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Projective Ring Line Encompassing Two-Qubits

Metod Saniga,¹ Michel Planat² and Petr Prajna³

¹Astronomical Institute, Slovak Academy of Sciences
SK-05960 Tatranská Lomnica, Slovak Republic
(msaniga@astro.sk)

²Institut FEMTO-ST, CNRS, Département LPMO, 32 Avenue de l'Observatoire
F-25044 Besançon Cedex, France
(michel.planat@femto-st.fr)

³J. Heyrovský Institute of Physical Chemistry, Academy of Sciences of the Czech Republic
Dolejškova 3, CZ-182 23 Prague 8, Czech Republic
(petr.pracna@jh-inst.cas.cz)

Abstract

The projective line over the (non-commutative) ring of two-by-two matrices with coefficients in $GF(2)$ is found to *fully* accommodate the algebra of 15 operators — generalized Pauli matrices — characterizing two-qubit systems. The relevant sub-configuration consists of 15 points each of which is either simultaneously distant or simultaneously neighbor to (any) two given distant points of the line. The operators can be identified with the points in such a one-to-one manner that their commutation relations are *exactly* reproduced by the underlying geometry of the points, with the ring geometrical notions of neighbor/distant answering, respectively, to the operational ones of commuting/non-commuting. This remarkable configuration can be viewed in two principally different ways accounting, respectively, for the basic 9+6 and 10+5 factorizations of the algebra of the observables. First, as a disjoint union of the projective line over $GF(2) \times GF(2)$ (the “Mermin” part) and two lines over $GF(4)$ passing through the two selected points, the latter omitted. Second, as the generalized quadrangle of order two, with its ovoids and/or spreads standing for (maximum) sets of five mutually non-commuting operators and/or groups of five maximally commuting subsets of three operators each. These findings open up rather unexpected vistas for an algebraic geometrical modelling of finite-dimensional quantum systems and give their numerous applications a wholly new perspective.

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Projective lines defined over finite associative rings with unity/identity^{1–7} have recently been recognized to be an important novel tool for getting a deeper insight into the underlying algebraic geometrical structure of finite dimensional quantum systems.^{8–10} Focusing almost uniquely on the two-qubit case, i.e., the set of 15 operators/generalized four-by-four Pauli spin matrices, of particular importance turned out to be the lines defined over the direct product of the simplest Galois fields, $GF(2) \times GF(2) \times \dots \times GF(2)$. Here, the line defined over $GF(2) \times GF(2)$ plays a prominent role in grasping qualitatively the basic structure of so-called Mermin squares,^{9,10} i.e., three-by-three arrays in certain remarkable 9 + 6 split-ups of the algebra of operators, whereas the line over $GF(2) \times GF(2) \times GF(2)$ reflects some of the basic features of a specific 8 + 7 (“cube-and-kernel”) factorization of the set.¹⁰ Motivated by these partial findings, we started our quest for such a ring line that would provide us with a complete picture of the algebra of all the 15 operators/matrices. After examining a large number of lines defined over commutative rings,^{6,7} we gradually realized that a proper candidate is likely to be found in the non-commutative domain and

this, indeed, turned out to be a right move. It is, as we shall demonstrate in sufficient detail, the projective line defined over the full two-by-two matrix ring with entries in $GF(2)$ — the unique simple non-commutative ring of order 16 featuring six units (invertible elements) and ten zero-divisors.¹¹ Having in mind the conceptual rather than formal side of the task, we shall try to reduce the technicalities of the exposition to a minimum, referring instead the interested reader to the relevant literature.

We first recall the concept of a projective ring line.^{1–7} Given an associative ring R with unity/identity^{12–14} and $GL(2, R)$, the general linear group of invertible two-by-two matrices with entries in R , a pair $(a, b) \in R^2$ is called admissible over R if there exist $c, d \in R$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R). \quad (1)$$

The projective line over R , usually denoted as $P_1(R)$, is the set of equivalence classes of ordered pairs $(\varrho a, \varrho b)$, where ϱ is a unit of R and (a, b) is admissible. Two points $X := (\varrho a, \varrho b)$ and $Y := (\varrho c, \varrho d)$ of the line are called *distant* or *neighbor* according as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, R) \quad \text{or} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \notin GL(2, R), \quad (2)$$

respectively. $GL(2, R)$ has an important property of acting transitively on a set of three pairwise distant points; that is, given any two triples of mutually distant points there exists an element of $GL(2, R)$ transforming one triple into the other.

The projective line we are exclusively interested in here is the one defined over the full two-by-two matrix ring with $GF(2)$ -valued coefficients, i. e.,

$$R = M_2(GF(2)) \equiv \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in GF(2) \right\}. \quad (3)$$

Labelling these matrices as follows

$$\begin{aligned} 1 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & 2 &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, & 3 &\equiv \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & 4 &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \\ 5 &\equiv \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & 6 &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, & 7 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, & 8 &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ 9 &\equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & 10 &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & 11 &\equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & 12 &\equiv \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \\ 13 &\equiv \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, & 14 &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & 15 &\equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & 0 &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned} \quad (4)$$

one can readily verify that addition and multiplication in $M_2(GF(2))$ is carried out as shown in Table 1.¹⁵ Checking first for admissibility (Eq. (1)) and then grouping the admissible pairs left-proportional by a unit into equivalence classes (of cardinality six each), we find that $P_1(M_2(GF(2)))$ ¹ possesses altogether 35 points, with the following representatives of each equivalence class (see Refs. 6–8 for more details about this methodology and a number of illustrative examples of a projective ring line):

$$\begin{aligned} &(1, 1), (1, 2), (1, 9), (1, 11), (1, 12), (1, 13), \\ &(1, 0), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (1, 8), (1, 10), (1, 14), (1, 15), \\ &(0, 1), (3, 1), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1), (10, 1), (14, 1), (15, 1), \\ &(3, 4), (3, 10), (3, 14), (5, 4), (5, 10), (5, 14), (6, 4), (6, 10), (6, 14). \end{aligned} \quad (5)$$

From the multiplication table one can easily recognize that the representatives in the first row of the last equation have both entries units (1 being, obviously, unity/multiplicative identity), those of the second and third row have one entry unit(y) and the other a zero-divisor, whilst all pairs in the last row feature zero-divisors in both the entries. At this point we are ready to shown which “portion” of $P_1(M_2(GF(2)))$ is the proper algebraic geometrical setting of two-qubits.

¹This line has been found to have a distinguished footing among non-commutative ring lines for it fundamentally differs from its two commutative counterparts.¹¹

Table 1: Addition (*top*) and multiplication (*bottom*) in $M_2(GF(2))$.

+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	3	2	5	4	7	6	9	8	11	10	13	12	15	14
2	2	3	0	1	6	7	4	5	10	11	8	9	14	15	12	13
3	3	2	1	0	7	6	5	4	11	10	9	8	15	14	13	12
4	4	5	6	7	0	1	2	3	12	13	14	15	8	9	10	11
5	5	4	7	6	1	0	3	2	13	12	15	14	9	8	11	10
6	6	7	4	5	2	3	0	1	14	15	12	13	10	11	8	9
7	7	6	5	4	3	2	1	0	15	14	13	12	11	10	9	8
8	8	9	10	11	12	13	14	15	0	1	2	3	4	5	6	7
9	9	8	11	10	13	12	15	14	1	0	3	2	5	4	7	6
10	10	11	8	9	14	15	12	13	2	3	0	1	6	7	4	5
11	11	10	9	8	15	14	13	12	3	2	1	0	7	6	5	4
12	12	13	14	15	8	9	10	11	4	5	6	7	0	1	2	3
13	13	12	15	14	9	8	11	10	5	4	7	6	1	0	3	2
14	14	15	12	13	10	11	8	9	6	7	4	5	2	3	0	1
15	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0
×	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	1	3	7	5	6	4	14	12	15	13	9	11	8	10
3	0	3	3	0	3	0	0	3	6	5	5	6	5	6	6	5
4	0	4	4	0	4	0	0	4	14	10	10	14	10	14	14	10
5	0	5	6	3	0	5	6	3	6	3	0	5	6	3	0	5
6	0	6	5	3	3	5	6	0	0	6	5	3	3	5	6	0
7	0	7	7	0	7	0	0	7	8	15	15	8	15	8	8	15
8	0	8	15	7	7	15	8	0	0	8	15	7	7	15	8	0
9	0	9	13	4	3	10	14	7	8	1	5	12	11	2	6	15
10	0	10	14	4	0	10	14	4	14	4	0	10	14	4	0	10
11	0	11	12	7	4	15	8	3	6	13	10	1	2	9	14	5
12	0	12	11	7	3	15	8	4	14	2	5	9	13	1	6	10
13	0	13	9	4	7	10	14	3	6	11	15	2	1	12	8	5
14	0	14	10	4	4	10	14	0	0	14	10	4	4	10	14	0
15	0	15	8	7	0	15	8	7	8	7	0	15	8	7	0	15

To this end, we consider two distant points of the line. Taking into account the above-mentioned three-distant-transitivity of $GL(2, R)$, we can take these, without any loss of generality, to be the points $U := (1, 0)$ and $V := (0, 1)$. Next we pick up all those points of the line which are either simultaneously distant or simultaneously neighbor to U and V . Employing the left part of Eq. (2), we find the following six points

$$\begin{aligned}
 C_1 &= (1, 1), \quad C_2 = (1, 2), \quad C_3 = (1, 9), \\
 C_4 &= (1, 11), \quad C_5 = (1, 12), \quad C_6 = (1, 13),
 \end{aligned} \tag{6}$$

to belong to the first family, whereas the right part of Eq. (2) tells us that the second family comprises the following nine points

$$\begin{aligned}
 C_7 &= (3, 4), \quad C_8 = (3, 10), \quad C_9 = (3, 14), \\
 C_{10} &= (5, 4), \quad C_{11} = (5, 10), \quad C_{12} = (5, 14), \\
 C_{13} &= (6, 4), \quad C_{14} = (6, 10), \quad C_{15} = (6, 14).
 \end{aligned} \tag{7}$$

Making again use of Eq. (2), one finds that the points of our special subset of $P_1(M_2(GF(2)))$ are related with each other as shown in Table 2; from this table it can readily be discerned that

Table 2: The distant and neighbor (“+” and “−”, respectively) relation between the points of the configuration. The points are arranged in such a way that the last nine of them (i. e., C_7 to C_{15}) form the projective line over $GF(2) \times GF(2)$.^{8–10}

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	−	−	−	−	+	+	−	+	+	+	−	+	+	+	−
C_2	−	−	+	+	−	−	−	+	+	+	+	−	+	−	+
C_3	−	+	−	+	−	−	+	−	+	−	+	+	+	+	−
C_4	−	+	+	−	−	−	+	+	−	+	−	+	−	+	+
C_5	+	−	−	−	−	+	+	−	+	+	+	−	−	+	+
C_6	+	−	−	−	+	−	+	+	−	−	+	+	+	−	+
C_7	−	−	+	+	+	+	−	−	−	−	+	+	−	+	+
C_8	+	+	−	+	−	+	−	−	−	+	−	+	+	−	+
C_9	+	+	+	−	+	−	−	−	−	+	+	−	+	+	−
C_{10}	+	+	−	+	+	−	−	+	+	−	−	−	−	+	+
C_{11}	−	+	+	−	+	+	+	−	+	−	−	−	+	−	+
C_{12}	+	−	+	+	−	+	+	+	−	−	−	−	+	+	−
C_{13}	+	+	+	−	−	+	−	+	+	−	+	+	−	−	−
C_{14}	+	−	+	+	+	−	+	−	+	+	−	+	−	−	−
C_{15}	−	+	−	+	+	+	+	+	−	+	+	−	−	−	−

to every point of the configuration there are six neighbor and eight distant points, and that the maximum number of pairwise neighbor points is three. The final step is to identify these 15 points with the 15 generalized Pauli matrices/operators of two-qubits (see, e. g., Ref. 10, Eq. (1)) in the following way

$$\begin{aligned}
C_1 &= \sigma_z \otimes \sigma_x, & C_2 &= \sigma_y \otimes \sigma_y, & C_3 &= 1_2 \otimes \sigma_x, \\
C_4 &= \sigma_y \otimes \sigma_z, & C_5 &= \sigma_y \otimes 1_2, & C_6 &= \sigma_x \otimes \sigma_x, \\
C_7 &= \sigma_x \otimes \sigma_z, & C_8 &= \sigma_y \otimes \sigma_x, & C_9 &= \sigma_z \otimes \sigma_y, \\
C_{10} &= \sigma_x \otimes 1_2, & C_{11} &= \sigma_x \otimes \sigma_y, & C_{12} &= 1_2 \otimes \sigma_y, \\
C_{13} &= 1_2 \otimes \sigma_z, & C_{14} &= \sigma_z \otimes \sigma_z, & C_{15} &= \sigma_z \otimes 1_2,
\end{aligned} \tag{8}$$

where 1_2 is the 2×2 unit matrix, σ_x , σ_y and σ_z are the classical Pauli matrices and the symbol “ \otimes ” stands for the tensorial product of matrices, in order to readily verify that Table 2 gives the correct commutation relations between these operators with the symbols “+” and “−” now having the meaning of “non-commuting” and “commuting”, respectively. Slightly rephrased, *one and the same* “incidence matrix”, Table 2, pertains to *two distinct* configurations of a *completely different* origin: a set of points of the projective line over a particular finite ring, with the symbols “+”/“−” having the algebraic geometrical meaning of distant/neighbor, as well as a set of operators of four-dimensional Hilbert space, with the same symbols acquiring the operational meaning of non-commuting/commuting, respectively.

This remarkable configuration can be interpreted in two principally different ways, which account, respectively, for the basic 9+6 (Fig. 1, *left*) and 10+5 (Fig. 1, *right*) factorizations of the algebra of observables. The first is simply a disjoint union of the projective line over $GF(2) \times GF(2)$ and two lines over $GF(4)$ passing through the two selected points U and V , these latter omitted. As demonstrated in detail elsewhere,^{9,10} the line over $GF(2) \times GF(2)$ underlies the qualitative structure of Mermin’s magic squares, i. e., 3×3 arrays of nine observables commuting pairwise in each row and column and arranged so that their product properties contradict those of the assigned eigenvalues. The two lines over $GF(4)$ represent the remaining, bipartite part of the split-up, where three points/observables on each of the lines are mutually distant/non-commuting and every point/observable of one line is neighbor to/commutes with any point/observable of the other line (see Fig. 1, *left*). The second interpretation involves a *generalized quadrangle*, a rank two point-line incidence geometry where two points share at most one line and where for any point X and a line \mathcal{L} , $X \notin \mathcal{L}$, there exists exactly one line through X which intersect \mathcal{L} .^{16–20} The gener-

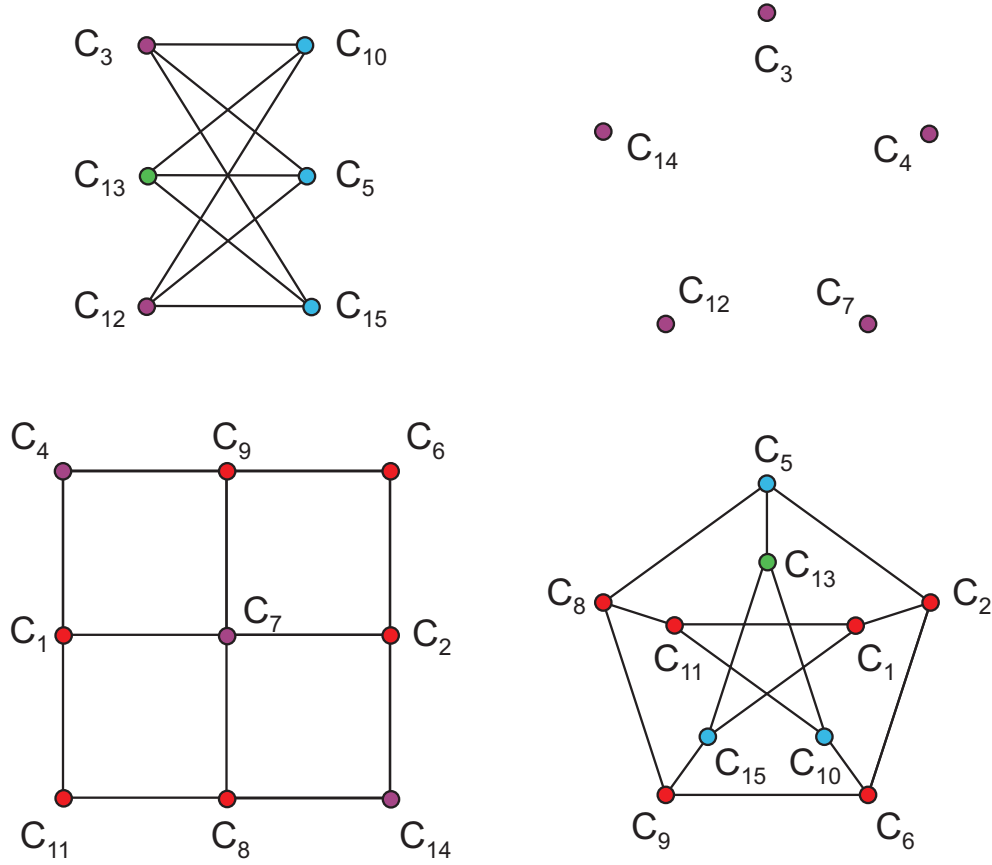


Figure 1: The two basic factorizations of the algebra of the 15 observables/operators of a two-qubit system. A 9+6 factorization (*left*) corresponds geometrically to the split-up of our sub-configuration of $P_1(M_2(GF(2)))$ into the projective line over $GF(2) \times GF(2)$ (*bottom*) and a couple of projective lines over $GF(4)$ having two points in common (*top*). A 10+5 one (*right*) corresponds, as also demonstrated in a different way in Fig. 2, to the partition of the generalized quadrangle into one of its ovoids (*top*) and the Petersen graph (*bottom*). In both the cases, two points/observables are joined by a line-segment only if they are neighbor/commute and the color is used to illustrate how the two factorizations relate to each other.

alized quadrangle associated with our observables is of order two, i. e., the one where every line contains three points and every point is on three lines. Such a quadrangle has, indeed, 15 points (and, because of its self-duality, the same number of lines), each of which is joined by a line with other six (Fig. 2, *left*). If one removes from this quadrangle one of its *ovoids*, i. e., a set of (five) points that has exactly one point in common with every line (Fig. 2, *middle*), one is left with the set of ten points that form the famous Petersen graph (Fig. 2, *right*);^{19,20} five points of an ovoid answer to nothing but the five mutually distant points of $P_1(GF(4))$ and, so, to the five (i. e., the maximum number of) mutually non-commuting observables of two-qubits. If, dually, one removes from the quadrangle a *spread*, i. e., a set of (five) pairwise disjoint lines that partition the point set, one gets the dual of the Petersen graph (Fig. 3); five lines of a spread represent nothing but the five maximum subsets of three mutually commuting operators each, whose associated bases are *mutually unbiased*.^{10,21} It is a straightforward exercise to associate the points of the quadrangle with the operators/observables C_i , Eq. (8), in such a way to recover Table 2, after substituting the “-”/“+” sign for any two points of the quadrangle which are/are not on a common line.

To complete this interesting algebraic geometrical picture of two-qubits there remains to be introduced one more important geometrical object. The attentive reader might have noticed that we have already employed two different kinds of the projective lines defined over rings of order four and characteristic two, viz. the line defined over the field $GF(4)$ and that defined over the

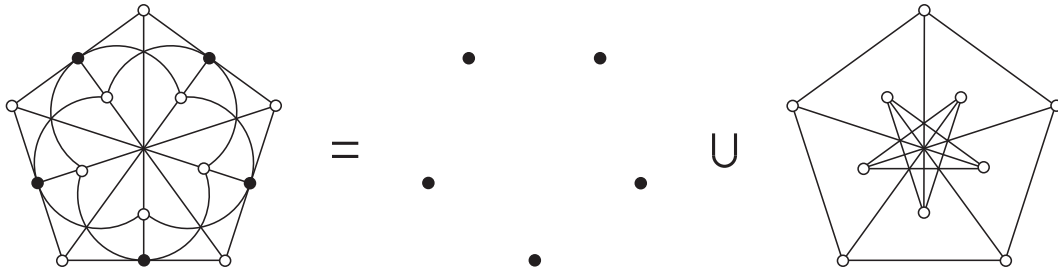


Figure 2: The generalized quadrangle of order two (*left*) and its factorization into an ovoid (*middle*) and the “Petersen” part (*right*). The lines of the quadrangle are illustrated by the straight segments as well as by the segments of circles; note that not every intersection of two segments counts for a point of the quadrangle.

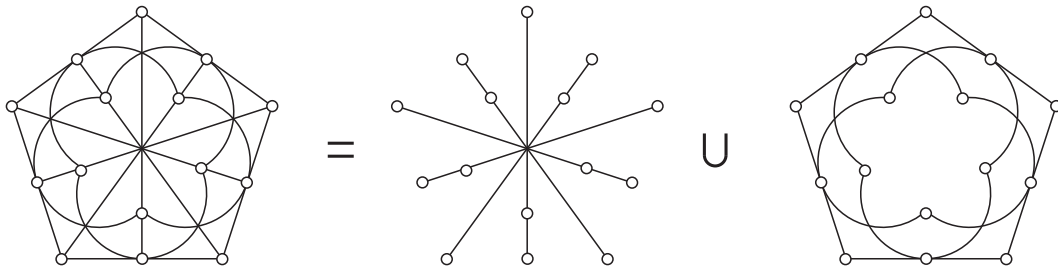


Figure 3: A dual view of the generalized quadrangle of order two (*left*) as a disjoint union of one of its spreads (*middle*) and a dual of the Petersen graph, or a 2-spread (*right*).

direct product ring $GF(2) \times GF(2)$; the former was seen to answer to an *ovoid* of the generalized quadrangle, alias to a set of five mutually non-commuting operators (Fig. 1, *right – top*), while the latter corresponds to a *grid* of nine points on six lines,² alias to a Mermin’s square of operators (Fig. 1, *left – bottom*). There, however, exists *one more* associative ring with unity of order four and characteristic two, namely the (local) factor ring of polynomials $GF(2)[x]/\langle x^2 \rangle$.^{12–14} This ring is also a subring of $M_2(GF(2))$, so the corresponding projective line is expected to play a role in our model, too. And this is indeed the case. As demonstrated, for example, in Ref. 6 (Table 3), the projective line $P_1(GF(2)[x]/\langle x^2 \rangle)$ features six points any of which is neighbor to one and distant to the remaining four points, comprising thus three pairs of neighbors. In the set of Pauli operators this configuration is present as the sextuple of operators commuting with a given operator; taking the latter to be, e. g., C_{13} , the six operators in question, as readily discerned from Table 2, are $\{C_4, C_5; C_7, C_{10}; C_{14}, C_{15}\}$, which indeed form three pairs of commuting members (these pairs being separated from each other by a semicolon). In the generalized quadrangle any such configuration resides as the sextuple of points collinear with a given point. A deeper understanding and a fuller appreciation of this observation is acquired after introducing the concept of a *geometric hyperplane*. A geometric hyperplane H of a finite geometry is a set of points such that every line of the geometry either contains exactly one point of H , or is completely contained in H .^{20, 22} It is easy to verify that for the generalized quadrangle of order two H is of one of the following *three kinds*:²² 1) H_{ov} , an ovoid (there are six such hyperplanes); 2) $H_{cl}(X)$, a set of points collinear with a given point X , the point itself *inclusive* (there are 15 such hyperplanes); and 3) H_{gr} , a grid as defined above (there are 10 such hyperplanes). One thus reveals a perfect parity between the three kinds of the geometric hyperplanes of the generalized quadrangle of order two and the three kinds of the projective lines over the rings of four elements and characteristic two embedded in our sub-configuration of $P_1(M_2(GF(2)))$, giving rise to the three kinds of the distinguished subsets of the Pauli operators of two-qubits, as summarized in Table 3. As a final note, it is worth mentioning that the generalized quadrangle of order two also resides in $P_1(M_2(GF(2)))$ as the projective line

²Also known as the *slim* generalized quadrangle of order $(2, 1)$; see, e. g., Refs. 18 and 20. In fact, *both* the configurations depicted in Fig. 1, *left*, are slim generalized quadrangles, one being the *dual* of the other.

Table 3: Three kinds of the distinguished subsets of the generalized Pauli operators of two-qubits (TQ) viewed as the geometric hyperplanes in the generalized quadrangle of order two (GQ) and/or as the projective lines over the rings of order four and characteristic two living in the projective line $P_1(M_2(GF(2)))$ (PL).

GQ	H_{ov}	$H_{cl}(X) \setminus \{X\}$	H_{gr}
PL	$P_1(GF(4))$	$P_1(GF(2)[x]/\langle x^2 \rangle)$	$P_1(GF(2) \times GF(2))$
TQ	set of five mutually non-commuting operators	set of six operators commuting with a given one	nine operators of a Mermin's square

over the so-called Jordan system of *symmetric* two-by-two matrices over $GF(2)$,²³ or, equivalently, as a generic hyperplane section of the Klein quadric in the 5-dimensional projective space over $GF(2)$.²²

We have demonstrated that the basic properties of a system of two interacting spin-1/2 particles are uniquely embodied in the (sub)geometry of a particular projective line, found to be equivalent to the generalized quadrangle of order two. As such systems are the simplest ones exhibiting phenomena like quantum entanglement and quantum non-locality and play, therefore, a crucial role in numerous applications like quantum cryptography, quantum coding, quantum cloning/teleportation and/or quantum computing to mention the most salient ones, our discovery thus not only offers a principally new geometrically-underlined insight into their intrinsic nature, but also gives their applications a wholly new perspective and opens up rather unexpected vistas for an algebraic geometrical modelling of their higher-dimensional counterparts.^{24,25}

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