Partial Durations: The Case of Fixed and Floating Rate Bonds

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Floating rate bonds are coupon-paying instruments generally indexed to interest parameters. At the trading date, the payment dates and indexation rules are known, while the value of future coupons is uncertain. In this perspective, floating rate bonds are generally seen as a portfolio of zero coupon bonds. Therefore, at each coupon payment date, the bond should be quoted at face value and its duration should match the maturity of the replicating zero coupon bond. However, the empirical evidence based on historical prices of real floating rate bonds shows that such an instrument is not systematically quoted at parity and its market risk profile could differ from that of the replicating zero coupon bond. The aim of this work is to study the duration of a floating rate bond using a partial modified duration approach after decomposing the floating rate bond into its main building blocks. The final goal is to capture the effective risk factors of these instruments in real financial markets in order to define synthetic risk measures that should be able to reflect the instrument's effective risk profile.

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JEL Codes: C02, C60, G10, G11

1. Introduction

Duration is a fundamental asset and liability management tool commonly used to measure interest rate risk exposure. For plain vanilla bonds, duration measures the ratio between the proportional change in the bond value to the infinitesimal parallel shift of the yield curve. As commonly known, the duration metric of a plain vanilla bond may provide inaccurate results as it measures the sensitivity of a bond price according to a single risk factor (namely, the yield) and is only of use if the yield curve shifts in parallel. If, for example, the yield curve is affected by a rotation, then such a measure is of little value. In addition, if a bond is callable or puttable (or an option is embedded) then the measure fails. Nevertheless, capital market practitioners are comfortable with duration because it represents a single, synthetic and powerful risk indicator. Important to underline in this context is the significant research on potential instruments to enhance the accuracy of estimates using the traditional duration metric. For example, duration accuracy can be

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improved by adding extra elements, such as convexity, or using duration vectors instead of a single measure.

Nevertheless, all possible upgrades of the traditional duration measure still consider a single yield curve as a unique risk factor influencing bond price behavior. This has been relatively true, especially for the Eurozone, up to the recent sovereign debt crisis. Bond portfolio risk frameworks based mainly on traditional duration have started showing clear signs of weakness in a macroeconomic scenario where there is no longer a single yield curve and therefore a unique and given source of risk. The price of a bond and/or the value of a fixed income portfolio is instead influenced by different components (the creditworthiness of issuers amongst others). Moreover, daily rate variations of different risk factors are far from correlated and could follow different paths. For instance, as a risk-off mood starts to prevail in the market, the risk-free rate drops and the credit spread jumps as investors sell riskier - and buy safer - bonds as a consequence of the typical 'flight to quality' movement. However, even in this context, being able to rely on a set of synthetic indicators reflecting a fixed income portfolio risk profile is still important for market practitioners who need to understand the potential impact on the entire portfolio value arising from different and often negatively correlated risk factor movements.

To achieve this goal, sophisticated computational engines enable obtaining more complex risk measures with a single click (i.e., Value at Risk and/or ALM sensitivity measures), which take into account different risk factors (in case of bonds, the risk-free rate coupled with the issuer's credit worthiness). These measures again represent a too synthetic single figure as the relevant moves are difficult to understand and attempting a possible match with traditional duration estimations could be even more challenging and/or misleading.

In this context, our present work aims to enrich the capital market practitioner's toolbox with a new set of sensitivity measures (Addessi et al. 2012). Starting from the partial derivatives approach embedded in the traditional duration framework, this could prove very useful to understanding the main causes of variations in the fixed-income portfolio value depending on several and different risk factors and applying this to a variable cash flow evaluation.

Specifically, the objective of our work is to bridge the gap in literature with respect to duration calculations using a single yield curve, suggesting the use of a risk factor matrix that can explain changes in the value of bonds in line with the variation of each risk factor identified and hypothesizing synchronous or uncorrelated variations. Indeed, this multidimensional approach, nevertheless based on Reitano's (1992a and 1992b) partial duration model and the yield single curve, is in line with current market scenarios where a single currency can usually be associated with a plurality of curves (particularly for the Eurozone where each issuing government uses a particular yield curve).

Our proposed approach enables:

- Enriching the set of analysis tools that can be used to interpret the effect (ex-ante and ex-post) of market movements on the market values of single instruments and/or portfolios.
- Critically evaluating the results obtained with more sophisticated measurement instruments (i.e., VAR) for which, however, an analysis by components can be particularly costly without always clarifying the potential market effects.

- Assessing and managing the effectiveness of derivatives in hedging only one of the risk factors (i.e., asset swaps to hedge the interest rate risk implicit in government bonds).

With reference to the floating rate instruments, we use Reitano's (1992a) original framework based on partial durations duly extended to take account of a variety of risk factors to enrich and extend Pacati's (2001) model developed in a single curve scenario.

In particular, our proposed approach overcomes, also from a purely mathematical perspective, the widespread assumption amongst market practitioners that a floating rate instrument is by construction immune from interest rate changes; an approach as yet unexplored in literature.

We test the validity of our proposed mathematical approach on real Italian market data.

2. Literature Review

Macaulay (1938) first defined duration to compare securities with different payment schedules based on their weighted average income stream. The various duration measures that followed can generally be divided into durations based on traditional bond mathematics and durations based on no-arbitrage or equilibrium-based bond pricing models.

The first duration measure, albeit not explicitly modeling the underlying term structure behavior, is attributed to Macaulay (1938). Subsequent studies such as Cooper's (1977) show that duration is not the ideal measure to predict the response of riskless asset values to changes in interest rates. The study indicates that a blind selection of portfolios based on their duration would be an unrealistically naïve strategy and that there are ways of enriching the concept of duration without losing the basic analytical framework from which it derives. Bierwag (1977), Bierwag and Kaufman (1977), and Khang (1979) assume specific characteristics of the term structure movement, such as changes in both the level and shape of the yield curve. Khang (1979) proposes different duration measures for specific changes in the yield curve.

More recent measures have been proposed by Chambers, Carleton and McEnally (1988), Ho (1992), Nawalkha and Chambers (1996, 1997), and Nawalkha, Soto and Zhang (2003). Chambers, Carleton and McEnally (1988) indicate that interest risk can only be measured by a vector of numbers as opposed to a single number and propose the Duration Vector. Ho (1992) proposes the Key Rate Duration, which associates the price sensitivity of a bond to multiple segments of the yield curve. Nawalkha and Chambers (1996) propose an M-Absolute duration measure that allows greater immunization compared to the traditional Fisher and Weil (1971) duration by selecting a bond portfolio clustered around its planning horizon date. Nawalkha and Chambers (1997) and Nawalkha, Soto and Zhang (2003) extend the Nawalkha and Chambers (1996) analysis into a multi-factor M-Vector.

The development of duration measures based on no-arbitrage or equilibrium-based bond pricing models coincides with Ingersoll, Skelton and Weil's (1978) appraisal of Macaulay's (1938) duration and developments in studies on bond pricing models. The seminal papers of Vasicek (1977) and Cox, Ingersoll and Ross (1985) gave birth to the no-arbitrage or

equilibrium-based bond pricing models. Ingersoll, Skelton and Weil (1978) provide an arbitrage-based criticism of the Macaulay (1938) duration. To resolve this criticism, duration measures are developed that take into account an explicit random process driving bond pricing models. Cox, Ingersoll and Ross (1979) formally derive a duration measure consistent with the general equilibrium conditions of the Cox, Ingersoll and Ross (1985) term structure model.

Moreover, Reitano (1992a) proposes partial duration as a measure applicable to quantifying the potential effect exerted by non-parallel changes in the yield curve. Reitano (1992b) introduces directional duration, assuming that the interest rate shift is not unique for the entire investment period and could move in different directions.

Reitano's (1992a) model does not however apply partial duration to floating rate bonds and is based on a single curve scenario, which does not enable taking due account of the proliferation of risk factors that we have witnessed since the Lehman crisis.

Other authors (see for example, Shirvani and Wilbratte 2005), propose an approximation formula for bond price volatility as an alternative to the well-known Macaulay (1938) formula to achieve more accurate approximations of the magnitude of coupon bond price changes in response to interest rate changes. In addition, their model resolves an insufficiency of the Macaulay model by providing a functional form that reflects the asymmetric effects of interest rate changes on bond prices. However, this approach does not consider the possibility of breaking down the duration into its partial *n* durations or breaking down the yield curve into its specific risk components.

Several authors (e.g., Pacati 2001) establish equivalency between Macaulay's duration applied to known cash flows and variable cash flows. Pacati (2001) attempts to evaluate the price and duration of a perfectly indexed bond with variable cash flow and demonstrate that it can be seen as a zero coupon bond portfolio and its duration is consequently equal to the maturity of the ZCB.

Combining Reitano's evolved model with Pacati's framework allows synthesizing and modeling the variations in the variable interest rate using partial duration and multiple risk factors, thus including the credit component of the issuing institution that in terms of pricing is represented by the spread compared to the risk-free rate and to fluctuations in bond prices that in times of market turmoil are anything but contained.

3. Fixed and Floating Rate Bond Framework: Theoretical Underpinnings

Partial duration (Reitano 1992a) allows quantifying the potential effect exerted by both parallel and non-parallel changes in the yield curve. If the yield curve has *n* time points, partial duration $D_k\left(0;\delta_k\right)$ referring to the kth yield curve point is:

$$D_k\left(0;\delta_k\right) = -\frac{1}{P} \cdot \frac{\partial P_k}{\partial \delta_k} \tag{1}$$

where $\partial \delta_k$ represents the infinitesimal variation to the kth yield curve point and ∂P_k is the resultant bond price variation. If the yield curve shifts in parallel, we can compute the

traditional *synthetic* duration, which can be approximated by the sum of the *n* partial durations (see equation (2)).

$$D(0;\delta) \cong \sum_{k=1}^{n} D_k(0;\delta_k)$$

(2)

More interesting is the hypothesis of not only an additive but also a rotary shift in the yield curve (i.e., yield curve twist) as shown in Figure 1. In this case, a partial duration framework could be useful. To compute the effect produced by a curve twist, we must consider two different (*n*-dimensional) information vectors: $\vec{\delta} = (\delta_1, \delta_2, \ldots, \delta_k, \ldots, \delta_n)$ and $\vec{Z} = (z_1, z_2, \ldots, z_k, \ldots, z_n)$, both defined on the same time schedule. The first vector represents the (usual) information on market yields, the second is the shift direction vector: its components $(z_k \in \Re)$ are used to graduate the movements in the yield curve. Figure 1 graphically shows a numeric example of yield curve twist. If $\vec{\delta} = (2\%, 3\%, 5\%)$ and $\vec{Z} = (1; -0,5; 3)$ the new yield vector is $\vec{\delta}* = (2\% + \Delta; 3\% - 0,5 \cdot \Delta; 5\% + 3 \cdot \Delta)$, where Δ represents the shift amplitude (3):

$$\vec{\delta}* = \vec{\delta} + \Delta$$

(3)

Figure 1. Yield curve twist: an example
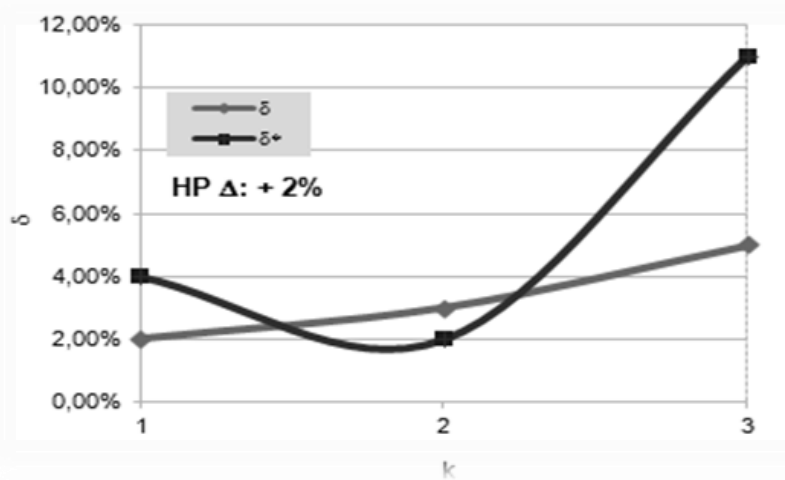
Given the bond price $P(0;\vec{\delta})$ and its duration $D(0;\vec{\delta})$, we can estimate the effect on the current value of a fixed income instrument using Taylor's expansion:

$$(\Delta P) \cong P(0;\vec{\delta}) \cdot \left(1 - D(0;\vec{\delta}) \cdot \Delta\vec{\delta}\right)$$

(4)

Using equations (3) and (4), the new current bond value - under the hypothesis that only the δ_k moves - is explained in equation (5):

$$P(\Delta\delta_k) \cong P(0;\vec{\delta}) \cdot \left(1 - D_k(0;\delta_k) \cdot \Delta\delta_k\right)$$

(5)

Its modified (partial) duration D_k^M is:

$$D_k^M \cong -\frac{P(\Delta\delta_k) - P(0;\vec{\delta})}{P(0;\vec{\delta}) \cdot \Delta\delta_k}$$

Both the duration and modified duration can be decomposed into the sum:

$$D(0;\delta) \cong z_1 D_1(\delta) + z_2 D_2(\delta) + \cdots + z_k D_k(\delta) + \cdots + z_n D_n(\delta)$$

(6)

where $z_k D_k(\delta)$ with k=1, 2, …, n represents the contribution on the duration of the k-th cash flow, relative to the direction $\vec{Z}$.

The main lesson from the recent Eurozone sovereign debt crisis is that even within a single currency area, the yield curve is no longer unique. It is therefore no longer possible to assume that each single government lends money based on the same yield curve, which is mainly influenced by the risk-free component. The assumption that the creditworthiness of an issuer is only a residual, essentially flat and not very volatile component, is no longer acceptable. Empirical evidence has incontrovertibly demonstrated that this is not the baseline scenario for traders even in the so-far quiet fixed income market.

In particular, Figure 2 shows that from the second half of 2011 up to the second half of 2013, against a progressive reduction in the risk-free component, a steady increase in the component occurred as a result of the principal component of the Italian government bond yield curve.

Figure 2: **The evolution of fixed income European markets: the case of Italy**
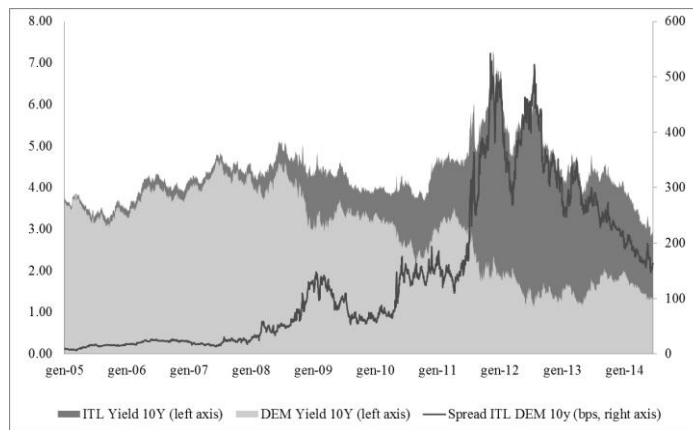
For this reason, we decompose the interest rate δ into its main components. Under this hypothesis, the traditional yield is a scalar obtained by multiplying two information vectors: $\vec{v}$ (different risk factor) and $\vec{w}^T$ (different weights). We assume a two dimensional $\vec{v}$ vector. In other words, the interest rate is the result of the interaction of the risk-free rate and spread (assumed as a proxy of issuer creditworthiness). If we assume $\vec{w}^T = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the interest rate becomes:

$$\delta = \vec{v} \cdot \vec{w}^T = \begin{pmatrix} \tau & \sigma \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{7}$$

where τ represents the risk-free rate component and σ the spread component. Using equation (7), the bond price becomes:

$$P(0;\delta) = \sum_{k=1}^{n} c_k \cdot \exp\{-\delta \cdot t_k\} = \sum_{k=1}^{n} c_k \cdot \exp\{-\tau \cdot t_k\} \cdot \exp\{-\sigma \cdot t_k\} \tag{8}$$

We use equation (1) to compute the partial derivatives of (8) with respect to both the risk-free rate component and the spread component, and obtain two different sensitivity measures:

$$\frac{\partial P(0;\delta)}{\partial \tau} = -\sum_{k=1}^{n} t_k \cdot c_k \cdot \exp\{-\tau \cdot t_k\} \cdot \exp\{-\sigma \cdot t_k\} \tag{9.a}$$

$$\frac{\partial P(0;\delta)}{\partial \sigma} = -\sum_{k=1}^{n} t_k \cdot c_k \cdot \exp\{-\tau \cdot t_k\} \cdot \exp\{-\sigma \cdot t_k\} \tag{9.b}$$

The proposed decomposition does not involve the introduction of new duration measures. Traditional duration can be used to identify *ex ante* the effect on a bond price caused by multiple risk factors and/or to check *ex post* the effective causes leading to a given market price movement. Indeed, using both the Taylor expansion (see equation (4)) and the proposed decomposition (see equation (7)), we can estimate the new price:

$$P(0;\delta + \Delta\delta) \cong P(0;\delta) + \frac{\partial P(0;\delta)}{\partial \tau} \cdot \Delta\tau + \frac{\partial P(0;\delta)}{\partial \sigma} \cdot \Delta\sigma \tag{10}$$

If we consider a yield curve, we can easily decompose the yield vector:

$$\vec{\delta} = V \cdot W$$

where $\vec{\delta}$ is the n-component yield vector, V is a $[n \times 2]$ matrix representing the two risk factors and W is a $[2 \times 1]$ matrix representing the different risk factor weights:

$$V = \begin{pmatrix} \tau_1 & \sigma_1 \\ \tau_2 & \sigma_2 \\ \vdots & \vdots \\ \tau_k & \sigma_k \\ \vdots & \vdots \\ \tau_n & \sigma_n \end{pmatrix} \tag{11.a}$$

$$\mathbf{W} = \begin{pmatrix} w_1 & w_2 \end{pmatrix}$$
(11.b)

$$\vec{\delta} = \begin{pmatrix} \tau_1 & \sigma_1 \\ \tau_2 & \sigma_2 \\ \vdots & \vdots \\ \tau_k & \sigma_k \\ \vdots & \vdots \\ \tau_n & \sigma_n \end{pmatrix} \cdot \begin{pmatrix} w_1 & w_2 \end{pmatrix}$$
(11)

According to equations (11.a), (11.b) and (11), the duration, defined as the first central moment, becomes:

$$D\left(0;\vec{\delta}\right) = \frac{1}{P\left(0;\vec{\delta}\right)} \cdot \left\{ \sum_{k=1}^{n} t_k \cdot c_k \cdot \exp\{-t_k \tau_k\} + \sum_{k=1}^{n} t_k \cdot c_k \cdot \exp\{-t_k \sigma_k\} \right\}$$
(12)

Focusing on the interaction between the risk factor decomposition and the partial duration framework as discussed above and using equation (8) to define the bond price, we can compute the first partial derivatives with respect to τ_k and σ_k (equations (9.a) and (9.b)) where the kth partial duration is (see equation 1):

$$D_k = -\frac{1}{P} \cdot \frac{\partial P}{\partial \delta_k} = \frac{t_k \cdot c_k \cdot \exp\{-t_k \cdot \delta_k\}}{P}$$

We can decompose the partial derivative (and thus the partial duration) according to the risk factor's decomposition explained in equation (11). The kth decomposed partial durations (DD_k) are:

$$DD_k(\tau) = \frac{t_k \cdot c_k \cdot \exp\{-\tau t_k\} \cdot \exp\{-\sigma t_k\}}{P}$$
(13a)

$$DD_k(\sigma) = \frac{t_k \cdot c_k \cdot \exp\{-\tau t_k\} \cdot \exp\{-\sigma t_k\}}{P}$$
(13b)

$$DD_k = \omega_1 \frac{t_k \cdot c_k \cdot \exp\{-\tau t_k\} \cdot \exp\{-\sigma t_k\}}{P} + \omega_2 \frac{t_k \cdot c_k \cdot \exp\{-\tau t_k\} \cdot \exp\{-\sigma t_k\}}{P}$$
(13)

Equations (13a) and (13b) show that for each t_k we need to consider two distinct partial durations. Interesting to note is that equations (13a) and (13b), representing the derived partial durations, are analytically the same but depend on different risk parameters: in our case, this allows separately evaluating the risk free and the credit effect on a bond price. The linear combination (see equation (13)) is useful to estimate the total effect of the two components on price. Extending this approach to a general approach with n different risk factors is intuitive.

In this way, we enrich Reitano's (1992a) approach with a new dimension: market risk factors. Analytically, this step is equivalent to transforming the k partial duration vector into a $[k \times n]$ matrix, where k is the number of periods corresponding to the bond cash flows and n is the number of risk factors influencing the behavior of the instrument's price.

As a consequence, the linear combination of the different matrix elements can help the analyst to quickly and intuitively consider the consequences of uncorrelated risk movements.

The shock price can then be estimated as:

$$P\left(0;\vec{\delta}+\Delta\delta_k\right) \cong P\left(0;\vec{\delta}\right) + \frac{\partial P\left(0;\vec{\delta}\right)}{\partial \tau_k} \cdot \Delta\tau_k + \frac{\partial P\left(0;\vec{\delta}\right)}{\partial \sigma_k} \cdot \Delta\sigma_k \qquad (14)$$

The overall effect on price will depend on the possible moves of the different risk factors. As Figure 3 shows, if both risk factors drop, the bond price increases, if both risk factors increase, the bond price drops. Forecasting the effect on the bond price of the different movements of the two risk factors is more difficult: the price may be unaffected if the risk-free and spread movements are balanced.

Figure 3: Possible price paths
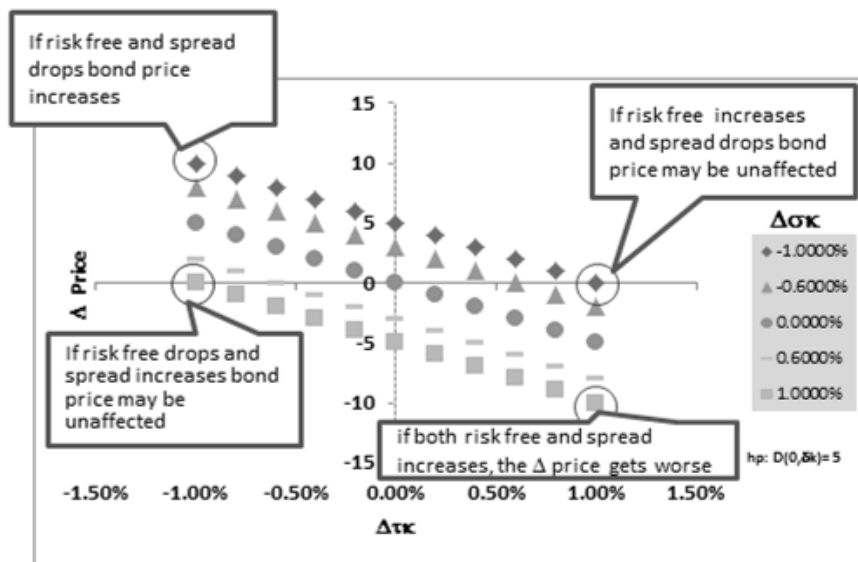
Applying partial durations and risk factor decomposition to the floating rate bond can be even more challenging.

Floating rate bonds, largely used in financial markets, are coupon-paying instruments linked to the interest rate term structure. At the trade date, the payment dates and indexation rules are known, while the amount of each future coupon is unknown and therefore uncertain. Moreover, floating rate bonds are often indexed to market parameters, which could be strongly correlated to the risk-free yield curve generally used to discount cash flows.

The aim of this section is to define an analytical framework that identifies a comprehensive set of duration measures for floating rate bonds.

Hence, we focus on a floating rate bond with the interest coupon indexed to a specific market parameter (i.e., 6-month Euribor rate). In this context, the same risk-free yield

curve will be used to estimate future interest payments and thereafter discounting the bond cash flows.

To achieve our objective, we decouple the entire floating bond price into its single building blocks as follows:

a. Fixed block corresponding to the fixed and known components of future cash flows
b. Floating block corresponding to unknown future payments.

The price of the floating rate bond will be given by the sum of prices computed separately for the above-mentioned components:

$$P\left(0;\vec{\delta}\right) = P_{Fix}\left(0;\vec{\delta}\right) + P_{Float}\left(0;\vec{\delta}\right) \tag{15}$$

where

$$P_{Fix}\left(0;\vec{\delta}\right) = \sum_{k=1}^{n} c_k^{Fix} \cdot \exp\{-\tau_k t_k\} \cdot \exp\{-\sigma_k t_k\}$$
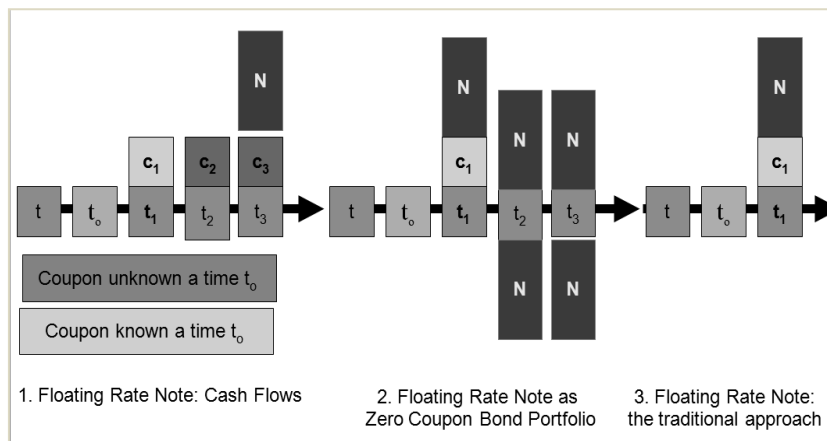
and

$$P_{Float}\left(0;\vec{\delta}\right) = \sum_{k=1}^{n} c_k^{Float} \cdot \exp\{-\tau_k t_k\} \cdot \exp\{-\sigma_k t_k\}$$

The fixed block can be analyzed using the set of duration measures described in the previous section. We thus focus on the floating block starting from Pacati's (2001) approach to estimate the duration of a floating rate bond in a risk-free environment.

In this perspective, floating rate bonds are generally seen as a portfolio of zero coupon bonds. This means that at each payment date, the bond should quote at par and its duration should match the maturity of the zero coupon expiring at the next coupon payment date. If the yield curve used to calculate future interest payments corresponds to that used in discounting the entire bond cash flows, a single zero coupon bond is enough to replicate the entire floating rate bond, as shown in Figure 4.

### Figure 4: Floating rate bond: evaluation framework within a risk-free environment

1. Floating Rate Note: Cash Flows
2. Floating Rate Note as Zero Coupon Bond Portfolio
3. Floating Rate Note: the traditional approach

Nevertheless, the empirical evidence based on a historical series of real floating rate bond prices shows that such an instrument does not systematically quote at par at each single coupon payment date.

Figure 5 shows that, especially in the second half of 2011 and during 2012, the instrument price settled at levels even lower than the 85 percentage points (left side) and also deviating from parity by nearly 15 percentage points in proximity to the coupon payment dates (right side).

Figure 5: Floating Rate Bond: empirical evidence
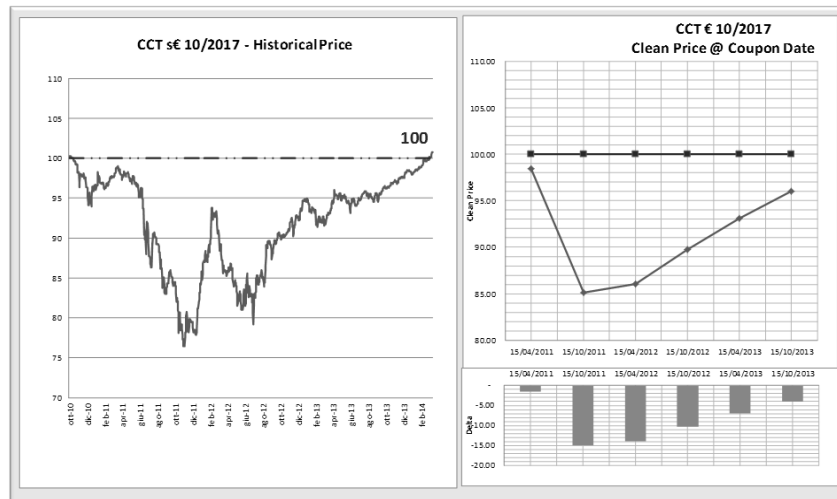
The entire market risk profile could therefore differ greatly from the risk of the replicating zero coupon bond. Differences may arise from indexation - which in effective markets is rarely synchronous – and the presence of different risk factors affecting the entire value of a floating rate bond.

Consequently, moving to a more flexible partial duration framework, the Pacati approach assumes that when estimating the sensitivity of a floating rate bond to the risk-free component of the yield curve, only a single measure can be referred to, namely, the maturity of the zero coupon bond expiring at the next coupon payment date.

Once the risk-free sensitivity measure has been identified, we focus on the spread sensitivity measure.

Intuitively, according to the duration framework rationale, it could be argued that the more relevant source of spread sensitivity will be given by the face value payable at the bond's final maturity.

Therefore, to determinate the spread sensitivity and given that we have already dealt with the risk-free sensitivity, we can first adjust the final cash flow as follows (assuming a notional amount of one unit):

$$c_n(0;\delta)_{spread} = \exp\{-\pi t_n\} \cdot \exp\{-\sigma t_n\} - \exp\{-\pi t_n\} = \exp\{-\pi t_n\} \cdot (\exp\{-\sigma t_n\} - 1) \tag{16}$$

We then calculate the partial derivatives:

$$\frac{\partial c_n(0;\delta)_{spread}}{\partial \tau} = -t_n \cdot \exp\{-\tau t_n\} \cdot (\exp\{-\sigma t_n\}-1) \tag{17}$$

$$\frac{\partial c_n(0;\delta)_{spread}}{\partial \sigma} = -t_n \cdot \exp\{-\tau t_n\} \cdot \exp\{-\sigma t_n\} \tag{18}$$

Finally, we move to a partial duration approach obtaining:

$$DD_{\tau,n} = \frac{-t_n \cdot \exp\{-\tau t_n\} \cdot \exp\{-\sigma t_n\}}{\exp\{-\tau t_n\} \cdot \exp\{-\sigma t_n\}} - \frac{-t_n \cdot \exp\{-\tau t_n\}}{\exp\{-\tau t_n\}} = 0 \tag{19}$$

$$DD_{\sigma,n} = \frac{-t_n \cdot \exp\{-\tau t_n\} \cdot \exp\{-\sigma t_n\}}{\exp\{-\tau t_n\} \cdot \exp\{-\sigma t_n\}} = -t_n \tag{20}$$

Equations (19) and (20) show that the adjusted last payment will have zero sensitivity to the risk-free component of the yield curve, being only exposed to the spread movements of the yield curve.

4. Some Applications: The Floating Rate Bond Case

4.1 Methodology

The aim of this section is to test the validity and potential uses of the DD_k approach.
To achieve this goal, we proceed with the following main steps:

1. Decompose the generic risk factor bond yield into its main components. This enables estimating the possible effects of different risk factors on a fixed income portfolio's value, enhancing the evaluation of the risk mitigation effect of derivatives such as interest rate swaps and, in case of partial risk hedging, aiming at hedging "only" the risk-free component.
2. Extend the traditional duration (Macauley 1938; Castellani et al. 1993; De Felice and Moriconi 1991) with reference to a partial duration framework (Reitano 1992a), which also allows taking into account the non-parallel movements of multiple risk factors.
3. Define a partial duration approach for a variable cash flow.

With reference to testing the Italian market data, it should be specified that given the proposed mathematical approach, the tests were conducted with the aim of verifying whether given a certain market scenario, the results of the approximations obtained with the presented framework were sufficiently in line with the data recorded ex post. Hence, in terms of the present paper, the more strictly statistical aspects inherent in the sample size and the testable hypotheses were not applied. However, our future intention is to test the soundness of the proposed approach to verify its correctness on a wider range of data.

4.2 Empirical Evidence

We use a floating rate bond - the Italian CCT€ with maturity of about 4 years - with semi-annual coupons indexed to the 6-month Euribor rate with the characteristics specified in Table 1.

Table 1: Financial characteristics of an Italian CCT€ used to test the DD_k model.

International Securities Identification Number	IT0004652175
Face Value	100
Variable Coupon	EUR 6M +0.80 bps
Maturity	15/10/2017
Issuer	Italy
Discount Curve	Italian Government Bond Yield Curve

We define the main risk factors as follows:
a. the **risk-free component** that determines the instrument's floating coupon (i.e., euro swap curve against 6 months);
b. the **credit-spread component** that reflects the issuer's credit profile (i.e., the difference between the swap curve and the Italian government bond yield curve).
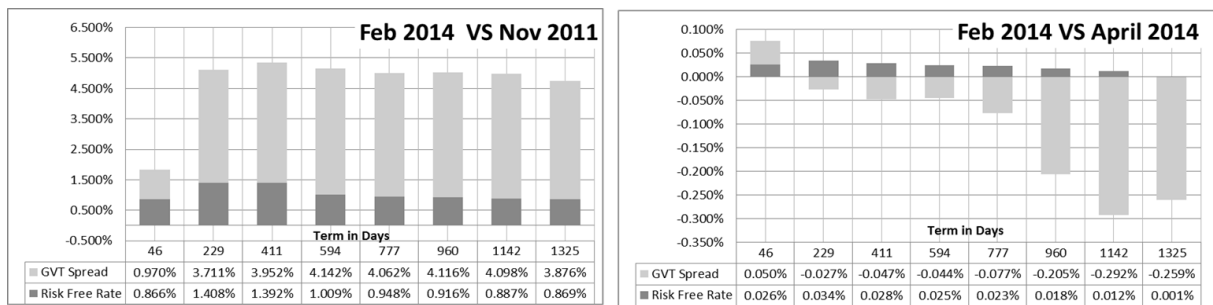
To establish the soundness of the partial duration approach to a floating rate bond, we estimate the expected change in price through the shift of generic points δ_ϕ, τ_ϕ and σ_ϕ, with ϕ representing the corresponding term to the duration, and compare it with the price exerted by the given market scenarios.
We focus on two different real market scenarios, decomposing the risk factors affecting the FRN price.

In the first scenario, we compare the Italian government yield curve on 28 February 2014 with the level of the Italian curve on 3 April 2014: the risk-free component shows an upward movement while the spread component, especially on the longer term, shows downward pressure. The risk factors show different movements not only in their amplitude, but also in the direction of the shifts.

In the second scenario, we compare the level registered by the Italian curve at the end of February 2014 with that registered on 7 November 2011. In this case, both risk components show an upward movement but the credit-spread component shows a larger movement, driven by the turmoil experienced by the European government bond markets at the end of 2011.

Table 2: Market scenarios

	46	229	411	594	777	960	1142	1325
Feb 2014 VS Nov 2011								
GVT Spread	0.970%	3.711%	3.952%	4.142%	4.062%	4.116%	4.098%	3.876%
Risk Free Rate	0.866%	1.408%	1.392%	1.009%	0.948%	0.916%	0.887%	0.869%

	46	229	411	594	777	960	1142	1325
Feb 2014 VS April 2014								
GVT Spread	0.050%	-0.027%	-0.047%	-0.044%	-0.077%	-0.205%	-0.292%	-0.259%
Risk Free Rate	0.026%	0.034%	0.028%	0.025%	0.023%	0.018%	0.012%	0.001%

As Table 2 shows, the scenarios adopted for the empirical analysis include:
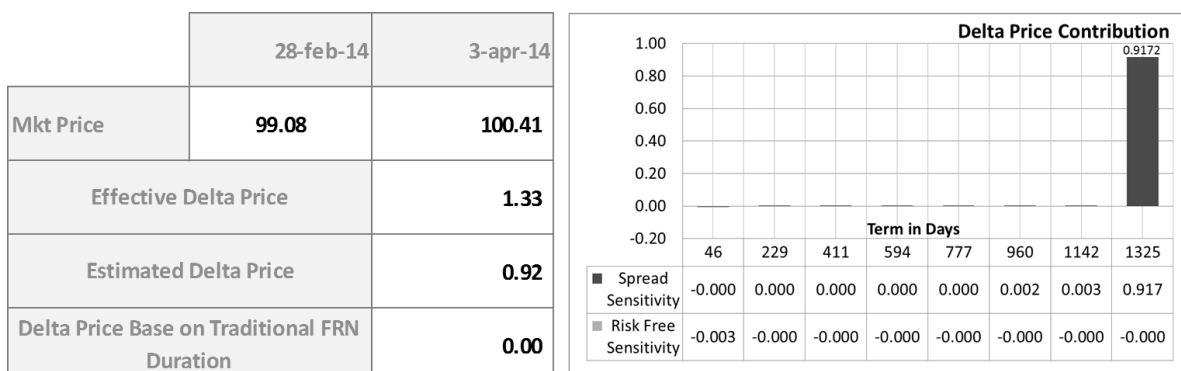
- In the November 2011 scenario, a positive change in both components of the yield curve. The shifts applied to the risk-free curve and the spread component move respectively in the 80/140 interval from 97 to 415 base points.
- In the April 2014 scenario, a positive change in the risk-free component (in the range from 3 to 18 bps) and a reduction in the spread component that reaches levels of - 30 bps in the medium term.

The adoption of variations differentiated by type of components (risk-free and spread) and for different maturities of the yield curve is essential to highlight the consistency of the partial durations approach.

Table 3 and Table 4 report the sensitivity measures calculated by applying our framework based on partial durations and risk factor decomposition.

As expected, in both cases, we observe that the traditional duration approach does not permit a valid estimation of the price sensitivity for a CCT€ (Buffa and Peccati 1989). The price variation we compute using the traditional approach in both scenarios is close to zero while the market price of the instrument varies significantly especially in the November 2011 scenario.

Table 3: Sensitivity Analysis February 2014 vs. April 2014

	28-feb-14	3-apr-14
Mkt Price	99.08	100.41
Effective Delta Price		1.33
Estimated Delta Price		0.92
Delta Price Base on Traditional FRN Duration		0.00

Delta Price Contribution

Term in Days	46	229	411	594	777	960	1142	1325
Spread Sensitivity	-0.000	0.000	0.000	0.000	0.000	0.002	0.003	0.917
Risk Free Sensitivity	-0.003	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000	-0.000

Table 4: Sensitivity Analysis February 2014 vs. November 2011

	28-feb-14	November 2011
Price	99.08	87.45
Effective Delta Price	-	11.63
Estimated Delta Price	-	14.06
Delta Price Base on Traditional FRN Duration		0.02

Delta Price Contribution

Term in Days	46	229	411	594	777	960	1142	1325
Spread Sensitivity	-0.0005	-0.0093	-0.0178	-0.0268	-0.0343	-0.0427	-0.0502	-13.7117
Risk Free Sensitivity	-0.1097	-0.0035	-0.0062	-0.0065	-0.0080	-0.0095	-0.0109	-0.0123

At the same time, the application of our framework returns good results: the price variation we compute in both scenarios is very similar to the effective price variation we observe when fully evaluating the instrument.

In particular, Tables 3 and 4 show that the analysis of the different risk factors and maturities of the yield curve in the partial approach allows highlighting the prevalence of the spread component that is mainly manifested - also in the floating rate instrument analyzed - in proximity to bond maturity. In both the April 2014 vs. the February 2014 scenario (Table 3) and in the April 2014 vs. November 2011 scenario (Table 4) the estimated price change, in addition to being in line with the actual change in the price recorded on the markets, is primarily determined by the variation of the curve due to the movement of the spread.

5. Summary and Conclusions

Duration is one of the most frequently used measures of bond interest rate risk exposure when the bond is a "plain vanilla" type. If wanting to estimate changes in a callable or puttable bond value, the duration approach fails. Nevertheless, capital market practitioners often refer to duration as it is synthetic and simple to use. Together with duration, researchers and practitioners use other powerful tools (such as convexity) to improve the accuracy of the price movement estimates. These duration upgrades still consider a single risk factor (yield curve) and only the possibility that the yield curve shift up or down but does not, for example, rotate.

The recent sovereign debt crisis has highlighted that more than one risk factor exists and must be considered: there is no longer a single yield curve and therefore a unique and given source of risk. Moreover, bond prices are influenced not only by the risk free rate but also, for example, by the creditworthiness of issuers.

In this context, it is important to consider a set of synthetic risk indicators to understand the potential effect of different and frequently uncorrelated risk factor movements on the portfolio value and to interpret the significance of measurements obtained by applying more sophisticated but often opaque approaches (e.g., VAR). This also enables obtaining a synthetic representation of the exposure to major risk factors and estimating the potential reactions in case of variations among the correlation factors.

In this work, we enrich the capital market practitioner's toolbox with a new set of sensitivity measures. We start from the partial derivatives approach embedded in the traditional duration framework and enhance it by considering several and different risk factors. Finally, we apply this approach to a variable cash flow evaluation.

In this way, the total or partial duration vectors operating in a simplified market scenario characterized by a single interest rate curve become matrices for the timely evaluation of the possible market effects resulting from a plurality of risk factors. In particular, our paper enhances the traditional frameworks based on duration while preserving their synthetic and powerful nature, rendering these even more suitable compared to more complex indicators such as VAR to interpret the effects of an increasingly complex market scenario on measuring the fair value of a portfolio.

We decompose the generic risk factor bond yield into its main components to estimate the possible effects of different risk factors on a fixed income portfolio's value, enhancing the evaluation of the risk mitigation effect of derivatives such as interest rate swaps and, in case of partial risk hedging, aiming at hedging only the risk-free component. We use the traditional duration (Macauley 1938; Castellani et al. 1993; De Felice and Moriconi 1991) and extend it to a partial duration framework (Reitano 1992b) to take into account the non-parallel movements of multiple risk factors. In the last part, we define a decomposed partial duration for a variable cash flow and test decomposed partial duration on market data to verify the real predictive capacity of the proposed approach.

Analyzing the market data, we conclude that the decomposed partial duration approach enables estimating the expected price change after a non-parallel shift of the curve, taking into account different risk factors and their different movements.

We "upgrade" the decomposed partial duration approach taking into account a more sophisticated representation of the floating component and analyze several risk factors (such as macroeconomic variables) as well as different types of indexations (such as inflation) to estimate price movements in a more accurate way.

The results obtained enable a simple and rigorous quantification of the contribution of each risk factor in determining the expected changes in the value of variable coupon bonds.

The innovative approach proposed is flexible and adaptable to changing market scenarios. If the risk factors undergo changes, then the weights of (11) can simply be changed to adapt the model to the new market conditions.

In the future, we aim to deepen the model to estimate the risk factors and corresponding weights, for example, by using the approach that Reinhart and Sack (2002) propose.

In addition, a further element of the study is implementing the DD_K model by considering a stochastic model of the yield curve (multi-factor) bond valuations and random cash flows.

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