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Pricing and Hedging Asian Basket Options with Quasi-Monte Carlo Simulations

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Abstract

In this article we consider the problem of pricing and hedging high-dimensional Asian basket options by Quasi-Monte Carlo simulation. We assume a Black-Scholes market with time-dependent volatilities and show how to compute the deltas by the aid of the Malliavin Calculus, extending the procedure employed by Montero and Kohatsu-Higa [1]. Efficient path-generation algorithms, such as Linear Transformation and Principal Component Analysis, exhibits a high computational cost in a market with time-dependent volatilities. We present a new and fast Cholesky algorithm for block matrices that makes the Linear Transformation even more convenient. Moreover, we propose a new-path generation technique based on a Kronecker Product Approximation. This construction returns the same accuracy of the Linear Transformation used for the computation of the deltas and the prices in the case of correlated asset returns while requiring a lower computational time. All these techniques can be easily employed for stochastic volatility models based on the mixture of multi-dimensional dynamics introduced by Brigo et al. [2, 3].

1 Introduction and Motivation

In a recent paper Dahl and Benth [4] have investigated the efficiency and the computational cost of the Principal Component Analysis (PCA) used in the Quasi-Monte Carlo (QMC) simulations for the pricing of high-dimensional Asian basket options in a multi-dimensional Black-Scholes (BS) model with constant volatilities. In particular they have shown the essential role of the Kronecker product for a fast implementation as well as for the analysis of variance (ANOVA) in order to identify the effective dimension (see below). Indeed the convergence rate of the QMC method is $O\left(N^{-1}\log^d N\right)$, where N is the number of simulation trials, and d the nominal dimension of the problem. This implies that the theoretically higher asymptotic convergence rate of QMC could not be achieved for practical purposes in high dimensions. On the other hand, some applications in finance (see Paskov and Traub [5]) have shown that QMC provides a higher accuracy than standard Monte Carlo (MC), even for high dimensions.

To explain the success of QMC in high dimensions Caflisch et al. [6] have introduced two notions of effective dimensions based on the ANOVA of the integrand function. Consider an integrand function f and a MC problem with nominal dimension d. Let $\mathcal{A} = \{1, \ldots, d\}$ denote the labels of the input variables of the function f: then the effective dimension of f, in the *superposition sense*, is the smallest integer d_S such that $\sum_{|u|\leq d_S} \sigma^2(f_u) \geq p\sigma^2(f)$, where f_u is a function with variables in the set $u \subseteq \mathcal{A}$, $\sigma^2(\cdot)$ denotes the variance of the given function, $|u|$ is the cardinality of the set and $0 \leq p \leq 1$, for instance ($p = 0.99$). The effective dimension of f, in the *truncation sense* is the smallest integer d_T such that $\sum_{u\subseteq\{1,2,\ldots,d_T\}} \sigma^2(f_u) = p\sigma^2(f)$. Essentially, the truncation dimension indicates the number of important variables which predominantly capture the given function f. The superposition dimension takes into account that for some integrands, the inputs might influence the outcome through their joint action within small groups.

The PCA decomposition only permits a dimension reduction without taking into account the particular payoff function of a European option. In contrast, Imai and Tan [7] have proposed a general dimension reduction construction, named Linear Transformation (LT), that depends on the payoff function and that minimizes the effective dimension in the truncation sense. They have shown that the LT approach is more accurate than the standard PCA, but has a higher computational cost. In a previous paper one of the authors [8] has discussed how to implement this technique quickly and – with a slower computer – has obtained computational times that are about 30 times smaller that those originally presented by Imai and Tan [7].

In the present study we consider time-dependent volatilities: as a consequence it is not possible to rely on the properties of the Kronecker product, and the problem is computationally more complex. In order to simplify the computational complexity, we present a fast Cholesky (CH) decomposition algorithm tailored for block matrices that remarkably reduces the computational cost. Moreover, we present a new path-generation technique based on the Kronecker Product Approximation (KPA) of the correlation matrix of the multi-dimensional Brownian path that returns a suboptimal ANOVA decomposition with a substantial advantage from the computational point of view.

Our numerical experiment consists in calculating the Randomized QMC (RQMC) estimation of the prices and the deltas of high-dimensional Asian basket options in a BS market with time-dependent volatilities. In order to compute the deltas, we extend to a multi-assets dependence the procedure employed by Montero and Kohatsu-Higa [1] in a single asset setting. This procedure is based on the Malliavin Calculus and allows a certain flexibility that can enhance the localization technique introduced by Fournié et al. [9]. As far as the computation of Asian options prices is concerned, the KPA and LT approaches are tested both in terms of accuracy and computational cost. We demonstrate that the LT construction becomes more efficient than the PCA even from the computational point of view, provided we use the CH algorithm that we present and the approach described in [8]. The KPA and the PCA constructions perform equally in terms of accuracy, with the former one requiring a considerably shorter computational time. However, the KPA and the LT display the same accuracies in the computation of the deltas. Moreover, we compare our simulation experiment – also using the standard CH and the PCA decomposition methods – with pseudo-random and Latin Hypercube Sampling (LHS) generators.

Remark finally that all the methods described here can accommodate a market with stochastic volatility where the evolution of the risky securities is modeled by a mixture of multi-dimensional dynamics as in the papers by Brigo et al. [2, 3]. It is noteworthy to say that none of these constructions can be applied to Heston-like multi-dimensional stochastic volatility models. In principle, we might still use the LT for the Euler discretization of the Heston model, but this could be no longer applicable with more realistic schemes that involve discrete random variables as proposed for instance by Alfonsi [10].

The paper is organized as follows: Section 2 describes Asian options. Section 3 discusses some path-generation techniques and in particular, presents the fast CH algorithm and the KPA construction. Section 4 shows the numerical tests for the Asian option pricing. Section 5 explains how to represent the deltas of Asian basket options as expected values with the aid of Malliavin Calculus and shows the estimated values by RQMC. Section 6 summarizes the most important results and concludes the paper.

2 Asian Basket Options

Assume a multi-dimensional BS market with M risky securities and one risk-free asset. Denote $\mathbf{B}(t) = (B_1(t), \ldots, B_M(t))$ an M-dimensional Brownian motion (BM) with correlated components and $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by this BM. Moreover, denote ρ_{ik} the constant instantaneous correlation between $B_i(t)$ and $B_k(t)$, $S_i(t)$ the i-th asset price at time t, $\sigma_i(t)$ the instantaneous time-dependent volatility of the i-th asset return and r the continuously compounded risk-free rate. In the risk-neutral probability, we assume that the dynamics of the risky assets are

$$dS_i(t) = rS_i(t)\,dt + \sigma_i(t)\,S_i(t)\,dB_i(t), \qquad i = 1, \ldots, M. \tag{1}$$

The solution of Equation (1) is

$$S_i(t) = S_i(0)\exp\left[\int_0^t \left(r - \frac{\sigma_i^2(s)}{2}\right) ds + \int_0^t \sigma_i(s)\,dB_i(s)\right], \quad i = 1, ..., M. \tag{2}$$

Discretely monitored Asian basket options are derivative contracts that depend on the arithmetic mean of the prices assumed by a linear combination of the underlying securities at precise times $t_1 < t_2 \cdots < t_N = T$, where T is the maturity of the contract. By the risk-neutral pricing formula (see for instance Lamberton and Lapeyre [11]) the fair price at time t of the contract is

$$a(t) = e^{r(T-t)}\mathbb{E}\left[\left(\sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij}\,S_i(t_j) - K\right)^+ \middle| \mathcal{F}_t\right], \tag{3}$$

with the assumption that $\sum_{i,j} w_{ij} = 1$.

Pricing Asian options by simulation hence requires the discrete averaging of the solution (2) at a finite set of times $\{t_1, \ldots, t_N\}$. This sampling procedure yields

$$S_i(t_j) = S_i(0)\exp\left[\int_0^{t_j}\left(r - \frac{\sigma_i^2(t)}{2}\right) dt + Z_i(t_j)\right] \quad i = 1, \ldots, M, j = 1, \ldots, N, \tag{4}$$

where the components of the vector

$$(Z_1(t_1), \ldots, Z_1(t_N);\, Z_2(t_1), \ldots, Z_2(t_N);\, \ldots;\, Z_M(t_1), \ldots, Z_M(t_N))^T$$

are $M \times N$ normal random variables with zero mean and the following covariance matrix

$$\Sigma_{MN} = \begin{pmatrix} \Sigma(t_1) & \Sigma(t_1) & \ldots & \Sigma(t_1) \\ \Sigma(t_1) & \Sigma(t_2) & \ldots & \Sigma(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma(t_1) & \Sigma(t_2) & \ldots & \Sigma(t_N) \end{pmatrix}, \tag{5}$$

where the elements of the $M \times M$ submatrices $\Sigma(t_n)$ are $(\Sigma(t_n))_{ik} = \int_0^{t_n} \rho_{ik}\sigma_i(s)\sigma_k(s)ds$ with $i, k = 1, \ldots, M$; $n = 1, \ldots, N$. This setting would allow time-dependent correlations as well. In the case of constant volatilities the covariance matrix is

$$\Sigma_{MN} = \begin{pmatrix} t_1\Sigma & t_1\Sigma & \ldots & t_1\Sigma \\ t_1\Sigma & t_2\Sigma & \ldots & t_2\Sigma \\ \vdots & \vdots & \ddots & \vdots \\ t_1\Sigma & t_2\Sigma & \ldots & t_N\Sigma \end{pmatrix}, \tag{6}$$

where now Σ denotes the $M \times M$ covariance matrix of the logarithmic returns of the assets. It follows from the last equation that the covariance matrix Σ_{MN} can be represented as $R \otimes \Sigma$, where $\otimes$ denotes the Kronecker product and R is the auto-covariance matrix of a single BM. This simplification is not possible in the case of time-dependent volatilities. We recall that the elements of R are

$$R_{ln} = t_l \wedge t_n, \quad l, n = 1, \ldots N. \tag{7}$$

R has the peculiarity to be invariant for a reflection about the diagonal.

Definition 1 (Boomerang Matrix)**.** *Let $B \in \mathbb{R}^{n_B \times n_B}$ be a square matrix and let $\mathbf{b} = (b_1, \ldots, b_{n_B}) \in \mathbb{R}^{n_B}$. B is a boomerang matrix if*

$$B_{hp} = b_{h \wedge p}, \quad h, p = 1, \ldots, n_B. \tag{8}$$

We call $\mathbf{b}$ the elementary vector associated to B.

As a consequence R is boomerang, and in general the auto-covariance matrix of every Gaussian process is boomerang. This definition can also be extended to block matrices as follows.

Definition 2 (Block Boomerang Matrix)**.** *Partition the rows and the columns of a square matrix $B \in \mathbb{R}^{n_B \times n_B}$ to obtain:*

$$B = \begin{pmatrix} B_{11} & \ldots & B_{1P} \\ \vdots & \ddots & \vdots \\ B_{P1} & \ldots & B_{PP} \end{pmatrix}, \tag{9}$$

where for $h, p = 1, \ldots, P$, $B_{hp} \in \mathbb{R}^{D \times D}$ designates the (h, p) square submatrix and $n_B = P \times D$. Given P matrices $B_1, \ldots, B_P$ with $B_h \in \mathbb{R}^{D \times D}, h = 1, \ldots, P$, B is a boomerang block matrix if

$$B_{hp} = B_{h \wedge p}, \quad h, p = 1, \ldots, n_B. \tag{10}$$

We call $\mathbf{b} = (B_1, \ldots, B_P)^T$ the elementary block vector associated to B.

From these definitions we have that Σ_{MN} is block boomerang.

The payoff at maturity of the Asian basket option now is $a(T) = (g(\mathbf{Z}) - K)^+$ with

$$g(\mathbf{Z}) = \sum_{k=1}^{M\times N} \exp\left(\mu_k + Z_k\right) \tag{11}$$

where $\mathbf{Z} \sim \mathcal{N}(0, \Sigma_{MN})$ and

$$\mu_k = \ln(w_{k_1 k_2} S_{k_1}(0)) + r t_{k_2} - \int_0^{t_{k_2}} \frac{\sigma_{k_1}^2(t)}{2}\, dt \tag{12}$$

with $k_1 = (k-1) \bmod M$; $k_2 = \lfloor (k-1)/M \rfloor + 1$; $k = 1, \ldots, M$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3 Path-generation Techniques

From the previous discussion it comes out that the pricing of Asian basket options by simulation requires an averaging on the sample trajectories of an M-dimensional BM. In general, if $\mathbf{Y} \sim \mathcal{N}(0, \Sigma_Y)$ and $\mathbf{X} \sim \mathcal{N}(0, I)$ are two N-dimensional Gaussian random vectors, we will alawys be able to write $\mathbf{Y} = C\mathbf{X}$, where C is a matrix such that:

$$\Sigma_Y = CC^T. \tag{13}$$

and the core problem consists in finding the matrix C. In our case Σ_Y coincides with Σ_{MN} of Equation (5). The accuracy of the standard MC method does not depend on the choice of the matrix C because the order of the random variables is not important. However, a choice of C that reduces the nominal dimension would improve the efficiency of the (R)QMC. In the following we discuss some possibilities.

3.1 Cholesky Construction

The CH decomposition simply finds the matrix C among all the *lower triangular matrices*. In the case of constant volatilities the matrix Σ_{MN} is the Kronecker product of R and Σ, and the Kronecker product shows compatibility with the CH decomposition (see Pitsianis and Van Loan [12]). Indeed, denote $C_{\Sigma_{MN}}$, C_R and C_Σ the CH matrices associated to Σ_{MN}, R and Σ respectively; we then have

$$C_{\Sigma_{MN}} = C_R \otimes C_\Sigma. \tag{14}$$

This now allows a remarkable reduction of the computational cost: it turns out in fact that a $O\left((M\times N)^3\right)$ computation is reduced to a $O\left(M^3\right) + O\left(N^3\right)$ one.

When time-dependent volatilities are considered, however, we can no longer use these properties of the Kronecker product. In any case, since Σ_{MN} is a block boomerang matrix, we can use the following result:

Proposition 1. *Let $B \in \mathbb{R}^{n_B \times n_B}$ be a block boomerang matrix and let $(B_1, \ldots, B_P)^T$, where $B_h \in \mathbb{R}^{D\times D}, h = 1, \ldots, P$ with $n_B = P \times D$, be its associated elementary block vector. C_B, the CH matrix associated to B, is given by:*

$$C_B = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ \vdots & C_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_1 & C_2 & \ldots & C_P \end{pmatrix} \tag{15}$$

where the $D \times D$ blocks C_h, $h = 1, \ldots, P$ are

$$C_h = \text{Chol}\left(B_h - B_{h-1}\right) \tag{16}$$

with Chol *denoting the CH factorization, and we assume $B_0 = 0$.*

Proof. Consider the h^{th} row of C_B and the m^{th} row of its transposed matrix; we then have

$$\begin{aligned}(C_1, \ldots, C_h, 0, \ldots, 0)^T \cdot \left(C_1^T, \ldots, C_m^T, 0, \ldots, 0\right)^T &= \sum_{l=1}^{h \wedge m} C_l C_l^T \\ &= \sum_{l=1}^{h \wedge m} (B_l - B_{l-1}) = B_{h \wedge m}\end{aligned}$$

and this concludes the proof. □

3.2 Principal Component Analysis

Acworth et al. [13] have proposed a path generation technique based on the PCA. Following this approach we consider the spectral decomposition of Σ_{MN}

$$\Sigma_{MN} = E \Lambda E^T = (E \Lambda^{1/2})(E \Lambda^{1/2})^T, \tag{17}$$

where Λ is the diagonal matrix of all the positive eigenvalues of Σ_{MN} sorted in decreasing order and E is the orthogonal matrix ($EE^T = I$) of all the associated eigenvectors. The matrix C solving Equation (13) is then $E\Lambda^{1/2}$. The amount of variance explained by the first k principal components is the ratio: $\frac{\sum_{i=1}^{k} \lambda_i}{\sum_{i=1}^{d} \lambda_i}$ where d is the rank of Σ_{MN}. The PCA construction permits the statistical ranking of the normal factors, while this is not possible by the CH decomposition. For the market with constant volatilities, the Kronecker product reduces this calculation into the computation of the eigenvalues and vectors of the two smaller matrices R and Σ. All these simplifications are no longer valid for the time-dependent volatilities. However, we can reduce the computational cost for the PCA decomposition in the following way.

Take M_1, M_2, M_3 and M_4, respectively $p \times p, p \times q, q \times p$ and $q \times q$ matrices, and suppose that M_1 and M_4 are invertible. Assume

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and define $S_1 = M_4 - M_3 M_1^{-1} M_2$ and $S_4 = M_1 - M_2 M_4^{-1} M_3$, the Schur complements of M_1 and M_4, respectively. Then by Schur's lemma the inverse M^{-1} is:

$$M^{-1} = \begin{pmatrix} S_4 & -M_1^{-1} M_2 S_1^{-1} \\ -M_4^{-1} S_4^{-1} & S_1^{-1} \end{pmatrix}. \tag{18}$$

Taking into account the previous result it is possible to prove the following proposition

Proposition 2. *Let $B \in \mathbb{R}^{n_B \times n_B}$ be a block boomerang matrix and let $(B_1, \ldots, B_P)^T$, where $B_h \in \mathbb{R}^{D \times D}, h = 1, \ldots, P$ with $n_B = P \times D$, be its associated elementary block vector. The inverse of B is symmetric block tri-diagonal. The blocks on the lower (and upper) diagonal are $T_l = -\left(B_{l+1} - B_l\right)^{-1}$, $l = 1, \ldots, P-1$ while* *those on the diagonal are* $D_m = (B_m - B_{m-1})^{-1}(B_{m+1} - B_{m-1})(B_{m+1} - B_m)^{-1}$, $m = 1, \dots, P$, *with the assumption that* $B_0 = B_{N+1} = 0$*:*

$$B^{-1} = \begin{pmatrix} D_1 & T_1 & 0 & \dots & 0 \\ T_1 & D_2 & T_2 & \ddots & \vdots \\ 0 & T_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & T_{P-1} \\ 0 & 0 & 0 & T_{P-1} & D_P \end{pmatrix} \tag{19}$$

This property can be used to reduce the computational cost of evaluating the PCA decomposition in the case of time-dependent volatilities and in general for multi-dimensional Gaussian processes. Indeed, if B is a non-singular square matrix then the eigenvalues of the B^{-1} are the reciprocal of the eigenvalues of B and the eigenvectors coincide.

3.3 Linear Transformation

Imai and Tan [7] have considered the following class of LT as a solution of (13):

$$C^{\mathrm{LT}} = C^{\mathrm{Ch}} A \tag{20}$$

where C^{Ch} is the CH matrix associated to the covariance matrix of the normal random vector to be generated, and A is an orthogonal matrix, i.e. $AA^T = I$. The matrix A is introduced with the main purpose of minimizing the effective dimension of a simulation problem in the truncation sense. Imai and Tan [7] have proposed to approximate an arbitrary function g, such that $(g - K)^+$ is the payoff function of a European derivative contract, with its first order Taylor expansion around $\hat{\epsilon}$

$$g(\epsilon) = g(\hat{\epsilon}) + \sum_{l=1}^{n} \frac{\partial g}{\partial \epsilon_l}\Big|_{\epsilon=\hat{\epsilon}} \Delta\epsilon_l. \tag{21}$$

The approximated function is linear in the standard normal random vector $\Delta\epsilon$. Considering an arbitrary point about which we form the expansion, such as $\hat{\epsilon} = \mathbf{0}$, we can derive the first column of the optimal orthogonal matrix A^*. It is possible to find the complete matrix by expanding g about different points and then compute the optimization algorithm. Imai and Tan [7] have set: $\hat{\epsilon}_1 = \mathbf{0} = (0, 0, \dots, 0)$, $\hat{\epsilon}_2 = (1, 0, \dots, 0), \dots, \hat{\epsilon}_n = (1, \dots, 1, 0)$, where the k-th point has $k-1$ non-zero components. The optimization can then be formulated as follows:

$$\max_{\mathbf{A}_{\cdot\mathbf{k}} \in \mathbf{R}^{\mathbf{n}}} \left(\frac{\partial g}{\partial \epsilon_k}\Big|_{\epsilon=\hat{\epsilon}_{\mathbf{k}}} \right)^2, \quad k = 1, \dots, n, \tag{22}$$

subject to $\|\mathbf{A}_{\cdot\mathbf{k}}\| = 1$ and $\mathbf{A}^*_{\cdot\mathbf{j}} \cdot \mathbf{A}_{\cdot\mathbf{k}} = 0$; $j = 1, \dots, k-1$; $k \leq n$. In the case of Asian basket options we have

$$g(\epsilon) = g(\hat{\epsilon}) + \sum_{l=1}^{NM} \left[\sum_{i=1}^{NM} \exp\left(\mu_i + \sum_{k=1}^{NM} C_{ik}\hat{\epsilon}_k \right) C_{il} \right] \Delta\epsilon_l. \tag{23}$$

Imai and Tan [7] have proved the following result:

Proposition 3. *Consider an Asian basket options in a BS model, define:*

$$\mathbf{d}^{(\mathbf{p})} = \left(e^{\left(\mu_1 + \sum_{k=1}^{p-1} C^*_{1k}\right)}, \dots, e^{\left(\mu_{MN} + \sum_{k=1}^{p-1} C^*_{MN,k}\right)} \right)^T \tag{24}$$

$$\mathbf{B}^{(\mathbf{p})} = \left(C^{\mathrm{Ch}}\right)^T (\mathbf{d}^{(\mathbf{p})}), \quad p = 1, \dots, MN. \tag{25}$$

Then the p-th column of the optimal matrix A^* is

$$\mathbf{A}^*_{\cdot\mathbf{p}} = \pm\frac{\mathbf{B}^{(\mathbf{p})}}{\|\mathbf{B}^{(\mathbf{p})}\|} \quad p = 1, \ldots, MN. \tag{26}$$

The matrices C^*_{ik}, $k < p$ have been already found at the $p-1$ previous steps. $\mathbf{A}_{\cdot\mathbf{p}}$ must be orthogonal to all the other columns. This feature can be easily obtained by an incremental QR decomposition as described in Sabino [8].

3.4 Kronecker Product Approximation

In a time-dependent volatilities BS market the covariance matrix Σ_{MN} has time-dependent blocks. The multi-dimensional BM is the unique source of risk in the BS market and the generation of the trajectories of the 1-dimensional BM does depend on the volatilities. Based on these considerations, we propose to find a constant covariance matrix among the assets H, in order to approximate, in an appropriate sense, the matrix Σ_{MN} as a Kronecker product of R and H. In the following we illustrate the proposed procedure that we label Kronecker Product Approximation (KPA). Pitsianis and Van Loan [12] have proved the following proposition.

Proposition 4. *Suppose $G \in \mathbb{R}^{m\times n}$ and $G_1 \in \mathbb{R}^{m_1\times n_1}$ with $m = m_1 m_2$ and $n = n_1 n_2$. Consider the problem of finding $G_2^* \in \mathbb{R}^{m_1\times n_1}$ that realizes the minimum*

$$\min_{G_2\in\mathbb{R}^{m_1\times n_1}} \| G - G_1 \otimes G_2 \|_F^2, \tag{27}$$

where $\|\cdot\|_F^2$ denotes the Frobenius norm. For fixed $h = 1, \ldots, m_2$ and $l = 1, \ldots, n_2$ denote $\mathcal{R}(G)_{hl}$ the $m_1 \times n_1$ matrix defined by the rows $h, h+m_2, h+2m_2, \ldots, h+(m_1-1)m_2$ and the columns $l, l+n_2, l+2n_2, \ldots, l+(n_1-1)n_2$ of the original matrix G. The elements of G_2^* which gives (27) are

$$(G_2^*)_{hl} = \frac{\mathrm{Tr}\left(\mathcal{R}(G)_{hl}^T G_1\right)}{\mathrm{Tr}\left(G_1 G_1^T\right)} \quad h = 1, \ldots, m_2,\, l = 1, \ldots, n_2, \tag{28}$$

where Tr *indicates the trace of a matrix.*

In our setting, we have $G = \Sigma_{MN}$, $G_1 = R$ and $G_2 = H$. We note that for any $i, j = 1, \ldots, N$, $\mathcal{R}(\Sigma_{MN}))_{ij}$ is a $N \times N$ boomerang matrix. Moreover, given two general $N \times N$ boomerang matrices A and B, by direct computations we can prove

$$\mathrm{Tr}(A^T B) = \mathrm{Tr}(AB) = \sum_{j=1}^{N} \left(2(N-j)+1\right) a_{jj} b_{jj}. \tag{29}$$

Then we perform the PCA decomposition of $R \otimes H$ relying on the properties of the Kronecker product. However, if we use the PCA decomposition of the matrix $F = R \otimes H$ we do not get the required path. In order to produce the required trajectory we take

$$\mathbf{Z} = C^{\mathrm{KPA}}\epsilon = C_{\Sigma_{MN}}(C_F)^{-1} E_H \Lambda_H^{1/2} \epsilon \tag{30}$$

where $C_{\Sigma_{MN}}$ and C_F are the CH matrices associated to Σ_{MN} and F, respectively, and $E_H \Lambda_H^{1/2}$ is the PCA decomposition of F. The matrix C^{KPA} produces the correct covariance matrix; indeed, denoting $P = E_H \Lambda_H^{1/2}$, we have

$$C^{\mathrm{KPA}}\left(C^{\mathrm{KPA}}\right)^T = C_{\Sigma_{MN}}(C_F)^{-1} P P^T \left[(C_F)^{-1}\right]^T C_{\Sigma_{MN}}^T = C_{\Sigma_{MN}} C_{\Sigma_{MN}}^T = \Sigma_{MN}$$

Table 1: Inputs Parameters

$S_i(0)$	$=$	$100, \quad \forall i = 1\dots, N$
K	$\subset$	$\{90, 100, 110\}$
r	$=$	4%
T	$=$	1
$\sigma_i(0)$	$=$	$10\% + \frac{i-1}{9} 40\% \quad i = 1\dots, N$
$\sigma_i(+\infty)$	$=$	$9\% \quad \forall i = 1\dots, N$
τ_i	$=$	$1.5 \quad \forall i = 1\dots, N$
ρ_{ij}	$\subset$	$\{0, 40\} \quad i, j = 1\dots, N$

Table 2: Effective Dimensions. Time-dependent Volatilities

$\rho = 0\%$

Ch	PCA	LT	KPA
$d_T > 1900$	$d_T = 14$	$d_T = 10$	$d_T = 19$

$\rho = 40\%$

Ch	PCA	LT	KPA
$d_T > 1900$	$d_T = 9$	$d_T = 8$	$d_T = 11$

because $PP^T = C_F C_F^T = F$. Our fundamental assumption is that the principal components of $\mathbf{Z}$ are not so different from those of the normal random vector $\mathbf{Z}'$ whose covariance matrix is F. We expect that the KPA decomposition would produce an effective dimension higher than the effective dimension obtained by the PCA decomposition, but with an advantage from the computational point of view. Due to properties of the Kronecker product, Equation (30) becomes

$$\mathbf{Z} = C_{\Sigma_{MN}} \left(C_R^{-1} \otimes C_H^{-1} \right) E_H \Lambda_H^{1/2} \epsilon, \tag{31}$$

where C_R and C_H are the CH matrices of R and H, respectively. This matrix multiplication can be carried out quickly by block-matrices multiplication and knowing that, due to the Propositions 1 and 2, C_R^{-1} is a sparse bi-diagonal matrix.

4 Computing the Option Price

We will now estimate the fair price of an Asian option on a basket of $M = 10$ underlying assets with $N = 250$ sampled points in the BS model with time-dependent volatilities having the following expression

$$\sigma_i(t) = \hat{\sigma}_i(0) \exp\left(-t/\tau_i\right) + \sigma_i(+\infty), \quad i = 1, \dots M. \tag{32}$$

The parameters chosen for the simulation are listed in Table 1 ($\hat{\sigma}_i(0)$ is then $\sigma_i(0) - \sigma_i(+\infty)$). We implement the numerical investigation in two parts: first we test the effectiveness of the path-generation constructions on dimension reduction and compute their computational times, and then we compare the accuracy of the simulation.

Table 3: Computational Times in Seconds

Constant Volatilities

$\rho = 0\%$			$\rho = 40\%$		
Ch	PCA	LT	Ch	PCA	LT
0.60	25.77	71.14	0.59	25.55	71.02

Time-dependent Volatilities

$\rho = 0\%$				$\rho = 40\%$			
Ch	PCA	LT	KPA	Ch	PCA	LT	KPA
0.62	565.77	71.65	28.25	0.62	568.55	71.20	28.33

The Table 2 shows the effective dimensions obtained by all the path-generation methods ($p = 0.99$). The LT construction provides the lowest effective dimension, while the PCA decomposition performs almost as well as the LT approach for the correlation case only, and the KPA returns a slightly higher effective dimension. The CH decomposition collects 98.58% and 98.70% of the total variance for $d_T \approx 2000$ for the uncorrelated and correlated cases, respectively. To have a more accurate comparison, Table 3 displays the elapsed times computed in Sabino [8] using an *ad hoc* incremental QR algorithm for the LT and assuming constant volatilities each equal to $\sigma_i(0)$ of Table 1. The computation is implemented in MATLAB running on a laptop with an Intel Pentium M, processor 1.60 GHz and 1 GB of RAM. We compute 50 optimal columns for the LT technique. The CH algorithm for block boomerang matrices has almost the same cost as the one relying on the properties of the Kronecker product. As a consequence, the LT also requires almost the same computational cost, while the PCA needs a time almost 20 times higher because we can no longer rely on the properties of the Kronecker product. In contrast, the KPA has almost the same computational time as the PCA in the constant volatility case and is the best performing path-generation technique from the computational time point of view. We have applied Proposition 2 to implement the PCA and computed the eigenvalues and eigenvectors of Σ_{MN} relying only on the *sparse* function of MATLAB. It is noteworthy to say that there exist algorithms tailored for the computation of the eigenvalues and eigenvectors of tri-diagonal symmetric block matrices that can further reduce the computational time.

In the second part of our investigation we launch a simulation in order to estimate the Asian option price using 10 replications each of 8192 random points following the strategy in Imai and Tan [7]. We use different random generators: standard MC, LHS and RQMC generators. Concerning the computational times of the price estimation, the CPU ratio between LHS and RQMC is almost 1 while standard MC is 1.33 faster. Moreover, the LT construction needs a time that is almost 1/30 of the total computational time of the LHS or RQMC simulation. As a RQMC generator we use a Matoušek scrambled version (see Matoušek [14]) of the 50-dimensional Sobol sequence satisfying Sobol's property A (see Sobol [15]). We pad the remaining random components out with LHS. This hybrid strategy is intended to investigate the effective improvement of the decomposition methods when coupled with (R)QMC. Indeed, it can be proven that LHS gives good vari-

Table 4: Estimated At-the Money Prices and Errors.

		$\rho = 0$		$\rho = 40\%$	
		Price	RMSE	Price	RMSE
MC	Ch	3.18200	0.01300	5.18900	0.02600
	KPA	3.12400	0.01300	5.19400	0.02600
	PCA	3.10600	0.01300	5.20100	0.02600
	LT	3.11100	0.01300	5.24200	0.02600
		Price	RMSE	Price	RMSE
LHS	Ch	3.12200	0.00750	5.20000	0.01200
	KPA	3.12440	0.00550	5.20950	0.00340
	PCA	3.12010	0.00540	5.20070	0.00320
	LT	3.12200	0.00290	5.20090	0.00120
		Price	RMSE	Price	RMSE
RQMC	Ch	3.11240	0.00550	5.19500	0.01500
	KPA	3.12240	0.00086	5.20080	0.00054
	PCA	3.12140	0.00078	5.20080	0.00069
	LT	3.12230	0.00021	5.20080	0.00019

ance reductions when the target function is the sum of one-dimensional functions (see Stein [16]). On the other hand, the LT method is conceived to capture the lower effective dimension in the truncation sense for linear combinations. As a consequence, we should already observe a high accuracy when running the simulation using LHS combined with LT. We expect the KPA technique to produce a suboptimal decomposition in the sense of ANOVA, with the advantage of a lower computational effort. Our setting is thought to test how large is the improvement given by all the factorizations. Tables 4 and 5 present the results of our investigation. The prices in Table 4 are all in statistical agreement. Those obtained with the CH decomposition are almost not sensitive to the random number generator. KPA, PCA and LT all provide good improvements both for the LHS and RQMC implementations for all the strike prices. The LT has an evident advantage compared to the PCA and KPA constructions in the uncorrelated case. In contrast, we observe that the KPA and PCA-based simulations give almost the same accuracy, both assuming uncorrelated and correlated asset returns. Considering the total computational cost and accuracy we observe that the KPA performs better than the standard PCA. Moreover, all these constructions can be employed in stochastic and local volatility models that are based on the mixture of multi-dimensional dynamics for basket options as done in Brigo et al. [2].

5 Computing the Sensitivities

In the financial jargon, a Greek is the derivative of an option price with respect to a parameter. A Greek is therefore a measure of the sensitivity of the price with respect to one of its parameters. The deltas (Δ's) are the components of the gradient of the discounted expected outcome of the option with respect to the initial values of the assets. The problem of computing the Greeks in finance has been studied by several authors. In the following we extend the methodology employed by Montero and Kohatsu-Higa [1], based on the use of Malliavin Calculus, to the multi-assets case. The main difficulties of this extension lie in the fact that the assets are now correlated and the formulas in Montero and Kohatsu-Higa [1] cannot be directly extended to the multi-dimensional case. Moreover, the localization technique, introduced by Fournié et al [9], should generally control all the components of the multi-dimensional BM to improve the accuracy of the estimation. We write the dynamics (1) with respect to a M-dimensional BM $\mathbf{W}(t)$ with uncorrelated components

$$dS_i(t) = rS_i(t)dt + S_i(t)\sigma_i(t)\sum_{m=1}^{M}\alpha_{im}(t)dW_m(t) \quad i = 1,\dots,M, \tag{33}$$

where $\sum_{m=1}^{M}\alpha_{im}\alpha_{km} = \rho_{ik}$ and we have defined $\sigma_{im}(t) = \sigma_i(t)\sum_{m=1}^{M}\alpha_{im}$.

The Malliavin calculus is a theory of variational stochastic calculus and provides the mechanics to compute derivatives and integration by parts of random variables (see Nualart [17] for more on Malliavin Calculus).

Denote by $D_s^1,\dots,D_s^M$ the Malliavin derivatives with respect to the components of $\mathbf{W}(t)$, while $\delta^{\mathrm{Sk}} = \sum_{m=1}^{M}\delta_m^{\mathrm{Sk}}$ represents the Skorohod integral with δ_m^{Sk} indicating the Skorohod integral on the single $W_m(t)$. The domains of the Malliavin derivatives and the Skorohod integral are denoted by $\mathbb{D}^{1,2}$ and $\mathrm{dom}(\delta^{\mathrm{Sk}})$, respectively, while δ^{Kr} indicates the Kronecker delta. We prove the following proposition.

Proposition 5. *Denote* $\mathbf{x} = \mathbf{S}(0)$*, and* G_k *the partial derivative*

$$G_k = \frac{\partial m(T)}{\partial x_k} = \frac{\sum_{j=1}^{N} w_{kj}S_k(t_j)}{x_k}, \quad k = 1,\dots,M, \tag{34}$$

where $m(T) = \sum_{j=1}^{N} w_{kj}S_k(t_j)$*. Knowing that* $a(T) \in \mathbb{D}^{1,2}$*, the k-th delta (the k-th component of the gradient) is*

$$\Delta_k = \frac{\partial a(0)}{\partial x_k} = e^{-rT}\mathbb{E}\left[a'(T)G_k\right] = e^{-rT}\mathbb{E}\left[a(T)\sum_{m=1}^{M}\delta_m^{\mathrm{Sk}}(G_k u_m)\right], \tag{35}$$

where $\mathbf{u} = (u_1,\dots,u_M) \in \mathrm{dom}(\delta^{\mathrm{Sk}})$*,* $\mathbf{z} = (z_1,\dots,z_m) \in \mathrm{dom}(\delta^{\mathrm{Sk}})$*,* $G_k\mathbf{u} \in \mathrm{dom}(\delta^{\mathrm{Sk}})$ *and*

$$\begin{aligned} \frac{z_m(s)}{\sum_{h=1}^{M}\int_0^T z_h(s)D_s^h m(T)ds} &= u_m(s) \\ \sum_{h=1}^{M}\int_0^T z_h(s)D_s^h m(T)ds &\neq 0, \quad \text{a.s.} \end{aligned}$$

Proof. Compute

$$D_s^h a(T) = a'(T)D_s^h m(T) \quad h = 1,\dots,M. \tag{36}$$

Multiply the above equation by $z_h(t)$ – so that $\mathbf{z} \in \mathrm{dom}(\delta^{\mathrm{Sk}})$ – and by G_k; then sum for all $h = 1,\dots,M$ and integrate:

$$\sum_{h=1}^{M}\int_0^T G_k z_h(s)D_s^h a(T)ds = \sum_{h=1}^{M}\int_0^T G_k z_h(s)a'(T)D_s^h m(T)ds. \tag{37}$$

Due to the definition of $\mathbf{u}$ and to the fact that $a'(T)G_k$ does not depend on s we can write

$$a'(T)G_k = \sum_{m=1}^{M} \int_0^T u_m(s) G_k D_s^m a(T))ds. \quad k = 1, \ldots, M \tag{38}$$

Finally compute the expected value of both sides of (38)

$$\mathbb{E}\left[a'(T)G_k\right] = \mathbb{E}\left[\sum_{m=1}^{M} \int_0^T u_m(s) G_k D_s^m a(T) ds\right]. \tag{39}$$

so that by duality

$$\Delta_k = \mathbb{E}\left[a(T))\delta^{\mathrm{Sk}}(G_k \mathbf{u})\right] \quad k = 1, \ldots, M. \tag{40}$$

and this concludes the proof. $\square$

Proposition 5 allows a certain flexibility in choosing either the process $\mathbf{u}$, or better $\mathbf{z}$. We consider $z_h = \alpha_k \delta_{hk}^{\mathrm{Kr}};\ h, k = 1, \ldots, M,\ \alpha_k = 1, \forall k$. Namely, in order to compute the k-th delta we consider only the k-th term of the Skorohod integral reducing the computational cost. In particular, this choice is motivated by the fact that in this way the localization technique needs to control only $\delta_k^{\mathrm{Sk}}(\cdot)$ and then only the k-th component of $\mathbf{W}(t)$. Then we define L_k and calculate for $k = 1, \ldots, M$

$$\begin{aligned} L_k &= \int_0^T D_s^k m(T) ds = \sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij} S_i(t_j) \int_0^{t_j} \sigma_{ik}(s) ds, && (41)\\ \int_0^T D_s^k G_k ds &= \sum_{j=1}^{N} w_{jk} S_k(t_j) \int_0^{t_j} \sigma_{kk}(s) ds = \sum_{j=1}^{N} w_{jk} S_k(t_j) \int_0^{t_j} \sigma_k(s) ds, && (42)\\ \int_0^T D_s^k L_k ds &= \sum_{j=1}^{N} w_{ij} S_i(t_j) \left(\int_0^{t_j} \sigma_{ik} ds\right)^2, && (43) \end{aligned}$$

and hence

$$\Delta_k = \mathbb{E}\left[a(T)\delta_k^{\mathrm{Sk}}\left(\frac{G_k}{L_K}\right)\right], \quad k = 1, \ldots, M. \tag{44}$$

Due to the properties of the Skorohod integral we have for $k = 1, \ldots, M$

$$\delta_k\left(\frac{G_k}{L_K}\right) = \frac{G_k}{L_K} W_k(T) - \frac{1}{L_k^2}\left(L_k \int_0^T D_s^k G_k ds - G_k \int_0^T D_s^k L_k ds\right), \tag{45}$$

With another choice of $\mathbf{z}$, for instance $z_h = \alpha_h$, Δ_k would depend linearly on the whole M-dimensional BM, making the localization technique less efficient.

We investigate the applicability of the RQMC approach to estimate the expected value in Equation (44) for $k = 1, \ldots, M$. We assume the same input parameters as in Section 4 and generate the trajectories (the values $S_i(t_j), i = 1, \ldots, M, j = 1, \ldots, N$) in Equations (41-43) as in Section 4. Moreover, we consider α_{im} as the elements of the CH matrix associated to $\rho_{im},\ i, m = 1, \ldots, M$. Table 5 compares the deltas obtained with RQMC only with the same number of scenarios as in Section 4. We apply the same LT construction used to estimate the price of the option and not the one for the integrand function in Equation (44).

Table 5: At-the-money estimated Δ's (10^{-2}) and errors (10^{-4}) with RQMC.

$\rho = 0\%$

LT		KPA		PCA		CH	
Δ	RMSE	Δ	RMSE	Δ	RMSE	Δ	RMSE
6.1832	0.80	6.1820	1.10	6.1808	0.86	6.2060	1.50
6.2024	0.75	6.2126	0.90	6.2016	0.86	6.2250	1.10
6.2305	0.85	6.2340	0.87	6.2341	0.99	6.2530	1.20
6.2667	0.75	6.2701	0.82	6.2699	0.91	6.2830	1.40
6.3081	0.60	6.3133	0.92	6.3093	0.96	6.3270	1.20
6.3569	0.55	6.3595	0.97	6.3598	0.83	6.3750	1.10
6.4107	0.50	6.4141	0.93	6.4103	0.78	6.4329	1.20
6.4709	0.50	6.4744	0.91	6.4677	0.84	6.4920	1.40
6.5338	0.50	6.5390	0.93	6.5325	0.93	6.5530	1.30
6.6001	0.65	6.6060	0.78	6.6000	0.96	6.6120	1.10

$\rho = 40\%$

LT		KPA		PCA		CH	
Δ	RMSE	Δ	RMSE	Δ	RMSE	Δ	RMSE
5.47830	0.056	5.48410	0.110	5.48050	0.130	5.46767	1.100
5.53510	0.062	5.54050	0.110	5.53740	0.140	5.52457	1.200
5.59430	0.054	5.60020	0.110	5.59660	0.130	5.58730	1.200
5.65440	0.062	5.66120	0.120	5.65680	0.130	5.64031	1.000
5.71680	0.075	5.72330	0.130	5.71790	0.140	5.70994	1.100
5.78130	0.082	5.78850	0.120	5.78410	0.120	5.77038	1.300
5.84840	0.077	5.85320	0.096	5.85060	0.120	5.83233	1.200
5.91560	0.082	5.92110	0.110	5.91790	0.130	5.90019	1.100
5.98490	0.052	5.99070	0.093	5.98680	0.120	5.97059	1.000
6.05470	0.059	6.06050	0.110	6.05680	0.130	6.04613	1.200

This would not seem to be the optimal choice, but if we would have applied the LT for the integrand function in Equation (44) $M = 10$ decomposition matrices (one for each delta) would be required. This setting would have increased the CPU time to obtain the LT to at least 1/3 (even higher due to the larger number of terms to compute) of the total time making the estimation less convenient. Table 5 shows that the PCA, LT and KPA approaches perform almost equally in terms of RMSEs, with the LT giving only slightly better results in the uncorrelated case. In terms of computational cost the KPA performs better than the PCA. The CH construction displays RMSEs that are even 10 times higher. As explained before, g can be considered a good approximation for the payoff function in Equation (44) but in the Malliavin expression $a(T)$ is multiplied by a random weight that depends on the Gaussian vector $\mathbf{Z}$. In contrast, the PCA and the KPA concentrate most of the variation in the first dimensions of $\mathbf{Z}$. This is the explanation of the almost equal accuracy of the LT, the PCA and the KPA.

6 Conclusions

We have considered the problem of computing the fair price and the deltas of high-dimensional Asian basket options in a BS market with time-dependent volatilities. In order to extend the QMC superiority to high dimensions it is necessary to employ path-generation techniques with the main purpose to reduce the nominal dimension. The LT and the PCA constructions try to accomplish this task by the concept of ANOVA. In the case of time-dependent volatilities in the BS economy the computational cost of the LT and the PCA cannot be reduced relying on the properties of the Kronecker product and the computation is more complex. We have presented a new and fast CH algorithm for block matrices that remarkably reduces the computational burden making the LT construction even more convenient than the PCA. We have introduced a new path-generation technique, named KPA, that in the applied setting, is as accurate as the PCA and is even more convenient with respect to the computational cost. In addition, we proved that the KPA enhances RQMC for the estimation of the fair price and the calculation of the deltas of Asian basket options in a BS model with time-dependent volatilities. In this setting the KPA provides the same accuracy of the LT in the case of correlated asset returns and in the estimation of the deltas. Finally, concerning the computation of the sensitivities, we have extended the procedure adopted by Montero and Kohatsu-Higa [1], based on the Malliavin Calculus, to the multi-assets case. All these results can be easily applied to stochastic and local volatility models that are based on the mixture of multi-dimensional dynamics for basket options, as done in Brigo et al. [2].

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