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A group theoretic characterization of Buekenhout–Metz unitals in $\mathrm{PG}(2, q^2)$ containing conics

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Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$, $q = p^h$ and let G be the group of projectivities of $\mathrm{PG}(2, q^2)$ stabilizing $\mathcal{U}$. In this paper we prove that $\mathcal{U}$ is a Buekenhout–Metz unital containing conics and q is odd if, and only if, there exists a point A of $\mathcal{U}$ such that the stabilizer of A in G contains an elementary Abelian p-group of order q^2 with no non-identity elations.

1. Introduction

Baker and Ebert [2] and Hirschfeld and Szőnyi [6] independently discovered an orthogonal Buekenhout–Metz unital in $\mathrm{PG}(2, q^2)$, $q = p^h$, q odd, which is the union of q conics of a hyperosculating pencil with base a point A. We call such a unital Buekenhout–Metz of *BEHS-type*. These are the only Buekenhout–Metz unitals containing conics. In [1] Abatangelo and Larato determine the linear collineation group Γ stabilizing a Buekenhout–Metz unital of BEHS-type and prove that this group has the following properties:

(i) the order of Γ is $2q^3(q-1)$;
(ii) Γ is transitive on the points of the unital different from A;
(iii) the stabilizer of a point of the unital, different from A, in Γ is a cyclic group of order $2(q-1)$;
(iv) Γ is the semidirect product of a normal elementary Abelian subgroup of order q^3 with a cyclic subgroup of order $2(q-1)$.

They also prove that Γ has an elementary Abelian p-group of order q^2, with no non-identity elations, that stabilizes every conic of $\mathcal{U}$. Further, they show that, if the group of projectivities G preserving a unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ with q odd satisfies these four conditions, then $\mathcal{U}$ is a Buekenhout–Metz unital of BEHS-type. Ebert and Wantz [5] prove that a unital $\mathcal{U}$ is orthogonal Buekenhout–Metz if and only if the group of projectivities stabilizing $\mathcal{U}$ contains a semidirect product $S \rtimes R$ where S has order q^3 and R has order $q-1$. Also, S is Abelian if and only if $\mathcal{U}$ is of BEHS-type, in which case q is necessarily odd and S is elementary Abelian.

In this paper we obtain the following group theoretic characterization of Buekenhout–Metz unitals of BEHS-type.

Theorem 1.1. *Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$, with $q = p^h$, and let G be the group of projectivities stabilizing $\mathcal{U}$. If there exists a point A of $\mathcal{U}$ such that the stabilizer of A in G contains an elementary Abelian p-group of order q^2 with no non-identity elations, then $\mathcal{U}$ is a Buekenhout–Metz unital of BEHS-type and q is odd.*

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2. Preliminary results

Let $\mathrm{PG}(2, q^2)$, $q = p^h$, be the projective plane over the Galois field $\mathrm{GF}(q^2)$. A *unital* in $\mathrm{PG}(2, q^2)$ is a set $\mathcal{U}$ of $q^3 + 1$ points meeting every line of $\mathrm{PG}(2, q^2)$ in either 1 or $q + 1$ points. Lines meeting a unital $\mathcal{U}$ in 1 or $q + 1$ points are called *tangent* or *secant* lines to $\mathcal{U}$. Through each point of $\mathcal{U}$ there pass q^2 secant lines and one tangent line. Through each point P not on $\mathcal{U}$ there pass $q^2 - q$ secant lines and $q + 1$ tangent lines; the points of contact of the tangent lines are called the *feet* of P.

An example is the *non-degenerate Hermitian curve* or *classical* unital, that is, the set of the absolute points of a non-degenerate unitary polarity of $\mathrm{PG}(2, q^2)$. For more information on unitals in projective planes, see [3].

Consider the polynomial $x^2 - r$, irreducible over $\mathrm{GF}(q)$, and $t \in \mathrm{GF}(q^2)$ such that $t^2 - r = 0$. Let a be an element in $\mathrm{GF}(q^2)$ and let Γ_a be the conic of $\mathrm{PG}(2, q^2)$ with equation $x_1x_3 - x_2^2 + ax_3^2 = 0$. The set

$$\mathcal{U} = \bigcup_{a \in t\mathrm{GF}(q)} \Gamma_a$$

is an orthogonal Buekenhout–Metz unital in $\mathrm{PG}(2, q^2)$ of BEHS-type; see [6]. Observe that $\mathcal{U}$ is the the union of q conics of a hyperosculating pencil with base $(1, 0, 0)$.

A *central* collineation of $\mathrm{PG}(2, q^2)$ is a collineation α fixing every point of a line ℓ (the *axis* of α) and fixing every line through a point C (the *center* of α). If $C \in \ell$, then α is an *elation*; otherwise α is a *homology*. It is known that given a line ℓ and three distinct collinear points C, P, P' of $\mathrm{PG}(2, q^2)$, with $P, P' \notin \ell$ and both different from C, there is a unique central collineation with axis ℓ and center C mapping P onto P'.

Note that a non-identity homology f of $\mathrm{PG}(2, q^2)$ stabilizing a unital $\mathcal{U}$ has as center a point V not on $\mathcal{U}$ and as axis a secant line ℓ to $\mathcal{U}$. Suppose by way of contradiction that V is on $\mathcal{U}$. Let P be a point of $\ell \cap \mathcal{U}$. The line VP is a secant line to $\mathcal{U}$, hence for any point Q on $(\mathcal{U} \cap VP) \setminus \{V, P\}$ we have that $|\langle f \rangle| = |Orb_{\langle f \rangle}(Q)|\, |Stab_{\langle f \rangle}(Q)|$. Since $Stab_{\langle f \rangle}(Q)$ is the trivial subgroup, it follows that $|\langle f \rangle|$ divides $q - 1$. Let m be a secant line to $\mathcal{U}$ through V such that $\ell \cap m \notin \mathcal{U}$. For any point R on $m \cap \mathcal{U}$ different from V we have that $|\langle f \rangle| = |Orb_{\langle f \rangle}(R)|$, therefore $|\langle f \rangle|$ divides q. As q and $q - 1$ are relatively prime, $|\langle f \rangle| = 1$ and f is the identity, a contradiction. Suppose now that ℓ is a tangent line to $\mathcal{U}$. The line ℓ contains at most one of the feet of V; so there exists one of the feet of V, say T, not on ℓ. Since VT is the tangent line to $\mathcal{U}$ at T, it follows that $f(T) = T$, thus f is the identity, again a contradiction.

From now on we identify, unambiguously, a projectivity of $\mathrm{PG}(2, q^2)$ with its matrix representation with respect to a frame of the plane. Then a group of projectivities of the plane is identified by a group of 3×3 matrices.

3. Characterization

Let $\mathcal{U}$ be a unital in $\mathrm{PG}(2, q^2)$, $q = p^h$, and let A be a point of $\mathcal{U}$ with tangent line ℓ_∞. Throughout the paper we will denote by G the linear collineation group preserving $\mathcal{U}$ and by G_A an elementary Abelian p-group of order q^2, with no non-identity elations, contained in the stabilizer of A in G. Let L_∞ be the group of projectivities of the line ℓ_∞ into itself. Every element $f \in G_A$ induces a projectivity f_∞ of L_∞. Consider the homomorphism

$$\Psi : f \in G_A \longrightarrow f_\infty \in L_\infty.$$

An element $g \in Ker\Psi$ induces the identity map on ℓ_∞, hence g is a perspectivity with axis ℓ_∞. Since g cannot be a non-identity homology (see Section 2) and G_A has no non-identity elations, it follows that g is the identity. The map Ψ is then a monomorphism.

Proposition 3.1. *If f is a non-identity element of G_A, then f_∞ has A as a unique fixed point.*

Proof. Let P be a point of ℓ_∞ different from A. There exists an element $h \in G_A$ such that $h(P) \neq P$. Indeed, suppose on the contrary that P is fixed by every element of G_A. In such a case $\Psi(G_A)$ is a subgroup of the stabilizer L_{AP} of both A and P in L_∞. The groups $\Psi(G_A)$ and L_{AP} have size q^2 and $q^2 - 1$, respectively, a contradiction. Since G_A is an Abelian group, for every element $f \in G_A$ we have that

$$f_\infty(P) = (h_\infty^{-1} \circ f_\infty \circ h_\infty)(P).$$

If $f_\infty(P) = P$ then $(h_\infty^{-1} \circ f_\infty \circ h_\infty)(P) = P$; hence f_∞ fixes the three distinct points A, P and $h(P)$, so it is the identity. Therefore $f \in Ker\Psi$; so f is the identity. It follows that, for every non-identity element f of G_A, the map f_∞ has A as unique fixed point. □

Proposition 3.2. *The group G_A has a sharply transitive action on the points of ℓ_∞ different from A.*

Proof. If P is a point of ℓ_∞ different from A, then

$$|G_A| = |Orb_{G_A}(P)|\, |Stab_{G_A}(P)|.$$

From the previous proposition $Stab_{G_A}(P)$ is trivial; thus $Orb_{G_A}(P)$ has size q^2. The assertion follows. □

By dualizing the previous arguments it can be shown that G_A has a sharply transitive action on the lines through A different from ℓ_∞. It follows that every non-identity element f of G_A has A as unique fixed point and ℓ_∞ as unique fixed line.

Proposition 3.3. *The group G_A stabilizes every conic of a hyperosculating pencil with base A containing the line ℓ_∞ counted twice.*

Proof. We may assume, without loss of generality, that $A = (1, 0, 0)$ and that ℓ_∞ has equation $x_3 = 0$. A non-identity element $f \in G_A$ has A as unique fixed point and ℓ_∞ as unique fixed line; so it is given by

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix},$$

for some $a, b, c \in \mathrm{GF}(q^2)$.

Every elementary Abelian p-group of order q^2 is isomorphic to the additive group of $\mathrm{GF}(q^2)$. So there exists an isomorphism

$$\Phi : x \in \mathrm{GF}(q^2) \longrightarrow \begin{pmatrix} 1 & \alpha(x) & \gamma(x) \\ 0 & 1 & \beta(x) \\ 0 & 0 & 1 \end{pmatrix} \in G_A,$$

where α, β and γ are mappings of $\mathrm{GF}(q^2)$ into itself such that $\alpha(0) = \beta(0) = \gamma(0) = 0$. From the condition $\Phi(x + y) = \Phi(x)\Phi(y)$, it follows that

$$\begin{aligned} \alpha(x+y) &= \alpha(x) + \alpha(y), \\ \beta(x+y) &= \beta(x) + \beta(y), \\ \gamma(x+y) &= \gamma(x) + \gamma(y) + \alpha(x)\beta(y), \end{aligned} \tag{1}$$

for any x, y in $\mathrm{GF}(q^2)$.

The functions α, β and γ, as any map of $\mathrm{GF}(q^2)$ into itself, are polynomial functions. Also, α and γ are additive maps; hence

$$\alpha(x) = \sum_{i=1}^{u} a_i x^{p^i}, \qquad \beta(x) = \sum_{j=1}^{v} b_j x^{p^j},$$

for some integers u and v and some elements a_i and b_j in $\mathrm{GF}(q^2)$.

Let

$$\gamma(x) = \sum_{k=1}^{t} c_k x^k;$$

it follows from (1) that

$$\sum_{k=1}^{t} c_k (x+y)^k = \sum_{k=1}^{t} c_k x^k + \sum_{k=1}^{t} c_k y^k + \sum_{i,j} a_i b_j x^{p^i} y^{p^j}.$$

Therefore

$$\alpha(x) = a x^{p^n}, \qquad \beta(x) = b x^{p^n}, \qquad \gamma(x) = \frac{ab}{2} x^{2p^n},$$

for a suitable integer n and for some elements $a, b \in \mathrm{GF}(q^2)$. We may assume that the point $P = (0, 0, 1)$ belongs to $\mathcal{U}$ and if f is the previously defined non-identity element of G_A, then $f(P) = (\gamma(s), \beta(s), 1) \in \mathcal{U}$ for some $s \in \mathrm{GF}(q^2)$. The points A, P and $f(P)$ are non-collinear points, since G_A has a sharply transitive action on the lines through A different from ℓ_∞. So $f(P)$ is on a line through A, different from ℓ_∞ and from AP. If $f(P)$ is on the line $x_1 = 0$, then $\gamma(s) = 0$ and since $\beta(s) \neq 0$, then $a = 0$ and hence $\alpha(s) = 0$. It follows that f has $B(0, 1, 0)$ as a fixed point, a contradiction. Therefore, by appropriately choosing s, we may assume that $f(P) = (1, 1, 1)$ and hence

$$f = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $a = \frac{2}{s^{p^n}}$, $b = \frac{1}{s^{p^n}}$, and so

$$G_A = \left\{ \begin{pmatrix} 1 & \dfrac{2}{s^{p^n}} x^{p^n} & \dfrac{1}{s^{2p^n}} x^{2p^n} \\ 0 & 1 & \dfrac{1}{s^{p^n}} x^{p^n} \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathrm{GF}(q^2) \right\}.$$

Since the map $x \mapsto x^{p^n}$ of $GF(q^2)$ is an automorphism, it follows that

$$G_A = \left\{ \begin{pmatrix} 1 & 2d & d^2 \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} : d \in GF(q^2) \right\}.$$

Finally, observe that G_A stabilizes every conic of the hyperosculating pencil $\mathscr{P}$ with equation $x_1x_3 - x_2^2 + wx_3^2 = 0$, with $w \in GF(q^2) \cup \{\infty\}$. Since $\mathscr{P}$ contains the line ℓ_∞ counted twice and has the point A as base, the assertion follows. $\square$

Proof of Theorem 1.1. From the previous result, the unital $\mathcal{U}$ is the union of q conics $\Gamma_1, \ldots, \Gamma_q$ of $\mathscr{P}$ with equations $x_1x_3 - x_2^2 + w_ix_3^2 = 0$, $i = 1, \ldots, q$. For q even, the tangents to Γ_1 all contain a common point N, the nucleus of Γ_1. Thus there would be $q^2 + 1$ tangents to $\mathcal{U}$ on N, a contradiction. Hence q must be odd (see also [3, Chapter 4]). Let P be a point of Γ_i. Since the secant lines through P to Γ_i are also secant to $\mathcal{U}$, it follows that the tangent line to Γ_i at P coincides with the tangent line to $\mathcal{U}$ at P. Hence the points of Γ_j, for any $j \neq i$, are all internal points with respect to Γ_i. From the equations of Γ_i and Γ_j, it follows that $w_i - w_j$ is a non-square in $GF(q^2)$. Without loss of generality we may assume that the point $(1, 1, 1)$ belongs to $\mathcal{U}$; so the conic with equation $x_1x_3 - x_2^2 = 0$ is contained in $\mathcal{U}$ and then the set $W = \{w_1, \ldots, w_q\}$ is a q-set containing 0 with the property that the difference of any two distinct elements is always a non-square. From [4] it follows that, considering $GF(q^2)$ in the usual way as the affine plane $AG(2, q)$, the set W is a line through the origin. Thus W is a set of the form $tGF(q)$, with t a non-square in $GF(q^2)$. Then $\mathcal{U}$ is a Buekenhout–Metz unital of BEHS-type. $\square$

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