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The Gribov problem in noncommutative QED

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Abstract: It is shown that in the noncommutative version of QED (NCQED) Gribov copies induced by the noncommutativity of space-time appear in the Landau gauge. This is a genuine effect of noncommutative geometry which disappears when the noncommutative parameter vanishes.

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1 Introduction

Space-time noncommutativity emerges at Plank scale in gedanken experiments when one tries to conjugate Quantum Mechanics and General Relativity [1–3]. The result is the failure of the classical description of space-time as a pseudo-Riemannian manifold and the appearance of uncertainty principles which are compatible with non-commuting coordinates. In different contexts, both string theory [4] and loop quantum gravity [5, 6], predict the appearance of space-time noncommutativity at a fundamental level. It is therefore natural to investigate how gauge theories have to be modified in order to accomplish with the quantum structure of space-time in extreme energy regimes.

In gauge theory one of the fundamental problems to be solved, in order to be able to perform perturbative computations of physical quantities, is the overcounting of the degrees of freedom related to gauge invariance (see, for instance, the detailed analysis in [8]). The Faddeev-Popov gauge fixing procedure allows for perturbative computations around the trivial vacuum $A_\mu = 0$. On the other hand, the existence of a proper gauge transformation preserving the gauge-fixing, would spoil the whole quantization procedure. In [9] Gribov showed that in non-Abelian gauge theories (on flat topologically trivial space-times) a *proper gauge fixing* is not possible. Moreover, Singer [10] showed that if Gribov ambiguities occur in the Coulomb gauge, they occur in all the gauge fixing conditions[1] involving

[1] Other gauge fixings are possible such as the axial gauge, the temporal gauge, etc., nevertheless these choices have their own problems (see, for instance, [8]).

derivatives of the gauge field. The problem is generally addressed in the Landau gauge and in this paper we will stick to the latter, although the Gribov-Zwanziger modification of the gauge action to cure the problem of Gribov copies has been recently extended from the Landau gauge to general R_ξ gauges [11].

In the path integral formalism, a Gribov copy close to the identity of the gauge group corresponds to a smooth zero mode of the Faddeev-Popov (**FP**) operator. In order to define the path integral in the presence of Gribov copies close to the identity, the most successful method is to restrict the path-integral to the neighborhood of $A_\mu = 0$ in the functional space of transverse gauge potentials, where the FP operator is positive (see, in particular, [9, 12–18, 20]).When the space-time metric is flat, this approach coincides with the usual perturbation theory and, at the same time, it takes into account the infrared effects related to the (partial) elimination of the Gribov copies [12–15, 21, 22]. If one computes the propagator corresponding to such a restriction, one finds the famous Gribov form factor for the propagator

$$G^{\mathrm{G-Z}}(p) \sim \frac{p^2}{p^4 + \gamma^4}, \tag{1.1}$$

where the dimensional constant γ is related with the size of the Gribov horizon. Although at high momenta such a propagator recovers the usual one $\sim 1/p^2$, the infrared behavior is drastically different. When one takes into account the presence of suitable condensates [23–25] the agreement with lattice data is excellent [26, 27]. This approach allowed to solve (see [28]) the well known sign problem of the Casimir energy and force in the MIT-bag model. Thus, in a sense, the Gribov problem is not just a problem since, as the whole (refined) Gribov-Zwanziger approach shows, it also suggests in a natural way a solution which allows to go far beyond perturbation theory[2] in a very successful way.

On the other hand, on a space-time with curved metric and/or non-trivial topology the situation can be much more complicated since also Abelian gauge theories can have smooth zero modes of the Faddeev-Popov operator [29], the maximally (super)symmetric vacuum can be *outside* the Gribov region [30] and the modular region could shrink to zero [31, 32]. For these reasons, it is natural to wonder whether the presence of noncommutativity can induce Gribov copies even in U(1) gauge theories, which, because of noncommutativity of the product, develop self-interaction terms, thus behaving as non-Abelian gauge theories, with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star. \tag{1.2}$$

Noncommutativity thus modifies the covariant derivative (see eq. (2.7) below) in the same way as non-Abelian gauge symmetry and it is then natural to expect a non-trivial equation for copies. This issue is extremely important since, as we already noticed, noncommutative geometry at a fundamental level has been shown to manifest in many different approaches to quantum gravity. Consequently, it is mandatory to investigate whether noncommutative geometry induces novel features in QFT which prevent from using the standard perturbative techniques.

[2]In particular, both in the case of the glueballs mass spectrum and in the case of (the solution of) the sign problem in the Casimir energy and force in the MIT bag model, perturbation theory is obviously not sufficient to get the correct answers as non-perturbative physics is needed.

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The paper is organized as follows. In section 2 we establish the equation for infinitesimal Gribov copies for noncommutative QED. In section 3 we investigate the existence of exact solutions for particularly simple gauge potentials and we actually show that we may have an infinite number of genuine noncommutative solutions. In section 4 we discuss our results and draw some conclusions.

2 Equation for Gribov copies

Let us fix the notations: for each two functions $f(x)$ and $g(x)$ the noncommutative Moyal star product $f \star g$ is defined as follows

$$(f \star g)(x) = f(x) \exp\left\{\frac{i}{2}\,\theta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\right\} g(x), \tag{2.1}$$

where the indices[3] $\rho, \sigma = 1, \ldots, d$ and d is the dimension of the space-time, which we assume Euclidean. The antisymmetric matrix θ, has the following nonzero components

$$\theta_{1,2} = -\theta_{2,1} = \theta_1,\ \theta_{3,4} = -\theta_{4,3} = \theta_2,\ \ldots\ ,\theta_{d-1,d} = -\theta_{d,d-1} = \theta_{d/2}, \tag{2.2}$$

where θ_i are real deformation parameters, in principle all different from each other, characterizing noncommutativity. A rescaling could be performed in order to make all parameters equal but we keep them different because it wouldn't simplify the calculations in the multidimensional case analyzed in section 3.3. When $\theta_i \to 0$, the star product $\star$ goes to the standard commutative point-wise product of f and g.

2.1 Gauge transformation

Under the U(1) gauge transformation in NCQED the gauge field A transforms as follows

$$A \to A'_\mu[\alpha] = U \star A_\mu \star U^\dagger + i\, U \star \partial_\mu U^\dagger, \qquad U \equiv \exp_\star(i\alpha)\,, \tag{2.3}$$

where the star exponent of an arbitrary function f is by definition

$$\exp_\star(f) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star \ldots \star f}_{n \text{ times}}, \tag{2.4}$$

and α is some function of x considered as a parameter of the transformation. It is worth noting that in the commutative limit $\theta \to 0$, the gauge transformation eq. (2.3) reduces to the standard Abelian gauge transformation

$$A \to A'_\mu[\alpha] = A_\mu + \partial_\mu \alpha + \mathcal{O}(\theta). \tag{2.5}$$

For us of crucial importance will be the infinitesimal form of the gauge transformation eq. (2.3):

$$A \to A'_\mu[\alpha] = A_\mu + D_\mu \alpha + \mathcal{O}(\alpha), \tag{2.6}$$

where the covariant derivative D_μ appears due to *non commutativity* and is given by

$$D_\mu f = \partial_\mu f + i\,(f \star A_\mu - A_\mu \star f) \tag{2.7}$$

for an arbitrary function $f(x)$.

[3] Although it is not necessary, let us assume, that our space time is even dimensional, since we are interested in $d = 2$ and $d = 4$.

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2.2 Zero mode equation in general

Let us fix the gauge to be the Landau gauge,

$$\partial^\mu A_\mu = 0. \tag{2.8}$$

In commutative QED this gauge fixing condition fixes the gauge completely, indeed, under suitable regularity conditions at the boundary, if A_μ satisfies eq. (2.8), the transformed field under eq. (2.5) automatically does not

$$\partial^\mu A'_\mu[\alpha] \neq 0, \quad \alpha \neq 0. \tag{2.9}$$

In other words the equation

$$\partial^\mu A'_\mu[\alpha] = 0 \tag{2.10}$$

called also the "zero mode equation" has only the trivial solution $\alpha = 0$. However, in a more general setting, for example non-Abelian gauge theories, the zero mode equation eq. (2.10) may have nontrivial solutions $\alpha \neq 0$ called *zero modes*.

The lack or the presence of zero modes means correspondingly the possibility or impossibility to foliate the functional space of all possible gauge fields into the orbits of the gauge group in a way, that each gauge orbit intersects the gauge fixing hypersurface eq. (2.8) just once (in order to be able to integrate over the representatives of each equivalence class). The latter situation leads to an overcounting of degrees of freedom when one performs the functional integral over A, creating the Gribov problem in the infrared regime.

The main goal of the present research is to figure out whether the zero mode equation eq. (2.10) can exhibit nontrivial solutions in the case of NCQED, i.e. when $A'[\alpha]$ is given by eq. (2.3).

2.3 Infinitesimal Gribov copies

The *infinitesimal zero mode equation* which corresponds to the infinitesimal gauge transformations eq. (2.6) is of special interest since it has direct relation with the Faddeev-Popov ghost action and with the Gribov-Zwanziger term.

Substituting eq. (2.6) in the general formula eq. (2.10) we obtain

$$\partial^\mu D_\mu \alpha = 0. \tag{2.11}$$

Let us understand the structure of this equation from the mathematical point of view. Substituting the expression of the covariant derivative eq. (2.7) and the star product eq. (2.1) into eq. (2.11) we arrive at the following zero mode equation written in terms of α and its derivatives

$$-\partial^2\alpha + \underbrace{iA_\mu \exp\left\{\frac{i}{2}\theta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\right\}(\partial^\mu\alpha) - i(\partial^\mu\alpha)\exp\left\{\frac{i}{2}\theta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\right\}A_\mu}_{\text{nonlocal terms}} = 0. \tag{2.12}$$

The presence of nonlocal terms implies that, differently form QCD, this is not a differential equation and its resolution is a very hard task. However, in order to say whether we have Gribov copies or not we only need to understand whether it has nontrivial solutions $\alpha \neq 0$.

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On performing the Fourier transform of α and A one can rewrite the pseudo-differential equation eq. (2.12) as a homogenous Fredholm equation of the second kind. After some simple computations we obtain indeed

$$
\begin{aligned}
0 &= -\partial^2\alpha + iA_\mu \exp\left\{\frac{i}{2}\,\theta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\right\}(\partial^\mu\alpha) - i(\partial^\mu\alpha)\exp\left\{\frac{i}{2}\,\theta^{\rho\sigma}\overleftarrow{\partial}_\rho\overrightarrow{\partial}_\sigma\right\}A_\mu \\
&= \int d^dk e^{ikx}\left\{-k^2\hat{\alpha}(k) + 2i\int d^dq \sin\left(-\frac{1}{2}\theta^{\rho\sigma}q_\rho k_\sigma\right)k^\mu \hat{A}_\mu(q)\hat{\alpha}(k-q)\right\} \qquad (2.13)
\end{aligned}
$$

which is equivalent to the following integral equation

$$
k^2\hat{\alpha}(k) + 2i\int d^dq \sin\left(\frac{1}{2}\theta^{\rho\sigma}q_\rho k_\sigma\right)k^\mu\hat{A}_\mu(q)\hat{\alpha}(k-q) = 0. \qquad (2.14)
$$

Changing the integration variable $q \to k-q$ we finally arrive at

$$
\hat{\alpha}(k) = \int d^dq\ Q(q,k)\ \hat{\alpha}(q), \qquad (2.15)
$$

which is a homogeneous Fredholm equation of the second kind, with the kernel Q given by[4]

$$
Q(q,k) = -\frac{2i\,k^\mu\hat{A}_\mu(k-q)}{k^2}\sin\left(\frac{1}{2}\,\theta^{\rho\sigma}q_\rho k_\sigma\right). \qquad (2.16)
$$

It is possible to recast the integral equation eq. (2.15) in such a way that the corresponding integral operator becomes manifestly symmetric. To this purpose we notice that, due to the gauge fixing condition, $\hat{A}_\mu(k)k^\mu = 0$, one can replace k_μ by $(k_\mu + q_\mu)/2$ in eq. (2.16). Upon making the change of variable $\beta = |k|\cdot\alpha$, with $|k| \equiv \sqrt{k_\mu k^\mu}$ and multiplying both sides of eq. (2.15) by k, we arrive at the following equivalent integral equation:

$$
\hat{\beta}(k) = \int d^dq\ P(q,k)\ \hat{\beta}(q), \qquad (2.17)
$$

$$
P(q,k) = -\frac{i\ (k^\mu + q^\mu)\,\hat{A}_\mu(k-q)}{|k||q|}\sin\left(\frac{1}{2}\,\theta^{\rho\sigma}q_\rho k_\sigma\right). \qquad (2.18)
$$

In order to show that the linear integral operator P defined by the kernel eq. (2.18) is formally self-adjoint it is necessary and sufficient to show that the corresponding kernel satisfies

$$
P^*(q,k) = P(k,q), \qquad (2.19)
$$

where "*" means complex conjugation. Now we recall that we are interested in *real* gauge potentials $A(x)$, which is equivalent to impose

$$
\hat{A}^*(k) = \hat{A}(-k) \qquad (2.20)
$$

[4]Let us notice that, in the Coulomb gauge $\partial_j A^j = 0, j = 1,\ldots,d-1$ the equation for the copies is formally the same as eq. (2.14), provided we replace k^2 with $\vec{k}^2$ and $k^\mu\hat{A}_\mu(q)$ with $k^j\hat{A}_j(q)$. Obviously the solutions will be different. In particular none of the Ansätze we shall make in the rest of the paper can be easily adapted to the Coulomb gauge. We shall comment more on this aspect in the discussion session.

on the corresponding Fourier transform. Performing complex conjugation of eq. (2.18) and using eq. (2.20) and skew symmetry of θ we immediately obtain

$$\left(\frac{i\,(k^\mu+q^\mu)\,\hat{A}_\mu(k-q)}{|k||q|}\sin\left(\frac{\theta^{\rho\sigma}}{2}\,q_\rho k_\sigma\right)\right)^* = \frac{i\,(q^\mu+k^\mu)\,\hat{A}_\mu(q-k)}{|q||k|}\sin\left(\frac{\theta^{\rho\sigma}}{2}\,k_\rho q_\sigma\right) \tag{2.21}$$

that is exactly the equality eq. (2.19).

In principle self-adjoint operators have an infinite set of eigenfunctions and eigenvalues, however since we are in the infinite dimensional situation a lot depends on the properties of the kernel eq. (2.18). If for some particular $\hat{A}(k) = B(k)$, a complete set of eigenfunctions ψ_n with eigenvalues λ_n exists, we obtain

$$\psi_n(k) = \frac{1}{\lambda_n}\int d^d q\, P(q,k)|_{A=B}\psi_n(q), \quad n = 1,2\dots \tag{2.22}$$

The latter implies that we have an infinite set of gauge potentials B/λ_n, $n = 1,2\dots$ and each of them exhibits zero modes $\alpha = \psi_n$.

2.4 The Henyey approach

As it is done in standard QCD, also in this case one can follow the Henyey strategy [33] where one fixes the form of the zero modes $\hat{\alpha}(k)$ and solves for the gauge potential.

It is worth emphasizing here that, in the standard commutative case, the Henyey strategy to fixing the form of the copies and to solving for the gauge potential gives rise to algebraic equations which can be easily solved while, in the present noncommutative case, even following such a strategy leads to a rather non-trivial equation, due to the non-locality appearing in noncommutative geometry. Therefore we will only sketch the strategy here but we will follow a different approach in the next section.

As we will show, in the noncommutative Henyey approach, instead of a homogeneous Fredholm equation of the second kind we obtain a non-homogeneous Fredholm equation of the first kind, with Hermitian integral kernel.

Since $\hat{A}_\mu$ has d components constrained by the gauge fixing condition, but the integral equation is an equation for one unknown function, we are free to choose some particular parametrization of $\hat{A}$. Let us make the following Ansatz:

$$\hat{A}_\mu(q) = i\,a(q)G_\mu(q) \qquad \text{with}\, G_\mu(q) = -\tilde{\theta}_{\mu\nu}q^\nu \tag{2.23}$$

with $\tilde{\theta}$ the inverse matrix of the matrix θ. This obviously satisfies the gauge condition $q^\mu\hat{A}_\mu(q) = 0$. The presence of the imaginary unit in the first line of eq. (2.23) will become clear soon.

Substituting the Ansatz eq. (2.23) in eq. (2.14) we get the following (non homogeneous) Fredholm equation of the first kind for the unknown function $a(q)$:

$$\int d^d q\, R(q,k)a(q) = f(k), \tag{2.24}$$

where

$$f(k) = k^2 \hat{\alpha}(k), \quad \text{and}$$
$$R(q,k) = 2\sin\left(\frac{1}{2}\,\theta^{\rho\sigma} k_\rho q_\sigma\right) k^\mu G_\mu\ \hat{\alpha}(k-q). \tag{2.25}$$

Note that

$$k^\mu G_\mu(q) = -k^\mu \tilde{\theta}_{\mu\nu} q^\nu \tag{2.26}$$

is *real* and skew symmetric with respect to the exchange $k \leftrightarrow q$:

$$k^\mu G_\mu(q) = -q^\mu G_\mu(k) \tag{2.27}$$

therefore the combination

$$\sin\left(\frac{1}{2}\,\theta^{\rho\sigma} k_\rho q_\sigma\right) k^\mu G_\mu$$

is real and symmetric with respect to the mentioned exchange. Since we are interested in *real* $\alpha(x)$, the corresponding Fourier transform $\hat{\alpha}(k)$ satisfies $(\hat{\alpha}(k))^* = \hat{\alpha}(-k)$. Summarizing all observations listed above, we conclude that the kernel $R(k,q)$defined by eq. (2.25) satisfies

$$(R(k,q))^* = R(q,k), \tag{2.28}$$

therefore the corresponding linear integral operator R is *self-adjoint*. Exactly for this reason the imaginary unit in the Ansatz eq. (2.23) is needed.

In conclusion, in order to solve eq. (2.24) for the potential it is sufficient to show that for some particular choice of α the inverse integral operator R^{-1} exists, so that $a = R^{-1} f$ will give us the gauge potential, for which α is a zero mode.

3 Some exact solutions

The question of the existence of Gribov copies is equivalent to the question of the existence of eigenvectors of the self-adjoint operator defined by eq. (2.18). No matter how we choose the gauge potential A_μ, this operator *does not* belong to the Hilbert-Schmidt class, since

$$\int dq\, dk\, |P(q,k)|^2 = \infty, \tag{3.1}$$

so one can not say a priori whether there exists at least one gauge potential exhibiting zero modes! Neither one can say the opposite. The aim of this section is to demonstrate that gauge potentials that give solutions of eq. (2.18) *do* exist.

For the scope of the present section we will not use the property of Hermiticity of P, therefore it will be more convenient for the forthcoming computations to resort to the original form of the zero mode equation eq. (2.14), which we rewrite as

$$k^2 \hat{\alpha}(k) + 2ik^\mu \int d^d q\, \sin\left(\frac{1}{2}\theta^{\sigma\rho} q_\rho k_\sigma\right) \hat{A}_\mu(k-q)\, \hat{\alpha}(q) = 0. \tag{3.2}$$

We notice that if we consider gauge potentials $\hat{A}_\mu$ which are proportional to derivatives of $\delta(k)$, eq. (3.2) becomes a differential equation for $\hat{\alpha}(k)$.

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3.1 The simplest situation

First we try the following Ansatz

$$A_\mu = Q\tilde{\theta}_{\mu\nu}x^\nu \tag{3.3}$$

with Q some constant to be fixed. The Fourier transform reads

$$\hat{A}_\mu(k) = iQ\tilde{\theta}_{\mu\nu}\partial^\nu\delta(k). \tag{3.4}$$

This potential obviously satisfies the gauge fixing condition $\partial^\mu A_\mu$ in coordinate space, while in momentum space we deal with a distribution, therefore we have to specify in which sense the equality

$$k^\mu \hat{A}_\mu(k) = 0 \tag{3.5}$$

holds. Let us fix the space of probe functions $\hat{\alpha}(k)$ to be the Schwartz space of infinitely smooth functions decaying at infinity faster than any arbitrary power. For an arbitrary Schwartzian function $\hat{\alpha}(k)$ one must have

$$\int d^dk\,\hat{\alpha}(k)\,k^\mu\hat{A}_\mu = 0 \tag{3.6}$$

which is satisfied by eq. (3.4) it being for arbitrary $\hat{\alpha}(k)$

$$\begin{aligned}\int d^dk\,\hat{\alpha}(k)\,k^\mu iQ\tilde{\theta}_{\mu\nu}\partial^\nu\delta(k) &\equiv -iQ\tilde{\theta}_{\mu\nu}\left[\partial^\nu(k^\mu\hat{\alpha}(k))\right]\big|_{k=0}\\ = -iQ(\underbrace{\tilde{\theta}_{\mu\nu}\delta^{\mu\nu}}_{0}\,\hat{\alpha}(0) + \tilde{\theta}_{\mu\nu}(\partial^\nu f)\big|_{k=0}\cdot\underbrace{k^\mu\big|_{k=0}}_{0}) &= 0.\end{aligned} \tag{3.7}$$

The reason to search for zero modes $\alpha(x)$ belonging to the Schwarz space is twofold. On one side we observe that, in the commutative case, the zero mode equation is a Laplace equation which, unless one specifies boundary conditions, may have nontrivial solutions. Indeed each linear function solves it. On the other side the Green function of the corresponding Laplacian gives a singular solution, that decreases at infinity for $d > 2$. In order to get rid of these irrelevant solutions (in the commutative case there is no Gribov problem!) we impose regularity of α at each finite point and vanishing at infinity, which are both satisfied by Schwarz functions. In this class of functions the commutative zero mode equation has just the trivial solution $\alpha = 0$.

Substituting the ansatz (3.4) in the equation (3.2) and using

$$-2Qk^\mu\tilde{\theta}_{\mu\nu}\int d^dq\,\sin\left(\frac{1}{2}\theta^{\sigma\rho}q_\rho k_\sigma\right)\hat{\alpha}(q)\,{}^q\partial^\nu\delta(k-q) = Qk^2\hat{\alpha}(k), \tag{3.8}$$

we arrive at the following algebraic equation

$$(1+Q)k^2\hat{\alpha}(k) = 0, \tag{3.9}$$

which exhibits nontrivial solutions. Indeed if (and only if)

$$Q = -1, \tag{3.10}$$

for arbitrary even space-time dimension, any *arbitrary* function $\hat{\alpha}(k)$ is a solution! Unfortunately, although we found nontrivial solutions of eq. (3.2), this particular gauge potential has a peculiar feature. One may show [34, 35] that it is invariant under gauge transformations (2.3) and therefore we do not have Gribov copies.

Nevertheless this potential is of interest. First of all, the existence of such a gauge invariant connection is a purely noncommutative feature [34, 35] (also see [36] where such a connection has been used to study NCQED as a nonlocal matrix model) and does not exist in the commutative limit. Second, its smooth approximations may be used in principle to search solutions of the integral equation eq. (3.2).

3.2 The next to the simplest situation

To simplify the presentation let us consider the two dimensional case. Here we have only one noncommutative parameter, $\theta_{12} = -\theta_{21} = \theta$. The next to the simplest gauge potential leading to a viable differential equation is the following one:[5]

$$A_\mu(x) \propto \tilde{\theta}_{\mu\nu} x^\nu x^2, \tag{3.11}$$

which, being in two dimensions, can be further simplified to the form

$$A_\mu(x) = Q\varepsilon_{\mu\nu} x^\nu x^2, \tag{3.12}$$

with Q some constant to be determined and $\varepsilon_{\mu\nu}$ the Levi-Civita tensor in two dimensions. The corresponding Fourier transform reads

$$\hat{A}_\mu(k) = iQ\varepsilon_{\mu\nu} \Box\, \partial^\nu \delta(k). \tag{3.13}$$

It is worth emphasizing here that the gauge potential in eq. (3.13) can be approximated as closely as one wants replacing the δ-function with a Gaussian (obviously, if one would use the Gaussian from the very beginning the Gribov copies equation would not be solvable anymore). Hence, the present example is not only interesting in itself, since it also shows that there is a whole family of smooth gauge potentials which are arbitrarily close to having smooth normalizable Gribov copies.

In what follows we will refer to Q as the amplitude of the potential. In spatial coordinates it obviously satisfies the Landau gauge fixing condition and for consistency we check whether it satisfies the gauge fixing condition (3.5) in the above mentioned "distributive" sense. For an arbitrary probe function $\hat{\alpha}(k)$ one obtains

$$\begin{aligned}
&\int d^d k\, \hat{\alpha}(k) k^\mu \hat{A}_\mu = iQ\varepsilon_{\mu\nu} \int d^d k\, \hat{\alpha}\, k^\mu \Box\, \partial^\nu \delta(k) = -iQ\varepsilon_{\mu\nu} \left[\Box\, \partial^\nu \left(\hat{\alpha} k^\mu\right)\right]\Big|_{k=0} \\
&= -iQ(\varepsilon_{\mu\nu} \left(\Box \partial^\nu \hat{\alpha}\right)\Big|_{k=0} \underbrace{k^\mu\Big|_{k=0}}_{0} + 2(\underbrace{\varepsilon_{\mu\nu} \partial^\mu \partial^\nu}_{0}\, \hat{\alpha})\Big|_{k=0} + (\Box \hat{\alpha})\Big|_{k=0} \underbrace{\varepsilon_{\mu\nu} \delta^{\mu\nu}}_{0}) = 0.
\end{aligned}$$

[5]In principle one may also consider quadratic potentials, however for technical reasons we prefer to deal with rotationally invariant potentials, therefore the next to the simplest potential which we consider is cubic. Indeed, in $d = 2$ rotationally invariant quadratic in x gauge fields do not exist.

Let us now substitute the potential eq. (3.13) in the integral equation eq. (3.2) in order to derive a partial differential equation for the zero modes $\hat{\alpha}(k)$. We obtain

$$
\begin{aligned}
&Qk^{\mu}\epsilon_{\mu\nu}\int d^dq\,({}^q\Box\,{}^q\partial^{\nu}\,\delta(q-k))\,\sin\left(\frac{1}{2}\theta^{\sigma\rho}q_{\rho}k_{\sigma}\right)\hat{\alpha}(q) \\
&= -Qk^{\mu}\varepsilon_{\mu\nu}\left\{{}^q\Box\,{}^q\partial^{\nu}\left[\sin\left(\frac{1}{2}\theta^{\sigma\rho}q_{\rho}k_{\sigma}\right)\hat{\alpha}(q)\right]\Bigg|_{q=k}\right\} \\
&= \frac{Q\theta}{8}\left(\theta^2k^4\hat{\alpha}-4k^2\Box\hat{\alpha}-8\,\varepsilon^{\mu\nu}\varepsilon^{\eta\lambda}k_{\mu}k_{\eta}\partial_{\nu}\partial_{\lambda}\hat{\alpha}\right)
\end{aligned}
\tag{3.14}
$$

hence the zero modes $\hat{\alpha}(k)$have to satisfy the partial differential equation given below:

$$
\left(-4k^2\Box-8\,\varepsilon^{\mu\nu}\varepsilon^{\eta\lambda}k_{\mu}k_{\eta}\partial_{\nu}\partial_{\lambda}-\frac{4k^2}{Q\theta}+\theta^2k^4\right)\hat{\alpha}(k)=0. \tag{3.15}
$$

We notice that, since in two dimensions $\varepsilon^{\mu\nu}$ is a universal tensor, this equation is rotationally invariant, therefore it makes sense to rewrite it in polar coordinates (r,ϕ) given by

$$
\begin{cases} k_1 = r\cos\phi \\ k_2 = r\sin\phi \end{cases}. \tag{3.16}
$$

One may easily see that in polar coordinates eq. (3.15) reads

$$
r^2\hat{\alpha}_{rr}+3r\hat{\alpha}_r+\frac{1}{Q\theta}r^2\hat{\alpha}-\frac{\theta^2}{4}r^4\hat{\alpha}+3\hat{\alpha}_{\phi\phi}=0, \tag{3.17}
$$

therefore it exhibits separation of variables. Let us look for a solution in the following form

$$
\alpha(\phi,r)=\Phi(\phi)\,f(r), \tag{3.18}
$$

where the functions Φ and f satisfy the following *ordinary* differential equations

$$
\begin{cases} -\Phi_{\phi\phi}=\lambda\Phi, \\ r^2f_{rr}+3rf_r+\left(-3\lambda+\frac{1}{Q\theta}r^2-\frac{\theta^2}{4}r^4\right)f=0. \end{cases} \tag{3.19}
$$

The former is just the equation of the simple harmonic motion, while the latter is the confluent hypergeometric equation whose properties are very well studied, see e.g. [38] for a review. Let us specify the boundary conditions as follows:

$$
\begin{cases} \Phi(0)=\Phi(2\pi), \\ |f(0)|<\infty, \quad f(r)\to 0,\ \text{when}\ r\to\infty. \end{cases} \tag{3.20}
$$

Below we will see that each function satisfying the boundary conditions given by eq. (3.20) belongs to the Schwarz space. We also notice that the deformation parameter θ enters the equation eq. (3.19) in the combinations $Q\theta$ and θ^2, where Q is arbitrary. Therefore without loss of generality one may consider only $\theta>0$, since the opposite sign of θ corresponds to the opposite sign of the arbitrary amplitude Q. From the angular boundary conditions we see that

$$\lambda_n = n^2, n = 0, \pm 1, \pm 2, \dots \tag{3.21}$$

so that the general solution for the angular equation eq. (3.19) is of the form

$$\Phi(\phi) = \tilde{c}_1 \cos(n\phi) + \tilde{c}_2 \sin(n\phi), \tag{3.22}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are arbitrary constants. The general solution of the radial equation for $\lambda = \lambda_n$ is given by

$$\begin{aligned} f(r) &= r^{\sqrt{3n^2+1}-1} \exp\left(-\frac{r^2\theta}{4}\right)\left(c_1\, M\left(a, c, \frac{\theta\, r^2}{2}\right) + c_2\, U\left(a, c, \frac{\theta\, r^2}{2}\right)\right), \\ \text{with} \quad a &= \frac{1}{2} + \frac{1}{2}\sqrt{3n^2+1} - \frac{1}{2\theta^2 Q}, \qquad c = 1 + \sqrt{3n^2+1} \end{aligned} \tag{3.23}$$

U and M are Kummer functions and c_1 and c_2 are arbitrary constants.

We notice however that the boundary condition for the radial dependence can be satisfied if and only if the number a defined by eq. (3.23) is a non positive integer

$$a = -m, \quad m = 0, 1, 2, \dots \tag{3.24}$$

In that case Kummer functions reduce to Laguerre polynomials.

Thus, the solution regular both at zero and infinity (see [38]) is

$$f(r) = C\, r^{\sqrt{3n^2+1}-1} \exp\left(-\frac{r^2\theta}{4}\right) L_m^{\sqrt{3n^2+1}}\left(\frac{\theta\, r^2}{2}\right), \tag{3.25}$$

where C is an arbitrary constant, and $L_n^a(z)$ stands for generalized Laguerre polynomial. This solution exists when the amplitude Q takes one of the discrete values

$$Q_{nm} = \frac{1}{\theta^2(\sqrt{3n^2+1} + 2m + 1)}, \quad n = 0, \pm 1, \pm 2, \dots, \quad m = 0, 1, 2, \dots \tag{3.26}$$

The general form of the zero modes, when the amplitude Q belongs to the discrete set defined above, is

$$\begin{aligned} &\hat{\alpha}_{nm}(r, \phi) = (C_1 \cos(n\phi) + C_2 \sin(n\phi))\, r^{\sqrt{3n^2+1}-1} \exp\left(-\frac{r^2\theta}{4}\right) L_m^{\sqrt{3n^2+1}}\left(\frac{\theta\, r^2}{2}\right) \\ &\quad \text{where} \quad C_1, C_2 \text{ are real if } n \text{ is even} \\ &\quad\quad \text{and} \quad C_1, C_2 \text{ are purely imaginary if } n \text{ is odd.} \end{aligned} \tag{3.27}$$

The restriction on arbitrary constants comes from the requirement to have real $\alpha(x)$. Indeed, in order to satisfy this requirement, the corresponding Fourier transform $\hat{\alpha}$ must satisfy $(\hat{\alpha}(k))^* = \hat{\alpha}(-k)$. From another side the reflection $k \to -k$ is equivalent to the shift $\phi \to \phi + \pi$ in polar coordinates. One may easily check that the radial dependence is real, and the linear combination of sine and cosine appearing in (3.27) satisfies

$$(C_1 \cos(n(\phi+\pi)) + C_2 \sin(n(\phi+\pi))) = (C_1 \cos(n\phi) + C_2 \sin(n\phi))^* \tag{3.28}$$

if and only if the restriction described in (3.27) is imposed.

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l	p_l	P_l
1	0	1
2	1	2
3	4	7
4	15	26
5	56	97
6	209	362
...	...	...

Table 1. The table shows the first six numbers from the A001353 sequence.

3.2.1 How many zero modes do we have?

In this subsection we discuss how many linearly independent solutions for a given amplitude (3.26) we have, connecting our results with number theory. From the solution (3.27) we can guarantee at least two. It is remarkable that for some special subset of the amplitudes Q_{nm} we can have more than two, or more precisely the following statement holds:

Theorem. *For arbitrarily large number N there exists such an amplitude Q_{nm}, that the number of linearly independent zero modes corresponding to this potential is greater than N.*

Before we go ahead to prove the existence of "many" solutions, let us remind some facts from number theory. The key property allowing such an existence is due to the fact that there exist infinitely many natural numbers p_l, $l = 1, 2, \ldots$ such that

$$3p_l^2 + 1 \quad \text{is a perfect square,} \tag{3.29}$$

i.e. there exists such a natural number P_l, that

$$3p_l^2 + 1 = P_l^2, \quad l = 1, 2, \ldots \tag{3.30}$$

This sequence, called "A001353", is well known and studied (see [37] and refs therein). We have

$$p_1 = 0,\, p_2 = 1,\, p_l = 4p_{l-1} - p_{l-2}, \quad l = 3, 4, \ldots \tag{3.31}$$

and the corresponding P_l are given by

$$P_l = 2p_l - p_{l-1}. \tag{3.32}$$

We also notice that all $P_{2k}, k = 1, 2, \ldots$ are *even* natural integers, therefore

$$J_k \equiv \frac{P_{2k}}{2}, \quad k = 1, 2, \ldots \tag{3.33}$$

are natural integers. The first six numbers from the sequence A001353 are presented in table 1.

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k	m	$\sqrt{3n^2+1}$	n
1	$J_N - 1$	2	1
2	$J_N - 13$	26	15
3	$J_N - 181$	362	209
...	...	...	...
k	$J_N - J_K$	P_K	p_{2k}
...	...	...	...
N	0	P_N	p_{2N}

Table 2. This table summarizes the construction of N different solutions for a given amplitude, as described in section 3.2.1.

We are now ready to prove the above stated theorem. For a given (arbitrary large) N, we choose the amplitude $Q = Q_N$ where

$$Q_N = \frac{1}{\theta^2 \left(2J_N + 1\right)}. \tag{3.34}$$

This implies that the parameter a appearing in eq. (3.23)

$$a = \frac{1}{2} + \frac{\sqrt{3n^2+1}}{2} - \frac{1}{2\theta^2 Q} = -J_N + \frac{\sqrt{3n^2+1}}{2} \tag{3.35}$$

is a nonpositive integer if and only if $\frac{\sqrt{3n^2+1}}{2}$ is a positive integer not greater than J_N. Let us now substitute $n = n_k$ given by

$$n_k = p_{2k}, \quad k = 1, \ldots, N. \tag{3.36}$$

We obtain

$$a = -J_N + J_k, \quad k = 1, \ldots, N, \tag{3.37}$$

and for each k from 1 to N we have two linearly independent solutions (with sine and cosine), given by (3.27), where the Laguerre polynomial is labelled by $m = -J_N + J_k$, and the parameter $\sqrt{3n^2+1} = P_k$. Finally we conclude that we have found the amplitude $Q = Q_N$ for the gauge potential in (3.12), that exhibits $2N$ linearly independent zero modes. *QED*

The logic of our construction is summarized in table 2.

3.2.2 Examples

Let us illustrate the results of the previous section with some examples.

- N=1. The corresponding amplitude Q is given by (see eq. (3.34))

$$Q_1 = \frac{1}{3\theta^2}. \tag{3.38}$$

There are just two linearly independent solutions

$$m = 0, \quad n = 1 \quad \text{angular dependence} = \text{cosine}$$

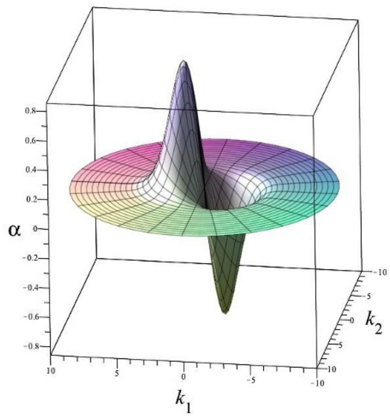

Figure 1. Shape of $i\hat{\alpha}$, for N=1 and angular dependence of cosine type. The deformation parameter θ is chosen to be equal to one.

$$m = 0, \quad n = 1 \quad \text{angular dependence} = \text{sine}$$

In figure 1 we plot the corresponding potential for the cosine case.[6]

- N=2. The corresponding amplitude is given by (see eq. (3.34))

$$Q_2 = \frac{1}{27\theta^2}. \tag{3.39}$$

There are four linearly independent solutions

$$m = 12, \quad n = 1 \quad \text{angular dependence} = \text{cosine}$$

$$m = 12, \quad n = 1 \quad \text{angular dependence} = \text{sine}$$

$$m = 0, \quad n = 15 \quad \text{angular dependence} = \text{cosine}$$

$$m = 0, \quad n = 15 \quad \text{angular dependence} = \text{sine}$$

In figure 2 we plot the corresponding potential with angular dependence of cosine type.

3.3 Multidimensional generalization

Let us consider now the generalization to arbitrary dimensions of the solutions discussed above.

Although the potential given by eq. (3.11) satisfies the Landau gauge fixing condition in arbitrary even dimensions, it does not lead to a simple partial differential equation where one can easily separate variables. However, if one considers the space $\mathbb{R}^d$ as a direct product of $d/2$ orthogonal planes, the tensor $\epsilon^{\mu\nu}$ is invariant under rotation in each plane. This

[6]Since for odd n the Fourier image $\hat{\alpha}(k)$ is purely imaginary, we multiplied it by i to build a plot.

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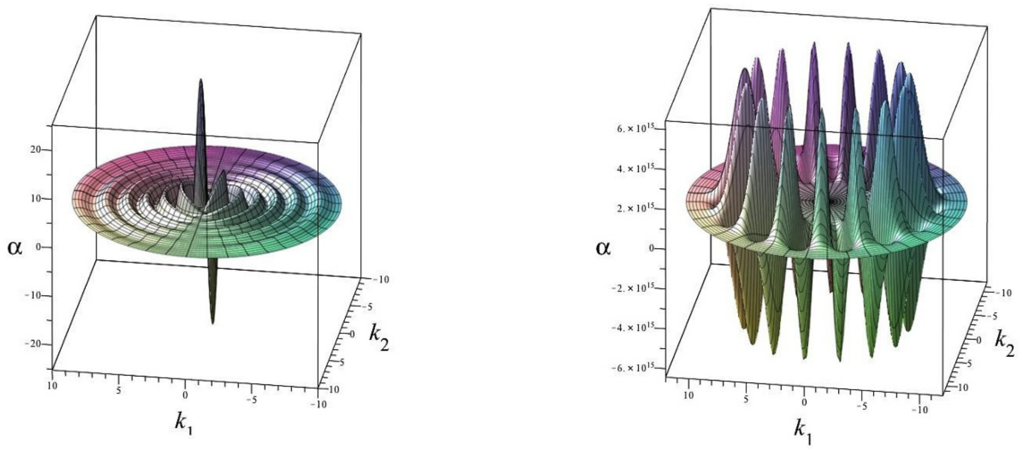

Figure 2. Shape of $i\hat{\alpha}$, for N=2. The first plot represents the case $m = 12$, while the second one corresponds to $m = 0$ of cosine type. The deformation parameter θ is chosen to be equal to one.

observation suggests to define the potential by reproducing the two dimensional structure in each plane:

$$\begin{cases} A_1(x) = Q_I x_2(x_1^2 + x_2^2) \\ A_2(x) = -Q_I x_1(x_1^2 + x_2^2) \\ A_3(x) = Q_{II} x_3(x_3^2 + x_4^2) \\ A_4(x) = -Q_{II} x_4(x_3^2 + x_4^2) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases} \tag{3.40}$$

where $Q_I, Q_{II}, \ldots$ are in general different constants.

Taking the Fourier transform of eq. (3.40) and substituting the result in the integral equation (3.2), after carrying out similar computations to the ones that we did in the $d = 2$ case, we arrive at the following partial differential equation

$$[Q_I D_I + Q_{II} D_{II} + \ldots]\, \hat{\alpha} = (k_I^2 + k_{II}^2 + \ldots)\hat{\alpha}, \tag{3.41}$$

where we used the following notations

$$\begin{aligned} k_I^2 &= k_1^2 + k_2^2, \\ k_{II}^2 &= k_3^2 + k_4^2, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} D_I &= -\theta_1 \left(k_I^2 \left(\partial_1^2 + \partial_2^2\right) + 2k_1^2\partial_2^2 + 2k_2^2\partial_1^2 - 4k_1k_2\partial_{12}^2 - \frac{\theta_1^2}{4} k_I^4 \right) \\ D_{II} &= -\theta_2 \left(k_{II}^2 \left(\partial_3^2 + \partial_4^2\right) + 2k_3^2\partial_4^2 + 2k_4^2\partial_3^2 - 4k_3k_4\partial_{34}^2 - \frac{\theta_2^2}{4} k_{II}^4 \right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{3.43}$$

with θ_i, $i = 1, \ldots, d/2$ the different noncommutative parameters of each two-dimensional plane, as in eq. (2.2). One may easily see that eq.(3.41) is a sum of $d/2$ equations

$$Q_I D_I \hat{\alpha} = \hat{\alpha} k_I^2, \tag{3.44}$$

$$Q_{II} D_{II} \hat{\alpha} = \hat{\alpha} k_{II}^2, \tag{3.45}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{3.46}$$

where each equation is of the form (3.15).

If functions $\hat{\alpha}_I(k_1, k_2), \hat{\alpha}_{II}(k_3, k_4), \ldots,$ are solutions of *two dimensional* equations eq. (3.44), eq. (3.45), ..., then their product

$$\hat{\alpha} = \hat{\alpha}_I(k_1, k_2) \cdot \hat{\alpha}_{II}(k_3, k_4) \cdot \ldots \tag{3.47}$$

solves each of the equations eq. (3.44), eq. (3.45), ... as well as their sum eq. (3.41). But we have already constructed solutions for the two dimensional case belonging to the Schwarz space $\mathcal{S}(\mathbb{R}^2)$ hence the product eq. (3.47) automatically belongs to the Schwarz space $\mathcal{S}(\mathbb{R}^d)$.

4 Discussion and outlook

In the present paper we have shown that an infinite number of Gribov copies exists in noncommutative QED, and this is a genuine noncommutative effect. As already recalled in the introduction and well known in the literature on noncommutative gauge theory, NCQED behaves like a non-Abelian gauge theory because of the non-trivial deformation of the covariant derivative eq. (2.7). Therefore the existence of a nontrivial equation for copies was expected and not surprising. The main result of the paper is that such equation has solutions, which we compute, and they are an infinite number. The intrinsic interest of this observation is that it shows how noncommutative geometry can give rise to a global obstruction preventing a proper gauge-fixing already in Abelian gauge theories. Consequently, the intriguing possibility which is naturally suggested by the present analysis is to extend to the case of NCQED the Gribov-Zwanziger restriction. Indeed, the Gribov-Zwanziger restriction would yield to the following modification of the NCQED propagator

$$G^{\mathrm{G-Z}}(p) \sim \frac{p^2}{p^4 + \gamma^4} \tag{4.1}$$

with γ depending on the noncommutative parameter θ.

However, having proved that the noncommutativity of space-time induces (infinitely many) Gribov copies also in NCQED (as we did in the present paper) is not enough to justify the Gribov-Zwanziger approach.

In non-Abelian gauge theory on flat topologically trivial space-times, the Gribov-Zwanziger approach is based on the following fundamental results [16]:

1) The Gribov region is bounded in every direction (in the functional space of transverse gauge potentials).

2) The Faddeev-Popov determinant changes sign at the Gribov horizon.

3) Every gauge orbit passes inside the Gribov horizon.

The importance of the last result lies in the fact that it justifies the restriction of functional integration to the Gribov region (since what is left outside the Gribov horizon is just a copy of something inside it and so no relevant configuration is lost).

In order to justify the Gribov-Zwanziger restriction in NCQED we should generalize the analysis of [16, 17] to the noncommutative case. This is a highly non-trivial technical task since many of the arguments used in these references to prove the properties **1)**, **2)** and especially **3)** make heavy use of the theory of local elliptic PDEs while, in the noncommutative case, the Gribov copy equation becomes non-local. We hope to come back in a future publication on these important issues.

As a final remark, it is natural to wonder whether the present results can be extended to the Coulomb gauge as well. In the footnote 4 we have described the equation of the copies in such a case. Formally, some of the copies constructed here can also be used to construct copies in the Coulomb gauge (for instance, a copy in the Landau gauge in a two dimensional non-commutative space-time can be trivially promoted to a copy in the Coulomb gauge in 3+1 dimensions). However, the physical interpretation is rather subtle. Indeed, one of the fundamental properties of the Moyal star product is the 'democracy' between all space-time coordinates. Obviously, such democracy is not respected by the Coulomb gauge (one well known consequence being the existence of residual gauge transformations in this gauge). Thus, in the non-commutative setting, not only one should declare the Euclidean time as 'special' but its non-commutative partner as well and this would make the analysis much more complicated than in the Landau case. We hope to come back on this interesting issue in a future investigation.

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