Nonlinear Dyn (2012) 69:531–537
DOI 10.1007/s11071-011-0284-z

ORIGINAL PAPER

A hyperchaotic system without equilibrium

Zenghui Wang · Shijian Cang · Elisha Oketch Ochola · Yanxia Sun

Received: 20 January 2011 / Accepted: 24 November 2011 / Published online: 17 December 2011

Abstract This article introduces a new chaotic system of 4-D autonomous ordinary differential equations, which has no equilibrium. This system shows a hyper-chaotic attractor. There is no sink in this system as there is no equilibrium. The proposed system is investigated through numerical simulations and analyses including time phase portraits, Lyapunov exponents, and Poincaré maps. There is little difference between this chaotic system and other chaotic systems with one or several equilibria shown by phase portraits, Lyapunov exponents and time series methods, but the Poincaré maps show this system is a chaotic system with more complicated dynamics. Moreover, the circuit realization is also presented.

Z. Wang (✉)
School of Engineering, University of South Africa, Florida 1710, South Africa
e-mail: wangzengh@gmail.com

S. Cang
Department of Industry Design, Tianjin University of Science and Technology, Tianjin 300222, P.R. China
e-mail: csj98231@tust.edu.cn

E.O. Ochola
School of Computing, University of South Africa, Pretoria 0003, South Africa
e-mail: Ocholeo@unisa.ac.za

Y. Sun
F'SATI & Department of Electrical Engineering, Tshwane University of Technology, Pretoria 0001, South Africa
e-mail: sunyanxia@gmail.com

Keywords Hyperchaos · Chaos · Lyapunov exponents · Poincaré map · Equilibrium

1 Introduction

Recently, it has been found that chaos is useful in many application fields such as engineering, medicine, secure communications, and so on. Up to now, a lot of research results have been achieved such as in references [1–3]. Many chaotic systems have been discovered. The most famous chaotic system is Lorenz system from meteorology [4]. Another one is the Chen system [1], which was generated via a state feedback to the second equation in Lorenz system. Both Lorenz system and Chen system possess a two-wing attractor. Another example is the chaotic attractor with one equilibrium [5]. There are many other multiwing chaotic systems such as three wings and/or four wings [6, 7]. The equilibrium is very important for showing chaotic attractors, especially for showing multiple wings or scrolls.

In the theory of nonlinear systems, the equilibrium plays important roles. One of the most important methods to analyze chaotic system is Shil'nikov method, which has been used to check whether one system is chaotic or not [8], and construct chaos system for autonomous systems [5]. On the other hand, it should be noted that one commonly used analytic criterion for proving chaos in autonomous systems is based on the fundamental work of Šil'nikov [9, 10], and its subsequent embellishment and slight extension [8, 11]. This is known as the Šil'nikov method or Šil'nikov criterion today, and its role is in some sense equivalent to that of the Li–Yorke lemma in the discrete setting [12, 13]. Dynamics around equilibria or hyperbolic saddle focus plays important role to create chaotic attractors.

In a 3-D nonlinear dynamical system, a saddle point is an equilibrium point on which the equivalent linearized model has at least one eigenvalue in the stable region and one in the unstable region. In the same system, a saddle point is called the saddle point of index 1, if one of the eigenvalues is unstable and others are stable. Also, a saddle point of index 2 is a saddle point with one stable eigenvalue and two unstable ones. For the multiscroll chaotic systems, it is shown that scrolls are generated only around the saddle points of index 2. Moreover, the saddle points of index 1 are responsible only for connecting scrolls [8, 14, 15]. The logical question is what will happen if there is no equilibrium in the chaotic systems. Although one chaotic system without equilibrium was proposed in [16], no analysis was proposed. In this paper, a new hyperchaotic system without equilibrium is proposed and analyzed. It is analyzed through phase portraits, Lyapunov exponents, and Poincare maps. Finally, the circuit realization is presented.

2 New hyperchaotic system without equilibrium

The new 4-D chaotic system is described by a system of first-order ordinary differential equations:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + yz + axzw, \\ \dot{z} &= 1 - y^2, \\ \dot{w} &= z + bxz + cxyz. \end{aligned} \tag{1}$$

Here, a, b, and c are constant parameters of the system. If there are equilibria for system (1), they can be obtained by solving $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$, and $\dot{w} = 0$, that is,

$$y = 0, \tag{2}$$

$$-x - yz + axzw = 0, \tag{3}$$

$$1 - y^2 = 0, \tag{4}$$

$$z + bxz + cxyz = 0. \tag{5}$$

As can be seen from (4), $y = \pm 1$ which is inconsistent with (2). Hence, in system (1), there is no equilibrium and there are no characteristics as the common chaotic systems such as pitchfork bifurcation, Hopf bifurcation, and so on. Moreover, there is no sink for this system as there is no equilibrium.

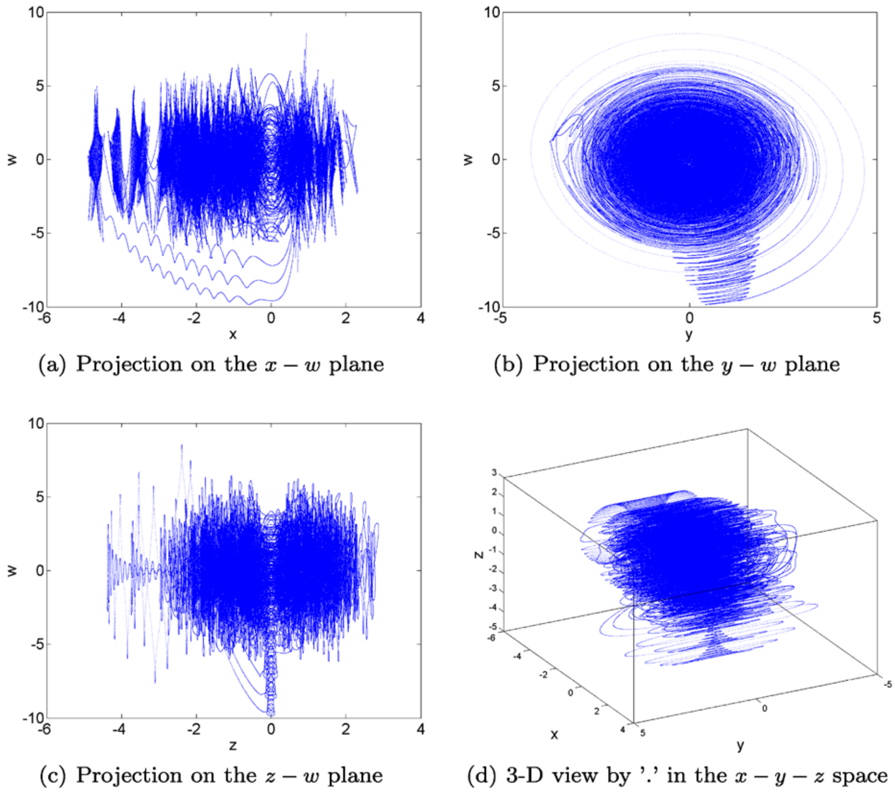
2.1 Hyperchaotic system without equilibrium

When $a = 8$, $b = -2.5$, and $c = -30$, the initial condition is $(0.1, 0.1, 0.1, 0.1)$ and the simulation time is 800 s, the 3-D phase portraits are shown in Fig. 1. The projections of the phase portrait on x–w, y–w, and z–w planes are shown in Fig. 1(a)–(c), respectively, and the x–y–z 3-D chaotic attractor is shown in Fig. 1(d).

If the efficient QR based method [17] is used to calculate the Lyapunov exponents, the simulation time is set as 15000 s and the sampling time-step is 0.0005 s, the Lyapunov exponents of system (1) are $\lambda_1 = 0.87$, $\lambda_2 = 0.03$, $\lambda_3 = 0.00$, and $\lambda_4 = -1.01$ which show the system is hyperchaotic.

2.2 Poincaré map of the four-wing chaotic attractor

As an important analysis technique, the Poincaré map can reflect bifurcation and folding properties of chaos. When $a = 8, b = -2.5$, and $c = -30$, one may take $x = 0$ and $y = 0$ as crossing planes, respectively. Figure 2 shows the Poincaré map in the y–z–w 3-D space when $x = 0$. Figure 3 shows the Poincaré mapping in the x–z–w 3-D space and on several other planes when $y = 0$. It is clear that some sheets are folded and indicates that the system has extremely rich dynamics. As can be seen from Fig. 2 and Fig. 3, there is no regular limbs, which further indicates that the system has extremely rich dynamics and it is different from the normal chaotic systems with one or more equilibria. The reason for the differences of Poincaré maps between this hyperchaotic system without equilibrium and other normal chaotic or hyperchaotic systems are that there is no limitation of equilibriums for the non-equilibrium chaotic attractor. In [8, 14, 15], it is shown that scrolls are generated only around the saddle points of index 2. Moreover, the saddle points of index 1 are responsible only for connecting scrolls. The results in these references mean that the attractors always surround the equilibriums. However, there is different dynamics in the proposed system as there is not equilibrium.



(a) Projection on the $x-w$ plane

(b) Projection on the $y-w$ plane

(c) Projection on the $z-w$ plane

(d) 3-D view by '.' in the $x-y-z$ space

Fig. 1 Hyperchaotic attractor without equilibrium, with $a=8$, $c=-2.5$, and $c=-30$

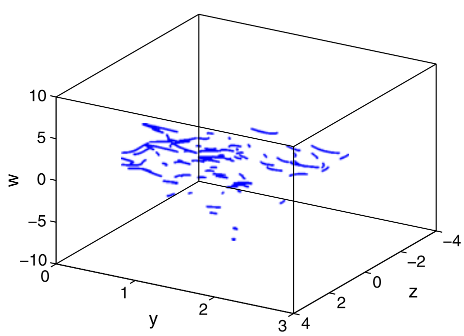


Fig. 2 Poincaré map in the y–z–w 3-D space $x=0$ with $a=8$, $b=-2.5$, and $c=-30$

2.3 Frequency spectral analysis

Frequency spectra can be used to analyze chaotic attractors since they can reveal how random signals are. The frequency spectra of signals, generated numerically from the hyperchaotic systems (1) proposed in this paper, are shown in Fig. 4. For calculation purposes, the Runge–Kutta method is used to solve all systems, with sampling time-step 0.0005 s, running time 0–800 s, and all spectra are normalized. As can be seen from Fig. 4, the frequency spectra are continuous and have noise-like background, which means the system (1) is chaotic. The chaos of the signal of states of system (1) can also be shown by their time sequences. The time sequences of the variables x, y, z and w of the hyperchaotic attractor (1), with $a=8, b=-2.5$ and $c=-30$, are shown in Fig. 5(a)–(d), respectively. As can be seen from Fig. 5, the time sequences of the system variables are neither sink nor periodic, and similar to stochastic signal which is one of the properties of chaos. As can be seen from Fig. 4, the frequency spectra of states x and z, corresponding to Fig. 4(a) and Fig. 4(c) are narrower than the frequency spectra of states y and w, corresponding to Fig. 4(b) and Fig. 4(d). This result is caused by simpler parts about $\dot{x}$ and $\dot{z}$ of system (1) which only include variable y.

2.4 Circuit realization

In this section, an electronic circuit is designed to realize system (1). To prevent the operational amplifiers and analog multipliers from saturating, the transfor-

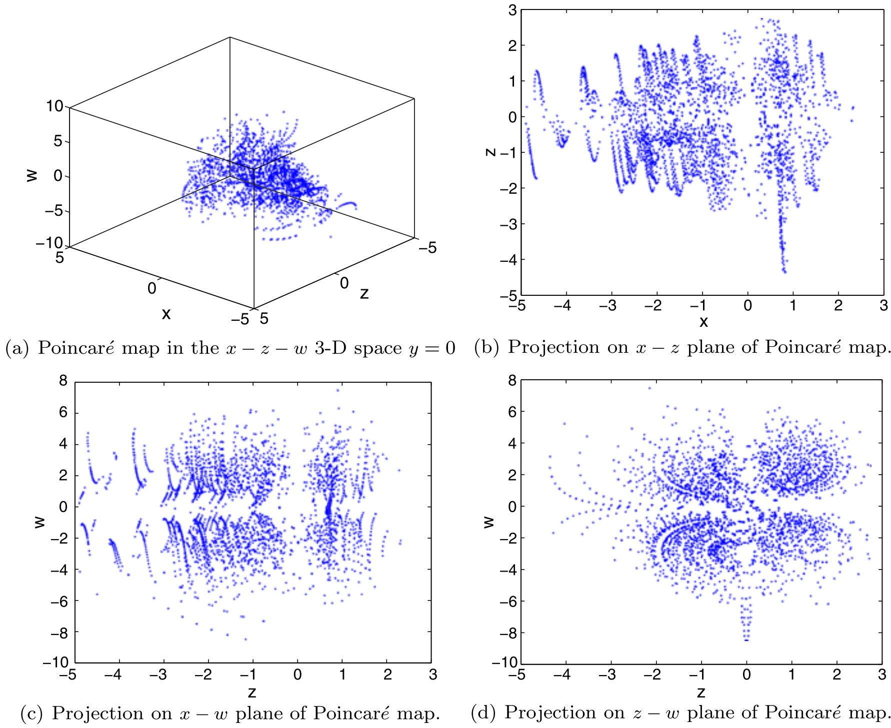


Fig. 3 Hyperchaotic attractor Poincaré mappings of system (1) when $y=0$ with $a=8$, $b=-2.5$, and $c=-30$

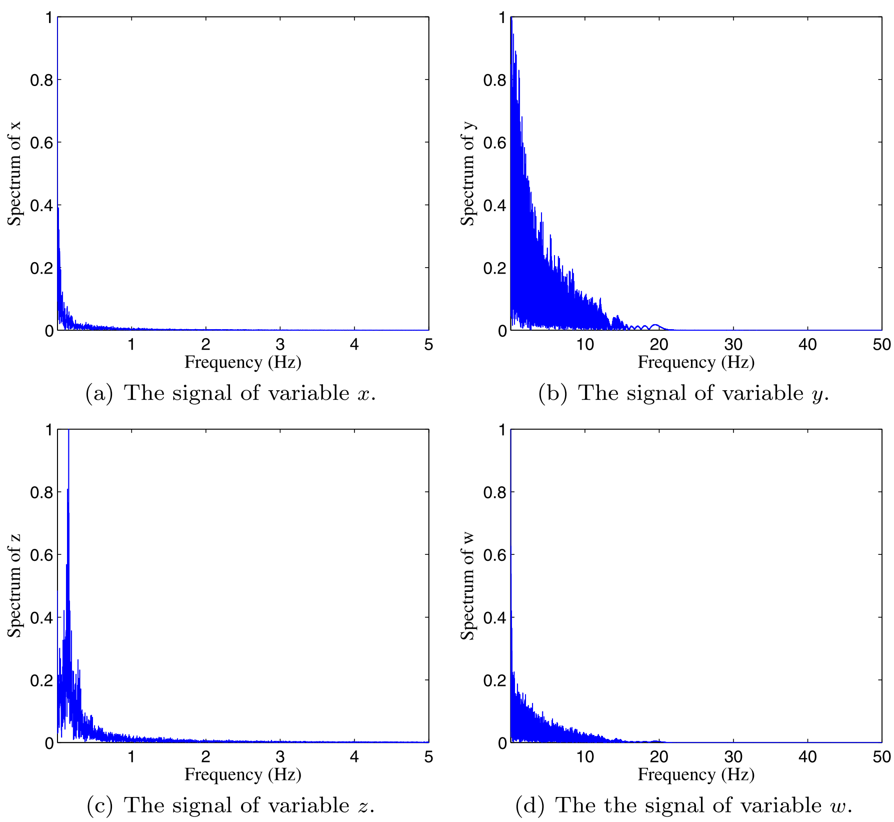
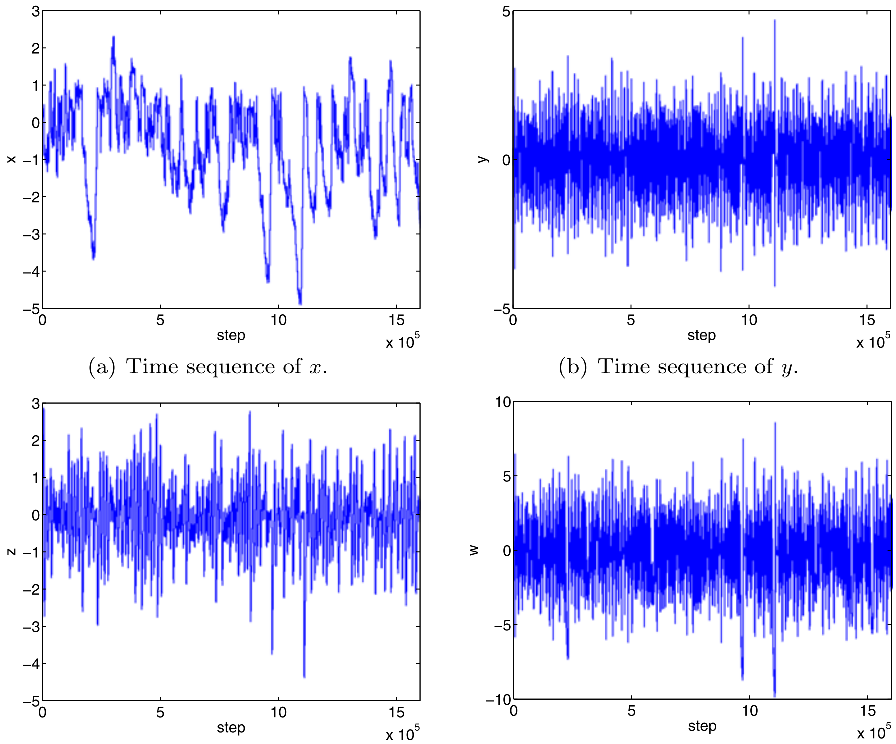


Fig. 4 The frequency spectra generated numerically from the proposed hyperchaotic system (1), with $a=8$, $b=-2.5$, and $c=-30$

Fig. 5 The time sequence of the hyperchaotic attractor (1) with $a = 8$, $b = -2.5$, and $c = -30$



(a) Time sequence of x.
(b) Time sequence of y.
(c) Time sequence of z.
(d) Time sequence of w.

mation $x = 2x_m$, $y = 2y_m$, $z = 2z_m$ and $w = 2w_m$ can be used and the minification is 2.

The system (1) will be changed to

$$\begin{aligned}
\dot{x}_m &= y_m, \\
\dot{y}_m &= -x_m + 2y_m z_m + 4a x_m z_m w_m, \\
\dot{z}_m &= 0.5 - 2y_m^2, \\
\dot{w}_m &= z_m + 2b x_m z_m + 4c x_m y_m z_m
\end{aligned} \tag{6}$$

which has the similar properties with system (1). The designed circuit realizing (6) is presented in Fig. 6.

If the value of capacitor is not small enough, the frequency of the signal will be too small to be displayed on the oscilloscope. Here, all capacitors are taken to be 10 nF and all operational amplifiers and analog multipliers are chosen as LF347 and AD633JN, respectively. The value of the resistor can be set as

$$\begin{aligned}
&R_{11} = R_{13} = R_{21} = R_{24} = R_{31} = R_{33} = R_{41} \\
&\quad = R_{43} = 100\ \text{k}\Omega, \\
&R_{12} = R_{42} = 100\ \Omega, \qquad R_{22} = 6.25\ \text{k}\Omega, \\
&R_{23} = 200\ \Omega, \qquad R_{25} = 100\ \text{M}\Omega, \qquad R_{32} = 50\ \Omega, \\
&R_{34} = 50\ \text{M}\Omega, \qquad R_{45} = 20\ \text{M}\Omega, \\
&R_{44} = 5/6\ \text{M}\Omega.
\end{aligned}$$

The circuit is realized in EWB and the simulation results of phase diagrams on the y_m–w_m and z_m–w_m planes are shown in Fig. 7 which are similar with the ones of Fig. 1.

3 Conclusion

A hyperchaotic attractor has been analyzed in this article. The analyses have shown that there is no equilibrium and no sink for this hyper-chaotic system. Other numerical methods have also been used to prove that the nonlinear system shows chaotic dynamics. These numerical methods include phase portraits, Lyapunov exponents, time series, and Poincaré maps. There is little difference between this chaotic system without equilibrium and other chaotic systems with one or several equilibria if only phase portraits, Lyapunov exponents and time series methods are used. However, the Poincaré maps indicates that the system without the

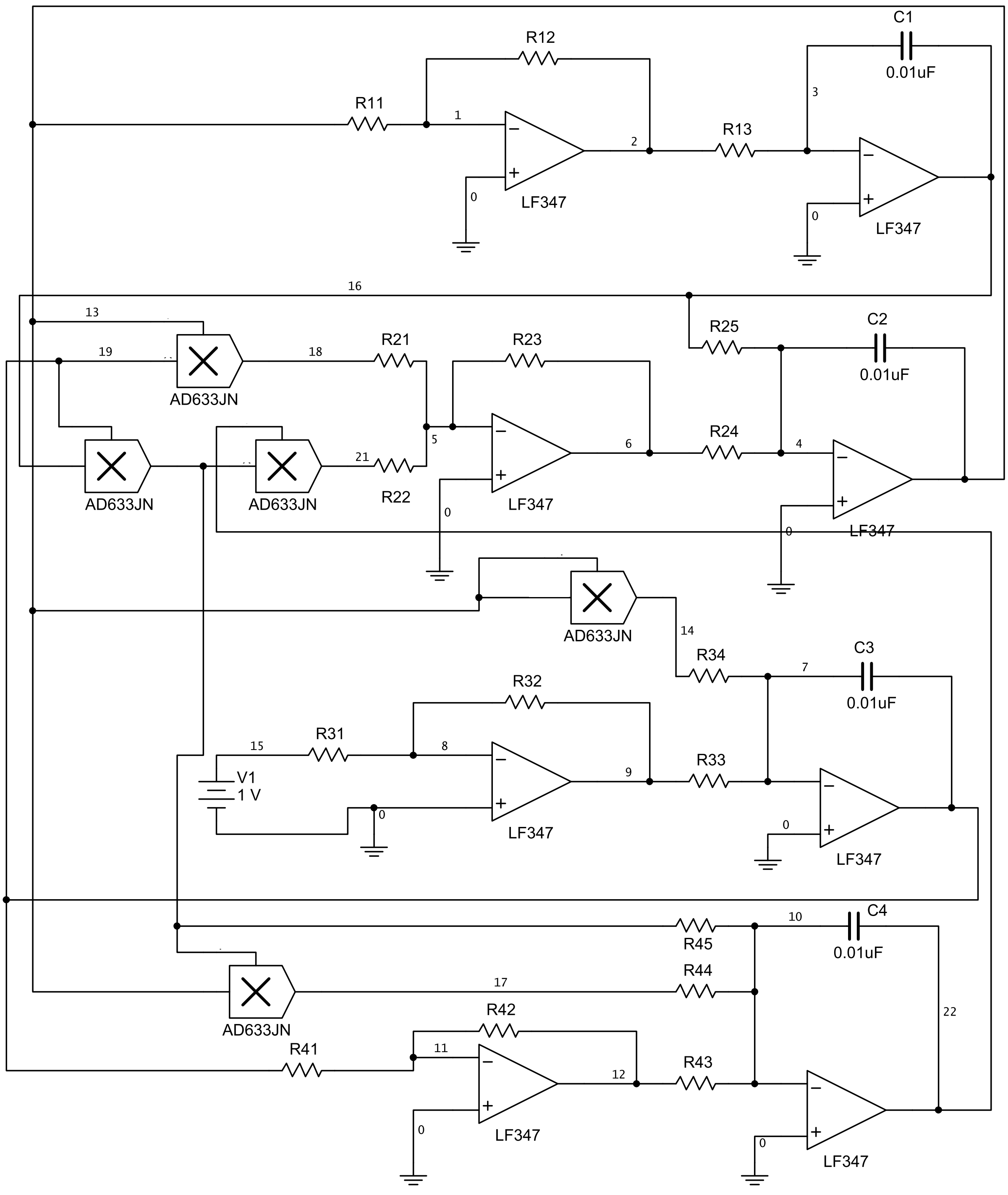
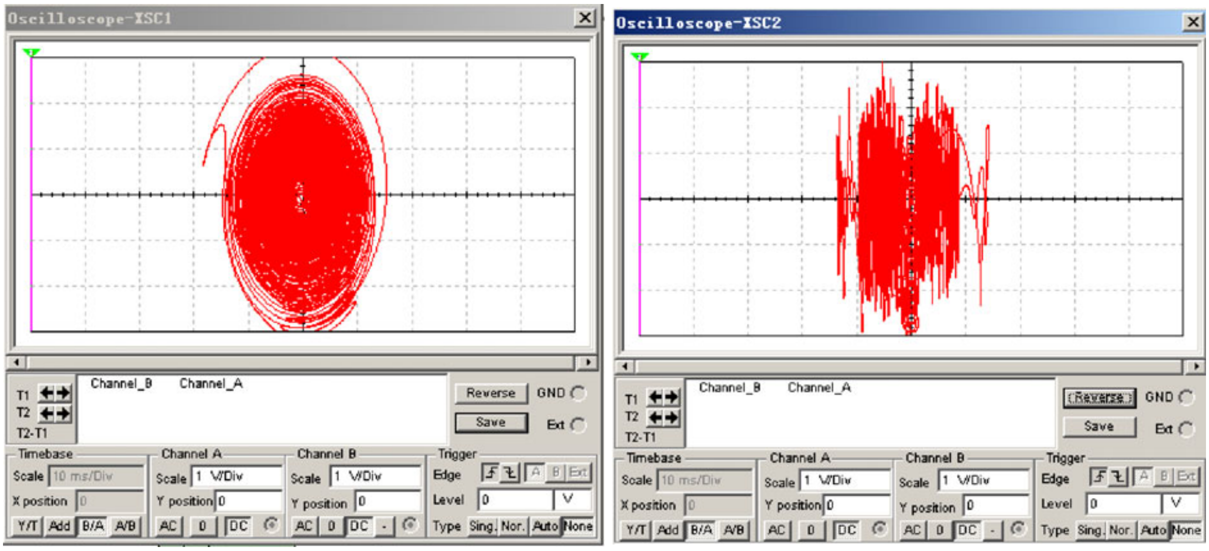


Fig. 6 Circuitry realization of the chaotic system (6)

limitation of equilibrium has extremely richer dynamics than the normal hyperchaotic systems. The potential significance to study this hyperchaotic system is that the proposed hyperchaotic system can be used in



(a) Projection on the $y - w$ plane

(b) Projection on the $z - w$ plane

Fig. 7 Phase diagram of (6)

some engineering applications especially in the area of information encryption as there is no limitation of equilibrium. Moreover, the circuit realization was also proposed. Future research work will focus on the theory and the application about the chaotic system without equilibrium.

References

1. Chen, G., Yu, X: Chaos Control: Theory and Applications. Springer, Berlin (2003)
2. Mahmoud, G.M., Mahmoud, E.E., Farghaly, A.A., Aly, S.A.: Chaotic synchronization of two complex nonlinear oscillators. Chaos Solitons Fractals, **42**, 2858–2864 (2009)
3. Sun, Y., Qi, G., Wang, Z., van Wyk, B.J.: Bifurcation analysis of Qi 3-D four-wing chaotic system. Acta Phys. Pol. B, **41**, 767–778 (2010)
4. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
5. Zhou, T.S., Chen, G., Yang, Q.G.: Constructing a new chaotic system based on the Silnikov criterion. Chaos Solitons Fractals, **19**, 985–993 (2004)
6. Chen, Z., Yang, Y., Yuan, Z.: A single three-wing or four-wing chaotic attractor generated from a three-dimensional smooth quadratic autonomous system. Chaos Solitons Fractals, **38**, 1187–1196 (2008)
7. Wang, Z., Qi, G., Sun, Y., van Wyk, B.J., van Wyk, M.A.: A new type of four-wing chaotic attractors in 3-D quadratic autonomous systems. Nonlinear Dyn. **60**, 443–457 (2010)
8. Silva, C.P.: Shil'nikov's theorem—a tutorial. IEEE Trans. Circuits Syst. I **40**, 675–682 (1993)
9. Šil'nikov, L.: A case of the existence of a countable number of periodic motions. Sov. Math. Dokl. **6**, 163–166 (1965)
10. Šil'nikov, L.: A case of the existence of a countable number of periodic motions. Math. USSR Sb. **10**, 91–102 (1970)
11. Tresser, C.: On some theorems of L. P. Ši'lnikov and some applications. Preprint, University of Nice, France (1982); also abridged form in Ann. Inst. Henri Poincaré **49**, 441–461 (1984)
12. Kennedy, J., Kocak, S., Yorke, J.A.: A chaos lemma. Am. Math. Mon. **108**, 411–423 (2001)
13. Li, T.Y., Yorke, J.A.: Period three implies chaos. Am. Math. Mon. **82**, 985–992 (1975)
14. Chua, L.O., Komuro, M., Matsumoto, T.: The double scroll family. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **33**, 1072–1118 (1986)
15. Lü, J., Chen, G., Yu, X, Leung, H.: Design and analysis of multi-scroll chaotic attractors from saturated function series. IEEE Trans. Circuits Syst. I, Fundam. Theory Appl. **51**, 2476–2490 (2004)
16. Sprott, J.C.: Some simple chaotic flows. Phys. Rev. E, Stat. Nonlinear Soft Matter Phys. **50**, 647–650 (1994)
17. von Bremen, H.F., Udwadia, F.E., Proskurowski, W.: An efficient QR based method for the computation of Lyapunov exponents. Physica D **101**, 1–16 (1997)