Mass spectrum from stochastic Lévy-Schrödinger relativistic equations: possible qualitative predictions in QCD

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Starting from the relation between the kinetic energy of a free Lévy-Schrödinger particle and the logarithmic characteristic of the underlying stochastic process, we show that it is possible to get a precise relation between renormalizable field theories and a specific Lévy process. This subsequently leads to a particular cut-off in the perturbative diagrams and can produce a phenomenological mass spectrum that allows an interpretation of quarks and leptons distributed in the three families of the standard model.

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I. INTRODUCTION AND NOTATIONS

In this note we adopt the space-time relativistic approach of Feynman's propagators (for bosons and fermions) instead of the canonical Lagrangian-Hamiltonian quantized field theory. The rationale for this choice is that for the development of our basic ideas the former alternative is better suited to exhibit the connection between the propagator of quantum mechanics and the underlying Lévy processes. More precisely, the relativistic Feynman propagators are here linked to a dynamical theory based on a particular Lévy process: a point, already discussed in a previous paper [1], which is here analyzed thoroughly with the purpose of deducing its consequences for the basic interactions among the fundamental constituents, namely quarks, leptons, gluons, photons etc. To this end we first recall that a Lévy process is a stochastic process $X(t),\ t \geq 0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- $X(0) = 0, \quad \mathbb{P}$-q.o.
- $X(t)$ has independent and stationary increments: for each n and for very choice of $0 \leq t_1 < t_2 < \ldots < t_n < +\infty$ the increments $X(t_{j+1}) - X(t_j)$ are independent and $X(t_{j+1}) - X(t_j) \stackrel{d}{=} X(t_{j+1} - X(t_j)$;
- $X(t)$ is stochastically continuous: for every $a > 0$ and for every s

$$\lim_{t \to s} \mathbb{P}\left(|X(t) - X(s)| > a\right) = 0.$$

To simplify the notation we will consider in the following only one-dimensional Lévy processes (an n-dimensional extension, however, would not be a very difficult task): it is well known [2–4] that all its laws are infinitely divisible, but we will be mainly interested in the non stable (and in particular non Gaussian) case. In other words the characteristic function of the process Δt-increment is $[\varphi(u)]^{\Delta t/\tau}$ where φ is infinitely divisible, but not stable[1], and τ is a time scale parameter. The transition probability density $p(2|1) = p(x_2, t_2|x_1, t_1)$ of a particle moving from the space-time point 1 to 2 then is

$$p(2|1) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} du\, [\varphi(u)]^{(t_2 - t_1)/\tau} e^{-iu(x_2 - x_1)} \tag{1}$$

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[1] A law φ is said to be *infinitely divisible* if for every n it exists a characteristic function φ_n such that $\varphi = \varphi_n^n$; on the other hand it is said to be *stable* when for every $c > 0$ it is always possible to find $a > 0$ and $b \in \mathbf{R}$ such that $e^{ibu}\varphi(au) = [\varphi(u)]^c$. Every stable law is also infinitely divisible.

In analogy with the non relativistic Wiener case, we then obtain for the motion of a free particle the Feynman propagator $\mathcal{K}(2|1) = \mathcal{K}(x_2, t_2|x_1, t_1)$ as

$$\mathcal{K}(2|1) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} du\, [\varphi(u)]^{i(t_2-t_1)/\tau} e^{-iu(x_2-x_1)} \tag{2}$$

and the corresponding wave function evolution is

$$\psi(x,t) = \int_{-\infty}^{+\infty} dx'\, \mathcal{K}(x,t|x',t')\psi(x',t'). \tag{3}$$

From (2) and (3) we easily obtain [1]

$$i\partial_t\psi = -\frac{1}{\tau}\eta(\partial_x)\psi$$

where $\eta = \log\varphi$ and $\eta(\partial_x)$ is a pseudodifferential operator with symbol $\eta(u)$ [3, 5–7] which plays the role of the generator of the semigroup $T_t = e^{t\eta(\partial_x)/\tau}$ operating on a Banach space of measurable, bounded functions [3, 5–7].

It is very well known [2, 3], on the other hand, that φ represents an infinitely divisible law if and only if $\eta(u) = \log\varphi(u)$ satisfies the Lévy–Khintchin formula

$$\eta(u) = i\gamma u - \frac{\beta^2 u^2}{2} + \int_{\mathbb{R}} \left[e^{iux} - 1 - iux\, I_{[-1,1]}(x)\right]\, \nu(dx) \tag{4}$$

where $\gamma, \beta \in \mathbb{R}$, $I_{[-1,1]}(x)$ is the indicator of $[-1, 1]$, and $\nu(dx)$ is a Lévy measure, namely a measure on $\mathbb{R}$ such that $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} (x^2 \wedge 1)\, \nu(dx) < +\infty.$$

In the most common cases of centered and symmetric laws the equation (4) simplifies in

$$\eta(u) = -\frac{\beta^2 u^2}{2} + \int_{\mathbb{R}} (\cos ux - 1)\, \nu(dx) \tag{5}$$

and $\eta(u)$ becomes even and real. As a consequence the corresponding operator $\eta(\partial_x)$ is self–adjoint and acts on propagators and wave functions according to the Lévy–Schrödinger integro–differential equation

$$i\partial_t\psi(x,t) = -\frac{1}{\tau}\,\eta(\partial_x)\psi(x,t) = -\frac{\beta^2}{2\tau}\,\partial_x^2\psi(x,t) - \frac{1}{\tau}\int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)]\, \nu(dy) \tag{6}$$

The integral term accounts for the jumps in the trajectories of the underlying stochastic process, while an action α with $\beta^2 = \alpha\tau/m$ weights the usual differential term of the Schrödinger equation. For $\beta = 0$ a pure jump Lévy–Schrödinger equation is obtained

$$i\partial_t\psi(x,t) = -\frac{1}{\tau}\int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)]\, \nu(dy). \tag{7}$$

II. STATIONARY SOLUTIONS FOR THE FREE PARTICLE

The free equation (6) admits simple stationary solutions: if we consider

$$\psi(x,t) = e^{-iE_0t/\alpha}\phi(x), \qquad \alpha = \frac{m\beta^2}{\tau}$$

we then have

$$E_0\phi(x) = -\frac{\alpha^2}{2m}\,\phi''(x) - \frac{\alpha}{\tau}\int_{\mathbb{R}} [\phi(x+y) - \phi(x)]\, \nu(dy), \tag{8}$$

and for a plane wave $\phi(x) = e^{\pm iux}$ from (5) – namely with a symmetric Lévy noise – we get

$$E_0\phi(x) = -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}}\left(e^{\pm iuy} - 1\right)\,\nu(dy)\right]e^{\pm iux} = -\frac{\alpha}{\tau}\left[-\frac{\beta^2u^2}{2} + \int_{\mathbb{R}}(\cos uy - 1)\,\nu(dy)\right]\phi(x) = -\frac{\alpha}{\tau}\eta(u)\phi(x)$$

which is satisfied when $E_0 = -\alpha\eta(u)/\tau$. Hence, by taking $p = \alpha u$ as a momentum variable, we obtain [1] the relevant equation

$$E_0 = -\frac{\alpha}{\tau}\,\eta\left(\frac{p}{\alpha}\right) \tag{9}$$

which connects the kinetic energy of a forceless particle to the logarithmic characteristic of a Lévy process.

III. RELATIVISTIC QUANTUM MECHANICS

Let us take now in particular the non stable law

$$\eta(u) = 1 - \sqrt{1 + a^2u^2}\,. \tag{10}$$

With the following identification of the parameters

$$\alpha = \hbar, \qquad \frac{\hbar}{\tau} = mc^2, \qquad a = \frac{\hbar}{mc}, \qquad p = \hbar u.$$

we are led to the formula

$$E_0 = -mc^2\eta\left(\frac{p}{\hbar}\right) = E - mc^2 = \sqrt{m^2c^4 + p^2c^2} - mc^2 \tag{11}$$

which is the well–known relativistic kinetic energy for a particle of mass m. The Schrödinger equation of a relativistic free-particle is then easily obtained from (11) by reinterpreting as usual E and p respectively as the operators $i\hbar\partial_t$ and $-i\hbar\partial_x$

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - \hbar^2c^2\partial_x^2}\,\psi(x,t), \tag{12}$$

It is easy to check that this derives also from (6) after absorbing the mass energy term $-mc^2$ of (11) into a phase factor $e^{imc^2t/\hbar}$. Remark that in three dimensions (12) would read

$$i\hbar\partial_t\psi(x,t) = \sqrt{m^2c^4 - \hbar^2c^2\boldsymbol{\nabla}^2}\,\psi(x,t) \tag{13}$$

It has been shown [3, 8] that the Lévy process behind the equations (12) and (13) is a pure jump process [1, 3] with an absolutely continuous Lévy measure $\nu(dx) = W(x)dx$ such that

$$W(x) = \frac{1}{\pi|x|}\,K_1\left(\frac{|x|}{a}\right) = \frac{1}{\pi|x|}\,K_1\left(\frac{mc}{\hbar}\,|x|\right) \tag{14}$$

(K_ν are the modified Bessel functions [9]), that in three dimensions becomes

$$W(\boldsymbol{x}) = \frac{1}{2a\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{|\boldsymbol{x}|}{a}\right) = \frac{mc}{2\hbar\pi^2|\boldsymbol{x}|^2}\,K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{x}|\right) \tag{15}$$

while from (7) the equation (12) takes the form

$$i\hbar\partial_t\psi(x,t) = -mc^2\int_{\mathbb{R}}\frac{\psi(x+y,t) - \psi(x,t)}{\pi|y|}\,K_1\left(\frac{mc}{\hbar}\,|y|\right)dy \tag{16}$$

and in three dimensions is

$$i\hbar\partial_t\psi(\boldsymbol{x},t) = -mc^2\int_{\mathbb{R}^3}\frac{\psi(\boldsymbol{x}+\boldsymbol{y},t) - \psi(\boldsymbol{x},t)}{2\pi^2|\boldsymbol{y}|^2}\,\frac{mc}{\hbar}K_2\left(\frac{mc}{\hbar}\,|\boldsymbol{y}|\right)d^3\boldsymbol{y} \tag{17}$$

From the equation (13) – by means of well known standard procedures [10] – one also derives (always for the free particle) the Klein–Gordon and Dirac equations in three dimensions both for scalar, and for spinor wave functions ψ, namely respectively

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\psi \;=\; 0, \tag{18}$$

$$\left(i\gamma_\mu\partial^\mu - \frac{mc}{\hbar}\right)\psi \;=\; 0. \tag{19}$$

The corresponding Klein–Gordon and Dirac propagators in their turn satisfy the inhomogeneous equations (here with $\hbar = c = 1$)

$$\left(\Box_2 - m^2\right)\mathcal{K}_{KG}(2|1) \;=\; \delta^4(2|1) \tag{20}$$

$$(i\gamma_\mu\partial_2^\mu - m)\,\mathcal{K}_D(2|1) \;=\; i\,\delta^4(2|1) \tag{21}$$

with $\delta^4(2|1) = \delta(t_2 - t_1)\delta^3(\boldsymbol{x}_2 - \boldsymbol{x}_1)$. Let us finally remark that these relativistic quantum wave equations have been recently of particular interest [11] also in the field of quantum optical phenomena and of quantum information.

IV. INFINITE DIVISIBILITY–PRESERVING MODIFICATIONS

From the relativistic kinetic energy E_0 of a point particle of rest mass m

$$\frac{E_0(\boldsymbol{p})}{mc^2} = \sqrt{1 + \frac{\boldsymbol{p}^2}{m^2c^2}} - 1. \tag{22}$$

with the identifications

$$\eta = -\frac{E_0}{mc^2}, \qquad \boldsymbol{u} = \frac{\boldsymbol{p}}{amc}$$

we obtain

$$\eta(\boldsymbol{u}) = 1 - \sqrt{1 + a^2\boldsymbol{u}^2}$$

which of course coincides with the equation (10) extended to three dimensions. We take now a class of transformations of $\eta(\boldsymbol{u})$ characterized by the fact that they preserve the infinite divisibility, while producing changes in the forceless particle equations of motion with respect to the usual ones (18) and (19). To this end we modify the energy-momentum formula as follows

$$E(\boldsymbol{p}) = mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{m^2c^2} + f\left(\frac{p^2}{m^2c^2}\right)} \tag{23}$$

where f is a – possibly small – dimensionless, smooth function of the relativistic scalar p^2/m^2c^2. Of course this modification entails that p^2 no longer coincides with m^2c^2 since the standard energy-momentum relation is now changed into

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2 + m^2c^2 f\left(\frac{p^2}{m^2c^2}\right). \tag{24}$$

As we will see in the following, this also implies that the mass no longer is m: it will take instead one or more values depending on the choice of f. As a matter of fact, it could appear to be preposterous to introduce a function f of an argument which after all is a constant (albeit different from 1). However we will show that this artifice will lend us the possibility of having both a mass spectrum, and a new wave equation when – in the next section – we will quantize our classical relations. Moreover it will be argued in the following that we will choose f in such a way that the corresponding modified logarithmic characteristic η will remain infinitely divisible: a feature that is instrumental to keep a viable connection to a suitable underlying Lévy process.

To see that, we first remark that (24) defines the total particle energy E in an implicit form. To find it explicitly we first rewrite (24) in a dimensionless form as

$$\frac{p^2}{m^2c^2} = 1 + f\left(\frac{p^2}{m^2c^2}\right),$$

and then, by taking $g(x) = x - f(x)$, we just observe that the former equation requires that x be solution of $g(x) = 1$, namely

$$g\left(\frac{p^2}{m^2c^2}\right) = \frac{p^2}{m^2c^2} - f\left(\frac{p^2}{m^2c^2}\right) = 1.$$

Note that f and g should be considered universal functions, and that the following conditions hold

$$f(1) = 0 \qquad\qquad g(1) = 1$$

If then $g^{-1}(1)$ represents one of the (possibly many) solutions of this equation, we could write

$$\frac{p^2}{m^2c^2} = g^{-1}(1) = x(g)\Big|_{g=1}$$

so that we have

$$p^2 = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2g^{-1}(1)$$

which can be interpreted as a simple mass re-scaling, from m to one of the (possibly many) values $M = m\sqrt{g^{-1}(1)}$. The new hamiltonian then is

$$E(\boldsymbol{p}) = \sqrt{m^2c^4g^{-1}(1) + \boldsymbol{p}^2c^2} = Mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{M^2c^2}} \tag{25}$$

and its kinetic part (by applying the same re-scaling also to the subtracted rest mass term) is

$$E_0(\boldsymbol{p}) = E(\boldsymbol{p}) - mc^2\sqrt{g^{-1}(1)} = Mc^2\sqrt{1 + \frac{\boldsymbol{p}^2}{M^2c^2}} - Mc^2.$$

Hence the main consequence of our modification consists of a re-scaling of the mass value ($m \to M$) at a purely classical level. This fact is apparently welcome, because its straightforward consequence is that the new associated logarithmic characteristic η is trivially again infinitely divisible, and hence still produces acceptable Lévy processes. But there is more: since $g^{-1}(1)$ can take several different real and positive values, by means of our modification (23) we have introduced a mass spectrum: in the rest frame of the particle we indeed have now

$$M = E_{cm}/c^2 = m\sqrt{g^{-1}(1)} \tag{26}$$

V. QUANTUM EQUATIONS OF MOTION

From the modified energy formula one derives a relativistic Schrödinger equation (for instance by means of the formal substitutions $E \to i\hbar\partial_t$ and $\boldsymbol{p} \to -i\hbar\boldsymbol{\nabla}$):

$$i\hbar\partial_t\psi(x,t) = mc^2\sqrt{1 - \frac{\hbar^2}{m^2c^2}\boldsymbol{\nabla}^2 + f\left(\frac{\Box}{m^2c^2}\right)}\,\psi(x,t) \tag{27}$$

where the square root pseudo-differential operator satisfies the constraints exposed in the Section I. From (27) one easily obtains in the usual manner (from here on $\hbar = c = 1$)

$$\left[\Box - m^2 f\left(\frac{1}{m^2}\Box\right) - m^2\right]\psi \;=\; 0, \tag{28}$$

$$\left[\Box_2 - m^2 f\left(\frac{1}{m^2}\Box_2\right) - m^2\right]\mathcal{K}_{KG}(2|1) \;=\; \delta^4(2|1) \tag{29}$$

and, by standard methods [10], the *modified* Dirac spinor equations

$$\left[i\gamma_\mu\partial^\mu - m\sqrt{1 + f\left(\frac{1}{m^2}\Box\right)}\right]\psi \;=\; 0 \tag{30}$$

$$\left[i\gamma_\mu\partial_2^\mu - m\sqrt{1 + f\left(\frac{1}{m^2}\Box_2\right)}\right]\mathcal{K}_D(2|1) \;=\; i\delta^4(2|1) \tag{31}$$

In the momentum space (with Fourier transforms in four dimensions) these equations become much simpler: more precisely we have

$$\mathcal{K}_{KG}(p^2) = \frac{1}{p^2 - m^2\left[1 + f(p^2/m^2)\right] + i\epsilon} \tag{32}$$

$$\mathcal{K}_D(p^2) = \frac{1}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} \tag{33}$$

We notice that $\mathcal{K}_D(2|1)$ is in our case simply related to the $\mathcal{K}_{KG}(2|1)$ (like in the usual case) as

$$\mathcal{K}_D(2|1) = i\left[\, i\partial\!\!\!/_2 + m\sqrt{1 + f(\Box_2/m^2)}\,\right]\mathcal{K}_{KG}(2|1) \tag{34}$$

VI. PHENOMENOLOGY: QUARK AND LEPTON MASSES

The equations (29) and (31) generalize the well known propagator equations (20) and (21) deriving from QED and QCD at zero order (in absence of interaction terms). The Standard Model (SM)[2] $SU_c(3) \times SU_L(2) \times U(1)$ treats both strong, and electro-weak interactions: within this scheme the modified $\eta(\boldsymbol{u})$ leads to new interesting consequences. We begin by considering the Feynman rules in perturbation theory in presence of the modified zero order propagator for both spin $\frac{1}{2}$ (quarks and leptons) and spin 1 (gluons, vector weak interacting Bosons). The amplitude A for a fermion that propagates from vertex X to vertex Y, if expanded, looks as follows: $A = A^{(0)} + A^{(1)} + A^{(2)} + \ldots$ The lowest order is

$$A^{(0)} = Y \frac{i}{\gamma^\mu p_\mu - m\sqrt{1 + f(p^2/m^2)} + i\epsilon} X. \tag{35}$$

It is then possible that the Fermion emits and reabsorbs a virtual vector boson[3] from X to Y:

$$\begin{aligned} A^{(1)} &= 4\pi g_s^2 Y \int d^4k \, \frac{\gamma^\mu}{\gamma^\rho p_\rho - m\sqrt{1 + f(p^2/m^2)}} \, \frac{1}{(p-k)^2} \\ &\quad \times \frac{1}{k^\nu \gamma_\nu - m\sqrt{1 + f(k^2/m^2)} + i\epsilon} \, \frac{\gamma_\mu}{\gamma^\rho p_\rho - m\sqrt{1 + f(p^2/m^2)}} X \end{aligned} \tag{36}$$

We choose now $f(x)$ in such a way that it makes finite the integral

$$C = \gamma^\mu \int \frac{d^4k}{\gamma^\rho k_\rho - m\sqrt{1 + f\left(k^2/m^2\right)} + i\epsilon} \, \frac{1}{(p-k)^2} \, \gamma_\mu \tag{37}$$

One may notice that $f(x)$ behaves as a smooth *cut-off* in a procedure of regularization at each order in QCD (and QED). The integral C is an invariant of the form $C = \tilde{A}(p^2)p\!\!\!/ + \tilde{B}(p^2)$ and its integrand is also present as a factor in higher order terms, thus producing convergence.

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[2] For future developments we recall that the Lagrangian density of QCD is, up to gauge fixing terms:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_q \overline{\psi}{}^q_i [i\gamma^\mu (D_\mu)_{ij} - m_q \delta_{ij}] \psi^q_j$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu$, and the insertion of interaction terms is done with the minimal interaction by substituting the simple derivative ∂_μ with the covariant one D_μ where we have respectively for QED and QCD

$$\begin{aligned} D_\mu &\equiv \partial_\mu - ieA_\mu \\ (D_\mu)_{ij} &\equiv \delta_{ij}\partial_\mu - ig_s T^a_{ij} A^a_\mu. \end{aligned}$$

Here g_s is the QCD coupling constant, T^a_{ij} and f_{abc} are the $SU(3)$ color matrices and structure constants respectively, and A^a_μ the eight Yang–Mills gluon fields; ψ^q_i are the Dirac 4-spinors associated with each quark field of color i and flavor q.

[3] The presence of *gammas* in the numerator of formulae (36) and (37) is typical of QED. More elaborated numerators may be present in non-abelian theories (in particular QCD); however they appear totally unessential for our subsequent developments and purposes.

The search of poles of the fermion propagators can be done in the following way [12]: one considers the contributions of the perturbative expansion of the amplitude $A(p^2)$ (here we always understand $f = f(p^2/m^2)$):

$$\begin{aligned} A(p^2) \;&=\; Y\left\{\frac{1}{\not p - m\sqrt{1+f}} + \frac{1}{\not p - m\sqrt{1+f}}C\frac{1}{\not p - m\sqrt{1+f}}\right. \\ &\qquad\left. + \frac{1}{\not p - m\sqrt{1+f}}C\frac{1}{\not p - m\sqrt{1+f}}C\frac{1}{\not p - m\sqrt{1+f}} + \ldots\right\}X \end{aligned} \tag{38}$$

and using the formula

$$\frac{1}{A-B} = \frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \ldots \tag{39}$$

one obtains the approximate expression

$$A \simeq Y\frac{1}{\not p - m\sqrt{1+f} - C}X = Y\frac{1}{\not p - m\sqrt{1+f} - \tilde{A}\not p - \tilde{B}}X \tag{40}$$

and looks for possible poles which – after rationalizing equation (40) – are solutions of the equation

$$\left[1 - \tilde{A}(p^2)\right]^2 p^2 - \left[m\sqrt{1+f(p^2/m^2)} + \tilde{B}(p^2)\right]^2 = 0 \tag{41}$$

A. Hypothesis for an approximate evaluation of the mass spectrum

Let us now focus our attention on QCD. We may consider the approximation $\tilde{A}(p^2) \simeq \tilde{A}(m^2)$ and $\tilde{B}(p^2) \simeq \tilde{B}(m^2)$ which follows from the assumption

$$\tilde{A} \ll 1 \qquad \text{and} \qquad \tilde{B} \ll m\sqrt{1+f(p^2/m^2)} \tag{42}$$

We then obtain the equation

$$p^2 = \left[\frac{m\sqrt{1+f(p^2/m^2)} + \tilde{B}(m^2)}{1 - \tilde{A}(m^2)}\right]^2 = m^2_{exp} \tag{43}$$

where the experimental masses m_{exp} are still represented in an implicit form. However note that in the limit $g_s \to 0$, $\tilde{A}$ and $\tilde{B} \to 0$ one achieves the equation

$$p^2 = m^2\left[1 + f(p^2/m^2)\right] \tag{44}$$

which coincides with the *classical* equation (24). At this point we notice that the simplest choice of $f(x)$ that makes the integral C finite (integrand convergent) is a polynomial of third degree:

$$f(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \lambda_3 x^3 \tag{45}$$

The equation (44) then becomes

$$x - 1 - f(x) = -\lambda_3(x-1)(x-x_+)(x-x_-) = 0 \qquad\qquad \left(x = \frac{p^2}{m^2}\right) \tag{46}$$

with

$$f(1) = 0 \qquad\qquad \lambda_0 = -\lambda_1 - \lambda_2 - \lambda_3 \tag{47}$$

where

$$x_\pm = \frac{1}{2\lambda_3}\left(-\lambda_2 - \lambda_3 \pm \sqrt{\Delta}\right) \qquad\qquad \Delta = (\lambda_2 - \lambda_3)^2 - 4\lambda_1\lambda_3 - 4\lambda_3^2 + 4\lambda_3 \tag{48}$$

and we finally get

$$\frac{\lambda_2}{\lambda_3} = -(1 + x_+ + x_-) \tag{49}$$

$$\frac{\lambda_1 - 1}{\lambda_3} = x_+ + x_- + x_+ x_- \tag{50}$$

From the previous formulae one achieves the following interesting result: the convergence of C determines a possible phenomenological function $f(x)$ that produces a mass spectrum of three fermion particles (quarks in QCD).

The connections with the possible experimental physical masses are $M_1 = m$, $M_2 = m\sqrt{x_+}$, $M_3 = m\sqrt{x_-}$. If the three poles in the free (zero order) propagator are real and positive (with proper residues), with appropriate values of the λ's, they allow the interpretation of physical basic masses of fermions (quark or leptons) belonging to the three different families of the Standard Model. To be more specific we get two different propagators for quarks, one with charge $-\frac{1}{3}$ (d, s, b quarks) and another with charge $+\frac{2}{3}$ (u, c, t quarks). Similarly for charged leptons (charge -1 and spin $\frac{1}{2}$) we get one propagator.

VII. CONCLUSIONS

We have proposed a modification of the classical relativistic hamiltonian that allows the presence of several masses without changing its basic structure. This modification does not affect the infinite divisibility of the laws that are at the basis of the correspondence between stochastic processes and Lévy-Schrödinger equations. However we discovered that the mentioned modification suggests a reformulation of the relativistic equations for wave functions and propagators in such a way that a suitable choice of the background noise produces a convergence in the perturbative contributions. To this purpose we remarked that a modification – with respect to the one given by equation (11) – of the logarithmic characteristic $\eta(\boldsymbol{u})$, by the insertion of a cut-off $f(x)$, allows to proceed to regularization first, and then to renormalization of the two-point function of QCD. There are three parameters in our phenomenological $f(x)$ which is a third degree polynomial; the latter appears as the simplest choice that produces convergence in the integrals representing high order contributions to the fermion and boson propagators. Such parameters create three different poles in the zero-order propagators and allow the interpretation of a physical system with three different masses under precise constraints on $f(x)$. The masses might be related to the three families of the Standard Model.

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