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Review Article

A Brief Review on Mathematical Tools Applicable to Quantum Computing for Modelling and Optimization Problems in Engineering

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Abstract

Since its emergence, quantum computing has enabled a wide spectrum of new possibilities and advantages, including its efficiency in accelerating computational processes exponentially. This has directed much research towards completely novel ways of solving a wide variety of engineering problems, especially through describing quantum versions of many mathematical tools such as Fourier and Laplace transforms, differential equations, systems of linear equations, and optimization techniques, among others. Exploration and development in this direction will revolutionize the world of engineering. In this manuscript, we review the state of the art of these emerging techniques from the perspective of quantum computer development and performance optimization, with a focus on the most common mathematical tools that support engineering applications. This review focuses on the application of these mathematical tools to quantum computer development and performance improvement/optimization. It also identifies the challenges and limitations related to the exploitation of quantum computing and outlines the main opportunities for future contributions. This review aims at offering a valuable reference for researchers in fields of engineering that are likely to turn to quantum computing for solutions.

Keywords:
Quantum Computing;
Fourier Transform; Laplace Transform;
Differential Equations;
Systems of Equations;
Optimization.

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1- Introduction

Since their inception, mathematics have been a central component of business, architecture, psychology, biology, chemistry, physics, engineering, and computing science, among others. Mathematics are an ever-growing set of tools for conceptualizing, modeling, and designing both concrete and abstract constructions and hypothetical situations, as well as calculating exact solutions to technical problems encountered in everyday life. Since the latter half of the 19th century, mathematical tools have propelled spectacular advances in the physical sciences, most notably in relativity and quantum mechanics. The latter is now being harnessed to drive the development of quantum computing, the design of quantum computers, and the optimization of their performance, as well as adapting engineering tools and concepts to the new quantum reality. Among these tools, the Fourier transform, the Laplace transform, differential equations, systems of equations, and optimization techniques have attracted the attention of a rapidly growing number of quantum computing researchers over the past few years (Figure 1), due to their historical importance in modeling and solving engineering problems.

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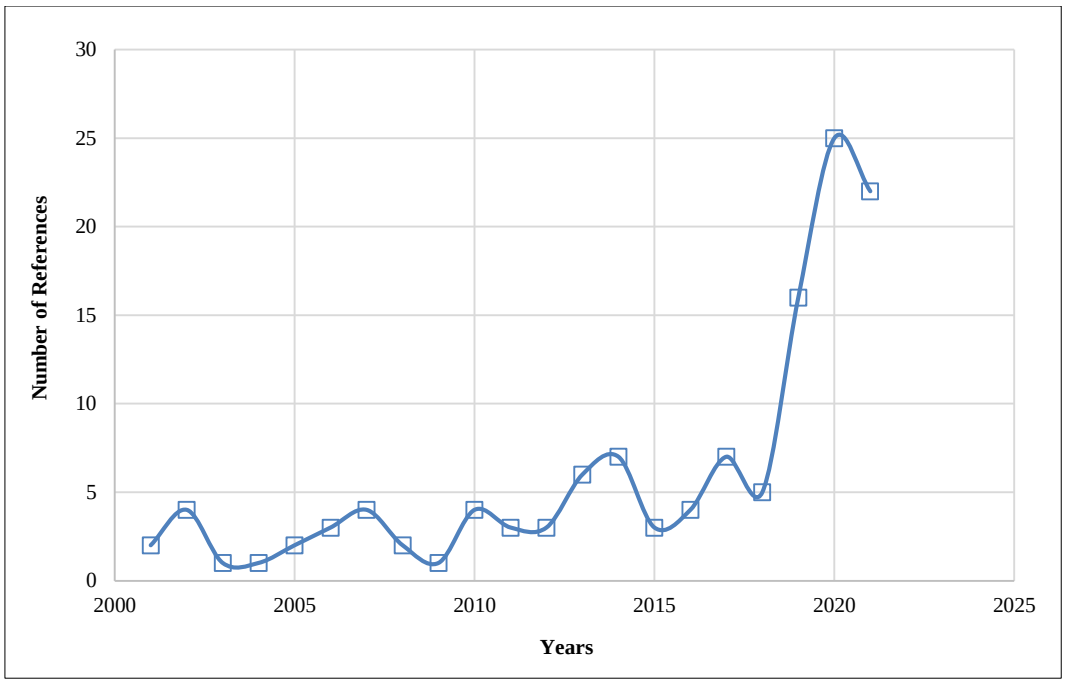

Figure 1. Trend of the number of research publications focused on the use of advanced mathematical tools in quantum computing

Though still in early development, quantum computing is gaining increasing interest among scientists due to its promise of higher speed and efficiency compared to conventional computing. A quantum computer, unlike classical ones, works with quantum bits (qubits), which are based on the principle of quantum superposition, in which 0 and 1 are superposed. This leads to an infinite range of possible states based on 0 and 1, with continuously changing weights [1, 2]. A qubit state can be manipulated with quantum operators (gates) such as X, Y, and Z gates, the Hadamard gate, and rotation gates (among others), which can be combined to build quantum circuits [3].

In anticipation of qubit-based computing becoming technically feasible and sufficiently reliable for practical use, much work has been devoted to reformulating classical computational algorithms and techniques into quantum versions using the principles of quantum mechanics. These contributions include the definition and development of quantum Fourier and Laplace transforms (QFT and q-Laplace) and quantum differential equations (QDE), in addition to novel methods for solving QDEs and systems of equations, and the efforts undertaken to develop quantum tools for solving engineering problems that generally have been approached using the abovementioned mathematical tools. The power of quantum computing would open the door to optimizations that are not practicable on conventional computers and significantly speed up most classical approaches. This would revolutionize the field of optimization by leading to the resolution of NP-complete problems.

All the attention given to quantum computing has led to the contributions mentioned in this review and certainly to other contributions in progress. The considered mathematical tools are unavoidable in science and engineering, and the use of their quantum versions with all their advantages will revolutionize several engineering fields. Despite all the work done in this context, only a few reviews have been published, considering only one direction at a time, and none of them has considered regrouping and discussed all the considered mathematical tools of engineering. Hence, the objective of our review is to offer to researchers interested in quantum computing a recent reference that gathers the work done in relation to these thematic tools so that they know what has already been done and use it towards new contributions.

The flowchart methodology of this research is presented in Figure 2. We summarize studies and other contributions to the adaptation of Fourier and Laplace transforms, differential equations, systems of equations, and optimization techniques to their qubit-supported counterparts. The focus on these specific tools is due to the increasing research interests illustrated in Figure 3 for both measurement, applications, and problem solving. This research also features several applications of the quantum version of these tools to real-world problems, including that of developing quantum computers.

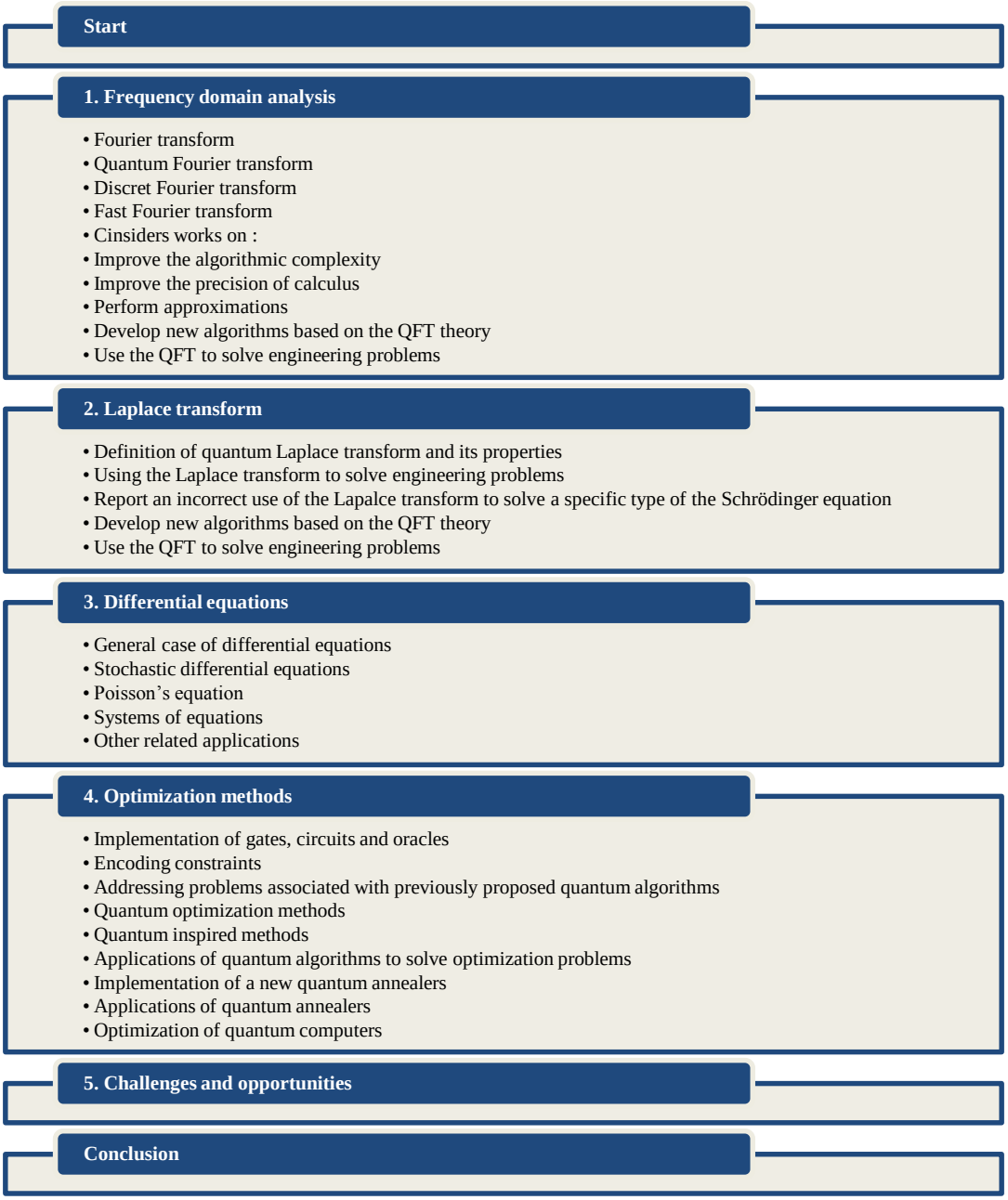

Figure 2. The flowchart of this research

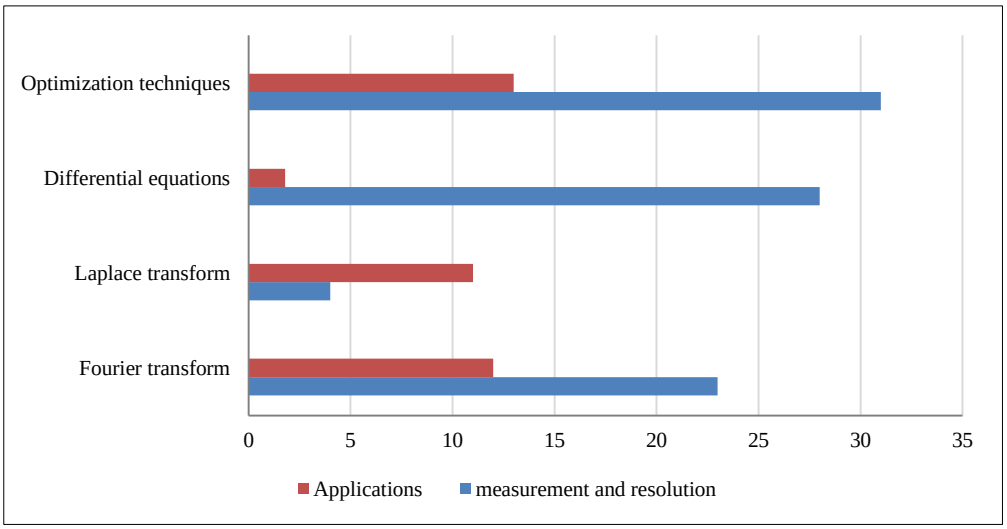

Figure 3. Contributions classified by topic and general research objective

2- Frequency Domain Analysis

2-1- Fourier Transform

The Fourier transform is a mathematical application introduced by Jean Baptiste Joseph Fourier in 1824 [4]. In his work entitled "*Analytical Theory of Heat*", this industrial revolution era physicist postulated that the solution of the equation representing the diffusion of heat can be written as a series comprising an infinite sum of weighted complex exponentials, now called the Fourier transform and written as an integral of weighted complex exponentials. The Fourier transform has since allowed the development of various techniques for the processing and analysis of signals, images, and sounds, in addition to being used very often to solve differential and partial differential equations. Table 1 presents a summary and classification of the contributions related to the Fourier transform in quantum computing to be reported in the current section according to the focus and main objective of the study.

Table 1. Classification of contributions towards the Fourier transform according to the main objective of the study

Mathematic tool	References	Objective
Fourier transform	[5-8, 10, 19, 20, 23-25, 28, 29]	Improving the algorithmic complexity of the QFT
	[11-14, 18, 19]	Improving the precision while performing the QFT
	[9, 17, 22]	Performing an approximation of the QFT
	[15, 16]	Developing new algorithms that use the QFT theory
	[9, 19, 26, 27, 29-36]	Using the QFT to solve engineering problems

2-2- Quantum Fourier Transform

In recent years, with the advent of quantum computing, the quantum Fourier transform (QFT) was defined and used to speed up the execution of arithmetic operations on a quantum computer and became the backbone of many quantum algorithms, such as order finding, integer factorization, solving systems of linear equations, and discrete logarithm computation, which are some classical examples of its growing use. These first inclusions of the QFT in quantum algorithms led to a succession of promising solutions to problems once considered unsolvable on conventional computers. Two main branches of research have emerged from these developments: (i) improving the algorithmic complexity of the QFT, and (ii) exploring and developing new applications and new algorithms using QFT theory. The main contributions to the latter branch are mentioned below chronologically.

Akis & Ferry (2001) proposed a quantum wave-guide array generator to speed up the execution of quantum algorithms by accelerating the Fourier transform process [5]. This generator is based on a parallel approach that favours the use of the analog properties of waves over the qubit concept [6] and on the QFT being formed by the Walsh-Hadamard transform H and phase shifts $P(0)$ which are sufficient to generate the unitary group $U(2)$ and from which its universality can be deduced. It was later shown by Tyson (2003) that the Operator-Schmidt decompositions of the QFT can be computed on C-N1 circle times C-N2 for all N-1, N-2 greater than or equal to 2 [7]. In all cases, the bipartite communication cost of exact computation of the QFT was maximal. Considering the QFT as a C-M1 circle times C-M2 => C-N1 circle times C-N2 operator with M1M2 = N1N2, a generalized Schmidt operator decomposition was described and a class of unitary operators on C-n circle times C-n was constructed, for which the operator-Schmidt decompositions are evaluated using the discrete Fourier transform [7].

A decomposition of the QFT as a product of non-selective Hadamard transformation monitored by multi qubit gates corresponding to square and higher roots of controlled-NOT gates was then proposed and a comparison of various spin-based quantum computing schemes to implement the QFT was provided with the actual time-costs and accuracy [8]. Then, Takahashi et al. (2007) described a method for constructing an efficient quantum circuit of length $O(n)$ and size $O(n \log n)$ was claimed to provide a good approximation of the QFT on a linear nearest neighbor architecture that decreases the size of the Fowler quantum circuit for the Shor factoring algorithm in the same situation [9].

An algorithm based on singular value decomposition, called “Quantum-SVD”, then followed to compute the QFT of non-uniformly distributed quantum states [10]. Quantum-SVD has the same complexity as the standard QFT but with precision orders of magnitude higher than obtainable by the interpolation approach. Not long after, the QFT was formulated by Cao et al. (2011) in a general qubit quantum system [11]. The higher dimensional QFT allows direct generalization of the quantum circuit and the phase estimation algorithm into a single d-dimensional quantum system. At nearly the same time, Lacroix & Semay (2011) proved that the Fourier transform of the eigen functions, computed in the configuration space, can be obtained easily with good accuracy in the physical domain of the moment space [12].

Van den Nest (2013) introduced a new class of quantum circuits, called “normalizer circuits”, based on polynomial time for conventional simulations of the QFT [13], with connections between the normalizers and Shor’s factoring algorithm and demonstration of the impossibility of achieving quantum factoring as a normalizer circuit. A few years later, Ruiz-Perez and Garcia-Escartin (2017) reviewed and commented on variations of the existing QTF adders and multipliers [14]. Then, they proposed circuit modifications that extend their capabilities by performing modular and non-modular arithmetic operations and working with signed integers [14].

An original approach to designing QFTs for solving complex problems based on a spin-torque architecture that uses a circuit based on optimal-depth Clifford+T gates was proposed in Kulkarni and Kaushik [15]. That same year, Grigoryan & Agaian (2019) presented a novel paired transform-based algorithm to compute the standard r-qubit discrete Fourier transform, using a method that calculates the QFT with the minimal number of stages implemented with only r Hadamard gates and structuring parallel computation in r steps [16]. Meanwhile, Nam et al. (2020) described an approach based on quantum circuits and reusing a special quantum state that induces the phase gradient transformation offering an approximation of the QFT with T gates numbering $O(n\log(n))$ [17].

Jaffea et al. [18] (2020) performed a quantum Fourier analysis and explored bounds on the QFT, which yield to the description of an uncertainty principle for relative entropy. Meanwhile, an efficient digital-analog quantum algorithm to compute the QFT was introduced by Martin et al. (2020) [19]. The proposed digital-analog quantum computing provides means of keeping the fidelity of QFT implementation under control, and thereby enhancing the applicability of other quantum algorithms such as Shor's algorithm or the HHL algorithm. More recently, Sakk (2021) presented a review that examines the structure and the implementation of the QFT's [20]. Consequently, Sakk (2021) proposed a novel universal permutation operator approach with efficient tensor products of quantum operators with the aim of reducing computational complexity [20].

2-3- Discrete Fourier Transform

In the context of a discrete and periodical signal, instead of using the continuous Fourier transform, we shall use the discrete Fourier transform (DFT) and thus the definition and implementation of the quantum version of the DFT is mandatory. Obada et al. (2013) built a new two-qubit gate by proposing a new physical algorithm that implements an N-bit discrete quantum Fourier transform by coupling superconducting qubits to a single-mode superconducting cavity [21]. Then, they built a physical protocol that couples superconducting transom qubits to a superconducting strip line resonator to implement new one-qubit and two-qubit N-bit discrete QFT. The proposed protocol has been validated experimentally in comparison with previous schemes [22].

In fixed Fibonacci (2+1) topological quantum field theory, approximation is thought to be essential in the implementation of the quantum Fourier transforms by braiding conformational blocks [23]. Roetteler & Beth (2008) proved that an approximation of the quantum Fourier transforms can be obtained by pruning the discrete Fourier transform (DFT) twiddle factors [24]. This transformation has been interpreted theoretically for all levels of pruning between none and maximal (leading to the Hadamard transform). Accardi & Boukas (2015) described a method of computing the Fourier transform of quantum random variables by splitting formulae for the multi-dimensional Heisenberg group [25]. And most recently, Rioux (2022) showed that the QFT is a unitary operator by providing a Mathcad implementation of the DFT and the QFT [26].

2-4- Fast Fourier Transform

The fast Fourier transform (FFT) is one of the most important and successful numerical algorithms devised in the 20th century for computing the DFT. It has found application in numerous areas of computer science and engineering, notably in digital signal processing.

Decades ago, Stroud (2010) described an implementation of the FFT algorithm in an entangled system of multilevel atoms using wave-packet control of ion internal states in the linear ion-trap scheme [27]. More recently, Camps et al. (2020) showed that the QFT can be computed by decomposition of the diagonal factors of the FFT matrix into matrices having a Kronecker product structure [28]. This structure allows the QFT algorithm to reach exponential speeds on a quantum computer compared to the FFT algorithm on a conventional computer.

Asaka et al. (2020) [29] presented the quantum version of the FFT (QFFT) as a transformation of a tensor product of quantum states. It can be implemented as a quantum circuit by combining several fundamental arithmetic operators without generating any garbage bits. One of its major advantages, besides being very fast, is that it is applicable to all problems that can be solved by the conventional FFT. In addition, the quantum FFT is more efficient than the fast QFT in terms of data storage efficiency [29].

2-5- Applications

The applications of the Quantum Fourier transform, and its variants, are very diverse. Heszler (2006) constructed and studied a gedanken model of an analog optical computer for general purpose Fourier transform with considering cases of entangled or non-entangled photons appearing at the input [30]. Barak and Ben-Aryeh (2007) used the 1, 2, and 3-qubit discrete Fourier transforms for computing the phase estimation and order finessing of optical systems based on the Cooley-Tukey algorithm [31]. The applications of the Fourier transform have also been seen in computer simulations of Fourier complex associations, associative memories, optical processors and linkage of results to holographic properties of bacteriorhodopsin-based thin films [32]. As well as in the application of QFT to quantum visual information processing to develop a comprehensive QFT framework based on a quantum visual representation model [32].

To make QFT-based algorithms work optimally, Hen (2014) introduced a set of subroutines used them in the construction of an adiabatic algorithm that uses the QFT [33]. The adiabatic evolutions proposed have the distinction of not sacrificing any of the initial computational complexity of the algorithm. Further, Greene and Batista (2017) introduced a powerful computational approach called tensor-train split-operator Fourier transform (TT-SOFT) for simulations of quantum dynamics of polyatomic systems [34]. This method avoids the problem of exponential scaling of representations based on grids of integral ranks.

On the other hand, Further, Grigoryan & Agaian (2020) described a new approach to image and discrete signal processing by quantum computing [35]. This approach uses the Fourier transform qubit representation after mapping data into the unit circle. Finally, eight quantum-inspired numerical analysis algorithms based on heuristics have been presented, combining classical ideas-finite-differences, spectral methods with efficient encoding of quantum registers, and well-known algorithms such as the QFT to provide exponential increases in speed over other classical algorithms such as finite-difference, FFT and Monte Carlo integration [36].

3- Laplace Transform

The Laplace transform is a simple yet very powerful mathematical tool that facilitates solving complex problems. It is widely used in control engineering, power systems engineering, electrical circuit analysis, mathematics, optics, signal processing, and physics, especially quantum mechanics and quantum systems. Its use is encountered frequently in the development of quantum computing and quantum computers. In Table 2, we summarize and classify the publications related to the Laplace transform in quantum computing that will be reported in the current section in accordance with the orientation and the main objective of the work.

Table 2. Classification of contributions towards the Laplace transform according to the main objective of the study

Mathematical tool	References	Objective
Laplace transform	[40, 45- 47]	Definition of quantum Laplace transform and its properties
	[38-48]	Using the Laplace transform to solve engineering problems
	48	Report an incorrect use of the Laplace transform to solve a specific type of the Schrödinger equation

Englefield (1968) used the Laplace transform method to examine the problem of the one-dimensional quantum harmonic oscillator [37]. Further, it has been shown that the wave function which defines a quantum mechanical system can be represented by the Laplace transform of a certain distribution and the result of a certain subclass [38]. This theory was tested by Hong-Yi et al. (2004) on a hydrogen-like model atom [39]. A decade later, Tsaur & Wang (2014) used the completeness relation constituted by the coherent state and the proper kernel of the bosonic creation operator to establish a one-to-one correspondence between the z-transform and the quantum-mechanical transform from representation by number states $|n\rangle$ to Bargmann representation [40]. This approach produced a quantum-mechanical version of the different properties of the z-transform, and thus operator descriptions allowing the incorporation of these properties into Fock space.

In addition, the Laplace transform can be used to solve differential equations with variable coefficients, which is very useful in various branches of physics and quantum mechanics. For example, Das and Arda (2015) proposed a universal Laplace transform scheme to solve the Schrödinger wave equation [41]. The proposed scheme has a closed form that makes it suitable for all known solvable models.

The Laplace transform is particularly efficient for solving linear ordinary differential equations. For example, Al-Omari (2017) addressed the N-dimensional second order Schrödinger equation with pseudo-harmonic potential. The proposed approach uses the Laplace transform to reduce the considered equation to a first order differential equation and then uses the convolution theorem to determine the exact solutions of the boundary state [42]. The generalized Morse potential also was studied briefly using this approach. In addition to some verified special cases, variations of energy eigenvalues E_n as a function of the dimension N are studied. Later, the authors described q-analogues of the natural transform along with their applications to three families of q-Bessel functions [43]. Their validity is shown also in case of q-Sumudu and q-Laplace transforms. Next, Al-Omari et al. (2018) discussed q-analogs of Laplace-type integrals for various types of q-special functions, notably: Fox's Hq-functions, the first, second and third kinds of q-Bessel function, and the q-Struve functions [44], with results for hyperbolic sine (cosine) functions. A general quantum Laplace transform $\mathcal{L}_\beta$ of the general quantum difference operator defined by $D_\beta f(t) = \frac{\left(f(\beta(t))-f(t)\right)}{\beta(t)-t}$ followed, where β is a strictly increasing continuous function [45]. Its properties are discussed and the β-Laplace transform of some fundamental functions is computed, which allows solving of the β-difference equations, and the inverse β-Laplace transform $\mathcal{L}_\beta^{-1}$ was also presented. Then, using q-integral definition on quantum analogs, Alp & Sarikaya (2023) described the $\bar{q}$-Laplace transform and some of its properties along with formulas of q-Laplace transform and applications [46].

Sumudu and Laplace transforms of the first and second kind have been defined in quantum calculus for functions of several variables, and some interesting relationships and identities for these transforms are described [47]. Correlations among Aleph functions and the above-mentioned integral transforms in quantum calculus have been derived. Finally, de Castro (2020) reported and addressed an incorrect use of Laplace transforms of the time-independent Schrödinger equation as well as its inverse [48]. The author established a clear and detailed list of errors that can occur when an inappropriate method is used for a specific eigenvalue problem [48].

4- Differential Equations

Linear differential equations (LDEs) are ubiquitous in science and engineering. They are used to model many important problems including dynamic system behavior and time functions as well as challenges raised by quantum systems and the development of quantum computers. The task of solving LDEs is overwhelming for conventional computers. But, once again, quantum computing brings its advantages to overcome the shortcomings of classical methods. In the section below, we summarize both aspects of this endeavor: developing quantum algorithms for solving differential equations and implementing differential equations for the development of quantum computers. The various contributions to the resolution of differential equations are categorized in Table 3 based on the relevant type of equation.

Table 3. Classification of contributions towards solving differential equations according to the considered type in the study

Mathematical tool	References	Objective
Differential equations	[49-59, 68, 69, 76]	General case of differential equations
	[60-64]	Stochastic differential equations
	[65-67]	Poisson's equation
	[70-75]	Systems of equations

4-1- Linear and Nonlinear Differential Equations

In the context, Iyer & McCune (2002) computed and studied a polynomial ring of univariate quantum differential operators [49] within conservation laws computed symbolically [50]. In a later work, Berry (2014) described a direct algorithmic method of computing conservation laws for polynomial systems of nonlinear partial differential equations (PDEs) in multiple dimensions. The author ensured the existence of a sequence of conserved densities by determining the necessary conditions of the method's parameters. This approach works on a panoply of PDEs, using linear combinations of scaling homogeneous terms with undetermined coefficients to construct the densities, the Euler operator to determine the coefficients and the homotopic operator to compute the fluxes. Since many classical physical systems are described adequately by sparse inhomogeneous linear differential equations, the use of high-order methods to improve the efficiency of quantum simulation algorithms was extended to the type of differential equations mentioned above. Berry (2014) provides a gate-based quantum algorithm to efficiently solve the LDE problem under certain conditions [51], thus providing a time vector $x(t)$ satisfying the following constraint:

$$dx(t)Idt = Mx(t) + b \tag{1}$$

Where, M is a square matrix of size N, b is a vector of size N, and $x(0)$ is the initial vector.

This algorithm has proven to be very efficient based on extensive experimentations and it offers an exponential acceleration of the execution as compared to its conventional counterparts.

A few years later, Berry et al. (2017) presented a quantum algorithm for systems of linear ordinary differential equations with constant coefficients. The proposed algorithm generates a quantum state that is proportional to the solution within a required final time, and thus offers an exponential improvement over previous quantum algorithms in terms of complexity. It is important to mention that the algorithm is polynomial in logarithms of the inverse error, and unlike finite difference methods, it does not require any additional assumptions to guarantee numerical stability [52].

Not long after, the extended simple equation method and the novel $(\frac{G'}{G})$-expansion method were studied as solutions to nonlinear partial differential equations [53]. Both techniques depend on the auxiliary equations, defined by Equations 2 and 3:

$$\frac{d\phi}{d\xi} = \alpha + \lambda\phi + \mu\,\phi^2 \tag{2}$$

$$\left(\frac{G'}{G}\right)' = \alpha + \lambda\left(\frac{G'}{G}\right) + (v-1)\left(\frac{G'}{G}\right)^2 \tag{3}$$

which are special cases of the Riccati equation. Applied to the two-dimensional nonlinear Kadomtsev-Petviashvili (KP) Burger's equation in quantum plasma, these methods give exact traveling wave solutions, and are therefore considered powerful, effective, and straightforward means of solving nonlinear partial differential equations.

Srivastava & Sundararaghavan (2019) used a quantum annealer to solve second-order differential equations by recasting a finite element model in the form of an Ising Hamiltonian [54]. A graph-coloring methodology called the "box algorithm" is then used to search iteratively for solutions in a subspace of weak solutions defined graphically. Its usefulness is demonstrated by solving a truss mechanics problem on a D-Wave quantum computer.

A quantum algorithm based on spectral methods for linear ordinary differential equations has been developed by Childs and Liu (2020) to provide a global approximation alternative to finite difference methods [55]. This allows solving of time-dependent initial and boundary value problems with poly log d, log(1/ε) complexity. Meanwhile, Xin et al. (2020) designed an algorithm composed exclusively of a universal set of quantum gates and its corresponding quantum circuit to solve LDEs. Its efficiency was validated by simulations on a 4x4 linear differential equation using a 4-qubit nuclear magnetic resonance quantum information processor [56].

More recently, Kyriienko et al. (2021) proposed a quantum algorithm for implementing a spectral method of solving differential equations in a high-dimensional feature space [57]. This approach uses feature map encoding to define functions in terms of expected values of parametrized circuits and a hybrid quantum-classical workflow to satisfy differential equations and specify boundary conditions. Function derivatives are represented automatically in analytical form as differentiable quantum circuits. This avoids inaccurate finite difference procedures for the computation of gradients.

Two recent approaches to using digital quantum computers to solve differential equations are basis encoding and fixed-point arithmetic, which represent and solve high-order Runge-Kutta methods as optimization problems on quantum annealers [58]. Circuits simulated as two-dimensional linear ordinary differential equations and tested in conjunction with a sixth-order Gauss-Legendre collocation method on a D-Wave 2000Q system showed a good agreement with the reference solution.

A hybrid algorithm based on the advanced cuckoo search algorithm and adaptive Gaussian quantum particle swarm optimization (AGQPSO) was proposed by Kumar et al. (2021) to solve first and second order differential equations with initial and/or boundary conditions by transforming these equations into unconstrained/bound constrained optimization problems [59]. The algorithm efficiently solves several bound constrained optimization problems of different dimensions considered as benchmarks.

4-2- Stochastic Differential Equations

Another mathematical tool of great interest is the stochastic differential equation (SDE). SDEs allow modeling of uncertain phenomena and are used in various fields of natural and social sciences. Since SDEs rarely lend themselves to analytical solution and are usually solved numerically using huge conventional computing resources, they provide strong motivation to harness the extra speed offered by quantum computing.

More than 20 years ago, several one-step calculus schemes were proposed and tested as weak solutions of Lipschitzian quantum stochastic differential equations (QSDEs) for operator-valued processes associated with the creation, annihilation, and gauge operators of quantum field theory [60]. Hudson-Parthasarathy formulation was used on the assumption that the matrix elements of the solution are sufficiently differentiable. Based on the weak convergence criteria for Ito's stochastic differential equation, it was shown that the Euler-Maruyama scheme is a special case of Euler schemes. To provide an alternative to convolution cocycles on coalgebra, the existence and uniqueness of theorems for QSDEs with non-trivial initial conditions were proven for the case of coefficients with completely bounded columns [61]. Necessary and sufficient conditions were also defined for a conjugate pair of stochastic quantum cocycles in a finite dimensional operator space to satisfy such QSDEs.

Several years later, the performance of the IBMqx4 Tenerife quantum computer as a solver of QSDE models of the interaction of a laser-driven two-level atom with an electromagnetic field in the vacuum state was investigated with comparison of the master equation and quantum filtering equations to existing theory [62]. More recently, Kubo et al. (2021) used the variational quantum simulation to propose a quantum-classical hybrid algorithm to solve SDEs [63]. The steps of the proposed algorithm consists in first approximating the target SDE using a trinomial tree structure with discretization, then formulating it as the time function of a quantum state integrating the probability distributions of the variables, and finally (unlike in previous studies in which the square root of the probability distribution amplitude is taken), integrating the probability distribution directly into the quantum state amplitudes, which allows the building of simple quantum circuits that simulate the time course of the general SDE state. This study also proposes a quantum acceleration scheme for evaluating the expected values of SDE variables and thus provides a new direction for SDE-based simulations on quantum computers. Meanwhile, An et al. (2021) resolved stochastic differential equations via quantum algorithms that achieve quadratic acceleration using multilevel Monte Carlo methods in a general framework [64]. One such algorithm was validated in computational tests and can be used in a variety of applications arising in the mathematics of finance. A similar degree of improvement was achieved in the case of another quantum algorithm, based on sub-linear binomial sampling for the binomial option pricing model.

4-3- The Poisson Equation

The Poisson equation appears in many areas of science and engineering such as analysis of electrical potential and fluid mechanics. Successful development of a quantum solver for this type of differential equation should open avenues to new approaches to solving structural systems on quantum computers, such as those involving Toeplitz matrices. A successful and efficient description of quantum programs for Poisson matrix simulation, approximation of trigonometric functions and calculation of eigenvalue reciprocals, could lead to the resolution of many problematic partial differential equations.

Cao et al. (2013) described one efficient quantum algorithm and scalable quantum circuit for solving the Poisson equation [65]. This algorithm generates a quantum state that encodes the numerical solution with nearly linear cost as the problem dimension increases, in contrast to conventional algorithms, of which the cost scale increases exponentially or faster. Arrazola et al. (2019) designed a quantum algorithm that prepares states to encode solutions of linear inhomogeneous partial differential equations [66]. The inputs of the proposed algorithm are polynomial differential operators and their partial derivatives, and the output is a quantum state of which the wave function is proportional to a specific solution. This approach avoids the use of expensive switch approximations by deriving exact decompositions for the gates employed in the Hamiltonian simulation, which reduces the number of gates by several orders of magnitude. In addition, machine learning methods are used to find explicit circuits that prepare the required resource states. Two applications provide validation of this approach: solving the Poisson equation in electrostatics and one-dimensional integration.

The Harrow-Hassidim-Lloyd (HHL) quantum algorithm, which can solve linear system problems, was used by Wang et al. (2020) to describe a quantum Poisson equation solver as well as a complete and modular circuit design [67]. This is an important step towards practical applications of the current circuits as fast Poisson solvers in hybrid classical/quantum devices.

4-4- Other Related Applications

4-4-1-Emulating Linear Differential Operators

Szkopek et al. (2005) used the abrams-Lloyd algorithm to investigate how a quantum processor, if ever built, could be used to emulate linear differential operators [68]. That would allow an exponential reduction in memory used for gate operations. The ground state energy of two or more particles, for example in the Schrödinger equation, could be estimated using fewer quantum mechanical gates than classical gates. In the meantime, the falling body in a resisting medium problem has been revisited with q-calculus for the first time by describing the q-differential equations modeling the vertical velocity and distance fallen [69]. Exact expressions for velocity and distance were thus established, leading to a reduction in gate operation compared to the corresponding classical Newtonian mechanics solutions when the quantum parameter q tends to one.

4-4-2- Solving Systems of Equations

Solving systems of linear equations is a common task in almost all fields of science and engineering, an indispensable tool in applied mathematics and usable as a basis for solving more complex problems such as partial differential equations, optimization problems, and eigen problems, among others. In real-life situations involving large data sets, conventional computers are often overwhelmed by such tasks, since the algorithms used are iterative and require processing times proportional to the number of variables and the number of constraints. To overcome this problem, much research has focused on exploiting the advantages and the execution speed offered by quantum computing.

Harrow et al. (2009) showed that a quantum computer, short of calculating a complete solution $\vec{x}$ of size N, can approximate the value of a function by computing some function of the solution in a time that scales logarithmically in N, and polynomially in the number of conditions κ and the desired accuracy [70]. For the Harrow-Hassidim-Lloyd (HHL) quantum algorithm, the execution time of $\vec{x}^t M\vec{x}$ is polynomial in $log(N)$ and κ. The HHL algorithm solves linear system problems with exponential speed compared to conventional methods and has been harnessed in many important quantum computing algorithms. Various systems of linear equations of size 2×2 have since been solved experimentally using the HHL algorithm, notably with the quantum circuit optimized and compiled into a linear optical lattice. With four photonic qubits and four controlled logic gates, solution accuracy ranged from 0.825 to 0.993 [71].

Lee et al. (2019) described a hybrid quantum algorithm for solving systems of linear equations [72]. The proposed algorithm is an improved version of the HHL algorithm that reduces the circuit depth by passing conventional information after the quantum phase estimation algorithm while getting identical results. It can be even more accurate than the HHL algorithm on specific systems of linear equations. The HHL algorithm has been used also to define a quantum Poisson equation solver as well as a complete and modular circuit design [67], which is an important advancement towards practical applications of current circuits as fast Poisson solvers in hybrid classical/quantum devices.

4-4-3- Solving Quadratic of Equations

Another challenging computation problem is the solving of systems of m quadratic equations in n variables over a finite field [73]. This has been addressed by designing and analyzing a quantum algorithm based on the concept of extended linearization (XL) combined with brute force search and/or Grover's algorithm [74].

4-4-4- Solving Systems of Polynomial Equations

Cheng et al. (2019) reported the attempts to solve general systems of polynomial equations by methods based on quantum annealing [75]. This approach has been validated using a system of second-order polynomial equations solved on a commercially available quantum annealing computer. Its application to linear regression has been demonstrated, and an iterative annealing process has been defined and proven to be efficient at solving a linear system with a tolerance of 10^{-8}.

4-4-5- Solving Systems of Linear Equations

A solver that uses quantum mechanics for systems of linear equations also has been implemented [76]. The basic idea of this solver is to introduce an equivalent optimization problem of which the objective function defines an electrostatic potential. This solver was validated convincingly using three simple but different linear systems with increasing complexity.

4-4-6- Solving Regression Equations

Linear regression is one of the most common and essential methods of analysis in mathematical statistics. Recently, Li et al. (2021) proposed a new simplified quantum scheme for solving the linear regression equation based on the sparsity-independent quantum singular value estimation algorithm [77]. This scheme uses the general model of the linear regression problem with the least-squares method that reduces the time complexity from $o(Nn^2)$ to $o\left(\sqrt{N}\log(n)\right)$.

5- Optimization Methods

Mathematical optimization is a set of techniques and methods used to justify decisions in many fields, including physics, biology, engineering, economics, and business. Each situation is modelled as a mathematical model or program task consisting of maximizing or minimizing an objective function under a set of constraints to be satisfied, expressed in terms of decision variables. The constructed model is then solved using an exact or approximate approach, depending on the size and numerical complexity of the model and method applied. The characteristics of the computer used for the execution of these algorithms play a crucial role in the execution speed and efficiency of these methods. The faster the computer is, the less complex and more solvable the optimization problem becomes. The additional speed of quantum computers makes them attractive for implementing quantum versions of the state-of-the-art optimization algorithms. Resolution of NP-complete optimization problems would be a paradigm-shifting breakthrough. Table 4 summarizes and classifies the contributions towards solving optimization problems that will be presented in this section.

Table 4. Classification of contributions towards solving optimization problems according to the main objective of the study

Mathematical tool	References	Objective
Optimization techniques	[102, 114, 117]	Implementation of gates, circuits and oracles
	[91, 120, 121]	Encoding constraints
	[99, 100, 111, 115, 116]	addressing problems associated with previously proposed quantum algorithms
	[77, 78, 80-82,84, 87-90,94, 95, 97, 98, 104, 106-108, 112, 113]	Quantum optimization methods
	[86, 91, 96, 103, 110]	Quantum inspired methods
	[83-85,89, 90, 93, 95, 98, 109, 112, 116, 117, 122]	Applications of quantum algorithms to solve optimization problems
	[121]	Implementation of a new quantum annealers
	[118, 119, 123, 124]	Applications of quantum annealers
	[125-138]	Optimization of quantum computers

5-1- Quantum Optimization

One of the first quantum versions of a classical optimization algorithm is the quantum-inspired evolutionary algorithm proposed by Han and Kim (2002) two decades ago [78]. In this algorithm, one Q-bit is used for probabilistic representation while another operates as a string of Q-bits. A Q-gate also was introduced as the operator of variation to guide individuals towards better solutions and thus avoid premature convergence. Compared to the conventional genetic algorithm, the Q version performed adequately without precipitous convergence.

Jiao et al. (2008) described a quantum-inspired immune clonal algorithm (QICA) [79] to address the global optimization problem defined by,

$$\text{minimize } f(x), x = (x_1, \dots, x_n) \in S \cap F \tag{4}$$

where, $f(x)$ is the objective function to be minimized, $S = \prod_{i=1}^{n}[\underline{x_i}, \overline{x_i}\,] \subseteq \mathbb{R}^n$ is the search space in which, $\underline{x_i}$ and $\overline{x_i}$ are lower and upper bounds, respectively, of the variable x_i, $i = \overline{1, n,}$ and, F is the feasible region including all $\overline{x} \in \mathbb{R}^n$ that satisfy each one of the m ($m \in \mathbb{N}$) constraints $g_j(x) \leq 0, j = \overline{1, m}$.

In the proposed approach QICA antibodies are propagated, organized into subpopulation groups, and then represented by multi-state gene quantum bits [79]. The general quantum rotation gate strategy and dynamic adjustment angle mechanism are then employed to accelerate the convergence, and the NOT quantum gate is applied to ensure quantum mutation and thereby prevent premature convergences. The QICA converges in theory on the global optimum, and experimental results show that it performs much better than other state-of-the-art genetic algorithms in terms of solution quality and computational cost.

A successful stochastic quantum computing method named BBW is introduced by Baritompa et al. (2005) to solve unconstrained global optimization problems using Grover's algorithm [80]. This later was revisited by Liu and Koehler (2010) in search of a global optimal solution to discrete optimization problems [81]. The original BBW algorithm was redesigned to achieve a significant increase in speed and thereby extend static BBW scheduling (from 33 to 43 points) and hence its applicability. The method was made dynamic at the same time. The power of these modifications has been demonstrated in experiments.

Zheng and Yamashiro (2010) inspected the integration of quantum operators and evolutionary algorithms, including the use of the basic quantum-inspired evolutionary algorithm to then establish a *quantum differential evolutionary algorithm* for the permutation flow shop scheduling problem [82]. This new algorithm was shown to be an efficient solver of the problem in comparison with benchmark algorithms. The applicability of the *quantum-behaved particle swarm optimization algorithm* to the economic load dispatch problem has been validated in two simulations and found effective at solving the economic load balancing problem [83], giving solutions superior to those of the improved particle swarm and other optimization algorithms.

Zhisheng (2010) investigated the efficiency of the quantum-behaved particle swarm optimization (QPSO) algorithm (a fusion of quantum computing theory and the particle swarm optimization algorithm) in the economic load distribution of a power system [83]. In this case, a particle is represented in terms of several determined probability states and the particle update operation is performed by the quantum rotation gates. The algorithm was found effective for solving the economic distribution problem, outperforming the improved particle swarm and other optimization algorithms.

Since the optimization of reactor core fuel assembly reloading is an issue of great interest in the nuclear power industry, an evolutionary-inspired quantum algorithm combining basic concepts of *population-based incremental learning* with quantum bits and linear superposition of states, named QPBIL, has been proposed to meet this challenge [84]. The effectiveness of the QPBIL has been proven for the optimization of Angra 1 cycle 7, outperforming currently available techniques based on artificial intelligence.

Karimi et al. (2012) computed the median adiabatic times from the minimal deviation during quantum optimization of a class of NP-hard Ising spin glass instances with up to 128 binary variables [85]. In this case, if computational time were defined on adiabatic time scales (seconds), quantum optimization would be considerably more efficient than classical solvers. Later, Layeb (2013) successfully applied a hybrid quantum-inspired harmony search algorithm to quantum inspired operators such as measurement and interference [86]. Shang et al. (2014) proposed a *quantum immune clonal coevolution algorithm,* based on an artificial immune system, quantum evolutionary computation and coevolution strategy (QICCA) to solve dynamic multi-objective optimization problems such as nominality of uniformity, local optimality and non-convergence [87], which frequently overwhelm more classical algorithms. The QICCA improves the search capability by employing integral cloning and invokes quantum theory to design a quantum update. In addition, the co-evolution strategy is integrated into the holistic operation, and the co-evolutionary competitive operation and the co-evolutionary cooperative operation are designed to improve the uniformity, diversity, and convergence performance of the solutions. The algorithm is efficient, with a strong ability to evolve convergent, diverse, and uniformly distributed Pareto fronts compared to ICADMO and DBM.

A quantum algorithm for combinatorial optimization that produces approximate solutions is described by Farhi et al. (2014) [88]. This algorithm is based on a quantum circuit of unitary gates of which the locality never exceeds that of the objective function for which an optimum is sought. This approach was illustrated for the MaxCut problem for graphs with bounded degree. This work was continued by applying the QAOA algorithm to the combinatorial problem of bounded occurrence Max E3LIN2 [89]. Solutions were thus obtained more efficiently than from two well-known conventional algorithms. A QAOA variant that considers an instance of the maximum 2-satisfiability problem (MAX-2-SAT) generates a state with high overlap with the optimal state [90]. This was validated in tests on a set of hard instances with comparison to CFLLS optimized annealing times.

Zouache et al. (2016) attempts to solve discrete optimization problems by adapting the firefly approach led to a new algorithm called *quantum-inspired firefly algorithm with particle swarm optimization* [91]. The two basic concepts of quantum computing used in this algorithm to improve the control of solutions diversity are superposition states of Q-bit and quantum measurement. Fireflies are represented discretely, and the attraction between them is computed by a variant of the well-known Hamming distance. Finally, two strategies are combined to provide cooperation while exploring the search space. The efficiency of the proposed algorithm was tested on several instances of the multidimensional knapsack problem modeled by,

$$\begin{cases} maximize\ \sum_{i=1}^{n} p_i x_i \qquad subject\ to, \\ \sum_{i=1}^{n} a_{ji} x_i \le c_j, x_i \in \{0,1\}, i = \overline{1,n}, j = \overline{1,m} \end{cases} \tag{5}$$

where, $p_i$and a_{ji} are fixed value parameters satisfying $p_i \geq 0$ and $0 \leq a_{ji} \leq b_j, b_j \geq 0$ for all $i = \overline{1,n}, j = \overline{1,m}$.

The experimental results validated the efficiency of the QIFAPSO and also demonstrated that it was competitive and generally better than existing methods.

In another direction, Hen & Spedalieri (2016) proposed a novel method for encoding constrained optimization problems on quantum recyclers with the aim of eliminating the need for penalty terms and thus reducing the number of couplers required as well as removing the need for minor embedding [92]. The authors claim that their approach reduces the number of physical qubits required to implement optimization procedures on quantum recyclers and has proven its usefulness in quantum computing.

Ranjbar et al. (2016) reformulated the computation of generalized Ramsey numbers as a combinatorial optimization problem and then provided a quantum algorithm based on the adiabatic algorithm to address the problem [93]. This method allows the determination of described Ramsey numbers (I-m, I-n) for trees of order m, n = 6, 7, 8, most of which were previously unknown. Palittapongarnpim et al. (2017) used the ddifferential evolution algorithms to circumvent the stagnation problem of nonconvex optimization [94]. The mean of the objective function is considered to improve the fidelity of quantum control for a noisy system, and heuristics for early termination of executions and for adaptive selection of search subspaces are used. This implementation is massively parallel and vectorized to further reduce execution time. The proposed algorithm has been shown to achieve higher fidelity and scalability than obtained with greedy algorithms.

In Chiang (2017) [95], the power economic dispatch problem of generators with multiple fuel options was using a method based on quantum-behaved particle swarm optimization with multiple updating, described as highly effective. This involves conducting a refined search and avoiding the deformations associated with the augmented Lagrange function (thanks to multiple updating) and thus ensuring convergence towards the optimal solution. Further, particle motion is modelled in a novel approach to quantum behavior, based on the Schrödinger equation and the Monte Carlo method and consisting of three basic mathematical steps [96]. The nonlinear problem was thus solved more effectively compared to other approaches. Sayed et al. (2019) combined the quantum representation of the search space, interference and quantum operators, with the multiverse optimization algorithm to provide a hybrid algorithm called quantum multiverse optimization to find the optimal trade-off between natural resource exploration and extraction [97]. In tests on 50 unimodal and multimodal benchmark functions, the proposed algorithm was found superior to well-known algorithms at solving complex numerical optimization problems.

One of the important applications of quantum computing is hybrid variational classical quantum algorithms for optimizations. Quantum circuits have been designed to implement two such algorithms, one of which is variational, both being proposed for training neural networks and claimed to achieve exponential increases in the number of samples processable and polynomial increases in the sample size compared to conventional training algorithms [98]. These algorithms also return the training weight information so that the outputs can be used directly to solve other problems.

In the application of quantum computing to energy system optimization problems, the use of open-source software tools as a first step has been described along with limitations of state-of-the-art quantum computers in this field [99].

Metallurgy raises a major challenge for quantum computing, and a data-driven optimization approach based on game strategies has been proposed to help identify states of matter with the desired properties [100]. To solve the problem of spatial dimension truncation in parameterized optical circuits, an algorithm for recursive computing of the exact matrix elements of Gaussian operators and their gradient with respect to parameterization is claimed to be several times faster than the state of the art [101]. This direct computing is achieved without the need to combine several other operators, and the gradient can be implemented in any variant of the gradient descent algorithm. Quantum applications in optical hardware research, machine learning, optical data processing, device discovery and design are supported.

One of the very important contributions is that of van Apeldoorn et al. (2020), in which the steps that must be taken to accelerate the solution of convex optimization problems have been determined for implementing a separation oracle

using $\tilde{\mathcal{O}}(1)$ quantum queries to a membership oracle, thus providing an exponential increase in speed over conventional Omega(n) membership querying techniques [102]. $\tilde{\mathcal{O}}(n)$ quantum queries to a membership oracle suffice to implement an optimization oracle. Multiple lower bounds are determined: if an interior point of the convex set is known, the algorithm requires $\Omega(\sqrt{n})$ membership queries, otherwise $\Omega(\sqrt{n})$ separation queries.

A new quantum-inspired evolutionary algorithm proposed for the robot path planning problem is reportedly suitable for solving large scale optimization problems for both complex static and dynamic environments and outperforms conventional genetic algorithms to a considerable degree [103].

Alexandru et al. (2020) provides new quantum schemes that speed up several general-purpose numerical optimization methods for minimizing a function $f : \mathbb{R}^n \to \mathbb{R}$ [104]. These provide nearly quadratic acceleration of many techniques for global optimization under a Lipschitz constraint, including the backtracking line search, a non-negligible element in quasi-Newton optimization algorithms.

Also, to avoid getting stuck in a local optimum in case studies using the original version of the quantum evolutionary algorithm, combining two or more methods can lead to a new approach with augmented features [105]. A quantum version of *enhanced colliding body optimization* uses quantum particle behavior laws in the delta potential well and the Schrödinger uncertainty principle along with Newtonian collision laws and an advanced solution space search approach to balance natural resource exploration and extraction. Tilting the agents toward the solution found best at each iteration improves the chance of finding an optimal solution, whereas the efficient randomization induced by the uncertainty laws allows the algorithm to leave the local optimum intact.

Another algorithm in which a memetic evolution operator based on the shuffling process and a quantum evolution operator that uses an adaptive selection mechanism for different potential sinks designed to complete the balance between global search and local search has been proposed to solve global optimization problems [106]. Based on Newton's law of gravity, the gravitational center and the geometric center are incorporated as essential elements to improve the population diversity. Tested on several optimization problems, the proposed method is shown to outperform the 11 most advanced algorithms.

A recent work describes an extended version of the quantum existence test, called the *constrained quantum relation test* (CQRT) [107]. The CQRT can be used to extend the quantum extreme value search algorithm (QEVSA) to a constrained quantum optimization algorithm, which greatly reduces complexity, achieves maximal exponential speed, and high classical and quantum certainty.

Khan et al. (2021) formulated a quantum version of the meta-heuristic optimization algorithm known as the beetle antenna search and applied it to financial portfolio optimization using real stock market data [108]. The QBAS is claimed to outperform algorithms such as particle swarm optimization and the genetic algorithm. Recently, Warren (2021) presented seven arguments demonstrating that symmetric *traveling salesman problems* constitute a benchmark for quantum computation of combinatorial optimization problems for all types of quantum hardware [109]. Wang & Wang (2021) proposed a quantum-inspired differential evolution algorithm with gray wolf optimizer to improve the diversity and convergence performance in high-dimensional cases for 0-1 knapsack problems, based on quantum gates and superposition states, among other principles [110]. Adaptive differential evolution mutation operations, differential evolution crossover operations, and quantum observation are used to produce new solutions as test individuals. Selection operations are employed to determine the best solution among the stored individuals and the test individuals created by the mutation and crossover operations. In the case where the test individuals are worse than the current individuals, the adaptive grey wolf optimizer and the quantum rotation gate are used to conserve the diversity of the population along with speeding up the search for the global optimal solution. Experimental results for 0-1 knapsack problems confirm the superior efficiency and global search capability of this algorithm, especially for high-dimensional situations.

To solve the problems associated with the differential evolution algorithm, such as low solution efficiency, insufficient diversity in the subsequent search phase, slow convergence speed, and the possibility of search stagnation, Deng et al. (2021) proposed an improved form combining the computational features of the quantum evolution algorithm and the divide-and-conquer aspect of the cooperative coevolution evolution algorithm [111]. Quantum coding of chromosomes is used to enhance population diversity, along with quantum rotation to boost convergence speed. The CC framework is used to split high-dimensional, large-scale complex optimization problems into low-dimensional optimization subproblems. The resulting model is found to have better convergence accuracy and stability, specifically a high capacity to optimize high-dimensional complex functions.

The quantum approximate optimization algorithm introduced by Farhi et al. [88, 89] has been used and improved in several works, for example, to allow alternation between more general families of operators and thereby support the representation of a larger set of states [112]. This could have a long-term impact on a broad array of application areas and would be particularly useful for approximate and exact optimization problems with hard constraints. In addition to introducing the quantum alternating operator ansatz, design criteria for mixing operators are established, mappings for eight problems are detailed, and a compendium with brief descriptions of mappings for a diverse array of problems is

provided. In Pagano et al. [113], a low-depth quantum approximate optimization algorithm was implemented using an analog quantum simulator to estimate the ground-state energy of the transverse field Ising model with long-range interactions and tunable range, and to optimize the corresponding combinatorial classical problem by sampling the output with high-fidelity, single-shot, individual qubit measurements. Performance withstands scale-up adequately, and the runtime is nearly independent of the number of qubits.

A new field of quantum circuit architecture design for approximate optimization algorithms has been opened in which the Gibbs objective function is defined and has been shown to be superior to the energy expectation value for use as an objective function in variational parameter tuning [114]. An ansatz architecture search algorithm was then described to investigate the discrete space of similar quantum circuit architectures. With the resulting better ansatz, applying these modifications to a full-graph Ising model gave a median relative improvement of 244.7% in the probability of finding a low-energy state while using 33.3% fewer two-qubit gates. In the case of Ising models on a 2D grid, the probability was improved by 44.4% (median) with a 20.8% reduction in the number of two-qubit gates.

Algorithm selection for optimal performance has long been studied in practical computer science, and has been addressed in the quantum context, notably problem instance detection, for which approximate optimization has been found superior to the Goemans and Williamson algorithm on the Max-Cut problem [115], with cross-validated accuracy well above 96%. Although machine learning was used in the simulations, the authors believe that their methods will be applicable to a broad range of classical heuristics and to real (noisy) situations.

Egger et al. (2021) discussed how to initialize quantum optimization with a state that corresponds to the solution of a relaxation of a combinatorial optimization problem. They also analyzed. The required characteristics of the algorithm used for this purpose [116]. The quantum algorithm ideally inherits the performance guarantees of the classical algorithm. This is illustrated in the context of portfolio optimization, for which results reveal that hot starting the approximate optimization is beneficial at shallow depth. Similarly, the recursive QAO algorithm for Max-Cut problems exhibits a systematic enhancement of the cut size obtained for fully connected graphs with random weights when Goemans-Williamson random rounding is used in a hot start. The authors claim that this technique can be applied easily to other random rounding schemes and optimization problems.

Finally, application of Google's Sycamore superconducting qubit quantum processor to combinatorial approximate optimization has been demonstrated recently [117]. Its performance is tested on the hardware's native planar connectivity graph, with the Sherrington-Kirkpatrick model and Max-Cut, non-native problems of which the implementation requires extensive compilation. In the latter case, the approximation ratio obtained is independent of the problem size, and performance improves with circuit depth. For problems requiring compilation, performance decreases with the problem size. This suggests that for problems on non-native graphs, quantum processors should be evaluated using approximate optimization algorithms.

5-2- Quantum Annealing

The recent advances in quantum technology have been followed by the development and construction of experimental programmable quantum annealers that show great potential for solving unconstrained binary quadratic optimization problems and consequently many combinatorial optimization problems of practical interest much faster than their classical analogs. A considerable advantage of quantum annealing is the possibility of solving optimization problems formulated with integer variables, such as the travelling salesman problem and the satisfiability problem.

Among the first applications of quantum annealing was a method of determining the extrema of multidimensional functions [118]. Based on an extension of classical and simulated annealing, this approach allowed avoidance of local minima. Unlike some of its predecessors, it did not require approximation of a wave function. It was applied successfully to the problem of finding the lowest energy configurations of Lennard-Jones clusters of up to 19 particles (about 105 local minima). This encouraged researchers to consider this method as a complement to the widely used simulated annealing technique and to apply and develop it in other contexts. Decades later, programming a quantum annealer was described in an overview of quantum annealing along with exploration of three potential application areas [119].

Vyskocil & Djidjev (2019) proposed a new constraint implementation approach based on combinatorial design and mixed integer linear programming (MILP) for obtaining better embedding of constraints of the type $\sum xi = k$ for binary variables x_i [120]. This approach is scalable for any number of variables and uses a linear number of auxiliary variables for a fixed k. In May 2018, Fujitsu launched its Digital Annealer service inspired by quantum computing. A strategy of using this service to solve real-world combinatorial optimization problems for clients, which involves formulating the problem for unconstrained binary quadratic optimization, was later explained [121].

More recently, an innovative approach was proposed by Ohzeki (2020), to solve a large-scale optimization problem on the chimera graph using a method well-known in statistical mechanics, namely the Hubbard-Stratonovich transformation [122]. This method supports a fully connected Ising model without embedding on the chimera graph and provides non-trivial results. Its efficiency was shown in experiments on several partition problems involving the

resolution of linear equations and on a realistic traffic flow optimization problem in the Japanese cities of Sendai and Kyoto. Abel et al. (2021) presented a detailed comparison of quantum annealing to classical optimization techniques, namely thermal annealing, Nelder-Mead, and gradient descent has been published [123]. This study shows that the thermal annealing method is slightly more efficient at discovering the global minimum, while the Nelder-Mead and gradient descent methods are very likely to be trapped in local minima. However, despite its current size limitations, quantum annealing is clearly the best method of minimizing potentials quickly and efficiently.

The D-Wave 2000Q quantum annealing processor has been used to inspect the diagonal thermal properties of the canonical one-dimensional transverse-field Ising model [124]. The authors of this study found that the quantum processor could not establish the expectation values predicted correctly by the quantum Monte Carlo method, making it an unsuitable candidate for thermal sampling.

5-3- Optimization of Quantum Computers

With the expectation that the computing power of qubit-based systems will eventually surpass that of conventional supercomputers, the development of machines that can harness this capacity to solve real-world problems is forging ahead, and ever more research is being devoted to optimizing quantum computer design and performance to maximize efficiency.

To address the problem of numerical design from a practical perspective, universal criteria are needed to evaluate what constitutes optimal design, and whether and to what extent quantum technologies can improve on conventional methods in terms of computational time. Actual performance of conventional and quantum designs has been compared experimentally in the case of two modern problems, namely the training of support vector machines and the construction of vector quantizers. The empirical evidence matches the theoretical expectations and supports the consistency of the design method [125]. In the application of genetic algorithms to solve quantum computing problems including the Bell-CHSH inequality and entanglement measures, the C++ bit set class can be utilized to represent any type of data, providing flexibility to solve optimizations [126].

In protected quantum computing, McDonald & Katzgraber (2013) proposed an optimization model that uses a generic approach based on evolutionary algorithms intended for quantum gate computing [124]. The proposed model determines efficiently and quickly the optimal braids for quantum gate computation, even when error-only optimization is performed, while allowing the user to optimize for the relative utilities of precision and/or length [127]. Sheng-Hao et al. (2014) discussed the optimization of the projected entangled pair state in a 2-dimensional quantum system and the connection between updating the associated tensor and computing the physical state observable for the tensor network algorithm, in terms of improvement of computational resources utilization [128]. Then, they described an optimized algorithm that constitutes a powerful tool for studying quantum phase transitions and critical phenomena within the thermodynamic limits of the highly correlated two-dimensional electronic quantum lattice system.

Quantum volume as a metric to compare the power of near-term quantum devices, variational algorithms, and mapping from fermions to qubits has been explained in Moll et al. (2018) [129]. Coupled-cluster and heuristic trial wave-functions can be efficient for determining molecular ground states, whereas hyperparameters in stochastic synthesis are major factors in the determination of the optimal program. Bayesian optimization in BayeSyn is therefore used to adjust these parameters and generate an optimized quantum program. Jayashree et al. (2018) used a reversible binary coded decimal adder with its unified reversible addition/subtraction circuit design that reduces garbage outputs to zero in an n-digit reversible addition/subtraction circuit to take further advantage of the superior performance parameter optimization thus made possible [130]. Later, Shaydulin et al. (2019) demonstrated the efficiency of the hybrid decomposition-based approaches for solving optimization problems for applications related to community detection [131].

In recent years, simulation-driven automated design optimization processes have become widespread in computer-aided product development and have therefore been the focus of quantum conceptual framework intended to strengthen them and reduce their computational complexity [132]. In this case, it is assumed that quantum computers will be part of hybrid high-performance computing platforms to be used as application-specific accelerator devices.

In a review now two years old, Li et al. (2020) summarized the latest breakthroughs in quantum optimization and/or learning algorithm concepts and recent progress [133]. Experiments demonstrating the strong competitiveness of quantum smart algorithms with conventional smart algorithms and their great potential for simulating quantum computing are also included. That same year, a new method completely different from Shor's algorithm was presented by Wang et al. (2020) [134]. The proposed method has the potential of a D-Wave quantum computer for RSA decryption and effective factorization of all integers up to 10,000 with the major advantage of a significant reduction of the local field coefficient h and the coupling term J, respectively by at least 33% and 26%, thus further improving the stability of the qubit chains and increasing the upper bound on integer factorization. A better RSA decryption index was obtained for factorizing the 20-bit integer 1028171, for which Shor's algorithm requires about 40 qubits, far beyond the capacity of general-purpose quantum computers.

Furthermore, Liu et al. (2020) designed a hybrid model based on echo state network architecture and a quantum neural network [135]. The proposed model uses the particle swarm optimization for parameter optimization to maintain its stability and outperform benchmark algorithms in time series prediction tasks, thus enhancing the performance of the echo state network. Liao et al. (2021) designed a mechanism for harnessing quantum optimization algorithms to identify the optimal parameters of quantum neural networks for specific tasks with coherent encoding of the QNN cost function on the relative phases of a superposition state in the network parameter Hilbert space [136]. An iterative quantum optimization framework using adaptively selected Hamiltonians is applied to tune the parameters. This technique is expected to provide greater speedup than Grover's algorithm and thus alleviate the sterile plateau problem.

In Medvidović & Carleo [137], a method has been described for simulating layered quantum circuits made of parameterized gates, a key architecture for many variational quantum algorithms suitable for short-range quantum computers. Neural network parameterization of the wave function of multiple qubits is used, with focus on states relevant to the quantum approximate optimization algorithm. Experiments performed on large circuits have revealed a requirement for 54 qubits at four layers, implementing about 324 RZZ gates and 216 RX gates without consuming large-scale computing resources. This approach provides accurate simulations on large systems at previously unexplored parameter values and allows planning of the next generation of experiments in anticipation of the noisy intermediate-scale quantum era.

Finally, Xiao et al. (2021) used an improved stochastic program synthesis and Bayesian optimization in BayeSyn. This allows the automatic generation of quantum programs from high-level languages under a set of constraints [138]. Stochastic synthesis can efficiently generate a competitive program at reduced cost from high-dimensional program spaces.

6- Challenges and Opportunities

All the attention given to quantum computing has led to all the contributions mentioned in this review and certainly to other contributions in progress. The considered mathematical tools are crucial in the engineering field and the use of their quantum versions with all their advantages will revolutionize the science of engineering.

Despite all the work done in this context, only a few reviews have been published considering only one direction at a time and none of them have considered regrouping all the considered basic mathematical tools of engineering. Hence the objective of our review: offer researchers interested in quantum computing a recent reference that gathers the work done in relation to these thematic tools and thus know what has already been done and use it to lead to new contributions.

Figure 4 summarizes the distribution of the most discussed research topics in the previous two decades in the consulted literature.

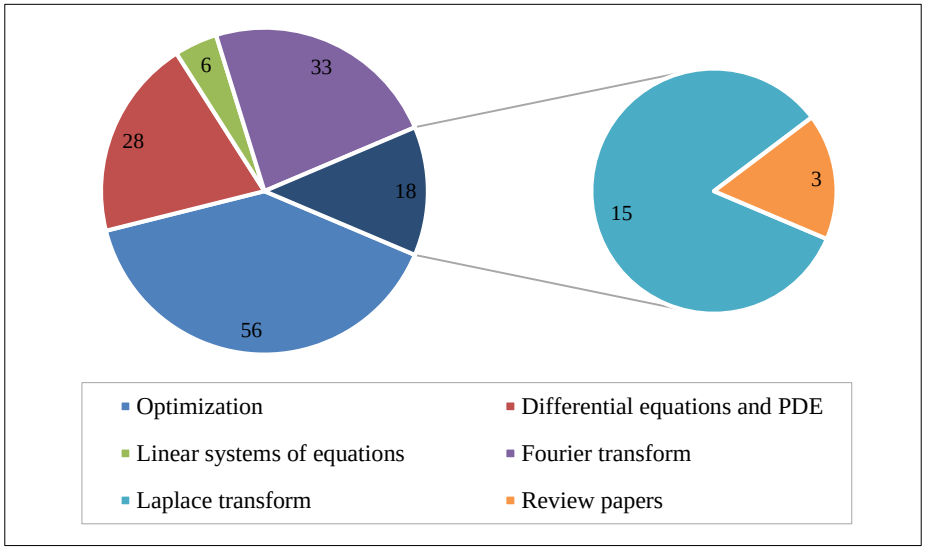

Figure 4. Distribution of research topics found in the consulted literature

It is quite clear that most of the works were dedicated to the design of quantum optimization methods. These methods have been designed and successfully applied to solve some specific problems. However, these methods have not been generalized or adapted to solve other problems.

Other researchers have focused on improving and optimizing the performance of quantum computers using classical optimization methods, which is a very important research direction to consider since there is a direct relationship between the quality of the performance of quantum computers and that of quantum algorithms. The next most recurrent research

direction is the Fourier transform (FT) which aimed mainly at the definition, factorization, implementation, and application of the quantum version of the FT. The most recent of these contributions led to the implementation of very fast quantum algorithms for the computation of the QFT and the DFT and even used them as a basis for other quantum algorithms with real applications.

In the context of solving differential equations, we found several research works that have allowed the development of quantum approaches to solve efficiently 1st and 2nd order linear equations under certain conditions even in a high-dimensional feature space, stochastic equations, and Poisson equations. However, no such method has been designed for nonlinear differential equations. The use and application of the latter methods as well as those ensuring the efficient solution of the systems of equations has not yet been explored, which will be of great interest in various fields. Little work has been dedicated to the quantum version of the Laplace transform. In addition, most of the works consulted have been aimed at using the Laplace transform and/or one of its variants to tackle a specific task. Since the Laplace transform is a very powerful tool that has already been used to facilitate the resolution of complex problems, it will be very interesting to devote further attention to the development of its quantum version.

The challenges encountered and to be taken into consideration, and the main opportunities for contributions are listed in the following.

6-1- Challenges

Quantum algorithms and mathematical tools are being developed on the assumption that quantum computers will become available for research as well as industrial use. However, daunting challenges remain, including accuracy and fault tolerance, which could take years of work to ensure the reliability of qubits [139]. A major obstacle is the high sensitivity of physical quantum states to noise, which inevitably causes execution errors by any algorithm. Some noise sources are inherent in the current materials platforms [140].

Progress is rapid in this field, and it is difficult to predict the point at which users will decide that the advantages of quantum computers are worth the imposition of having to accept and adapt to ever-changing ways of using them. In addition, due to the highly advanced nature of quantum processing of information, these computers might raise serious security challenges for current cryptographic algorithms. Much effort needs to be devoted to developing means of ensuring a safe transition from conventional computers to their quantum successors.

6-2- Opportunities

With the effervescence of the scientific research in this field, the need for quantum development tools has increased, and a few cloud quantum systems offering access to qubit-based services for testing theoretical developments have become available, which is fueling an increase in the number of subject areas in the literature [141]. Developing and adapting hardware to meet the needs of quantum research and development appears to represent a huge opportunity for ambitious engineers interested in entering this field.

Benchmarking quantum systems will also become necessary when quantum computers reach a level of functionality sufficient to tempt investors in commercialization. Standards will be needed for evaluating and rating the performance of the developed concepts and their impact in the marketplace. The promotion of quantum computers currently rests entirely on claims of tremendous increases in speed and efficiency for certain types of problem solving. However, since real-world problems may or may not be solved advantageously on a quantum computer, it will be important to examine the possibility of adapting conventional algorithms to the new hardware technology, which may be a challenge and an opportunity at the same time.

Meanwhile, as research continues to advance, improving on what has been achieved so far will provide much opportunity, for example, optimizing quantum algorithms by reducing computational resources given the limited number of qubits available for validation [2, 3, 142]. On the other hand, the door remains wide open for novel contributions. In fact, most of the newest mathematical tools for engineering appear to be focused on optimization methods, the Fourier transform, and differential or partial differential equations, as illustrated in Figure 2. Other engineering-related tools might require adaptation and further development.

7- Conclusion

Quantum computing is getting closer to reality as more and more mathematical tools used in various fields of engineering find their quantum equivalents, including Laplace and Fourier transforms, differential and partial differential equations, systems of linear equations, and optimization techniques. In this literature review, for the first time, more than two decades of application of these mathematical tools are reviewed, with emphasis on work devoted to their implementation for the development of quantum computers, performance improvement, and optimization algorithms. The contributions explored in this study have highlighted and pointed the importance and power of quantum computing to solve very complicated real-world problems with high efficiency and speed. It also allowed us to detect the challenges

and limitations encountered with quantum computers, such as their availability for research and industrial use, accuracy, fault tolerance, and reliability of qubits. Furthermore, it opened up major development directions and opportunities in the field. We believe that this synthesis will be a valuable baseline for researchers in the engineering fields who are ready to rely on quantum computing for solutions.

8- Declarations

8-1- Author Contributions

Conceptualization: N.Z.; methodology: N.Z. and Y.M., software: Y.M. and N.Z.; validation: N.Z. and M.T.; formal analysis: N.Z., Y.M. and M.T.; investigation: Y.M., resources: N.Z.; data curation: N.Z., M.T. and A.R.; writing—original draft preparation: Y.M. and N.Z.; writing—review and editing: N.Z., H.B., M.T. and A.R.; visualization: N.Z.; supervision: N.Z. and H.B.; project administration: N.Z. All the authors read and agreed to the published version of the manuscript.

8-2- Data Availability Statement

Data sharing is not applicable to this article.

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8-5- Institutional Review Board Statement

Not applicable.

8-6- Informed Consent Statement

Not applicable.

8-7- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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