Bosonic Spectral Action Induced from Anomaly Cancelation

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Abstract

We show how (a slight modification of) the noncommutative geometry bosonic spectral action can be obtained by the cancelation of the scale anomaly of the fermionic action. In this sense the standard model coupled with gravity is induced by the quantum nature of the fermions. The regularization used is very natural in noncommutative geometry and puts the bosonic and fermionic action on a similar footing.

1   Introduction

Classical general relativity is a *geometrical* theory describing how the curvature of space-time influences the motion of classical bodies. The standard model of elementary particles is on the other side a quantum field theory and the difficulties in reconciling the two are well known. The noncommutative geometry programme [1, 2] aims at a generalization of ordinary geometry along the lines of the one made to describe quantum mechanically the phase space. The programme is based on a transcription of ordinary (commutative) geometry in algebraic terms, based on the duality between commutative C^*-algebras and topological spaces. The setting is then generalized to noncommutative algebras which may or may not be matrix algebras over an ordinary space. In the former case one talks of *almost commutative spaces*. The geometrical information on the space is given by the spectral data defined by the *spectral triple*, comprised of an $*$-algebra $\mathcal{A}$, a fermionic Hilbert space on which the algebra is represented by bounded operators and a Hermitian (generalized) Dirac operator D_0. Geometry is then translated into the spectral properties of these operators. All usual concepts obtain an algebraic equivalent: integrals generalize to traces, differential forms are operators obtained commuting functions with the Dirac operator, and a dictionary translating the concept ordinary geometry in this language is being built. The setting is solid and it generalizes naturally to the case in which the algebra is noncommutative (hence the name of noncommutative geometry), and an underlying point geometry may not exist. Details can be found in [1, 3, 4, 5], which by now are classic descriptions of noncommutative geometry.

Already at the classical level the construction requires the presence of fermions. While it is still impossible to "hear the shape of a fermionic drum" because of isospectral manifolds [6], the Dirac operator carries more information than the Laplacian [7]. This gives a centrality to fermions in geometry. The elements of the algebra of functions on a manifold are "bosonic", and they capture only the topology of the space (via continuity of the functions). The full geometrical information requires necessarily the presence of fermions and their Hilbert space on which the Dirac operator is defined.

Connes' approach to the standard model is the attempt to understand which kind of (noncommutative) geometry gives rise to the standard model of elementary particles coupled with gravity. The most complete formulation of this is given by the *spectral action*, described in the next section. The starting point is an almost commutative geometry product of the algebra of functions on ordinary spacetime times a finite dimensional matrix algebra. It comprises of a bosonic and a fermionic part, which are treated somewhat differently, and it reproduces the Lagrangian of the fermions of the standard model in a curved background, and contains all required terms for the Higgs mechanism of symmetry breaking. Its input are the masses (and mixings) of the fermions, and the coupling constants at low energy. The action must be read in the Wilsonian sense and undergoes

renormalization, which is done in the usual way. The result is the full action of the standard model coupled with gravity, with some extra phenomenologically relevant terms. The mass of the Higgs can be calculated in terms of the other parameters of the theory, and while its value (170 GeV) may have been recently excluded by *Fermilab* data, it is still surprising that a purely geometrical theory is capable to make specific predictions which are of the correct order of magnitude.

The purpose of this paper is to show that the bosonic spectral action is a consequence of the fermionic action and the cancelation of the scale anomaly. We see that the spectral action is a *quantum* effect of the fermionic action, and its regularization. A fact already noted in a different context in [8, 9, 10]. The crucial aspect is the cancelation of the anomaly which develops under a *spectral* regularization of the fermionic action. Our calculation is totally general and comes prior to the application to the standard model. We will therefore be very general in our treatment of the action and comment on the standard model where appropriate. In a sense we explicitly show that the spectral action is induced perturbatively by the action for matter, which is the old idea of Sakharov [11] (for a modern review see [12]). In fact it has been already shown by Yu. Novozhilov and D. Vassilevich [13] that this anomaly induces quantum gravity.

In section 2 we briefly introduce the spectral action and discuss its properties under scale invariance. In section 3 we discuss anomalies in the present context. In section 4 we show with an explicit calculation how a slightly modified version of the bosonic spectral action is the term required to cancel the scale anomaly, and in the following section we show explicitly the slight modifications, which amount to a change of some coefficients. A final section contains some final remarks.

2   Spectral Physics and Scale Invariance

The point of view that we will take here is that the main characteristics of the standard model coupled with gravity can be obtained from the study of the spectral properties of a suitable algebra of functions on spacetime (the fields) and a generalized Dirac operator. We will first review the main aspects of the spectral action, stressing the differences between the fermionic and bosonic parts, and then discuss scale invariance in this context.

2.1   The Spectral Action

The spectral action, in the spirit of noncommutative geometry, depends on the spectral data of the space, defined by a spectral triple. In the description of the standard model of [14, 15, 16, 17] the space is the tensor product of ordinary (Euclidean) spacetime by an inner space described by a finite dimensional matrix algebra. The algebra of this

extended spacetime acts as operators on an Hilbert space which comprises the fermions[*]. The metric properties of the space, as well as the differential structure and the action, depend on the operator D_0. This operator "fluctuates" with the addition of a Hermitian one-form which we will generically indicate with A and that can be expressed in terms of the algebra of functions which defines spacetime:

$$
\begin{aligned}
D &= D_0 + A \\
A &= \sum_i a_i[D_0, b_i]
\end{aligned}
\tag{2.1}
$$

with $a_i, b_i \in \mathcal{A}$. In the case of the standard model plus gravity the connection A comprises both the Levi-Civita and the gauge connections. In this case the geometry is the product of the continuous (commutative) spacetime times a noncommutative inner space described by a finite dimensional algebra, i.e.:

$$
\begin{aligned}
\mathcal{A} &= C(M) \otimes \mathcal{A}_F \\
\mathcal{A}_F &= \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})
\end{aligned}
\tag{2.2}
$$

where $C(M)$ is the algebra of continuous functions on spacetime M, $\mathbb{H}$ is the algebra of quaternions (whose unitary subgroup is $SU(2)$) and $M_3(\mathbb{C})$ is the algebra of 3×3 complex valued matrices. The unimodular (unitary and unit determinant) elements of $\mathcal{A}$ form the standard model group $U(1) \times SU(2) \times SU(3)$. This algebra is represented on the Hilbert space

$$
\mathcal{H} = L_2(sp(M)) \otimes \mathcal{H}_F
\tag{2.3}
$$

the tensor product of spinors on M, times a finite dimensional space which comprises all fermions, in three generations. Also the Dirac operator has two parts

$$
D_0 = D_M \otimes \mathbb{1} + \gamma_5 \otimes D_F
\tag{2.4}
$$

where D_M is the ordinary Dirac operator on M, and D_F is a finite matrix which carries the information of the values of the masses of the fermions and their Cabibbo mixings (including that of neutrinos).

Although the successes of the spectral action are obviously related to the standard model, in the following we will be more general, and our considerations will be valid for any spectral triple with a representation on the fermionic Hilbert. Given the ingredients of the triple, the spectral action comprises of two parts, one bosonic and one fermionic

$$
S = S_B + S_F = \operatorname{Tr} \chi \left(\frac{D^2}{\Lambda^2} \right) + \langle \psi | D \psi \rangle
\tag{2.5}
$$

---

[*]A projection may be necessary to avoid fermion doubling [18, 19, 17] but this is not essential at this stage. Likewise the real structure J and the chirality γ, which are otherwise crucial [15], play no role in this discussion.

where Tr is the usual operator trace, Λ is the energy cutoff of renormalization and χ is a positive function. Its particular shape is not essential as long as $\chi(0) = 1$ and $\chi(x) = 0$ for $x \gtrsim 1$. The fermionic part of the action is the usual integral over spacetime of the expectation value of the Dirac operator. The bosonic action contains the renormalization cutoff in its very definition, and therefore it must be considered in the Wilson renormalization scheme. On the other side the fermionic part is in general divergent and it must be renormalized as well.

The bosonic spectral action is a sum of residues [20] and can be expanded in a power series in terms of Λ^{-1} as

$$S_B = \sum_n f_n \, a_n(D^2/\Lambda^2) \tag{2.6}$$

where the f_n are the momenta of χ

$$
\begin{aligned}
f_0 &= \int_0^\infty \mathrm{d}x \, x\chi(x) \\
f_2 &= \int_0^\infty \mathrm{d}x \, \chi(x) \\
f_{2n+4} &= (-1)^n \partial_x^n \chi(x)\Big|_{x=0} \qquad n \geq 0
\end{aligned}
\tag{2.7}
$$

while the a_n are the Seeley-de Witt coefficients [20] which in this case vanish for n odd. We now give the form of the first three a's as functions of the terms of the square of the Dirac operator, using essentially the notations of [21] (see also [22]). Consider a D^2 of the form

$$D^2 = g^{\mu\nu}\partial_\mu\partial_\nu \mathbb{1} + \alpha^\mu\partial_\mu + \beta \tag{2.8}$$

then define

$$
\begin{aligned}
\omega_\mu &= \frac{1}{2}g_{\mu\nu}\left(\alpha^\nu + g^{\sigma\rho}\Gamma^\nu_{\sigma\rho}\mathbb{1}\right) \\
\Omega_{\mu\nu} &= \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu,\omega_\nu] \\
E &= \beta - g^{\mu\nu}\left(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu - \Gamma^\rho_{\mu\nu}\omega_\rho\right)
\end{aligned}
\tag{2.9}
$$

then

$$
\begin{aligned}
a_0 &= \frac{\Lambda^4}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,\mathbb{1}_F \\
a_2 &= \frac{\Lambda^2}{16\pi^2}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\left(-\frac{R}{6}+E\right) \\
a_4 &= \frac{1}{16\pi^2}\frac{1}{360}\int \mathrm{d}x^4\sqrt{g}\,\mathrm{tr}\,(-12\nabla^\mu\nabla_\mu R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} \\
&\qquad +2R_{\mu\nu\sigma\rho}R^{\mu\nu\sigma\rho} - 60RE + 180E^2 + 60\nabla^\mu\nabla_\mu E + 30\Omega_{\mu\nu}\Omega^{\mu\nu})
\end{aligned}
\tag{2.10}
$$

where by tr we indicate the trace over the inner indices of the finite algebra $\mathcal{A}_F$.

The action with the spectral triple described by the data (2.2)-(2.4) reproduces correctly [17] the standard model coupled with gravity and it has predictive power in relation to the Higgs mass for example, and it has been applied recently to cosmology as well [23, 24]. We refrain to write in full glory all of the terms of the action which takes a full page and can be found in [17, Sect. 4.1].

However the action is basically a classical quantity, and the renormalization is performed, especially in the fermionic sector, using standard field theory techniques. The model has the three coupling constants equal at the renormalization point, as is the case of $SU(5)$ non-supersymmetric unification, and hence some of the predictions are similar to the ones of the this theory.

2.2 The different nature of the Bosonic and Fermionic Actions

As they stand the bosonic and the fermionic parts of the action (2.5) are very different. The bosonic one is always finite and it depends on the cutoff Λ. It is the usual trace of an operator and it does not diverge because of the presence of the function χ which regularizes. In the case of χ being the characteristic function of the interval, i.e.

$$\chi(x) = \begin{cases} 0 & x < 0 \\ 1 & x \in [0,1] \\ 0 & x > 1 \end{cases} \tag{2.11}$$

or a smooth version of it, the bosonic spectral action simply counts the eigenvalues of D which are less than the cutoff Λ.

The fermionic action on the contrary is divergent, and will require renormalization. It is formulated as an usual integral, which in this context (in four dimensions) is the Dixmier trace:

$$\int \mathrm{d}x f = \mathrm{Tr}_\omega |D|^{-4} f \tag{2.12}$$

where the Dixmier trace of an operator O with eigenvalues o_n (ordered in decreasing order, repeated in case of degeneracy) is:

$$\mathrm{Tr}_\omega O = \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=0}^{N} o_n \tag{2.13}$$

The integral/Dixmier trace has to be regularized. Since the action is written as an usual integral, the renormalization analysis can be done in a variety of ways. In this process however some quantum symmetries can be lost, and the theory can develop an anomaly. In Sect. 4.1 we will use a regularization which is rather similar to the one used for the fermionic part. It remains the fact that the different treatment of the two parts of the action seems *ad hoc*, and it would be desirable to have them to be part of a more uniform approach.

2.3 Scale Invariance in the Spectral Action

The standard model is classically invariant against scale transformations if one ignores the mass terms, which can be done at high energy. The lack of full invariance can also be compensated by the introduction of a *dilaton* field, or by giving nontrivial transformation properties to the masses under a scale transformations. In this paper will only discuss the case of a global rescaling by a constant parameter.

We want to define our theory to be invariant under rescaling defined as

$$
\begin{aligned}
x^\mu &\rightarrow e^\phi x^\mu \\
\psi &\rightarrow e^{-\frac{3}{2}\phi}\psi \\
D &\rightarrow e^{-\frac{1}{2}\phi}De^{-\frac{1}{2}\phi}
\end{aligned}
\tag{2.14}
$$

where for the scope of this paper e^ϕ is a constant real parameter. In future we hope to discuss the case of ϕ being a (dilaton) field.

Note that since the rescaling involves also the matrix part of D, we must also rescale the masses of the fermions. This is tantamount to a change of the unit of measurement and, in the absence of a dimensional scale, is an exact symmetry of the classical theory. This classical symmetry can however develop an anomaly, namely not be a symmetry of the (renormalized) quantum theory anymore. In the next section we will discuss the presence of an anomaly due to the breaking of this symmetry.

3 Anomalies

In the present context we have an anomaly: a classical theory is invariant for a symmetry, but the quantum theory, due to unavoidable regularization, does not possess this symmetry anymore. If also the quantum theory is required to be symmetric then the symmetry can be restored by the addition of extra terms in the action. A textbook introduction to anomalies can be found in [25].

As explained in the previous chapter, the notion of scale anomaly is attached to the dilatation of both coordinates, fields and mass-like parameters according to their dimensionalities, Eq. (2.14). Evidently, in the absence of UV divergences, there is no scale anomaly which therefore can be correlated to rescaling of a cutoff in the theory. In general the dilatation need not be constant, and the quantum field corresponding is called the *dilaton*.

There is the adjacent notion of Weyl or conformal anomaly, which is closely related. It is based on the symmetry against local Weyl dilatation of the metric accompanied by an appropriate transformation of the dilaton field dressing all mass-like vertices in order

to make homogeneous the entire transformation of the Dirac Lagrangian, that is

$$
\begin{aligned}
g^{\mu\nu} &\to \mathrm{e}^{2\alpha} g^{\mu\nu} \\
\psi &\to \mathrm{e}^{-\frac{3}{2}\alpha}\psi \\
D &\to \mathrm{e}^{-\frac{1}{2}\alpha}D\mathrm{e}^{-\frac{1}{2}\alpha}
\end{aligned}
\tag{3.1}
$$

while x^μ is left untouched and in this case α is local function of x. Scale and Weyl anomalies are closely and directly related.

In the functional integral the proper measure to use is the sum over all configurations of $\tilde{\psi} = (-g)^{1/4}\psi$ and $\tilde{\bar{\psi}} = (-g)^{1/4}\bar{\psi}$, which we will indicate as $[\mathrm{d}\psi][\mathrm{d}\bar{\psi}]$, the partition function (which we define below) is formally invariant for the scale (or Weyl) transformation, but the regularization procedure spoils this formal invariance, giving rise to the anomaly.

In spite of the fact that the generator of Weyl dilatation is localized, the transformation of quantum action reveals an anomalous breaking of the symmetry (see the history in [26]). The reason is that the Dirac operator is unbounded whereas any local transformation of fields and/or operators is singular as an integral operator, and they don't commute. When calculating the determinant of the product of Dirac operator and its local Weyl transformation one cannot just factorize it to prove the essential invariance of the fermionic quantum action, first one has to make the product finite and therefore perform a regularization. As the above mentioned operators don't commute their regularization may entail non-factorizability - a non-commutative residue [27], which can be interpreted as a conformal non-invariance of the measure in the path integral approach [28]. Symbolically one can present the anomalous action for fermions as,

$$
||\mathrm{e}^{-\frac{1}{2}\phi}D\mathrm{e}^{-\frac{1}{2}\phi}||_{Reg} = ||\mathrm{e}^{-\phi}||_{Reg} \times ||D||_{Reg} \times \exp(-S_{anom}(\text{external fields})).
\tag{3.2}
$$

In the next section we will apply this procedure in the concrete example of the spectral action.

4   Bosonic Action from Scale Anomaly for Fermions

In this section, which forms the central part of the paper, we argue that the bosonic part of the action can be seen as emerging naturally from the regulated fermionic action as the term necessary to compensate the scale anomaly.

Although most of discussion about the renormalization of the spectral action (2.5) has been concentrated on its bosonic part, here we start from the fermionic action which for the purposes of this section we write as

$$
S_\psi = \int \mathrm{d}x\bar{\psi}D\psi
\tag{4.1}
$$

In the following we will analyze its *quantum* behaviour under scale transformations.

4.1 Regularization of the Fermionic Action

The action (4.1) appears in the partition function of the theory:

$$Z(D) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}]e^{-S_\psi} = \det(D) \times \mathrm{const},\qquad(4.2)$$

where the last equality is of course just formal because the expression is divergent and needs regularizing. The writing of the fermionic action in this form (as a Pfaffian) is instrumental in the solution of the fermion doubling problem [18, 17].

The regularization can be done in several ways but in the spirit of noncommutative geometry and the spectral action the most natural one is a truncation of the spectrum of the Dirac operator. This regularization scheme has been introduced by one of us together with L. Bonora and R. Gamboa-Saravi in [8, 9, 10]. The energy cutoff is enforced by considering only the first N eigenvalues of D. Consider the projector

$$P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n| \qquad(4.3)$$

where λ_n are the eigenvalues of D in increasing order (repeated according to possible multiplicities), and $|\lambda_n\rangle$ a corresponding orthonormal basis. The integer N is a function of the cutoff and is defined as

$$N = \max n \text{ such that } \lambda_n \leq \Lambda \qquad(4.4)$$

This means that we are effectively using the Nth eigenvalue as cutoff.

We define the regularized partition function[†]

$$Z_\Lambda(D) = \prod_{n=0}^{N} \lambda_n = \det\left(\mathbb{1} - P_N + P_N\frac{D}{\Lambda}P_N\right) \qquad(4.5)$$

In this way we can define the fermionic action in an intrinsic way, without reference to the Dixmier trace (integral) in a formulation which is purely spectral.

The regularized partition function Z_Λ has a well defined meaning. Expressing ψ and $\bar{\psi}$ as

$$\psi = \sum_{n=0}^{\infty} a_n |\lambda_n\rangle$$
$$\bar{\psi} = \sum_{n=0}^{\infty} b_n |\lambda_n\rangle \qquad(4.6)$$

[†]Although P_N commutes with D we prefer to use a more symmetric notation.

with a_n and b_n anticommuting (Grassman) quantities. Then Z_Λ becomes (performing the integration over Grassman variables for the last step)

$$Z_\Lambda(D) = \int \prod_{n=0}^{N} \mathrm{d}a_n \mathrm{d}b_n \mathrm{e}^{-\sum_{n=0}^{N} b_n \frac{\lambda_n}{\Lambda} a_n} = \det(D_N) \tag{4.7}$$

where we defined
$$D_N = 1 - P_N + P_N \frac{D}{\Lambda} P_N. \tag{4.8}$$

In the basis in which D/Λ is diagonal it corresponds to set to 1 all eigenvalues larger than 1. Note that D_N is dimensionless and depends on Λ both explicitly and intrinsically via the dependence of N and P_N.

Since P_N commutes with D. It is possible to give an explicit functional expression to the projector in terms of the cutoff:

$$P_N = \Theta\left(1 - \frac{D^2}{\Lambda^2}\right) = \int \mathrm{d}\alpha \frac{1}{2\pi\mathrm{i}(\alpha - \mathrm{i}\epsilon)} \mathrm{e}^{\mathrm{i}\alpha\left(1 - \frac{D^2}{\Lambda^2}\right)} \tag{4.9}$$

where Θ is the Heaviside step function.

4.2   Cancelation of the Anomaly and the Bosonic Action

The regulated determinant is not invariant under scale transformation, and we are in the case of (3.2). Accordingly the regulated partition function develops an anomaly. We have therefore to add another term to the action which will cancel this anomaly.

The action S_F is invariant under (2.14) but the partition function is not, thus we need to add another term to the action to compensate this lack of invariance at the quantum level. This calculation has been performed in [29] in the QCD context, and applied to gravity in [13].

Let us see in a very heuristic way with ϕ constant why the effective action S_{eff} is nothing but the spectral action with the function χ being a sharp cutoff. In this case N is just a number of eigenvalues smaller that Λ, and thereby

$$\mathrm{Tr}\,\chi\left(\frac{D^2}{\Lambda^2}\right) = \mathrm{Tr}\,P_N = N \tag{4.10}$$

It is worth recalling again that the integer N depends on the cutoff Λ, on the Dirac operator D and also on the function χ which we have chosen to be a sharp cutoff.

Then the compensating term – the effective action, will be defined by

$$Z_{\mathrm{inv}\Lambda}(D) = Z_\Lambda(D) \int \mathrm{d}\phi\, \mathrm{e}^{-S_{\mathrm{anom}}} \tag{4.11}$$

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where the effective action will be depending on N and hence the cutoff Λ and on ϕ. Define

$$Z_{\mathrm{inv}\Lambda}(D) = \int \mathrm{d}\phi Z_{\Lambda}(\mathrm{e}^{-\frac{1}{2}\phi}D\mathrm{e}^{-\frac{1}{2}\phi}) \tag{4.12}$$

then

$$S_{\mathrm{anom}} = \log Z_{\Lambda}^{-1}(D)Z_{\Lambda}(\mathrm{e}^{-\frac{1}{2}\phi}D\mathrm{e}^{-\frac{1}{2}\phi}) \tag{4.13}$$

Let us designate

$$Z_t = Z_{\Lambda}(\mathrm{e}^{-\frac{t}{2}\phi}D\mathrm{e}^{-\frac{t}{2}\phi}) \tag{4.14}$$

therefore $Z_0 = Z_{\Lambda}(D)$ and

$$Z_{\mathrm{inv}N}(D)Z_{\Lambda}^{-1}(D) = \int \mathrm{d}\phi\frac{Z_1}{Z_0} \tag{4.15}$$

and hence

$$S_{\mathrm{eff}} = -\int_0^1 \mathrm{d}t\partial_t \log Z_t = -\int_0^1 \mathrm{d}t\frac{\partial_t Z_t}{Z_t} \tag{4.16}$$

We have the following relation that can easily proven:

$$D_N^{-1} = (1 - P_N + P_N D P_N)^{-1} = 1 - P_N + P_N D^{-1} P_N \tag{4.17}$$

and

$$
\begin{aligned}
\partial_t Z_t &= \partial_t \det(\mathrm{e}^{-\frac{t}{2}\phi}D\mathrm{e}^{-\frac{t}{2}\phi})_N \\
&= \partial_t \mathrm{e}^{\mathrm{tr}\,\log(1-P_N+\mathrm{e}^{-\frac{t}{2}\phi}D_N\mathrm{e}^{-\frac{t}{2}\phi})} \\
&= \mathrm{Tr}\,(\partial_t \log(1 - P_N + \mathrm{e}^{-\frac{t}{2}}D_N\mathrm{e}^{-\frac{t}{2}\phi})Z_t \\
&= -\mathrm{Tr}\,((1 - P_N + \mathrm{e}^{-\frac{t}{2}\phi}D_N\mathrm{e}^{-\frac{t}{2}\phi})^{-1}\phi\mathrm{e}^{-\frac{t}{2}\phi}D_N\mathrm{e}^{-\frac{t}{2}\phi})Z_t \\
&= -\phi Z_t\,\mathrm{tr}\,P_N
\end{aligned}
\tag{4.18}
$$

and therefore

$$S_{\mathrm{anom}} = \int_0^1 \mathrm{d}t\,\phi\,\mathrm{tr}\,P_N \tag{4.19}$$

which is indeed a structure very similar to the spectral action in the case χ as in (2.11).

5   The scale invariant Spectral Action

The calculations of the modified spectral action are very similar to the ones for the regular spectral action and were done by Chamseddine and Connes in [21] for the more general case of a x dependent ϕ. We can read the modifications to the spectral action from their work simply setting to zero the derivatives of ϕ and then carrying out the integral in (4.19).

The rescaled action with the new Dirac operator, in this case of constant rescaling, gives just a correction of the Seeley-de Witt coefficient of a very simple kind

$$a_n \to a'_n = e^{(4-n)\phi} a_n \tag{5.20}$$

while the coefficients f_n in (2.7) for the case of a χ the characteristic function of the interval are:

$$f_0 = \frac{1}{2} \ ; \ f_2 = 1 \ ; \ f_4 = 1 \ ; f_n = 0, \ n > 4 \tag{5.21}$$

The fermionic action remains invariant.

We can now perform easily the integral in t of (4.19) noting that t appears always together with ϕ, therefore with the change of variables $t' = \phi t$ we have that

$$S_{\text{anom}} = \int_0^\phi dt' \sum_n e^{(4-n)t'} a_n f_n = \frac{1}{8}(e^{4\phi} - 1)a_0 + \frac{1}{2}(e^{2\phi} - 1)a_2 + \phi a_4 \tag{5.22}$$

A different cutoff function will give some slightly different coefficient with the appearance of higher Seeley-de Witt coefficient. We see that the changes from the spectral action are rather small, the constant ϕ appears in multiplicative factors. It will play a role in the full renormalized theory where the dependence on ϕ can be eliminated at the expense of the fundamental scale Λ, given the phenomenological input of the cosmological constant and the electroweak scale. We leave this to another project.

6 Final Remarks

There are two obvious directions of development of the ideas of this paper. On one side one can apply this to the detailed spectral action for the standard model coupled with gravity [17]. Since the structure of that spectral action is very similar to the one discussed here, we expect the same coefficients to appear, but we have not checked this. The second development is the gauging of the symmetry, i.e. consider ϕ to be a dilaton field. This dilaton may play an important role in the inflationary epoch and have a role for the solution of hierarchy problem [30, 31, 32]. In this case however the field would not commute with D, and in particular we would have that $e^{-\frac{1}{2}\phi} D^2 e^{-\frac{1}{2}\phi} \neq (e^{-\frac{1}{2}\phi} D e^{-\frac{1}{2}\phi})^2$ and this will change things. Moreover terms with derivatives of ϕ would appear, like the kinetic term for the dilaton, and therefore the details of the calculations will change, causing probably changes in the coefficients of the expansion. The conceptual framework will however remain unchanged.

We have seen how the cancelation of anomalies induces the spectral action, and hence gravity at a quantum level. We have used only global scale invariance (and a rescaling of the masses), in other words the statement of the invariance of the theory is just invariance

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under a change of the unit of measurement. This is symmetry is classically exact, but the presence of a cutoff scale spoils it. What is interesting from the point of view of noncommutative geometry is that this scheme favors a sort of "fermion predominance", i.e. the natural fundamental fields are the fermions, moving in a fixed background, which is fixed since the action does not contain the terms for the self-interaction of the gauge and gravitational degrees of freedom. But quantization, and the ensuing anomaly, induce the spectral action, which contains the gauge and gravitational interaction. In some sense *matter was created before light!*

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