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OPTIMAL NETWORKS FOR MASS TRANSPORTATION PROBLEMS

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Abstract. In the framework of transport theory, we are interested in the following optimization problem: given the distributions μ^+ of working people and μ^- of their working places in an urban area, build a transportation network (such as a railway or an underground system) which minimizes a functional depending on the geometry of the network through a particular cost function. The functional is defined as the Wasserstein distance of μ^+ from μ^- with respect to a metric which depends on the transportation network.

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1. Introduction

Optimal Transportation Theory was first developed by Monge in 1781 in [12] where he raised the following question: given two mass distributions f^+ and f^-, minimize the transport cost

$$\int_{\mathbb{R}^N} |x - t(x)| f^+(x) \, \mathrm{d}x$$

among all *transport maps* t, *i.e.* measurable maps such that the mass balance condition

$$\int_{t^{-1}(B)} f^+(x) \, \mathrm{d}x = \int_B f^-(y) \, \mathrm{d}y$$

holds for every Borel set B. Because of its strong non-linearity, Monge's formulation did not lead to significant advances up to 1940, when Kantorovich proposed his own formulation (see [10, 11]).

In modern notation, given two finite positive Borel measures μ^+ and μ^- on $\mathbb{R}^N$ such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$, Kantorovich was interested to minimize

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| \, \mathrm{d}\mu(x, y)$$

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among all *transport plans* μ, *i.e.* positive Borel measures on $\mathbb{R}^N \times \mathbb{R}^N$ such that $\pi^+_\# \mu = \mu^+$ and $\pi^-_\# \mu = \mu^-$, where by $\#$ we denoted the *push-forward* operator (*i.e.* $h_\# \mu(E) = \mu(h^{-1}(E))$). It is easy to see that if t is a transport map between $\mu^+ = f^+ \mathcal{L}^N$ and $\mu^- = f^- \mathcal{L}^N$, then $(\mathrm{Id} \times t)_\# \mu^+$ is a transport plan. So, Kantorovich's problem is a weak formulation of Monge's one.

Of course, one can take, instead of $\mathbb{R}^N$ and the cost function given by the Euclidean modulus, a generic pair of metric spaces X and Y and a positive lower semicontinuous cost function $c : X \times Y \to \mathbb{R}$, so that the Kantorovich problem reads:

$$\min \left\{ \int_{X \times Y} c(x,y) \, \mathrm{d}\mu(x,y) \ : \ \pi^+_\# \mu = \mu^+, \ \pi^-_\# \mu = \mu^- \right\} \cdot \tag{1.1}$$

We stress the fact that μ^+ and μ^- must have the same mass, otherwise there are no transport plans.

If we set $X = Y$ and take as cost function the distance d in X, then the minimal value in (1.1) is called *Wasserstein distance* (of power 1) between μ^+ and μ^-. In this case, we shall write $W_d(\mu^+, \mu^-)$.

For other details on transportation problems on networks we refer the interested reader to [2,3,5–7,13].

2. The optimal network problem

We consider a bounded connected open subset Ω with Lipschitz boundary of $\mathbb{R}^N$ (the urban area) with $N > 1$ and two positive finite measures μ^+ and μ^- on $K := \overline{\Omega}$ (the distributions of working people and of working places). We assume that μ^+ and μ^- have the same mass that we normalize both equal 1, that is μ^+ and μ^- are probability measures on K.

In this section we introduce the optimization problem for transportation networks: to every "urban network" Σ we may associate a suitable "cost function" d_Σ which takes into account the geometry of Σ as well as the costs for customers to move with their own means and by means of the network. The cost functional will be then

$$T(\Sigma) = W_{d_\Sigma}(\mu^+, \mu^-)$$

so that the optimization problem we deal with is

$$\min\{T(\Sigma) \ : \ \Sigma \text{ "admissible network"}\} \cdot \tag{2.2}$$

The main result of this paper is to prove that, under suitable and very mild assumptions, and taking as admissible networks all connected, compact one-dimensional subsets Σ of K, the optimization problem (2.2) admits a solution. The tools we use to obtain the existence result are a suitable relaxation procedure to define the function d_Σ (Th. 4.2) and a generalization of the Gołab theorem (Th. 3.3), also obtained by Dal Maso and Toader in [8].

In order to introduce the distance d_Σ we consider a function $J : [0, +\infty]^3 \to [0, +\infty]$. For a given path γ in K the parameter a in $J(a, b, c)$ measures the length of γ outside Σ, b measures the length of γ inside Σ, while c represents the total length of Σ. The cost $J(a, b, c)$ is then the cost of a customer who travels for a length a by his own means and for a length b on the network, being c the length of the latter. For instance we could take $J(a, b, c) = A(a) + B(b) + C(c)$ and then the function $A(t)$ is the cost for traveling a length t by one's own means, $B(t)$ is the price of a ticket to cover the length t on Σ and $C(t)$ represents the cost of a network of length t.

For every closed connected subset Σ in K, we define the cost function d_Σ as

$$d_\Sigma(x,y) := \inf \left\{ J\left(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)\right) \ : \ \gamma \in \mathscr{C}_{x,y} \right\},$$

where $\mathscr{C}_{x,y}$ is the class of all closed connected subsets of K containing x and y. The optimization problem we consider is then (2.2) where we take as *admissible networks* all closed connected subsets Σ of K with

$\mathcal{H}^1(\Sigma) < +\infty$. We also define, for every closed connected subset γ of K

$$L_\Sigma(\gamma) := J\left(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)\right).$$

We assume that J satisfies the following conditions:

- J is lower semicontinuous;
- J is non-decreasing, *i.e.*

$$a_1 \le a_2, b_1 \le b_2, c_1 \le c_2 \Longrightarrow J(a_1, b_1, c_1) \le J(a_2, b_2, c_2);$$

- $J(a, b, c) \ge G(c)$ with $G(c) \to +\infty$ when $c \to +\infty$;
- J is continuous in its first variable.

A *curve* joining two points $x, y \in K$ is an element of the set

$$\mathscr{C}_{x,y} := \{\gamma \text{ closed connected}, \{x, y\} \subseteq \gamma \subseteq K\}$$

while an element of $\mathscr{C}$ will be, by definition, a closed connected set in K:

$$\mathscr{C} := \{\gamma \text{ closed connected}, \gamma \subseteq K\}\cdot$$

We associate to every admissible network $\Sigma \in \mathscr{C}$ the cost function

$$d_\Sigma(x, y) = \inf\{L_\Sigma(\gamma) \ : \ \gamma \in \mathscr{C}_{x,y}\}\cdot$$

We are interested in the functional T given by

$$\Sigma \mapsto T(\Sigma) := W_{d_\Sigma}(\mu^+, \mu^-)$$

which is defined on the class $\mathscr{C}$, where the Wasserstein distance is defined in the introduction.

Finally by $\overline{L}_\Sigma^{x,y}$ we denote the lower semicontinuous envelope of L_Σ with respect to the Hausdorff convergence on $\mathscr{C}_{x,y}$ (see Sect. 3 for the main definitions). In other words, for every $\gamma \in \mathscr{C}_{x,y}$ we set

$$\overline{L}_\Sigma^{x,y}(\gamma) = \begin{cases} \min\left\{\liminf_n L_\Sigma(\gamma_n) \ : \gamma_n \to \gamma, \gamma_n \in \mathscr{C}_{x,y}\right\} & \text{if } \gamma \in \mathscr{C}_{x,y} \\ +\infty & \text{if } \gamma \notin \mathscr{C}_{x,y}, \end{cases}$$

where we fix the condition $x, y \in \gamma$. Moreover, we define $\overline{L}_\Sigma$ as

$$\overline{L}_\Sigma(\gamma) = \min\left\{\liminf_{n\to+\infty} L_\Sigma(\gamma_n) \ : \gamma_n \to \gamma, \gamma_n \in \mathscr{C}\right\},$$

that is to say, the lower semicontinuous envelope of L_Σ with respect to the Hausdorff convergence on the class of closed connected sets of K.

3. The Gołab theorem and its extensions

In this section X will be a set endowed with a distance function d, *i.e.* (X, d) is a metric space. We assume for simplicity X to be *compact*. By $\mathscr{C}(X)$ we indicate the class of all closed subsets of X.

Given two closed subsets C and D, the *Hausdorff distance* between them is defined by

$$d_{\mathcal{H}}(C, D) := 1 \wedge \inf\{r \in [0, +\infty[\ : \ C \subseteq D_r, D \subseteq C_r\}$$

where

$$C_r := \{x \in X \ : \ d(x, C) < r\}\cdot$$

It is easy to see that $d_{\mathcal{H}}$ is a distance on $\mathscr{C}(X)$, so $(\mathscr{C}(X), d_{\mathcal{H}})$ is a metric space. We remark the following well-known facts (see for example [1]):

- (X, d) compact $\Longrightarrow (\mathscr{C}(X), d_{\mathcal{H}})$ compact;
- (X, d) complete $\Longrightarrow (\mathscr{C}(X), d_{\mathcal{H}})$ complete.

In the rest of the paper we will use the notation $C_n \to C$ to indicate the convergence of a sequence $\{C_n\}_{n\in\mathbb{N}}$ to C with respect to the distance $d_{\mathcal{H}}$.

Proposition 3.1. *Let $\{C_n\}_{n\in\mathbb{N}}$ be a sequence of compact connected subsets in X such that $C_n \to C$ for some compact subset C. Then C is connected.*

Proof. Suppose, on the contrary, that there exist two closed non-void separated subsets F_1 and F_2 such that $C = F_1 \cup F_2$. Since F_1 and F_2 are compact, $d(F_1, F_2) = d > 0$. Let us choose $\varepsilon = d/4$. By the definition of Hausdorff convergence, there exists a positive integer N such that

$$n \geq N \Longrightarrow C_n \subseteq (C)_\varepsilon, \ C \subseteq (C_n)_\varepsilon.$$

Since C_N is connected, we must have either $C_N \subseteq (F_1)_\varepsilon$ or $C_N \subseteq (F_2)_\varepsilon$. Let us suppose, for example, that $C_N \subseteq (F_1)_\varepsilon$. On one side by the Hausdorff convergence it is $F_2 \subseteq (C_N)_\varepsilon$, on the other by the choice of ε we have $(C_N)_\varepsilon \cap F_2 = \varnothing$, a contradiction. □

The *Hausdorff 1-dimensional measure* in (X, d) of a Borel set B is defined by

$$\mathcal{H}^1(B) := \lim_{\delta\to 0^+} \mathcal{H}^{1,\delta}(B),$$

where

$$\mathcal{H}^{1,\delta}(B) := \inf \left\{ \sum_{n\in\mathbb{N}} \operatorname{diam} B_n : \operatorname{diam} B_n < \delta, B \subseteq \bigcup_{n\in\mathbb{N}} B_i \right\}\cdot$$

The measure $\mathcal{H}^1$ is Borel regular and if (X, d) is the 1-dimensional Euclidean space, then $\mathcal{H}^1$ is just the Lebesgue measure $\mathcal{L}^1$.

The Gołab classical theorem states that in a metric space, the measure $\mathcal{H}^1$ is sequentially lower semicontinuous with respect to the Hausdorff convergence over the class of all compact connected subsets of X.

Theorem 3.2 (Gołab)**.** *Let X be a metric space. If $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of compact connected subsets of X and $C_n \to C$ for some compact connected subset C, then*

$$\mathcal{H}^1(C) \leq \liminf_{n\to+\infty} \mathcal{H}^1(C_n). \tag{3.3}$$

Actually, this result can be strengthened.

Theorem 3.3. *Let X be a metric space, $\{\Gamma_n\}_{n\in\mathbb{N}}$ and $\{\Sigma_n\}_{n\in\mathbb{N}}$ be two sequences of compact subsets such that $\Gamma_n \to \Gamma$ and $\Sigma_n \to \Sigma$ for some compact subsets Γ and Σ. Let us also suppose that Γ_n is connected for all $n \in \mathbb{N}$. Then*

$$\mathcal{H}^1(\Gamma \setminus \Sigma) \leq \liminf_{n\to+\infty} \mathcal{H}^1(\Gamma_n \setminus \Sigma_n). \tag{3.4}$$

A proof of this result has been given by Dal Maso and Toader in [8]; for sake of completeness, we include the proof here below. It is in fact based on the following two rectifiability theorems whose proof can be found in [1].

Theorem 3.4. *Let X be a metric space and C a closed connected subset of finite length, i.e. $\mathcal{H}^1(C) < +\infty$. Then C is compact and connected by injective rectifiable curves.*

Theorem 3.5. *Let C be a closed connected subset in a metric space X such that $\mathcal{H}^1(C) < +\infty$. Then there exists a sequence of Lipschitz curves $\{\gamma_n\}_{n\in\mathbb{N}}$, $\gamma_n : [0,1] \to C$, such that*

$$\mathcal{H}^1(C \setminus \bigcup_{n\in\mathbb{N}} \gamma_n([0,1])) = 0.$$

The first step in the proof of Theorem 3.3 is a localized form of the Gołab classical theorem. To this aim we need the following lemma.

Lemma 3.6. *Let C be a closed connected subset of X and let $x \in C$. If $r \in [0, \frac{1}{2}\operatorname{diam} C]$, then*

$$\mathcal{H}^1(C \cap B_r(x)) \geq r.$$

Proof. See for instance Lemma 4.4.2 of [1] or Lemma 3.4 of [9]. □

Remark 3.7. Lemma 3.6 yields the following estimate from below for the upper density:

$$\overline{\theta}(C,x) := \limsup_{r\to 0^+} \frac{\mathcal{H}^1(C \cap B_r(x))}{2r} \geq \frac{1}{2}\cdot$$

We recall that for every measure μ the *upper density* is defined by

$$\overline{\theta}(\mu,x) := \limsup_{r\to 0^+} \frac{\mu(B_r(x))}{2r}\cdot$$

We also recall that $\overline{\theta}(\mu,x) \geq t$ for all $x \in X$ implies $\mu(B) \geq t\mathcal{H}^1(B)$ for every Borel set B (see Th. 2.4.1 in [1]).

We are now in a position to obtain the localized version of the Gołab theorem.

Theorem 3.8. *Let X be a metric space. If $\{C_n\}_{n\in\mathbb{N}}$ is a sequence of compact connected subsets of X such that $C_n \to C$ for some compact connected subset C, then for every open subset U of X*

$$\mathcal{H}^1(C \cap U) \leq \liminf_{n\to+\infty} \mathcal{H}^1(C_n \cap U).$$

Proof. We can suppose that $L := \lim_n \mathcal{H}^1(C_n \cap U)$ exists, is finite and $\mathcal{H}^1(C_n \cap U) \leq L + 1$. Let $d_n = \operatorname{diam}(C_n \cap U)$. We can suppose up to a subsequence that $d_n \to d > 0$. Let us consider the sequence of Borel measures defined by

$$\mu_n(B) := \mathcal{H}^1(B \cap C_n \cap U)$$

for every Borel set B. Up to a subsequence we can assume that $\mu_n \rightharpoonup^* \mu$ for a suitable μ. We choose $x \in C \cap U$ and $r' < r < \operatorname{diam}(C \cap U)/2$. Then, by Lemma 3.6,

$$\mu(B_r(x)) \geq \mu\left(\overline{B}_{r'}(x)\right) \geq \limsup_{n\to+\infty} \mu_n\left(\overline{B}_{r'}(x)\right) = \limsup_{n\to+\infty} \mathcal{H}^1\left(C_n \cap \overline{B}_{r'}(x) \cap U\right) \geq r'.$$

Since r' was chosen arbitrarily we get

$$\mu(B_r(x)) \geq r$$

for every $x \in C \cap U$ and $r < \operatorname{diam}(C \cap U)/2$. This implies $\overline{\theta}(C,x) \geq 1/2$. By Remark 3.7

$$\mathcal{H}^1(C \cap U) \leq 2\mu(X) \leq 2\liminf_{n\to+\infty} \mu_n(X) = 2\liminf_{n\to+\infty} \mathcal{H}^1(C_n \cap U) = 2L.$$

By Theorem 3.5 for $\mathcal{H}^1$-almost all $x_0 \in C \cap U$ there exists a Lipschitz curve γ whose range is in $C \cap U$ such that $x_0 = \gamma(t_0)$ and $t_0 \in]0,1[$. We can also suppose that

$$\lim_{h\to 0^+} \frac{d(\gamma(t_0+h),\gamma(t_0-h))}{2|h|} = 1.$$

We choose arbitrarily $\sigma \in]0,1[$. If h is small, then

$$d(\gamma(t_0+h),\gamma(t_0-h)) \geq (2-\sigma)|h|$$

and

$$(1-\sigma)|h| \leq d(\gamma(t_0 \pm h),\gamma(t_0)) \leq (1+\sigma)|h|.$$

Let us also suppose that $|h| < \sigma/(1+\sigma)$ and put

$$y := \gamma(t_0-h), \quad z := \gamma(t_0+h), \quad r := \max\{d(y,x_0), d(z,x_0)\}\cdot$$

We get

$$r < (1+\sigma)|h| < \sigma, \quad d(y,z) \geq (2-\sigma)|h| \geq \frac{2-\sigma}{2+\sigma} r.$$

Let $r' := (1+\sigma)r$. Since $C_n \to C$, then (see Prop. 4.4.3 in [1]) there exist subsequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ such that $y_n, z_n \in C_n \cap U$, $y_n \to y$ and $z_n \to z$. One must have $y_n, z_n \in B_{r'}(x_0)$ for n large enough and

$$\mu_n\left(\overline{B_{r'}(x)}\right) = \mathcal{H}^1\left(C_n \cap \overline{B_{r'}(x)} \cap U\right) \geq d(z,y_n).$$

Taking the limsup

$$\begin{aligned}\mu\left(\overline{B_{r'}(x)}\right) &\geq \limsup_{n\to+\infty} \mathcal{H}^1\left(C_n \cap \overline{B_{r'}(x)} \cap U\right) \geq \limsup_{n\to+\infty} d(z,y_n)\\ &= d(z,y) \geq \frac{2-\sigma}{2+\sigma} r = \frac{2-\sigma}{(2+\sigma)(1+\sigma)} r'.\end{aligned}$$

Since σ was arbitrary, we get $\overline{\theta}(\mu, x_0) \geq 1$ for $\mathcal{H}^1$-almost all $x_0 \in C \cap U$. Then, by Remark 3.7

$$\mathcal{H}^1(C \cap U) \leq \mu(X) \leq \liminf_{n\to+\infty} \mu_n(X) = \liminf_{n\to+\infty} \mathcal{H}^1(C_n \cap U). \qquad \square$$

Proof of Theorem 3.3. Let $A = \Gamma \cap \Sigma$. Thanks to the equality

$$\bigcup_{\varepsilon>0} (\Gamma \setminus \overline{A}_\varepsilon) = \Gamma \setminus \Sigma$$

we have

$$\lim_{\varepsilon\to 0^+} \mathcal{H}^1(\Gamma \setminus \overline{A}_\varepsilon) = \mathcal{H}^1(\Gamma \setminus \Sigma).$$

Recalling that the following inclusion of sets holds for large values of n

$$\Gamma_n \setminus \overline{A}_\varepsilon \subseteq \Gamma_n \setminus A_n \subseteq \Gamma_n \setminus \Sigma_n$$

by the localized form of Gołab theorem (Th. 3.8) we deduce

$$\mathcal{H}^1\left(\Gamma \setminus \overline{A}_\varepsilon\right) \leq \liminf_{n\to+\infty} \mathcal{H}^1\left(\Gamma_n \setminus \overline{A}_\varepsilon\right) \leq \liminf_{n\to+\infty} \mathcal{H}^1\left(\Gamma_n \setminus \Sigma_n\right).$$

Taking the limit as $\varepsilon \to 0^+$, we obtain

$$\mathcal{H}^1(\Gamma \setminus \Sigma) \leq \liminf_{n \to +\infty} \mathcal{H}^1(\Gamma_n \setminus \Sigma_n). \qquad \square$$

Remark 3.9. It is easy to see that if the number of connected components of C_n is bounded from above by a positive integer independent on n, then the localized form of Gołab theorem is still valid. All details can be found in [8].

4. Relaxation of the cost function

We can give an explicit expression for the lower semicontinuous envelopes $\overline{L}_\Sigma$ and $\overline{L}_\Sigma^{x,y}$ in terms of J. In order to achieve this result it is useful to introduce the function:

$$\overline{J}(a,b,c) = \inf\{J(a+t, b-t, c) \ : \ 0 \leq t \leq b\}\cdot$$

The following lemma is an important step to establish Theorem 4.2.

Lemma 4.1. *Let γ and Σ be closed connected subsets of K. Let also suppose that Σ has a finite length. Then for every $t \in [0, \mathcal{H}^1(\gamma \cap \Sigma)]$ we can find a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}$ such that*

- $\gamma_n \to \gamma$;
- $\lim_n \mathcal{H}^1(\gamma_n) = \mathcal{H}^1(\gamma)$;
- $\mathcal{H}^1(\gamma_n \cap \Sigma) \nearrow \mathcal{H}^1(\gamma \cap \Sigma) - t$.

Moreover, if $x, y \in \gamma$ then the sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ can be chosen in $\mathscr{C}_{x,y}$.

Proof. The set $\gamma \cap \Sigma$ is closed and with a finite length. By the second rectifiability result (Th. 3.5) it follows the existence of a sequence of curves $\sigma_n \in \mathrm{Lip}([0,1], K)$ such that

$$\mathcal{H}^1\left((\gamma \cap \Sigma) \setminus \bigcup_{n\in\mathbb{N}} \sigma_n([0,1])\right) = 0.$$

We can also suppose that the subsets $\sigma_n([0,1])$ are disjoint up to subsets of negligible length. Fix a sufficiently small $\delta > 0$ and choose a sequence of intervals $I_n = [a_n, b_n]$ such that

$$\sum_{n\in\mathbb{N}} \mathcal{H}^1(\sigma_n(I_n)) = t + \delta.$$

For every sequence $\underline{v} = \{v_n\}_{n\in\mathbb{N}}$ of unit vectors of $\mathbb{R}^N$ such that v_n is not tangent to $\gamma \cap \Sigma$ in $\sigma_n(a_n)$ and $\sigma_n(b_n)$, and every sequence $\underline{\varepsilon} = \{\varepsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers, let us consider

$$\begin{aligned} A_{\underline{v},\underline{\varepsilon}} &= \bigcup_{n\in\mathbb{N}} \sigma_n([0,a_n] \cup [b_n,1]), \\ B_{\underline{v},\underline{\varepsilon}} &= \bigcup_{n\in\mathbb{N}} (\sigma_n(a_n) + \varepsilon_n V_n), \\ C_{\underline{v},\underline{\varepsilon}} &= \bigcup_{n\in\mathbb{N}} (v_n + \sigma_n(I_n)), \\ D_{\underline{v},\underline{\varepsilon}} &= \bigcup_{n\in\mathbb{N}} (\sigma_n(b_n) + \varepsilon_n V_n) \\ \gamma_{\underline{v},\underline{\varepsilon}} &= (\gamma \setminus \Sigma) \cup A_{\underline{v},\underline{\varepsilon}} \cup B_{\underline{v},\underline{\varepsilon}} \cup C_{\underline{v},\underline{\varepsilon}} \cup D_{\underline{v},\underline{\varepsilon}} \end{aligned}$$

where $V_n = \{tv_n \ : \ t \in [0,1]\}$ (see Fig. 1).

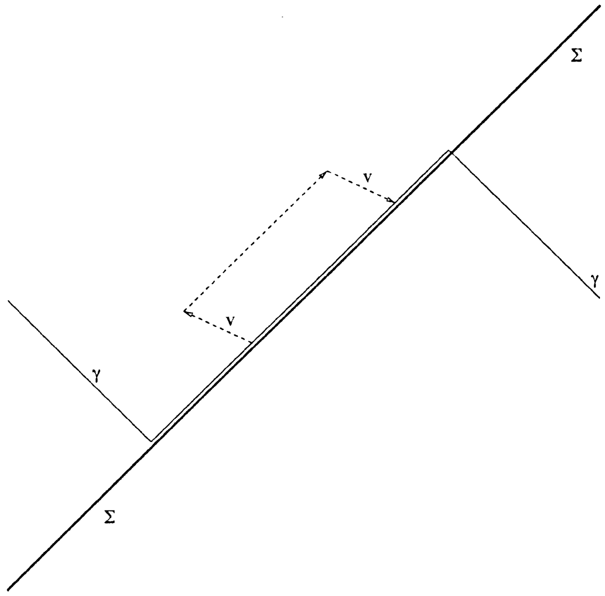

FIGURE 1. The approximating curves γ_n.

Since Σ is closed and with a finite length, the class of $\gamma_{\underline{v},\underline{\varepsilon}}$ that have not $\mathcal{H}^1$-negligible intersection with Σ is at most countable. Out of that set we can choose sequences $\delta_m \searrow 0$, and $\{\gamma_{\underline{v}_m,\underline{\varepsilon}_m}\}_{m\in\mathbb{N}}$ such that $\|\underline{\varepsilon}_m\| \searrow 0$, where by $\|\underline{\varepsilon}\|$ we denote the quantity $\sum_n \varepsilon_n$. The sequence $\{\gamma_{\underline{v}_m,\underline{\varepsilon}_m}\}_{m\in\mathbb{N}}$ is the one we were looking for. □

Theorem 4.2. *For every closed connected subset $\gamma \in \mathscr{C}_{x,y}$ we have*

$$\overline{L}_\Sigma^{x,y}(\gamma) = \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

Moreover, if $\gamma \in \mathscr{C}_{x,y}$ then

$$\overline{L}_\Sigma^{x,y}(\gamma) = \overline{L}_\Sigma(\gamma).$$

Proof. Let γ be a fixed curve in $\mathscr{C}_{x,y}$. First we establish that

$$\overline{L}_\Sigma^{x,y}(\gamma) \geq \overline{J}(\mathcal{H}^1(\gamma \setminus \Sigma), \mathcal{H}^1(\gamma \cap \Sigma), \mathcal{H}^1(\Sigma)).$$

It is enough to show that for every sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}_{x,y}$ converging to γ with respect to the Hausdorff metric, there exists $t \in [0, \mathcal{H}^1(\gamma \cap \Sigma)]$ such that

$$J(\mathcal{H}^1(\gamma \setminus \Sigma) + t, \mathcal{H}^1(\gamma \cap \Sigma) - t, \mathcal{H}^1(\Sigma)) \leq \liminf_{n\to+\infty} L_\Sigma(\gamma_n).$$

Up to a subsequence we can suppose the following equalities hold true:

$$\begin{aligned} \liminf_{n\to+\infty} L_\Sigma(\gamma_n) &= \lim_{n\to+\infty} L_\Sigma(\gamma_n), \\ \liminf_{n\to+\infty} \mathcal{H}^1(\gamma_n) &= \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n), \\ \liminf_{n\to+\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma) &= \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n \setminus \Sigma). \end{aligned}$$

Moreover, by Gołab theorems (Ths. 3.2 and 3.3)

$$\mathcal{H}^1(\gamma) \le \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n),$$
$$\mathcal{H}^1(\gamma\setminus\Sigma) \le \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n\setminus\Sigma).$$

Choose $t = \lim_n \mathcal{H}^1(\gamma_n\setminus\Sigma) - \mathcal{H}^1(\gamma\setminus\Sigma)$. Then $\mathcal{H}^1(\gamma\setminus\Sigma) + t = \lim_n \mathcal{H}^1(\gamma_n\setminus\Sigma)$. We have

$$\begin{aligned}\mathcal{H}^1(\gamma_n) &= \mathcal{H}^1(\gamma_n\setminus\Sigma) + \mathcal{H}^1(\gamma_n\cap\Sigma)\\ &= [\mathcal{H}^1(\gamma_n\setminus\Sigma) - t] + [\mathcal{H}^1(\gamma_n\cap\Sigma) + t].\end{aligned}$$

Taking the limit as $n\to+\infty$ gives

$$\mathcal{H}^1(\gamma) \le \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n) = \left[\mathcal{H}^1(\gamma\setminus\Sigma) + t\right] + \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n\cap\Sigma)$$

so that

$$\mathcal{H}^1(\gamma\cap\Sigma) - t \le \lim_{n\to+\infty} \mathcal{H}^1(\gamma_n\cap\Sigma).$$

It follows by the semicontinuity and monotonicity of J in the first two variables

$$J\left(\mathcal{H}^1(\gamma\setminus\Sigma)+t, \mathcal{H}^1(\gamma\cap\Sigma)-t, \mathcal{H}^1(\Sigma)\right) \le \liminf_{n\to+\infty} J\left(\mathcal{H}^1(\gamma_n\setminus\Sigma), \mathcal{H}^1(\gamma_n\cap\Sigma), \mathcal{H}^1(\Sigma)\right).$$

Now, we have to establish the opposite inequality:

$$\overline{L}_\Sigma^{x,y}(\gamma) \le \overline{J}\left(\mathcal{H}^1(\gamma\setminus\Sigma), \mathcal{H}^1(\gamma\cap\Sigma), \mathcal{H}^1(\Sigma)\right).$$

In the same way as before, it is enough to show that for every $t\in[0,\mathcal{H}^1(\gamma\cap\Sigma)]$ we can find a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ in $\mathscr{C}_{x,y}$ which converges to γ such that

$$\liminf_{n\to+\infty} L_\Sigma(\gamma_n) \le J\left(\mathcal{H}^1(\gamma\setminus\Sigma)+t, \mathcal{H}^1(\gamma\cap\Sigma)-t, \mathcal{H}^1(\Sigma)\right).$$

Given t, let $\{\gamma_n\}_{n\in\mathbb{N}}$ be the sequence given by Lemma 4.1. Then we get

$$\lim_{n\to+\infty} \mathcal{H}^1(\gamma_n\setminus\Sigma) = \mathcal{H}^1(\gamma) - \mathcal{H}^1(\gamma\cap\Sigma) + t = \mathcal{H}^1(\gamma\setminus\Sigma) + t.$$

Thanks to $\mathcal{H}^1(\gamma_n\cap\Sigma) \le \mathcal{H}^1(\gamma\cap\Sigma) - t$, we have

$$J\left(\mathcal{H}^1(\gamma_n\setminus\Sigma), \mathcal{H}^1(\gamma_n\cap\Sigma), \mathcal{H}^1(\Sigma)\right) \le J\left(\mathcal{H}^1(\gamma_n\setminus\Sigma), \mathcal{H}^1(\gamma\cap\Sigma)-t, \mathcal{H}^1(\Sigma)\right)$$

and by the continuity of J in the first variable

$$\liminf_{n\to+\infty} J\left(\mathcal{H}^1(\gamma_n\setminus\Sigma), \mathcal{H}^1(\gamma_n\cap\Sigma), \mathcal{H}^1(\Sigma)\right) \le J\left(\mathcal{H}^1(\gamma\setminus\Sigma)+t, \mathcal{H}^1(\gamma\cap\Sigma)-t, \mathcal{H}^1(\Sigma)\right)$$

which implies the inequality we looked for. The proof of the second statement of the theorem is analogous and hence omitted. □

The next proposition is a consequence of Theorem 4.2.

Proposition 4.3. *For every $x, y\in K$ we have*

$$d_\Sigma(x,y) = \inf\left\{\overline{L}_\Sigma(\gamma) \ : \ \gamma\in\mathscr{C}_{x,y}\right\}\cdot$$

Proof. By a general result of relaxation theory (see for instance [4]), the infimum of a function is the same as the infimum of its lower semicontinuous envelope, so

$$d_\Sigma(x,y) = \inf\left\{\overline{L}_\Sigma^{x,y}(\gamma) \ : \ \gamma \in \mathscr{C}_{x,y}\right\}.$$

It is then enough to prove that

$$\inf\left\{\overline{L}_\Sigma^{x,y}(\gamma) \ : \ \gamma \in \mathscr{C}_{x,y}\right\} = \inf\left\{\overline{L}_\Sigma(\gamma) \ : \ \gamma \in \mathscr{C}_{x,y}\right\},$$

which is a consequence of Theorem 4.2. □

It is more convenient to introduce the function whose variables a, b, c now represent the length $\mathcal{H}^1(\gamma \setminus \Sigma)$ covered by one's own means, the path length $\mathcal{H}^1(\gamma)$, and the length of the network $\mathcal{H}^1(\Sigma)$:

$$\Theta(a,b,c) = \overline{J}(a, b-a, c).$$

Obviously, Θ satisfies

$$\Theta\left(\mathcal{H}^1(\gamma\setminus\Sigma), \mathcal{H}^1(\gamma), \mathcal{H}^1(\Sigma)\right) = \overline{J}\left(\mathcal{H}^1(\gamma\setminus\Sigma), \mathcal{H}^1(\gamma\cap\Sigma), \mathcal{H}^1(\Sigma)\right).$$

We now study some properties of Θ.

Proposition 4.4. Θ *is monotone, non-decreasing with respect to each of its variables.*

Proof. The monotonicity in the third variable is straightforward. The one in the first variable can be obtained observing that

$$\Theta(a,b,c) = \inf_{a \le s \le b} J(s, b-s, c) \tag{4.5}$$

and that the right-hand side of (4.5) is a non-decreasing function of a. The monotonicity in the second variable is obtained in a similar way, still relying on (4.5) and paying attention to the sets where the infimum is taken. □

Proposition 4.5. Θ *is lower semicontinuous.*

Proof. We have to show that

$$\Theta(a,b,c) \le \liminf_{n\to+\infty} \Theta(a_n, b_n, c_n)$$

when $a_n \to a$, $b_n \to b$ and $c_n \to c$. Let us consider for every real positive number ε and for every positive integer n a real number s_n such that $a_n \le s_n \le b_n$ and

$$J(s_n, b_n - s_n, c_n) \le \Theta(a_n, b_n, c_n) + \varepsilon.$$

Up to a subsequence, we can suppose that

$$\liminf_{n\to+\infty} \Theta(a_n, b_n, c_n) = \lim_{n\to+\infty} \Theta(a_n, b_n, c_n).$$

We can also suppose that $s_n \to s$, where $a \le s \le b$. Thanks to the semicontinuity of J

$$\Theta(a,b,c) \le J(s, b-s, c) \le \liminf_{n\to+\infty} J(s_n, b_n - s_n, c_n) \le \liminf_{n\to+\infty} \Theta(a_n, b_n, c_n) + \varepsilon.$$

Letting $\varepsilon \to 0^+$ yields the desired inequality. □

5. Existence theorem

In this section we continue to develop the tools we will use to prove Theorem 5.6.

Proposition 5.1. *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in K such that $x_n \to x$ and $y_n \to y$. If $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a sequence of closed connected sets such that $\Sigma_n \to \Sigma$, then*

$$d_\Sigma(x,y) \le \liminf_{n\to+\infty} d_{\Sigma_n}(x_n,y_n). \tag{5.6}$$

Proof. First, up to a subsequence, we can suppose that

$$\liminf_{n\to+\infty} d_{\Sigma_n}(x_n,y_n) = \lim_{n\to+\infty} d_{\Sigma_n}(x_n,y_n).$$

Given $\varepsilon > 0$, we choose a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \in \mathscr{C}_{x_n,y_n}$ and

$$\Theta(\mathcal{H}^1(\gamma_n \setminus \Sigma_n), \mathcal{H}^1(\gamma_n), \mathcal{H}^1(\Sigma_n)) \le d_{\Sigma_n}(x_n,y_n) + \varepsilon.$$

Up to a subsequence we can suppose that $\gamma_n \to \gamma$ (it is easy to check that $x_n \to x$ and $y_n \to y$ imply $\gamma \in \mathscr{C}_{x,y}$) and

$$\mathcal{H}^1(\gamma \setminus \Sigma) \le \lim_n \mathcal{H}^1(\gamma_n \setminus \Sigma_n),$$
$$\mathcal{H}^1(\gamma) \le \lim_n \mathcal{H}^1(\gamma_n),$$
$$\mathcal{H}^1(\Sigma) \le \lim_n \mathcal{H}^1(\Sigma_n).$$

Using the semicontinuity and monotonicity of Θ (Props. 4.4 and 4.5), we obtain

$$\begin{aligned}
d_\Sigma(x,y) &\le \Theta\left(\mathcal{H}^1(\gamma\setminus\Sigma), \mathcal{H}^1(\gamma), \mathcal{H}^1(\Sigma)\right)\\
&\le \Theta\left(\lim_{n\to+\infty}\mathcal{H}^1(\gamma_n\setminus\Sigma_n), \lim_{n\to+\infty}\mathcal{H}^1(\gamma_n), \lim_{n\to+\infty}\mathcal{H}^1(\Sigma_n)\right)\\
&\le \liminf_{n\to+\infty}\Theta\left(\mathcal{H}^1(\gamma_n\setminus\Sigma_n), \mathcal{H}^1(\gamma_n), \mathcal{H}^1(\Sigma_n)\right)\\
&\le \liminf_{n\to+\infty} d_{\Sigma_n}(x_n,y_n) + \varepsilon.
\end{aligned}$$

The arbitrary choice of ε gives then inequality (5.6). □

As a consequence of Proposition 5.1 we have the following corollary.

Corollary 5.2. *Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be sequences in K such that $x_n \to x$ and $y_n \to y$. If Σ is a closed connected set, then*

$$d_\Sigma(x,y) \le \liminf_{n\to+\infty} d_\Sigma(x_n,y_n).$$

In other words, d_Σ is a lower semicontinuous function on $K \times K$.

Proposition 5.5 will play a crucial role in the proof of our main existence result. We split its proof in the next two lemmas for convenience.

Lemma 5.3. *Let X be a compact metric space, $\{f_n\}_{n\in\mathbb{N}}$ a sequence of positive real valued functions defined on X. Let also g be a continuous positive real valued function defined on X. Then, the following statements are equivalent:*

(1) $\forall\varepsilon > 0\ \exists N\ :\ \forall n \ge N\ \forall x \in X \quad g(x) \le f_n(x) + \varepsilon$;
(2) $\forall x \in X\ \forall x_n \to x \quad g(x) \le \liminf_n f_n(x_n)$.

Proof.

- Let $x_n \to x$. Then

$$g(x_n) = f_n(x_n) + (g(x_n) - f_n(x_n)) \leq f_n(x_n) + \varepsilon.$$

By the continuity of g, taking the lower limit we achieve

$$g(x) \leq \liminf_{n\to+\infty} f_n(x_n) + \varepsilon. \tag{5.7}$$

Then (1) $\Rightarrow$ (2) is established when $\varepsilon \to 0^+$.

- Let us now prove that (2) $\Rightarrow$ (1). Suppose on the contrary that there exists a positive ε and an increasing sequence of positive integers $\{n_k\}_k$ such that

$$g(x_{n_k}) \geq f_{n_k}(x_{n_k}) + \varepsilon \tag{5.8}$$

for a suitable x_{n_k}. Thanks to the compactness of X we can suppose up to a subsequence that $x_{n_k} \to x$. Define

$$x_n = \begin{cases} x_{n_k} & \text{if } n = n_k \text{ for some } k \\ x & \text{otherwise.} \end{cases}$$

Then $x_n \to x$, and $g(x) \leq \liminf_n f_n(x_n)$. From (5.8) it follows,

$$g(x) \geq \liminf_{k\to+\infty} f_{n_k}(x_{n_k}) + \varepsilon \geq \liminf_{n\to+\infty} f_n(x_n) + \varepsilon \geq g(x) + \varepsilon$$

which is false. □

Lemma 5.4. *Let f be a lower semicontinuous function defined on a metric space (X, d) which ranges in $[0, +\infty]$. Then the set of functions $\{g_t \ : \ t \geq 0\}$ defined by*

$$g_t(x) = \inf\{f(y) + td(x, y) \ : \ y \in X\}$$

satisfies the following properties:

- $g_t \geq 0$;
- g_t *is t-Lipschitz continuous;*
- $g_t(x) \nearrow f(x)$ *as* $t \to +\infty$.

Proof. See Lemma 1.3.1 of [1] or Proposition 1.3.7 of [4]. □

Proposition 5.5. *Let $\{f_n\}_{n\in\mathbb{N}}$ and f be non-negative lower semicontinuous functions, all defined on a compact metric space (X, d). Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative measures on X such that $\mu_n \rightharpoonup^* \mu$. Suppose that*

$$\forall x \in X \ \forall x_n \to x \quad f(x) \leq \liminf_{n\to+\infty} f_n(x_n).$$

Then

$$\int_X f \ \mathrm{d}\mu \leq \liminf_{n\to+\infty} \int_X f_n \ \mathrm{d}\mu_n.$$

Proof. Let ψ be a continuous function with compact support such that $0 \leq \psi \leq 1$. Let g_t be the function of Lemma 5.4; since g_t satisfies the hypothesis of Lemma 5.3 with $g = g_t$, we have $g_t \leq f_n + \varepsilon$ for n large enough and then

$$\int_X g_t \psi \,\mathrm{d}\mu = \lim_{n \to +\infty} \int_X g_t \psi \,\mathrm{d}\mu_n \leq \liminf_{n \to +\infty} \int_X f_n \,\mathrm{d}\mu_n.$$

Taking the supremum in t and ψ, we obtain

$$\int_X f \,\mathrm{d}\mu \leq \liminf_{n \to +\infty} \int_X f_n \,\mathrm{d}\mu_n. \qquad \square$$

We may now state and prove our existence result.

Theorem 5.6. *The problem*

$$\min\{T(\Sigma) \;:\; \Sigma \in \mathscr{C}\}$$

admits a solution.

Proof. First, let us prove that for every $l > 0$ the class

$$\mathscr{D}_l := \{\Sigma \;:\; \Sigma \in \mathscr{C},\ \mathcal{H}^1(\Sigma) \leq l\}$$

is a compact subset of the metric space $(\mathscr{C}(K), d_{\mathcal{H}})$. Since $(\mathscr{C}(K), d_{\mathcal{H}})$ is a compact space, it is enough to show that $\mathscr{D}_l$ is closed. We already know that the Hausdorff limit of a sequence of closed connected set is a closed connected set. If $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a sequence of closed connected sets such that $\mathcal{H}^1(\Sigma_n) \leq l$

$$\Sigma_n \to \Sigma \Longrightarrow \mathcal{H}^1(\Sigma) \leq \liminf_{n \to +\infty} \mathcal{H}^1(\Sigma_n) \leq l$$

by Gołab theorem (Th. 3.2).

Second, by our assumption on the function J

$$d_\Sigma(x, y) \geq G\left(\mathcal{H}^1(\Sigma)\right)$$

so that

$$T(\Sigma) \geq G\left(\mathcal{H}^1(\Sigma)\right).$$

Then, if $\{\Sigma_n\}_{n\in\mathbb{N}}$ is a minimizing sequence, the sequence of 1-dimensional Hausdorff measures $\{\mathcal{H}^1(\Sigma_n)\}_{n\in\mathbb{N}}$ must be bounded, *i.e.* $\mathcal{H}^1(\Sigma_n) \leq l$, for some $l > 0$.

If we prove that the functional $\Sigma \mapsto T(\Sigma)$ is sequentially lower semicontinuous on the class $\mathscr{D}_l$, then the existence of an optimal Σ will be a consequence of the fact that a sequentially lower semicontinuous function takes a minimum on a compact metric space. Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{D}_l$ such that $\Sigma_n \to \Sigma$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be an optimal transport plan for the transport problem

$$\min\left\{\int_{K\times K} d_{\Sigma_n}(x, y)\mathrm{d}\mu \;:\; \pi^+_\#\mu = \mu^+, \pi^-_\#\mu = \mu^-\right\}.$$

Up to a subsequence we can suppose $\mu_n \rightharpoonup^* \mu$ for a suitable μ. It is easy to see that μ is a transport plan between μ^+ and μ^-.

Since by Proposition 5.1 $d_\Sigma(x, y) \leq \liminf_n d_{\Sigma_n}(x_n, y_n)$ for all $x_n \to x$ and $y_n \to y$, by Lemma 5.5 we have

$$\int_{K\times K} d_\Sigma(x, y) \,\mathrm{d}\mu \leq \liminf_{n \to +\infty} \int_{K\times K} d_{\Sigma_n}(x, y) \,\mathrm{d}\mu_n. \tag{5.9}$$

Then by (5.9) we have

$$T(\Sigma) \le \int_{K\times K} d_\Sigma(x,y)\ \mathrm{d}\mu \le \liminf_{n\to+\infty} \int_{K\times K} d_{\Sigma_n}(x,y)\ \mathrm{d}\mu_n = \liminf_{n\to+\infty} T(\Sigma_n). \qquad \square$$

We end with the following remark.

Remark 5.7. Note that if Σ_n is a minimizing sequence, then the measure μ obtained in the proof of Theorem 5.6 is an optimal transport plan for the transport problem

$$\min\left\{\int_{K\times K} d_\Sigma(x,y)\ \mathrm{d}\mu\ :\ \pi^+_\#\mu = \mu^+, \pi^-_\#\mu = \mu^-\right\}.$$

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