Gamma[\uparrow] \models_l \varphi \to \psi[\uparrow]$. We show that $\Gamma \models_l \exists\varphi \to \psi$. Let $\mathcal{M}$ be a model, $w \in W$ and α be an assignment. Assume that $\mathcal{M}, w, \alpha \Vdash \Gamma$. We need to show that $\mathcal{M}, w, \alpha \Vdash \exists\varphi \to \psi$. Let $v \geq w$ s.t. $\mathcal{M}, v, \alpha \Vdash \exists\varphi$. We need to show that $\mathcal{M}, v, \alpha \Vdash \psi$. As $\mathcal{M}, v, \alpha \Vdash \exists\varphi$, we get that there is a $d \in D$ s.t. $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$. Moreover, as $w \leq v$ and $\mathcal{M}, w, \alpha \Vdash \Gamma$ we get $\mathcal{M}, v, \alpha \Vdash \Gamma$ by Lemma 9.7.1. As in the previous case, we can prove that $\mathcal{M}, v, (d .: \alpha) \Vdash \Gamma[\uparrow]$. So, with our assumption $\Gamma[\uparrow] \models_l \varphi \to \psi[\uparrow]$ we obtain $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi \to \psi[\uparrow]$. It now suffices to use the reflexivity of $\leq$ and the fact that $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$ to obtain $\mathcal{M}, v, (d .: \alpha) \Vdash \psi[\uparrow]$. As we did for Γ, we can prove that the latter is equivalent to $\mathcal{M}, v, \alpha \Vdash \psi$ and we are done.

(2) (⌨) The proofs for the axioms are identical to the ones above, and the ones for the propositional rules (MP) and (sDN) are identical to the propositional Theorem 8.9.1. We are consequently left to prove that (Gen) and (EC) are sound:

(Gen) Assume that $\Gamma[\uparrow] \models_g \varphi$. We show that $\Gamma \models_g \forall\varphi$. Let $\mathcal{M}$ be a model and α be an assignment. Assume that $\mathcal{M}, \alpha \Vdash \Gamma$. We need to show that $\mathcal{M}, \alpha \Vdash \forall\varphi$. Let $w \in W$ and $d \in D$. We need to show that $\mathcal{M}, w, (d .: \alpha) \Vdash \varphi$. As $\mathcal{M}, \alpha \Vdash \Gamma$ we get $\mathcal{M}, (d .: \alpha) \Vdash \Gamma[\uparrow]$ as explained in the case for (Gen) in (1) just above. Consequently, as $\Gamma[\uparrow] \models_g \varphi$ and $\mathcal{M}, (d .: \alpha) \Vdash \Gamma[\uparrow]$, we get that $\mathcal{M}, (d .: \alpha) \Vdash \varphi$. In particular, we obtain $\mathcal{M}, w, (d .: \alpha) \Vdash \varphi$. This gives us $\mathcal{M}, w, \alpha \Vdash \forall\varphi$ as d is arbitrary.

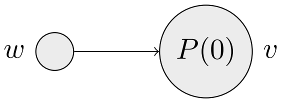
(EC) Assume that $\Gamma[\uparrow] \models_g \varphi \to \psi[\uparrow]$. We show that $\Gamma \models_g \exists\varphi \to \psi$. Let $\mathcal{M}$ be a model and α be an assignment. Assume that $\mathcal{M}, \alpha \Vdash \Gamma$. Let $w \in W$. We need to show that $\mathcal{M}, w, \alpha \Vdash \exists\varphi \to \psi$. Let $v \geq w$ such that $\mathcal{M}, v, \alpha \Vdash \exists\varphi$. We need to show that $\mathcal{M}, v, \alpha \Vdash \psi$. As $\mathcal{M}, v, \alpha \Vdash \exists\varphi$, we get that there is a $d \in D$ s.t. $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$. Moreover, as $\mathcal{M}, \alpha \Vdash \Gamma$ we get $\mathcal{M}, (d .: \alpha) \Vdash \Gamma[\uparrow]$, as explained above. So, with our assumption $\Gamma[\uparrow] \models_g \varphi \to \psi[\uparrow]$ we obtain $\mathcal{M}, (d .: \alpha) \Vdash \varphi \to \psi[\uparrow]$. In particular we have $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi \to \psi[\uparrow]$. It now suffices to use the reflexivity of $\leq$ and the fact that $\mathcal{M}, v, (d .: \alpha) \Vdash \varphi$ to obtain $\mathcal{M}, v, (d .: \alpha) \Vdash \psi[\uparrow]$. As we did for Γ, we can prove that the latter is equivalent to $\mathcal{M}, v, \alpha \Vdash \psi$.We picked an arbitrary v, giving us $\mathcal{M}, w, \alpha \Vdash \exists\varphi \to \psi$, and an arbitrary w, giving us our goal: $\mathcal{M}, \alpha \Vdash \exists\varphi \to \psi$.

■

Now that we have a proof of Theorem 9.8.1, we turn to the proof of all the claims we left pending in Section 9.4 and Section 9.5.

First, we treat Claim 9.4.1, which shows the extensional difference between FOwBIL and FOsBIL.

Proof. (⌨) We prove that $P(0) \nvdash_{\mathsf{w}} \neg{\sim}P(0)$. To do so, we use the contrapositive of soundness and are left with proving that $P(0) \not\models_l \neg{\sim}P(0)$. Consider the model $\mathcal{M}_0$ where $D = \{0\}$ and reflexive arrows are not depicted:



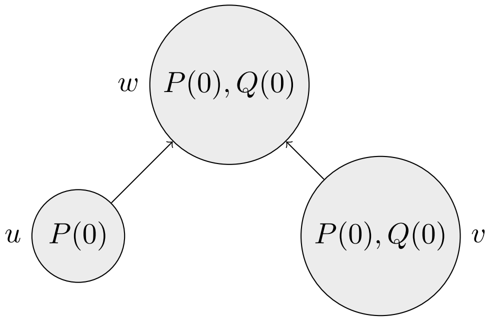
We clearly have $\mathcal{M}_0, v, \alpha \Vdash P(0)$ for $\alpha(0) = 0$, as $I_{pred}(v, P) = \{0\}$. We also have $\mathcal{M}_0, v, \alpha \nVdash \neg{\sim}P(0)$ as $\mathcal{M}_0, v, \alpha \Vdash {\sim}P(0)$ because $\mathcal{M}_0, w, \alpha \nVdash P(0)$ and $w \leq v$. Indeed, we have $\mathcal{M}_0, w, \alpha \nVdash P(0)$ as $I_{pred}(w, P) = \emptyset$. So, we obtain $P(0) \not\models_l \neg{\sim}P(0)$. ■

Second, we prove Claim 9.5.1 exhibiting a counterexample to the deduction theorem for FOsBIL.

Proof. (⌨) To prove $\nvdash_{\mathsf{s}} [\emptyset \mid P(0) \to \neg{\sim}\, P(0)]$ we use soundness and prove $\not\models_g P(0) \to \neg{\sim}P(0)$. Here again, Consider the model $\mathcal{M}_0$. We have $\mathcal{M}_0, v, \alpha \Vdash P(0)$ for $\alpha(0) = 0$, but we also obtain $\mathcal{M}_0, v, \alpha \nVdash \neg{\sim}P(0)$. Consequently, we have that there is a successor of v, i.e. v itself as $v \leq v$, such that $\mathcal{M}_0, v, \alpha \Vdash P(0)$ and $\mathcal{M}_0, v, \alpha \nVdash \neg{\sim}P(0)$. Thus, we obtain $\mathcal{M}_0, v, \alpha \nVdash P(0)\neg{\sim}P(0)$. So, we have $\not\models_g P(0) \to \neg{\sim}P(0)$. ■

Third, we give a proof for Claim 9.5.2, which shows both the failure of the dual deduction theorem for FOsBIL and the fact that the rule (DMP) cannot be safely added to FOsBIH.

Proof. (⌨) To prove $\nvdash_{\mathsf{s}} [P(0) \mid Q(0), \neg{\sim}{\sim}Q(0)]$ we use soundness and prove that $P(0) \not\models_g Q(0), \neg{\sim}{\sim}Q(0)$. Consider the model $\mathcal{M}_1$ where $D = \{0\}$:



First we have that $\mathcal{M}_1, \alpha \Vdash p$ for $\alpha(0) = 0$. We also have $\mathcal{M}_1, u, \alpha \nVdash Q(0)$ as $I_{pred}(u, Q) = \{0\}$. But we also have $\mathcal{M}_1, u, \alpha \nVdash \neg{\sim}{\sim}Q(0)$. In fact $\mathcal{M}_1, w, \alpha \Vdash {\sim}{\sim}Q(0)$ as $v \leq w$ and $\mathcal{M}_1, v, \alpha \nVdash {\sim}Q(0)$: its only predecessor is itself, and is such that $I_{pred}(v, Q) = \{0\}$. Consequently we have $P(0) \not\models_g Q(0) \vee \neg{\sim}{\sim}Q(0)$. ■

To summarize, soundness for FOwBIL and FOsBIL as given above is sufficient to show with absolute certainty that we are in presence of two different logics, both on an extensional and meta-level.

Still, soundness is only half of the work done to find a *corresponding* semantics for each logic. To obtain a corresponding semantics, we also need to prove strong completeness.

Unfortunately, we could not find a formalised proof of strong completeness for FOwBIL with respect to the local semantic consequence relation. In fact, we could neither adapt similar proofs found in the literature for neighboring logics nor find an original one.

Despite this state of things, we can explore various situations by assuming some results to hold. In Coq, this translates into using the tactic `admit`, which declares the statement we are currently trying to prove as holding. In the next section, we make use of this tactic and give an overview of the various possible situations and their consequences.

9.9 If weak is local, then strong is global

In this section, we consider three types of scenarios. First, we rule out some scenarios, as they lead to contradictions with results we solidly established. Second, we explore the expected scenario, where FOwBIL is strongly complete with respect to the local semantic consequence relation. Assuming this using `admit`, we show that FOsBIL is strongly complete with respect to the global semantic consequence relation. Third, and finally, we consider a last type of unorthodox alternative.

9.9.1 The impossibles

We show that three scenarios contradict the results we formally established this far.

First, FOwBIL cannot be strongly complete with respect to the global consequence relation.

$$\text{(A)} \quad \Gamma \models_g \Delta \quad \text{entails} \quad \vdash_{\mathsf{w}} [\Gamma \mid \Delta]$$

Why? The logic FOwBIL satisfies the deduction theorem, while the global consequence relation does clearly not: $P(0) \models_g \neg{\sim}P(0)$ but $\not\models_g P(0) \to \neg{\sim}P(0)$. So, the global consequence relation is not a semantics for FOwBIL.

Second, FOsBIL cannot be sound with respect to the local consequence relation.

$$\text{(B)} \quad \vdash_{\mathsf{s}} [\Gamma \mid \Delta] \quad \text{entails} \quad \Gamma \models_l \Delta$$

Why? We can prove $P(0) \vdash_{\mathsf{s}} \neg{\sim}P(0)$ (Claim 9.5.1), but clearly $P(0) \not\models_l \neg{\sim}P(0)$ as shown in the proof of Claim 9.4.1 in the previous section. So, the local consequence relation is not a semantics for FOsBIL.

Third, FOwBIL cannot be sound with respect to the local consequence relation *on the class of rooted frames.*

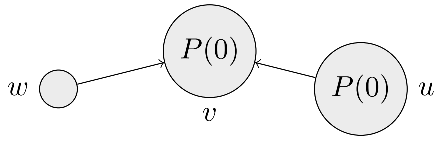
$$\text{(C)} \quad \Gamma \models_l^r \Delta \quad \text{entails} \quad \vdash_{\mathsf{w}} [\Gamma \mid \Delta]$$

Why? As we show below, we can prove $\models_l^r {\sim}P(0) \vee \neg{\sim}P(0)$, while we have $\not\models_l {\sim}P(0) \vee \neg{\sim}P(0)$ which entails $\nvdash_{\mathsf{w}} [\emptyset \mid {\sim}P(0) \vee \neg{\sim}P(0)]$ via soundness (Theorem 9.8.1 (1)). So, the local consequence relation on the class of rooted frames is not a semantics for FOwBIL. To show the correctness of our claim that (C) does not hold, we first prove that $\models_l^r {\sim}P(0) \vee \neg{\sim}P(0)$.

Lemma 9.9.1. Let $\mathcal{F} = (W, \leq)$ be a rooted Kripke frame. For any domain D, interpretation functions I_{fun} and I_{pred}, and assignment α, we have that $(W, \leq, D, I_{fun}, I_{pred}), \alpha \Vdash {\sim}P(0) \vee \neg{\sim}P(0)$.

Proof. (⌨) Let r be the root of $\mathcal{F}$, I_{fun} and I_{pred} interpretation functions, α an assignment, $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$ and $w \in W$. As $\mathcal{F}$ is rooted we have that $r \leq w$. If $\alpha(0) \in I_{pred}(r, P)$ then persistence and rootedness give $\mathcal{M}, v, \alpha \Vdash P(0)$ for every $v \in W$, hence $\mathcal{M}, w, \alpha \Vdash \neg{\sim}P(0)$. If $\alpha(0) \notin I_{pred}(r, P)$ then we get $\mathcal{M}, w, \alpha \Vdash {\sim}P(0)$ by rootedness. In each case we obtain $\mathcal{M}, w, \alpha \Vdash {\sim}P(0) \vee \neg{\sim}P(0)$. ■

Thus the formula ${\sim}P(0) \vee \neg{\sim}P(0)$ is valid on the class of rooted Kripke frames. Second, we show that there is a Kripke model $\mathcal{M}_2$ and an assignment α such that $\mathcal{M}_2, \alpha \nVdash {\sim}P(0) \vee \neg{\sim}P(0)$ (⌨). Consider the following model where reflexive arrows are omitted:



We have that $\mathcal{M}_2, u, \alpha \nVdash {\sim}P(0)$ for $\alpha(0) = 0$ as the only predecessor of u is itself and $\mathcal{M}_2, u, \alpha \Vdash P(0)$. Moreover we have that $\mathcal{M}_2, w, \alpha \nVdash P(0)$, hence $\mathcal{M}_2, v, \alpha \Vdash {\sim}P(0)$ which in turn implies $\mathcal{M}_2, u, \alpha \nVdash \neg{\sim}P(0)$. Consequently $\mathcal{M}_2, u, \alpha \nVdash {\sim}P(0) \vee \neg{\sim}P(0)$.

With these scenarios ruled out, we can turn to the expected one, which follows the propositional case.

9.9.2 The expected one

We proved in Coq that in the propositional case $\mathsf{wBI}\mathbf{L}$ and $\mathsf{sBI}\mathbf{L}$ correspond respectively to the local and global consequence relations. We expect the same correspondence to hold in the first-order case. However, as mentioned above, we could not find a way to correct Rauszer's completeness proof to obtain a formally verified proof.

While we expand on this failure in the next section, here we exhibit a formalised relative result pertaining to the expected correspondence: if weak is local, then strong is global.

Theorem 9.9.1. The statements (1) and (2) below are such that if (1) holds then (2) holds.

$$\begin{array}{lllll} (1) & \Gamma \models_l \Delta & \text{implies} & \vdash_{\mathsf{w}} [\Gamma \mid \Delta] \\ (2) & \Gamma \models_g \Delta & \text{implies} & \vdash_{\mathsf{s}} [\Gamma \mid \Delta] \end{array}$$

Proof. (⌨) Assume (1). (⌨) We prove (2). To do so, we prove the contrapositive of (2) by assuming $\nvdash_{\mathsf{s}} [\Gamma \mid \Delta]$ and proving $\Gamma \not\models_g \Delta$. Note that $\nvdash_{\mathsf{s}} [\Gamma \mid (\neg\sim)^\omega\Gamma]$, as the rule (sDN) can be applied at will in $\mathsf{FOsBI}\mathbf{H}$. So, as we have $\nvdash_{\mathsf{s}} [\Gamma \mid (\neg\sim)^\omega\Gamma]$ and $\nvdash_{\mathsf{s}} [\Gamma \mid \Delta]$, we must have $\nvdash_{\mathsf{s}} [(\neg\sim)^\omega\Gamma \mid \Delta]$ by the transitivity expressed in Theorem 9.3.1. Thus, we get $\nvdash_{\mathsf{w}} [(\neg\sim)^\omega\Gamma \mid \Delta]$ by the contrapositive of Theorem 9.4.1. Using the contrapositive of (1) on the last statement we get $(\neg\sim)^\omega\Gamma \not\models_l \Delta$. Thus, there is model $\mathcal{M} = (W, \leq, D, I_{fun}, I_{pred})$, a world $w \in W$ and an assignment α such that $\mathcal{M}, w, \alpha \Vdash (\neg\sim)^\omega\Gamma$ and $\mathcal{M}, w, \alpha \nVdash \delta$ for all $\delta \in \Delta$. Now, consider the restriction of $\mathcal{M}$ in w, i.e. the model $\mathcal{M}^w = (W^w, \leq^w, D, I_{fun}, I^w_{pred})$ as defined in Definition 9.7.2. By Lemma 9.7.1, we have that $\mathcal{M}^w, w, \alpha \Vdash (\neg\sim)^\omega\Gamma$ and for all $\delta \in \Delta$ we have $\mathcal{M}^w, w, \alpha \Vdash \delta$. If we prove that $\mathcal{M}^w, \alpha \Vdash \Gamma$, then we are done as we would then have exhibited a Γ-model $\mathcal{M}^w$ which has one point w that is a δ-point for no $\delta \in \Delta$, hence $\Gamma \not\models_g \Delta$. We thus proceed to show that $\mathcal{M}^w, \alpha \Vdash \Gamma$. Let $v \in W^w$ and $\gamma \in \Gamma$. We need to show $\mathcal{M}^w, v, \alpha \Vdash \gamma$. As $v \in W^w$ we know that there is a chain $wR_1 \dots R_n v$. We straightforwardly obtain that $\mathcal{M}^w, w, \alpha \Vdash (\neg\sim)^n\gamma$ as $\mathcal{M}^w, w, \alpha \Vdash (\neg\sim)^\omega\Gamma$. We finally obtain that $\mathcal{M}^w, v, \alpha \Vdash \gamma$ using Lemma 9.7.6 with $\mathcal{M}^w, w, \alpha \Vdash (\neg\sim)^n\gamma$ and the existence of the chain $wR_1 \dots R_n v$. As v and γ are arbitrary we get $\mathcal{M}^w, \alpha \Vdash \Gamma$. ■

This proof is similar in spirit to the proof of Theorem 8.9.1, where we use the completeness of $\mathsf{wBI}\mathbf{L}$ to obtain a modified model showing that the global consequence relation does not hold here.

In this scenario, where we `admit` (1) above, we have that $\mathsf{FOwBI}\mathbf{L}$ and $\mathsf{FOsBI}\mathbf{L}$ are conservative extensions of constant domains first-order intuitionistic logic, as expected given Rauszer [122] and Crolard's results [25]. As a consequence, we can confirm that first-order bi-intuitionistic logics are not conservative extensions of first-order intuitionistic logic [122, 92].

Theorem 9.9.2. Let $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_\mathbf{I}}$. For $i \in \{\mathsf{w}, \mathsf{s}\}$ the following holds:

$$\vdash_i [\Gamma \mid \Delta] \quad \text{iff} \quad \vdash_{\mathsf{CD}\mathbf{H}} [\Gamma \mid \Delta]$$

Proof ✍. ($\Rightarrow$) Assume $\vdash_{\mathsf{CD}\mathbf{H}} [\Gamma \mid \Delta]$. As $\mathsf{CD}\mathbf{H}$, $\mathsf{FOwBI}\mathbf{H}$ and $\mathsf{FOsBI}\mathbf{H}$ all share the same rules, and all the axioms of $\mathsf{CD}\mathbf{LH}$ are either axioms of $\mathsf{FOwBI}\mathbf{H}$ and $\mathsf{FOsBI}\mathbf{H}$ or are provable, like CD, we get that for any proof of $[\Gamma \mid \Delta]$ in $\mathsf{CD}\mathbf{LH}$ there is a proof in both $\mathsf{FOwBI}\mathbf{H}$ and $\mathsf{FOsBI}\mathbf{H}$. So, we get $\vdash_i [\Gamma \mid \Delta]$ for $i \in \{\mathsf{w}, \mathsf{s}\}$.
($\Leftarrow$) Assume $\vdash_i [\Gamma \mid \Delta]$ for $i \in \{\mathsf{w}, \mathsf{s}\}$. By Theorem 9.4.1 in both cases we get $\vdash_{\mathsf{s}} [\Gamma \mid \Delta]$. By Theorem 9.9.1 (2) and our assumption (1), we get that $\Gamma \models_g \Delta$. Given that $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_\mathbf{I}}$, we have that $\Gamma \models_g \Delta$ holds in the Kripke semantics for constant domains first-order intuitionistic logic $\mathsf{CD}\mathbf{L}$. However, by Lemma 9.7.7 we get that $\Gamma \models_l \Delta$ in the first-order Kripke semantics for $\mathsf{CD}\mathbf{L}$. By completeness of the $\mathsf{CD}\mathbf{L}$ with respect to this semantics [81] we finally get $\vdash_{\mathsf{CD}\mathbf{H}} [\Gamma \mid \Delta]$. ■

9.9.3 The surprising one

Another type of scenario, encompassing an infinity of concrete scenarios, is possible: FOwBIL is not complete with respect to the local semantic consequence relation, and FOsBIL is not complete with respect to the global semantic consequence relation. In this situation, three main possibilities can be conceived.

First, we may need to change the class of models or frames to obtain a corresponding semantics for these logics. We already know that the class of rooted frames is not a possible candidate, but there could be other classes we have not thought about so far.

Second, it may be that these logics are complete for different, more exotic, semantic consequence relations on the given Kripke semantics. Then, we would need to investigate the known possibilities.

Third, these logics could simply be Kripke incomplete: no change on the class of models, on the class of frames, or on semantic consequence relations will give birth to a corresponding semantics for these logics. Then, we would need to investigate different semantics.

However, this would be a very surprising situation given the propositional case. But given that we do not have a formal proof settling this issue, we cannot rule this situation out.

Our overview of impossible, expected, and surprising scenarios is completed. Now, we proceed to provide explanations about our failure to obtain a proof for the strong completeness result of FOwBIL. There, we exhibit frictions between a Lindenbaum lemma and existence lemmas, which are all at the heart of a well-known proof technique we present.

9.10 Frictions: Lindenbaum extensions and existence lemmas

A common proof technique for completeness results for propositional modal logics involves four components: a Lindenbaum lemma, a canonical model, a truth lemma, and an existence lemma. With the *Lindenbaum lemma*, we show how to obtain from sets of formulas a Lindenbaum extension, i.e. a *saturated* set of formulas given a certain notion of saturation. These saturated sets are then used as the points of a *canonical model*. Then the *truth lemma* aims at relating elementhood and forcing in the canonical model: if a formula is in a saturated set, then it is forced in the point corresponding to this set in the canonical model. This lemma is usually proven by induction on the structure of formulas, and when the connective under consideration is not locally defined in the semantics, i.e. involves the accessibility relation, *existence lemmas* are used to guarantee the existence of the required points for these cases. For example, in the modal case we show that if $\Box\varphi \notin w$ for a saturated set w, then there is a successor v of w in the canonical model such that $\varphi \notin v$.

We tried to follow this proof technique for our completeness proof for FOwBIL. Consequently, we first prove a Lindenbaum lemma on *closed* pairs, leading to a Lindenbaum extension which is $\exists\forall$-complete. Second, we define a canonical model based on $\exists\forall$-complete pairs. Third, we attempt to prove the truth lemma but need to `admit` two existence lemmas as we could not find a proof for them.

9.10.1 A Lindenbaum lemma on closed pairs

Here, we prove a Lindenbaum lemma on *potentially infinite closed* pairs leading to $\exists\forall$-complete pairs, as defined below.

To build these Lindenbaum extensions, we first need an encoding (Definition 6.1.1) which we assume is given.

Hypothesis 9.10.1. There is an encoding of $Form_{\mathbb{L}_{\mathbf{BI}}}$.

We fix this encoding and call it *encode*0. We then define the encoding we use in our Lindenbaum lemma.

Definition 9.10.1. We define $encode : \varphi \mapsto S\ (encode0(\varphi))$.

As in the propositional case, *encode* is injective and makes 0 the encoding of no formula. Second, we need a decoding function, as defined in Definition 6.1.4.

Lemma 9.10.1. *There is a decoding function for encode.*

Proof. (⌨) Identical to the proof of Lemma 6.1.3. ■

This lemma allows us to fix *decode*, a decoding function of *encode*. With these functions in hand, we can define a selection function crucial to our Lindenbaum extension.

Definition 9.10.2. We define the selection function sel which takes as inputs a pair of sets of formulas and a natural number, and outputs a pair of sets of formulas.

$$\mathsf{sel}([\Gamma \mid \Delta], n) = \begin{cases} [\Gamma \mid \Delta] & \text{if } decode(n) = None \\ [\Gamma, \forall\varphi \mid \Delta] & \text{if } decode(n) = Some\ \forall\varphi \text{ and } \nvdash_{\mathsf{w}} [\Gamma, \forall\varphi \mid \Delta] \\ [\Gamma \mid \Delta, \forall\varphi, \varphi[k .: id]] & \text{if } decode(n) = Some\ \forall\varphi \text{ and } \vdash_{\mathsf{w}} [\Gamma, \forall\varphi \mid \Delta] \\ & \text{and } k \notin FV(\Gamma \cup \Delta \cup \{\forall\varphi\}) \\ [\Gamma, \exists\varphi, \varphi[k .: id] \mid \Delta] & \text{if } decode(n) = Some\ \exists\varphi \text{ and } \nvdash_{\mathsf{w}} [\Gamma, \exists\varphi \mid \Delta] \\ & \text{and } k \notin FV(\Gamma \cup \Delta \cup \{\exists\varphi\}) \\ [\Gamma \mid \Delta, \exists\varphi] & \text{if } decode(n) = Some\ \exists\varphi \text{ and } \vdash_{\mathsf{w}} [\Gamma, \exists\varphi \mid \Delta] \\ [\Gamma, \varphi \mid \Delta] & \text{if } decode(n) = Some\ \varphi \text{ and } \nvdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta] \\ [\Gamma \mid \Delta, \varphi] & \text{if } decode(n) = Some\ \varphi \text{ and } \vdash_{\mathsf{w}} [\Gamma, \varphi \mid \Delta] \end{cases}$$

While the above definition may look complicated, it is in essence rather simple. If n is the encoding of no formula, then sel outputs the pair it was given (case 1). If n is the encoding of some formula ψ, then we need to look at the structure of ψ (cases 2-7).

If $\psi := \forall\varphi$, then we need to verify whether adding $\forall\varphi$ on the left of the pair makes the pair unprovable (cases 2-3). If so, simply add $\forall\varphi$ to the left of the pair (case 2). If not, add it to the right and make sure that $\forall\varphi$ is witnessed through an instantiation $\varphi[k .: id]$ for k a "fresh" variable in the context (case 3).

If $\psi := \exists\varphi$, then we need to verify whether adding $\exists\varphi$ on the left of the pair makes the pair unprovable (cases 4-5). If so, add $\exists\varphi$ to the left and make sure that $\exists\varphi$ is witnessed through an instantiation $\varphi[k .: id]$ for k a "fresh" variable in the context (case 4). If not, simply add $\exists\varphi$ to the left of the pair (case 5).

Finally, if ψ is any other formula, check for the unprovability of the pair after the addition of ψ on the left, and accordingly add ψ on the left or right (cases 6-7).

The function sel, which is inspired from Gabbay et al. [51, p.142], serves three purposes. First, if the addition of formula φ on the left of the pair $[\Gamma \mid \Delta]$ preserves the unprovability of the pair, then extend the *left* of the pair with φ. If not, extend the *right* of the pair with φ. Second, if φ is of the form $\forall\psi$ and its addition on the left does not preserve the unprovability of the pair, then extend the right-hand side of the pair not only with $\forall\psi$ but also with $\psi[k .: id]$ for some $k \in \mathbb{N}$ which is unused in the context. This way, we make sure that if a universally quantified formula on the right, then it is *instantiated* safely, using a natural number that has not been used so far. Third, if φ is of the form $\exists\psi$ and its addition on the left preserves the unprovability of the pair, then extend the left-hand side of the pair not only with $\exists\psi$ but also with $\psi[k .: id]$ for some $k \in \mathbb{N}$ which is unused in the context. This way, we instantiate existential formulas on the left-hand side of the pair in a safe way.

Then, we can define the function outputting the Lindenbaum extension of a pair.

Definition 9.10.3. We define the function Lindf, which takes as inputs a pair of sets of formulas and a natural number, and outputs a pair of sets of formulas, inductively on natural numbers.

- $\mathsf{Lindf}([\Gamma \mid \Delta], 0) = [\Gamma \mid \Delta]$
- $\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\,,\ (S\ m)\,)$

We define the *Lindenbaum extension* of a pair $[\Gamma \mid \Delta]$ as follows.

$$\mathsf{Lind}([\Gamma \mid \Delta]) = (\ \bigcup_{n:=0}^{\infty} \mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], n))\ ,\ \bigcup_{n:=0}^{\infty} \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], n))\ )$$

Here again, behind heavy notations lie simple ideas. Before detailing the above, let us recall that the projection functions $\mathsf{proj1}$ and $\mathsf{proj2}$ are defined in Chapter 3. Now, the function Lindf takes a pair $[\Gamma \mid \Delta]$ and a natural number n, and proceeds to extend step-by-step from 0 to n the pair $[\Gamma \mid \Delta]$ using at each step the function sel. With this function in hand, it is rather straightforward to obtain the Lindenbaum extension of a pair $[\Gamma \mid \Delta]$ by making the union of the left components of $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ for all $n \in \mathbb{N}$, as well as the union of all the right components of $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ for all $n \in \mathbb{N}$.

In the propositional case, we took a pair $[\Gamma \mid \Delta]$, extended Γ into a prime theory Γ', and then defined the extension Δ' of Δ as the set of formulas not deduced Γ'. So, this construction first focused on a step-by-step extension of the left-hand side of the pair, and then brutally extended the right-hand side of the pair in one go. Here instead, we build *both sides* of the pair step-by-step. The main reason for that is our need to instantiate universally quantified formulas on the right. If we did a brutal last step extending at once the right-hand side of the pair, the instantiation of universal formulas would be hard to track down. This hardness notably comes from the fact that while in classical logic we would use the drinker paradox, i.e. the formula $\exists x(\varphi \to \forall x\varphi)$, to ensure us of a witness, this formula is not a theorem of bi-intuitionistic logic.

In the remaining of this subsection, we proceed to show that the Lindenbaum extension of a *closed* pair $[\Gamma \mid \Delta]$, i.e. such that $FV(\Gamma \cup \Delta) = \emptyset$, satisfies all of the properties defined below.

Definition 9.10.4. Let $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_{\mathbf{BI}}}$ We say that $[\Gamma \mid \Delta]$ is:

1. **rich** if for every $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\exists\varphi \in \Gamma$ then $\varphi[t .: id] \in \Gamma$ for some $t \in Term_{\mathcal{S}}$;
2. **co-rich** if for every $\varphi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\forall\varphi \in \Delta$ then $\varphi[t .: id] \in \Delta$ for some $t \in Term_{\mathcal{S}}$;
3. **complete** as in Definition 8.6.1;
4. **∃∀-complete** if it is complete, rich and co-rich.

Lemma 9.10.2 (Lindenbaum Lemma)**.** If $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$ and $FV(\Gamma \cup \Delta) = \emptyset$, then there exist $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$ such that:

1. $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$;
2. $\nvdash_{\mathsf{w}} [\Gamma' \mid \bot]$;
3. $[\Gamma' \mid \Delta']$ is complete;
4. if $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi]$ then $\varphi \in \Gamma'$;
5. if $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \psi]$ then $\varphi \in \Gamma'$ or $\psi \in \Gamma'$;
6. $[\Gamma' \mid \Delta']$ is rich;
7. $[\Gamma' \mid \Delta']$ is co-rich;

8. $[\Gamma' \mid \Delta']$ is $\exists\forall$-complete.

Proof. We claim that $\mathsf{Lind}([\Gamma \mid \Delta])$ is such a pair. For convenience, let Γ' and Δ' be such that $\mathsf{Lind}([\Gamma \mid \Delta]) = [\Gamma' \mid \Delta']$. We proceed to prove each of the items.

1. We need to show $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. To do so, we first prove that at each step of the construction $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ we have an unprovable pair (⌨).

   So, we show that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], n)$ by induction on n. If $n := 0$, then we have that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], 0)$ by assumption as $\mathsf{Lindf}([\Gamma \mid \Delta], 0) = [\Gamma \mid \Delta]$. If $n := S\ m$, then we need to show that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], S\ m)$. We make a first case distinction on $decode(S\ m)$.

   (1) If $decode(S\ m) = None$, then we have the following chain of equalities.

   $$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\,,\,(S\ m)\,) = \mathsf{Lindf}([\Gamma \mid \Delta], m)$$

   However, we get by induction hypothesis that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], m)$ so we are done.

   (2) If $decode(S\ m) = Some(\varphi)$, then we need to make a case distinction on the structure of φ.

   (2-1) If $\varphi := \forall\psi$, then note that $\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\,,\,(S\ m)\,)$. There, depending on whether adding $\forall\psi$ on the left of $\mathsf{Lindf}([\Gamma \mid \Delta], m)$, we have two cases.

   (2-1-1) If $\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, then we have the following equality.

   $$\mathsf{sel}(\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\,,\,(S\ m)\,) = [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

   Then, our assumption $\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$ gives us our goal.

   (2-1-2) If $\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, then we have that $\mathsf{sel}(\,\mathsf{Lindf}([\Gamma \mid \Delta], m)\,,\,(S\ m)\,)$ is equal to the following, for some $k \in \mathbb{N}$ such that k is unused in both side of the pair $\mathsf{Lindf}([\Gamma \mid \Delta], m)$ and in $\forall\psi$.

   $$[\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\psi[k .: id]\,,\,\forall\psi\,]$$

   Note that such a k exists, as we started from a *closed* pair $[\Gamma \mid \Delta]$ and have so far used up to m variables in instantiating quantifiers, which leaves us with an infinity of unused variables. In fact, in our formalisation we pick the smallest natural number such that it is unused in the context. We still need to prove that $\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\psi[k .: id]\,,\,\forall\psi\,]$. For a contradiction assume the following.

   $$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\psi[k .: id]\,,\,\forall\psi\,]$$

   Given that k is a fresh variable in this context, the formula $\psi[k .: id]$ is equivalent here to $\forall\psi$ (see our formalised result `Gen_unused`) . So, our assuming the above ultimately amounts to assuming the following.

   $$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi\,]$$

   To obtain $\vdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], m)$, it suffices to cut the formula $\forall\psi$ (see our formalised result `Cut`) using the two following statements.

   $$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi\,]$$

   $$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\,\forall\psi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

   But this is a contradiction by our induction hypothesis.

(2-2) If $\varphi := \exists\psi$, then note that $\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ )$. There, depending on whether adding $\exists\psi$ on the left of $\mathsf{Lindf}([\Gamma \mid \Delta], m)$, we have two cases.

(2-2-1) Assume that we have the following.

$$\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

Then, we obtain that $\mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ )$ is equal to the following.

$$[\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \psi[k \mathrel{.:} id]\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

For a contradiction, we assume the following statement.

$$\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \psi[k \mathrel{.:} id]\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

Then, by definition we have that there is a finite $\Delta'' \subseteq \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))$ such that $\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \psi[k \mathrel{.:} id]\ ,\ \exists\psi \vdash_{\mathsf{w}} \bigvee\Delta''$. Thus, we can apply Theorem 9.5.1 to obtain $\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi \vdash_{\mathsf{w}} \psi[k \mathrel{.:} id] \to \bigvee\Delta''$. Now, given that k is unused in this context, it allows us to use an alternative to the rule (EC) (see our formalised result `EC_unused`) which gives us $\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi \vdash_{\mathsf{w}} \exists\psi \to \bigvee\Delta''$. So, we can reuse Theorem 9.5.1 to obtain $\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\ ,\ \exists\psi \vdash_{\mathsf{w}} \bigvee\Delta''$. However, this contradicts our assumption displayed below.

$$\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

Indeed, the left-hand sides of these pairs are identical given that repetitions do distinguish sets.

(2-2-2) If $\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, then we have the following equality.

$$\mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ ) = [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\,]$$

For a contradiction assume the following.

$$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\,]$$

Given that $\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, we can cut the formula $\exists\psi$ to obtain $\vdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], m)$, which contradicts our induction hypothesis. So, we have $\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \exists\psi\,]$.

(2-3) If φ is not universally or existentially quantified, then note that we have the following equality.

$$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ )$$

There, depending on whether adding φ on the left of $\mathsf{Lindf}([\Gamma \mid \Delta], m)$, we have two cases.

(2-3-1) If $\nvdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \varphi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, then we obtain the following equality.

$$\mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ ) = [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \varphi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$$

By our assumption we have that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], (S\ m))$.

(2-3-2) If $\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \varphi\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$, then the following equality holds.

$$\mathsf{sel}(\ \mathsf{Lindf}([\Gamma \mid \Delta], m)\ ,\ (S\ m)\ ) = [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ \mid\ \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\ ,\ \varphi\,]$$

For a contradiction assume the following.

$$\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\, \varphi\,]$$

We can then obtain $\vdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], m)$ by cutting the formula φ using the above and $\vdash_{\mathsf{w}} [\,\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,,\, \varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))\,]$. However, this contradicts our induction hypothesis.

In all possible cases we concluded that $\nvdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], (S\ m))$, so we are done. With this result in hand we can show that $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ (⌨). Assume for a contradiction that $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. By the definition of Γ' and Δ', and the fact that we can reduce the statement $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ to $\vdash_{\mathsf{w}} [\Gamma'' \mid \Delta'']$ where Γ'' and Δ'' are finite, notably using Lemma 9.3.2. The finiteness of Γ'' and Δ'' entails that there must be a $n \in \mathbb{N}$ such that $\mathsf{Lindf}([\Gamma \mid \Delta], n)$ extends $[\Gamma'' \mid \Delta'']$. Consequently, we obtain $\vdash_{\mathsf{w}} \mathsf{Lindf}([\Gamma \mid \Delta], n)$ which contradicts our result above. So, we obtain $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$.

2. (⌨) We need to show $\nvdash_{\mathsf{w}} [\Gamma' \mid \bot]$. Assume for a contradiction that $\vdash_{\mathsf{w}} [\Gamma' \mid \bot]$. As a consequence, we easily get $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ which contradicts item 1 above.

3. (⌨) We need to show that $[\Gamma' \mid \Delta']$ is complete. Let $\varphi \in \mathit{Form}_{\mathbb{L}_{\mathbf{BI}}}$. We know there is a $n \in \mathbb{N}$ such that $\mathit{decode}(n) = \varphi$. By case analysis on the structure of φ, we can show that the latter either appears on the left or the right of $\mathsf{Lindf}([\Gamma \mid \Delta], n)$. As a consequence, it must appear on the right or left of $\mathsf{Lind}([\Gamma \mid \Delta]) = [\Gamma' \mid \Delta']$ by definition. So, we have that $\varphi \in \Gamma'$ or $\varphi \in \Delta'$.

4. (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi]$. We need to show $\varphi \in \Gamma'$. Given that $[\Gamma' \mid \Delta']$ is complete by item 2, we have that $\varphi \in \Gamma'$ or $\varphi \in \Delta'$. In the former case we are directly done, so let us consider the latter. If $\varphi \in \Delta'$, we get that $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ as $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi]$. So, the pair $[\Gamma' \mid \Delta']$ is provable, which is in contradiction with item 1.

5. (⌨) Assume $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \psi]$. We need to show $\varphi \in \Gamma'$ or $\psi \in \Gamma'$. Assume for *reductio* that $\varphi \notin \Gamma'$ or $\psi \notin \Gamma'$. As $[\Gamma' \mid \Delta']$ is complete by item 2, we get that $\varphi \in \Delta'$ and $\psi \in \Delta'$. However, we directly obtain $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ as $\{\varphi, \psi\} \subseteq \Delta'$ and $\vdash_{\mathsf{w}} [\Gamma' \mid \bigvee\{\varphi, \psi\}]$ i.e. $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi \vee \psi]$. This is in contradiction with item 1. So, we have that $\varphi \in \Gamma'$ or $\psi \in \Gamma'$.

6. (⌨) We need to show that $[\Gamma' \mid \Delta']$ is rich. Let $\exists\varphi$ such that $\exists\varphi \in \Gamma'$. Let $n = \mathit{decode}(\exists\varphi)$ and consider $\mathsf{Lindf}([\Gamma \mid \Delta], n)$. Note that $n = S\ m$ for some m, as 0 is the encoding of no formula by definition. Thus, we have the following equality.

$$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\ m))$$

Now, we need to check whether adding $\exists\varphi$ to the left of the pair $\mathsf{Lindf}([\Gamma \mid \Delta], m)$ makes the pair provable. If it does, i.e. we have that $\vdash_{\mathsf{w}} [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \exists\varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$, we get the following.

$$\mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\ m)) = [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \exists\varphi]$$

So, as a consequence we would have that $\exists\varphi$ is in Δ' by definition of $\mathsf{Lind}([\Gamma \mid \Delta])$. But that implies that $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ as $\exists\varphi \in \Gamma$ by assumption, which is a contradiction as shown in item 1. Now, assume the following.

$$\nvdash_{\mathsf{w}} [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \exists\varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$$

Then we obtain the following equality for some k satisfying the condition below.

$$\mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\,m)) = [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \exists\varphi, \varphi[k .: \mathit{id}] \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$$

$$k \notin \mathit{FV}(\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \cup \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \cup \{\exists\varphi\})$$

As a consequence, we have that $\varphi[k .: \mathit{id}] \in \Gamma'$ by definition of $\mathsf{Lind}([\Gamma \mid \Delta])$.

7. (⌨) We need to show that $[\Gamma' \mid \Delta']$ is co-rich. Let $\forall\varphi$ such that $\forall\varphi \in \Delta'$. Let $n = \mathit{decode}(\forall\varphi)$ and consider $\mathsf{Lindf}([\Gamma \mid \Delta], n)$. Note that $n = S\ m$ for some m, as 0 is the encoding of no formula by definition. Thus, we have that the following equality holds.

$$\mathsf{Lindf}([\Gamma \mid \Delta], (S\ m)) = \mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\ m))$$

Now, we need to check whether adding $\forall\varphi$ to the left of the pair $\mathsf{Lindf}([\Gamma \mid \Delta], m)$ makes the pair provable. If it does, i.e. we have that $\vdash_{\mathsf{w}} [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \forall\varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$, we get the following equality for some k satisfying the condition below.

$$\mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\ m)) = [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \forall\varphi, \varphi[k .: id]]$$

$$k \notin FV(\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \cup \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m)) \cup \{\exists\varphi\})$$

As a consequence, we have that $\varphi[k .: id] \in \Delta'$ by definition of $\mathsf{Lind}([\Gamma \mid \Delta])$. If we have that $\nvdash_{\mathsf{w}} [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \forall\varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$, then the following holds.

$$\mathsf{sel}(\mathsf{Lindf}([\Gamma \mid \Delta], m), (S\ m)) = [\mathsf{proj1}(\mathsf{Lindf}([\Gamma \mid \Delta], m)), \forall\varphi \mid \mathsf{proj2}(\mathsf{Lindf}([\Gamma \mid \Delta], m))]$$

So, as a consequence we would have that $\forall\varphi$ is in Γ' by definition of $\mathsf{Lind}([\Gamma \mid \Delta])$. But that implies that $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ as $\forall\varphi \in \Delta$ by assumption, which is a contradiction as shown in item 1.

8. (⌨) We need to show that $[\Gamma' \mid \Delta']$ is $\exists\forall$-complete. We get the latter as the above items show that $[\Gamma' \mid \Delta']$ is complete, rich and co-rich.

■

So, with the above lemma we have that the Lindenbaum extension of a closed and unprovable pair $[\Gamma \mid \Delta]$ is $\exists\forall$-complete.

Note that our requirement that the initial pair is closed is a way to make sure that we have enough "safe" variables to use in the process of witnessing universal formulas on the right and existential formulas on the left, while staying in the same signature.

9.10.2 Towards a truth lemma through existence lemmas

At this point, we know that unprovable closed pairs can be extended in an unprovable $\exists\forall$-complete pairs. So, we can follow the proof technique outlined at the beginning of this section and define a canonical model with unprovable $\exists\forall$-complete pairs as points.

Definition 9.10.5. The canonical model $\mathcal{M}^c = (W^c, \leq^c, D^c, I^c_{fun}, I^c_{pred})$ is defined in the following way:

1. $W^c = \{[\Gamma \mid \Delta] : [\Gamma \mid \Delta]$ is $\exists\forall$-complete and $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]\}$;

2. $[\Gamma_1 \mid \Delta_1] \leq^c [\Gamma_2 \mid \Delta_2]$ iff $\Gamma_1 \subseteq \Gamma_2$;

3. $D^c = Term_{\mathcal{S}}$;

4. $I^c_{fun}(f) = \{(t_1, \ldots, t_n) \mapsto f(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in D^c\}$, for every n-ary function symbol f;

5. $I^c_{pred}([\Gamma \mid \Delta], P) = \{(t_1, \ldots, t_n) \in D^c \mid P(t_1, \ldots, t_n) \in \Gamma\}$, for every $[\Gamma \mid \Delta]$ and n-ary relation symbol P.

The canonical assignment α^c is defined in the following way: $\alpha^c = id$.

Before turning to the topic of the truth lemma and its proof, we obtain a handy result about the canonical assignment. In essence, it shows that terms are interpreted as themselves in the domain D^c.

Lemma 9.10.3. For every $t \in Terms_{\mathcal{S}}$, we have $\overline{\alpha^c}(t) = t$.

Proof. (⌨) We reason by induction on the structure of t:

- $t := x$: then $\overline{\alpha^c}(x) = \alpha^c(x) = x$ by definition of α^c.
- $t := f(t_1, \dots, t_n)$: we have that $\overline{\alpha^c}(f(t_1, \dots, t_n)) = I^c_{fun}(f)(\overline{\alpha^c}(t_1), \dots, \overline{\alpha^c}(t_n))$. By induction hypothesis we have that $\overline{\alpha^c}(t_i) = t_i$ for $1 \leq i \leq n$. Consequently $\overline{\alpha^c}(f(t_1, \dots, t_n)) = I^c_{fun}(f)(t_1, \dots, t_n) = f(t_1, \dots, t_n)$ by definition of I^c_{fun}.

■

In addition to this lemma, we can prove that in the first-order case the pairs constituting the canonical model are also prime and closed under deducibility.

Lemma 9.10.4. Let $\Gamma \cup \Delta \subseteq Form_{\mathbb{L}_{\mathbf{BI}}}$. If $[\Gamma \mid \Delta]$ is $\exists\forall$-complete and $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$, then $[\Gamma \mid \Delta]$ is prime and closed under deducibility.

Proof. Identical to the proof of Lemma 8.8.3 (⌨) (⌨). ■

As explained at the beginning of Section 9.10, to prove the truth lemma we require existence lemmas for connectives that are not locally defined in the semantics. In the bi-intuitionistic case, there are two such connectives: $\rightarrow$ and $\prec$. So, we need an existence lemma for each to let the truth lemma go through. Unfortunately, while so far, all the results and constructions of this section are formalised, we could not find a proof for the existence lemmas below.

Hypothesis 9.10.2 (Implication Existence Lemma)**.** For all $[\Gamma \mid \Delta] \in W^c$ and $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\varphi \rightarrow \psi \in \Delta$ then there is a $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ and $\varphi \in \Gamma'$ and $\psi \in \Delta$.

Hypothesis 9.10.3 (Exclusion Existence Lemma)**.** For all $[\Gamma \mid \Delta] \in W^c$ and $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\varphi \prec \psi \in \Gamma$ then there is a $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma' \mid \Delta'] \leq^c [\Gamma \mid \Delta]$ and $\varphi \in \Gamma'$ and $\psi \in \Delta$.

Still, in Coq we can declare these statements as `Hypothesis` (see ⌨ and ⌨). So, in this context we can proceed to hypothetical reasoning using these lemmas. In fact, through their use we can prove the truth lemma.

Lemma 9.10.5 (Truth Lemma)**.** For every $[\Gamma \mid \Delta] \in W^c$:

$$\varphi \in \Gamma \quad \text{iff} \quad \mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi.$$

Proof. (⌨) By induction on φ:

- $\varphi := P(t_1, \dots, t_n)$: we have $P(t_1, \dots, t_n) \in \Gamma$ iff $(t_1, \dots, t_n) \in I^c_{pred}([\Gamma \mid \Delta], P)$ by definition of the canonical model. The last element is equivalent to $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash P(t_1, \dots, t_n)$ by definition and Lemma 9.10.3.
- $\psi := \top$: we have that $\top \in \Gamma$ as otherwise $\top \in \Delta$ by completeness, which leads to the contradiction $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$ as $\Gamma \vdash_{\mathsf{w}} \top$. In addition to that, we have $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \top$ by definition. So, we have $\top \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \top$ trivially.
- $\psi := \bot$: we have that $\bot \notin \Gamma$ as otherwise $\bot \in \Gamma$ by completeness, which leads to the contradiction $\vdash_{\mathsf{w}} [\Gamma' \mid \Delta']$. In addition to that, we have $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \nVdash \bot$ by definition. So, we have $\bot \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \bot$ trivially.
- $\varphi := \varphi_1 \wedge \varphi_2$: we have that $\varphi_1 \wedge \varphi_2 \in \Gamma$ iff $\varphi_1 \in \Gamma$ and $\varphi_2 \in \Gamma$, as $[\Gamma \mid \Delta]$ is $\exists\forall$-complete hence Γ is deductively closed by Lemma 9.10.4. By induction hypothesis this holds if and only if $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1$ and $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_2$. Then $\varphi_1 \wedge \varphi_2 \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \wedge \varphi_2$.

- $\varphi := \varphi_1 \vee \varphi_2$: we have that $\varphi_1 \vee \varphi_2 \in \Gamma$ iff $[\varphi_1 \in \Gamma$ or $\varphi_2 \in \Gamma]$ by Lemma 9.10.4. By induction hypothesis this holds if and only if $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1$ or $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_2$. Then $\varphi_1 \vee \varphi_2 \in \Gamma$ iff $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \vee \varphi_2$.

- $\varphi := \varphi_1 \to \varphi_2$: ($\Rightarrow$) Assume $\varphi_1 \to \varphi_2 \in \Gamma$. We need to show that $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \to \varphi_2$. Let $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$, and assume $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_1$. We need to show that $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_2$. By the induction hypothesis, it is sufficient to show that $\varphi_2 \in \Gamma'$. From $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_1$ we obtain $\varphi_1 \in \Gamma'$ by induction hypothesis. Also, as $[\Gamma \mid \Delta] \leq^c, \alpha^c[\Gamma' \mid \Delta']$, we have $\varphi_1 \to \varphi_2 \in \Gamma \subseteq \Gamma'$. It is straightforward to prove that $\Gamma' \vdash_{\mathsf{w}} \varphi_2$ using (MP). As $[\Gamma' \mid \Delta']$ is complete and unprovable we get $\varphi_2 \in \Gamma'$. The induction hypothesis gives us $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_2$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \to \varphi_2$. Assume for reductio that $\varphi_1 \to \varphi_2 \notin \Gamma$. Then by Hypothesis 9.10.2 we get that there is $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ and $\varphi_1 \in \Gamma'$ and $\varphi_2 \in \Delta_2$. By induction hypothesis we get $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \nVdash \varphi_2$. But as $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ and $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \to \varphi_2$ we reached a contradiction. So $\varphi_1 \to \varphi_2 \in \Gamma$.

- $\varphi := \varphi_1 \mathrel{-\!\!<} \varphi_2$: ($\Rightarrow$) Assume $\varphi_1 \mathrel{-\!\!<} \varphi_2 \in \Gamma$. By Hypothesis 9.10.3 we get that there is $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma' \mid \Delta'] \leq^c [\Gamma \mid \Delta]$ and $\varphi_1 \in \Gamma'$ and $\varphi_2 \in \Delta_2$. Consequently we get $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \nVdash \varphi_2$ by induction hypothesis. But we have that $[\Gamma' \mid \Delta'] \leq^c [\Gamma \mid \Delta]$, so $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \mathrel{-\!\!<} \varphi_2$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi_1 \mathrel{-\!\!<} \varphi_2$. Then, there is $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma' \mid \Delta'] \leq [\Gamma \mid \Delta]$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \varphi_1$ and $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \nVdash \varphi_2$. By induction hypothesis we obtain that $\varphi_1 \in \Gamma'$ and $\varphi_2 \notin \Gamma'$. Note that $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi_1 \to (\varphi_2 \vee (\varphi_1 \mathrel{-\!\!<} \varphi_2))]$ as the formula on the right is an instance of A_{11}. Consequently, given that $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi_1]$ as $\varphi_1 \in \Gamma'$, and $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi_1 \to (\varphi_2 \vee (\varphi_1 \mathrel{-\!\!<} \varphi_2))]$, we obtain via (MP) that $\vdash_{\mathsf{w}} [\Gamma' \mid \varphi_2 \vee (\varphi_1 \mathrel{-\!\!<} \varphi_2)]$. Then we use the closure under deducibility and the primeness of Γ', obtained via Lemma 9.10.4, to get that $\varphi_2 \in \Gamma'$ or $\varphi_1 \mathrel{-\!\!<} \varphi_2 \in \Gamma'$. As the case where $\varphi_2 \in \Gamma'$ leads to a contradiction, we have that $\varphi_1 \mathrel{-\!\!<} \varphi_2 \in \Gamma'$. But $\Gamma' \subseteq \Gamma$ as $[\Gamma' \mid \Delta'] \leq [\Gamma \mid \Delta]$, so $\varphi_1 \mathrel{-\!\!<} \varphi_2 \in \Gamma$.

- $\varphi := \forall\varphi$: ($\Rightarrow$) Assume $\forall\varphi \in \Gamma$. We need to show that $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \forall\varphi$. Let $d \in D^c$. We need to show $\mathcal{M}^c, [\Gamma \mid \Delta], (d \mathrel{.:} \alpha^c) \Vdash \varphi$. Note that $d \in \mathit{Term}_{\mathcal{S}} = D^c$. Using $\forall\varphi \in \Gamma$ and the fact that $[\Gamma \mid \Delta]$ is $\exists\forall$-complete we get that $\varphi[d \mathrel{.:} \mathit{id}] \in \Gamma$. Thus, we apply the induction hypothesis to obtain $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi[d \mathrel{.:} \mathit{id}]$. In turn, we apply Lemma 9.7.5 to get $\mathcal{M}^c, [\Gamma \mid \Delta], (\overline{\alpha^c} \circ (d \mathrel{.:} \mathit{id})) \Vdash \varphi$. We use Lemma 9.7.3 to obtain $\mathcal{M}^c, [\Gamma \mid \Delta], (d \mathrel{.:} \alpha^c) \Vdash \varphi$. So, we need to prove that for all $n \in \mathbb{N}$ we have $(\overline{\alpha^c} \circ (d \mathrel{.:} \mathit{id}))(n) = (d \mathrel{.:} \alpha^c)(n)$. We prove this by case distinction on n. If $n := 0$ we have the following chain of equalities.

  $$(\overline{\alpha^c} \circ (d \mathrel{.:} \mathit{id}))(0) = \overline{\alpha^c}((d \mathrel{.:} \mathit{id})(0)) = \overline{\alpha^c}(d) = d = (d \mathrel{.:} \alpha^c)(0)$$

  If $n := S\ m$, then we have the following chain of equalities.

  $$(\overline{\alpha^c} \circ (d \mathrel{.:} \mathit{id}))(S\ m) = \overline{\alpha^c}((d \mathrel{.:} \mathit{id})(S\ m)) = \overline{\alpha^c}(\mathit{id}(m)) = \overline{\alpha^c}(m) = (d \mathrel{.:} \alpha^c)(S\ m)$$

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \forall\varphi$. Assume for reductio that $\forall\varphi \notin \Gamma$. Then we get that $\forall\varphi \in \Delta$. As the pair $[\Gamma \mid \Delta]$ is co-rich we get that there is a $n \in \mathbb{N}$ such that $\varphi[n \mathrel{.:} \mathit{id}] \in \Delta$. By unprovability of the pair $[\Gamma \mid \Delta]$ we thus get $\varphi[n \mathrel{.:} \mathit{id}] \notin \Gamma$. By induction hypothesis we obtain $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \nVdash \varphi[n \mathrel{.:} \mathit{id}]$. But this is a contradiction as it implies that $\mathcal{M}^c, [\Gamma \mid \Delta], (n \mathrel{.:} \alpha^c) \nVdash \varphi$ as explained above, while we have $\mathcal{M}^c, [\Gamma \mid \Delta], (n \mathrel{.:} \alpha^c) \Vdash \varphi$ from $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \forall\varphi$.

- $\varphi := \exists\varphi$: ($\Rightarrow$) Assume $\exists\varphi \in \Gamma$. Thus, as $[\Gamma \mid \Delta]$ is rich there is $n \in \mathbb{N}$ such that $\varphi[n .\!: id] \in \Gamma$. By induction hypothesis we get $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi[n .\!: id]$. This implies $\mathcal{M}^c, [\Gamma \mid \Delta], (n .\!: \alpha^c) \Vdash \varphi$ as argued above. Hence $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \exists\varphi$.

  ($\Leftarrow$) Assume $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \exists\varphi$. Thus we get that there is a $d \in D^c$ such that $\mathcal{M}^c, [\Gamma \mid \Delta], (d .\!: \alpha^c) \Vdash \varphi$. Note that $d \in Terms_{\mathcal{S}} = D^c$. We reason as above to get that $\mathcal{M}^c, [\Gamma \mid \Delta], \alpha^c \Vdash \varphi[n .\!: id]$. By induction hypothesis we obtain $\varphi[n .\!: id] \in \Gamma$. Consequently, we get $\exists\varphi \in \Gamma$ as the latter is closed under deduction.

■

While we are still reasoning in an hypothetical way, under the assumption that Hypothesis 9.10.2 and Hypothesis 9.10.3 hold, we obtain a completeness result for FOwBIL on *closed* pairs. This result makes crucial use of the Lemma 9.10.5.

Theorem 9.10.1. Let $[\Gamma \mid \Delta]$ be such that $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$ and $FV(\Gamma \cup \Delta) = \emptyset$. The following holds:

$$\Gamma \models_l \Delta \quad \text{implies} \quad \vdash_{\mathsf{w}} [\Gamma \mid \Delta]$$

Proof. (⌨) Assume $\nvdash_{\mathsf{w}} [\Gamma \mid \Delta]$. Lemma 9.10.2 gives us a $\exists\forall$-complete pair $[\Gamma' \mid \Delta']$ such that $\nvdash_{\mathsf{w}} [\Gamma' \mid \Delta']$, where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Moreover there is no $\delta \in \Delta$ such that $\delta \in \Gamma'$, so by Lemma 9.10.5 we obtain that in the canonical model of Definition 9.10.5 the following holds: $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \nVdash \delta$ for every $\delta \in \Delta$, while $\mathcal{M}^c, [\Gamma' \mid \Delta'], \alpha^c \Vdash \Gamma$. Consequently, we have that $\Gamma \not\models_l \Delta$. ■

To summarize, by admitting Hypothesis 9.10.2 and Hypothesis 9.10.3 we can prove the truth lemma, which in turn gives us completeness on closed pairs for FOwBIL. In this hypothetical setting, we can use the well-established Theorem 9.9.1, which also holds for closed pairs, to get the full picture: weak is local, strong is global. So, by assuming that Hypothesis 9.10.2 and Hypothesis 9.10.3 hold, we retrieve a configuration identical to the propositional one.

9.10.3 Frictions

There is a burning question we have not answered: why couldn't we prove these hypotheses? In this subsection, we provide some insights into the issues we encountered.

First, we recall how the propositional version of Hypothesis 9.10.2 is proved. Note that this result is embedded in the truth lemma in Section 8.8, but is not explicitly proved as a separate lemma.

Lemma 9.10.6. For all $[\Gamma \mid \Delta] \in W^c$ and $\varphi, \psi \in Form_{\mathbb{L}_{\mathbf{BI}}}$, if $\varphi \to \psi \in \Delta$ then there is a $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ and $\varphi \in \Gamma'$ and $\psi \in \Delta$.

Proof. Assume $\varphi \to \psi \in \Delta$. Then, we notably have that $\Gamma \nvdash_{\mathsf{w}} \varphi \to \psi$ as $[\Gamma \mid \Delta]$ is unprovable. Consequently, we get by $\Gamma, \varphi \nvdash_{\mathsf{w}} \psi$ Theorem 8.6.1. Then, using the Lindenbaum Lemma 8.8.2 we obtain a complete and unprovable pair $[\Gamma' \mid \Delta']$ such that $\Gamma \cup \{\varphi\} \subseteq \Gamma'$ and $\{\psi\} \subseteq \Delta'$. We easily obtain that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ as $\Gamma \subseteq \Gamma'$. So, we found $[\Gamma' \mid \Delta'] \in W^c$ such that $[\Gamma \mid \Delta] \leq^c [\Gamma' \mid \Delta']$ and $\varphi \in \Gamma'$ and $\psi \in \Delta$. ■

The argument is rather straightforward and involves elements for which we have first-order equivalents. So, what goes wrong when we try to port this proof to the first-order case? In essence, the Lindenbaum Lemma 9.10.2 is not applicable in this context. Indeed, the initial pair $[\Gamma \mid \Delta]$ is an element of the canonical model, i.e. an unprovable $\exists\forall$-complete pair. So, the Lindenbaum lemma is inapplicable on the unprovable pair $[\Gamma, \varphi \mid \psi]$ as it is *not closed*. Indeed, for all formula φ, including the ones with "free" variables, either $\varphi \in \Gamma$ or $\sim\varphi \in \Gamma$ by Lemma 9.10.4 and the fact that $\varphi \vee \sim\varphi$ is a theorem.

So, in the first-order case, we cannot perform the step using the Lindenbaum lemma as in the propositional case.

Naturally, one could be tempted to simply *release this restriction* on the Lindenbaum lemma: start from an unprovable pair $[\Gamma \mid \Delta]$ (not necessarily closed), and extend it. However, in this situation we could encounter a pair that already contains all natural numbers as "free" variables, making any instantiation of an existential or universal formula potentially unsafe: it could break unprovability. In a way, we need to have an infinite stock of natural numbers which are safe to use for instantiations.

In a similar spirit, one could think about *extending the signature* with a copy of natural numbers. Then, we could consider any unprovable pair $[\Gamma \mid \Delta]$ in a signature $\mathcal{S}_0$ and extend it in an unprovable $\exists\forall$-complete pair in the signature $\mathcal{S}_1$, which contains one more copy of natural numbers than $\mathcal{S}_0$ does. This trick of shifting to another signature does indeed give us enough safe unused variables for all instantiations. However, we encounter a similar issue as in the initial case: the new Lindenbaum Lemma becomes inapplicable in the given context. In this situation, the canonical model is built on unprovable and $\exists\forall$-complete pairs in the extended signature $\mathcal{S}_1$. So, when we consider $[\Gamma \mid \Delta] \in W^c$, we are facing such a pair. Consequently, when we try to apply the Lindenbaum lemma to $[\Gamma, \varphi \mid \psi]$ we are stuck: this pair is expressed in $\mathcal{S}_1$ and not $\mathcal{S}_0$. Here again, we are unable to proceed further.

It seems that to obtain a completeness proof through a canonical model we need to consider another proof technique. Grigory Olkhovikov informed us in May 2022 via private communication of a discovery of his: Dieter Klemke may have a correct proof of completeness for first-order bi-intuitionistic logic using a canonical model [81]. There, he defines a notion of "universal bush", which seems to be a version of the Lindenbaum Lemma where families of extensions are created at each step, giving more controlled machinery. Unfortunately, we could not attempt at formalising his proof in the time we had left for this dissertation, notably because the article is written in German. We intend to investigate this direction in further work.

9.11 Why Rauszer's proofs are erroneous

We do not possess a formalised completeness proof for FOwBIL, because of the frictions exhibited above, but we collected enough results to explain why Rauszer's proofs are erroneous. The errors these proofs contain are similar to the ones mentioned in Section 8.11.

The first pertains to an issue already pointed out in the propositional case: behind bi-intuitionistic logic lie two distinct logics, understood as consequence relations, i.e. a weak (wBIL) and strong (sBIL) version. This distinction also shows in the first-order case, separating FOBIL into FOwBIL and FOsBIL as we showed in Section 9.8.1. In Rauszer's proofs, both logics are confused as properties proper to each logic are used indistinguishably, as explained in Section 8.11.

Second, the structure of the first-order Kripke frames used in both proofs to show completeness leads to a contradiction. These frames are *rooted*, i.e. they contain a point, the *root*, from which all other points are reachable through the accessibility relation. Thus, the use of these models in the completeness proofs shows completeness with respect to the class of *rooted* first-order Kripke frames. However, this class of frames validates the formula $\sim P(0) \vee \neg\sim P(0)$ (Lemma 9.9.1), which is not a theorem of FOwBIL or FOsBIL as argued in the impossible scenarios of Section 9.9.

Finally, the first proof exploits the Lemma 8.11.1 from Gabbay [49], showing the completeness of the fragment of FOBIL containing the logical symbols $\wedge, \rightarrow, \neg$ and $\forall$ with respect to the class of first-order Kripke frames with constant domains. Rauszer added Lemma 8.11.2, which is dual to Gabbay's lemma, showing that the fragment of FOBIL containing the dual of each of the symbols above is also complete in some way with respect to the same class of frames. These results are then combined to show that **FOBIL** *with all symbols* is complete with respect to this class. However, the application of these lemmas in this context is incorrect as symbols from *both fragments* are interleaved in the formulas under consideration.

9.12 Conclusion

The use of generalized Hilbert calculi here again allowed us to clearly distinguish two logics, **FOsBIL** and **FOwBIL**, emerging from the obviously different rules (wDN) and (sDN). The logics **FOwBIL** and **FOsBIL** are distinguishable not only on an extensional level but also on the meta-level through the deduction-detachment theorem, its dual and the dual of the rule (MP). All these distinctions are identical to the propositional ones, distinguishing **wBIL** and **sBIL**. So, the *same* phenomenon occurs in the first-order bi-intuitionistic logic, stressing the need to consider logics as consequence relations in this context as well.

As we have shown, the logics **FOwBIL** and **FOsBIL** are respectively sound for the local and global semantic consequence relations on the class of all first-order Kripke frames. This result is sufficient to show with certainty not only the differences between the two logics, but also the incompleteness of the latter with respect to the local and global relations on the class of rooted frames.

We investigated various scenarios for completeness, which led us to the educated guess that **FOwBIL** and **FOsBIL** must be respectively complete with respect to the local and global semantic consequence relations on the class of all first-order Kripke frames. Not only we ruled out some plausible scenarios, where rooted frames are considered, for example, but we showed that if "**FOwBIL** is local" then "**FOsBIL** is global". Thus, we were only left with finding a proof for the completeness of **FOwBIL** to complete the picture.

Unfortunately, we exhibited in a final moment the issues we encountered when trying to adapt a common proof technique for completeness to the case of **FOwBIL**. In essence, frictions between the Lindenbaum lemma and existence lemmas, pertaining to the unavailability of unused variables, prevent us from pushing the proof technique through. So, we could only complete a proof of completeness for **FOwBIL** by admitting the aforementioned existence lemmas without proving them.

There are two lines of work we wish to pursue next. First, we intend to build on the work on first-order bi-intuitionistic logics we formalised this far by formalising and/or adapting Klemke's proof [81]. With such a proof in hand, we would finally have solid, formalised, and reliable foundations for these logics. Second, we would like to investigate the possibility of obtaining a first-order bi-intuitionistic logic that does not deduce the constant domain axiom. As explained in Section 9.6, we cannot escape this theorem in the given setting. However, there are two conceivable ways of avoiding it. A first approach would consist in using an existence predicate E [132], determining which terms refer to existing entities, and which allows a better control over the quantifiers: the latter are defined on existing entities, and not on all entities. A second approach follows a line of research developed by van Benthem [157] and conceiving of quantifiers as modal operators. There, we can define *substructural* quantifiers that are weaker than the one we defined in this dissertation.

Chapter 10

Afterwords on Bi-Intuitionistic Logics

The lesson we learnt is clear: for the last fifty years, we grounded our work on bi-intuitionistic logic on unreliable foundations. For this long, we have been relying on Cecylia Rauszer's seminal work on this logic, which unfortunately contains critical mistakes.

Both in the propositional and first-order case, we can clearly identify the source of Rauszer's mistakes: behind what is usually conceived of as a unique logic, lies two different logics. This phenomenon, which happens in both propositional and first-order bi-intuitionistic logic, can be traced back to the use of traditional axiomatic calculi. With these calculi logics are not consequence relations but sets of theorems, leading to a striking ambiguity in rules like (DN) and (Nec) in the bi-intuitionistic and modal case. It is in this ambiguity that the confusion between two distinct rules, hence two distinct logics, nests.

We eradicated this confusion by distinguishing a *weak* and *strong* logic in each case: the propositional wBIL and sBIL, and the first-order FOwBIL and FOsBIL. To do so we relied on the use of generalized Hilbert calculi, which are aimed at capturing consequence relations by taking consecutions of the shape $\Gamma \vdash \varphi$ as first-class citizens. In this setting the rule (DN) splits into two rules (wDN) and (sDN), defining the weak and strong logics respectively. This is extremely similar to the modal case, where (Nec) splits into (wNec) and (sNec), giving birth to the two logics wKL and sKL. The similarities with modal logic do not stop here, as the weak and strong bi-intuitionistic logics differ from each other analogously to how wKL and sKL do. First, wBIL and sBIL as well as FOwBIL and FOsBIL interact extensionally as sets in an identical way to wKL and sKL: while they share the same set of theorems, the former is a subset of the latter. Second, some meta-level features like the deduction theorem, and its dual in the bi-intuitionistic case, are not shared by both logics. More precisely, in each case the weak logic, i.e. wKL, wBIL and FOwBIL, satisfies the (dual) deduction theorem, while the strong logic, i.e. sKL, sBIL and FOsBIL, only satisfies a modified version of it. Finally, the weak logic corresponds to the local (Kripke) semantic consequence relation, while the strong logic corresponds to the global semantic consequence relation. This well-known fact in the modal case was ported to the propositional bi-intuitionistic case with absolute certainty, but could not be entirely obtained in the first-order case. However, we expect this correspondence to hold for first-order bi-intuitionistic logic as it is the most plausible scenario out of the ones we considered.

To prevent our work from being the source of another fifty years of confusions we formalised all our results in Coq. As a consequence, we formalised rather common elements in a (modal) logician's life: axiomatic systems, results about them, Kripke semantics, Lindenbaum lemma, canonical model constructions, soundness and completeness results. By doing so, we have shown that the technology of interactive theorem provers is ready for use: much of the results published about (modal) logic are amenable to formalisation. This statement even holds for first-order results, now that there are readily available libraries formalising first-order syntax like the one we used in this dissertation.

Still, our results and their formalisation could be improved in at least two ways. First, a completeness proof for FOwBIL needs to be found. We did our best to adapt the usual proofs via canonical model construction, but could not succeed because of the frictions we exhibited. However, we can hope to formalise the proof contained in Klemke's article [81]. Unfortunately, this article is in German, a language we do not speak. So, we intend to collaborate with a German-speaking researcher to explore the formalisability of this proof. Second, we can investigate the level of non-constructivity of our proofs. We found convenient to use classical logic to complete our proofs, most of the time through the use of the law of the excluded middle. However, it could very well be that our proofs, notably completeness ones, could rely on less powerful non-constructive principles. Thus, by relying on our formalisation we can contribute to an already existing program of reverse mathematics on completeness proofs via canonical model constructions [47].

Part III

Provability Logics

Chapter 11

Forewords on Provability Logics

Modal logics are extremely popular because of the multiplicity of interpretations for their characteristic modal connectives. For example, the well-known unary connective $\Box$ received numerous readings: *alethic*, where $\Box\varphi$ is read as "φ is *necessary*" [85] ; *epistemic*, where $\Box\varphi$ is read as "φ is *known*" [74] ; *doxastic*, where $\Box\varphi$ is read as "φ is *believed*" [74] ; *deontic*, where $\Box\varphi$ is read as "φ is *obligatory*" [160, 161] ; *temporal*, where $\Box\varphi$ is read as "φ holds *in the future*" or as "φ holds *in the past*" [113, 114]; etc. Following the variations of the notion $\Box$ is interpreted by, these interpretations get divided into many refinements. Thus, we have a myriad of alethic logics (commonly called "modal logics", for simplicity), epistemic logics, doxastic logics, deontic logics, and temporal logics. Modal logics are consequently extraordinarily useful tools to study in a formal way notions of primary importance.

Among these interpretations, one holds a specific place: the interpretation in terms of provability in a formal theory of arithmetic (Peano, Heyting, etc.) with a provability predicate Prov. While the interpretations presented above require us to read the $\Box$ connective informally, the interpretation of $\Box\varphi$ as "φ is provable in a theory of arithmetic" relies on formal connections between the so-called *provability* logics and the formal theories of arithmetic. More precisely, some provability logics are tightly connected to specific theories of arithmetic through *translations* f from the modal to the arithmetic language notably respecting the equality $f(\Box\varphi) = \mathsf{Prov}(\ulcorner f(\varphi)\urcorner)$, where $\ulcorner f(\varphi)\urcorner$ is the Gödel number of the arithmetical formula $f(\varphi)$ (i.e. a natural number encoding this formula). The connection expresses itself through such translations in the following way: a formula φ is provable in the provability logic if and only if its translation $f(\varphi)$ is provable in the arithmetic theory. While such a connection was first hinted at by Gödel [56], Solovay inaugurated the formal treatment of this type of connection by proving its existence between the logic $\mathsf{GL}\mathbf{L}$ and the theory of Peano arithmetic PA [139]. Research aimed at finding such connections was notably led on Heyting arithmetic HA, whose corresponding provability logic was for long unknown: it is only very recently that Mojtahedi discovered it [97].

Provability logics are modal logics. Consequently, provability logics axiomatically extend basic modal logics (classical or intuitionistic) with additional axioms. There are several axioms, named after Gödel, Löb and Grzegorczyk, characterizing specific provability logics:

GL	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$
Go	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \Box\varphi$
Grz	$\Box(\Box(\varphi \to \Box\varphi) \to \varphi) \to \varphi$

When added to the *classical* modal logic $\mathsf{wK}\mathbf{L}$, defined right after Definition 4.1.4, the above axioms give birth to the logics $\mathsf{GL}\mathbf{L}$, $\mathsf{Go}\mathbf{L}$ and $\mathsf{Grz}\mathbf{L}$. Note that these logics are usually designated as GL, Go and Grz [2, 5, 15, 16, 18]. However, we preserve the uniformity in this dissertation by using the suffix $\mathbf{L}$ when naming a logic. Furthermore, when added to the *intuitionistic* modal logic iK**L**, presented right before Definition 13.2.2, the above axioms give birth to the logics iGL**L**, iGo**L** and iGrz**L**.

In this dissertation, we focus on the proof theory of some of the provability logics we mentioned: GL**L** and iGL**L**. In general, the proof theory of provability logics is well-developed. In the literature there are calculi of all sorts for these logics: Hilbert calculi [2, 14, 139], sequent calculi [5, 18, 62, 90, 128, 156, 158], tableaux calculi [2, 123], labelled calculi [44, 99, 100], tree-hypersequent calculi [108], cyclic sequent calculi [134]. We restrict our attention to sequent calculi notably because of their interesting history, which we give below.

Following Gentzen [53, 54], the literature abounds with proofs of cut-admissibility for various sequent calculi based on the size of the cut-formula and the height of the premise proofs. But these measures looked, at first sight, inadequate for proving cut-elimination for the standard set-based sequent calculus GLS for classical provability logic GL**L**, where sequents are built from *sets of formulas*. So, Leivant first modified Gentzen's approach by using the so-called "secondary grade", a different induction measure from the traditional height of the premise proofs [90]. Shortly after, Valentini detected that one of Leivant's proofs was incorrect, and proceeded to introduce a third novel measure called "width", and showed that cut-elimination for GLS could be obtained via a triple induction over size, height, and width [156].

A substantial controversy arose when it was (erroneously) claimed that Valentini's proofs contained a gap [96] and various authors provided alternative proofs of cut-elimination in response [17, 95, 99, 100, 128]. The question was resolved in Valentini's favour [61], with all proofs later verified using an interactive theorem prover Isabelle/HOL [31].

The cut-elimination proof for the logic Go**L** (due to Goré and Ramanayake [62] via a deeper analysis of the structure of derivations, and subsequently by Savateev and Shamkanov [129] via non-well founded-proofs) is even more intricate. The proof theory of *classical* provability logics can therefore be described as complicated. While it is barely developed [158], we expect the proof theory of *intuitionistic* provability logics to be at least as convoluted, given that sequent calculi for intuitionistic logics are usually more technical to deal with.

This part of our dissertation contains investigations located at the heart of these intricate proof-theoretic issues. Because of their nature, we decided to formalise our results in Coq. First, their proof-theoretic nature demands a formalised treatment. Indeed, proof theory is a discipline in which results are obtained through the tedious analysis of hundreds of cases. As a consequence, most pen-and-paper proofs elide cases for the sake of readability, leaving comfortable nests for mistakes. By formalising our results we avoid the creation of such nests, thereby producing reliable proof theory. Second, the issues we consider here already have a troubled history. This assessment leads to a duty of reliability of the results produced, hence to the formalisation of the latter.

We divide this part of the dissertation on provability logics into two chapters. In Chapter 12 we focus on the proof theory of the *classical* provability logic characterized by the axiom *GL*. Then, we tackle the proof theory of the *intuitionistic* counterpart to this logic in Chapter 13.

Chapter 12

Classical Provability Logic

For this chapter the following sections of the Toolbox I are required: Section 2.1, Section 5.1 and the entirety of Chapter 4.

The results of this chapter are extensions of the article "Cut-Elimination for Provability Logic by Terminating Proof-Search: Formalised and Deconstructed Using Coq." [63], written jointly with Rajeev Goré and Revantha Ramanayake.

Their formalisation can be found here: https://github.com/ianshil/PhD_thesis/tree/main/Cut_Elim_GLS.

12.1 Introduction

As mentioned above, propositional *classical* modal provability logics are axiomatically captured by extending the generalized Hilbert calculus $\mathsf{wK}\mathbf{H}$ with one of the axioms GL, Go and Grz. The resulting systems define the logics $\mathsf{GL}\mathbf{L}$, $\mathsf{Go}\mathbf{L}$ and $\mathsf{Grz}\mathbf{L}$, where the $\Box$ connective can be interpreted as the mathematical notion of being "provable" in Peano Arithmetic [15, 139].

While the provability interpretation of these logics is now well-understood, their proof theory is intricate and somewhat controversial, as explained in the above forewords. More precisely, the obtention cut-admissibility for a sequent calculus for the logic $\mathsf{GL}\mathbf{L}$ has been a technical and troubled quest. However, it was settled by Goré, Ramanayake and Dawson [61, 31] who thoroughly showed that Valentini's proof, via a triple induction over size, height, and width, was correct [156].

Recently, Brighton [19] provided yet another proof of cut-admissibility for GLS which is significantly simpler than any of the existing proofs of cut-admissibility in the literature. It uses a double induction with the traditional size of the cut-formula as primary measure. The secondary measure is called the "maximum height of regress trees" and it is a novel measure defined using a backward proof-search procedure for GLS called RGL, based on regress trees / regressants.

Backward proof-search can often be employed to obtain cut-free completeness with respect to the Kripke semantics of a logic. However, cut-elimination is *not* a result directly obtained by the use of backward proof-search. For this reason, Brighton's method is intriguing from a structural proof-theoretic perspective. Even more so because, from a tableaux perspective, the RGL calculus is nothing but the backward proof-search decision procedure for $\mathsf{GL}\mathbf{L}$ that is well-known to be cut-free complete with respect to the Kripke semantics of $\mathsf{GL}\mathbf{L}$. Unfortunately, Brighton's arguments are clouded by various issues that become apparent when studying them in detail.

In what follows, we explain why Brighton's use of a set-based sequent calculus leads to confusion, and explain how this can be clarified using multisets. We then show that the special calculus RGL on regress trees can be replaced by a standard proof-search procedure PSGLS on GLS itself. Putting this all together, we replace Brighton's detour through tautology elimination [77] with a direct proof of cut-admissibility for $\mathsf{GL}\mathbf{L}$ making use of the maximum height of a derivation (the existence of the latter follows from the termination of backward proof-search). Noting that Brighton's proof seems to ignore the structure of the given cut-free derivations of the premises, and since such a shortcoming undermines cut-elimination as a procedure that manipulates the given derivations to produce a cut-free derivation, we take particular care to highlight the local nature of our transformations.

This chapter is organized as follows. We introduce the syntax, generalized Hilbert calculus, and Kripke semantics for GLL in Section 12.2. In Section 12.3 we explain how the set-based calculus leads to a confusion. Then, we introduce the multiset-based calculus GLS in Section 12.4. Section 12.5 exhibits common proof-theoretical properties of GLS. In Section 12.6 we define the calculus PSGLS, which captures a backward proof-search procedure on GLS. Furthermore, we show that sequents have a derivation of maximal height in PSGLS. Section 12.7 contains our main result, i.e. cut-elimination, which we obtain by proving in cut-admissibility by local transformations on proofs. We finally conclude in Section 12.8.

12.2 Basics of GLL

Throughout this chapter we work with the propositional modal language $\mathbb{L}_{\mathsf{CM}}$ defined in Section 2.1, which has as set of connectives $\{\bot, \to, \Box\}$. For a set $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, we define $\boxtimes\Gamma = \{\varphi_1, \Box\varphi_1, \ldots, \varphi_n, \Box\varphi_n\}$. The last syntactic element we need to specify is the notion of *size of a formula*, defined recursively on the structure of formulas, which counts the number of symbols present in a formula.

Definition 12.2.1. The size $size(\varphi)$ of a formula φ is defined as follows:

$$\begin{array}{rcl} size(\bot) = size(p) & = & 1 \\ size(\psi \to \chi) & = & size(\psi) + size(\chi) + 1 \\ size(\Box\psi) & = & size(\psi) + 1 \end{array}$$

Next, we define the generalized Hilbert calculus $\mathsf{GLH} = (\mathcal{A}_{\mathsf{KL}} \cup \{GL\}, \{(\mathrm{MP}), (\mathrm{wNec})\})$ which extends the calculus wKH from Section 4.1 with the axiom GL. We define the logic GLL as the set $\{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{GLH}} \varphi\}$.

To define a Kripke semantics for GLL, we reuse the notions of frame and model from Section 5.1. However, we consider a specific class of frames.

Definition 12.2.2. A GL-Kripke frame is a Kripke frame $\mathcal{F} = (W, R)$ such that R is:

- *transitive*, i.e. for all $w, v, u \in W$ if $[wRv$ and $vRu]$ then wRu;
- *converse well-founded*, i.e. there are no infinite ascending sequences, that is sequences of the form $w_1 R w_2 R w_3 R \ldots$.

A GL-Kripke model $\mathcal{M}$ is a model $\mathcal{M} = (\mathcal{F}, I)$, where $\mathcal{F}$ is a GL-Kripke frame.

Note that converse well-founded frames are necessarily irreflexive, else we would have an infinite ascending sequence of the form $wRwRwR\ldots$. For short, we talk about GL-frames and GL-models.

We reuse and recall the notion of forcing from Definition 5.1.4.

Definition 12.2.3. Given a GL-model $\mathcal{M} = (W, R, I)$, we define the forcing relation as follows:

$$\begin{array}{lll} \mathcal{M}, w \Vdash p & \text{iff} & w \in I(p) \\ \mathcal{M}, w \Vdash \bot & & \text{never} \\ \mathcal{M}, w \Vdash \varphi \to \psi & \text{iff} & \text{if } \mathcal{M}, w \Vdash \varphi \text{ then } \mathcal{M}, w \Vdash \psi \\ \mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \text{for all } v \text{ such that } wRv \text{ we have } \mathcal{M}, v \Vdash \varphi \end{array}$$

With these models and a definition of forcing in hand, we can define restrictions of the local and global semantic consequence relations from Definition 5.1.3 to GL-models, respectively writing $\Gamma \models_l^{GL} \varphi$ and $\Gamma \models_g^{GL} \varphi$ to designate them.

One can easily show that GLL is sound for the local semantic consequence relation:

Theorem 12.2.1. $\Gamma \vdash_{\mathsf{GLH}} \varphi$ implies $\Gamma \models_l^{GL} \varphi$.

Proof. It suffices to prove that the axioms of GLH are valid on GL-frames, and that rules of GLH preserve local semantic consecutions. For the former, we refer to Boolos' work [16, Chapter 4]. For the latter, we leave the straightforward proof to the reader. ■

Consequently, we know that GLL cannot be complete with respect to the global semantic consequence relation, as $p \not\models_l^{GL} \Box p$ but $p \models_g^{GL} \Box p$. Unfortunately, we also have that GLL is not complete with respect to the local semantic consequence relation. To show this, we use a result from Boolos [16, p.102][159, Section 3.3] relying on the notion of compactness defined in Definition 3.0.1.

Theorem 12.2.2. The local semantic consequence relation $\models_l^{GL}$ is not compact.

Proof ✍. We show that $\Gamma \models_l^{GL} \bot$ for the following infinite set Γ.

$$\Gamma = \{\Diamond p_0, \Box(p_0 \to \Diamond p_1), \Box(p_1 \to \Diamond p_2), \Box(p_2 \to \Diamond p_3), \dots, \Box(p_n \to \Diamond p_{n+1}), \dots\}$$

As is well known in the classical setting, $\Diamond$ and $\Box$ are De Morgan dual: $\Diamond$ is definable as $\neg\Box\neg$. Indeed, the usual semantic definition of this connective is as follows: $\mathcal{M}, w \Vdash \Diamond\varphi$ if and only if *there exists* a $v \in W$ such that wRv and $\mathcal{M}, v \Vdash \varphi$. With this reading in mind, one can rather straightforwardly see that the forcing of the set Γ enforces the existence of an infinite ascending chain. Assume for a model $\mathcal{M}$ and a point w that $\mathcal{M}, w \Vdash \Gamma$. As $\mathcal{M}, w_0 \Vdash \Diamond p_0$ there is a w_1 such that $w_0 R w_1$ and $\mathcal{M}, w_1 \Vdash p_0$. Now, as $\mathcal{M}, w_0 \Vdash \Box(p_0 \to \Diamond p_1)$ we get that $\mathcal{M}, w_1 \Vdash \Diamond p_1$ entailing the existence of a w_2 such that $w_1 R w_2$ and $\mathcal{M}, w_2 \Vdash p_1$. In turn, given that $\mathcal{M}, w_0 \Vdash \Box(p_1 \to \Diamond p_2)$ and $w_0 R w_2$ (by using transitivity on $w_0 R w_1$ and $w_1 R w_2$) we get that $\mathcal{M}, w_2 \Vdash \Diamond p_2$. It then clearly appears that each formula $\Box(p_n \to \Diamond p_{n+1})$ implies the existence of another element in the chain $w_0 R w_1 R w_2 \dots R w_n$. Consequently, the infinite set Γ entails the presence of an infinite ascending chain, which is impossible in a GL-frame. So, as no point in no GL-model can force Γ, we trivially get that $\Gamma \models_l^{GL} \bot$.

Furthermore, it is straightforward to see that any finite subset Γ' of Γ can be forced in a model: it suffices to have a model with a finite chain described by the finite number of formulas in Γ'.

So, we have that $\Gamma \models_l^{GL} \bot$ but there is no finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_l^{GL} \bot$. Consequently $\models_l^{GL}$ is not compact. ■

Now, we can show straightforwardly that GLL is compact, or *finitary* as in Definition 3.0.2. So, we have that GLL is not complete with respect to the local semantic consequence relation.

Still, we can find a weaker connection between these relations.

Theorem 12.2.3. $\emptyset \vdash_{\mathsf{GLH}} \varphi$ if and only if $\emptyset \models_l^{GL} \varphi$.

Proof. Such a proof was notably given by Segerberg [133]. ■

In the remainder of this chapter, we focus on the proof theory of GLL. More specifically, we discuss *sequent calculi* for this logic.

12.3 Various issues with the method used by Brighton

In 2016 Brighton provided a new and much simpler proof of cut-admissibility for the canonical sequent calculus for GLL. Although his work is extremely appealing, we have already mentioned that the argument and the proof technique supporting it require further clarification. Let us exhibit the two main elements that appeared through the formalisation process to be responsible for this unclarity.

First, as the sequents that are used are based on sets, the rule for implication on the right, presented below on the left, is just a notation for the rule on the right where the comma is interpreted as set union.

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi} \qquad \frac{\{\varphi\} \cup \Gamma \Rightarrow \Delta \cup \{\psi\}}{\Gamma \Rightarrow \Delta \cup \{\varphi \to \psi\}} \ (\to\mathrm{R_{Set}})$$

However, it is well-known that $(\to\mathrm{R_{Set}})$ contains an implicit contraction [61]. As a consequence, $(\to\mathrm{R_{Set}})$ could be reapplied as many times as one wants above $\Rightarrow p \to q$ on the formula $p \to q$. That implies the existence of derivations of all heights for this sequent, as shown below.

$$\dfrac{\dfrac{\{p\} \cup \emptyset \Rightarrow \emptyset \cup \{q\} \cup \{p \to q\}}{\{p\} \cup \emptyset \Rightarrow \emptyset \cup \{q\} \cup \{p \to q\}} \ (\to\mathrm{R_{Set}})}{\emptyset \Rightarrow \emptyset \cup \{p \to q\}} \ (\to\mathrm{R_{Set}})$$

Brighton's argument requires (and proves) that all sequents have a derivation of maximum height - this would contradict our observation above. For his argument to hold, it must therefore be the case that Brighton is not using the usual interpretation for the rules $(\to\mathrm{R})$ and $(\to\mathrm{L})$.

The only reasonable option seems to be that Brighton intends for the comma to be interpreted as *disjoint union*. This amounts to the following rule.

$$\frac{\{\varphi\} \cup \Gamma \Rightarrow (\Delta \setminus \{\varphi \to \psi\}) \cup \{\psi\}}{\Gamma \Rightarrow \Delta \cup \{\varphi \to \psi\}} \ (\to\mathrm{R_{Dis}})$$

If that was the case, a proof that the calculus is complete for GLL under this interpretation is required. Moreover, further issues arise with this interpretation.

For example, it is not true in general that the premise of the sequent $\Gamma \Rightarrow \Delta, \psi \to \chi$ via the rule $(\to\mathrm{R_{Dis}})$ is $\psi, \Gamma \Rightarrow \Delta, \chi$ (Case 2 of Theorem 1 of Brighton's article). Indeed, if $\psi \to \chi \in \Delta$ then $\psi, \Gamma \Rightarrow \Delta, \chi$ and $\psi, \Gamma \Rightarrow (\Delta \setminus \{\psi \to \chi\}), \chi$ would be different. This issue seems repairable, at the cost of numerous hours of work. However, the situation is undesirable given the sensitivity of structural proof theory to small syntactic details and especially given the history of cut-elimination for GLL.

Second, Brighton provides an unusual argument for the admissibility of cut. To obtain a proof of the latter, Brighton proves a result equivalent to it in the case of classical calculi: tautology elimination. More precisely, this lemma has the following shape: if $\varphi \to \varphi, \Gamma \Rightarrow \Delta$ is provable then so is $\Gamma \Rightarrow \Delta$. On inspection, it is clear that a procedure for tautology elimination can easily be turned into a procedure for cut-elimination, and *vice versa*. Given the proximity between these results, arguing for the admissibility of cut by proving tautology elimination seems to be an unnecessary detour.

12.4 The sequent calculus GLS

We define the sequent calculus GLS, manipulating sequents in $\mathbb{L}_{\mathbf{CM}}$ and given in Figure 12.1.

For the rule (IdP), $(\bot\mathrm{L})$, $(\to\mathrm{L})$ and $(\to\mathrm{R})$ we define the notion of principal formula following the modal case given in Section 4.2. In a rule instance of (GLR), the formula $\Box\psi$

$$\dfrac{}{p, \Gamma \Rightarrow \Delta, p}\ (\mathrm{IdP}) \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\ (\bot\mathrm{L})$$

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to\mathrm{L}) \qquad \dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\mathrm{R})$$

$$\dfrac{\boxtimes\Gamma, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi}\ (\mathrm{GLR})$$

Figure 12.1: The sequent calculus GLS. Here, Φ and Ψ do not contain any boxed formulas.

has an extremely unusual behavior. Indeed, it changes polarity by jumping from the right-hand side of the conclusion to the left-hand side of the premise *without being altered.* As this formula holds a noticeable role in the rule it is given a specific name: the *diagonal formula* [126].

Example 12.4.1. The following are examples of derivations in GLS. Note that while the first and second examples are derivations, the third is a proof.

$$p \Rightarrow q \to r \qquad \dfrac{p, q \Rightarrow r, r}{p \Rightarrow q \to r, r}\ (\to\mathrm{R}) \qquad \dfrac{\dfrac{}{\Box p, p, \Box p \Rightarrow p}\ (\mathrm{IdP})}{\Box p \Rightarrow \Box p, q}\ (\mathrm{GLR})$$

Example 12.4.2. A special example of derivation in GLS is the following:

$$\dfrac{\dfrac{\Box\varphi \to \varphi, \Box(\Box\varphi \to \varphi), \varphi, \varphi, \Box\varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi}{\Box(\Box\varphi \to \varphi), \varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi, \Box\varphi}\ (\mathrm{GLR}) \qquad \Box(\Box\varphi \to \varphi), \varphi, \varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi}{\Box\varphi \to \varphi, \Box(\Box\varphi \to \varphi), \varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi}\ (\to\mathrm{L})$$

By noticing the identity modulo formula multiplicities between the topmost and the lowest sequents, it appears that the sequence of application of rules in the above could be iterated indefinitely on the topmost sequent. This notably means that the "naive" backward proof-search procedure, i.e. which allows to apply any rule upwards without restriction, in GLS does not terminate.

Finally, we recall the *additive* cut rule from Section 4.2. In this rule we call φ the *cut-formula.*

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ (\mathrm{cut})$$

By adding the rule (cut) to GLS, giving us the calculus GLS + (cut), we obtain a sequent calculus for GLL [126].

Theorem 12.4.1. For all $\varphi \in Form_{\mathbb{L}_{\mathbf{CM}}}$ we have: $(\Gamma, \varphi) \in \mathsf{GLL}$ iff there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \Rightarrow \varphi$ is provable in GLS + (cut)

12.5 Properties of GLS

In this section, we prove that GLS satisfies properties commonly used in proof theory.

First, through a proof-theoretical property, we highlight a distinction between our pen-and-paper approach and our formalisation in Coq. While we consider here sequents based on *finite multisets*, in our formalisation sequents are based on *lists.* The crucial difference between the two pertains to *order*: in lists the order of elements matters, in multisets it does not. Consequently, if we consider the sequents $p, r \Rightarrow q$ and $r, p \Rightarrow q$, then we are facing *identical* multiset-sequents but *different* list-sequents. This situation may worry our reader: did we really formalise the results we present in this chapter? Strictly speaking, the answer is "no". However, we defined the rules of GLS such that *exchange* is admissible (see Definition 4.2.1). Below, `list_exch_L s se` encodes the fact that `se` is obtained from the sequent `s` by permuting two sub-lists in the list representing its antecedent, and `list_exch_R s se` encodes the same idea in the succedent.

Lemma 12.5.1 (Admissibility of exchange)**.** For all $\Gamma, \Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Delta, \Delta_0, \Delta_1, \Delta_2$ and Δ_3:

(i) If $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \Delta$ has a proof in GLS, then $\Gamma_0, \Gamma_2, \Gamma_1, \Gamma_3 \Rightarrow \Delta$ has a proof in GLS.

(ii) If $\Gamma \Rightarrow \Delta_0, \Delta_1, \Delta_2, \Delta_3$ has a proof in GLS, then $\Gamma \Rightarrow \Delta_0, \Delta_2, \Delta_1, \Delta_3$ has a proof GLS.

```
Lemma GLS_adm_list_exch_LR : forall s (D0: GLS_prv s),
  (forall se, ((list_exch_L s se) -> (GLS_prv se)) *
              ((list_exch_R s se) -> (GLS_prv se))).
```

Proof. This result is immediate for multiset-sequents. For list-sequents, we refer to our formalisation (⌨). ■

This result informs us that the order in which formulas are presented in a list does not matter in the formalised calculus GLS. Consequently, we are ensured that the latter calculus on list-sequents mimics adequately the pen-and-paper version of GLS on multiset-sequents presented here.

Second, we prove that all identities, i.e. sequents of the shape $\varphi, \Gamma \Rightarrow \Delta, \varphi$ for any formula φ, are provable in GLS.

Lemma 12.5.2. For all φ, Γ and Δ, the sequent $\varphi, \Gamma \Rightarrow \Delta, \varphi$ has a proof.

Proof. (⌨) We reason by induction on the structure of φ.

- $\varphi = p$: We immediately get a proof of $p, \Gamma \Rightarrow \Delta, p$ through the rule (IdP).
- $\varphi = \bot$: A proof of $\bot, \Gamma \Rightarrow \Delta, \bot$ is directly obtained using the ($\bot$L).
- $\varphi = \psi \to \chi$: By induction hypothesis, we have a proof of both $\psi, \Gamma \Rightarrow \Delta, \chi, \psi$ and $\psi, \chi, \Gamma \Rightarrow \Delta, \chi$. So, using these proofs we build the following:

$$\dfrac{\dfrac{\psi, \Gamma \Rightarrow \Delta, \chi, \psi \quad \psi, \chi, \Gamma \Rightarrow \Delta, \chi}{\psi, \psi \to \chi, \Gamma \Rightarrow \Delta, \chi}\ (\to\mathrm{L})}{\psi \to \chi, \Gamma \Rightarrow \Delta, \psi \to \chi}\ (\to\mathrm{R})$$

- $\varphi = \Box\psi$: We need to give a proof of $\Box\psi, \Gamma \Rightarrow \Delta, \Box\psi$. To do so, we first divide Γ into its subset of boxed formulas $\Box\Gamma_0$ and its set of remaining formulas Γ_1. Note that each set can be empty. By the induction hypothesis, we have a proof of the sequent $\psi, \Box\psi, \Gamma_0, \Box\Gamma_0, \Box\psi \Rightarrow \psi$, as ψ appears on both sides of this sequent. Thus, we use this proof to build the following.

$$\dfrac{\psi, \Box\psi, \Gamma_0, \Box\Gamma_0, \Box\psi \Rightarrow \psi}{\Box\psi, \Box\Gamma_0, \Gamma_1 \Rightarrow \Delta, \Box\psi}\ (\mathrm{GLR})$$

■

Third, we prove that the rules of weakening on the left and right are height-preserving admissible in GLS.

Lemma 12.5.3 (Height-preserving admissibility of weakening)**.** For all Γ, Δ, φ and ψ:

(i) If $\Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}$ in GLS, then $\Gamma \Rightarrow \Delta, \varphi$ has a proof $\mathfrak{p}'$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

(ii) If $\Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}$ in GLS, then $\varphi, \Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}'$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

Proof. We prove each statement independently.

(i) (⌨) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\Gamma \Rightarrow \Delta, \varphi$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ is an instance of an initial sequent.

  (b) $r = (\text{GLR})$: then r is

$$\frac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi}\ (\text{GLR})$$

  Either φ is boxed and then we can apply (GLR) on the premise to obtain $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_1, \Psi$ where $\Box\Delta_1 = \Box\Delta_0, \varphi$. Or φ is not boxed and then we can apply (GLR) on the premise to obtain a proof of $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi_0$ where $\Psi_0 = \Psi, \varphi$. In both cases, we get that the height of the proof is preserved.

  (c) $r = (\to\text{R})$: then r is

$$\frac{\chi, \Gamma \Rightarrow \Delta_0, \psi}{\Gamma \Rightarrow \Delta_0, \chi \to \psi}\ (\to\text{R})$$

  Then we can apply the induction hypothesis on the premise to get a proof of $\chi, \Gamma \Rightarrow \Delta_0, \varphi, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\to\text{R})$ to get a proof of $\Gamma \Rightarrow \Delta_0, \varphi, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

  (d) $r = (\to\text{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\text{L})$$

  Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p}) - 1$ of $\Gamma_0 \Rightarrow \Delta, \varphi, \chi$ and $\psi, \Gamma_0 \Rightarrow \Delta, \varphi$. We can consequently apply the rule $(\to\text{L})$ to obtain a proof of $\chi \to \psi, \Gamma_0 \Rightarrow \Delta, \varphi$ of height less than or equal to $h(\mathfrak{p}) - 1$.

(ii) (⌨) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\varphi, \Gamma \Rightarrow \Delta$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

  (a) $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\varphi, \Gamma \Rightarrow \Delta$ is an instance of an initial sequent.

  (b) $r = (\text{GLR})$: then r is

$$\frac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi}\ (\text{GLR})$$

  Either φ is boxed, i.e. $\varphi = \Box\chi$ for some χ, and then we can apply the induction hypothesis twice to obtain a proof of $\chi, \Box\chi, \boxtimes\Gamma_0, \Box\psi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the (GLR) rule to obtain a proof of $\Phi, \Box\chi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi$ of height less than or equal to $h(\mathfrak{p})$. Or φ is not boxed and then we can apply (GLR) on the premise to obtain a proof of $\Phi_0, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi$ of height $h(\mathfrak{p})$, where $\Phi_0 = \Phi, \varphi$.

  (c) $r = (\to\text{R})$: then r is

$$\frac{\chi, \Gamma \Rightarrow \Delta_0, \psi}{\Gamma \Rightarrow \Delta_0, \chi \to \psi}\ (\to\text{R})$$

  Then we can apply the induction hypothesis on the premise to get a proof of $\chi, \Gamma, \varphi \Rightarrow \Delta_0, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\to\text{R})$ to get a proof of $\Gamma, \varphi \Rightarrow \Delta_0, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

(d) $r = (\to\mathrm{L})$: then r is

$$\frac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\mathrm{L})$$

Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p}) - 1$ of $\Gamma_0, \varphi \Rightarrow \Delta, \chi$ and $\psi, \Gamma_0, \varphi \Rightarrow \Delta$. We can consequently apply the rule to obtain a proof of $\chi \to \psi, \Gamma_0, \varphi \Rightarrow \Delta$ of height less than or equal to $h(\mathfrak{p})$.

■

Fourth, we prove that the implication rules $(\to\mathrm{R})$ and $(\to\mathrm{L})$ are height-preserving invertible.

Lemma 12.5.4 (Height-preserving invertibility of the implication rules)**.** For all Γ, Δ, φ and ψ:

(i) If $\varphi \to \psi, \Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}$ in GLS, then $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ have proofs $\mathfrak{p}'$ and $\mathfrak{p}''$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$ and $h(\mathfrak{p}'') \leq h(\mathfrak{p})$.

(ii) If $\Gamma \Rightarrow \Delta, \varphi \to \psi$ has a proof $\mathfrak{p}$ in GLS, then $\varphi, \Gamma \Rightarrow \Delta, \psi$ has a proof $\mathfrak{p}'$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

Proof. (⌨) We prove (i) and (ii) simultaneously by strong induction on the height of the given proof.

(i) Let $\mathfrak{p}$ be a proof of $\varphi \to \psi, \Gamma \Rightarrow \Delta$. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ are instances of an initial sequent.

(b) r is (GLR): then r is

$$\frac{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\varphi \to \psi, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi}\ (\mathrm{GLR})$$

Then we can apply (GLR) on the premise to obtain a proof of $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi, \varphi$ of height $h(\mathfrak{p})$. Now, to get a proof for the other desired sequent $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi$, we need to distinguish two cases. If ψ is boxed, i.e. there is a δ such that $\psi = \Box\delta$, then we apply Lemma 12.5.3(ii) twice on $\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ to get a proof of $\delta, \Box\delta, \boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply (GLR) on the latter to obtain a proof of $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi$ of height less than or equal to $h(\mathfrak{p})$. If ψ is not boxed, then we can simply apply (GLR) on $\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ to obtain a proof of $\Phi, \psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi$ of height less than or equal to $h(\mathfrak{p})$.

(c) r is $(\to\mathrm{R})$:

$$\frac{\chi, \varphi \to \psi, \Gamma \Rightarrow \Delta, \delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta, \chi \to \delta}\ (\to\mathrm{R})$$

Applying the induction hypothesis to the proof of $\chi, \varphi \to \psi, \Gamma \Rightarrow \Delta, \delta$ we obtain a proof of height less than or equal to $h(\mathfrak{p}) - 1$ for

$$\chi, \Gamma \Rightarrow \Delta, \delta, \varphi \tag{12.1}$$

$$\psi, \chi, \Gamma \Rightarrow \Delta, \delta \tag{12.2}$$

Applying $(\to\mathrm{R})$ to the proof of (12.2) yields a proof of $\psi, \Gamma \Rightarrow \Delta, \chi \to \delta$ of height less than or equal to $h(\mathfrak{p})$. Also, we can apply $(\to\mathrm{R})$ to the proof of (12.1) to obtain a proof of $\Gamma \Rightarrow \Delta, \chi \to \delta, \varphi$ of height less than or equal to $h(\mathfrak{p})$. We have obtained proofs for the desired sequents.

(d) Now suppose that r is ($\rightarrow$L). If $\varphi \rightarrow \psi$ is principal in r then the premises are $\Gamma \Rightarrow \Delta, \varphi$ and $\psi, \Gamma \Rightarrow \Delta$ so we are done. If $\varphi \rightarrow \psi$ is not principal in r then we have the following.

$$\dfrac{\varphi \rightarrow \psi, \Gamma \Rightarrow \Delta, \chi \qquad \delta, \varphi \rightarrow \psi, \Gamma \Rightarrow \Delta}{\varphi \rightarrow \psi, \chi \rightarrow \delta, \Gamma \Rightarrow \Delta} \ (\rightarrow\mathrm{L})$$

Applying the induction hypothesis to the proofs of the premises we can obtain proofs of height less than or equal to $h(\mathfrak{p}) - 1$ of

$$\Gamma \Rightarrow \Delta, \chi, \varphi \tag{12.3}$$
$$\psi, \Gamma \Rightarrow \Delta, \chi \tag{12.4}$$
$$\delta, \Gamma \Rightarrow \Delta, \varphi \tag{12.5}$$
$$\psi, \delta, \Gamma \Rightarrow \Delta \tag{12.6}$$

From the proofs of (12.3) and (12.5) we get a proof of height less than or equal to $h(\mathfrak{p})$ for one of the desired sequents.

$$\dfrac{\Gamma \Rightarrow \Delta, \chi, \varphi \qquad \delta, \Gamma \Rightarrow \Delta, \varphi}{\chi \rightarrow \delta, \Gamma \Rightarrow \Delta, \varphi} \ (\rightarrow\mathrm{L})$$

Also, from the proofs of (12.4) and (12.6):

$$\dfrac{\psi, \Gamma \Rightarrow \Delta, \chi \qquad \psi, \delta, \Gamma \Rightarrow \Delta}{\psi, \chi \rightarrow \delta, \Gamma \Rightarrow \Delta} \ (\rightarrow\mathrm{L})$$

So we have obtained a proof of height less than or equal to $h(\mathfrak{p})$ for the desired sequents.

(ii) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi$. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: then either $p \in \Gamma \cap \Delta$ or $\bot \in \Gamma$. In all cases $\Gamma, \varphi \Rightarrow \Delta, \psi$ is an instance of an initial sequent.

(b) r is (GLR): then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi, \varphi \rightarrow \psi} \ (\mathrm{GLR})$$

We need to distinguish two cases. If φ is boxed, i.e. there is a δ such that $\varphi = \Box\delta$, then we apply Lemma 12.5.3(i) twice on the proof of $\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ to get a proof of $\delta, \Box\delta, \boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply (GLR) on the latter to obtain a proof of $\Phi, \varphi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi, \psi$ of height less than or equal to $h(\mathfrak{p})$. If φ is not boxed, then we can simply apply (GLR) on $\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$ to obtain a proof of $\Phi, \varphi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi, \psi$ of height $h(\mathfrak{p})$.

(c) Suppose that r is ($\rightarrow$R). If $\varphi \rightarrow \psi$ is principal in r then the premise is $\varphi, \Gamma \Rightarrow \Delta, \psi$ so we are done. If $\varphi \rightarrow \psi$ is not principal in r then we have the following.

$$\dfrac{\chi, \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi, \delta}{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi, \chi \rightarrow \delta} \ (\rightarrow\mathrm{R})$$

Applying the induction hypothesis to the proof of $\chi, \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi, \delta$ we can obtain a proof of $\varphi, \chi, \Gamma \Rightarrow \Delta, \delta, \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Then we can apply ($\rightarrow$R) to get a proof of $\varphi, \Gamma \Rightarrow \Delta, \psi, \chi \rightarrow \delta$ of height less than or equal to $h(\mathfrak{p})$, so we have obtained a proof for the desired sequent.

(d) Finally, suppose that r is ($\rightarrow$L).

$$\dfrac{\Gamma \Rightarrow \Delta, \varphi \rightarrow \psi, \chi \qquad \delta, \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi}{\chi \rightarrow \delta, \Gamma \Rightarrow \Delta, \varphi \rightarrow \psi} \ (\rightarrow\mathrm{L})$$

Applying the induction hypothesis to the proofs of the premises we can obtain proofs of height less than or equal to $h(\mathfrak{p}) - 1$ for

$$\varphi, \Gamma \Rightarrow \Delta, \chi, \psi \tag{12.7}$$

$$\varphi, \delta, \Gamma \Rightarrow \Delta, \psi \tag{12.8}$$

Then proceed:

$$\frac{\varphi, \Gamma \Rightarrow \Delta, \chi, \psi \qquad \varphi, \delta, \Gamma \Rightarrow \Delta, \psi}{\varphi, \chi \to \delta, \Gamma \Rightarrow \Delta, \psi}\ (\to\mathrm{L})$$

So we have obtained proofs of height less than or equal to $h(\mathfrak{p})$ for the desired sequent.

■

Fifth, we proceed to show that the left and right rules of contraction are height-preserving admissible.

Lemma 12.5.5 (Height-preserving admissibility of contraction)**.** For all Γ, Δ, φ and ψ:

(i) If $\Gamma \Rightarrow \Delta, \varphi, \varphi$ has a proof $\mathfrak{p}$ in GLS, then $\Gamma \Rightarrow \Delta, \varphi$ has a proof $\mathfrak{p}'$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

(ii) If $\varphi, \varphi, \Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}$ in GLS, then $\varphi, \Gamma \Rightarrow \Delta$ has a proof $\mathfrak{p}'$ in GLS such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

Proof. (⌨) We prove (i) and (ii) simultaneously by strong induction on the height of the given proof.

(i) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \Delta, \varphi, \varphi$ of height n. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: either $p \in \Gamma \cap (\Delta \cup \{\varphi, \varphi\})$ or $\bot \in \Gamma$. In all cases $\Gamma \Rightarrow \Delta, \varphi$ is an instance of an initial sequent.

(b) $r = (\mathrm{GLR})$: then r is

$$\frac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi}\ (\mathrm{GLR})$$

Either φ is not boxed (i.e. $\varphi \in \Psi$) and then we can apply (GLR) on the premise to obtain a proof of height $h(\mathfrak{p})$ of $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi_0$ where $\Psi_0, \varphi = \Psi$. Or φ is boxed. If $\varphi \in \Box\Delta_0$ then we can apply (GLR) on the premise to obtain a proof of height $h(\mathfrak{p})$ of $\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_1, \Psi$ where $\Box\Delta_1, \varphi = \Box\Delta_0$. If φ is the diagonal formula $\Box\psi$ we consequently have that the second $\Box\psi$ appearing in the conclusion is an element of $\Box\Delta_0$ and then we can apply the rule on the proof of the premise using $\Box\Delta_1$ instead of $\Box\Delta_0$.

(c) $r = (\to\mathrm{R})$: then r is

$$\frac{\chi, \Gamma_0 \Rightarrow \Delta_0, \psi}{\Gamma_0 \Rightarrow \Delta_0, \chi \to \psi}\ (\to\mathrm{R})$$

Either $\varphi = \chi \to \psi$, and then we use Lemma 12.5.4(ii) on the premise of the form $\chi, \Gamma_0 \Rightarrow \Delta_1, \chi \to \psi, \psi$ to obtain a proof $\chi, \chi, \Gamma_0 \Rightarrow \Delta_1, \psi, \psi$ of height less than or equal to $h(\mathfrak{p} - 1)$. Then we can apply the induction hypothesis to contract on both sides and obtain a proof of $\chi, \Gamma_0 \Rightarrow \Delta_1, \psi$ of height less than or equal to $h(\mathfrak{p} - 1)$. It consequently suffices to apply $(\to\mathrm{R})$ on the latter to get a proof of $\Gamma_0 \Rightarrow \Delta_1, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$. Or $\varphi \neq \chi \to \psi$, and then $\{\varphi, \varphi\} \subseteq \Delta_0$. Thus we can apply the induction hypothesis on the premise to get a proof of height less than or equal to $h(\mathfrak{p} - 1)$ of $\chi, \Gamma \Rightarrow \Delta_1, \psi$ where $\Delta_1, \varphi = \Delta_0$. We can thus apply the rule $(\to\mathrm{R})$ to get a proof of $\Gamma \Rightarrow \Delta_1, \chi \to \psi$ of height less than or equal to $h(\mathfrak{p})$.

(d) $r = (\to\mathrm{L})$: then r is

$$\dfrac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\mathrm{L})$$

Then we can apply the induction hypothesis on both premises to get proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of $\Gamma_0 \Rightarrow \Delta_0, \chi$ and $\psi, \Gamma_0 \Rightarrow \Delta_0$ where $\Delta_0, \varphi = \Delta$. We can consequently apply the rule to obtain a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_0 \Rightarrow \Delta_0$.

(ii) Let $\mathfrak{p}$ be a proof of $\varphi, \varphi, \Gamma \Rightarrow \Delta$ of height n. We consider the last rule r applied in $\mathfrak{p}$:

(a) $r \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$: either $p \in (\{\varphi, \varphi\} \cup \Gamma) \cap \Delta$ or $\bot \in \{\varphi, \varphi\} \cup \Gamma$. In all cases $\varphi, \Gamma \Rightarrow \Delta$ is an instance of an initial sequent.

(b) $r = (\mathrm{GLR})$: then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi}\ (\mathrm{GLR})$$

Either φ is boxed, i.e. there is a χ such that $\varphi = \Box\chi$, and consequently the premise is of the form $\chi, \Box\chi, \chi, \Box\chi, \boxtimes\Gamma_1, \Box\psi \Rightarrow \psi$, where $\Gamma_1, \chi, \chi = \Gamma_0$. Then we can apply the induction hypothesis twice to obtain a proof of height less than or equal to $h(\mathfrak{p}-1)$ of $\chi, \Box\chi, \boxtimes\Gamma_1, \Box\psi \Rightarrow \psi$. We can then apply the (GLR) rule to obtain a proof of height less than or equal to $h(\mathfrak{p})$ of $\Phi, \Box\chi, \Box\Gamma_1 \Rightarrow \Box\psi, \Box\Delta_0, \Psi$. Or φ is not boxed and then we can apply (GLR) on the premise to obtain a proof of of height $h(\mathfrak{p})$ of $\Phi_0, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi$ where $\Phi_0, \varphi = \Phi$.

(c) $r = (\to\mathrm{R})$: then r is

$$\dfrac{\chi, \Gamma_0 \Rightarrow \Delta_0, \psi}{\Gamma_0 \Rightarrow \Delta_0, \chi \to \psi}\ (\to\mathrm{R})$$

Then $\{\varphi, \varphi\} \subseteq \Gamma_0$ and we can apply the induction hypothesis on the premise to get a proof of height less than or equal to $h(\mathfrak{p} - 1)$ of $\chi, \Gamma_1 \Rightarrow \Delta_0, \psi$ where $\Gamma_1, \varphi = \Gamma_0$. We can thus apply the rule $(\to\mathrm{R})$ to get a proof of height less than or equal to $h(\mathfrak{p})$ of $\Gamma_1 \Rightarrow \Delta_0, \chi \to \psi$.

(d) $r = (\to\mathrm{L})$: then r is

$$\dfrac{\Gamma_0 \Rightarrow \Delta, \chi \qquad \psi, \Gamma_0 \Rightarrow \Delta}{\chi \to \psi, \Gamma_0 \Rightarrow \Delta}\ (\to\mathrm{L})$$

Either $\varphi = \chi \to \psi$, and then we use Lemma 12.5.4(i) on both premises to get proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of:

$$\Gamma_1 \Rightarrow \Delta, \chi, \chi \tag{12.9}$$

$$\psi, \Gamma_1 \Rightarrow \Delta, \chi \tag{12.10}$$

$$\psi, \Gamma_1 \Rightarrow \Delta, \chi \tag{12.11}$$

$$\psi, \psi, \Gamma_1 \Rightarrow \Delta \tag{12.12}$$

where $\Gamma_1, \chi \to \psi = \Gamma_0$. Then we can apply the induction hypothesis on the proofs of (12.9) and (12.12) to obtain proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of $\Gamma_1 \Rightarrow \Delta, \chi$ and, respectively, $\psi, \Gamma_1 \Rightarrow \Delta$. It consequently suffices to apply $(\to\mathrm{L})$ on these sequents get a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_1 \Rightarrow \Delta$. Or $\varphi \neq \chi \to \psi$, and then $\{\varphi, \varphi\} \subseteq \Delta_0$. Thus we can apply the induction hypothesis on the premise to get proofs of height less than or equal to $h(\mathfrak{p} - 1)$ of $\chi, \Gamma_1 \Rightarrow \Delta$ and $\Gamma_1 \Rightarrow \Delta, \psi$, where $\Gamma_1, \varphi = \Gamma_0$. We can thus apply the rule $(\to\mathrm{L})$ to get a proof of height less than or equal to $h(\mathfrak{p})$ of $\chi \to \psi, \Gamma_1 \Rightarrow \Delta$.

■

All the properties we proved in this section are not only canonical properties in proof theory, but they are used as either crucial lemmas in later proofs or preliminary lemmas to crucial lemmas.

In the following section, we introduce a sequent calculus embodying a proof-search procedure for GLS which terminates. This allows us to define the maximum height of a derivation of a sequent with respect to this procedure. Later on this notion of maximum height of a derivation constitutes the secondary induction measure in the proof of admissibility of cut.

12.6 PSGLS: a terminating proof-search

In this section, we define the sequent calculus PSGLS which intends to embody a proof-search procedure on GLS. We aim at obtaining two results about this calculus. First, we proceed to show the existence of a derivation of maximal height in PSGLS for each sequent, which allows us to define a measure mhd on sequents used in our cut-admissibility proof. Second, we justify our use of the prefix PS in PSGLS by showing that the latter helps decide the provability of sequents in GLS, as discussed in Section 4.2.

Let us turn to the definition of PSGLS. The latter restricts the rules of GLS in the following way.

(IdentBox) An additional identity rule (IdB), derivable in GLS as shown in Lemma 12.5.2, is introduced.

$$\dfrac{}{\Box\varphi, \Gamma \Rightarrow \Delta, \Box\varphi}\ (\text{IdB})$$

(NoInit) The conclusion of the rule (GLR) is not permitted to be an instance of either (IdP) or (⊥L) or (IdB).

Conjointly, these restrictions aim at avoiding repetitions along a branch of a sequent which is either an identity described by (IdP) or (IdB), or an instance of (⊥L), as in Example 12.4.2. More precisely, restriction (NoInit) disallows the destruction of a formula upwards in presence of a sequent that is obviously provable, while (IdentBox) allows to designate the latter as provable. In fact, by showing that no loop can appear in a branch of a PSGLS derivation, we concretely show that the only type of loop present in PSGLS are loops on provable sequents.

Next, we proceed to show that each sequent has a derivation of maximum height in PSGLS (something that does not hold of GLS, as witnessed by Example 12.4.2). This crucial result is not thoroughly proved in Brighton's work.

In general, it is easy to prove that if there is a measure that decreases, given a well-founded order, upwards through the rules of a sequent calculus **S**, then each sequent has a derivation of maximum height in **S**. We intend to apply this strategy to show that each sequent has a derivation of maximum height in PSGLS. To do so, we need to define a measure on sequents.

Definition 12.6.1. We define:

1. the *number of occurrences of* "→" $imp(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$.
2. the *usable boxes* $ub(\Gamma \Rightarrow \chi)$ of $\Gamma \Rightarrow \Delta$ as:
$$ub(\Gamma \Rightarrow \Delta) := \{\Box\varphi \mid \Box\varphi \in \text{Subf}(\Gamma \cup \Delta)\} \setminus \{\Box\varphi \mid \Box\varphi \in \Gamma\}$$
3. the *number of usable boxes* $\beta(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$ as $\beta(\Gamma \Rightarrow \Delta) = \text{Card}(ub(\Gamma \Rightarrow \Delta))$, where $\text{Card}(\Phi)$ is the cardinality of the set Φ.
4. the *measure* $\Theta(\Gamma \Rightarrow \Delta)$ of $\Gamma \Rightarrow \Delta$ as

$$\Theta(\Gamma \Rightarrow \Delta) := (\beta(\Gamma \Rightarrow \Delta), \mathit{imp}(\Gamma \Rightarrow \Delta))$$

The notion of usable boxes of a sequent $\Gamma \Rightarrow \Delta$ is the set of boxed formulas of $\Gamma \Rightarrow \Delta$ minus the boxed formulas elements of Γ, i.e. appearing on the “top-level”. Intuitively, this notion captures the set of boxed formulas of a sequent s which might be the diagonal formula of an instance of (GLR) in a derivation of s in PSGLS. Consequently, the number of usable boxes gives an upper bound to the number of applications of (GLR) in a branch of a derivation of s in PSGLS.

We proceed to prove that the measure Θ decreases on the usual lexicographic ordering on n-tuples, which is well-known to be well-founded, upwards through the rules of PSGLS.

Lemma 12.6.1. Let s_0 and $s_1, ..., s_n$ be sequents. If there is an instance of a rule r of PSGLS of the following form, then $\Theta(s_i) < \Theta(s_0)$ for $1 \leq i \leq n$.

$$\frac{s_1 \quad \dots \quad s_n}{s_0}\ r$$

Proof. (⌨) We reason by case analysis on r:

1. If r is (IdP) or (IdB) or ($\bot$L), then we are done as there is no premise.
2. If r is ($\to$R), then it must have the following form.

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}\ (\to\text{R})$$

   Then we distinguish two cases. If φ is boxed, then $\{\Box\psi \mid \Box\psi \in \Gamma\} \subseteq \{\Box\psi \mid \Box\psi \in \Gamma \cup \{\varphi\}\}$. As a consequence, we have that $\mathit{ub}(\Gamma, \varphi \Rightarrow \Delta, \psi) \subseteq \mathit{ub}(\Gamma \Rightarrow \Delta, \varphi \to \psi)$ hence $\beta(\Gamma, \varphi \Rightarrow \Delta, \psi) \leq \beta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$. If $\beta(\Gamma, \varphi \Rightarrow \Delta, \psi) < \beta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$ then we are done. If $\beta(\Gamma, \varphi \Rightarrow \Delta, \psi) = \beta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$, then we can see that $\mathit{imp}(\Gamma, \varphi \Rightarrow \Delta, \psi) = \mathit{imp}(\Gamma \Rightarrow \Delta, \varphi \to \psi) - 1$ hence $\Theta(\Gamma, \varphi \Rightarrow \Delta, \psi) < \Theta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$. If φ is not boxed, then obviously we get that $\beta(\Gamma, \varphi \Rightarrow \Delta, \psi) = \beta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$ but also $\mathit{imp}(\Gamma, \varphi \Rightarrow \Delta, \psi) = \mathit{imp}(\Gamma \Rightarrow \Delta, \varphi \to \psi) - 1$ hence $\Theta(\Gamma, \varphi \Rightarrow \Delta, \psi) < \Theta(\Gamma \Rightarrow \Delta, \varphi \to \psi)$.

3. If r is ($\to$L), then it must have the following form.

$$\frac{\Gamma \Rightarrow \Delta, \varphi \qquad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta}\ (\to\text{L})$$

   We can easily establish that $\Theta(\Gamma \Rightarrow \Delta, \varphi) < \Theta(\varphi \to \psi, \Gamma \Rightarrow \Delta)$ as one implication symbol is deleted while the cardinality of usable boxes stays the same. To prove that $\Theta(\psi, \Gamma \Rightarrow \Delta) < \Theta(\varphi \to \psi, \Gamma \Rightarrow \Delta)$ we reason as in (2).

4. If r is (GLR) then it must have the following form.

$$\frac{\boxtimes\Gamma, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi}\ (\text{GLR})$$

   Clearly, we have that $\{\Box\varphi \mid \Box\varphi \in \text{Subf}(\boxtimes\Gamma \cup \{\Box\psi\} \cup \{\psi\})\} \subseteq \{\Box\varphi \mid \Box\varphi \in \text{Subf}(\Phi \cup \Box\Gamma \cup \{\Box\psi\} \cup \Box\Delta \cup \Psi)\}$. Also, given that we consider a derivation in PSGLS, we can note that (IdB) is not applicable on $\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi$ by assumption, hence $\Box\psi \notin \Box\Gamma$. Consequently, we get $\{\Box\varphi \mid \Box\varphi \in \Phi \cup \Box\Gamma\} \subset \{\Box\varphi \mid \Box\varphi \in \boxtimes\Gamma \cup \{\Box\psi\}\}$. An easy set-theoretic argument leads to $\mathit{ub}(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) \subset \mathit{ub}(\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi)$. As a consequence we obtain $\beta(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) < \beta(\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi)$, hence $\Theta(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) < \Theta(\Phi, \Box\Gamma \Rightarrow \Box\psi, \Box\Delta, \Psi)$.

■

Next, we prove that each sequent has a (finite) list of premises through rules of PSGLS.

Lemma 12.6.2. For all sequent s there is a list $Prems(s)$ such that for all s', s' is a premise of the conclusion s for an instance of a rule r in PSGLS iff s' is in $Prems(s)$.

Proof. (⌨) The proof idea is similar to the one of Lemma 4.2.5. ■

Conjointly, the two previous lemmas imply the existence of a derivation in PSGLS of maximum height for all sequents. Indeed, for a sequent s we can explore its space of all possible derivation in PSGLS by repetitively using Lemma 12.6.2, potentially infinitely many times. These repetitive applications of Lemma 12.6.2 are building derivations starting from the root. But as Θ decreases upwards in rule applications as shown in Lemma 12.6.1, each application of Lemma 12.6.2 gives birth to a list of sequents with a decreased measure. Thus, we only need to apply Lemma 12.6.2 finitely many times above s to exhaust the space of all possible derivation in PSGLS. Next, we formally prove this theorem.

Theorem 12.6.1. Every sequent s has a PSGLS derivation of maximum height.

```
Theorem PSGLS_termin :
 forall s, existsT2 (DMax: PSGLS_drv s), (is_mhd DMax).
```

Proof. We reason by strong induction on the ordered pair $\Theta(s)$. As the applicability of the rules of PSGLS is decidable (⌨), we distinguish two cases:

(I) No PSGLS rule is applicable to s. Then the derivation of maximum height sought after is simply the derivation constituted of s solely, which is the only derivation for s.

(II) Some PSGLS rule is applicable to s. Either only initial rules are applicable, in which case the derivation of maximum height sought after is simply the derivation of height 1 constituted of the application of the applicable initial rule to s. Or, some other rules than the initial rules are applicable. Then by Lemma 12.6.2 there is a list $Prems(s)$ of all sequents s_0 such that there is an application of a PSGLS rule r with s as conclusion of r and s_0 as premise of r. By Lemma 12.6.1 we know that every element s_0 in the list $Prems(s)$ is such that $\Theta(s_0) < \Theta(s)$. Consequently, the induction hypothesis allows us to consider the derivation of maximum height of all the sequents in $Prems(s)$. As $Prems(s)$ is finite, there must be an element s_{max} of $Prems(s)$ such that its derivation of maximum height is higher than the derivation of maximum height of all sequents in $Prems(s)$ or of the same height. It thus suffices to pick that s_{max}, use its derivation of maximum height, and apply the appropriate rule to obtain s as a conclusion: this is by choice the derivation of maximum height of s. ■

Here, `DMax` is a *derivation*, the existence of which is guaranteed by the constructive existential quantifier `existsT2`. This quantifier not only requires us to construct a witnessing term but also to provide a proof that the witness is of the correct type. The function `is_mhd` returns the constructive Coq proposition `True` if and only if its argument, `DMax`, is a derivation of maximum height.

As the previous lemma implies the existence of a derivation $\mathfrak{D}$ of maximum height in PSGLS, we are entitled to let $\mathrm{mhd}(s)$ denote the height of $\mathfrak{D}$. Similarly to Brighton, we later use $\mathrm{mhd}(s)$ as the secondary induction measure used in the proof of admissibility of cut.

Before proving the only property we need from $\mathrm{mhd}(s)$, let us interpret the previous lemma from two points of view.

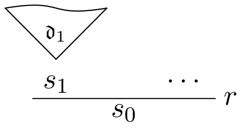
First, we consider the point of view of the proof-search procedure underlying PSGLS. The existence of a derivation of maximum height for each sequent in PSGLS shows that in the backward application of rules of PSGLS on a sequent, i.e. the carrying of the proof-search procedure, a halting point has to be encountered. As a consequence, the proof-search procedure is *terminating*.

Second, we consider the termination of PSGLS from a semantic point of view. It has to be noted that loops like in Example 12.4.2 emerge from the use of the rule (GLR). In addition to that, each rule application of (GLR) semantically corresponds to a "jump" from one point in the frame to another. Given that the GL-frames are irreflexive we know that these jumps are effectively transitioning from one point to a *different* one. Furthermore, the converse well-foundedness of the accessibility relation of GL-frames entails that we can only do finitely many such jumps. This semantic property of converse well-foundedness consequently manifests itself proof-theoretically by allowing only finitely many applications of the rule (GLR) above a sequent without *creating a contradiction.* How does a contradiction manifest itself in this setting? Through an identity sequent or an instance of ($\bot$L), assuming the semantic interpretation of a sequent $\Gamma \Rightarrow \Delta$ as $\mathcal{M}, w \Vdash \Gamma$ and $\mathcal{M}, w \nVdash \Delta$. In the case of GLS, it so happens that the identities on boxes and propositional variables are sufficient to detect the contradictions pertaining to an identity. Following a suggestion by Stepan Kuznetsov at the conference Advances in Modal Logic 2022, we believe that a similar proof-theoretic manifestation of converse well-foundedness is at play in the cyclic sequent calculus for GLL defined by Shamkanov [134]. This sequent calculus is nothing but the usual sequent calculus for K4L [126], accounting for transitivity, enhanced with cycles, accounting for well-foundedness.

While this is the essence of the content of the previous lemma, we effectively only use the fact that mhd(s) decreases upwards in the rules of PSGLS.

Lemma 12.6.3. If r is a rule instance from PSGLS with conclusion s_0 and s_1 as one of the premises, then $\text{mhd}(s_1) < \text{mhd}(s_0)$.

Proof. (⌨) Suppose that $\text{mhd}(s_1) \geq \text{mhd}(s_0)$. Let $\mathfrak{d}_0$ and $\mathfrak{d}_1$ be the derivations of, respectively, s_0 and s_1 witnessing Theorem 12.6.1. Then the following $\mathfrak{d}_2$ is derivation of s_0 of height $\text{mhd}(s_1) + 1$.



Because of the maximality of $\mathfrak{d}_0$, we get that the height of $\mathfrak{d}_0$ is greater than the height of $\mathfrak{d}_2$, i.e. $\text{mhd}(s_1)+1 \leq \text{mhd}(s_0)$. As our initial assumption implies that $\text{mhd}(s_1)+1 > \text{mhd}(s_0)$, we reached a contradiction. ■

Coq is constructive, so how does it allow a proof by contradiction? It can do a proof by contradiction (without having to introduce classical axioms) when dealing with an expression of the decidable fragment. Here, $\text{mhd}(s_1) < \text{mhd}(s_0)$ can be decided because mhd is computable.

Now that the mhd notion is defined, we show that the PS prefix in PSGLS is justified: we can use the latter to decide the provability of sequents in GLS.

To do so, we first prove straightforwardly that a sequent is provable in PSGLS if and only if it is provable in GLS.

Proposition 12.6.1. For all Γ and Δ: $\Gamma \Rightarrow \Delta$ has a proof in PSGLS iff $\Gamma \Rightarrow \Delta$ has a proof in GLS.

Proof. (⌨) From left to right it suffices to notice that all rules of PSGLS but (IdB) are rules of GLS, and that the latter rule is admissible in GLS by Lemma 12.5.2. (⌨) From right to left, we notice that all rules of GLS but (GLR) are rules of PSGLS. Indeed, (GLR) in GLS is not restricted, while it is PSGLS. However, if the application of (GLR) in GLS violates the conditions imposed in PSGLS, we can clearly prove the conclusion of this application via (IdP), ($\bot$L) or (IdB). ■

So, according to the above general description, it suffices to prove that PSGLS can be used to decide the provability of sequents in GLS to show that the former deserves its prefix. We obtain these results through the existence for each sequent of a derivation of maximum height in PSGLS.

Second, we obtain a decision procedure for the provability of sequents in PSGLS using Lemma 12.6.3.

Theorem 12.6.2. Provability in PSGLS is decidable.

Proof. (⌨) We prove the statement by strong induction on the mhd of sequents. So, we assume that the result holds for all sequent s' such that $\mathrm{mhd}(s') < \mathrm{mhd}(s)$ and proceed to show that the result holds for s. By Lemma 12.6.2 we can consider the finite list $Prems(s)$ of all sequents s_0 such that there is an application of a PSGLS rule r with s as conclusion of r and s_0 as premise of r. By Lemma 12.6.3 we know that any s_0 in $Prems(s)$ is such that $\mathrm{mhd}(s_0) < \mathrm{mhd}(s)$. So, we can apply the induction hypothesis on all of the sequents in $Prems(s)$. It then suffices to check whether there is a rule r with conclusion s and premises in $Prems(s)$ and provable, which is doable in finite time as there are only finitely many rules applicable to s. If there is such a rule and premises, then s is provable. If not, s is not provable. ■

As an immediate corollary, we obtain that the provability of sequents in GLS is decidable.

Corollary 12.6.1. Provability in GLS is decidable.

Proof. (⌨) By Proposition 12.6.1, the results holds for GLS if and only if it holds for PSGLS. Consequently, Theorem 12.6.2 gives us the desired result. ■

So, PSGLS embodies a proof-search procedure on GLS according to our general definition. Note that we formalised the latter result in `Type`, so we can extract a program effectively deciding sequents in GLS.

In the next section, we use the notion of mhd, obtained through the termination of PSGLS, to establish cut-elimination for GLS.

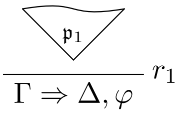
12.7 Cut-elimination for GLS

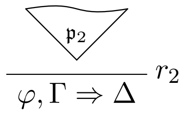
To prove cut-elimination, our main theorem, we first prove through local proof transformations that additive cut is admissible.

Theorem 12.7.1. The additive cut rule is admissible in GLS.

```
Theorem GLS_cut_adm : forall φ Γ0 Γ1 Δ0 Δ1,
 (GLS_prv (Γ0++Γ1,Δ0++φ::Δ1) * GLS_prv (Γ0++φ::Γ1,Δ0++Δ1))
                  -> GLS_prv (Γ0++Γ1,Δ0++Δ1).
```

Proof. (⌨) Let $\mathfrak{p}_1$ (with last rule r_1) and $\mathfrak{p}_2$ (with last rule r_2) be proofs in GLS of $\Gamma \Rightarrow \Delta, \varphi$ and $\varphi, \Gamma \Rightarrow \Delta$ respectively, as shown below.





It suffices to show that there is a proof in GLS of $\Gamma \Rightarrow \Delta$. We reason by strong primary induction (PI) on the size of the cut-formula φ, giving the primary inductive hypothesis (PIH), and strong secondary induction (SI) on $\mathrm{mhd}(s)$ of the conclusion of a cut, giving the secondary inductive hypothesis (SIH).

There are five cases to consider for r_1: one for each rule in GLS. We separate them by using Roman numerals. The SIH is invoked in all of the following cases: (III-a), (III-b-1), (III-b-2), (IV) and (V-a-2).

(I) $\mathbf{r_1}$ =(IdP) : If φ is not principal in r_1, then the latter must have the following form.

$$\dfrac{}{\Gamma_0, p \Rightarrow \Delta_0, p, \varphi}\;(\text{IdP})$$

where $\Gamma_0, p = \Gamma$ and $\Delta_0, p = \Delta$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, p \Rightarrow \Delta_0, p$, and is an instance of an initial sequent. So we are done.

If φ principal in r_1, i.e. $\varphi = p$, then $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, p \Rightarrow \Delta$. Thus, the conclusion of r_2 is of the form $\Gamma_0, p, p \Rightarrow \Delta$. We can consequently apply Lemma 12.5.5 (ii) to obtain a proof of $\Gamma_0, p \Rightarrow \Delta$.

(II) $\mathbf{r_1}$ =(⊥L): Then r_1 must have the following form.

$$\dfrac{}{\Gamma_0, \bot \Rightarrow \Delta, \varphi}\;(\bot\text{L})$$

where $\Gamma_0, \bot = \Gamma$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Gamma_0, \bot \Rightarrow \Delta$, and is an instance of an initial sequent. So we are done.

(III) $\mathbf{r_1}$ =(→R) : We distinguish two cases.
(III-a) If φ is not principal in r_1, then the latter must have the following form.

$$\dfrac{\Gamma, \psi \Rightarrow \Delta_0, \chi, \varphi}{\Gamma \Rightarrow \Delta_0, \psi \to \chi, \varphi}\;(\to\text{R})$$

where $\Delta_0, \psi \to \chi = \Delta$. Thus, we have that the sequent $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\Gamma \Rightarrow \Delta_0, \psi \to \chi$ and $\varphi, \Gamma \Rightarrow \Delta_0, \psi \to \chi$. We can apply Lemma 12.5.4 (ii) on the proof of the latter to get a proof of $\varphi, \Gamma, \psi \Rightarrow \Delta_0, \chi$. Thus proceed as follows.

$$\dfrac{\dfrac{\Gamma, \psi \Rightarrow \Delta_0, \chi, \varphi \qquad \varphi, \Gamma, \psi \Rightarrow \Delta_0, \chi}{\Gamma, \psi \Rightarrow \Delta_0, \chi}\;\text{SIH}}{\Gamma \Rightarrow \Delta_0, \psi \to \chi}\;(\to\text{R})$$

Note that the use of SIH is justified here since the last rule in this proof is an instance of (→R) in PSGLS and hence $\text{mhd}(\Gamma, \psi \Rightarrow \Delta_0, \chi) < \text{mhd}(\Gamma \Rightarrow \Delta_0, \psi \to \chi)$ by Lemma 12.6.3.
(III-b) If φ principal in r_1, i.e. $\varphi = \psi \to \chi$, then r_1 must have the following form.

$$\dfrac{\psi, \Gamma \Rightarrow \Delta, \chi}{\Gamma \Rightarrow \Delta, \psi \to \chi}\;(\to\text{R})$$

The conclusion of r_2 must be of the form $\psi \to \chi, \Gamma \Rightarrow \Delta$. In that case, we distinguish two further cases. In the first case, $\psi \to \chi$ is principal in r_2. Consequently, the latter must have the following form.

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \chi, \Gamma \Rightarrow \Delta}{\psi \to \chi, \Gamma \Rightarrow \Delta}\;(\to\text{L})$$

Proceed as follows.

$$\dfrac{\Gamma \Rightarrow \Delta, \psi \qquad \dfrac{\psi, \Gamma \Rightarrow \Delta, \chi \qquad \dfrac{\chi, \Gamma \Rightarrow \Delta}{\chi, \psi, \Gamma \Rightarrow \Delta}\;\text{Lem.12.5.3 (ii)}}{\psi, \Gamma \Rightarrow \Delta}\;\text{PIH}}{\Gamma \Rightarrow \Delta}\;\text{PIH}$$

In the second case, $\psi \to \chi$ is not principal in r_2. In the cases where r_2 is one of (IdP) and (⊥L) proceed respectively as in **(I)** and **(II)** when the cut-formula is not principal in the rule considered. We are left with the cases where r_2 is one of (→R), (→L) and (GLR).
(III-b-1) If r_2 is (→R) then it must have the following form.

$$\dfrac{\psi \to \chi, \delta, \Gamma \Rightarrow \Delta_0, \gamma}{\psi \to \chi, \Gamma \Rightarrow \Delta_0, \delta \to \gamma}\ (\to\text{R})$$

where $\Delta_0, \delta \to \gamma = \Delta$. In that case, note that the provable sequent $\Gamma \Rightarrow \Delta, \psi \to \chi$ is of the form $\Gamma \Rightarrow \Delta_0, \delta \to \gamma, \psi \to \chi$. We can use Lemma 12.5.4 (ii) on the proof of the latter to get a proof of $\delta, \Gamma \Rightarrow \Delta_0, \gamma, \psi \to \chi$. Proceed as follows.

$$\dfrac{\begin{array}{c} \delta, \Gamma \Rightarrow \Delta_0, \gamma, \psi \to \chi \qquad \psi \to \chi, \delta, \Gamma \Rightarrow \Delta_0, \gamma \\ \hdashline \delta, \Gamma \Rightarrow \Delta_0, \gamma \end{array}\ \text{SIH}}{\Gamma \Rightarrow \Delta_0, \delta \to \gamma}\ (\to\text{R})$$

Note that the use of SIH is justified here as the last rule in this proof is effectively an instance of ($\to$R) in PSGLS, hence $\text{mhd}(\Gamma, D \Rightarrow \Delta_0, \gamma) < \text{mhd}(\Gamma \Rightarrow \Delta_0, \delta \to \gamma)$ by Lemma 12.6.3.

(III-b-2) If r_2 is ($\to$L) then it must have the following form.

$$\dfrac{\psi \to \chi, \Gamma_0 \Rightarrow \Delta, \delta \qquad \psi \to \chi, \gamma, \Gamma_0 \Rightarrow \Delta}{\psi \to \chi, \delta \to \gamma, \Gamma_0 \Rightarrow \Delta}\ (\to\text{L})$$

where $\Gamma_0, \delta \to \gamma = \Gamma$. In that case, note that the provable sequent $\Gamma \Rightarrow \Delta, \psi \to \chi$ is of the form $\Gamma_0, \delta \to \gamma \Rightarrow \Delta, \psi \to \chi$. We can use Lemma 12.5.4 (i) on the proof of the latter to get proofs of both $\Gamma_0 \Rightarrow \Delta, \delta, \psi \to \chi$ and $\Gamma_0, \gamma \Rightarrow \Delta, \psi \to \chi$. Thus proceed as follows.

$$\dfrac{\begin{array}{c} \Gamma_0 \Rightarrow \Delta, \delta, \psi \to \chi \qquad \psi \to \chi, \Gamma_0 \Rightarrow \Delta, \delta \\ \hdashline \Gamma_0 \Rightarrow \Delta, \delta \end{array}\ \text{SIH} \qquad \begin{array}{c} \Gamma_0, \gamma \Rightarrow \Delta, \psi \to \chi \qquad \psi \to \chi, \gamma, \Gamma_0 \Rightarrow \Delta \\ \hdashline \Gamma_0, \gamma \Rightarrow \Delta \end{array}\ \text{SIH}}{\Gamma_0, \delta \to \gamma \Rightarrow \Delta}\ (\to\text{L})$$

Note that both uses of SIH are justified here as the last rule in this proof is effectively an instance of ($\to$L) in PSGLS, hence $\text{mhd}(\Gamma_0 \Rightarrow \Delta, \delta) < \text{mhd}(\Gamma_0, \delta \to \gamma \Rightarrow \Delta)$ and $\text{mhd}(\Gamma_0, \gamma \Rightarrow \Delta) < \text{mhd}(\Gamma_0, \delta \to \gamma \Rightarrow \Delta)$ by Lemma 12.6.3.

(III-b-3) If r_2 is (GLR) then it must have the following form.

$$\dfrac{\boxtimes\Gamma_0, \Box\delta \Rightarrow \delta}{\Phi, \psi \to \chi, \Box\Gamma_0 \Rightarrow \Box\delta, \Box\Delta_0, \Psi}\ (\text{GLR})$$

where $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\delta, \Box\Delta_0, \Psi = \Delta$. In that case, note that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\delta, \Box\Delta_0, \Psi$. To obtain a proof of the latter, we apply the rule (GLR) on the premise of r_2 without weakening $\psi \to \chi$:

$$\dfrac{\boxtimes\Gamma_0, \Box\delta \Rightarrow \delta}{\Phi, \Box\Gamma_0 \Rightarrow \Box\delta, \Box\Delta_0, \Psi}\ (\text{GLR})$$

(IV) $\mathbf{r_1}$ =($\to$L): Then r_1 must have the following form.

$$\dfrac{\Gamma_0 \Rightarrow \Delta, \psi, \varphi \qquad \chi, \Gamma_0 \Rightarrow \Delta, \varphi}{\psi \to \chi, \Gamma_0 \Rightarrow \Delta, \varphi}\ (\to\text{L})$$

where $\psi \to \chi, \Gamma_0 = \Gamma$. Thus, we have that the sequents $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\psi \to \chi, \Gamma_0 \Rightarrow \Delta$ and $\varphi, \psi \to \chi, \Gamma_0 \Rightarrow \Delta$. It thus suffices to apply Lemma 12.5.4 (i) on the proof of the latter to obtain proofs of both $\varphi, \Gamma_0 \Rightarrow \Delta, \psi$ and $\varphi, \chi, \Gamma_0 \Rightarrow \Delta$, and then proceed as follows.

$$\dfrac{\begin{array}{c} \Gamma_0 \Rightarrow \Delta, \psi, \varphi \qquad \varphi, \Gamma_0 \Rightarrow \Delta, \psi \\ \hdashline \Gamma_0 \Rightarrow \Delta, \psi \end{array}\ \text{SIH} \qquad \begin{array}{c} \chi, \Gamma_0 \Rightarrow \Delta, \varphi \qquad \varphi, \chi, \Gamma_0 \Rightarrow \Delta \\ \hdashline \chi, \Gamma_0 \Rightarrow \Delta \end{array}\ \text{SIH}}{\psi \to \chi, \Gamma_0 \Rightarrow \Delta}\ (\to\text{L})$$

Note that both uses of SIH are justified here as the last rule in this proof is effectively an instance of ($\to$L) in PSGLS, hence $\text{mhd}(\Gamma_0 \Rightarrow \Delta, \psi) < \text{mhd}(\psi \to \chi, \Gamma_0 \Rightarrow \Delta)$ and $\text{mhd}(\chi, \Gamma_0 \Rightarrow \Delta) < \text{mhd}(\psi \to \chi, \Gamma_0 \Rightarrow \Delta)$ by Lemma 12.6.3.

(V) $\mathbf{r_1}$ =(GLR): Then we distinguish two cases.

(V-a) φ is the diagonal formula in r_1:

$$\dfrac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi, \Box\Delta_0, \Psi}\ \text{(GLR)}$$

where $\varphi = \Box\psi$ and $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\Delta_0, \Psi = \Delta$. Thus, we have that the sequents $\Gamma \Rightarrow \Delta$ and $\varphi, \Gamma \Rightarrow \Delta$ are respectively of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\Delta_0, \Psi$ and $\Box\psi, \Phi, \Box\Gamma_0 \Rightarrow \Box\Delta_0, \Psi$. We now consider r_2. If r_2 is one of (IdP), ($\bot$L), ($\to$R) and ($\to$L) then respectively proceed as in (I), (II), (III) and (IV) when the cut-formula is not principal in the rules considered by using SIH. We are consequently left to consider the case when r_2 is (GLR). Then r_2 is of the following form:

$$\dfrac{\psi, \Box\psi, \boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\Phi, \Box\psi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_1, \Psi}\ \text{(GLR)}$$

where $\Box\chi, \Box\Delta_1 = \Box\Delta_0$. In this situation, we distinguish two sub-cases.

(V-a-1) One of the rules (IdP), ($\bot$L) or (IdB) is applicable to $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_1, \Psi$, then we are done for the two first cases as it suffices to apply the corresponding rules to obtain a proof of the conclusion of the cut-rule. For the case of (IdB) it suffices to apply Lemma 12.5.2.

(V-a-2) None of these rules is applicable to $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_1, \Psi$ (NoInit). Then, proceed as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\Box\Gamma_0 \Rightarrow \Box\psi}\ \text{(GLR)}}{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi, \Box\psi}\ \text{Lem.12.5.3} \qquad \dfrac{\dfrac{\boxtimes\Gamma_0, \Box\psi \Rightarrow \psi}{\boxtimes\Gamma_0, \Box\psi, \Box\chi \Rightarrow \chi, \psi}\ \text{Lem.12.5.3} \qquad \psi, \Box\psi, \boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\boxtimes\Gamma_0, \Box\psi, \Box\chi \Rightarrow \chi}\ \text{PIH}}{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}\ \text{SIH}}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_1, \Psi}\ \text{(GLR)}$$

Note that the use of SIH is justified here as the assumption NoInit ensures that the last rule in this proof is effectively an instance of (GLR) in PSGLS, hence mhd($\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi$) $<$ mhd($\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_1, \Psi$) by Lemma 12.6.3.

(V-b) φ is not the diagonal formula in r_1:

$$\dfrac{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \varphi, \Box\Delta_0, \Psi}\ \text{(GLR)}$$

where $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\chi, \Box\Delta_0, \Psi = \Delta$. In that case, note that the sequent $\Gamma \Rightarrow \Delta$ is of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi$. To obtain a proof of the latter, we apply the rule (GLR) on the premise of r_1 without weakening φ:

$$\dfrac{\boxtimes\Gamma_0, \Box\chi \Rightarrow \chi}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi, \Box\Delta_0, \Psi}\ \text{(GLR)}$$

■

Before turning to cut-elimination let us comment on the need to use additive cuts in the previous proof. To justify a cut through SIH, we need to link the sequent-conclusion of the initial cut to the sequent-conclusion of the newly created cut by a chain of rule applications which make mhd decrease upwards. Now, contraction and weakening can increase mhd upwards. While it is obvious for contraction, this fact is surprising for weakening. Ultimately, weakening can increase mhd upwards as it can delete on the left a boxed formula $\Box\varphi$ whose role is to block another potential application of (GLR) on $\Box\varphi$. So, in the mhd technique we cannot use contraction or weakening in the chain linking the two sequent-conclusion, forbidding us from considering multiplicative cuts. The use of additive cuts allows us to circumvent this difficulty. This sensitivity of the proof technique is surprising as both calculi admit weakening and contraction, making additive and multiplicative cuts equivalent.

It is commonly accepted that a purely syntactic proof of cut-admissibility, based on local proof transformations, provides a cut-elimination procedure: eliminate topmost cuts first. So, the above proof-theoretically establishes that cuts are eliminable in the calculus

GLS extended with (cut). To effectively prove this statement in Coq we explicitly encode the additive cut rule as follows:

$$\frac{(\Gamma 0{+}{+}\Gamma 1, \Delta 0{+}{+}\varphi{:}{:}\Delta 1) \qquad (\Gamma 0{+}{+}\varphi{:}{:}\Gamma 1, \Delta 0{+}{+}\Delta 1)}{(\Gamma 0{+}{+}\Gamma 1, \Delta 0{+}{+}\Delta 1)}$$

With this rule in hand, we can encode the set of rules `GLS_cut_rules` as `GLS_rules` enhanced with (cut), i.e. the calculus GLS + (cut). We can finally turn to the elimination of additive cuts:

Theorem 12.7.2. The additive cut rule is eliminable from GLS + (cut).

```
Theorem GLS_cut_elimination : forall s,
 (GLS_cut_prv s) -> (GLS_prv s).
```

Proof. (⌨) Let `p` be the proof in GLS + (cut) of the sequent s. We prove the statement by induction on the structure of `p`. If the last rule applied is a rule in GLS, then it suffices to apply the induction hypothesis on the premises and then the rule. If the last rule applied is (cut), then we use the induction hypothesis on both premises and then Theorem 12.7.1. ■

The above theorem shows that given a proof in GLS + (cut) of a sequent, that is `GLS_cut_prv s`, we can transform this proof directly to obtain a proof in GLS of the same sequent. Given that this theorem is in fact a constructive function based on elements defined on `Type`, we can use the extraction feature of Coq and obtain a cut-eliminating Haskell program.

12.8 Conclusion

In this chapter, we introduced GLS, a multiset-based sequent calculus for the logic GLL, which avoids the confusions of the set-based calculus used by Brighton. We showed that this calculus enjoys the canonical proof-theoretical properties of exchange, weakening, and contraction.

Then, we introduced a second sequent calculus: PSGLS. In this calculus each sequent has a derivation of maximal height, allowing us to define the measure mhd, i.e. the height of this maximal derivation. We made use of this measure to provide a decision procedure for the provability of sequents in GLS using PSGLS, justifying our use of the prefix PS: PSGLS embodies a terminating backward proof-search procedure for GLS.

The mhd measure was crucially involved in the proof technique used to obtain cut-admissibility and first described by Brighton [19]. A cut-elimination result for GLS was consequently obtained as the proof of cut-admissibility only relied on steps locally transforming given proofs. From the formalisation of this result, we extracted a Haskell program effectively eliminating cuts from cut-containing GLS proofs.

In essence, we have seen how the termination of backward proof-search can be exploited to obtain cut-elimination. This phenomenon, relying on the mhd technique, is particularly interesting because the termination of backward proof-search is close to a semantic proof of completeness, and the latter is typically much simpler to achieve than a proof of cut-elimination. Investigations in the range of applicability of this technique are consequently in order, with a focus on calculi possessing a terminating proof-search procedure. The next chapter constitutes a step in this direction.

Before closing the current chapter, let us defuse a potential misreading of our results. Indeed, our work may appear to beg the following question: if we first need to show semantic cut-free completeness to use this technique, then we already know that every instance of cut is admissible, so, what is the point? Note that this misses the mark. We chose to introduce PSGLS to clarify the role of terminating proof-search in the argument, and to demonstrate that the additional notion of regress tree was not essential. In particular, we did not have to show that PSGLS was complete for our purposes.

Chapter 13

Intuitionistic Provability Logic

For this chapter the following sections of the Toolbox I are required: Section 2.1, Section 5.1, the introduction of Chapter 4 and Section 4.2.

The results of this chapter are extensions of the article "Direct elimination of additive-cuts in GL4ip: verified and extracted." [65], written jointly with Rajeev Goré.

Their formalisation can be found here: https://github.com/ianshil/PhD_thesis/tree/main/Cut_Elim_iGLS.

13.1 Introduction

As shown in the previous Chapter 12, proofs of cut-admissibility for classical modal provability logics are usually not trivial. More specifically, the standard double-induction on the size of the cut-formula and the height of the derivation do not suffice. To solve this problem, there are now two known approaches. First, Valentini's introduction of a third complex parameter called "width" [156] in addition to these two traditional induction measures [156] led to a proof of cut-admissibility, as confirmed by Goré and Ramanayake [61] despite controversies. Second, the mhd technique introduced by Brighton [19] and rectified in Chapter 12 constitutes another pathway to cut-admissibility.

Recently, van der Giessen and Iemhoff [158] showed that the proof theory of intuitionistic provability logics is also complicated. They gave a cut-free sequent calculus GL3ip for intuitionistic provability logic extending the standard G3ip [153] calculus for intuitionistic logic with the following well-known rule:

$$\frac{\Gamma, \Box\Gamma, \Box\varphi \Rightarrow \varphi}{\Phi, \Box\Delta, \Box\Gamma \Rightarrow \Box\varphi}\ (\text{GLR})$$

Similarly to G3ip, the admissibility of the rules of weakening and contraction can easily be shown for GL3ip. However, the admissibility of cut encounters the same problems as for GLS, leading van der Giessen and Iemhoff to successfully adapt the technique developed by Valentini, thus obtaining a direct proof of cut-admissibility for intuitionistic provability logic.

However, GL3ip cannot support a simple terminating backward proof-search strategy because its left-implication rule, inherited from G3ip and shown below, allows trivial cycles up the left premise as is well known:

$$\frac{\Gamma, \varphi \to \psi \Rightarrow \varphi \quad X, \psi \Rightarrow \chi}{\Gamma, \varphi \to \psi \Rightarrow \chi}\ (\to\text{L})$$

To solve this problem and characterize a terminating proof-search procedure, they follow Vorob'ev [162, 163], Dyckhoff [41] and Hudelmaier [76] (see Dyckhoff's paper for historical developments [42]) and define the calculus GL4ip by both slightly modifying the rule (GLR) and mimicking G4ip by replacing (→L) with a collection of rules sensitive to the form of the formula φ in $\varphi \to \psi$. To prove cut-admissibility they show that GL3ip and GL4ip are equivalent, in that they prove the same sequents.

They point out that although the calculus GL4ip enjoys terminating backward proof-search, the existence of a direct proof of cut-admissibility is doubtful: all standard methods fail, including Valentini's. While a direct and syntactic proof of cut-admissibility usually leads to a straightforward algorithm for cut-elimination, here the only potential cut-elimination algorithm for GL4ip is quite convoluted: (1) take a GL4ip proof containing cuts; (2) transform it to a GL3ip proof containing cuts ; (3) apply the cut-elimination procedure for GL3ip to obtain a cut-free GL3ip proof ; (4) transform the cut-free GL3ip proof into a cut-free GL4ip proof. In particular, the steps from (2) to (3), which rely on Valentini's complicated argument, and from (3) to (4), which involve intricate transformations, are anything but trivial. This indirection, coupled with the intricacies mentioned, can only lead to a painful and obscure algorithm for cut-elimination for GL4ip.

Naturally, the following question comes to mind: can we eliminate the indirection from GL4ip+(cut) to GL3ip+(cut) to GL3ip to GL4ip, and obtain a direct cut-elimination procedure for GL4ip? If so, then we would be able to extract the associated computer program for cut-elimination from a Coq formalisation using the automatic program extraction facilities of Coq.

Here, we answer this question positively by porting the second approach to cut-admissibility for the classical provability logic **GLL**, i.e. the mhd technique, to the intuitionistic case. Consequently, to obtain a direct syntactic proof of cut-admissibility for GL4ip we follow the argumentative structure laid out in Chapter 12. First, we show the admissibility in GL4ip of the structural rules by adapting the arguments from Dyckhoff and Negri [43]. Second, we define a proof-search procedure PSGL4ip on GL4ip. Furthermore, we develop a thorough termination argument by defining a local measure on sequents and a well-founded relation along which this measure decreases upwards in the proof-search. Finally, we directly prove cut-admissibility for GL4ip using the *mhd proof technique*, which makes use of the termination of PSGL4ip to attribute a *m*aximum *h*eight of *d*erivations to each sequent [19]. We use this number as an induction measure in an argument involving local and syntactic transformations, allowing us to exhibit and hence extract a cut-elimination procedure.

This chapter is organized as follows. We introduce the syntax, generalized Hilbert calculus, and Kripke semantics for i**GLL** in Section 13.2. Then, we introduce the multiset-based calculus GL4ip in Section 13.3. Section 13.4 exhibits common proof-theoretical properties of GL4ip. In Section 13.5 we define the calculus PSGL4ip, which captures a backward proof-search procedure on GL4ip. Furthermore, we show that sequents have a derivation of maximal height in PSGL4ip. Section 13.6 contains our main result, i.e. cut-elimination, which we obtain by proving in cut-admissibility by local transformations on proofs. We finally conclude in Section 13.7.

13.2 Basics of iGLL

In this chapter we consider a larger set of connectives than $\mathcal{C}_{\mathbf{CM}}$ defined in Section 2.1, i.e. the **I**ntuitionistic **M**odal set of connectives $\mathcal{C}_{\mathbf{IM}} = (Con_{\mathbf{IM}}, Ar_{Con_{\mathbf{IM}}})$ where:

- $Con_{\mathbf{IM}} = \{\bot, \wedge, \vee, \to, \Box\}$;

- $Ar_{Con_{\mathbf{IM}}}$ is such that:

$$\begin{array}{rcl} Ar_{Con_{\mathbf{BI}}}(\bot) & = & 0 \\ Ar_{Con_{\mathbf{IM}}}(\wedge) = Ar_{Con_{\mathbf{IM}}}(\vee) = Ar_{Con_{\mathbf{IM}}}(\to) & = & 2 \\ Ar_{Con_{\mathbf{IM}}}(\Box) & = & 1 \end{array}$$

We can thus define the propositional language $\mathbb{L}_{\mathbf{IM}} = (\mathbb{V}, \mathcal{C}_{\mathbf{IM}})$ and obtain its infix grammar:

$$\varphi ::= p \in \mathbb{V} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi$$

The reason we consider this extended set of connectives is that, unlike in classical logic, in intuitionistic logic we cannot define $\wedge$ and $\vee$ from the connectives of $\mathcal{C}_{\mathbf{CM}}$. We recall that for a set $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$, $\boxtimes\Gamma$ is the set $\{\varphi_1, \Box\varphi_1, \ldots, \varphi_n, \Box\varphi_n\}$. The last syntactic element we need to specify is the notion of *weight of a formula*, defined recursively on the structure of formulas.

Definition 13.2.1. The *weight* $w(\varphi)$ of a formula φ is defined as follows:

$$\begin{aligned} weight(\bot) = weight(p) &= 1 \\ weight(\psi \vee \chi) = weight(\psi \to \chi) &= weight(\psi) + weight(\chi) + 1 \\ weight(\psi \wedge \chi) &= weight(\psi) + weight(\chi) + 2 \end{aligned}$$

This notion is different from the size of formula defined in Definition 12.2.1, as it does not simply count the number of symbols appearing in a formula. More specifically, the weight differs from the size in one place: the clause for conjunction. The main reason behind this oddity is that we use the notion of weight to show the termination of a proof-search procedure on a given set of rules, and without giving a heavier weight to conjunction the argument would not go through. So, the peculiarity of this definition originates from nothing else but technical details.

Next, we define the generalized Hilbert calculus $\mathsf{iGL}\mathbf{H} = (\mathcal{A}_{iK} \cup \{GL\}, \{(\mathrm{MP}), (\mathrm{wNec})\})$ where $\mathcal{A}_{iK}$, presented below, is the set of axioms for $\mathsf{iK}\mathbf{L}$. The latter is the intuitionistic variant of the basic normal modal logic $\mathsf{wK}\mathbf{L}$, and is captured by the generalized Hilbert calculus $\mathsf{iK}\mathbf{H} = (\mathcal{A}_{iK}, \{(\mathrm{MP}), (\mathrm{wNec})\})$. The logic $\mathsf{iGL}\mathbf{L}$ is defined as the set $\{(\Gamma, \varphi) \mid \Gamma \vdash_{\mathsf{iGL}\mathbf{H}} \varphi\}$.

Definition 13.2.2. We define the set of axioms $\mathcal{A}_{iK}$ below:

A_1	$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$	A_7	$(\varphi \to \psi) \to ((\varphi \to \chi) \to (\varphi \to (\psi \wedge \chi)))$
A_2	$\varphi \to (\varphi \vee \psi)$	A_8	$(\varphi \to (\psi \to \chi)) \to ((\varphi \wedge \psi) \to \chi)$
A_3	$\psi \to (\varphi \vee \psi)$	A_9	$((\varphi \wedge \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
A_4	$(\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$	A_{10}	$(\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$
A_5	$(\varphi \wedge \psi) \to \varphi$	A_{11}	$\bot \to \varphi$
A_6	$(\varphi \wedge \psi) \to \psi$	A_{12}	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

The Kripke semantics for $\mathsf{iGL}\mathbf{L}$ has an intuitionistic and modal flavor. As a consequence, two relations of accessibility are required to deal with each dimension: R for the modal and $\leq$ for the intuitionistic. In addition to that, one needs to add further constraints on these relations to make the persistence lemma hold [138, Section 3.3]. As discussed by Litak [91, Section 2.1], requiring that the relation composition of $\leq$ and R is a subset of R, in symbols "$\leq; R \subseteq R$", is an adequate constraint. Consequently, we obtain the following notions of frames and models.

Definition 13.2.3. An iGL-Kripke frame is a Kripke frame $\mathcal{F} = (W, \leq, R)$ such that:

- $\leq$ is *reflexive*, i.e. for all $w \in W$ we have $w \leq w$;
- $\leq$ is *transitive*, i.e. for all $w, v, u \in W$ if $[w \leq v$ and $v \leq u]$ then $w \leq u$;
- $(\leq; R) \subseteq R$ where ; is relation composition;
- R is *transitive*, i.e. for all $w, v, u \in W$ if $[wRv$ and $vRu]$ then wRu;
- R is *converse well-founded*, i.e. there are no infinite ascending sequences, that is sequences of the form $w_1 R w_2 R w_3 R \ldots$.

An iGL-Kripke model $\mathcal{M}$ is a model $\mathcal{M} = (\mathcal{F}, I)$, where $\mathcal{F}$ is a iGL-Kripke frame and $I : \mathbb{V} \to \mathsf{Pow}(W)$ is an interpretation function obeying persistence: for every $v, w \in W$ with $w \leq v$ and $p \in \mathbb{V}$, if $w \in I(p)$ then $v \in I(p)$.

Here again, such frames are necessarily irreflexive in R, else we would have a sequence of the form $wRwRwR\ldots$. For short, we talk about iGL-frames and iGL-models.

To define forcing in this context we combine the forcing notions we used in a modal (Definition 5.1.4) and intuitionistic case (Definition 8.7.2).

Definition 13.2.4. Given a, iGL-model $\mathcal{M} = (W, \leq, R, I)$, we define the forcing relation as follows:

$$\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in I(p) \\
\mathcal{M}, w \Vdash \bot & & \text{never} \\
\mathcal{M}, w \Vdash \varphi \wedge \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ and } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \varphi \text{ or } \mathcal{M}, w \Vdash \psi \\
\mathcal{M}, w \Vdash \varphi \to \psi & \text{iff} & \text{for all } v \text{ s.t. } w \leq v, \text{ if } \mathcal{M}, v \Vdash \varphi \text{ then } \mathcal{M}, v \Vdash \psi \\
\mathcal{M}, w \Vdash \Box\varphi & \text{iff} & \text{for all } v \text{ s.t. } wRv \text{ we have } \mathcal{M}, v \Vdash \varphi
\end{array}$$

The local and global semantic consequence relation from Definition 5.1.3 can be restricted to iGL-models. We write $\Gamma \models_l^{iGL} \varphi$ and $\Gamma \models_g^{iGL} \varphi$ to designate them. The situation is identical to the classical case GL**L**: (1) iGL**L** is sound for the local semantic consequence relation, and hence cannot be complete for the global semantic consequence relation ; (2) iGL**L** is compact while the local semantic consequence relation is not, hence the former cannot be complete with respect to the latter ; (3) iGL**L** and the local semantic consequence relation coincide on their sets of theorems.

The result (1) is rather straightforward to obtain as it suffices to check that axioms are valid and that rules preserve local semantic consecutions.

The second result (2) follows from an adaptation of the counterexample for the GL**L** case exhibited in Theorem 12.2.2.

$$\Gamma = \{\Diamond p_0, \Box(p_0 \to \Diamond p_1), \Box(p_1 \to \Diamond p_2), \Box(p_2 \to \Diamond p_3), \ldots, \Box(p_n \to \Diamond p_{n+1}), \ldots\}$$

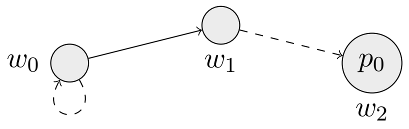
This set cannot be used in the current context as we are not anymore in a *classical* setting, where $\Diamond$ is definable as $\neg\Box\neg$, but in an *intuitionistic* setting, where this definability is known to be lost [138]. More precisely, we have that $\Diamond\varphi$ entails $\neg\Box\neg\varphi$, but not the other way around. So, without $\Diamond$ we cannot simply reuse the set Γ above. However, as pointed out by Rosalie Iemhoff in private communication, it is in fact sufficient to use the weaker combination $\neg\Box\neg$ instead of $\Diamond$.

Theorem 13.2.1. The local semantic consequence relation $\models_l^{iGL}$ is not compact.

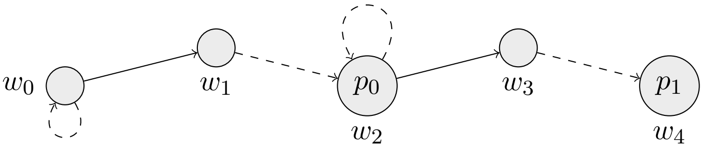
Proof ✍. We show that $\Gamma \models_l^{iGL} \bot$ for the following infinite set Γ.

$$\Gamma' = \{\neg\Box\neg p_0, \Box(p_0 \to \neg\Box\neg p_1), \Box(p_1 \to \neg\Box\neg p_2), \ldots, \Box(p_n \to \neg\Box\neg p_{n+1}), \ldots\}$$

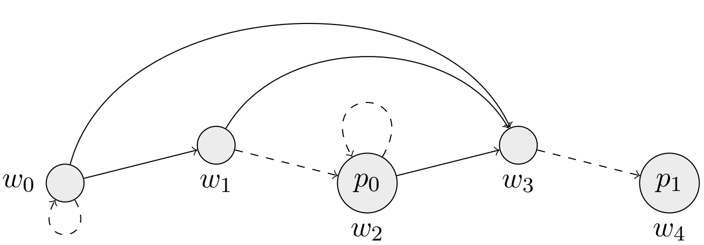
Here again, we get that the forcing of this set Γ' enforces the existence of an infinite ascending chain. Assume for a model $\mathcal{M}$ and a point w that $\mathcal{M}, w \Vdash \Gamma'$. As $\mathcal{M}, w_0 \Vdash \neg\Box\neg p_0$, we notably get $\mathcal{M}, w_0 \nVdash \Box\neg p_0$ by reflexivity of $\leq$. So, we have that there is a w_1 such that $w_0 R w_1$ and $\mathcal{M}, w_1 \nVdash \neg p_0$. As a consequence, we get that there is a w_2 such that $w_1 \leq w_2$ and $\mathcal{M}, w_2 \Vdash p_0$. So, we have the following setting, where dashed arrows represent $\leq$ and plain arrows represent R.



Now, as $\mathcal{M}, w_0 \Vdash \Box(p_0 \to \neg\Box\neg p_1)$ we get that $\mathcal{M}, w_1 \Vdash p_0 \to \neg\Box\neg p_1$, hence $\mathcal{M}, w_2 \Vdash \neg\Box\neg p_1$ as $w_1 \leq w_2$ and $\mathcal{M}, w_2 \Vdash p_0$. Here again, $\mathcal{M}, w_2 \Vdash \neg\Box\neg p_1$ entails the existence of two further worlds w_3 and w_4 as described below.



But how do we proceed from here? We have $\mathcal{M}, w_0 \Vdash \Box(p_1 \to \neg\Box\neg p_2)$ but no obvious way to reach w_3 from w_0 through an R relation. However, we need to remember that $(\leq; R) \subseteq R$. Thus, we get that $w_1 R w_3$ as $w_1 \leq w_2$ and $w_2 R w_3$. In addition to that, given that R is transitive, we obtain $w_0 R w_3$ from $w_0 R w_1$ and $w_1 R w_3$. The situation is as follows.



Thus, we are able to connect w_0 and w_3 in the appropriate way to use the formula $p_1 \to \neg\Box\neg p_2$ in w_3 in a similar way as we did with $p_0 \to \neg\Box\neg p_1$ in w_1. It then clearly appears that each formula $\Box(p_n \to \neg\Box\neg p_{n+1})$ implies the existence of another element in the chain $w_0 R w_1 R w_3 \dots R w_{2n+1}$. Consequently, the infinite set Γ' entails the presence of an infinite ascending chain, which is impossible in a iGL-frame. So, as no point in no iGL-model can force Γ, we trivially get that $\Gamma \models_l^{iGL} \bot$.

Furthermore, it is straightforward to see that any finite subset Γ' of Γ can be forced in a model: it suffices to have a model with a finite chain described by the finite number of formulas in Γ'.

So, we have that $\Gamma \models_l^{iGL} \bot$ but there is no finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \models_l^{iGL} \bot$. Consequently $\models_l^{iGL}$ is not compact. ■

The last result (3) is due to Ursini [155].

Theorem 13.2.2. $\emptyset \vdash_{\mathsf{iGLH}} \varphi$ if and only if $\emptyset \models_l^{iGL} \varphi$.

In the remainder of this chapter, we focus on the proof theory of iGL**L**. More specifically, we discuss *sequent calculi* for this logic.

13.3 The sequent calculus GL4ip

The sequent calculus GL4ip is given in Figure 13.1. While GL4ip manipulates *sequents*, it has to be noted that these sequents are of a different nature from the sequents manipulated in GLS: the former are of the shape $\Gamma \Rightarrow \varphi$ while the latter are of the shape $\Gamma \Rightarrow \Delta$, where $\Gamma, \Delta \subseteq Form_{\mathbb{L}_{\mathbf{IM}}}$ and $\varphi \in Form_{\mathbb{L}_{\mathbf{IM}}}$. This restriction, only allowing a single formula in the succedent instead of a set of formulas, was already imposed by Gentzen in his sequent calculus LJ for intuitionistic logic [53, 154].

When defining rules we put the label naming the rule on the left of the horizontal line for aesthetic reasons, while the label appears on the right of the line in the remaining of the article.

For the rule (IdP), ($\bot$L) and ($\to$R) we define the notion of principal formula following the modal case given in Section 4.2. For the rules ($\wedge$R), ($\wedge$L), ($\vee$R$_i$) and ($\vee$L) the

$$(\bot\mathrm{L})\ \frac{}{\bot, \Gamma \Rightarrow \chi} \qquad (\mathrm{IdP})\ \frac{}{\Gamma, p \Rightarrow p}$$

$$(\wedge\mathrm{L})\ \frac{\Gamma, \varphi, \psi \Rightarrow \chi}{\Gamma, \varphi \wedge \psi \Rightarrow \chi} \qquad (\wedge\mathrm{R})\ \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$$

$$(\vee\mathrm{L})\ \frac{\Gamma, \varphi \Rightarrow \chi \qquad \Gamma, \psi \Rightarrow \chi}{\Gamma, \varphi \vee \psi \Rightarrow \chi} \qquad (\vee\mathrm{R}_i)\ \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (i \in \{1, 2\})$$

$$(p\to\mathrm{L})\ \frac{\Gamma, p, \varphi \Rightarrow \chi}{\Gamma, p, p \to \varphi \Rightarrow \chi} \qquad (\to\mathrm{R})\ \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$$

$$(\Box\to\mathrm{L})\ \frac{\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi \qquad \Phi, \Box\Gamma, \psi \Rightarrow \chi}{\Phi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi} \qquad (\mathrm{GLR})\ \frac{\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi}{\Phi, \Box\Gamma \Rightarrow \Box\varphi}$$

$$(\wedge\to\mathrm{L})\ \frac{\Gamma, \varphi \to (\psi \to \chi) \Rightarrow \delta}{\Gamma, (\varphi \wedge \psi) \to \chi \Rightarrow \delta} \qquad (\vee\to\mathrm{L})\ \frac{\Gamma, \varphi \to \chi, \psi \to \chi \Rightarrow \delta}{\Gamma, (\varphi \vee \psi) \to \chi \Rightarrow \delta}$$

$$(\to\to\mathrm{L})\ \frac{\Gamma, \psi \to \chi \Rightarrow \varphi \to \psi \qquad \Gamma, \chi \Rightarrow \delta}{\Gamma, (\varphi \to \psi) \to \chi \Rightarrow \delta}$$

Figure 13.1: The sequent calculus GL4ip. Here, Φ does not contain any (top-level) boxed formula.

principal formula of that instance is defined as usual. In a rule instance of ($p\to$L), both a propositional variable instantiating p and the formula instantiating the featured $p \to \varphi$ are principal formulas of that instance. In a rule instance of ($\wedge\to$L), ($\vee\to$L), ($\to\to$L) or ($\Box\to$L), the formula instantiating respectively $(\varphi \wedge \psi) \to \chi$, $(\varphi \vee \psi) \to \chi$, $(\varphi \to \psi) \to \chi$ or $\Box\varphi \to \psi$ is the principal formula of that instance. Finally, we recall that in a rule instance of (GLR), the formula $\Box\varphi$ is called the *diagonal formula* [126].

Example 13.3.1. The following are examples of derivations in GL4ip. Note that while the first and second examples are derivations, the third is a proof.

$$p \Rightarrow q \to r \qquad \frac{\Rightarrow p}{\Rightarrow p \vee (p \to \bot)}\ (\vee\mathrm{R}_1) \qquad \cfrac{\cfrac{}{\Box p, p, \Box p \Rightarrow p}\ (\mathrm{IdP})}{\Box p \Rightarrow \Box p}\ (\mathrm{GLR})$$

Example 13.3.2. A special example of a derivation in GL4ip is the following:

$$\frac{\Box\varphi \to \varphi, \Box(\Box\varphi \to \varphi), \varphi, \varphi, \Box\varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi \qquad \Box(\Box\varphi \to \varphi), \varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi}{\Box\varphi \to \varphi, \Box(\Box\varphi \to \varphi), \varphi, \Box\varphi, \Box\varphi \Rightarrow \varphi}\ (\Box\to\mathrm{L})$$

The conclusion and left premise are identical modulo formula multiplicities, so the rule ($\Box\to$L) can be infinitely applied upwards on the left branch. As a consequence, the naive backward proof-search procedure in GL4ip does not terminate.

Finally, we restrict the additive cut rule, presented in Section 4.2, to sequents with a single formula in the succedent. In this rule φ is called the *cut-formula*.

$$\frac{\Gamma \Rightarrow \varphi \qquad \varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi}\ (\mathrm{cut})$$

The addition of the rule (cut) to GL4ip, giving us the calculus GL4ip + (cut), we obtain a sequent calculus for iGL**L** [158].

Theorem 13.3.1. *For all $\varphi \in Form_{\mathbb{L}_{\mathbf{IM}}}$ we have: $(\Gamma, \varphi) \in$ iGL**L** iff there is a finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \Rightarrow \varphi$ is provable in GL4ip + (cut)*

13.4 A path to contraction for GL4ip

In a similar way to our formalisation of the calculus GLS, the calculus GL4ip is encoded with sequents using lists and not multisets. Despite this distance between our formalisation and the pen-and-paper definition, list-sequents from the former mimic multiset-sequents from the latter thanks to the way we formalised our rules, as explained in Section 4.2. Below, `exch s se` encodes the fact that `se` is obtained from the sequent `s` by permuting two sub-lists in the list representing its antecedent.

Lemma 13.4.1 (Admissibility of exchange)**.** For all $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \varphi, \psi$ and χ, if we have that $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3 \Rightarrow \chi$ is provable in GL4ip, then so is $\Gamma_0, \Gamma_2, \Gamma_1, \Gamma_3 \Rightarrow \chi$.

```
Lemma GL4ip_adm_exch : forall s, (GL4ip_prv s) ->
        (forall se, (exch s se) -> (GL4ip_prv se)).
```

Proof. (⌨) This result is immediate for multiset-sequents. For list-sequents, we refer to our *long* formalisation (the file GL4ip_exch.v has 5000 lines of code!). ■

Given the above lemma, we allow ourselves to consider that the left-hand side of sequents is indeed a multiset. The remaining of this section extends the work of Dyckhoff and Negri [43] on G4ip to the sequent calculus GL4ip. Thus, the proofs they developed are embedded in our proofs and hence formalised. Most lemmata are proven by straightforward inductions on the structure of formulas or derivations.

First, we prove that the rule of weakening on the left is height-preserving admissible in GL4ip.

Lemma 13.4.2 (Height-preserving admissibility of weakening)**.** For all Γ, φ and ψ, if $\Gamma \Rightarrow \psi$ has a proof $\mathfrak{p}$ in GL4ip, then $\Gamma, \varphi \Rightarrow \psi$ has a proof $\mathfrak{p}'$ in GL4ip such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

Proof. (⌨) Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \psi$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\Gamma, \varphi \Rightarrow \psi$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma$ or $\bot \in \Gamma$. In all cases $\Gamma, \varphi \Rightarrow \psi$ is an instance of an initial sequent.

2. $r = (\wedge\text{R})$: then r is

$$\frac{\Gamma \Rightarrow \psi_0 \qquad \Gamma \Rightarrow \psi_1}{\Gamma \Rightarrow \psi_0 \wedge \psi_1}\ (\wedge\text{R})$$

   Then we can apply the induction hypothesis on the premises to get proofs of $\Gamma, \varphi \Rightarrow \psi_0$ and $\Gamma, \varphi \Rightarrow \psi_1$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\wedge\text{R})$ to get a proof of $\Gamma, \varphi \Rightarrow \psi_0 \wedge \psi_1$ of height less than or equal to $h(\mathfrak{p})$.

3. $r = (\wedge\text{L})$: then r is

$$\frac{\chi_0, \chi_1\Gamma_0 \Rightarrow \psi}{\chi_0 \wedge \chi_1\Gamma_0 \Rightarrow \psi}\ (\wedge\text{L})$$

   Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0, \chi_1\Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\wedge\text{L})$ to get a proof of $\chi_0 \wedge \chi_1\Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

4. $r = (\vee\text{R}_i)$ for $i \in \{1, 2\}$: then r is

$$\frac{\Gamma \Rightarrow \psi_i}{\Gamma \Rightarrow \psi_1 \vee \psi_2}\ (\vee\text{R}_i)$$

Then we can apply the induction hypothesis on the premise to get a proof of $\Gamma, \varphi \Rightarrow \psi_i$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\vee$R$_1$) to get a proof of $\Gamma, \varphi \Rightarrow \psi_1 \vee \psi_2$ of height less than or equal to $h(\mathfrak{p})$.

5. $r = (\vee\mathrm{L})$: then r is

$$\frac{\chi_0, \Gamma_0 \Rightarrow \psi \qquad \chi_1, \Gamma_0 \Rightarrow \psi}{\chi_0 \vee \chi_1, \Gamma_0 \Rightarrow \psi} \ (\vee\mathrm{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\chi_0, \Gamma_0, \varphi \Rightarrow \psi$ and $\chi_0, \Gamma_1, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\vee$L) to get a proof of $\chi_0 \vee \chi_1, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

6. $r = (\to\mathrm{R})$: then r is

$$\frac{\psi_0 \Gamma \Rightarrow \psi_1}{\Gamma \Rightarrow \psi_0 \to \psi_1} \ (\to\mathrm{R})$$

Then we can apply the induction hypothesis on the premise to get a proof $\psi_0, \Gamma, \varphi \Rightarrow \psi_1$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\to$R) to get a proof of $\Gamma, \varphi \Rightarrow \psi_0 \to \psi_1$ of height less than or equal to $h(\mathfrak{p})$.

7. $r = (p{\to}\mathrm{L})$: then r is

$$\frac{p, \chi, \Gamma_0 \Rightarrow \psi}{p, p \to \chi, \Gamma_0 \Rightarrow \psi} \ (p{\to}\mathrm{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $p, \chi, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($p{\to}$L) to get a proof of $p, p \to \chi, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

8. $r = (\wedge{\to}\mathrm{L})$: then r is

$$\frac{\chi_0 \to (\chi_1 \to \chi_2), \Gamma_0 \Rightarrow \psi}{(\chi_0 \wedge \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \psi} \ (\wedge{\to}\mathrm{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0 \to (\chi_1 \to \chi_2), \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\wedge{\to}$L) to get a proof of $(\chi_0 \wedge \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

9. $r = (\vee{\to}\mathrm{L})$: then r is

$$\frac{\chi_0 \to \chi_2, \chi_1 \to \chi_2, \Gamma_0 \Rightarrow \psi}{(\chi_0 \vee \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \psi} \ (\vee{\to}\mathrm{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0 \to \chi_2, \chi_1 \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\vee{\to}$L) to get a proof of $(\chi_0 \vee \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

10. $r = ({\to}{\to}\mathrm{L})$: then r is

$$\frac{\chi_1 \to \chi_2, \Gamma_0 \Rightarrow \chi_0 \to \chi_1 \qquad \chi_2, \Gamma_0 \Rightarrow \psi}{(\chi_0 \to \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \psi} \ ({\to}{\to}\mathrm{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\chi_1 \to \chi_2, \Gamma_0, \varphi \Rightarrow \chi_0 \to \chi_1$ and $\chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule (${\to}{\to}$L) to get a proof of $(\chi_0 \to \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

11. $r = (\Box{\to}\mathrm{L})$: then r is

$$\frac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0 \qquad \chi_1, \Phi, \Box\Gamma_0 \Rightarrow \psi}{\Box\chi_0 \to \chi_1, \Phi, \Box\Gamma_0 \Rightarrow \psi}\ (\Box{\to}\mathrm{L})$$

    Either φ is boxed, i.e. $\varphi = \Box\varphi_0$ for some φ_0, and then we can first apply the induction hypothesis twice on the left premise to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$ of height less than or equal to $h(\mathfrak{p}) - 1$. Second, we apply the induction hypothesis on the right premise to obtain a proof of $\chi_1, \Phi, \Box\varphi_0, \Box\Gamma_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the $(\Box{\to}\mathrm{L})$ rule to obtain a proof of $\Box\chi_0 \to \chi_1, \Phi, \Box\Gamma_0, \Box\varphi_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$. Or φ is not boxed and then we apply the induction hypothesis on the right premise to obtain a proof of $\chi_1, \Phi, \varphi, \Box\Gamma_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Then we can apply $(\Box{\to}\mathrm{L})$ on the initial left premise and the obtained sequent to obtain a proof of $\Box\chi_0 \to \chi_1, \Phi, \varphi, \Box\Gamma_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

12. $r = (\mathrm{GLR})$: then r is

$$\frac{\boxtimes\Gamma_0, \Box\psi_0 \Rightarrow \psi_0}{\Phi, \Box\Gamma_0 \Rightarrow \Box\psi_0}\ (\mathrm{GLR})$$

    Either φ is boxed, i.e. $\varphi = \Box\chi$ for some χ, and then we can apply the induction hypothesis twice to obtain a proof of $\chi, \Box\chi, \boxtimes\Gamma_0, \Box\psi_0 \Rightarrow \psi_0$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the (GLR) rule to obtain a proof of $\Phi, \Box\chi, \boxtimes\Gamma_0 \Rightarrow \Box\psi_0$ of height less than or equal to $h(\mathfrak{p})$. Or φ is not boxed and then we can apply (GLR) on the premise to obtain a proof of $\Phi_0, \Box\Gamma_0 \Rightarrow \Box\psi_0$ of height $h(\mathfrak{p})$, where $\Phi_0 = \Phi, \varphi$.

■

Second, we show the height-preserving invertibility in GL4ip of a majority of rules.

Lemma 13.4.3 (Height-preserving invertibility of rules). The rules $(\wedge\mathrm{R})$, $(\wedge\mathrm{L})$, $(\vee\mathrm{L})$, $({\to}\mathrm{R})$, $(p{\to}\mathrm{L})$, $(\wedge{\to}\mathrm{L})$, $(\vee{\to}\mathrm{L})$ are height-preserving invertible.

Proof. Each of these height-preserving invertibility results is obtained through a proof by strong induction on the height of the given derivation, and on an analysis of the last rule applied. All their proofs require the application of Lemma 13.4.2. Their formalisations can be found in the following list: $(\wedge\mathrm{R})$ ⌨, $(\wedge\mathrm{L})$ ⌨, $(\vee\mathrm{L})$ ⌨, $(p{\to}\mathrm{L})$ ⌨, $(\wedge{\to}\mathrm{L})$ ⌨, $(\vee{\to}\mathrm{L})$ ⌨. We only consider the height-preserving invertibility of $({\to}\mathrm{R})$ as an example (⌨).

Let $\mathfrak{p}$ be a proof of $\Gamma \Rightarrow \varphi \to \psi$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\Gamma, \varphi \Rightarrow \psi$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r \in \{(\mathrm{IdP}), (\wedge\mathrm{R}), (\vee\mathrm{R}_1), (\vee\mathrm{R}_2), (\mathrm{GLR})\}$: Then we reach a contradiction, as the application of these rules implies that the succedent of the sequent $\Gamma \Rightarrow \varphi \to \psi$ is a formula other than an implication.

2. $r = (\bot\mathrm{L})$: then either $\bot \in \Gamma$, and as a consequence $\Gamma, \varphi \Rightarrow \psi$ is an instance of $(\bot\mathrm{L})$.

3. $r = (\wedge\mathrm{L})$: then r is

$$\frac{\chi_0, \chi_1\Gamma_0 \Rightarrow \varphi \to \psi}{\chi_0 \wedge \chi_1\Gamma_0 \Rightarrow \varphi \to \psi}\ (\wedge\mathrm{L})$$

    Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0, \chi_1\Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\wedge\mathrm{L})$ to get a proof of $\chi_0 \wedge \chi_1\Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

4. $r = (\vee\text{L})$: then r is

$$\frac{\chi_0, \Gamma_0 \Rightarrow \varphi \to \psi \qquad \chi_1, \Gamma_0 \Rightarrow \varphi \to \psi}{\chi_0 \vee \chi_1, \Gamma_0 \Rightarrow \varphi \to \psi} (\vee\text{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of the sequents $\chi_0, \Gamma_0, \varphi \Rightarrow \psi$ and $\chi_0, \Gamma_1, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\vee\text{L})$ to get a proof of $\chi_0 \vee \chi_1, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

5. $r = (\to\text{R})$: then r is

$$\frac{\varphi, \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi} (\to\text{R})$$

Then we directly have a proof of $\Gamma, \varphi \Rightarrow \psi$ of height $h(\mathfrak{p}) - 1$.

6. $r = (p{\to}\text{L})$: then r is

$$\frac{p, \chi, \Gamma_0 \Rightarrow \varphi \to \psi}{p, p \to \chi, \Gamma_0 \Rightarrow \varphi \to \psi} (p{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $p, \chi, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(p{\to}\text{L})$ to get a proof of $p, p \to \chi, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

7. $r = (\wedge{\to}\text{L})$: then r is

$$\frac{\chi_0 \to (\chi_1 \to \chi_2), \Gamma_0 \Rightarrow \varphi \to \psi}{(\chi_0 \wedge \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \varphi \to \psi} (\wedge{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0 \to (\chi_1 \to \chi_2), \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\wedge{\to}\text{L})$ to get a proof of $(\chi_0 \wedge \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

8. $r = (\vee{\to}\text{L})$: then r is

$$\frac{\chi_0 \to \chi_2, \chi_1 \to \chi_2, \Gamma_0 \Rightarrow \varphi \to \psi}{(\chi_0 \vee \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \varphi \to \psi} (\vee{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0 \to \chi_2, \chi_1 \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\vee{\to}\text{L})$ to get a proof of $(\chi_0 \vee \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

9. $r = ({\to}{\to}\text{L})$: then r is

$$\frac{\chi_1 \to \chi_2, \Gamma_0 \Rightarrow \chi_0 \to \chi_1 \qquad \chi_2, \Gamma_0 \Rightarrow \varphi \to \psi}{(\chi_0 \to \chi_1) \to \chi_2, \Gamma_0 \Rightarrow \varphi \to \psi} ({\to}{\to}\text{L})$$

Then we can first apply the induction hypothesis on the right premise to get a proof of $\chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Second, we apply Lemma 13.4.2 on the left premise to obtain a proof of $\chi_1 \to \chi_2, \Gamma_0, \varphi \Rightarrow \chi_0 \to \chi_1$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $({\to}{\to}\text{L})$ to get a proof of $(\chi_0 \to \chi_1) \to \chi_2, \Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

10. $r = (\Box{\to}\text{L})$: then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0 \qquad \chi_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi \to \psi}{\Box\chi_0 \to \chi_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi \to \psi}\ (\Box{\to}\mathrm{L})$$

First, we apply the induction hypothesis on the right premise to get a proof of $\chi_1, \Phi, \Box\Gamma_0, \varphi \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})-1$. Now, we need to distinguish two cases. If φ is boxed, i.e. $\varphi = \Box\varphi_0$ for some φ_0, then we apply Lemma 13.4.2 twice on the left premise to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the $(\Box{\to}\mathrm{L})$ rule to obtain a proof of $\Box\chi_0 \to \chi_1, \Phi, \Box\Gamma_0, \Box\varphi_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$. Or φ is not boxed and then we can straightforwardly apply $(\Box{\to}\mathrm{L})$ on the initial left premise and the obtained sequent $\chi_1, \Phi, \Box\Gamma_0, \varphi \Rightarrow \psi$ to obtain a proof of $\Box\chi_0 \to \chi_1, \Phi, \varphi, \Box\Gamma_0 \Rightarrow \psi$ of height less than or equal to $h(\mathfrak{p})$.

■

Third, we prove that all identities, i.e. sequents of the shape $\varphi, \Gamma \Rightarrow \varphi$ for any formula φ, are provable in GL4ip. Note that the equivalent to this lemma for GLS, i.e. Lemma 12.5.2, could be proved *first* in Section 12.5. Here, we needed to prove preliminary lemmas.

Lemma 13.4.4. For all Γ and φ, the sequent $\varphi, \Gamma \Rightarrow \varphi$ has a proof.

Proof. (⌨) Let φ be a formula and $n \in \mathbb{N}$ be such that $\mathit{weight}(\varphi) = n$. We prove by strong induction on n that for Γ we have that the sequent $\varphi, \Gamma \Rightarrow \varphi$ has a proof. We consider the structure of φ.

1. $\varphi := p$: We immediately get a proof of $p, \Gamma \Rightarrow p$ through the rule (IdP).

2. $\varphi := \bot$: A proof of $\bot, \Gamma \Rightarrow \bot$ is directly obtained using the $(\bot\mathrm{L})$.

3. $\varphi := \psi \wedge \chi$: Given that $\mathit{weight}(\psi) < \mathit{weight}(\psi \wedge \chi)$ and $\mathit{weight}(\chi) < \mathit{weight}(\psi \wedge \chi)$, by induction hypothesis we get a proof of both $\psi, \chi, \Gamma \Rightarrow \psi$ and $\psi, \chi, \Gamma \Rightarrow \chi$. So, using these proofs we build the following:

$$\dfrac{\dfrac{\psi, \chi, \Gamma \Rightarrow \psi \quad \psi, \chi, \Gamma \Rightarrow \chi}{\psi, \chi, \Gamma \Rightarrow \psi \wedge \chi}\ (\wedge\mathrm{R})}{\psi \wedge \chi, \Gamma \Rightarrow \psi \wedge \chi}\ (\wedge\mathrm{L})$$

4. $\varphi := \psi \vee \chi$: Given that $\mathit{weight}(\psi) < \mathit{weight}(\psi \vee \chi)$ and $\mathit{weight}(\chi) < \mathit{weight}(\psi \vee \chi)$, by induction hypothesis we get a proof of both $\psi\Gamma \Rightarrow \psi$ and $\chi, \Gamma \Rightarrow \chi$. So, using these proofs we build the following:

$$\dfrac{\dfrac{\psi, \Gamma \Rightarrow \psi}{\psi, \Gamma \Rightarrow \psi \vee \chi}\ (\vee\mathrm{R}_1) \quad \dfrac{\chi, \Gamma \Rightarrow \chi}{\chi, \Gamma \Rightarrow \psi \vee \chi}\ (\vee\mathrm{R}_2)}{\psi \vee \chi, \Gamma \Rightarrow \psi \vee \chi}\ (\vee\mathrm{L})$$

5. $\varphi := \psi \to \chi$: Consider the following.

$$\dfrac{\psi, \psi \to \chi, \Gamma \Rightarrow \chi}{\psi \to \chi, \Gamma \Rightarrow \psi \to \chi}\ ({\to}\mathrm{R})$$

   Thus, if we find a proof of $\psi, \psi \to \chi, \Gamma \Rightarrow \chi$ we are done. To obtain such a proof, we consider the structure of ψ.

   (a) $\psi := p$: Then we need to give a proof of $p, p \to \chi, \Gamma \Rightarrow \chi$. Given that $\mathit{weight}(\chi) < \mathit{weight}(p \to \chi)$, by induction hypothesis we get a proof of $p, \chi, \Gamma \Rightarrow \chi$. So, using this proof we build the following:

$$\dfrac{p, \chi, \Gamma \Rightarrow \chi}{p, p \to \chi, \Gamma \Rightarrow \chi}\ (p{\to}\mathrm{L})$$

(b) $\psi := \bot$: Then we need to give a proof of $\bot, \bot \to \chi, \Gamma \Rightarrow \chi$. This is immediately obtained through an application of ($\bot$L).

(c) $\psi := \psi_0 \wedge \psi_1$: Then we need to give a proof of $\psi_0 \wedge \psi_1, (\psi_0 \wedge \psi_1) \to \chi, \Gamma \Rightarrow \chi$. Now, note that $weight(\psi_0 \to (\psi_1 \to \chi)) < weight((\psi_0 \wedge \psi_1) \to \chi)$. This is the case as $\wedge$ adds 2 to the weight of a formula while $\to$ adds only 1. This justifies the following equations, founding our inequality.

$$weight(\psi_0 \to (\psi_1 \to \chi)) = weight(\psi_0) + weight(\psi_1) + weight(\chi) + 2$$

$$weight((\psi_0 \wedge \psi_1) \to \chi) = weight(\psi_0) + weight(\psi_1) + weight(\chi) + 3$$

Consequently, by induction hypothesis we get a proof of $\psi_0 \to (\psi_1 \to \chi), \Gamma \Rightarrow \psi_0 \to (\psi_1 \to \chi)$. We can thus apply Lemma 13.4.3, more precisely the invertibility of the rule ($\to$R), twice to obtain a proof of $\psi_0, \psi_1, \psi_0 \to (\psi_1 \to \chi), \Gamma \Rightarrow \chi$. Then, using this proof we build the following:

$$\dfrac{\dfrac{\psi_0, \psi_1, \psi_0 \to (\psi_1 \to \chi), \Gamma \Rightarrow \chi}{\psi_0, \psi_1, (\psi_0 \wedge \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\wedge\!\to\!\text{L})}{\psi_0 \wedge \psi_1, (\psi_0 \wedge \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\wedge\text{L})$$

(d) $\psi := \psi_0 \vee \psi_1$: Then we need to give a proof of $\psi_0 \vee \psi_1, (\psi_0 \vee \psi_1) \to \chi, \Gamma \Rightarrow \chi$. Given that $weight(\psi_0 \to \chi) < weight((\psi_0 \vee \psi_1) \to \chi)$ and $weight(\psi_1 \to \chi) < weight((\psi_0 \vee \psi_1) \to \chi)$, by induction hypothesis we get proofs of $\psi_0 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \psi_0 \to \chi$ and $\psi_1 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \psi_1 \to \chi$. We can thus apply Lemma 13.4.3, more precisely the invertibility of the rule ($\to$R), once on each proof to obtain proofs of $\psi_0, \psi_0 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \chi$ and $\psi_1, \psi_0 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \chi$. Then, using this proof we build the following:

$$\dfrac{\dfrac{\psi_0, \psi_0 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \chi}{\psi_0, (\psi_0 \vee \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\wedge\!\to\!\text{L}) \qquad \dfrac{\psi_1, \psi_0 \to \chi, \psi_1 \to \chi, \Gamma \Rightarrow \chi}{\psi_1, (\psi_0 \vee \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\wedge\!\to\!\text{L})}{\psi_0 \vee \psi_1, (\psi_0 \vee \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\wedge\text{L})$$

(e) $\psi := \psi_0 \to \psi_1$: Then we need to give a proof of $\psi_0 \to \psi_1, (\psi_0 \to \psi_1) \to \chi, \Gamma \Rightarrow \chi$. Given that $weight(\psi_0 \to \psi_1) < weight((\psi_0 \to \psi_1) \to \chi)$ and $weight(\chi) < weight((\psi_0 \to \psi_1) \to \chi)$, by induction hypothesis we get proofs of $\psi_0 \to \psi_1, \psi_1 \to \chi, \Gamma \Rightarrow \psi_0 \to \psi_1$ and $\psi_0 \to \psi_1, \chi, \Gamma \Rightarrow \chi$. So, using these proofs we build the following:

$$\dfrac{\psi_0 \to \psi_1, \psi_1 \to \chi, \Gamma \Rightarrow \psi_0 \to \psi_1 \qquad \psi_0 \to \psi_1, \chi, \Gamma \Rightarrow \chi}{\psi_0 \to \psi_1, (\psi_0 \to \psi_1) \to \chi, \Gamma \Rightarrow \chi}\,(\to\!\to\!\text{L})$$

(f) $\psi := \Box\psi_0$: Then we need to give a proof of $\Box\psi_0, \Box\psi_0 \to \chi, \Gamma \Rightarrow \chi$. Given that $weight(\psi_0) < weight(\Box\psi_0 \to \chi)$ and $weight(\chi) < weight(\Box\psi_0 \to \chi)$, by induction hypothesis we get proofs of $\psi_0, \Box\psi_0, \Box\psi_0, \boxtimes\Gamma_0 \Rightarrow \psi_0$ and $\Box\psi_0, \chi, \Box\Gamma_0, \Gamma_1 \Rightarrow \chi$ where $\Gamma = \Box\Gamma_0, \Gamma_1$. So, using these proofs we build the following:

$$\dfrac{\psi_0, \Box\psi_0, \Box\psi_0, \boxtimes\Gamma_0 \Rightarrow \psi_0 \qquad \Box\psi_0, \chi, \Box\Gamma_0, \Gamma_1 \Rightarrow \chi}{\Box\psi_0, \Box\psi_0 \to \chi, \Box\Gamma_0, \Gamma_1 \Rightarrow \chi}\,(\Box\!\to\!\text{L})$$

6. $\varphi := \Box\psi$: We need to give a proof of $\Box\psi, \Gamma \Rightarrow \Box\psi$. To do so, we first divide Γ into its subset of boxed formulas $\Box\Gamma_0$ and its set of remaining formulas Γ_1. Note that each set can be empty. By induction hypothesis, we have a proof of the sequent $\psi, \Box\psi, \Gamma_0, \Box\Gamma_0, \Box\psi \Rightarrow \psi$, as ψ appears on both side of this sequent and $weight(\psi) < weight(\Box\psi)$. Thus, we use this proof to build the following.

$$\dfrac{\psi, \Box\psi, \Gamma_0, \Box\Gamma_0, \Box\psi \Rightarrow \psi}{\Box\psi, \Box\Gamma_0, \Gamma_1 \Rightarrow \Box\psi}\,(\text{GLR})$$

■

Fourth, we can show that the height-preserving invertibility of the rules ($\to\to$L) and ($\Box\to$L) holds for the right premise:

Lemma 13.4.5 (Height-preserving right-invertibility of rules)**.** For all Γ, φ, ψ, δ and χ:

(i) If $\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi$ has a proof $\mathfrak{p}$ in GL4ip, then $\Gamma, \delta \Rightarrow \chi$ has a proof $\mathfrak{p}'$ in GL4ip such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

(ii) If $\Gamma, \Box\varphi \to \psi \Rightarrow \chi$ has a proof $\mathfrak{p}$ in GL4ip, then $\Gamma, \psi \Rightarrow \chi$ has a proof $\mathfrak{p}'$ in GL4ip such that $h(\mathfrak{p}') \leq h(\mathfrak{p})$.

Proof. Both of these height-preserving right-invertibility results are obtained through a proof by strong induction on the height of the given derivation, and on an analysis of the last rule applied. Their proofs require the application of Lemma 13.4.2. We only consider the height-preserving right-invertibility of ($\Box\to$L) as an example (⌨), and refer to our formalisation for item (i) (⌨).

Let $\mathfrak{p}$ be a proof of $\Gamma, \Box\varphi \to \psi \Rightarrow \chi$ of height n. We prove by strong induction on n that there is a proof $\mathfrak{p}'$ of $\Gamma, \psi \Rightarrow \chi$ such that $h(\mathfrak{p}') \leq n$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r \in \{(\text{IdP}), (\bot\text{L})\}$: then either $p \in \Gamma$ or $\bot \in \Gamma$. In all cases $\Gamma, \psi \Rightarrow \chi$ is an instance of an initial sequent.

2. $r = (\wedge\text{R})$: then r is

$$\frac{\Gamma, \Box\varphi \to \psi \Rightarrow \chi_0 \qquad \Gamma, \Box\varphi \to \psi \Rightarrow \chi_1}{\Gamma, \Box\varphi \to \psi \Rightarrow \chi_0 \wedge \chi_1}\ (\wedge\text{R})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\Gamma, \psi \Rightarrow \chi_0$ and $\Gamma, \psi \Rightarrow \chi_1$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\wedge$R) to get a proof of $\Gamma, \psi \Rightarrow \chi_0 \wedge \chi_1$ of height less than or equal to $h(\mathfrak{p})$.

3. $r = (\wedge\text{L})$: then r is

$$\frac{\gamma_0, \gamma_1, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{\gamma_0 \wedge \gamma_1, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\wedge\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0, \gamma_1, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\wedge$L) to get a proof of $\gamma_0 \wedge \gamma_1, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

4. $r = (\vee\text{R}_i)$ for $i \in \{1, 2\}$: then r is

$$\frac{\Gamma, \Box\varphi \to \psi \Rightarrow \chi_i}{\Gamma, \Box\varphi \to \psi \Rightarrow \chi_1 \vee \chi_2}\ (\vee\text{R}_i)$$

Then we can apply the induction hypothesis on the premise to get a proof of $\Gamma, \psi \Rightarrow \chi_i$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\vee\text{R}_i$) to get a proof of $\Gamma, \psi \Rightarrow \chi_1 \vee \chi_2$ of height less than or equal to $h(\mathfrak{p})$.

5. $r = (\vee\text{L})$: then r is

$$\frac{\gamma_0, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi \qquad \gamma_1, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{\gamma_0 \vee \gamma_1, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\vee\text{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\gamma_0, \Gamma_0, \psi \Rightarrow \chi$ and $\gamma_1, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule ($\vee$L) to get a proof of $\gamma_0 \vee \gamma_1, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

6. $r = (\to\text{R})$: then r is

$$\frac{\chi_0, \Gamma, \Box\varphi \to \psi \Rightarrow \chi_1}{\Gamma, \Box\varphi \to \psi \Rightarrow \chi_0 \to \chi_1}\ (\to\text{R})$$

Then we can apply the induction hypothesis on the premise to get a proof $\chi_0, \Gamma, \psi \Rightarrow \chi_1$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\to\text{R})$ to get a proof of $\Gamma, \psi \Rightarrow \chi_0 \to \chi_1$ of height less than or equal to $h(\mathfrak{p})$.

7. $r = (p{\to}\text{L})$: then r is

$$\frac{p, \gamma, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{p, p \to \gamma, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (p{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $p, \gamma, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(p{\to}\text{L})$ to get a proof of $p, p \to \gamma, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

8. $r = (\wedge{\to}\text{L})$: then r is

$$\frac{\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\wedge{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\wedge{\to}\text{L})$ to get a proof of $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

9. $r = (\vee{\to}\text{L})$: then r is

$$\frac{\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\vee{\to}\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $(\vee{\to}\text{L})$ to get a proof of $(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

10. $r = ({\to}{\to}\text{L})$: then r is

$$\frac{\gamma_1 \to \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \gamma_0 \to \gamma_1 \qquad \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ ({\to}{\to}\text{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\gamma_1 \to \gamma_2, \Gamma_0, \psi \Rightarrow \gamma_0 \to \gamma_1$ and $\gamma_2, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can thus apply the rule $({\to}{\to}\text{L})$ to get a proof of $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

11. $r = (\Box{\to}\text{L})$: Then we consider two cases. If $\Box\varphi \to \psi$ is principal in the rule application, then r is

$$\frac{\boxtimes\Gamma_0, \Box\varphi \Rightarrow \varphi \qquad \psi, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\Phi, \Box\Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\Box{\to}\text{L})$$

In this case, we are given a proof of $\psi, \Phi, \Box\Gamma_0 \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$, so we are done. If $\Box\varphi \to \psi$ is not principal in the rule application, then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi, \Box\Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \Box\varphi \to \psi \Rightarrow \chi}\ (\Box{\to}\mathrm{L})$$

In this case, we apply the induction hypothesis on the right premise to obtain a proof of $\gamma_1, \Phi, \Box\Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Note that $(\Box{\to}\mathrm{L})$ is not necessarily applicable on $\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$ and $\gamma_1, \Phi, \Box\Gamma_0, \psi \Rightarrow \chi$, as if ψ is boxed it needs to appear in the first premise. If ψ is not boxed, we can apply $(\Box{\to}\mathrm{L})$ on the two mentioned sequents to obtain a proof of $\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \psi \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$. If ψ is boxed, i.e. $\psi = \Box\psi_0$, then we apply twice Lemma 13.4.2 on the proof of $\boxtimes\Gamma_0, \Box\varphi \Rightarrow \varphi$ to obtain a proof of $\psi_0, \Box\psi_0 \boxtimes \Gamma_0, \Box\varphi \Rightarrow \varphi$ of height less than or equal to $h(\mathfrak{p}) - 1$. Consequently, we apply $(\Box{\to}\mathrm{L})$ on$\psi_0, \Box\psi_0 \boxtimes \Gamma_0, \Box\varphi \Rightarrow \varphi$ and $\gamma_1, \Phi, \Box\Gamma_0, \Box\psi_0 \Rightarrow \chi$ to obtain a proof of $\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \Box\psi_0 \Rightarrow \chi$ of height less than or equal to $h(\mathfrak{p})$.

12. $r = (\mathrm{GLR})$: then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0}{\Phi, \Box\Gamma_0, \Box\varphi \to \psi \Rightarrow \Box\chi_0}\ (\mathrm{GLR})$$

Either ψ is boxed, i.e. $\psi = \Box\psi_0$, and then we can apply the induction hypothesis twice to obtain a proof of $\psi_0, \Box\psi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$ of height less than or equal to $h(\mathfrak{p}) - 1$. We can then apply the (GLR) rule to obtain a proof of $\Phi, \Box\psi, \boxtimes\Gamma_0 \Rightarrow \Box\chi_0$ of height less than or equal to $h(\mathfrak{p})$. Or ψ is not boxed and then we can apply (GLR) on the premise to obtain a proof of $\Phi_0, \Box\Gamma_0 \Rightarrow \Box\chi_0$ of height $h(\mathfrak{p})$, where $\Phi_0 = \Phi, \psi$.

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Fifth, we need to show that the usual left-implication rule is admissible. This lemma will allow us to prove Lemma 13.4.7, pertaining to the rule $({\to}{\to}\mathrm{L})$, which is key for the admissibility of contraction.

Lemma 13.4.6. The rule $({\to}\mathrm{L})$ is admissible in GL4ip:

$$\dfrac{\Gamma \Rightarrow \varphi \qquad \Gamma, \psi \Rightarrow \chi}{\Gamma, \varphi \to \psi \Rightarrow \chi}\ ({\to}\mathrm{L})$$

Proof. (⌨) We prove by strong induction on n that for all $\Gamma, \Delta \subseteq Form_{\mathbb{L}_{\mathbf{IM}}}$ and $\varphi, \psi, \chi \in Form_{\mathbb{L}_{\mathbf{IM}}}$, if we have a proof of $\Gamma \Rightarrow \varphi$ of height n and a proof of $\Gamma, \psi \Rightarrow \chi$, then we have a proof of $\Gamma, \varphi \to \psi \Rightarrow \chi$. Let $\mathfrak{p}$ be a proof of a sequent $\Gamma \Rightarrow \varphi$ of height n, and $\mathfrak{p}'$ a proof of a sequent $\Gamma, \psi \Rightarrow \chi$. We need to show that there is a proof of $\Gamma, \varphi \to \psi \Rightarrow \chi$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r = (\mathrm{IdP})$: then $\Gamma \Rightarrow \varphi$ is of the form $p, \Gamma_0 \Rightarrow p$. As a consequence, $\Gamma, \varphi \to \psi \Rightarrow \chi$ is of the form $p, \Gamma_0, p \to \psi \Rightarrow \chi$. Furthermore, we have a proof of $\Gamma, \psi \Rightarrow \chi$ which is of the form $p, \Gamma_0, \psi \Rightarrow \chi$. Then consider the following proof.

$$\dfrac{p, \Gamma_0, \psi \Rightarrow \chi}{p, \Gamma_0, p \to \psi \Rightarrow \chi}\ (p{\to}\mathrm{L})$$

2. $r = (\bot\mathrm{L})$: then $\bot \in \Gamma$, hence $\Gamma, \varphi \to \psi \Rightarrow \chi$ is an instance of $(\bot\mathrm{L})$.

3. $r = (\wedge\mathrm{R})$: then r is

$$\dfrac{\Gamma \Rightarrow \varphi_0 \qquad \Gamma \Rightarrow \varphi_1}{\Gamma \Rightarrow \varphi_0 \wedge \varphi_1}\ (\wedge\mathrm{R})$$

Then we can apply the induction hypothesis using the proofs of $\Gamma \Rightarrow \varphi_1$ and $\Gamma, \psi \Rightarrow \chi$ to get a proof of $\Gamma, \varphi_1 \to \psi \Rightarrow \chi$. In turn, we apply the induction hypothesis using $\Gamma \Rightarrow \varphi_0$ and $\Gamma, \varphi_1 \to \psi \Rightarrow \chi$ to obtain a proof of $\Gamma, \varphi_0 \to (\varphi_1 \to \psi) \Rightarrow \chi$. Then consider the following.

$$\dfrac{\Gamma, \varphi_0 \to (\varphi_1 \to \psi) \Rightarrow \chi}{\Gamma, (\varphi_0 \wedge \varphi_1) \to \psi \Rightarrow \chi}\ (\wedge\mathrm{R})$$

4. $r = (\wedge\mathrm{L})$: then r is

$$\dfrac{\gamma_0, \gamma_1\Gamma_0 \Rightarrow \varphi}{\gamma_0 \wedge \gamma_1\Gamma_0 \Rightarrow \varphi}\ (\wedge\mathrm{L})$$

We apply the invertibility of the rule $(\wedge\mathrm{L})$ proved in Lemma 13.4.3 to the proof of $\gamma_0 \wedge \gamma_1\Gamma_0, \psi \Rightarrow \chi$ to get a proof of $\gamma_0, \gamma_1\Gamma_0, \psi \Rightarrow \chi$. Then we can apply the induction hypothesis using the proofs of $\gamma_0, \gamma_1\Gamma_0 \Rightarrow \varphi$ and $\gamma_0, \gamma_1\Gamma_0, \psi \Rightarrow \chi$ to get a proof of $\gamma_0, \gamma_1\Gamma_0, \varphi \to \psi \Rightarrow \chi$. Thus it suffices to apply the rule $(\wedge\mathrm{L})$ to get a proof of $\gamma_0 \wedge \gamma_1\Gamma_0, \varphi \to \psi \Rightarrow \chi$.

5. $r = (\vee\mathrm{R}_i)$ for $i \in \{1, 2\}$: then r is

$$\dfrac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_1 \vee \varphi_2}\ (\vee\mathrm{R}_i)$$

Then we can apply the induction hypothesis using the proofs of $\Gamma \Rightarrow \varphi_i$ and $\Gamma, \psi \Rightarrow \chi$ to get a proof of $\Gamma, \varphi_i \to \psi \Rightarrow \chi$. We use this proof with Lemma 13.4.2 to get a proof of $\Gamma, \varphi_0 \to \psi, \varphi_1 \to \psi \Rightarrow \chi$. Then consider the following.

$$\dfrac{\Gamma, \varphi_0 \to \psi, \varphi_1 \to \psi \Rightarrow \chi}{\Gamma, (\varphi_0 \vee \varphi_1) \to \psi \Rightarrow \chi}\ (\wedge\mathrm{R})$$

6. $r = (\vee\mathrm{L})$: then r is

$$\dfrac{\gamma_0, \Gamma_0 \Rightarrow \varphi \qquad \gamma_1, \Gamma_0 \Rightarrow \varphi}{\gamma_0 \vee \gamma_1, \Gamma_0 \Rightarrow \varphi}\ (\vee\mathrm{L})$$

Note that the sequent $\Gamma, \psi \Rightarrow \chi$ has the form $\gamma_0 \vee \gamma_1, \Gamma_0, \psi \Rightarrow \chi$. We use the invertibility of $(\vee\mathrm{L})$ from Lemma 13.4.3 on the given proof of the latter to obtain proofs of $\gamma_0, \Gamma_0, \psi \Rightarrow \chi$ and $\gamma_1, \Gamma_0, \psi \Rightarrow \chi$. Then, we apply the induction hypothesis on the above premises and the proof of the last two sequents to get proofs of $\gamma_0, \Gamma_0, \varphi \to \psi \Rightarrow \chi$ and $\gamma_1, \Gamma_0, \varphi \to \psi \Rightarrow \chi$. We can thus apply the rule $(\vee\mathrm{L})$ to get a proof of $\gamma_0 \vee \gamma_1, \Gamma_0, \varphi \to \psi \Rightarrow \chi$.

7. $r = (\to\mathrm{R})$: then r is

$$\dfrac{\varphi_0, \Gamma \Rightarrow \varphi_1}{\Gamma \Rightarrow \varphi_0 \to \varphi_1}\ (\to\mathrm{R})$$

Note that the sequent $\Gamma, \varphi\psi \Rightarrow \chi$ has the form $\Gamma, (\varphi_0 \to \varphi_1) \to \psi \Rightarrow \chi$. Consider the following semi-proof.

$$\dfrac{\dfrac{\begin{array}{c}\varphi_0, \Gamma \Rightarrow \varphi_1 \\ \hdashline \varphi_0, \Gamma, \varphi_1 \to \psi \Rightarrow \varphi_1\end{array}\ \text{Lem. 13.4.2}}{\Gamma, \varphi_1 \to \psi \Rightarrow \varphi_0 \to \varphi_1}\ (\to\mathrm{R}) \qquad \Gamma, \psi \Rightarrow \chi}{\Gamma, (\varphi_0 \to \varphi_1) \to \psi \Rightarrow \chi}\ (\to\to\mathrm{L})$$

8. $r = (p{\to}\mathrm{L})$: then r is

$$\dfrac{p, \gamma, \Gamma_0 \Rightarrow \varphi}{p, p \to \gamma, \Gamma_0 \Rightarrow \varphi}\ (p{\to}\mathrm{L})$$

Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $p, p \to \gamma, \Gamma_0, \psi \Rightarrow \chi$. We use the invertibility of $(p{\to}\mathrm{L})$ on the proof of the latter to obtain a proof of $p, \gamma, \Gamma_0, \psi \Rightarrow \chi$. Then we apply the induction hypothesis on the proof of premise above and the proof of $p, \gamma, \Gamma_0, \psi \Rightarrow \chi$ to get a proof of $p, \gamma, \Gamma_0, \varphi \to \psi \Rightarrow \chi$. An application of the rule $(p{\to}\mathrm{L})$ gives a proof of $p, p \to \gamma, \Gamma_0, \varphi \to \psi \Rightarrow \chi$.

9. $r = (\wedge{\to}\mathrm{L})$: then r is

$$\frac{\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \varphi}{(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi}\ (\wedge{\to}\mathrm{L})$$

Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$. We use the invertibility of $(\wedge{\to}\mathrm{L})$ on the proof of the latter to obtain a proof of $\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, \psi \Rightarrow \chi$. Then we can apply the induction hypothesis on the premise and the latter sequent to get a proof of $\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, \varphi \to \psi \Rightarrow \chi$. We can thus apply the rule $(\wedge{\to}\mathrm{L})$ to get a proof of $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi$.

10. $r = (\vee{\to}\mathrm{L})$: then r is

$$\frac{\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \varphi}{(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi}\ (\vee{\to}\mathrm{L})$$

Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$. We use the invertibility of $(\vee{\to}\mathrm{L})$ on the proof of the latter to obtain a proof of $\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$. Then we can apply the induction hypothesis on the premise and the latter sequent to get a proof of $\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi$. We can thus apply the rule $(\wedge{\to}\mathrm{L})$ to get a proof of $(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi$.

11. $r = ({\to}{\to}\mathrm{L})$: then r is

$$\frac{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \gamma_2, \Gamma_0 \Rightarrow \varphi}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi}\ ({\to}{\to}\mathrm{L})$$

Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, \psi \Rightarrow \chi$. We use the right-invertibility of $({\to}{\to}\mathrm{L})$ from Lemma 13.4.5 on the proof of the latter to obtain a proof of $\gamma_2, \Gamma_0, \psi \Rightarrow \chi$. Then we can apply the induction hypothesis on the right premise above and the proof of the latter sequent to get a proof of $\gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi$. Consider the following semi-proof.

$$\frac{\dfrac{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1}{\gamma_1 \to \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \gamma_0 \to \gamma_1}\ \text{Lem. 13.4.2} \qquad \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, \varphi \to \psi \Rightarrow \chi}\ ({\to}{\to}\mathrm{L})$$

12. $r = (\Box{\to}\mathrm{L})$: Then r is

$$\frac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0 \Rightarrow \varphi}\ (\Box{\to}\mathrm{L})$$

Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \psi \Rightarrow \chi$. We use the right-invertibility of $(\Box{\to}\mathrm{L})$ from Lemma 13.4.5 on the proof of the latter to obtain a proof of $\Phi, \gamma_1, \Box\Gamma_0, \psi \Rightarrow \chi$. Then we can apply the induction hypothesis on the right premise above and the proof of the latter sequent to get a proof of $\Phi, \gamma_1, \Box\Gamma_0, \varphi \to \psi \Rightarrow \chi$. Consider the following.

$$\frac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \Phi, \gamma_1, \Box\Gamma_0, \varphi \to \psi \Rightarrow \chi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \varphi \to \psi \Rightarrow \chi}\ (\Box{\to}\mathrm{L})$$

13. $r =$ (GLR): then r is

$$\frac{\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0}{\Phi, \Box\Gamma_0 \Rightarrow \Box\varphi_0}\text{ (GLR)}$$

   Note that $\Gamma, \psi \Rightarrow \chi$ is of the form $\Phi, \Box\Gamma_0, \psi \Rightarrow \chi$. Consider the following.

$$\frac{\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0 \qquad \Phi, \Box\Gamma_0, \psi \Rightarrow \chi}{\Phi, \Box\Gamma_0, \Box\varphi_0 \to \psi \Rightarrow \chi}\text{ (}\Box\!\to\!\text{L)}$$

■

With the above lemma, we can prove a crucial result to prove that contraction is admissible.

Lemma 13.4.7. For all Γ, φ, ψ, δ and χ, if $\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi$ is provable in $\mathsf{GL4ip}$, then $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$ is provable in $\mathsf{GL4ip}$.

Proof. (⌨) We prove by strong induction on n that for all $\Gamma, \Delta \subseteq Form_{\mathbb{L}_{\mathbf{IM}}}$ and $\varphi, \psi, \delta, \chi \in Form_{\mathbb{L}_{\mathbf{IM}}}$, if we have a proof of $\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi$ of height n then we have a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. Let $\mathfrak{p}$ be a proof of a sequent $\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi$ of height n,. We need to show that there is a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r =$ (IdP): then $\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi$ is of the form $p, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow p$. As a consequence, $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$ is of the form $p, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow p$, which is an obvious instance of (IdP).

2. $r =$ ($\bot$L): then $\bot \in \Gamma$, hence $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$ is an instance of ($\bot$L).

3. $r =$ ($\wedge$R): then r is

$$\frac{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_0 \qquad \Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_1}{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_0 \wedge \chi_1}\text{ (}\wedge\text{R)}$$

   Then we can apply the induction hypothesis on both premises to get proofs of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_0$ and $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_1$. Then an application of ($\wedge$R) gives us a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_0 \wedge \chi_1$.

4. $r =$ ($\wedge$L): then r is

$$\frac{\gamma_0, \gamma_1, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{\gamma_0 \wedge \gamma_1, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}\text{ (}\wedge\text{L)}$$

   Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0, \gamma_1, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. Then an application of ($\wedge$L) gives us a proof of $\gamma_0 \wedge \gamma_1, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

5. $r =$ ($\vee$R$_i$) for $i \in \{1, 2\}$: then r is

$$\frac{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_i}{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_1 \vee \chi_2}\text{ (}\vee\text{R}_i\text{)}$$

   Then we can apply the induction hypothesis on the premise to get a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_i$. An application of ($\vee$R$_i$) gives us a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_1 \vee \chi_2$.

6. $r =$ ($\vee$L): then r is

$$\frac{\gamma_0, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi \qquad \gamma_1, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{\gamma_0 \vee \gamma_1, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi} \ (\vee\text{L})$$

Then we can apply the induction hypothesis on both premises to get proofs of $\gamma_0, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$ and $\gamma_1, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. An application of ($\vee$L) gives us a proof of $\gamma_0 \vee \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

7. $r = (\to\text{R})$: then r is

$$\frac{\chi_0, \Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_1}{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi_0 \to \chi_1} \ (\to\text{R})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\chi_0, \Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_1$. An application of ($\to$R) gives us a proof of $\Gamma, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi_0 \to \chi_1$.

8. $r = (p\to\text{L})$: then r is

$$\frac{p, \gamma, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{p, p \to \gamma, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi} \ (p\to\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $p, \gamma, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. An application of ($p\to$L) gives us a proof of $p, p \to \gamma, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

9. $r = (\wedge\to\text{L})$: then r is

$$\frac{\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi} \ (\wedge\to\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. An application of ($\wedge\to$L) gives us a proof of $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

10. $r = (\vee\to\text{L})$: then r is

$$\frac{\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi} \ (\vee\to\text{L})$$

Then we can apply the induction hypothesis on the premise to get a proof of $\gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. An application of ($\vee\to$L) gives us a proof of $(\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

11. $r = (\to\to\text{L})$: then we consider two cases. If $(\varphi \to \psi) \to \delta$ is principal in r, then the latter is:

$$\frac{\psi \to \delta, \Gamma \Rightarrow \varphi \to \psi \qquad \delta, \Gamma \Rightarrow \chi}{\Gamma, (\varphi \to \psi) \to \delta \Rightarrow \chi} \ (\to\to\text{L})$$

In this case we apply the invertibility of ($\to$R) from Lemma 13.4.3 on the left premise to obtain a proof of $\varphi, \psi \to \delta, \Gamma \Rightarrow \psi$. Then we use Lemma 13.4.2 twice on the right premise to obtain a proof of $\varphi, \psi \to \delta, \delta, \Gamma \Rightarrow \chi$. Consider the following use of the admissibility of the rule ($\to$L).

$$\frac{\varphi, \psi \to \delta, \Gamma \Rightarrow \psi \qquad \varphi, \psi \to \delta, \delta, \Gamma \Rightarrow \chi}{\varphi, \psi \to \delta, \psi \to \delta, \Gamma \Rightarrow \chi} \ \text{Lem. 13.4.6}$$

If $(\varphi \to \psi) \to \delta$ is not principal in r, then the latter is:

$$\dfrac{\gamma_1 \to \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \gamma_0 \to \gamma_1 \qquad \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}\ (\to\to\mathrm{L})$$

Then we can apply the induction hypothesis on the premises to get proofs of $\gamma_1 \to \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \gamma_0 \to \gamma_1$ and $\gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. An application of ($\to\to$L) gives us a proof of $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$.

12. $r = (\Box\!\to\!\mathrm{L})$: Then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi, \Box\Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

Then we can apply the induction hypothesis on the right premise to obtain a proof of $\gamma_1, \Phi, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi$. There are two cases to consider. If φ is boxed, i.e. $\varphi = \Box\varphi_0$, then we use Lemma 13.4.2 on the left premise to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$. In this case, consider the following.

$$\dfrac{\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

If φ is not boxed, then consider the following.

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi}{\Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

13. $r = (\mathrm{GLR})$: then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0}{\Phi, \Box\Gamma_0, (\varphi \to \psi) \to \delta \Rightarrow \Box\chi_0}\ (\mathrm{GLR})$$

There are two cases to consider. If φ is boxed, i.e. $\varphi = \Box\varphi_0$, then we use Lemma 13.4.2 on the premise to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$. In this case, a straightforward application of (GLR) gives us $\Phi, \Box\Gamma_0, \Box\varphi_0, \psi \to \delta, \psi \to \delta \Rightarrow \Box\chi_0$. If φ is not boxed, then we simply apply (GLR) on the premise to obtain $\Phi, \Box\Gamma_0, \varphi, \psi \to \delta, \psi \to \delta \Rightarrow \Box\chi_0$.

■

The building of the series of lemmas we proved in this section has one purpose: proving the admissibility of contraction in GL4ip. We finally turn to this result.

Lemma 13.4.8 (Admissibility of contraction). For all Γ, φ and χ, if $\varphi, \varphi, \Gamma \Rightarrow \chi$ is provable in GL4ip then $\varphi, \Gamma \Rightarrow \chi$ is provable in GL4ip.

Proof. (⌨) We reason by two strong inductions: first on the weight of the contracted formula φ (PIH), and second on the height of the proof $\mathfrak{p}$ of $\varphi, \varphi, \Gamma \Rightarrow \chi$ (SIH). We need to show that there is a proof of $\varphi, \Gamma \Rightarrow \chi$. We consider the last rule r applied in $\mathfrak{p}$:

1. $r = (\mathrm{IdP})$: then either $\varphi = p$ or $p \in \Gamma$. In both cases, $\varphi, \Gamma \Rightarrow \chi$ is an instance of (IdP).

2. $r = (\bot\mathrm{L})$: then either $\varphi = \bot$ or $\bot \in \Gamma$., hence $\varphi, \Gamma \Rightarrow \chi$ is an instance of ($\bot$L).

3. $r = (\wedge\mathrm{R})$: then r is

$$\dfrac{\varphi, \varphi, \Gamma \Rightarrow \chi_0 \qquad \varphi, \varphi, \Gamma \Rightarrow \chi_1}{\varphi, \varphi, \Gamma \Rightarrow \chi_0 \wedge \chi_1}\ (\wedge\mathrm{R})$$

Then we can apply the induction hypothesis SIH on both premises to get proofs of $\varphi, \Gamma \Rightarrow \chi_0$ and $\varphi, \Gamma \Rightarrow \chi_1$. Then an application of ($\wedge$R) gives us a proof of $\varphi, \Gamma \Rightarrow \chi_0 \wedge \chi_1$.

4. $r = (\wedge\text{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\frac{\varphi_0, \varphi_1, \varphi_0 \wedge \varphi_1, \Gamma \Rightarrow \chi}{\varphi_0 \wedge \varphi_1, \varphi_0 \wedge \varphi_1, \Gamma \Rightarrow \chi}\ (\wedge\text{L})$$

We use the invertibility of ($\wedge$L) from Lemma 13.4.3 to obtain a proof of $\varphi_0, \varphi_1, \varphi_0, \varphi_1, \Gamma \Rightarrow \chi$. Then we can use PIH twice on the proof of the latter to obtain a proof of $\varphi_0, \varphi_1, \Gamma \Rightarrow \chi$. A simple application of ($\wedge$L) gives us a proof of $\varphi_0 \wedge \varphi_1, \Gamma \Rightarrow \chi$. If φ is not principal in r, then the latter is

$$\frac{\varphi, \varphi, \gamma_0, \gamma_1, \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, \gamma_0 \wedge \gamma_1, \Gamma_0 \Rightarrow \chi}\ (\wedge\text{L})$$

Then we can use SIH on the proof of the premise to obtain a proof of $\varphi, \gamma_0, \gamma_1, \Gamma_0 \Rightarrow \chi$. A simple application of ($\wedge$L) gives us a proof of $\varphi, \gamma_0 \wedge \gamma_1, \Gamma_0 \Rightarrow \chi$.

5. $r = (\vee\text{R}_i)$ for $i \in \{1, 2\}$: then r is

$$\frac{\varphi, \varphi, \Gamma \Rightarrow \chi_i}{\varphi, \varphi, \Gamma \Rightarrow \chi_1 \vee \chi_2}\ (\vee\text{R}_i)$$

Then we can apply the induction hypothesis SIH on the premise to get a proof of $\varphi, \Gamma \Rightarrow \chi_i$. An application of ($\vee$R$_i$) gives us a proof of $\varphi, \Gamma \Rightarrow \chi_1 \vee \chi_2$.

6. $r = (\vee\text{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\frac{\varphi_0, \varphi_0 \vee \varphi_1, \Gamma \Rightarrow \chi \qquad \varphi_1, \varphi_0 \vee \varphi_1, \Gamma \Rightarrow \chi}{\varphi_0 \vee \varphi_1, \varphi_0 \vee \varphi_1, \Gamma \Rightarrow \chi}\ (\vee\text{L})$$

We use the invertibility of ($\vee$L) from Lemma 13.4.3 to obtain proofs of $\varphi_0, \varphi_0, \Gamma \Rightarrow \chi$ and $\varphi_1, \varphi_1, \Gamma \Rightarrow \chi$. Then we can use PIH twice on each of these proofs to obtain proofs of $\varphi_0, \Gamma \Rightarrow \chi$ and $\varphi_1, \Gamma \Rightarrow \chi$. A simple application of ($\vee$L) gives us a proof of $\varphi_0 \vee \varphi_1, \Gamma \Rightarrow \chi$. If φ is not principal in r, then the latter is

$$\frac{\varphi, \varphi, \gamma_0, \Gamma_0 \Rightarrow \chi \qquad \varphi, \varphi, \gamma_1, \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, \gamma_0 \vee \gamma_1, \Gamma_0 \Rightarrow \chi}\ (\vee\text{L})$$

Then we can use SIH on the proof of each premise to obtain proofs of $\varphi, \gamma_0, \Gamma_0 \Rightarrow \chi$ and $\varphi, \gamma_1, \Gamma_0 \Rightarrow \chi$. A simple application of ($\vee$L) gives us a proof of $\varphi, \gamma_0 \vee \gamma_1, \Gamma_0 \Rightarrow \chi$.

7. $r = (\to\text{R})$: then r is

$$\frac{\chi_0, \varphi, \varphi, \Gamma \Rightarrow \chi_1}{\varphi, \varphi, \Gamma \Rightarrow \chi_0 \to \chi_1}\ (\to\text{R})$$

Then we can apply the induction hypothesis SIH on the premise to get a proof of $\chi_0, \varphi, \Gamma \Rightarrow \chi_1$. An application of ($\to$R) gives us a proof of $\varphi, \Gamma \Rightarrow \chi_0 \to \chi_1$.

8. $r = (p\to\text{L})$: then there are three cases. If φ is principal in r and $\varphi = p$, then the latter is

$$\frac{p, p, \gamma, \Gamma_0 \Rightarrow \chi}{p, p, p \to \gamma, \Gamma_0 \Rightarrow \chi}\ (p\to\text{L})$$

Then we can use SIH on the proof of the premise to obtain a proof of $p, \gamma, \Gamma_0 \Rightarrow \chi$. A simple application of ($p\to$L) gives us a proof of $p, p \to \gamma, \Gamma_0 \Rightarrow \chi$. If φ is principal in r and $\varphi = p \to \varphi_0$, then the latter is

$$\frac{\varphi_0, p \to \varphi_0, p, \Gamma_0 \Rightarrow \chi}{p \to \varphi_0, p \to \varphi_0, p, \Gamma_0 \Rightarrow \chi}\ (p{\to}\mathrm{L})$$

We use the invertibility of ($p\to$L) from Lemma 13.4.3 to obtain a proof of $\varphi_0, \varphi_0, p, \Gamma_0 \Rightarrow \chi$. Then we can use PIH on the proof of the latter to obtain a proof of $\varphi_0, p, \Gamma_0 \Rightarrow \chi$. A simple application of ($p\to$L) gives us a proof of $p \to \varphi_0, p, \Gamma_0 \Rightarrow \chi$. If φ is not principal in r, then the latter is

$$\frac{\varphi, \varphi, p, \gamma, \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, p, p \to \gamma, \Gamma_0 \Rightarrow \chi}\ (p{\to}\mathrm{L})$$

Then we can use SIH on the proof of the premise to obtain a proof of $\varphi, p, \gamma, \Gamma_0 \Rightarrow \chi$. A simple application of ($p\to$L) gives us a proof of $\varphi, p, p \to \gamma, \Gamma_0 \Rightarrow \chi$.

9. $r = (\wedge{\to}\mathrm{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\frac{\varphi_0 \to (\varphi_1 \to \varphi_2), (\varphi_0 \wedge \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}{(\varphi_0 \wedge \varphi_1) \to \varphi_2, (\varphi_0 \wedge \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}\ (\wedge{\to}\mathrm{L})$$

We use the invertibility of ($\wedge\to$L) from Lemma 13.4.3 to obtain a proof of $\varphi_0 \to (\varphi_1 \to \varphi_2), \varphi_0 \to (\varphi_1 \to \varphi_2), \Gamma \Rightarrow \chi$. Now, note that $weight(\varphi_0 \to (\varphi_1 \to \varphi_2)) < weight((\varphi_0 \wedge \varphi_1) \to \varphi_2)$. So, we can use PIH on the proof of $\varphi_0 \to (\varphi_1 \to \varphi_2), \varphi_0 \to (\varphi_1 \to \varphi_2), \Gamma \Rightarrow \chi$ to obtain a proof of $\varphi_0 \to (\varphi_1 \to \varphi_2), \Gamma \Rightarrow \chi$. A simple application of ($\wedge\to$L) gives us a proof of $(\varphi_0 \wedge \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi$. If φ is not principal in r, then the latter is

$$\frac{\varphi, \varphi, \gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, (\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\ (\wedge{\to}\mathrm{L})$$

Then we can use SIH on the proof of the premise to obtain a proof of $\varphi, \gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \chi$. A simple application of ($\wedge\to$L) gives us a proof of $\varphi, (\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$.

10. $r = (\vee{\to}\mathrm{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\frac{\varphi_0 \to \varphi_2, \varphi_1 \to \varphi_2, (\varphi_0 \vee \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}{(\varphi_0 \vee \varphi_1) \to \varphi_2, (\varphi_0 \vee \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}\ (\wedge{\to}\mathrm{L})$$

We use the invertibility of ($\vee\to$L) from Lemma 13.4.3 to obtain a proof of $\varphi_0 \to \varphi_2, \varphi_1 \to \varphi_2, \varphi_0 \to \varphi_2, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \chi$. Then we use PIH twice on the proof of the latter to obtain a proof of $\varphi_0 \to \varphi_2, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \chi$. A simple application of ($\vee\to$L) gives us a proof of $(\varphi_0 \vee \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi$. If φ is not principal in r, then the latter is

$$\frac{\varphi, \varphi, \gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, (\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\ (\wedge{\to}\mathrm{L})$$

Then we can use SIH on the proof of the premise to obtain a proof of $\varphi, \gamma_0 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \chi$. A simple application of ($\vee\to$L) gives us a proof of $\varphi, (\gamma_0 \vee \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$.

11. $r = (\to{\to}\mathrm{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\dfrac{\varphi_1 \to \varphi_2, (\varphi_0 \to \varphi_1) \to \varphi_2, \Gamma \Rightarrow \varphi_0 \to \varphi_1 \qquad \varphi_2, (\varphi_0 \to \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}{(\varphi_0 \to \varphi_1) \to \varphi_2, (\varphi_0 \to \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}\ (\to\to\mathrm{L})$$

First, we use the right-invertibility of (→→L) from Lemma 13.4.5 on the right premise to obtain a proof of $\varphi_2, \varphi_2, \Gamma \Rightarrow \chi$. An application of PIH on the latter gives a proof of $\varphi_2, \Gamma \Rightarrow \chi$. Second, we use the key Lemma 13.4.7 on the right premise to obtain a proof of $\varphi_1 \to \varphi_2, \varphi_0, \varphi_1 \to \varphi_2, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_0 \to \varphi_1$. Then we use PIH twice to obtain a proof of $\varphi_0, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_0 \to \varphi_1$. The invertibility of (→R) from Lemma 13.4.3 gives us a proof of $\varphi_0, \varphi_0, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_1$. Then, we can use once again PIH to obtain a proof of $\varphi_0, \varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_1$. Finally, an application of the (→R) gives a proof of $\varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_0 \to \varphi_1$. Third, we can build the following semi-proof.

$$\dfrac{\varphi_1 \to \varphi_2, \Gamma \Rightarrow \varphi_0 \to \varphi_1 \qquad \varphi_2, \Gamma \Rightarrow \chi}{(\varphi_0 \to \varphi_1) \to \varphi_2, \Gamma \Rightarrow \chi}\ (\to\to\mathrm{L})$$

If φ is not principal in r, then the latter is

$$\dfrac{\varphi, \varphi, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \varphi, \varphi, \gamma_2, \Gamma_0 \Rightarrow \chi}{\varphi, \varphi, (\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\ (\to\to\mathrm{L})$$

Then we can use SIH on the proof of each premise to obtain proofs of $\varphi, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1$ and $\varphi, \gamma_2, \Gamma_0 \Rightarrow \chi$. A simple application of (→→L) gives us a proof of $\varphi, (\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$.

12. $r = (\Box\!\to\!\mathrm{L})$: then there are two cases. If φ is principal in r, then the latter is

$$\dfrac{\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0 \qquad \varphi_1, \Box\varphi_0 \to \varphi_1, \Phi, \Gamma_0 \Rightarrow \chi}{\Box\varphi_0 \to \varphi_1, \Box\varphi_0 \to \varphi_1, \Phi, \Gamma_0 \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

First, we use the right-invertibility of (□→L) from Lemma 13.4.5 on the right premise to obtain a proof of $\varphi_1, \varphi_1, \Gamma \Rightarrow \chi$. An application of PIH on the latter gives a proof of $\varphi_1, \Gamma \Rightarrow \chi$. Second, we can build the following semi-proof.

$$\dfrac{\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0 \qquad \varphi_1, \Gamma \Rightarrow \chi}{\Box\varphi_0 \to \varphi_1, \Gamma \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

If φ is not principal in r, then there are two subcases to consider. If φ is boxed, i.e. $\varphi = \Box\varphi_0$, then r is

$$\dfrac{\varphi_0, \Box\varphi_0, \varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Box\varphi_0, \Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\Box\varphi_0, \Box\varphi_0, \Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0 \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

First, we use SIH on the proof of the right premise to obtain a proof of the sequent $\gamma_1, \Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \chi$. Second, we use SIH on the proof of the sequent $\varphi_0, \Box\varphi_0, \varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$ to obtain a proof of $\varphi_0, \Box\varphi_0, \varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$. Third, we use PIH on the latter to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$. It has to be noted that we need to perform the second step before the third, as the second step is sensitive to the height of the proof consider, which may be lost in an application of PIH. Finally, we build the following semi-proof.

$$\dfrac{\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\Box\varphi_0, \Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0 \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

If φ is not boxed, then r is

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \varphi, \varphi, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\varphi, \varphi, \Phi, \Box\gamma_0 \to \gamma_1, \Box\Gamma_0 \Rightarrow \chi}\ (\Box\!\to\!\mathrm{L})$$

Then we can use SIH on the proof of the right premise to obtain a proof of the sequent $\gamma_1, \varphi, \Phi, \Box\Gamma_0 \Rightarrow \chi$. A simple application of ($\Box\!\rightarrow$L) gives us a proof of $\varphi, \Box\gamma_0 \rightarrow \gamma_1, \Gamma_0 \Rightarrow \chi$.

13. $r =$ (GLR): then there are two cases to consider. If φ is boxed, i.e. $\varphi = \Box\varphi_0$, then r is

$$\frac{\varphi_0, \Box\varphi_0, \varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0}{\Box\varphi_0, \Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0}\ \text{(GLR)}$$

First, we use SIH on the proof of the premise to obtain a proof of the sequent $\varphi_0, \Box\varphi_0, \varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$. Second, we use PIH on the proof of the latter to obtain a proof of $\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$. Then an application of (GLR) gives us a proof of $\Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0$. If φ is not boxed, then r is

$$\frac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0}{\varphi, \varphi, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0}\ (\Box\!\rightarrow\!\text{L})$$

It suffices to adequately apply (GLR) on the premise, i.e. by adding only one of the φ in the conclusion, to obtain a proof of $\varphi, \Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0$.

■

We are now in possession of the two main structural properties of GL4ip: weakening and contraction. While we make use of these properties only in Section 13.6, in the following we introduce a second calculus PSGL4ip which embodies a terminating non-deterministic backward proof-search procedure for GL4ip. This will allow us to define the maximum height of derivations for a sequent with respect to this procedure. Later on, this will constitute the secondary induction measure in the proof of admissibility of cut.

13.5 PSGL4ip: terminating backward proof-search

In this section, we define the sequent calculus PSGL4ip which embodies a proof-search procedure on GL4ip. This type of embodiment is explained in Section 4.2. Two results are targeted here. First, we proceed to show the existence of a derivation of maximal height in PSGL4ip for each sequent, which allows us to define a measure mhd on sequents used in our cut-admissibility proof. Second, here again we justify our use of the prefix PS in PSGL4ip by showing that the latter helps decide the provability of sequents in GL4ip.

The calculus PSGL4ip restricts the rules of GL4ip in the following way.

(Ident) The rule (IdP) is replaced by the identity rule (Id) on formulas of any weight shown. Note that it is derivable in GL4ip as shown in Lemma 13.4.4.

$$\frac{}{\varphi, \Gamma \Rightarrow \varphi}\ \text{(Id)}$$

(NoInit) The conclusion of no rule is permitted to be an instance of either (Id) or (⊥L).

These restrictions have an identical aim as the one found in the calculus PSGLS: avoiding repetitions of some sequents in a branch. In particular, with PSGL4ip we try to get rid of loops as presented in Example 13.3.2. By scrutinizing the latter, one can see that the identity that can be found in the left premise is an identity on φ. So, we need to consider identities on formulas of *all structures* in PSGL4ip, on the contrary to PSGLS where we could focus only on identities on boxed formulas and atoms. This is why the addition of the rule (Id) is required, and the addition of (IdB) would be insufficient. The restrictions constituting PSGL4ip differ from the ones constituting PSGLS in another significant way: here the restriction (NoInit) is imposed on *all rules*, and not only on (GLR). Why is this brutal extension necessary? Simply because identities are not preserved upwards in rule applications. Concretely, if we were to impose this restriction on the rules (GLR) and ($\Box{\to}$L), say, we would indeed avoid the loop presented in Example 13.3.2. However, we could very easily twist the example by interleaving a rule application at the beginning of each loop as follows:

$$\dfrac{\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi), \Box(\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi)), (\varphi\wedge\psi), (\varphi\wedge\psi), \Box(\varphi\wedge\psi)^3 \Rightarrow (\varphi\wedge\psi) \qquad \dots\Rightarrow\dots}{\dfrac{\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi), \Box(\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi)), \boldsymbol{\varphi}, \boldsymbol{\psi}, \Box(\varphi\wedge\psi)^2 \Rightarrow (\varphi\wedge\psi)}{\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi), \Box(\Box(\varphi\wedge\psi)\to(\varphi\wedge\psi)), \boldsymbol{(\varphi\wedge\psi)}, \Box(\varphi\wedge\psi)^2 \Rightarrow \boldsymbol{(\varphi\wedge\psi)}}\;(\wedge\text{L})}\;(\Box{\to}\text{L})$$

By breaking the conjunction $(\varphi\wedge\psi)$ using ($\wedge$L), we make sure that (Id), applicable to the bottom line on the formulas in bold font, is not anymore on the line right above. So, we transition upwards from a sequent on which ($\Box{\to}$L) is not applicable to one on which the rule is applicable. If we do not restrict the rule ($\wedge$L) in (NoInit) as well, we are thus still allowing loops.

To sum up, the restrictions (NoInit) and (Ident) disallow all loops and enable us to consider any sequent violating these conditions as provable. In fact, by showing that no loop can appear in a branch of a PSGL4ip derivation, we concretely show that the only type of loop present in GL4ip are loops on provable sequents.

In the remainder of this section, we proceed to show that no loop can exist in PSGL4ip. We do so by proving that each sequent has a derivation of maximum height in PSGL4ip. The existence of such derivations is ensured by the strict decreasing of a local measure on sequents upwards in the rules of PSGL4ip.

13.5.1 A well-founded order on $(\mathbb{N}\times\mathbb{N}\times \mathit{list}\,\mathbb{N})$

For the PSGLS case we could rely on a well-known well-founded order: the lexicographic order on natural numbers. Here, we need to build a well-founded order tailored to the measure we introduce below. This order is essentially a composition of other well-known orders. More precisely, we define a well-founded order on triples $(n, m, l) \in (\mathbb{N}\times\mathbb{N}\times \mathit{list}\,\mathbb{N})$ where $\mathit{list}\,\mathbb{N}$ is the set of all lists of natural numbers. The formalisation of the well-foundedness of the various orders presented here is due to Dominique Larchey-Wendling [135], who shared his work in a thread we started [135] in the historic mailing list on Coq: the coq-club.

A first order we require is the usual order on natural numbers. In the following, we use $<$ to designate this order. Then, we require and recall the general definition of a lexicographic order.

Definition 13.5.1 (Lexicographic order)**.** Let $(A_1, <_1), \cdots, (A_n, <_n)$ be a collection of sets A_i with respective (strict total) orders $<_i$ on these sets. We define the lexicographic order $<_{lex}^{(A_1,<_1),\cdots,(A_n,<_n)}$ as follows. For two n-tuples $(a_1, \cdots, a_n)$ and $(a'_1, \cdots, a'_n)$ of the Cartesian product $A_1 \times\cdots\times A_n$, we write $(a_1, \cdots, a_n) <_{lex}^{(A_1,<_1),\cdots,(A_n,<_n)} (a'_1, \cdots, a'_n)$ if there is a $1\leq j\leq n$ such that:

1. $a_p = a'_p$, for all $1 \leq p < j$
2. $a_j <_j a'_j$

Note that if $<_i$ is a well-founded relation for all $1 \leq i \leq n$, then $<_{lex}^{(A_1,<_1),\ldots,(A_n,<_n)}$ is also well-founded [104]. If $(A_i, <_i) = (A_j, <_j)$ for all $1 \leq i, j \leq n$, then we note $(A_i, <_i)^n$ the sequence $(A_1, <_1), \ldots, (A_n, <_n)$.

Now, we define the *shortlex* order, also called *breadth-first* [89] or *length-lexicographic* order, over lists of natural numbers $\ll$:

Definition 13.5.2 (Shortlex order)**.** The shortlex order over lists of natural numbers, noted $\ll$, is defined as follows. For two lists l_0 and l_1 of natural numbers, we say that $l_0 \ll l_1$ whenever one of the following conditions is satisfied:

1. $length(l_0) < length(l_1)$;
2. $length(l_0) = length(l_1) = n$ and $l_0 <_{lex}^{(\mathbb{N},<)^n} l_1$;

Intuitively, the shortlex order is ordering lists according to their length and follows the lexicographic order whenever length does not discriminate. This order is also well-founded (⌨).

Finally, we define the order $<^3$ on $(\mathbb{N} \times \mathbb{N} \times list\,\mathbb{N})$ as $<_{lex}^{(\mathbb{N},<),(\mathbb{N},<),(list(\mathbb{N}),\ll)}$. Given that $<$ and $\ll$ are well-founded orders, we get that $<^3$ is a well-founded order too (⌨).

13.5.2 A $(\mathbb{N} \times \mathbb{N} \times list\,\mathbb{N})$-measure on sequents

In what follows we use the term "measure" in an informal way, notably distinct from its use in measure theory [13, 143]. We proceed to attach to each sequent $\Gamma \Rightarrow \chi$ a measure $\Theta(\Gamma \Rightarrow \chi)$ which is a triple $(\alpha(\Gamma \Rightarrow \chi), \beta(\Gamma \Rightarrow \chi), \eta(\Gamma \Rightarrow \chi)) \in (\mathbb{N} \times \mathbb{N} \times list\,\mathbb{N})$. For simplicity, in the following paragraphs we consider a fixed sequent $\Gamma \Rightarrow \chi$ for which we define the triple and thus erase the mention of the sequent in the measures.

First, we focus on η. As $\Gamma \Rightarrow \chi$ is built from a finite multiset of formulas, it contains a formula of maximal weight. Let φ be that formula and $n = weight(\varphi)$. We can create a list of length n such that at each position m in the list (counting from right to left) for $1 \leq m \leq n$, we find the number of occurrences in $\Gamma \Rightarrow \chi$ of *topmost* formulas of weight m. Such a list gives the count of occurrences in $\Gamma \Rightarrow \chi$ of formulas of weight n in its leftmost (i.e. n-th) component, then of occurrences of formulas of weight $n-1$ in the next (i.e. $(n-1)$-th) component, and so on until we reach 1. We define η to be this unique list. For example, $\eta(p \wedge q, p \vee q \Rightarrow q \rightarrow p)$ is the list $[1, 2, 0, 0]$ because $p \wedge q$ is the maximal formula of weight 4, and it is the only formula with this weight occurring in the list, while both $p \vee q$ and $q \rightarrow p$ are of weight 3. Two things need to be noted about such lists. First, if no topmost occurrence of a formula is of weight $1 \leq k \leq n$, then a 0 appears in position k in the list. This is the case for the weight 2 in the example just given. Second, as in general no formula is of weight 0 we do not need to dedicate a position for this particular weight in our list.

Why do we need such a list? With this list, the shortlex order becomes an adequate substitute to the Dershowitz-Manna order [38] considered in Dyckhoff's work on G4ip. We recall this order, given two multisets Γ_0 and Γ_1, by quoting van der Giessen and Iemhoff [158]: "$\Gamma_0 \ll \Gamma_1$ if and only if Γ_0 is the result of replacing one or more formulas in Γ_1 by zero or more formulas of lower degree". As our use of the symbol $\ll$ for the shortlex order suggests, the shortlex order can replace the order given above to order finite multisets of formulas.

A similar list was independently formalised in Coq by Daniel Schepler in the study of the calculus G4ip which he, following Dyckhoff, calls LJT [130]. However, he does not involve this list in a termination argument: instead, he uses it to show the equivalence of G4ip and the usual natural deduction system for intuitionistic logic.

Second, we turn to β. On the contrary to the measure defined by Bílková [10] and used by van der Giessen and Iemhoff, which attributes a natural number to a sequent *appearing in a proof-search tree which depends on the root*, we use the *local* notion of "number of usable boxes" we defined in Definition 12.6.1. We port it to sequents with a single formula as succedent.

Definition 13.5.3. We define:

1. the *usable boxes* $ub(\Gamma \Rightarrow \chi)$ of $\Gamma \Rightarrow \chi$ as:

$$ub(\Gamma \Rightarrow \chi) := \{\Box\varphi \mid \ \Box\varphi \in \text{Subf}(\Gamma \cup \{\chi\})\ \} \ \setminus \ \{\Box\varphi \mid \Box\varphi \in \Gamma\}$$

2. the *number of usable boxes* $\beta(\Gamma \Rightarrow \chi)$ of $\Gamma \Rightarrow \chi$ as $\beta(\Gamma \Rightarrow \chi) = \text{Card}(ub(\Gamma \Rightarrow \chi))$.

Thus, the notion of usable boxes of $\Gamma \Rightarrow \chi$ is the set of boxed subformulas of $\Gamma \Rightarrow \chi$ minus the topmost boxed formulas in Γ. Intuitively, this notion captures the set of boxed formulas of a sequent s which might be the diagonal formula of an instance of (GLR) in a derivation of s in PSGL4ip.

Third, we finally consider α. As Γ is a finite multiset of formulas, the checking of whether or not $\Gamma \Rightarrow \chi$ is an instance of the rule (Id) or ($\bot$L) is decidable (⌨). So, we can constructively define the following test function:

$$\alpha(\Gamma \Rightarrow \chi) = \begin{cases} 0 & \text{if } \Gamma \Rightarrow \chi \text{ is an instance of (Id) or } (\bot\text{L}) \\ 1 & \text{otherwise} \end{cases}$$

13.5.3 Every rule of PSGL4ip reduces Θ upwards

We are now in possession of a precisely defined measure on sequents and a well-founded order on that type of measure. Next, we proceed to prove that the measure Θ decreases upwards through the rules of PSGL4ip on the $<^3$ ordering.

Lemma 13.5.1. For all sequents $s_0, s_1, ..., s_n$ and for all $1 \leq i \leq n$, if there is an instance of a rule r of PSGL4ip of the form below, then $\Theta(s_i) <^3 \Theta(s_0)$:

$$\frac{s_1 \quad \dots \quad s_n}{s_0}\ r$$

Proof. (⌨) We reason by case analysis on r:

1. If r is (Id) or ($\bot$L), then we are done as there is no premise.

2. If r is ($\wedge$R), ($\wedge$L), ($\vee$R$_1$), ($\vee$R$_2$), ($\vee$L), ($\to$R), ($p\to$L), ($\wedge\to$L), ($\vee\to$L) or ($\to\to$L), then we have that $\eta(s_0) \ll \eta(s_1)$ and $\eta(s_0) \ll \eta(s_2)$ (if it exists), as shown by Dyckhoff and Negri [43]. Obviously, α can only decrease upwards in these rules, as no rule of PSGL4ip with premises can be applied to an initial sequent. Also, it is not hard to convince oneself that the number of usable boxes can only decrease in these rules as the boxed formulas on the left of the sequent are preserved upwards and the set of boxed subformulas is either stable or loses elements. So we can easily deduce that Θ decreases on $<^3$ from the conclusion to the premises of these rules.

3. If r is (GLR) then it must have the following form.

$$\frac{\boxtimes\Gamma, \Box\psi \Rightarrow \psi}{\Phi, \Box\Gamma \Rightarrow \Box\psi}\ \text{(GLR)}$$

   Clearly, we have the following inclusion.

$$\{\Box\varphi \mid \Box\varphi \in \text{Subf}(\boxtimes\Gamma \cup \{\Box\psi\} \cup \{\psi\})\} \subseteq \{\Box\varphi \mid \Box\varphi \in \text{Subf}(\Phi \cup \Box\Gamma \cup \{\Box\psi\})\}$$

   Also, given that we consider a derivation in PSGL4ip, we can note that (Id) is not applicable on $\Phi, \Box\Gamma \Rightarrow \Box\psi$ by assumption, hence $\Box\psi \notin \Box\Gamma$. Consequently, we get $\{\Box\varphi \mid \Box\varphi \in \Phi \cup \Box\Gamma\} \subset \{\Box\varphi \mid \Box\varphi \in \boxtimes\Gamma \cup \{\Box\psi\}\}$. An easy set-theoretic argument leads to $ub(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) \subset ub(\Phi, \Box\Gamma \Rightarrow \Box\psi)$. As a consequence we obtain $\beta(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) < \beta(\Phi, \Box\Gamma \Rightarrow \Box\psi)$, hence $\Theta(\boxtimes\Gamma, \Box\psi \Rightarrow \psi) <^3 \Theta(\Phi, \Box\Gamma \Rightarrow \Box\psi)$.

4. If r is ($\Box\to$L) then it must have the following form.

$$\frac{\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi \qquad \Phi, \Box\Gamma, \psi \Rightarrow \chi}{\Phi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi}\ (\Box\to\text{L})$$

For the right premise we can straightforwardly see that both α and β either are stable or decrease upwards, and that $\eta(\Phi, \Box\Gamma, \psi \Rightarrow \chi) \ll \eta(\Phi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$. So, we obtain $\Theta(\Phi, \Box\Gamma, \psi \Rightarrow \chi) <^3 \Theta(\Phi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$. The case of the left premise is more complex but can be treated similarly to the (GLR) as follows. Note that $\{\Box\delta \mid \Box\delta \in \mathrm{Subf}(\boxtimes\Gamma \cup \{\Box\varphi\} \cup \{\varphi\})\} \subseteq \{\Box\delta \mid \Box\delta \in \mathrm{Subf}(\Phi \cup \Box\Gamma \cup \{\Box\varphi \to \psi\} \cup \{\psi\})\}$. We consider two cases.

In the first case, we have that $\Box\varphi \notin \Box\Gamma$. Then as in (GLR) we obtain $\{\Box\delta \mid \Box\delta \in \Phi \cup \Box\Gamma \cup \{\Box\varphi \to \psi\}\} \subset \{\Box\delta \mid \Box\delta \in \boxtimes\Gamma \cup \{\Box\varphi\}\}$ and consequently $\beta(\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi) < \beta(\Phi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$. So, regardless of the value of $\alpha(\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi)$, we obtain $\Theta(\boxtimes\Gamma, \Box\varphi \Rightarrow \varphi) <^3 \Theta(\Phi, \Box\varphi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$.

In the second case, we have that $\Box\varphi \in \Box\Gamma$. Then the rule application is of the following form:

$$\frac{\boxtimes\Gamma, \Box\varphi, \varphi, \Box\varphi \Rightarrow \varphi \qquad \Phi, \Box\varphi, \Box\Gamma, \psi \Rightarrow \chi}{\Phi, \Box\varphi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi}\ (\Box{\to}\mathrm{L})$$

Clearly, we get $\alpha(\boxtimes\Gamma, \Box\varphi, \varphi, \Box\varphi \Rightarrow \varphi) = 0$ as it is an instance of an initial sequent, hence $\alpha(\boxtimes\Gamma, \Box\varphi, \varphi, \Box\varphi \Rightarrow \varphi) < \alpha(\Phi, \Box\varphi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$. Consequently, we get $\Theta(\boxtimes\Gamma, \Box\varphi, \varphi, \Box\varphi \Rightarrow \varphi) <^3 \Theta(\Phi, \Box\varphi, \Box\Gamma, \Box\varphi \to \psi \Rightarrow \chi)$.

■

Note that contraction and weakening as rules allow Θ to *increase* upwards. While it is rather obvious for contraction, this statement for weakening is surprising. The key point here is to note that weakening allows the deletion of boxed formulas in the antecedent of sequents, leading to a potential increase in the number of usable boxes β: that is, weakening may remove some of the boxes that "block" some applications of (GLR) upwards and so the number of usable boxes increases.

Now, for convenience we define the order $\lhd$ on sequents as follows:

$$s_0 \lhd s_1 \text{ if and only if } \Theta(s_0) <^3 \Theta(s_1)$$

As $<^3$ is a well-founded order, it is obvious that $\lhd$ is so as well. As a consequence, we obtain a strong induction principle following the $\lhd$ order (⌨).

Theorem 13.5.1. For any property P on sequents, to prove the statement $\forall s P(s)$ it is sufficient to show that every sequent s_0 satisfies P under the assumption that all its $\lhd$-predecessors satisfy P.

```
Theorem less_than3_strong_inductionT:
forall (P : Seq -> Type),
(forall s0, (forall s1, ((s1 <3 s0) -> P s1)) -> P s0)
-> forall s, P s.
```

13.5.4 The existence of a derivation of maximum height

Here, we make use of the order $\lhd$ and the strong induction principle which comes with it to show that each sequent has a derivation of maximal height.

First, we prove that each sequent has a list of premises through rules of PSGL4ip.

Lemma 13.5.2. For all sequent s there is a list $Prems(s)$ such that for all s', s' is a premise of the conclusion s for an instance of a rule r in PSGL4ip iff s' is in $Prems(s)$.

Proof. (⌨) In essence we proceed as in the proof of Lemma 12.6.2: show that each rule can be applied backwards on s only in finitely many ways, generating a list. We put together the list of each rule to obtain our final list. Note that this list is effectively computable, as shown by formalisation. ■

Second, if we use the strong induction principle of $\lhd_3$ with Lemma 13.5.1 and Lemma 13.5.2, we can easily prove the existence of a derivation in PSGL4ip of maximum height for all sequents.

Theorem 13.5.2. Every sequent s has a PSGL4ip derivation of maximum height.

```
Theorem PSGL4ip_termin :
 forall s, existsT2 (DMax: PSGL4ip_drv s), (is_mhd DMax).
```

Proof. (⌨) We use `less_than3_strong_inductionT`, the strong induction principle on $\lhd_3$ from Theorem 13.5.1. As the applicability of the rules of PSGL4ip is decidable (⌨), we distinguish two cases:

(I) No PSGL4ip rule is applicable to s. Then the derivation of maximum height sought after is simply the derivation constituted of s solely, which is the only derivation for s.

(II) Some PSGL4ip rule is applicable to s. Either only initial rules are applicable, in which case the derivation of maximum height sought after is simply the derivation of height 1 constituted of the application of the applicable initial rule to s. Or, some other rules than the initial rules are applicable. By Lemma 13.5.2 we can consider the finite list $Prems(s)$ of all sequents s_{prem} such that there is an application of a PSGL4ip rule r with s as conclusion of r and s_{prem} as premise of r. By Lemma 13.5.1 we know that every element s_0 in the list $Prems(s)$ is such that $s_{prem} \lhd_3 s$. Consequently, the strong induction hypothesis allows us to consider the derivation of maximum height of all the sequents in $Prems(s)$. As $Prems(s)$ is finite, there must be an element s_{max} of $Prems(s)$ such that its derivation of maximum height is higher than the derivation of maximum height of all sequents in $Prems(s)$ or of the same height. It thus suffices to pick that s_{max}, use its derivation of maximum height, and apply the appropriate rule to obtain s as a conclusion: this is by choice the derivation of maximum height of s. ■

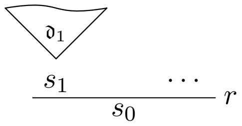
Here, `DMax` is a *derivation*, the existence of which is guaranteed by the constructive existential quantifier `existsT2`. This quantifier not only requires us to construct a witnessing term but also to provide a proof that the witness is of the correct type. The function `is_mhd` returns the constructive Coq proposition `True` if and only if its argument, `D`, is a derivation of maximum height.

As the previous lemma implies the *constructive* existence of a derivation $\mathfrak{d}$ of maximum height in PSGL4ip for any sequent s, we are entitled to let $\text{mhd}(s)$ denote the height of $\mathfrak{d}$. As in Section 12.6, we later use $\text{mhd}(s)$ as the secondary induction measure used in the proof of admissibility of cut. The previous lemma receives a similar interpretation as Lemma 13.5.2: it shows the termination of the proof-search embodied in PSGL4ip, and expresses the converse well-foundedness of iGL-frames.

Now we prove the sole result about $\text{mhd}(s)$ we effectively use: it decreases upwards in the rules of PSGL4ip.

Lemma 13.5.3. If r is a rule instance from PSGL4ip with conclusion s_0 and s_1 as one of the premises, then $\text{mhd}(s_1) < \text{mhd}(s_0)$.

Proof. (⌨) As $<$ and $=$ are decidable relations over natural numbers, we can reason by contradiction. So, suppose that $\text{mhd}(s_1) \geq \text{mhd}(s_0)$. Let $\mathfrak{d}_0$ be the derivation of s_0 of maximal height and let $\mathfrak{d}_1$ be the derivation of s_1 of maximal height as guaranteed by Theorem 13.5.2. If r is a rule instance from PSGL4ip with s_1 as one of the premises and with conclusion s_0, then $\mathfrak{d}_2$ as shown below is a derivation of s_0 of height greater than $\text{mhd}(s_1) + 1$:



The maximality of $\mathfrak{d}_0$ implies that the height of $\mathfrak{d}_0$ is greater than the height of $\mathfrak{d}_2$: thus $\text{mhd}(s_1) + 1 \leq \text{mhd}(s_0)$. As our initial assumption implies that $\text{mhd}(s_1) + 1 > \text{mhd}(s_0)$, we reached a contradiction. ■

13.5.5 Justifying the prefix PS

Now that the mhd notion is defined, we show that the PS prefix in PSGL4ip is justified: we can use the latter to decide the provability of sequents in GL4ip.

First, we prove that a sequent is provable in PSGL4ip if and only if it is provable in GL4ip.

Proposition 13.5.1. For all Γ and Δ: $\Gamma \Rightarrow \Delta$ has a proof in PSGL4ip iff $\Gamma \Rightarrow \Delta$ has a proof in GL4ip.

Proof. (⌨) From left to right it suffices to notice that all rules of PSGL4ip but (Id) are rules of GL4ip, and that the latter rule is admissible in GL4ip by Lemma 13.4.4. (⌨) From right to left, we notice that all rules of GL4ip are either rules of PSGL4ip or restricted in PSGL4ip. However, if the application of any unrestricted rule in GL4ip violates the conditions imposed in PSGL4ip, we can clearly prove the conclusion of this application via (Id) or ($\bot$L). ■

If we follow the general description given in Section 12.6, we are left to show that PSGL4ip can be used to decide the provability of sequents in GL4ip to show that the former deserves its prefix. This result is obtained through the existence for each sequent of a derivation of maximum height in PSGL4ip, leading to a decision procedure for the provability of sequents in PSGL4ip obtained using Lemma 13.5.3.

Theorem 13.5.3. Provability in PSGL4ip is decidable.

Proof. (⌨) We prove the statement by strong induction on the mhd of sequents. So, we assume that the result holds for all sequent s' such that $\text{mhd}(s') < \text{mhd}(s)$ and proceed to show that the result holds for s. By Lemma 13.5.2 we can consider the finite list $Prems(s)$ of all sequents s_0 such that there is an application of a PSGL4ip rule r with s as conclusion of r and s_0 as premise of r. By Lemma 13.5.3 we know that any s_0 in $Prems(s)$ is such that $\text{mhd}(s_0) < \text{mhd}(s)$. So, we can apply the induction hypothesis on all of the sequents in $Prems(s)$. It then suffices to check whether there is a rule r with conclusion s and premises in $Prems(s)$ and provable, which is doable in finite time as there are only finitely many rules applicable to s. If there is such a rule and premises, then s is provable. If not, s is not provable. ■

As an immediate corollary, we obtain that the provability of sequents in GL4ip is decidable.

Corollary 13.5.1. Provability in GL4ip is decidable.

Proof. (⌨) By Proposition 13.5.1, the results holds for GL4ip if and only if it holds for PSGL4ip. Consequently, Theorem 13.5.3 gives us the desired result. ■

So, PSGL4ip embodies a proof-search procedure on GL4ip according to the general definition given in Section 12.6. Here again, our result is formalised in `Type`, so we can extract a program effectively deciding sequents in GL4ip.

In the next section, we use the notion of mhd, obtained through the termination of PSGL4ip, to establish cut-elimination for GL4ip.

13.6 Cut-elimination for GL4ip

To reach cut-elimination, our main theorem, we first state and prove cut-admissibility in a purely syntactic way via local proof transformations. More precisely, we proceed to prove that the *additive*-cut rule is admissible, as in the **GLS** case.

Theorem 13.6.1. The additive cut rule is admissible in GL4ip.

```
Theorem GL4ip_cut_adm : forall φ Γ0 Γ1 χ,
 (GL4ip_prv (Γ0++Γ1,φ)  *  GL4ip_prv (Γ0++φ::Γ1,χ)) ->
              GL4ip_prv (Γ0++Γ1,χ).
```

Proof. (⌨) Let $\mathfrak{p}_1$ (with last rule r_1) and $\mathfrak{p}_2$ (with last rule r_2) be proofs in GL4ip of $\Gamma \Rightarrow \varphi$ and $\varphi, \Gamma \Rightarrow \chi$ respectively, as shown below.

It suffices to show that there is a proof in GL4ip of $\Gamma \Rightarrow \chi$. We reason by strong primary induction (PI) on the weight of the cut-formula φ, giving the primary inductive hypothesis (PIH). We also use a strong secondary induction (SI) on mhd(s) of the conclusion of a cut, giving the secondary inductive hypothesis (SIH).

We make a first case distinction: does $\Gamma \Rightarrow \chi$ violate (NoInit)? If it is the case, then this sequent is an instance of (Id) or ($\bot$L). So, we use Lemma 13.4.4 or apply ($\bot$L) to obtain a proof of $\Gamma \Rightarrow \chi$. If $\Gamma \Rightarrow \chi$ satisfies (NoInit), then it is not an instance of (Id) or ($\bot$L). In this case we consider r_1. In total, there are thirteen cases to consider for r_1: one for each rule in GL4ip. However, we can gather some of the cases together and reduce the number of cases to eight. We separate them by using Roman numerals.

(I) $\mathbf{r_1}$ =(IdP) : then we have that $\varphi = p$. Consequently, $\Gamma \Rightarrow \chi$ is of the form $\Gamma_0, p \Rightarrow \chi$. Also, the conclusion of r_2 is of the form $\Gamma_0, p, p \Rightarrow \chi$. We can apply the contraction Lemma 13.4.8 to obtain a proof of $\Gamma_0, p \Rightarrow \chi$.

(II) $\mathbf{r_1}$ =($\bot$L): Then r_1 must have the following form.

$$\dfrac{}{\Gamma_0, \bot \Rightarrow \varphi}\,(\bot\text{L})$$

where $\Gamma_0, \bot = \Gamma$. Thus, we have that the sequent $\Gamma \Rightarrow \chi$ is of the form $\Gamma_0, \bot \Rightarrow \chi$, and is an instance of $\bot$L. But this is in contradiction with (NoInit). So we are done.

(III) $\mathbf{r_1} \in$\{($\wedge$L), ($\vee$L), ($p{\to}$L), ($\wedge{\to}$L), ($\vee{\to}$L)\}: In all these cases, the cut formula is not principal in r_1 so it is preserved in the premise. Given that the rules considered are invertible, we simply take the conclusion of r_2 and use the corresponding invertibility lemma to destruct the principal formula of r_1. Then, we use SIH to cut on φ in the obtained premises, and apply r_1 on the conclusion of the cut. As an example, we consider the case of ($\wedge$L), where r_1 is of the following form and where $\Gamma_0, \psi \wedge \delta = \Gamma$:

$$\dfrac{\Gamma_0, \psi, \delta \Rightarrow \varphi}{\Gamma_0, \psi \wedge \delta \Rightarrow \varphi}\,(\wedge\text{L})$$

Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\varphi, \Gamma \Rightarrow \chi$ are respectively of the form $\Gamma_0, \psi \wedge \delta \Rightarrow \chi$ and $\varphi, \Gamma_0, \psi \wedge \delta \Rightarrow \chi$. Using the invertibility of ($\wedge$L), proven in Lemma 13.4.3, on $\varphi, \Gamma_0, \psi \wedge \delta \Rightarrow \chi$ we obtain a proof of the sequent $\varphi, \Gamma_0, \psi, \delta \Rightarrow \chi$. Then, we proceed as follows.

$$\dfrac{\dfrac{\Gamma_0, \psi, \delta \Rightarrow \varphi \qquad \varphi, \Gamma_0, \psi, \delta \Rightarrow \chi}{\Gamma_0, \psi, \delta \Rightarrow \chi}\,\text{SIH}}{\Gamma_0, \psi \wedge \delta \Rightarrow \chi}\,(\wedge\text{L})$$

Note that the use of SIH is justified here as the assumption (NoInit) ensures that the last rule in this proof is effectively an instance of $(\wedge\mathrm{L})$ in PSGL4ip, hence $\mathrm{mhd}(\Gamma_0, \psi, \delta \Rightarrow \chi) < \mathrm{mhd}(\Gamma_0, \psi \wedge \delta \Rightarrow \chi)$ by Lemma 13.5.3.

(IV) $\mathbf{r_1} \in\{(\wedge\mathbf{R}), (\vee\mathbf{R}_1), (\vee\mathbf{R}_2)\}$: In all these cases, the cut formula is principal in r_1 so it is deconstructed in the premise. Given that the corresponding left rules are invertible, we simply take the conclusion of r_2 and use the adequate invertibility lemma to destruct the cut formula. Then, we use PIH to cut on the obtained subformulas. As an example, we consider the case of $(\vee\mathrm{R}_1)$, where r_1 is of the following form and where $\psi \vee \delta = \varphi$:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi \vee \delta}\ (\vee\mathrm{R}_1)$$

Thus, we have that the sequents $\varphi, \Gamma \Rightarrow \chi$ is of the form $\psi \vee \delta, \Gamma \Rightarrow \chi$. Using the invertibility of $(\vee\mathrm{L})$, proven in Lemma 13.4.3, on $\psi \vee \delta, \Gamma \Rightarrow \chi$ we obtain a proof of the sequents $\psi, \Gamma \Rightarrow \chi$ and $\delta, \Gamma \Rightarrow \chi$. Then, we proceed as follows.

$$\frac{\Gamma \Rightarrow \psi \qquad \psi, \Gamma \Rightarrow \chi}{\Gamma \Rightarrow \chi}\ \mathrm{PIH}$$

(V) $\mathbf{r_1}$ =$(\rightarrow\mathbf{R})$: Then r_1 has the following form where $\varphi = \varphi_0 \rightarrow \varphi_1$:

$$\frac{\varphi_0, \Gamma \Rightarrow \varphi_1}{\Gamma \Rightarrow \varphi_0 \rightarrow \varphi_1}\ (\rightarrow\mathrm{R})$$

For the cases where $\varphi_0 \rightarrow \varphi_1$ is principal in r_2 and $r_2 \neq (\Box\!\rightarrow\!\mathrm{L})$, or where $r_2 \in \{(\mathrm{IdP}), (\bot\mathrm{L})\}$, we refer to Dyckhoff and Negri's proof [43] as the cuts produced in these cases involve the traditional induction hypothesis PIH. We are left with seven sub-cases.

(V-a) If r_2 is $(\rightarrow\mathrm{R})$ then it must have the following form.

$$\frac{\varphi_0 \rightarrow \varphi_1, \chi_0, \Gamma \Rightarrow \chi_1}{\varphi_0 \rightarrow \varphi_1, \Gamma \Rightarrow \chi_0 \rightarrow \chi_1}\ (\rightarrow\mathrm{R})$$

where $\chi_0 \rightarrow \chi_1 = \chi$. We can use Lemma 13.4.2 on the proof of $\Gamma \Rightarrow \varphi_0 \rightarrow \varphi_1$ to get a proof of $\chi_0, \Gamma \Rightarrow \varphi_0 \rightarrow \varphi_1$. Proceed as follows.

$$\frac{\dfrac{\chi_0, \Gamma \Rightarrow \varphi_0 \rightarrow \varphi_1 \qquad \varphi_0 \rightarrow \varphi_1, \chi_0, \Gamma \Rightarrow \chi_1}{\chi_0, \Gamma \Rightarrow \chi_1}\ \mathrm{SIH}}{\Gamma \Rightarrow \chi_0 \rightarrow \chi_1}\ (\rightarrow\mathrm{R})$$

Note that the use of SIH is justified here as the last rule in this proof is effectively an instance of $(\rightarrow\mathrm{R})$ in PSGL4ip, hence $\mathrm{mhd}(\chi_0, \Gamma \Rightarrow \chi_1) < \mathrm{mhd}(\Gamma \Rightarrow \chi_0 \rightarrow \chi_1)$ by Lemma 13.5.3.

(V-b) If r_2 is $(\wedge\mathrm{R})$ or $(\vee\mathrm{R}_i)$, then we simply use cut with the premise(s) of r_2 and the conclusion of r_1 using SIH. As an example, we consider the case of $(\vee\mathrm{R}_1)$, where r_2 has the form:

$$\frac{\varphi_0 \rightarrow \varphi_1, \Gamma \Rightarrow \chi_0}{\varphi_0 \rightarrow \varphi_1, \Gamma \Rightarrow \chi_0 \vee \chi_1}\ (\vee\mathrm{R}_1)$$

where $\chi_0 \vee \chi_1 = \chi$. Then we proceed as follows:

$$\frac{\dfrac{\Gamma \Rightarrow \varphi_0 \rightarrow \varphi_1 \qquad \varphi_0 \rightarrow \varphi_1, \Gamma \Rightarrow \chi_0}{\Gamma \Rightarrow \chi_0}\ \mathrm{SIH}}{\Gamma \Rightarrow \chi_0 \vee \chi_1}\ (\vee\mathrm{R}_1)$$

(V-c) If r_2 is $(\wedge\mathrm{L})$, $(\vee\mathrm{L})$, $(p\!\rightarrow\!\mathrm{L})$, $(\vee\!\rightarrow\!\mathrm{R})$ or $(\wedge\!\rightarrow\!\mathrm{R})$ where the cut formula is not principal in r_2, then we use the inversion lemma for r_2 on the conclusion of r_1, and then apply cut using SIH. As an example, we consider the case of $(\wedge\!\rightarrow\!\mathrm{L})$, where r_2 has the form:

$$\frac{\varphi_0 \rightarrow \varphi_1, \gamma_0 \rightarrow (\gamma_1 \rightarrow \gamma_2), \Gamma_0 \Rightarrow \chi}{\varphi_0 \rightarrow \varphi_1, (\gamma_0 \wedge \gamma_1) \rightarrow \gamma_2, \Gamma_0 \Rightarrow \chi}\ (\wedge\!\rightarrow\!\mathrm{L})$$

where $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 = \Gamma$. Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\Gamma \Rightarrow \varphi_0 \to \varphi_1$ are respectively of the form $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$ and $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Using the invertibility of $(\wedge{\to}\text{L})$, proven in Lemma 13.4.3, on $(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$ we obtain a proof of the sequent $\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Then, we proceed as follows.

$$\dfrac{\dfrac{\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1 \qquad \varphi_0 \to \varphi_1, \gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \chi}{\gamma_0 \to (\gamma_1 \to \gamma_2), \Gamma_0 \Rightarrow \chi}\,\text{SIH}}{(\gamma_0 \wedge \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\,(\wedge{\to}\text{L})$$

(V-d) If r_2 is $({\to}{\to}\text{L})$ where the cut formula is not principal in r_2, then it must have the following form where $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 = \Gamma$.

$$\dfrac{\varphi_0 \to \varphi_1, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \varphi_0 \to \varphi_1, \gamma_2, \Gamma_0 \Rightarrow \chi}{\varphi_0 \to \varphi_1, (\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\,({\to}{\to}\text{L})$$

Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\Gamma \Rightarrow \varphi_0 \to \varphi_1$ are respectively of the form $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$ and $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Using the right-invertibility of $({\to}{\to}\text{L})$, proven in Lemma 13.4.5, on the latter we obtain a proof of the sequent $\gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Then, we make a case distinction on whether the sequent $\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1$ is an instance of (Id) or $(\bot\text{L})$. If it is the case, then we proceed as follows.

$$\dfrac{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \dfrac{\gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1 \qquad \varphi_0 \to \varphi_1, \gamma_2, \Gamma_0 \Rightarrow \chi}{\gamma_2, \Gamma_0 \Rightarrow \chi}\,\text{SIH}}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\,({\to}{\to}\text{L})$$

Here the left branch is obviously provable either by invoking Lemma 13.4.4 or by applying $(\bot\text{L})$. If $\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma \to \gamma_1$ is not an instance of these rules, then consider the following proof of this sequent, where Lemma 13.4.7 deconstructs the implication $(\gamma_0 \to \gamma_1) \to \gamma_2$, Lemma 13.4.8 contracts $\gamma_1 \to \gamma_2$ and Lemma 13.4.3 is the invertibility of the rule $({\to}\text{R})$.

$$\dfrac{\dfrac{\dfrac{\dfrac{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1}{\gamma_0, \gamma_1 \to \gamma_2, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1}\,\text{Lem.13.4.7}}{\gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \varphi_0 \to \varphi_1}\,\text{Lem.13.4.8} \qquad \dfrac{\varphi_0 \to \varphi_1, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1}{\varphi_0 \to \varphi_1, \gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_1}\,\text{Lem.13.4.3}}{\gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_1}\,\text{SIH}}{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1}\,({\to}\text{R})$$

The crucial point here is to see that the use of SIH is justified, i.e. that $\text{mhd}(\gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_1) < \text{mhd}((\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi)$. This is the case as we made sure that the rule applications $({\to}{\to}\text{L})$ and $({\to}\text{R})$ are both instances of rules of PSGL4ip because their respective conclusions $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$ and $\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1$ are not instance of (Id) or $(\bot\text{L})$. So, we get that $\text{mhd}(\gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_1) < \text{mhd}(\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1) < \text{mhd}((\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi)$ by Lemma 13.5.3 hence $\text{mhd}(\gamma_0, \gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_1) < \text{mhd}((\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi)$ by transitivity of $<$. So, we are done. Note that the created cut could not be justified by usual induction on height, as Lemma 13.4.7 is not height-preserving.

(V-e) If r_2 is $(\Box{\to}\text{L})$ with the cut formula as principal formula, then it must have the following form, where $\Phi, \Box\Gamma_0 = \Gamma$ and $\varphi_0 = \Box\varphi_2$.

$$\dfrac{\boxtimes\Gamma_0, \Box\varphi_2 \Rightarrow \varphi_2 \qquad \varphi_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\Box\varphi_2 \to \varphi_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}\,(\Box{\to}\text{L})$$

Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\varphi_0, \Gamma \Rightarrow \varphi_1$ are respectively of the form $\Phi, \Box\Gamma_0 \Rightarrow \chi$ and $\Box\varphi_2, \Phi, \Box\Gamma_0 \Rightarrow \varphi_1$. Then, we proceed as follows.

$$\dfrac{\dfrac{\dfrac{\boxtimes\Gamma_0, \Box\varphi_2 \Rightarrow \varphi_2}{\Box\Gamma_0 \Rightarrow \Box\varphi_2}\,(\text{GLR})}{\Phi, \Box\Gamma_0 \Rightarrow \Box\varphi_2}\,\text{Lem.13.4.2} \qquad \dfrac{\Box\varphi_2, \Phi, \Box\Gamma_0 \Rightarrow \varphi_1 \qquad \dfrac{\varphi_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\varphi_1, \Box\varphi_2, \Phi, \Box\Gamma_0 \Rightarrow \chi}\,\text{Lem.13.4.2}}{\Box\varphi_2, \Phi, \Box\Gamma_0 \Rightarrow \chi}\,\text{PIH}}{\Phi, \Box\Gamma_0 \Rightarrow \chi}\,\text{PIH}$$

(V-f) If r_2 is $(\Box{\to}\text{L})$ with a principal formula different from the cut formula, then it must have the following form where $\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 = \Gamma$.

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \varphi_0 \to \varphi_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\varphi_0 \to \varphi_1, \Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}\;(\Box{\to}\mathrm{L})$$

Thus, we have that $\Gamma \Rightarrow \chi$ and $\Gamma \Rightarrow \varphi_0 \to \varphi_1$ are respectively of the form $\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi$ and $\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Using the right-invertibility of $(\Box{\to}\mathrm{L})$, proven in Lemma 13.4.5, on $\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$ we obtain a proof of $\gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi_0 \to \varphi_1$. Then, we proceed as follows.

$$\dfrac{\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \dfrac{\gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \varphi_0 \to \varphi_1 \qquad \gamma_1, \varphi_0 \to \varphi_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}{\gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}\;\mathrm{SIH}}{\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi}\;(\Box{\to}\mathrm{L})$$

Note that the use of SIH is justified here as the assumption (NoInit) ensures that the last rule in this proof is effectively an instance of $(\Box{\to}\mathrm{L})$ in PSGL4ip, hence $\mathrm{mhd}(\gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi) < \mathrm{mhd}(\Box\gamma_0 \to \gamma_1, \Phi, \Box\Gamma_0 \Rightarrow \chi)$ by Lemma 13.5.3.

(V-g) If r_2 is (GLR) then it must have the following form.

$$\dfrac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0}{\Phi, \varphi_0 \to \varphi_1, \Box\Gamma_0 \Rightarrow \Box\chi_0}\;(\mathrm{GLR})$$

where $\Phi, \Box\Gamma_0 = \Gamma$ and $\Box\chi_0 = \chi$. In that case, note that the sequent $\Gamma \Rightarrow \chi$ is of the form $\Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0$. To obtain a proof of the latter, we apply the rule (GLR) on the premise of r_2 without weakening $\psi \to \delta$:

$$\dfrac{\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0}{\Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0}\;(\mathrm{GLR})$$

(VI) $\mathbf{r_1}$ =($\to\to$L): Then r_1 is as follows, where $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 = \Gamma$.

$$\dfrac{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \gamma_2, \Gamma_0 \Rightarrow \varphi}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \varphi}\;({\to}{\to}\mathrm{L})$$

Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\varphi, \Gamma \Rightarrow \chi$ are respectively of the form $(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$ and $\varphi, (\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$. Using the right-invertibility of $({\to}{\to}\mathrm{L})$, proven in Lemma 13.4.5, on $\varphi, (\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi$ we obtain a proof of the sequent $\varphi, \gamma_2, \Gamma_0 \Rightarrow \chi$. Then, we proceed as follows.

$$\dfrac{\gamma_1 \to \gamma_2, \Gamma_0 \Rightarrow \gamma_0 \to \gamma_1 \qquad \dfrac{\gamma_2, \Gamma_0 \Rightarrow \varphi \qquad \varphi, \gamma_2, \Gamma_0 \Rightarrow \chi}{\gamma_2, \Gamma_0 \Rightarrow \chi}\;\mathrm{SIH}}{(\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi}\;({\to}{\to}\mathrm{L})$$

Note that the use of SIH is justified here as the assumption (NoInit) ensures that the last rule in this proof is effectively an instance of $({\to}{\to}\mathrm{L})$ in PSGL4ip, hence $\mathrm{mhd}(\gamma_2, \Gamma_0 \Rightarrow \chi) < \mathrm{mhd}((\gamma_0 \to \gamma_1) \to \gamma_2, \Gamma_0 \Rightarrow \chi)$ by Lemma 13.5.3.

(VII) $\mathbf{r_1}$ =($\Box\to$L): We proceed as in (V-f).

(VIII) $\mathbf{r_1}$ =(GLR): Then φ is the diagonal formula in r_1:

$$\dfrac{\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0}{\Phi, \Box\Gamma_0 \Rightarrow \Box\varphi_0}\;(\mathrm{GLR})$$

where $\varphi = \Box\varphi_0$ and $\Phi, \Box\Gamma_0 = \Gamma$. Thus, we have that the sequents $\Gamma \Rightarrow \chi$ and $\varphi, \Gamma \Rightarrow \chi$ are respectively of the form $\Phi, \Box\Gamma_0 \Rightarrow \chi$ and $\Box\varphi_0, \Phi, \Box\Gamma_0 \Rightarrow \chi$. We now consider r_2.

(VIII-a) If r_2 is one of (IdP), ($\bot$L), ($\wedge$R), ($\wedge$L), ($\vee$R$_1$), ($\vee$R$_2$), ($\vee$L), ($\to$R), ($p{\to}$L), ($\wedge{\to}$L), ($\vee{\to}$L) and (${\to}{\to}$L) then proceed similarly to the cases (I), (II), (III), (IV) and (VI), where the cut-formula is not principal in the rules considered by using SIH.

(VIII-b) If r_2 is ($\Box{\to}$L). Then r_2 is of the following form and where $\Box\gamma_0 \to \gamma_1, \Phi_0 = \Phi$:

$$\dfrac{\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0 \qquad \gamma_1, \Phi_0, \Box\varphi_0, \Box\Gamma_0 \Rightarrow \chi}{\Box\gamma_0 \to \gamma_1, \Phi_0, \Box\varphi_0, \Box\Gamma_0 \Rightarrow \chi}\;(\Box{\to}\mathrm{L})$$

We proceed as follows.

$$\begin{prooftree}
\AxiomC{$\mathfrak{p}$}
\noLine
\UnaryInfC{$\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$}
\AxiomC{$\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0$}
\RightLabel{(GLR)}
\UnaryInfC{$\Box\Gamma_0 \Rightarrow \Box\varphi_0$}
\dashedLine
\RightLabel{Lem.13.4.2}
\UnaryInfC{$\gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \Box\varphi_0$}
\AxiomC{$\gamma_1, \Phi_0, \Box\varphi_0, \Box\Gamma_0 \Rightarrow \chi$}
\dashedLine
\RightLabel{SIH}
\BinaryInfC{$\gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \chi$}
\RightLabel{($\Box\!\to$L)}
\BinaryInfC{$\Box\gamma_0 \to \gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \chi$}
\end{prooftree}$$

where $\mathfrak{p}$ is:

$$\begin{prooftree}
\AxiomC{$\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0$}
\RightLabel{(GLR)}
\UnaryInfC{$\Box\Gamma_0 \Rightarrow \Box\varphi_0$}
\dashedLine
\RightLabel{Lem.13.4.2}
\UnaryInfC{$\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \Box\varphi_0$}
\AxiomC{$\boxtimes\Gamma_0, \Box\varphi_0 \Rightarrow \varphi_0$}
\dashedLine
\RightLabel{Lem.13.4.2}
\UnaryInfC{$\boxtimes\Gamma_0, \Box\varphi_0, \Box\gamma_0 \Rightarrow \varphi_0$}
\AxiomC{$\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$}
\dashedLine
\RightLabel{PIH}
\BinaryInfC{$\boxtimes\Gamma_0, \Box\varphi_0, \Box\gamma_0 \Rightarrow \gamma_0$}
\dashedLine
\RightLabel{SIH}
\BinaryInfC{$\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$}
\end{prooftree}$$

Note that both uses of SIH are justified here as the assumption (NoInit) ensures that the last rule in this proof is effectively an instance of ($\Box\!\to$L) in PSGL4ip, hence mhd($\boxtimes\Gamma_0, \Box\gamma_0 \Rightarrow \gamma_0$) $<$ mhd($\Box\gamma_0 \to \gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \chi$) and mhd($\gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \chi$) $<$ mhd($\Box\gamma_0 \to \gamma_1, \Phi_0, \Box\Gamma_0 \Rightarrow \chi$) by Lemma 13.5.3.

(VIII-c) If r_2 is (GLR). Then r_2 is of the following form where $\Box\chi_0 = \chi$:

$$\begin{prooftree}
\AxiomC{$\varphi_0, \Box\varphi_0, \boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$}
\RightLabel{(GLR)}
\UnaryInfC{$\Phi, \Box\varphi_0, \Box\Gamma_0 \Rightarrow \Box\chi_0$}
\end{prooftree}$$

We proceed as follows where $\mathfrak{p}$ is taken from the case (VIII-b):

$$\begin{prooftree}
\AxiomC{$\mathfrak{p}$}
\noLine
\UnaryInfC{$\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$}
\RightLabel{(GLR)}
\UnaryInfC{$\Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0$}
\end{prooftree}$$

Note that the use of SIH is justified here as the assumption (NoInit) ensures that the last rule in this proof is effectively an instance of (GLR) in PSGL4ip, hence mhd($\boxtimes\Gamma_0, \Box\chi_0 \Rightarrow \chi_0$) $<$ mhd($\Phi, \Box\Gamma_0 \Rightarrow \Box\chi_0$) by Lemma 13.5.3. ■

The above theorem proof-theoretically establishes that cuts are eliminable in the calculus GL4ip extended with (cut). To effectively prove this statement in Coq we explicitly encode the version of the additive cut rule on sequents with a single formula as succedent as follows:

$$\dfrac{\texttt{(}\Gamma\texttt{0++}\Gamma\texttt{1,}\varphi\texttt{)} \qquad \texttt{(}\Gamma\texttt{0++}\varphi\texttt{::}\Gamma\texttt{1,}\chi\texttt{)}}{\texttt{(}\Gamma\texttt{0++}\Gamma\texttt{1,}\chi\texttt{)}}$$

With this rule in hand, we can encode the set of rules `GL4ip_cut_rules` as `GL4ip_rules` enhanced with (cut), i.e. the calculus GL4ip$+$(cut). We can finally turn to the elimination of additive cuts:

Theorem 13.6.2. The additive cut rule is eliminable from GL4ip $+$ (cut).

```
Theorem GL4ip_cut_elimination : forall s,
 (GL4ip_cut_prv s) -> (GL4ip_prv s).
```

Proof. (⌨) Let $\mathfrak{p}$ be the proof in GL4ip $+$ (cut) of the sequent s. We prove the statement by induction on the structure of $\mathfrak{p}$. If the last rule applied is a rule in GL4ip, then it suffices to apply the induction hypothesis on the premises and then the rule. If the last rule applied is (cut), then we use the induction hypothesis on both premises and then Theorem 13.6.1. ■

The above theorem shows that given a proof in GL4ip $+$ (cut) of a sequent, i.e. `GL4ip_cut_prv s`, we can transform this proof directly to obtain a proof in GL4ip of the same sequent. Given that this theorem is in fact a constructive function based on elements defined on `Type`, we can use the extraction feature of Coq and obtain a cut-eliminating Haskell program.

13.7 Conclusion

In this chapter, we ventured down the alley pointed at in the conclusion of Chapter 12: the range of applicability of the mhd technique. To do so, we introduced GL4ip, a multiset-based sequent calculus for the logic iGLL based on the terminating calculus G4ip for IL. We showed that this calculus enjoys the canonical proof-theoretical properties of exchange, weakening and, after some effort, contraction.

Then, we introduced a second sequent calculus: PSGL4ip. In this calculus each sequent has a derivation of maximal height, allowing us to define the measure mhd, i.e. the height of this maximal derivation. We made use of this measure to provide a decision procedure for the provability of sequents in GL4ip using PSGL4ip, justifying our use of the prefix PS: PSGL4ip embodies a terminating backward proof-search procedure for GL4ip.

The mhd measure was crucially involved in the proof technique used to obtain cut-admissibility. More than an alternative proof technique, the use of mhd in the case of GL4ip is to date the only known pathway to a direct proof of admissibility of cut: as admitted by van der Giessen and Iemhoff [158], all other available proof techniques fail. A cut-elimination result for GL4ip was consequently obtained as the proof of cut-admissibility only relied on steps locally transforming given proofs. From the formalisation of this result, we extracted a Haskell program that effectively eliminates cuts from cut-containing GL4ip proofs.

So, in addition to proving termination of proof-search using a local measure, while van der Giessen and Iemhoff use Bílková's non-local measure [10], and formalising on the way most of Dyckhoff and Negri's results on G4ip, we consequently addressed van der Giessen and Iemhoff's issue by providing a formalised direct proof of additive-cut admissibility for GL4ip. Crucially, this direct syntactic proof allows to obtain an extractable simple cut-elimination procedure for GL4ip hardly obtainable from the indirection in van der Giessen and Iemhoff's work.

Chapter 14

Afterwords on Provability Logics

As already mentioned, the proof theory of provability logics is extremely intricate. Indeed, in this context convoluted results such as cut-admissibility are commonly recognized as harder to obtain, because their proofs are even more intricate and involve non-trivial notions. For example, in the GLS case the proof of cut-admissibility contains a technical case involving the rule (GLR), which is solved by appealing in a crucial way to the unintuitive notion of "width" [156].

In this part, we confirmed the hardness of the treatment of provability logics via sequent calculi. However, we changed and shifted this hardness by using the *mhd proof technique* for cut-admissibility, which was recently discovered by Brighton [19].

First, we changed the hardness by replacing the unintuitive notion of "width" with a more easily graspable notion of "maximal height of derivations". The latter, shortened to "mhd", is defined using a terminating backward proof-search procedure which allows to exhibit for a given sequent a derivation of maximum height, hence bounding the height of all the possible derivations of this sequent. As we could see in the cases for wKS (see Subsection 4.2.1), GLS and GL4ip, reaching the definition of mhd is a non-trivial task: one has to show the termination of the backward proof-search through a local measure, and then use computationally heavy arguments to show the existence of a derivation of maximal height. Thus, the journey to the intuitive notion of maximal height of derivations is an intricate one, on the contrary to the unintuitive but easily definable notion of width.

Second, we shifted the hardness of the cut-admissibility proof by significantly simplifying the intricate (GLR) cases using the mhd proof technique. As seen above in the cut-admissibility proofs for wKS, GLS and GL4ip, the novelty of this technique consists in the binary induction measure it relies on: while the first component is the traditional "size of the cut formula", the second is mhd. Thus, as displayed in the proofs, with this technique we traded a simpler cut-admissibility proof for a harder-to-reach (but more intuitive) mhd induction measure.

On top of these benefits of clarity and simplicity, the mhd proof technique is interesting for two reasons.

The first reason is that this technique reverses the usual order of things by making cut-admissibility rely on the termination of backward proof-search. Indeed, one would usually prove that cut is admissible and then turn to the design of a proof-search procedure on the cut-free system and show its termination. This oddity is promising for a general treatment of cut admissibility via local transformations for calculi with a terminating backward proof-search.

The second reason is the sensitivity of the proof technique to the type of cut admitted. More precisely, this technique seems to only apply to *additive* cuts. To justify a cut through the secondary induction hypothesis, involving mhd, we need to link the sequent-conclusion of the initial cut to the sequent-conclusion of the newly created cut by a chain of rule applications that make mhd decrease upwards. However, we explained that contraction and weakening can increase mhd upwards. So, in the mhd technique we cannot use contraction or weakening in the chain linking the two sequent-conclusion, forbidding us from considering multiplicative cuts of the form below.

$$\frac{\Gamma_0 \Rightarrow \varphi \quad \Gamma_1 \Rightarrow \chi}{\Gamma_0, \Gamma_1 \Rightarrow \chi}$$

The use of additive cuts allows us to circumvent this difficulty. This sensitivity of the proof technique is surprising as all three calculi we considered admit weakening and contraction, making additive and multiplicative cuts equivalent.

All the proof-theoretic results presented in the last two chapters are formalised in Coq. As such, our work is a contribution to the dynamic field of formalised proof-theory [23, 27, 31, 149, 166] (see Reis' invited paper [124] for an interesting overview). However, our results are of a specific kind as they allow for *program extraction.* Let us recall that in Subsection 4.2.2, we formalised the results leading to the admissibility of cut for the sequent calculus wKS in such a way that we could extract a Haskell program, which should effectively eliminate cuts from proofs in wKS + (cut). For both GLS and GL4ip, we could formalise our results in a similar way, using `Type`, and thus obtain for each of these calculi a cut-elimination Haskell program, using the extraction feature of Coq. While we have neither tested nor tried to optimize any of these programs, their extraction is a promising milestone we managed to reach.

As further work, we are planning to tackle three lines of research.

First, we intend to pursue our exploration of the range of applicability of the mhd proof technique by considering calculi with terminating backward proof-search. Indeed, there are many sequent calculi for non-classical logics which enjoy terminating backward proof-search, and often in the intuitionistic case, they are based upon G4ip in some way. So, could we apply the mhd method to all these calculi? Answering this question would help in the generation of a general theory of cut-admissibility using the mhd proof technique.

Second, we intend to explore the limitation of the mhd proof technique by trying to apply it to sequent calculi with cycles. The calculi wKS, GLS, G4ip and GL4ip either contain no cycles or only contain *provable* cycles, i.e. cycles going through a provable sequent. As a consequence, the proof-search on these calculi only needs to get rid of provable cycles. This is done by imposing restrictions on the application of rules that, when violated, entail the provability of the sequent under consideration. For example, if a sequent violates the restrictions of the PSGL4ip calculus, then we know that either it is an instance of ($\bot$L) or (Id), which entails its provability. So, for every rule application of GL4ip we have the crucial case distinction, which we make use of in the admissibility of cut: either it is an instance of PSGL4ip, which makes mhd decrease, or its conclusion is obviously provable. A similar situation holds for GLS and PSGLS. Now, if we face a calculus containing unprovable cycles, such as the standard ones for modal logic K4**L** or S4**L**, then a terminating proof-search on this calculus needs to involve restrictions which, when violated, do not entail the provability of the sequent violating them. Then, the case distinction we mentioned above does not give us much when the sequent is violating the restrictions of the proof-search: its provability is not obvious. The applicability or adaptation of the mhd proof technique in this situation is still unclear and needs to be investigated.

Third, we intend to follow D'Abrera, Dawson and Goré [27] by testing and optimizing the Haskell programs we extracted from our formalisation. This way, we could experimentally determine the complexity of our cut-elimination algorithm, and try to obtain effectively usable and efficient programs performing cut-elimination.

Conclusion

In this dissertation, we contributed to the establishment of the formalised practice in logic in two different ways.

First, we provided reusable and adaptable formalisations in Coq of many of the tools a logician can use daily. Indeed, in Part I we presented general tools and results, which we instantiated (both in theory and in Coq) with propositional and first-order variants of the basic modal logic. We started in Chapter 2 by defining syntaxes for propositional and first-order logics, with the propositional modal and first-order modal language as examples. Then, we defined what logics are in Chapter 3. To capture logics we gave two main tools. First, we considered proof systems in Chapter 4. In particular, we considered two types of proof systems. In Section 4.1 we presented generalized Hilbert calculi, which we instantiated with the calculi wKH and sKH for the modal logics wKL and sKL. There, we showed that these calculi were closely related but different, as is well-known, notably because the deduction theorem holds in wKH but not in sKH. In Section 4.2 we presented sequent calculi, and gave the sequent calculus wKS for the logic wKL. We proceeded to obtain a cut-elimination result for wKS using the mhd proof technique. Second, to capture logics we also presented Kripke semantics in Chapter 5. More precisely, we gave Kripke semantics for propositional (Section 5.1) and first-order languages (Section 5.2). In particular, we gave a Kripke semantics for the modal propositional and modal first-order languages. Finally, we defined in Chapter 6 soundness and completeness. We thus proceeded to instantiate meta-theoretic proofs of these results between the local (resp. global) semantic consequence relation for the propositional modal language, and the generalized Hilbert calculus wKH (resp. sKH). These results notably involved the construction of a canonical model. It should appear clearly that most of these tools and results, which we formalised in Coq, are commonly used by logicians.

Second, we confirmed that the formalisation of results in logic does not solely consist in the formalisation of *known* results. Indeed, we proved and formalised results that are at the forefront of the research in bi-intuitionistic and provability logics.

In Part II we showed, both in the propositional and first-order case, that behind what has been so far considered as a unique bi-intuitionistic logic lie in fact two distinct bi-intuitionistic logics. This phenomenon is given an account of by using both generalized Hilbert calculi, and local and global semantic consequences of the Kripke semantics. For the propositional case in Chapter 8, we could show that the logic wBIL is sound and complete for the local semantic consequence relation, while the logic sBIL is sound and complete for the global semantic consequence relation. This clearly showed their differences. For the first-order case in Chapter 9, we could prove the difference between the logics FOwBIL and FOsBIL, using the soundness of the former (resp. latter) with the local (resp. global) semantic consequence relation. However, we did not manage to obtain a formalised completeness proofs for these logics, as we explained.

Finally, in Part III we obtained formalised cut-elimination results for GLS (Chapter 12) and $\mathsf{GL4ip}$ (Chapter 13), which are sequent calculi for, respectively, the classical provability logic GLL and the intuitionistic provability logic iGLL. These cut-elimination results involve a newly discovered proof technique for cut-admissibility called the “mhd proof technique”, which ultimately relies on the termination of backward proof-search. Thus, we notably explored the applicability of a new proof technique that reverses the usual order of things by first requiring a proof of the termination of the proof-search procedure, to then obtain a proof of cut-admissibility.

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