On the irreducibility of Deligne-Lusztig varieties
Cédric Bonnafé, Raphaël Rouquier

To cite this version:

HAL Id: hal-00016980
https://hal.archives-ouvertes.fr/hal-00016980v7
Submitted on 12 Apr 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
ON THE IRREDUCIBILITY OF DELIGNE-LUSZTIG VARIETIES

CÉDRIC BONNAFÉ & RAPHAËL ROUQUIER

Abstract. Let $G$ be a connected reductive algebraic group defined over an algebraic closure of a finite field and let $F: G \to G$ be an endomorphism such that $F^\delta$ is a Frobenius endomorphism for some $\delta \geq 1$. Let $P$ be a parabolic subgroup of $G$. We prove that the Deligne-Lusztig variety $\{gP \mid g^{-1}F(g) \in P \cdot F(P)\}$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Let $G$ be a connected reductive group over an algebraic closure of a finite field and let $F: G \to G$ be an endomorphism such that some power of $F$ is a Frobenius endomorphism of $G$. If $P$ is a parabolic subgroup of $G$, we set $X_P = \{gP \in G/P \mid g^{-1}F(g) \in P \cdot F(P)\}$. This is the Deligne-Lusztig variety associated to $P$. The aim of this note is to prove the following result:

Theorem A. Let $P$ be a parabolic subgroup of $G$. Then $X_P$ is irreducible if and only if $P$ is not contained in a proper $F$-stable parabolic subgroup of $G$.

Note that this result has been obtained independently by Lusztig (unpublished) and Digne and Michel [DiMi2, Proposition 8.4] in the case where $P$ is a Borel subgroup: both proofs are obtained by counting rational points of $X_P$ in terms of the Hecke algebra. We present here a geometric proof (inspired by an argument of Deligne [Lu, proof of Proposition 4.8]) which reduces the problem to the irreducibility of the Deligne-Lusztig variety associated to a Coxeter element: this case has been treated by Deligne and Lusztig [Lu, Proposition 4.8].

Before starting the proof of this Theorem, we first describe an equivalent statement. Let $B$ be an $F$-stable Borel subgroup of $G$, let $T$ be an $F$-stable maximal torus of $B$, let $W$ be the Weyl group of $G$ relative to $T$ and let $S$ be the set of simple reflections of $W$ with respect to $B$. We denote again by $F$ the automorphism of $W$ induced by $F$. Given $I \subseteq S$, let $W_I$ denote the standard parabolic subgroup of $W$ generated by $I$ and let $P_I = BW_IB$. We denote by $P_I$ the variety of parabolic subgroups of $G$ of type $I$ (i.e. conjugate to $P_I$) and by $B$ the variety of Borel subgroups of $G$ (i.e. $B = P_\emptyset$). For $w \in W$, we denote by $O_I(w)$ the $G$-orbit of $(P_I, wP_{F(I)})$ in $P_I \times P_{F(I)}$. Note that $O_I(w)$ depends only on the double coset $W_IwW_{F(I)}$. We define now

$$X_I(w) = \{P \in P_I \mid (P, F(P)) \in O_I(w)\}.$$

The group $G^F$ acts on $X_I(w)$ by conjugation. We set $O(w) = O_\emptyset(w)$ and $X(w) = X_\emptyset(w)$.
**Theorem A'**. Let $I \subset S$ and let $w \in W$. Then $X_I(w)$ is irreducible if and only if $W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$.

**Remark 1** - Let us explain why the Theorems A and A' are equivalent. Let $P_0$ be a parabolic subgroup of $G$. Let $I$ be its type and let $g_0 \in G$ be such that $P_0 = g_0 P_I$. Let $w \in W$ be such that $g_0^{-1} F(g_0) \in P_I w P_{F(I)}$. The pair $(I, W_I w W_{F(I)})$ is uniquely determined by $P_0$. Then, the map $X_{P_0} \to X_I(w)$, $g P_0 \mapsto g g_0^{-1} P_I$ is an isomorphism of varieties (indeed, it is straightforward that $g^{-1} F(g) \in P_0 \cdot F(P_0)$ if and only if $(g g_0)^{-1} F(g g_0) \in P_I w P_{F(I)}$).

Let $Q$ be a parabolic subgroup of $G$ containing $P$. Let $J$ be its type. Then $I \subset J$, $Q = g_0 P_J$ and $g_0^{-1} F(g_0) \in P_J w P_{F(I)}$. Now, $Q$ is $F$-stable if and only if $F(J) = J$ and $w \in W_J$. This shows the equivalence of the two Theorems.

**Remark 2** - The condition “$W_I w$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$” is equivalent to “$W_I w W_{F(I)}$ is not contained in a proper $F$-stable standard parabolic subgroup of $W$”.

The rest of this paper is devoted to the proof of Theorem A'. We fix a subset $I$ of $S$ and an element $w$ of $W$. We first recall two elementary facts. If $I \subset J$, let $\tau_{IJ} : P_J \to P_I$ be the morphism of varieties that sends $P \in P_J$ to the unique parabolic subgroup of type $J$ containing $P$. It is surjective. Moreover,

$$(1) \quad \tau_{IJ}(X_I(w)) \subset X_J(w)$$

and

$$(2) \quad \tau_{IJ}^{-1}(X_J(w)) = \bigcup_{W_I w W_{F(I)} \subset W_J w W_{F(J)}} X_I(x).$$

**First step: the “only if” part.** Assume that there exists a proper $F$-stable subset $J$ of $S$ such that $W_I w \subset W_J$. Then, by (1), we have $\tau_{IJ}(X_I(w)) \subset X_J(1) = F^P$. Since $G^F$ acts transitively on $P^F_J$, we get $\tau_{IJ}(X_I(w)) = X_J(1)$. This shows that $X_I(w)$ is not irreducible.

**Second step: reduction to Borel subgroups.** By the previous step, we can concentrate on the “if” part. So, from now on, we assume that $W_I w$ is not contained in a proper $F$-stable parabolic subgroup of $W$. Then, by (2), we have

$$\tau_{IJ}^{-1}(X_I(w)) = \bigcup_{x \in W_I w W_{F(I)}} X(x).$$

Let $v$ denote the longest element of $W_I w W_{F(I)}$. Then every element $x$ of the double coset $W_I w W_{F(I)}$ satisfies $x \leq v$ (here, $\leq$ denotes the Bruhat order on $W$): this follows for instance from the fact that $P_I w P_{F(I)}$ is irreducible and is equal to $\bigcup_{x \in W_I w W_{F(I)}} B w B$. In particular, $v$ is not contained in a proper $F$-stable parabolic subgroup of $W$.

Now, let $X' = \bigcup_{x \in W_I w W_{F(I)}} X(x)$. Then, since $X(v) = \bigcup_{x \leq v} X(x)$, we have

$$X(v) \subset X' \subset \overline{X(v)}.$$
I words, we may, and we will, assume that $I = \emptyset$.

**Third step: smooth compactification.** Let $(s_1, \ldots, s_n)$ be a finite sequence of elements of $S$. Let

$$
\hat{X}(s_1, \ldots, s_n) = \{(B_1, \ldots, B_n) \in B^n \mid (B_n, F(B_1)) \in \mathcal{O}(s_n) \}
$$

and $(B_i, B_{i+1}) \in \mathcal{O}(s_i)$ for $1 \leq i \leq n - 1$.

If $\ell(s_1 \cdots s_n) = n$, then $\hat{X}(s_1, \ldots, s_n)$ is a smooth compactification of $X(s_1 \cdots s_n)$ (see DeLu, Lemma 9.11): in this case,

$$
\text{(3) } X(s_1 \cdots s_n) \text{ is irreducible if and only if } \hat{X}(s_1, \ldots, s_n) \text{ is irreducible.}
$$

Note that $(B_1, \ldots, B) \in \hat{X}(s_1, \ldots, s_n)$. We denote by $\hat{X}^e(s_1, \ldots, s_n)$ the connected (i.e. irreducible) component of $\hat{X}(s_1, \ldots, s_n)$ containing $(B_1, \ldots, B)$. Let $H(s_1, \ldots, s_n) \subset G^F$ be the stabilizer of $\hat{X}^e(s_1, \ldots, s_n)$. Let us now prove the following fact:

$$
\text{(4) if } 1 \leq i_1 < \cdots < i_r \leq n, \text{ then } H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n).
$$

**Proof of (4).** The map $f : \hat{X}(s_{i_1}, \ldots, s_{i_r}) \longrightarrow \hat{X}(s_1, \ldots, s_n)$ defined by

$$
f(B_1, \ldots, B_1) = (B_1, \ldots, B_1, B_2, \ldots, B_{i_r}, F(B_1), \ldots, F(B_1))
$$

is a $G^F$-equivariant morphism of varieties. Moreover,

$$
f(B_1, \ldots, B) = (B_1, \ldots, B).
$$

In particular, $f(\hat{X}^e(s_{i_1}, \ldots, s_{i_r}))$ is contained in $\hat{X}^e(s_1, \ldots, s_n)$. This proves the expected inclusion between stabilizers. ■

**Last step: twisted Coxeter element.** The quotient variety

$$
G^F \backslash \{g \in G \mid g^{-1} F(g) \in B w B\}
$$

is irreducible (it is isomorphic to $B w B$ through the Lang map $G^F g \mapsto g^{-1} F(g)$), hence $G^F \backslash X(w)$ is irreducible as well. So,

$$
\text{(5) } G^F \text{ permutes transitively the irreducible components of } X(w).
$$

Let $w = s_1 \cdots s_n$ be a reduced decomposition of $W$ as a product of elements of $S$. By (3) and (5), it suffices to show that $H(s_1, \ldots, s_n) = G^F$. Since $w$ does not belong to any $F$-stable proper parabolic subgroup of $W$, there exists a sequence $1 \leq i_1 < \cdots < i_r \leq n$ such that $(s_{i_k})_{1 \leq k \leq r}$ is a family of representatives of $F$-orbits in $S$. By (4) we have $H(s_{i_1}, \ldots, s_{i_r}) \subset H(s_1, \ldots, s_n)$. But, by Lu, Proposition 4.8, $X(s_{i_1}, \ldots, s_{i_r})$ is irreducible so, again by (3) and (5), $H(s_{i_1}, \ldots, s_{i_r}) = G^F$. Therefore, $H(s_1, \ldots, s_n) = G^F$, as expected.
Acknowledgements. We thank F. Digne and J. Michel for fruitful discussions on these questions. We thank P. Deligne for the clarification of the scope of validity of the Theorem.

Références


Cédric Bonnafé: Laboratoire de Mathématiques de Besançon (CNRS: UMR 6623), Université de Franche-Comté, 16 Route de Gray, 25030 Besançon Cedex, France

E-mail address: bonnafe@math.univ-fcomte.fr

URL: http://www-math.univ-fcomte.fr/ppAnnu/CBONNAFE/


E-mail address: rouquier@maths.leeds.ac.uk

URL: http://www.maths.leeds.ac.uk/~rouquier/