ON THE CONSTRUCTION OF SUITABLE WEAK SOLUTIONS TO THE 3D NAVIER-STOKES EQUATIONS IN A BOUNDED DOMAIN BY AN ARTIFICIAL COMPRESSIBILITY METHOD

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Abstract. In this paper we will prove that suitable weak solutions of three dimensional Navier-Stokes equations in bounded domain can be constructed by a particular type of artificial compressibility approximation.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\Gamma = \partial\Omega$ and $T > 0$ be a fixed real number. We consider the three-dimensional Navier-Stokes equations with unit viscosity and zero external force:

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla)\, u + \nabla p = 0 \qquad & \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0 \qquad & \text{in } \Omega \times (0,T), \end{aligned} \tag{1.1}$$

and the Navier (slip without friction) boundary conditions for u, namely

$$\begin{aligned} u \cdot n = 0 \qquad & \text{on } \Gamma \times (0,T), \\ n \cdot \mathrm{D}u \cdot \tau = 0 \qquad & \text{on } \Gamma \times (0,T), \end{aligned} \tag{1.2}$$

where $\mathrm{D}u$ stands for the symmetric part of ∇u, n is the unit normal vector on Γ, while τ denotes any unit tangential vector on Γ (Recall also that for incompressible fluids $\Delta u = 2 \operatorname{div} \mathrm{D}u$, to compare with (1.5) in the compressible approximation).

The system (1.1) is coupled with a divergence-free and tangential to the boundary initial datum

$$u(x,0) = u_0(x) \qquad \text{on } \Omega \times \{t = 0\}. \tag{1.3}$$

For the initial value boundary problem (1.1)-(1.2) it is well-known that, for any tangential and divergence-free vector field $u_0 \in L^2(\Omega)$, there exists at least a global weak solution in the sense of Leray-Hopf, see [1, 6, 7, 10, 30, 33] for the generalization of the classical results of Hopf [23], which was obtained for the Dirichlet case.

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Remark 1.1. In the sequel it will be enough to assume that $\Gamma \in C^{1,1}$ and, to avoid technical complications, we also assume that the domain is simply connected and that it cannot be generated by revolution around a given axis (this could be relevant in the steady case, when dealing with the symmetric deformation tensor and Korn type inequalities), see [5].

On the other hand, the uniqueness and the regularity of the weak solutions represents a problem still open and very far to be understood. The best regularity result which is available for weak solutions of the three-dimensional Navier-Stokes equations is the Caffarelli-Kohn-Nirenberg theorem [11], which asserts that the velocity is smooth out of a set of parabolic Hausdorff dimension zero. For example it implies that there are not space-time curves of singularity in the velocity. However, the Caffarelli-Kohn-Nirenberg theorem holds only for a particular subclass of weak solutions, called in literature "*suitable weak solutions*," see Section 2 for the precise definitions. Roughly speaking a suitable weak solution is a particular Leray-Hopf weak solution which, in addition to the global energy inequality, satisfies in the sense of distributions also the following entropy-type inequality:

$$\partial_t\left(\frac{1}{2}|u|^2\right)+\nabla\cdot\left(\left(\frac{1}{2}|u|^2+p\right)u\right)-\Delta\left(\frac{1}{2}|u|^2\right)+|\nabla u|^2\leq 0. \tag{1.4}$$

Inequality (1.4) is often called in literature *generalized energy inequality* and is the main tool to prove the partial regularity theorems [11, 25]. Since uniqueness of Leray-Hopf weak solutions (within the same class of solutions) is not known, each method used to prove existence of weak solutions may possibly lead to a different weak solution. This in turn implies that it is a very interesting problem to understand which ones of the different approximation methods (in particular those important in applications and in the construction of numerical solutions), provide existence of suitable weak solutions. This question has been considered for many approximation methods [2, 3, 4, 9], and it is worth to point out that the solutions constructed by the Leray method [26] and the Leray-α variant turn out to be suitable. Particularly important are the results obtained by Guermond [20, 21] where it was proved that some special Galerkin methods lead to suitable weak solutions. However, understanding whether solutions obtained by general Galerkin methods (especially those ones based on Fourier series expansion) are suitable has been posed in [3] and it is still not completely solved.

In the spirit of understanding whether certain methods used in the numerical approximation produce suitable weak solutions (problem emphasized in [22]) in this note we prove that weak solutions obtained as a limit of a particular *artificial compressibility* method are so. More precisely, the approximation system we consider is the following:

$$\begin{aligned} \partial_t u^\varepsilon + (u^\varepsilon\cdot\nabla)\,u^\varepsilon + \frac{1}{2}u^\varepsilon \operatorname{div} u^\varepsilon - 2\operatorname{div}\mathrm{D}u^\varepsilon + \nabla p^\varepsilon = 0 \qquad & \text{in } \Omega\times(0,T),\\ -\varepsilon\,\Delta p^\varepsilon + \operatorname{div} u^\varepsilon = 0 \qquad & \text{in } \Omega\times(0,T), \end{aligned} \tag{1.5}$$

is coupled with Navier boundary conditions

$$\begin{aligned} u^\varepsilon\cdot n = 0 \qquad & \text{on } \Gamma\times(0,T),\\ n\cdot \mathrm{D}u^\varepsilon\cdot\tau = 0 \qquad & \text{on } \Gamma\times(0,T), \end{aligned} \tag{1.6}$$

and a tangential to the boundary initial datum

$$u^\varepsilon(x,0) = u_0^\varepsilon(x) \qquad \text{on } \Omega\times\{t=0\}. \tag{1.7}$$

Concerning the pressure, it is natural to impose on it Neumann boundary conditions,

$$\frac{\partial p^\varepsilon}{\partial n} = 0 \qquad \text{on } \Gamma\times(0,T), \tag{1.8}$$

and normalization by zero average over the domain:

$$\int_\Omega p^\varepsilon(x,t)\;dx = 0 \qquad \text{a.e. } t\in(0,T). \tag{1.9}$$

Remark 1.2. The nonlinear terms in (1.5) can be also be written in the following equivalent divergence-type form

$$nl(u^\varepsilon,u^\varepsilon) := (u^\varepsilon\cdot\nabla)\,u^\varepsilon + \frac{1}{2}\,u^\varepsilon\operatorname{div}u^\varepsilon = \operatorname{div}(u^\varepsilon\otimes u^\varepsilon) - \frac{1}{2}\,u^\varepsilon\operatorname{div}u^\varepsilon, \tag{1.10}$$

where as usual $(a\otimes b)_{ij} := a_i b_j$, for $i,j = 1,2,3$. The factor $\frac{1}{2}\,u^\varepsilon\operatorname{div}u^\varepsilon$, which vanishes in the incompressible limit, is needed to keep the usual energy estimates.

We recall that artificial compressibility methods were first studied by Chorin [13, 14] and Temam [31] and are relevant in numerical analysis since they relax the divergence-free constraint, which has a high computational cost. See also the recent results in [28] for the Cauchy problem with $-\varepsilon\,\Delta p^\varepsilon$ replaced by $\varepsilon\,p^\varepsilon$ (as in [5]) in equations (1.5). In addition, the case where the term $-\varepsilon\,\Delta p^\varepsilon$ in equations (1.5) is replaced by $\varepsilon\,\partial_t p^\varepsilon$ has been considered by different authors, see [32, 15]. Moreover, the convergence when $\varepsilon\to 0$ to a suitable weak solution in the whole space was considered in [16] for the scheme with the time derivative of the pressure.

Here, we focus especially on the fact that we have a domain with solid boundaries and the coupling with the Navier boundary conditions makes possible to obtain appropriate estimates on the pressure. The main theorem of this note is the following. See Section 2 for the notations and the main definitions.

Theorem 1.3. *Let Ω be bounded and of class $C^{1,1}$. Let $\{(u^\varepsilon, p^\varepsilon)\}_\varepsilon$ be a sequence of weak solutions of the initial value boundary problem* (1.5)-(1.8) *with*

$$\operatorname{div} u_0^\varepsilon = 0 \qquad \text{in } \Omega,$$

and $\{u_0^\varepsilon\}_\varepsilon$ bounded uniformly in $H^1_\tau(\Omega)$ such that

$$u_0^\varepsilon \to u_0 \qquad \text{strongly in } L^2(\Omega).$$

Then, the following ε-independent estimate on the pressure holds true

$$\exists\, C > 0 : \qquad \|p^\varepsilon\|_{L^{\frac{5}{3}}((0,T)\times\Omega)} \leq C, \qquad \forall\, \varepsilon > 0. \tag{1.11}$$

Moreover, there exists u, a Leray-Hopf weak solution of (1.1)-(1.2)*, and an associated pressure $p \in L^{\frac{5}{3}}((0,T)\times\Omega)$ such that*

$$u^\varepsilon \overset{*}{\rightharpoonup} u \qquad \text{weakly* in } L^\infty(0,T;L^2(\Omega)), \tag{1.12}$$

$$\nabla u^\varepsilon \rightharpoonup \nabla u \qquad \text{weakly in } L^\infty(0,T;L^2(\Omega)), \tag{1.13}$$

$$p^\varepsilon \rightharpoonup p \qquad \text{weakly in } L^{\frac{5}{3}}((0,T)\times\Omega). \tag{1.14}$$

Finally, (u,p) is a suitable weak solution to the Navier-Stokes equations (1.1)-(1.2)-(1.3).

The existence of weak solutions of (1.5)-(1.8) is quite standard and can be easily proved by smoothing suitably the non linear terms. The main obstacle to prove Theorem 1.3 is to get ε-independent estimates on the pressure. Since we do not have the divergence-free constraint we cannot deduce independent estimates on the pressure by using the classical elliptic equations associated to the pressure. Moreover, also methods based on the semigroup theory as in [29] seem not directly working here, since the approximation system does not immediately fit in that abstract framework. We will get the necessary a priori estimates on the pressure from the momentum equations as in the case of compressible Navier-Stokes equations [17, 27]. This approach has been introduced in [10] to study a class of non-Newtonian fluids and used also in [12] to address the analysis of models for the evolution of the turbulent kinetic energy, but without considering the question of the local energy inequality. In particular, we point out that it is in the pressure estimate that we need to employ the Navier boundary conditions, since they allow us to control the various term arising in the integration by parts.

Plan of the paper. The plan of the paper is the following. In Section 2 we recall the main definitions and the main tools we will use in the proof of Theorem 1.3. In Section 3 we prove the main a priori estimates needed in the proof of Theorem 1.3 and finally in Section 4 we prove Theorem 1.3.

2. Preliminaries

In this section we fix the notations and we recall the main definitions and tools we will use to prove Theorem 1.3. Given $\Omega \subset \mathbb{R}^3$, the space of C^∞ functions or vector fields on Ω tangential to the boundary will be denoted by $C^\infty_\tau(\Omega)$. We will denote with $L^p(\Omega)$ the standard Lebesgue spaces and with $\|\cdot\|_p$ their norm. The classical Sobolev space is denoted by $W^{1,2}(\Omega)$ and its norm by $\|\cdot\|_{k,p}$ and when $k=1$ and $p=2$ we denote $W^{k,p}(\Omega)$ with $H^1(\Omega)$. As usual we denote the L^2-scalar product by $(\,.\,,\,.\,)$. Finally, we denote by $L^2_{\sigma,\tau}(\Omega)$ and $H^1_{\sigma,\tau}(\Omega)$ the space of the tangential to the boundary and divergence-free vector fields respectively in $L^2(\Omega)$ and $H^1(\Omega)$. By "$\cdot$" we denote the scalar product between vectors, while by "$:$" we denote the complete contraction of second order tensors. Finally, given a Banach space we denote by $C_w([0,T];X)$ the space of continuous function from the interval $[0,T]$ to the space X endowed with the weak topology.

Let us start by giving the precise definition of Leray-Hopf weak solution for the initial value boundary problem (1.1)-(1.2).

Definition 2.1. *A vector field u is a Leray-Hopf weak solution to the Navier-Stokes equations* (1.1)-(1.2)-(1.3) *if the following proprieties hold true:*
1) The velocity u satisfies $u \in C_w(0,T;L^2_{\sigma,\tau}(\Omega)) \cap L^2(0,T;H^1_{\sigma,\tau}(\Omega))$;
2) The velocity u satisfies the following integral identity

$$\int_0^T \int_\Omega \Big(u\cdot\partial_t\phi - \nabla u : \nabla\phi - (u\cdot\nabla)\,u\cdot\phi\Big)\,dxds - \int_0^t\int_\Gamma u\cdot\nabla n\cdot\phi\,dSds = -\int_\Omega u_0\cdot\phi(0)\,dx,$$

for all smooth vector fields ϕ, divergence-free, tangential to the boundary, and such that $\phi(T)=0$;
3) The velocity u satisfies the global energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\,ds - \int_0^t\int_\Gamma u\cdot\nabla n\cdot u\,dSds \le \frac{1}{2}\|u_0\|_2^2. \tag{2.1}$$

Remark 2.2. In the above definition and in the sequel, we recall that

$$\int_\Gamma u\cdot\nabla n\cdot u\,dS := \int_\Gamma \sum_{i,j=1}^3 u_i u_j \frac{\partial n_i}{\partial x_j}\,dS.$$

and we also recall that the boundary condition (1.2) are encoded in the weak formulation, see also [24].

The next definition we recall is that of suitable weak solutions.

Definition 2.3. *A pair (u,p) is a suitable weak solution to the Navier-Stokes equations* (1.1)-(1.2)-(1.3) *if the following properties hold true:*
1) u is a Leray-Hopf weak solution with associated a pressure p such that

$$p \in L^{\frac{5}{3}}((0,T)\times\Omega); \tag{2.2}$$

2) The generalized energy inequality holds true

$$\begin{aligned}\int_0^T\int_\Omega |\nabla u|^2\phi\,dxds \leq \int_0^T\int_\Omega &\frac{|u^\varepsilon|^2}{2}\big(\phi_t+\Delta\phi\big)\,dxds\\ &+\Big(u^\varepsilon\frac{|u^\varepsilon|^2}{2}+p^\varepsilon u^\varepsilon\Big)\cdot\nabla\phi\,dxds,\end{aligned}\tag{2.3}$$

for all non-negative $\phi\in C_c^\infty((0,T)\times\Omega)$.

We want to point out that the real difference between the Leray-Hopf weak solutions and the suitable weak solutions relies in the local energy inequality. Indeed, it is always possible to associate to a Leray-Hopf weak solutions u a pressure p (modulo arbitrary functions of time) which belongs to the space $L^{\frac{5}{3}}((0,T)\times\Omega)$. This is standard in a setting without physical boundaries by using the Stokes system. In the case of Dirichlet boundary conditions this was first proved by Sohr and von Wahl in [29]. Finally, in the case of Navier boundary conditions this can be deduced again by the elliptic equations associated to the pressure p, see [8]. On the other hand, it is not known how the deduce the generalized energy inequality (1.4) since the regularity of a Leray-Hopf weak solution is not enough to justify the chain rule in the time derivative term and in the non linear term.

We pass now to the precise definitions concerning the compressible approximation we will consider.

Definition 2.4. *The couple* $(u^\varepsilon,p^\varepsilon)$ *is a weak solution to the compressible approximation* (1.5)-(1.6)-(1.7)-(1.8)-(1.9) *if:*
1) The velocity and pressure $(u^\varepsilon,p^\varepsilon)$ *satisfy*

$$\begin{aligned}&u^\varepsilon\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1_\tau(\Omega)),\\ &\sqrt{\varepsilon}\,\nabla p^\varepsilon\in L^2((0,T)\times\Omega);\end{aligned}$$

2) The velocity and pressure $(u^\varepsilon,p^\varepsilon)$ *satisfy the following integral identity*

$$\begin{aligned}\int_0^T\int_\Omega\Big[-u^\varepsilon\cdot\partial_t\phi+nl(u^\varepsilon,u^\varepsilon)\cdot\phi+2\mathrm{D}u^\varepsilon:\mathrm{D}\phi+\nabla p^\varepsilon\cdot\phi\\ +\varepsilon\,\nabla p^\varepsilon\cdot\nabla\psi+\operatorname{div}u^\varepsilon\,\psi\Big]\,dxds=\int_\Omega u_0^\varepsilon\cdot\phi(0)\,dx,\end{aligned}$$

for all smooth vector fields ϕ *tangential to the boundary, such that* $\phi(T)=0$ *and for all smooth scalar fields* ψ *with zero mean value;*
3) The velocity and pressure $(u^\varepsilon,p^\varepsilon)$ *satisfy the global energy inequality*

$$\frac{1}{2}\|u^\varepsilon(t)\|^2+\int_0^t\Big[2\,\|\mathrm{D}u^\varepsilon\|^2+\varepsilon\,\|\nabla p^\varepsilon\|^2\Big]\,ds\leq\frac{1}{2}\|u_0^\varepsilon\|^2\qquad\forall\,t\geq 0;\tag{2.4}$$

4) The velocity and pressure $(u^\varepsilon, p^\varepsilon)$ satisfy the local energy inequality

$$
\begin{aligned}
&\int_0^T\int_\Omega \Big(2|\mathrm{D}u^\varepsilon|^2+\varepsilon|\nabla p^\varepsilon|^2\Big)\phi\,dxds \le \int_0^T\int_\Omega \frac{|u^\varepsilon|^2}{2}\big(\phi_t+\Delta\phi\big)\,dxds\\
&+\int_0^T\int_\Omega \Big(u^\varepsilon\frac{|u^\varepsilon|^2}{2}+p^\varepsilon u^\varepsilon-\varepsilon\, p^\varepsilon\nabla p^\varepsilon+u^\varepsilon\ \operatorname{div}u^\varepsilon\Big)\cdot\nabla\phi\,dxds \qquad (2.5)\\
&\qquad+\int_0^T\int_\Omega u\otimes u:\nabla\phi\,dxds,
\end{aligned}
$$

for all non-negative $\phi\in C_c^\infty((0,T)\times\Omega)$.

Remark 2.5. The global and local energy estimates can be rewritten in a more useful form by performing some integration by parts. Observe that for any $v\in H^1_\tau(\Omega)$ it holds (cf. [5, Sec. 2]) that

$$
2\int_\Omega \mathrm{D}v:\mathrm{D}v\,dx=\|\nabla v\|^2+\|\operatorname{div}v\|^2-\int_\Gamma v\cdot\nabla n\cdot v\,dS.
$$

Moreover, by a standard compactness argument (see [5, Lem. 2.3]), it follows that there exists $c=c(\Omega)$ such that

$$
2\int_\Omega \mathrm{D}v:\mathrm{D}v\,dx\ge c\|\nabla v\|^2 \qquad \forall\, v\in H^1_\tau(\Omega),
$$

and the energy inequality (2.4) can be rewritten either as follows

$$
\begin{aligned}
\frac12\|u^\varepsilon(t)\|^2+\int_0^t\Big[\|\nabla u^\varepsilon\|^2+\|\operatorname{div}u^\varepsilon\|^2+\varepsilon\|\nabla p^\varepsilon\|^2&\\
-\int_\Gamma u^\varepsilon\cdot\nabla n\cdot u^\varepsilon\,dS\Big]\,ds\le\frac12\|u_0^\varepsilon\|^2&,
\end{aligned}
\qquad (2.6)
$$

or even as

$$
\frac12\|u^\varepsilon(t)\|^2+\int_0^t\Big[c\|\nabla u^\varepsilon\|^2+\varepsilon\|\nabla p^\varepsilon\|^2\Big]\,ds\le\frac12\|u_0^\varepsilon\|^2. \qquad (2.7)
$$

Concerning the local energy inequality (2.5), with several integration by parts we get the following equivalent formulation (which is more similar to the standard one for the incompressible case),

$$
\begin{aligned}
&\int_0^T\int_\Omega\Big(|\nabla u^\varepsilon|^2+|\operatorname{div}u^\varepsilon|^2+\varepsilon|\nabla p^\varepsilon|^2\Big)\phi\,dxds\le\int_0^T\int_\Omega\frac{|u^\varepsilon|^2}{2}(\phi_t+\Delta\phi)\,dxds\\
&+\int_0^T\int_\Omega\Big(u^\varepsilon\frac{|u^\varepsilon|^2}{2}+p^\varepsilon u^\varepsilon-\varepsilon\,p^\varepsilon\nabla p^\varepsilon-u^\varepsilon\ \operatorname{div}u^\varepsilon\Big)\cdot\nabla\phi\,dxds,
\end{aligned}
\qquad (2.8)
$$

for all non-negative $\phi\in C_c^\infty((0,T)\times\Omega)$.

The results in [10], when specialized to the case $r=2$ imply the following theorem and the local energy inequality follows from the improved regularity of the pressure $\nabla p^\varepsilon\in L^2((0,T)\times\Omega)$, which is valid (but not uniform) for all $\varepsilon>0$.

Theorem 2.6. *Let be given $u_0^\varepsilon \in L^2_\tau(\Omega)$, then there exists a weak solution to the compressible approximation* (1.5)-(1.6)-(1.7)-(1.8)-(1.9).

Finally, we recall the classical Aubin-Lions Lemma which will be useful to obtain compactness in time in the passage to the limit from (1.5) to (1.1).

Lemma 2.7. *Let X, B, and Y be reflexive Banach spaces. For $\varepsilon > 0$ let $\{u^\varepsilon\}_\varepsilon$ be a family of functions uniformly bounded in $L^p(0,T;X)$ with $p \geq 1$ and let $\{\partial_t u^\varepsilon\}_\varepsilon$ be uniformly bounded in $L^r(0,T;Y)$. with $r > 1$. If X is compactly embedded in B and B is continuously embedded in Y, then $\{u^\varepsilon\}_\varepsilon$ is relatively compact in $L^p(0,T;B)$.*

3. Some estimates on the pressure independent of ε

In this section we prove the ε-independent estimate (1.11) for the pressure, which represents the most relevant technical part to show the convergence towards a suitable weak solution.

Lemma 3.1. *Let $(u^\varepsilon, p^\varepsilon)$ be a weak solution of system* (1.5)-(1.7)-(1.8)-(1.9)*. Then, there exists a constant $C > 0$, independent of ε, such that*

$$\sup_{0<t<T} \varepsilon \, \|p^\varepsilon(t)\|_{\frac{5}{3}}^{\frac{5}{3}} + \int_0^T \|p^\varepsilon(s)\|_{\frac{5}{3}}^{\frac{5}{3}} \, ds \leq C.$$

Proof. Let $\alpha = \frac{5}{3}$ and let g^ε be the unique solution (normalized by a vanishing mean value) of the following Poisson problem with Neumann boundary conditions

$$\begin{cases} \Delta g^\varepsilon = |p^\varepsilon|^{\alpha-2} p^\varepsilon - \displaystyle\int_\Omega |p^\varepsilon|^{\alpha-2} p^\varepsilon \, dx \qquad \text{in } \Omega \times \{t\}, \\ \dfrac{\partial g^\varepsilon}{\partial n} = 0 \qquad \text{on } \Gamma \times \{t\}. \end{cases} \tag{3.1}$$

Remark 3.2. The number $\alpha - 2$ is negative but the expression we write is legitimate since we are not really dividing by zero in sets of positive measure. In fact, by using the regularity of p^ε from Definition 2.4 it turns out that the function $|p^\varepsilon|^{\frac{5}{3}-2} p^\varepsilon$ is well-defined and belongs at least to $L^3((0,T) \times \Omega)$, even if not uniformly in $\varepsilon > 0$.

We use now the vector field ∇g^ε as test function in the first equation of the system (1.5). Note that, ∇g^ε is tangential to the boundary and for each fixed $\varepsilon > 0$ and we have that ∇g^ε is smooth enough to make the integral below well-defined. Let $t \in (0,T)$, then we get (recall (1.10))

$$\int_0^t \int_\Omega \big(\partial_t u^\varepsilon + nl(u^\varepsilon, u^\varepsilon) - 2 \operatorname{div} \mathrm{D} u^\varepsilon + \nabla p^\varepsilon\big) \cdot \nabla g^\varepsilon \, dx ds = 0.$$

Integrating by parts the term involving the deformation gradient by using the Navier conditions (1.7) and the one involving the pressure and by using the definition of g^ε, the Neumann boundary conditions, and the fact that p^ε has zero average we get

$$\int_0^t\int_\Omega |p^\varepsilon|^{\frac{5}{3}}\,dxds=\int_0^t\int_\Omega\Big[\big(\partial_t u^\varepsilon+nl(u^\varepsilon,u^\varepsilon)\big)\cdot\nabla g^\varepsilon+2\,\mathrm{D}u^\varepsilon:\nabla^2 g^\varepsilon\Big]\,dxds. \tag{3.2}$$

Let us consider the term with the time derivative. We can use the Helmholtz decomposition to write

$$u^\varepsilon=Pu^\varepsilon+\nabla q.$$

where P is the Leray projector. Note that, since $u^\varepsilon\cdot n=0$ on $\Gamma\times(0,T)$ it holds that $n\cdot\nabla q=0$ on $\Gamma\times(0,T)$ as well. By using the second equation of (1.5) we have that $\Delta p^\varepsilon=\frac{1}{\varepsilon}\operatorname{div}(Pu^\varepsilon+\nabla q)=\frac{1}{\varepsilon}\operatorname{div}\nabla q=\frac{1}{\varepsilon}\Delta q$. This means that for a.e. $t\in(0,T)$ we have in a weak sense

$$\begin{cases}\Delta\left(p^\varepsilon-\dfrac{q}{\varepsilon}\right)=0 & \text{in}\qquad \Omega\times\{t\},\\ \dfrac{\partial}{\partial n}\left(p^\varepsilon-\dfrac{q}{\varepsilon}\right)=0 & \text{in}\qquad \Gamma\times\{t\}.\end{cases}$$

Then, by the uniqueness of the Neumann problem under the usual normalization of zero mean value, we get that

$$p^\varepsilon=\frac{q}{\varepsilon}.$$

Then, we observe that since $u^\varepsilon\cdot n=0$ on Γ and $\operatorname{div}Pu^\varepsilon=0$ in Ω, then

$$\begin{aligned}\int_0^t\int_\Omega\partial_t u^\varepsilon\cdot\nabla g^\varepsilon\,dxds&=-\int_0^t\int_\Omega(\operatorname{div}\partial_t u^\varepsilon)\,g^\varepsilon\,dxds=-\int_0^t\int_\Omega\Delta\partial_t q\,g^\varepsilon\,dxds\\&=-\varepsilon\int_0^t\int_\Omega\Delta\partial_t p^\varepsilon\,g^\varepsilon\,dxds.\end{aligned}$$

We then integrate by parts twice and by observing that $n\cdot\nabla p^\varepsilon=0$, (hence a fortiori also $n\cdot\nabla\partial_t p^\varepsilon=0$) and $n\cdot\nabla g^\varepsilon=0$ we obtain

$$\int_0^t\int_\Omega\partial_t u^\varepsilon\cdot\nabla g^\varepsilon\,dxds=-\varepsilon\int_0^t\int_\Omega\partial_t p^\varepsilon\,\Delta g^\varepsilon\,dxds.$$

Hence, by recalling the definition of g^ε via the boundary value problem (3.1) we get

$$\begin{aligned}\int_0^t\int_\Omega\partial_t u^\varepsilon\cdot\nabla g^\varepsilon\,dxds&=-\varepsilon\int_0^t\int_\Omega\partial_t p^\varepsilon(x,s)\;p^\varepsilon(x,s)|p^\varepsilon(x,s)|^{\alpha-2}\,dxds\\&=-\frac{\varepsilon}{\alpha}\int_0^t\left(\frac{d}{ds}\int_\Omega|p^\varepsilon(x,s)|^\alpha\,dx\right)\,ds,\end{aligned}$$

where we used again the fact that p^ε has zero average for any $t\in(0,T)$.

Then, by recalling that $\alpha=\frac{5}{3}$, by integrating in time the last term, and taking into account the fact that all the initial data $\{u_0^\varepsilon\}_\varepsilon$ are divergence-free (hence $p^\varepsilon(0)=0$) we get that the equation (3.2) reads now

$$\begin{aligned}\frac{3\varepsilon}{5}\|p^\varepsilon(t)\|_{\frac{5}{3}}^{\frac{5}{3}}+\int_0^t\|p^\varepsilon(s)\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds=\int_0^t\int_\Omega nl(u^\varepsilon,u^\varepsilon)\cdot\nabla g^\varepsilon\,dxds&\\ +2\int_0^t\int_\Omega \mathrm{D}u^\varepsilon:\nabla^2 g^\varepsilon\,dxds&\\ =:I_1+I_2.&\end{aligned}$$

We estimate the first integral from the right-hand side as follows:

$$\begin{aligned}I_1&\le\int_0^t\|nl(u^\varepsilon,u^\varepsilon)\|_{\frac{15}{14}}\|\nabla g^\varepsilon\|_{15}\,ds\\ &\le\int_0^t\|nl(u^\varepsilon,u^\varepsilon)\|_{\frac{15}{14}}\|D^2 g^\varepsilon\|_{\frac{5}{2}}\,ds\\ &\le\int_0^t\|nl(u^\varepsilon,u^\varepsilon)\|_{\frac{15}{14}}\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{2}{3}}\,ds\\ &\le C\int_0^t\|nl(u^\varepsilon,u^\varepsilon)\|_{\frac{15}{14}}^{\frac{5}{3}}\,ds+\frac18\int_0^t\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds,\end{aligned}$$

where we have used Hölder, Sobolev, and Young inequalities. Finally, by convex interpolation we have that if $u^\varepsilon\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, then the nonlinear term $nl(u^\varepsilon,u^\varepsilon)$ belongs to $L^{\frac{5}{3}}(0,T;L^{\frac{15}{14}}(\Omega))$.

Hence, we have proved that there exists a constant C_1, depending only on the L^2-norm of the initial data (through the energy inequality (2.7)) such that

$$I_1\le C+\frac14\int_0^t\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds.$$

Concerning the term I_2 we have, by using Hölder inequality and again the elliptic estimates

$$\begin{aligned}\int_0^t\int_\Omega|\nabla u^\varepsilon||\nabla^2 g^\varepsilon|\,ds&\le\int_0^t\|\nabla u^\varepsilon\|_{\frac{5}{3}}\|D^2 g^\varepsilon\|_{\frac{5}{2}}\,ds\\ &\le C(\Omega)\int_0^t\|\nabla u^\varepsilon\|_2\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{2}{3}}\,ds\\ &\le C(\Omega,T)\left(\int_0^t\|\nabla u^\varepsilon\|_2^2\right)^{\frac{6}{5}}\,ds+\frac14\int_0^t\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds\\ &\le C_2+\frac14\int_0^t\|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds,\end{aligned}$$

for a constant C_2, depending only on Ω, T, and the L^2-norm of the initial data (again through the energy inequality (2.7)). Collecting all estimates, we finally get

$$\frac{3\varepsilon}{5}\|p^\varepsilon(t)\|_{\frac{5}{3}}^{\frac{5}{3}}+\frac12\int_0^t\|p^\varepsilon(s)\|_{\frac{5}{3}}^{\frac{5}{3}}\,ds\le C_1+C_2,$$

ending the proof. By the way we also proved that

$$\varepsilon^{\frac{3}{5}} p^\varepsilon \text{ is bounded uniformly in } L^\infty(0,T;L^{\frac{5}{3}}(\Omega)),$$

even if this information will be not used in the sequel. □

4. Proof of the Main Theorem

By using the estimates from the previous section we can now prove Theorem 1.3 in an elementary way by using standard weak compactness method. By recalling (2.6)-(1.11) we have that, up to sub-sequences, there exist $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ and $p \in L^{\frac{5}{3}}((0,T)\times\Omega)$ such that

$$\begin{aligned} &\nabla u^\varepsilon \rightharpoonup \nabla u \qquad \text{weakly in } L^2(0,T\times\Omega),\\ &u^\varepsilon \overset{*}{\rightharpoonup} u \qquad \text{weakly* in } L^\infty(0,T;L^2(\Omega)),\\ &p^\varepsilon \rightharpoonup p \qquad \text{weakly in } L^{\frac{5}{3}}((0,T)\times\Omega). \end{aligned} \tag{4.1}$$

Moreover, by (2.6) if follows that $\sqrt{\varepsilon}\,\nabla p^\varepsilon$ is uniformly bounded in $L^2((0,T)\times\Omega)$, and then we have that $\varepsilon^{\frac{1}{2}+\delta}\,\nabla p^\varepsilon$ converges strongly to zero for all positive δ. Hence, in particular, we have

$$\varepsilon\,\nabla p^\varepsilon \to 0 \qquad \text{strongly in } L^2((0,T)\times\Omega). \tag{4.2}$$

It also follows that u is divergence-free. Indeed, by the second equation of (1.5) we get

$$\begin{aligned} \int_0^T\int_\Omega \operatorname{div} u^\varepsilon\,\psi\,dxds &= \sqrt{\varepsilon}\int_0^T < \sqrt{\varepsilon}\Delta p^\varepsilon, \psi >_{H^{-1},H^1_0}\,ds\\ &= \sqrt{\varepsilon}\int_0^T\int_\Omega \sqrt{\varepsilon}\nabla p^\varepsilon\cdot\nabla\psi\,dxds\\ &\leq \sqrt{\varepsilon}\int_0^T \|\sqrt{\varepsilon}\nabla p^\varepsilon\|_2\|\nabla\psi\|_2\,ds\\ &\leq \sqrt{\varepsilon}\|\sqrt{\varepsilon}\nabla p^\varepsilon\|_{L^2(0,T;L^2(\Omega))}\|\nabla\psi\|_{L^2(0,T;L^2(\Omega))}. \end{aligned}$$

By taking the supremum over the functions $\psi \in L^2(0,T;H^1_0(\Omega))$ and by using the estimate on p^ε from (2.6) we get that

$$\operatorname{div} u^\varepsilon \rightharpoonup 0 \qquad \text{weakly in } L^2(0,T;H^{-1}(\Omega)).$$

Then, by the uniqueness of weak limits we get that u is divergence-free.

The next step is to prove the strong convergence of u^ε in $L^2((0,T)\times\Omega)$. We can now use Aubin-Lions, Lemma 2.7, provided that we can show some estimates on the time derivative of the velocity, and these are usual obtained by comparison. We observe that p^ε is uniformly bounded in $L^{\frac{5}{3}}((0,T)\times\Omega))$, the nonlinear term $nl(u^\varepsilon,u^\varepsilon)$ is uniformly bounded in $L^{\frac{4}{3}}(0,T;H^{-1}(\Omega))$, and $\operatorname{div}\mathrm{D}u^\varepsilon$ is uniformly bounded in $L^2(0,T;H^{-1}(\Omega))$, hence we get that

$$\partial_t u^\varepsilon \in L^{\frac{4}{3}}(0,T;W^{-1,\frac{5}{3}}(\Omega)), \text{ uniformly with respect to } \varepsilon.$$

Then, by applying Lemma 2.7 with $X = W^{1,2}(\Omega)$, $B = L^2(\Omega)$, and $Y = W^{-1,\frac{5}{3}}(\Omega)$ we get

$$u^\varepsilon \to u \qquad \text{strongly in } L^2((0,T)\times\Omega). \tag{4.3}$$

Then, we investigate the convergence of Pu^ε and Qu^ε, where P is the Leray projector and $Q := I - P$. By applying P to the first equation of (1.5) we get that

$$\partial_t Pu^\varepsilon - P(\Delta u^\varepsilon) + P((u^\varepsilon\cdot\nabla)\,u^\varepsilon) - P\left(\frac{1}{2}\,(u^\varepsilon \operatorname{div} u^\varepsilon)\right) = 0.$$

From this equation we show, again by comparison, that $\partial_t Pu^\varepsilon$ is bounded in $L^{\frac{4}{3}}(0,T;(H^1_{\sigma,\tau}(\Omega))')$ uniformly with respect to ε, and since Pu^ε is bounded in $L^2(0,T;H^1_{\sigma,\tau}(\Omega))$ we can apply again Lemma 2.7 to obtain in a very standard way that

$$Pu^\varepsilon \to Pu = u \quad \text{strongly in } L^2((0,T)\times\Omega)).$$

Then, we have that

$$\begin{aligned}\|Qu^\varepsilon\|_2 &= \|u^\varepsilon - Pu^\varepsilon\|_2 = \|u^\varepsilon - u + u - Pu^\varepsilon\|_2\\ &\le \|u^\varepsilon - u\|_2 + \|Pu - Pu^\varepsilon\|_2 \to 0, \qquad \text{as } \varepsilon\to 0.\end{aligned}$$

This in turn implies,

$$Qu^\varepsilon \to 0 \text{ strongly in } L^2((0,T)\times\Omega)).$$

The next step is to prove that Qu^ε converge to 0 strongly in $L^{\frac{5}{2}}((0,T)\times\Omega))$. By using an interpolation inequality and a Sobolev embedding theorem together with the Poincaré inequality (valid for tangential vector fields, see [18]) we get

$$\|Qu^\varepsilon\|_{\frac{5}{2}}^{\frac{5}{2}} \le \|Qu^\varepsilon\|_2^{\frac{7}{4}}\|Qu^\varepsilon\|_6^{\frac{3}{4}} \le C\|Qu^\varepsilon\|_2^{\frac{7}{4}}\|\nabla Qu^\varepsilon\|_2^{\frac{3}{4}},$$

and due to the definition of $Qu^\varepsilon = u^\varepsilon - Pu^\varepsilon = \nabla q$. By integrating in time we get

$$\begin{aligned}\int_0^T \|Qu^\varepsilon(s)\|_{\frac{5}{2}}^{\frac{5}{2}}\,ds &\le c\int_0^T \|Qu^\varepsilon(s)\|_2^{\frac{7}{4}}\|\nabla u^\varepsilon(s)\|_2^{\frac{3}{4}}\,ds\\ &\le \sup_{0<t<T}\|Qu^\varepsilon(t)\|_2^{\frac{1}{2}}\left(\int_0^T \|Qu^\varepsilon(s)\|_2^2\,ds\right)^{\frac{5}{8}}\left(\int_0^T \|\nabla u^\varepsilon(s)\|_2^2\,ds\right)^{\frac{3}{8}}.\end{aligned}$$

Then, by using the regularity of $(u^\varepsilon, p^\varepsilon)$ and construction of the Helmholtz decomposition, it holds that $Qu^\varepsilon \in L^\infty(0,T;L^2(\Omega))$ and we get

$$Qu^\varepsilon \to 0 \qquad \text{strongly in } L^{\frac{5}{2}}((0,T)\times\Omega)). \tag{4.4}$$

The last convergence we need to prove concerns the term $\varepsilon\,\nabla p^\varepsilon$. By considering the second equation of (1.5) we get that

$$\varepsilon\,\Delta p^\varepsilon = \operatorname{div} u^\varepsilon = \operatorname{div} Qu^\varepsilon.$$

By standard using estimates on the elliptic equations with Neumann boundary conditions, see [19], we get the following estimate:

$$\varepsilon \, \|\nabla p^\varepsilon\|_{\frac{5}{2}} \leq \|Qu^\varepsilon\|_{\frac{5}{2}}.$$

By integrating in time and using (4.4) we get then

$$\varepsilon \left(\int_0^t \|\nabla p^\varepsilon\|_{\frac{5}{2}}^{\frac{5}{2}} \right)^{\frac{2}{5}} \to 0.$$

By using (4.1), (4.2), (4.3), and (2.6) it is straightforward to prove that u is a Leray-Hopf weak solution. In order to prove that (u,p) is a suitable weak solution of the Navier-Stokes equations (1.1)-(1.2) we only have to show is that (u,p) satisfies the local energy inequality (2.3). Since $\phi \geq 0$, from (2.8) we have that $(u^\varepsilon, p^\varepsilon)$ satisfies

$$\begin{aligned}\int_0^T \int_\Omega |\nabla u^\varepsilon|^2 \phi \, dxds \leq \int_0^T \int_\Omega \Big[& \frac{|u^\varepsilon|^2}{2} (\phi_t + \Delta\phi) \\ & + \big(u^\varepsilon \frac{|u^\varepsilon|^2}{2} + p^\varepsilon u^\varepsilon - p^\varepsilon \nabla p^\varepsilon - u^\varepsilon \operatorname{div} u^\varepsilon \big) \cdot \nabla\phi \Big] \, dxds.\end{aligned} \tag{4.5}$$

By weak lower semicontinuity of the L^2-norm, the fact that $\nabla u^\varepsilon \rightharpoonup \nabla u$ weakly in $L^2((0,T)\times\Omega)$, and since $\phi \geq 0$ we have that

$$\int_0^T \int_\Omega |\nabla u|^2 \, \phi \, dxds \leq \liminf_{\varepsilon\to 0} \int_0^T \int_\Omega |\nabla u^\varepsilon|^2 \, \phi \, dxds.$$

Since $u^\varepsilon \to u$ strongly in $L^2((0,T)\times\Omega)$, we get

$$\int_0^T \int_\Omega \frac{|u^\varepsilon|^2}{2} \big(\phi_t + \Delta\phi\big) \, dxds \to \int_0^T \int_\Omega \frac{|u|^2}{2} \big(\phi_t + \Delta\phi\big) \, dxds, \qquad \text{as } \varepsilon \to 0.$$

Next, by interpolation we also have that $u^\varepsilon \to u$ strongly in $L^2(0,T;L^3(\Omega))$ and that u^ε is bounded in $L^4(0,T;L^3(\Omega))$. Consequently, it also follows that

$$\int_0^T \int_\Omega u^\varepsilon \frac{|u^\varepsilon|^2}{2} \cdot \nabla\phi \, dxds \to \int_0^T \int_\Omega u \frac{|u|^2}{2} \cdot \phi \, dxds, \qquad \text{as } \varepsilon \to 0.$$

Now, we estimate the last two terms in (4.5). We start by estimating the term involving the pressure. We have that

$$p^\varepsilon \rightharpoonup p \text{ weakly in } L^{\frac{5}{3}}(\Omega \times (0,T)),$$

while by standard interpolation argument

$$u^\varepsilon \to u \text{ strongly in } L^{\frac{5}{2}}(\Omega \times (0,T)).$$

These in turn imply that

$$\int_0^T \int_\Omega p^\varepsilon u^\varepsilon \cdot \nabla\phi \, dxds \to \int_0^T \int_\Omega p \, u \cdot \nabla\phi \, dxds, \qquad \text{as } \varepsilon \to 0.$$

Concerning the integral

$$A := \varepsilon \int_0^T \int p^\varepsilon \nabla p^\varepsilon \cdot \nabla \phi \, dxds,$$

(quadratic in the pressure) we argue as follows: By using Hölder inequality we get that

$$\begin{aligned} |A| &\leq C\varepsilon \left(\int_0^T \|\nabla p^\varepsilon\|_{\frac{5}{2}}^{\frac{5}{2}} \right)^{\frac{2}{5}} \left(\int_0^T \|p^\varepsilon\|_{\frac{5}{3}}^{\frac{5}{3}} \right)^{\frac{3}{5}} \\ &\leq C\varepsilon \left(\int_0^T \|\nabla p^\varepsilon\|_{\frac{5}{2}}^{\frac{5}{2}} \right)^{\frac{2}{5}}, \end{aligned}$$

and also A goes to 0 as $\varepsilon \to 0$.

Finally, the we consider the last term, namely

$$B := -\int_0^T \int_\Omega u^\varepsilon \operatorname{div} u^\varepsilon \cdot \nabla \phi \, dxds,$$

and since $\operatorname{div} u^\varepsilon$ converge weakly to 0 in $L^2(0,T;H^{-1}(\Omega))$ and ∇u^ε is uniformly bounded in $L^2(0,T;H^1(\Omega))$ we get, by uniqueness of the weak limits, that

$$\operatorname{div} u^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2((0,T) \times \Omega). \tag{4.6}$$

By using the fact the $u^\varepsilon \to u$ strongly in $L^2((0,T) \times \Omega)$ we have that B goes to 0 when ε vanishes, hence that (2.3) is satisfied.

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