A Quasi-IRR for a Project without IRR

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Abstract

Discounted cash flows methods such as Net Present Value and Internal Rate of Return are often used interchangeably or even together for assessing value creation in industrial and engineering projects. Notwithstanding its difficulties of applicability and reliability, the internal rate of return (IRR) is commonly used in real-life applications. Among other problems, a project may have no real-valued IRR, a circumstance that may occur in projects which require shutting costs or imply an initial positive cash flow such as a down payment made by a client. This paper supplies a genuine IRR for a project which has no IRR. This seemingly paradoxical result is achieved by making use of a new approach to rate of return (Magni, 2010), whereby any project is associated with a unique return function which maps aggregate capitals into rates of return. Each rate of return is a weighted average of one-period (internal) rates of return, so it is called *Average Internal Rate of Return* (AIRR). We introduce a twin project which has a unique IRR and the same NPV as the original project's, and which is obtained through an appropriate minimization of the distance between the original project's cash flow stream and the twin project's. Given that the latter's IRR lies on the original project's return function, it represents an AIRR of the original project. And while it is not the IRR of the project, the measure presented is 'almost' the IRR of the project, so it is actually the "quasi-IRR" of the project.
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1 - Introduction

The net present value (NPV) and the internal rate of return (IRR), both conceived in the 1930s (Fisher, 1930; Boulding, 1935), are arguably the most widely used investment criteria in real-life applications (Remer and Nyeto, 1995a, 1995b; Graham and Harvey, 2001). However, in most cases, managers do not rely on one single investment criterion: NPV and IRR are often used together, as well as other criteria such as payback, residual income (e.g., EVA), return on investment, payback period (Remer et al., 1993; Lefely, 1996; Lindblom and Sjögren, 2009). The reason why IRR is widely used in any economic domain is that a relative measure of economic profitability (a percentage return) is easily understood, and is more intuitive than an absolute measure of worth (Evans and Forbes, 1993).

Differences between NPV and IRR have been recognized long since and have been debated in the literature extensively from various perspectives. In recent years, scholars have shown a renewed interest in this important issue. In particular, Hazen (2003) supplies an NPV-compatible decision criterion for both real-valued and complex-valued IRRs by associating their real parts with the real parts of the capital streams, so shedding new lights on the multiple-IRR problem. This problem is tackled by Hartman and Schafrick (2004) as well, who partition the graph of the NPV function in loaning part and borrowing part, so singling out the "relevant rate of return". Bosch, Montllor-Serrats and Tarrazon (2007) use payback coefficients to derive a normalized index compatible with the NPV, while Kierulff (2008) endorses the use of the Modified Internal Rate of Return. Osborne (2010) explicitly links all the IRRs (complex and real) to the NPV and Pierru (2010) gives complex rates a significant economic meaning. Percoco and Borgonovo (2012), using sensitivity analysis, focus on the key drivers of value creation and show that IRR and NPV provide different results. Ben-Horin and Kroll (2012) suggest that the multiple-IRR problem has limited relevance in practice.

Evidently, evaluators cannot rest upon a (real) IRR if the polynomial associated with the project's cash flows has no real roots. A necessary condition for this to occur is that the project's terminal cash flow has the same sign as the first cash flow. Some engineering projects can actually present patterns of cash flows which result in negative cash flows in the last part of the project's life: an investment that requires the removal of equipment or cleansing of a site in order to return it to its previous state (e.g., a nuclear plant) may have no real IRR. A similar situation occurs in natural-resource extractions (mining for coal, gold, ore, silver) where remediation and cleanup costs are common (see Hartman, 2007). Another situation is the case where a down payment is made by a client before an investment is made (e.g., production on commission).

This paper aims to supply an IRR even in those cases where an IRR does not exist. This seemingly paradoxical result is obtained by making use of a new paradigm of rate of return, named *Average Internal Rate of Return* (Magni, 2010, 2013). This approach is based on the finding that a project is uniquely associated (not with a return rate but) with a return function, which maps invested capitals to (real-valued) average rates of return. Any capital-rate pair lying on the return function's graph captures the project's economic profitability: no problem of existence arises and compatibility with NPV in every circumstance is ensured. Each value taken on by the function is a rate of return and is called *Average Internal Rate of Return* (AIRR). Any IRR is absorbed into the AIRR approach, for it is but a particular case of AIRR, implicitly associated with a capital automatically computed (in other words, the IRR lies on the return function).

This paper makes use of the AIRR approach to retrieve the IRR even in the case where an IRR does not exist. To achieve this objective, we introduce a twin project, which is a project characterized by three properties: (a) it has the same length and the same NPV as the project under consideration, (b) it has a unique real-valued IRR, (c) the distance of its cash flows stream from the project's cash flow stream is properly minimized. The latter property means that the twin project is the project which is the closest one to the original project. To show that such a rate correctly captures the project's economic profitability, we exploit Magni's (2010, 2013) results to show that this rate is just a value taken on by the original project's return function; in other words, it is an AIRR of the project. And given that it is the internal rate of return of the project which is 'as close as possible' to the original project, such a measure deserves the name of "quasi-IRR" of the project.

The structure of the paper is as follows. Section 1 presents the main definitions. Section 2 summarizes the AIRR approach. Section 3 introduces the minimization procedure necessary to obtain the quasi-IRR, which is the the IRR of the twin project which is closest to the original project. Section 4 shows that the quasi-IRR is just a particular case of AIRR. Two examples and some concluding remarks end the paper.

2 – Definitions

Consider a project P and let $\mathbf{a} = (a_0, a_1, ..., a_n)$ be its cash flow stream. Let r denote a rate of return and let $v = 1/(1+r)$ be the discount factor for evaluating the project. Let us assume that the classical feasibility condition $v > 0$ (i.e., $r > -1$) holds. The project's NPV, computed at the market rate r_m or, equivalently, at the discount market factor $v_m = 1/(1+r_m)$ is given by

$$NPV(\mathbf{a} \mid r_m) = \sum_{t=0}^{n} a_t \cdot (1+r_m)^{-t} \text{ or } NPV(\mathbf{a} \mid v_m) = \sum_{t=0}^{n} a_t \cdot v_m^t.$$

An internal rate of return (respectively, an internal discount factor) is any value r^* (respectively, v^*) such that

$$NPV(\mathbf{a} \mid r^*) = \sum_{t=0}^{n} a_t \cdot (1+r^*)^{-t} = 0 \text{ or } NPV(\mathbf{a} \mid v^*) = \sum_{t=0}^{n} a_t \cdot (v^*)^t = 0.$$

Let r^* be an IRR of project P. The capital that remains invested (or borrowed, if negative) in the project at time t (computed at the constant per-period internal rate r^*), which we denote by C_t^*, is defined as $C_t^* = -\sum_{h=0}^{t} a_h \cdot (1+r^*)^{t-h}$, or, recursively, as

$$C_t^* = C_{t-1}^* \cdot (1+r^*) - a_t. \quad (1)$$

Capital C_t^* has been variously named in the literature: *capital invested*, *unrecovered balance*, *unrecovered investment*, *outstanding capital*, *unrecovered investment balance* (see Magni, 2009). We also make use of the symbol $\mathbf{C}^* = (C_0^*, C_1^*, \ldots, C_n^*)$ to denote the capital stream implicitly determined by the internal rate of return r^*. The relation this capital stream bears to the project NPV passes through an excess-return term $G_t = C_{t-1}^*(r^* - r_m)$, $t = 1,2,\ldots,n$. which is obtained by the application of an excess return rate $(r^* - r_m)$ to the beginning-of-period outstanding capital. The NPV may be expressed as the present value, at the market rate, of the sum of the single period margins:

$$NPV(\mathbf{a} \mid v_m) = \sum_{t=1}^{n} G_t \cdot v_m^t = \sum_{t=1}^{n} C_{t-1}^*(r^* - r_m) \cdot v_m^t$$

(see Edwards and Bell, 1961; Peasnell, 1982; Peccati, 1989; Lohmann, 1988; Pressacco and Stucchi, 1997).

3 - Reconciling rate of return and Net Present Value

Consider the present value of the IRR-implied capital stream $PV(\mathbf{C}^* \mid r_m) := \sum_{t=1}^{n} C_{t-1}^* \cdot (1 + r_m)^{-(t-1)}$ or $PV(\mathbf{C}^* \mid v_m) := \sum_{t=1}^{n} C_{t-1}^* \cdot v_m^{t-1}$. Hazen (2003) rewrites (2) in the form

$$NPV(\mathbf{a} \mid v_m) = PV(\mathbf{C}^* \mid r_m) \cdot (r^* - r_m) \cdot v_m \tag{2}$$

and states the following

Hazen Theorem. *Suppose* r^* *is a real-valued IRR and* $PV(\mathbf{C}^* \mid r_m) > 0$ *(respectively,* 0*). Then,* $NPV(\mathbf{a} \mid r_m) > 0$ *if and only if* $r^* > r_m$ *(respectively,* $< r_m$*).*

(See Hazen, 2003, Theorem 3). The above theorem formally reconciles multiple IRRs and NPV.[4] Letting r^{*1}, r^{*2}, ... , r^{*p}, $p \le n$ be the project's real IRRs,the NPV can be written as $NPV(\mathbf{a}\,|\,r_m) = PV(\mathbf{C}^{*i}\,|\,r_m)(r^{*i} - r_m)/(1+r_m)$ for $i = 1,2,\ldots,p$, with obvious meaning of $PV(\mathbf{C}^{*i}\,|\,r_m)$. Each of the IRRs is then formally consistent with NPV. The multiple-IRR case has always been deemed an unfavorable case by scholars; much to the contrary, the mere fact that some projects have more than one IRR should have suggested scholars the inductive idea that a project *always* has more than one rate of return. This idea is at the basis of Magni's (2010, 2013) approach. Let $\mathbf{r}^* = (r_0^* = 0, r_1^*, r_2^*, \ldots, r_n^*)$, $\mathbf{v}^* = (v_{0,0}^* = 1, v_{0,1}^*, v_{0,2}^*, \ldots, v_{0,n}^*)$ be vectors of one-period rates and discount factors, respectively, where $v_{0,t}^* := (\prod_{h=0}^{t}(1+r_h^*))^{-1}$. This means that $NPV(\mathbf{a}\,|\,\mathbf{v}^*) = \sum_{t=0}^{n} a_h \cdot v_{0,t}^* = 0$. Vector $\mathbf{r}^*$ is an *internal return vector* (Weingartner, 1966; see also Peccati, 1989). The internal rate of return of the n-period project is just a particular case of internal return vector such that the one-period IRRs are constant: $\mathbf{r}^* = (r^*, r^*, \ldots, r^*)$. Evidently, there are infinite vectors $\mathbf{r}^*$ which are internal return vectors for project P. For any t, denoting by $R_t = (C_t + a_t) - C_{t-1}$ the project's return (with $R_t = r_t^* C_{t-1}$ if $C_{t-1} \ne 0$), the recursive relation (1) may be generalized as $C_t = C_{t-1} + R_t - a_t$ with the boundary conditions $C_0 = -a_0$ and $C_n = 0$. If $C_t \ne 0$ for all $t < n$, it is $r_t^* = R_t / C_{t-1}$, so the relation reduces to $C_t = C_{t-1}(1 + r_t^*) - a_t$. Owing to the boundary conditions, the sequence of the r_t^*'s is an internal return vector. The IRR-implied capital stream $\mathbf{C}^*$ is then only a particular case of a more general notion of capital stream

[4] Economically, the multiple-IRR problem remains. The choice of the correct rate of return should be made by exogenously fixing the capitals which are meaningful values of the economic resources actually deployed by the investor (see Altshuler and Magni, 2012; Magni, 2013).

$\mathbf{C}=(C_0, C_1, C_2, \ldots, C_{n-1})$: capital stream $\mathbf{C}^*$ is to capital stream $\mathbf{C}$ what sequence $(r^*, r^*, \ldots, r^*)$ is to sequence $(r_1^*, r_2^*, \ldots, r_n^*)$. Consider the equation

$$x \cdot (y - r_m) = k \tag{3}$$

where $k \in \mathrm{R}$.The term x takes on the meaning of aggregate capital, $(y - r_m)$ represents an excess return rate, and k represents the project's NPV referred to time 1. The equation admits infinitely many solutions, so there are infinite pairs (x, y) which fulfill the relation: they all lie on the graph of the function $y(x) = r_m + \frac{k}{x}$. To any vector $\mathbf{C}$ fulfilling the boundary conditions $C_0 = -a_0$ and $C_n = 0$, there corresponds an aggregate capital $x = \sum_{t=1}^{n} C_{t-1} v_m^{t-1}$. Hence, choosing $k = NPV_1(\mathbf{a} \mid r_m)$, we have

$$y(x) = r_m + \frac{NPV_1(\mathbf{a} \mid r_m)}{x}. \tag{4}$$

Magni (2010, 2013) shows that, for any x, the index $y(x)$ is a mean of internal period rates r_t^* which are weighted by capital coefficients expressing the relative weights of the interim capitals with respect to the overall capital x invested in the project:

$$y = \alpha_1 r_1^* + \alpha_2 r_2^* + \ldots + \alpha_n r_n^* \qquad \text{where} \quad \alpha_t := \frac{C_{t\text{-}1} \cdot v_m^{t-1}}{\sum_{t=1}^{n} C_{t\text{-}1} \cdot v_m^{t-1}}.$$

More generally, y can be written as "return on capital":

$$y = \frac{PV_1(\mathbf{R} \mid r_m)}{PV(\mathbf{C} \mid r_m)} \tag{6}$$

where $PV(\mathbf{C} \mid r_m) = \sum_{t=1}^{n} C_{t-1} v_m^{t-1}$.

Function $y(x)$ maps capitals into rates of return; in other words, each value taken on by the function is a rate of return corresponding to an investment of $x = PV(\mathbf{C} \mid r_m)$ dollars. Equations (7), (6), (5) correspond, respectively, to economic intuitions (i), (ii), and (iii) in Magni (2013). Their common value is called "Average Internal Rate of Return" (AIRR). Any point (x, y) on the graph of $y = y(x)$ represents the univocal association of an invested capital and its AIRR; the pair $(PV(\mathbf{C}^* \mid r_m), r^*)$ which identifies the IRR and its corresponding aggregate capital is only one possible pair among infinitely many ones.

Magni Theorem. *To every project* P *with cash flow stream* $\mathbf{a} = (a_0, a_1, ... a_n) \in \mathbb{R}^{n+1}$ *there corresponds a unique return function* $y = y(x)$, *which maps present values of capital stream to (weighted average) rates of return. For any capital stream* $\mathbf{C}$, *consider the aggregate capital* $x = PV(\mathbf{C} \mid r_m)$ *and let* y *be the associated rate of return (Average Internal Rate of Return). Then,*

$$NPV(\mathbf{a} \mid r_m) = \frac{PV(\mathbf{C} \mid r_m)(y - r_m)}{1 + r_m}. \qquad (7)$$

Furthermore, if $PV(\mathbf{C} \mid r_m) > 0$ *(respectively,* < 0*), then* $NPV(\mathbf{a} \mid r_m) > 0$ *if and only if* $y > r_m$, *(respectively,* $< r_m$*).*

(See also Magni, 2010, Theorem 2). Hazen Theorem is found back by choosing any $\mathbf{C} \in \mathbb{R}^n$ such that $PV(\mathbf{C} \mid r_m) = PV(\mathbf{C}^* \mid r_m)$. The return function $y(x)$ always exists and is uniquely associated to the project, regardless of whether a real IRR exists or not. And, if an internal rate of return r^* actually exists, the pair $(PV(\mathbf{C} \mid r_m), r^*)$ is just one point on the curve $y = y(x)$ (see Magni, 2010, Theorem 3). Any point $(x, y(x))$ represents the association of capital invested and return on that capital; depending on which capital stream is selected, a unique rate of return is derived and the ratio of the former to the latter captures economic profitability. Furthermore, the above theorem triggers a general definition of investment (borrowing), which enables the investor to determine the financial nature of the project on the basis of empirical evidence.

Definition 1. *Let* $\mathbf{C}$ *be any fixed capital stream for a project* P. *Then, a project is an investment (respectively, borrowing) if* $PV(\mathbf{C}\,|\,r_m) > 0$ *(respectively,* < 0*).* (See Figure 1.)

A project is not uniquely associated with a return *rate*, but with a return *function* (whose values are means of one-period return rates); it is the capital $x = PV(\mathbf{C}\,|\,r)$ that is associated with a unique return rate $y(x)$. Consider again eq. (4). It may be restated in the following form:

$$NPV(\mathbf{a}\,|\,r_m) = -x + \frac{x(1+y)}{1+r_m} \tag{8}$$

according to which project P is actually turned into a one-period project with cash flow stream $(-x, x(1+y)) \in \mathsf{R}^2$ and y is its internal rate of return. Choosing some $\mathbf{C}$, one picks $x = PV(\mathbf{C}\,|\,r_m)$ so that (9) becomes

$$NPV(\mathbf{a}\,|\,r_m) = -PV(\mathbf{C}\,|\,r_m) + PV(\mathbf{C}\,|\,r_m)(1+y)(/1+r_m)\,.$$

From a theoretical standpoint, for any fixed choice of r_m, project P's return function may be derived directly from project P (as shown in section 2) Therefore, in order to compute a particular rate of return (AIRR), the investor fixes any x for P: the AIRR ($=y$) is then consequently derived. From a practical standpoint, the most appropriate choice of x is the capital which is consistent with the value of the economic resources actually employed by the investor. This choice involves judgmental evaluation (Lindblom and Sjögren, 2009, and Magni, 2013, suggest the use of economic values). However, the AIRR approach comes to rescue of those evaluators who prefer to have an IRR, even if an IRR does not exist. In the next sections, we show that a significant IRR may actually be computed for a no-IRR project.

4 - Managing projects with no IRR with a quasi-IRR

In the previous section, we have derived a return function. In order to pick one particular value of P's return function we consider in this section the set of all those projects with a unique IRR having the same length and the same NPV as project P's. Within this set we choose the project P' whose cash flows stream $\mathbf{a}' = (a'_0, a'_1, \ldots, a'_n)$ has the minimum distance from project P's cash flow stream. Put it equivalently, we individuate a project P' through an adjustment process of P's cash flows, to the extent that, leaving the NPV unvaried, the resulting project has a unique IRR.

We will deal with a project P with no IRR such that there is (at least) a couple i, j of time indices such that $a_i \cdot a_j < 0$ (so excluding projects having the cash flows with the same sign).[5] We start by dealing with a two-period project in the following subsection and then give a view to introducing the general n-period extension.

4.1 - Two-period projects

Let P be a two-period project without feasible IRR and with cash flows of different sign. This means that P is characterized by a NPV which may be expressed as $NPV(\mathbf{a}\,|\,v) = a \cdot \left[(v - v_0)^2 + c/a\right]$ with $ac > 0$, $v_0 > 0$.[6] Project P is represented by a cash flow stream which can be framed as the product of the last cash flow a and a "normalized" cash flow vector $\vec{\mathbf{a}}$, that

[5] Projects with cash flows not alternating in sign are interpreted as arbitrage strategies. There is no IRR for these projects and there is no way of adjusting them without changing the sign of at least one cash flow. Obviously, the AIRR is always available as a rate of return, even for such a kind of projects.

[6] In detail, the conditions depends on the following reasons: (i) $a > 0$ and no real v^* means to deal with projects having strictly positive NPV for any value of the discount factor; on the contrary, $a < 0$ implies strictly negative NPV. We choose to examine in detail the problem with a positive a, but all considerations and theorems may be extended to the symmetric negative case; (ii) starting now from $a > 0$, we have three kinds of zeros, that is: (a) both negative: this characterizes a project having all positive cash flows, that is an arbitrage strategy (see the previous footnote); (b) one positive and one negative zero: this situation conflicts with the assumption of IRR unfeasibility; (c) both positive: again impossible if there are no feasible IRR. This implies positivity of c (in order to rule out positive zeros); (iii) at this point we need $v_0 > 0$, otherwise all the cash flows would be positive, giving again an unacceptable arbitrage strategy.

is, $\mathbf{a} = (av_0^2 + c, -2av_0, a) = a \circ \left(v_0^2 + c/a, -2v_0, 1\right) = a \circ \vec{\mathbf{a}}$. We look for a twin project P' with cash flow stream $\mathbf{a}'$ such that, for a fixed v_m the NPV of P and P' are equal (NPV-neutrality): $NPV(\mathbf{a}'\,|\,v_m) = NPV(\mathbf{a}\,|\,v_m)$. In addition we require that project P' has a unique feasible zero $v^* > 0$, so that the behaviour of the NPV twin function is analogous to that of the original NPV function:

$$NPV(\vec{\mathbf{a}}\,|\,v) = NPV(\mathbf{a}'\,|\,v)/a = \alpha(v - v^*)^2 = NPV(\vec{\mathbf{a}}'\,|\,v)$$

with $\alpha \cdot a > 0$. Henceforth, without losing generality, we suppose $a > 0$, so that α is positive. Note that there are two degrees of freedom in the choice of α and v^*. We can write the row cash flow vector $\mathbf{a}'$ in terms of the unknowns in the following, normalized, way: $\mathbf{a}' = a \circ (\alpha {v^*}^2, -2\alpha v^*, \alpha) = a \circ \vec{\mathbf{a}}'$. Let P_m be the value of the project corresponding to the fixed level v_m (greater than zero with the previous assumptions on a and c) of the market discount factor: $P_m/a = NPV(\vec{\mathbf{a}}\,|\,v) = (v_m - v_0)^2 + c/a$. For this fixed v_m, the NPV neutrality condition implies: $\alpha(v_m - v^*)^2 = P_m/a$. This gives the following solutions for v^* (we recall that both α and a are positive, so P_m is positive as well):

$$v^* = v_m + \sqrt{\frac{P_m}{a\alpha}} \qquad \forall 0 < \alpha < \frac{P_m}{av_m^2}. \tag{9}$$

$$v^* = \begin{cases} v_m - \sqrt{\dfrac{P_m}{a\alpha}} \\ v_m + \sqrt{\dfrac{P_m}{a\alpha}} \end{cases} \qquad \forall \alpha > \frac{P_m}{av_m^2}. \tag{10}$$

Note that $v^* = v^*(\alpha)$ depends on α, which means that there is a set of (two-period) projects with unique IRR having the same NPV as project P's. Within this set we choose the project with the feasible α which minimizes a proper measure of distance between the original normalized vector $\vec{\mathbf{a}}$ and the normalized twin vector $\vec{\mathbf{a}}'$:

$$\min_{\alpha}\sqrt{\sum_{h=0}^{2}\left(\frac{\vec{\mathbf{a}}'_h-\vec{\mathbf{a}}_h}{\vec{\mathbf{a}}_h}\right)^2} \tag{11}$$

The problem, expressed in terms of the unknowns, is equivalent to

$$\min_{\alpha}\left(\frac{v_0^2+\frac{c}{a}-\alpha v^{*2}}{v_0^2+\frac{c}{a}}\right)^2+\left(\frac{v_0-\alpha v^*}{v_0}\right)^2+(1-\alpha)^2.$$

Let $D^+(\alpha)$ be the distance corresponding to $v^*=v_m+\sqrt{\frac{P_m}{a\alpha}}$ and $D^-(\alpha)$ the one obtained with $v^*=v_m-\sqrt{\frac{P_m}{a\alpha}}$. We have

$$D^{\pm}(\alpha)=\left(\frac{v_0^2+\frac{c}{a}-\alpha\left(v_m\pm\sqrt{\frac{P_m}{a\alpha}}\right)^2}{v_0^2+\frac{c}{a}}\right)^2+\left(\frac{v_0-\alpha\left(v_m\pm\sqrt{\frac{P_m}{a\alpha}}\right)}{v_0}\right)^2+(1-\alpha)^2.$$

It is tedious but easy to show that $D^{\pm}(\alpha)=k_1\alpha^2\pm k_2\alpha^{3/2}+k_3\alpha\pm k_4\alpha^{1/2}+k_5$ where the constants $k_1,..,k_5$ are given by the following expressions:

$$k_1=1+\frac{v_m^2}{v_0^2}+\frac{a^2v_m^4}{(av_0^2+c)^2},$$

$$k_2=\frac{2\sqrt{P_m/a}v_m}{v_0^2}+\frac{4a^2\sqrt{P_m/a}v_m^3}{(av_0^2+c)^2}$$

$$k_3=-2+\frac{P_m}{av_0^2}-\frac{2v_m}{v_0}+\frac{-2acv_m^2+6aP_mv_m^2-2a^2v_0^2v_m^2}{(av_0^2+c)^2}$$

$$k_4=-\frac{2\sqrt{P_m/a}}{av_0}+\frac{-4ac\sqrt{P_m/a}v_m+4\sqrt{a}P_m^{3/2}v_m-4a^2\sqrt{P_m/a}v_0^2v_m}{(av_0^2+c)^2}$$

$$k_5 = 3 + \frac{-2cP_m + P_m^2 - 2aP_m v_0^2}{(av_0^2 + c)^2}.$$

In order to find α^*, the following problem may be solved numerically:

$$\alpha^* = \operatorname{argmin}\{D^+(\alpha), D^-(\alpha)\}. \qquad (12)$$

At this point, the cash flows of project P' may be expressed in terms of the solution α^* of the previous problem:

$$\mathbf{a}'(\alpha^*) = a \circ (\alpha^* \cdot [v^*(\alpha^*)]^2, -2\alpha^* \cdot v^*(\alpha^*), \alpha^*).$$

Thus, in what precedes, we have proved the following theorem.

Theorem 1. *The cash flows of the (two-period) twin project P' are:* $a_0' = a\alpha^* v_m^2 \oplus 2v_m\sqrt{a\alpha^* P_m} + P_m$, $a_1' = -2(a\alpha^* v_m \oplus \sqrt{a\alpha^* P_m})$, $a_2' = a\alpha^*$, *where* $\oplus = +$ *if* $v^*(\alpha^*) = v_m + \sqrt{P_m/(a\alpha^*)}$ *and* $\oplus = -$ *if* $v^*(\alpha^*) = v_m - \sqrt{P_m/(a\alpha^*)}$. *The internal rate of return of the twin project P' is* $r^*_{\alpha^*} = \dfrac{1}{v^*(\alpha^*)} - 1 = \dfrac{1}{v_m \oplus \sqrt{P_m/(a\alpha^*)}} - 1.$

As for the IRR-implied outstanding capitals of the twin project, it is straightforward to show that $c_0'^* = -a_0'$, $c_1'^* = c_0'^* \cdot (1 + r^*_{\alpha^*}) - a'_1$, $c_2'^* = c_1'^* \cdot (1 + r^*_{\alpha^*}) - a_2' = 0$. The following result is also important for what follows.

Theorem 2. *The PV of the IRR-implied capital stream of the (two-period) twin project at the evaluating discount factor v_m is*

$$PV_2(\mathbf{C}'^* \mid v_m) = -(P_m \oplus \sqrt{a\alpha^* P_m v_m}).$$

Proof. From (14),

$$PV_2(\mathbf{C}'^* \mid v_m) = -(a\alpha^* v^{*2}(\alpha^*)) - (a\alpha^* v^{*2}(\alpha^*))(1 + r^*_{\alpha^*})v_m + 2a\alpha^* v^*(\alpha^*)v_m$$
$$= (v_m - v^*(\alpha^*)) \cdot a\alpha^* v^*(\alpha^*)$$
$$= -a\alpha^*\left(\oplus\sqrt{P_m/(a\alpha^*)}\right)\left(v_m \oplus \sqrt{P_m/(a\alpha^*)}\right) = -\left(P_m \oplus \sqrt{a\alpha^* P_m} v_m\right)$$

4.2 - Multiperiod projects

Let us consider a multiperiod project without feasible real zeros. The net present value is a polynomial of degree n. Obviously, if n is even, the polynomial may be decomposed in the product of $n/2$ second degree polynomials, while if n is odd we have the product of a one-degree polynomial (with unfeasible zero) times the product of $(n-1)/2$ second degree polynomials. Letting a be, again, the last cash flow of project P, the NPV may be expressed in the form:

$$NPV(\mathbf{a} \mid v) = a\prod_{h=1}^{n/2}\left[(v - (v_0)_h)^2 + c_h\right]$$

or $NPV(\mathbf{a} \mid v) = a \cdot (v - (v_0)_0)\prod_{h=1}^{(n-1)/2}\left[(v - (v_0)_h)^2 + c_h\right]$ If we rule out projects with all the cash flows of the same sign, there is at least one second degree polynomial having abscissa of the vertex greater than 0. Among those projects, we choose the one that, coherently with the assumption of positivity of c_h, has the minimum distance from the horizontal axis, that is the minimum c_h. Skipping the deponents, this second degree polynomial has the form $P(v) = (v - v_0)^2 + c_h$ with $c_h > 0$ and $v_0 > 0$. Let $R(v)$ be the product of all the other polynomials (it has a $n-2$ degree) and $c_h = c/a$, so that the NPV of the project may be expressed as $NPV(\mathbf{a} \mid v) = a\left[(v - v_0)^2 + c/a\right] \cdot R(v) = Q(v)$. In order to adjust the original project in a simple way and to exploit the results obtained in the previous section, we leave unchanged the polynomial $R(v)$, and modify only the second degree polynomial in square brackets. The twin second degree polynomial is given by $P'(v)=\alpha(v-v^*)^2$. We let $Q'(v)$ be the *n*-degree twin polynomial, that is, $Q'(v)=a\cdot P'(v)\cdot R(v)=a\alpha(v-v^*)^2\cdot R(v)$. As previously done, we impose NPV neutrality for a fixed v_m for the second degree polynomial, that is: $\alpha(v_m-v^*)^2=(v_m-v_0)^2+c/a=P_m/a$. In order to grant NPV neutrality, for a fixed v_m, between projects Q and Q', we extend the NPV neutrality between the old projects P and P'. Formally, this means:

$$\begin{aligned}NPV(\mathbf{a}'\,|\,v_m)&=Q'(v_m)=a\alpha(v_m-v^*)^2\cdot R(v_m)\\&=aP'_m\cdot R(v_m)=P_m\cdot R(v_m)=a\left[(v_m-v_0)^2+c/a\right]\cdot R(v_m)=Q(v_m)\\&=NPV(\mathbf{a}\,|\,v_m).\end{aligned}$$

At this point, we can apply exactly the best fit condition (11) and use (9) and (10) in order to find α^* and $v^*(\alpha^*)$ so that the n-period twin project's cash-flow stream is determined, as well as the (unique) internal rate of return $r^*_{\alpha^*}=1/v^*(\alpha^*)-1$. As regards the present value of the capital stream of the project, we show that the following result holds.

Theorem 3. *The present value PV_n of the IRR-implied outstanding capitals of the n-period twin project, at the evaluating discount factor v_m, is the present value PV_2 of the IRR-implied outstanding capitals of the second degree polynomial times the NPV of the unmodified, $(n-2)$ degree, polynomial:*

$$PV_n(\mathbf{C}'^*\,|\,v_m)=a\cdot PV_2(\mathbf{C}'^*\,|\,v_m)\cdot R(v_m).$$

Proof. We remind (equation (3)) that the present value of the *n*-period project, evaluated at the market discount factor v_m, may be rewritten in terms of the present value of the capital stream of the twin project in the following way: $Q'(v_m)=PV_n(\mathbf{C}'^*\,|\,v_m)\cdot(r^*_{\alpha^*}-r_m)\cdot v_m$, where $r^*_{\alpha^*}$ denotes the IRR of the *n*-period twin project P'. On the other side, for the neutrality condition, we have $Q'(v_m) = Q(v_m) = P_m \cdot R(v_m) = P'_m \cdot R(v_m)$. Again, by (3), the present value of the 2-period adjusted project is expressed in terms of its IRR-implied outstanding capitals: $P'_m = a\alpha^*(v_m - v^*)^2 = a \cdot PV_2(\mathbf{C}'^* \mid v_m) \cdot (r^*_{\alpha^*} - r_m) \cdot v_m$. Putting these conditions together, we have

$$a \cdot PV_2(\mathbf{C}'^* \mid v_m) \cdot (r^*_{\alpha^*} - r_m) \cdot v_m \cdot R(v_m) = PV_n(\mathbf{C}'^* \mid v_m) \cdot (r^*_{\alpha^*} - r_m) \cdot v_m.$$

Hence, immediately, the conclusion.

5 - The quasi-IRR is a rate of return of the original project

The previous section has introduced the IRR of the twin project, which we have denoted by $r^*_{\alpha^*}$. It is the internal rate of return of the project which is the closest one to P, according to an appropriate measure of distance. Therefore, it represents the return on each dollar of capital invested in P', which amounts to $PV_n(\mathbf{C}'^* \mid r_m)$. In this section, we show that this measure is not only a rate of return for P', but it is a genuine rate of return for the original project P as well. Being a rate of return for P and being, technically, an IRR (of the twin project), we call this rate the "quasi-IRR" of P. We exploit Magni Theorem, which states that any project P is uniquely associated with a return function $y = y(x)$ that maps present value of capital streams to rates of return (AIRR). Let us then consider project P's return function and compute the counterimage of the quasi-IRR through $y(x)$: from $y(x^*) = r^*_{\alpha^*}$ we have $x^* = y^{-1}(r^*_{\alpha^*})$ and, using (5), one finds

$$x^* = \frac{NPV_1(\mathbf{a} \mid r_m)}{r^*_{\alpha^*} - r_m}. \tag{14}$$

Exploiting the NPV-neutrality condition one gets $x^* = NPV_1(\mathbf{a}' \mid r_m)/(r^*_{\alpha^*} - r_m)$

Hence,

$$x^* = \frac{(1+r_m)\cdot PV_n(\mathbf{C}'^* \mid r_m)(r^*_{\alpha^*} - r_m)v_m}{r^*_{\alpha^*} - r_m} = PV_n(\mathbf{C}'^* \mid r_m). \qquad (15)$$

which shows that the (aggregate) capital invested in project P corresponding to a rate of return of $r^*_{\alpha^*}$ coincides with the (aggregate) capital invested in the twin project.

By definition of return function, multiplying the quasi-IRR $r^*_{\alpha^*}$ by x^* one just obtains the corresponding project P's aggregate return: $PV(\mathbf{R}^* \mid r_m) = x^* \cdot r^*_{\alpha^*}$. Therefore, the quasi-IRR is the ordinate of the point $(x^*, r^*_{\alpha^*})$ lying on project P's return function (see Figure 2).

From Magni Theorem and eqs. (15)-(16) the following result obtains.

Theorem 4. *The quasi-IRR correctly captures the economic profitability of project P. In particular, project P is interpreted as an (aggregate) investment of x^* dollars which generate return at a rate of $r^*_{\alpha^*}$. The invested capital x^* coincides with the invested capital $PV_n(\mathbf{C}'^* \mid r_m)$ of the twin project P'.*

Resting on eq. (9), the information supplied to the investor is then as follows: project P may be interpreted as a one-period project whose capital invested is x^* and whose internal rate of return is $y(x^*) = r^*_{\alpha^*}$, which is the internal rate of return of P'. That is, we have kept in with the (seemingly paradoxical) requirement of computing an internal rate of return which expresses P's economic profitability even though P has no internal rate of return at all.

To sum up, we have found an AIRR for project P representing an *internal* rate of return in two senses: (i) it is the (unique) IRR of a project which strictly resembles project P; (ii) it is the internal rate of a one-period project whose outlay coincides with the aggregate capital invested in P.

Therefore, the rate of return $r^*_{\alpha^*}$ does deserve the label of "quasi-IRR" of project P.

6 - Examples

We will illustrate the meaning of the previous sections through two examples. Let us consider the project P with cash flow vector $\mathbf{a} = (4.925, -14, 10)$ without real zeros. Its present value may be written in the form $a\lfloor(v - v_0)^2 + c/a\rfloor$ with $a = 10$, $v_0 = 0.7$, $c = 0.025$. Let the market rate be $r_m = 0.1$ (10%) so that the market discounting factor is $v_m = 0.\overline{90}$ and the NPV, computed at the market rate, is 0.462. The value $\alpha^* = 1.00875$ (feasible, that is greater than $P_m/v_m^2 = 0.0559$) is the only real solution of the best fitting problem in (12) and $v^*(1.00875) = 0.695039$, in the form $v_m - \sqrt{P_m/10.0875}$, corresponds to a quasi-IRR of $r^*_{\alpha^*} = 43.8767\%$, which is the IRR of the twin project, whose cash flow stream is $\mathbf{a}' = (4.87307, -14.0224, 10.0875)$; its net present value may be written in the form $10.0875(v - 0.695039)^2$, resulting in 0.462 for $v = v_m$ (owing to NPV-neutrality). The return funtion of project P is $y(x) = 0.1 + 0.5084/x$; then, from $0.438767 = y(x^*)$ one gets $x^* = 1.5$. The project is (not a borrowing but) an investment and may be interpreted as a one-period investment of 1.5 dollars yielding a 43.8767% internal rate of return. From $x^* = PV(\mathbf{C} \mid 0.1) = C_0 + C_1/1.1$ one finds that the interim capital is equal to $C_1 = (1.5 + 4.925)(1.1) = 7.068$, so that the period rates are $r_1^* = -(7.068 - 14)/4.925 - 1 = 0.4075$ (borrowing rate) and $r_2^* = (0 + 10)/7.068 - 1 = 0.4148$ (investment rate), whose weighted mean is just $y = (0.4075) \cdot (-4.925)/1.5 + (0.4148) \cdot (7.068 \cdot 1{,}1^{-1})/1.5 = 0.438767$.

Coming now to a 4-period project, let us consider project P with cash flow stream $\mathbf{a} = (47.477, -231.49, 420.05, -336, 100)$. The present value of P may be framed as $a\lfloor(v - v_0)^2 + c/a\rfloor \cdot \lfloor(v - v_{0r})^2 + c_r\rfloor$ where the parameters are $a = 100$, $v_0 = 0.7$, $c = 0.25$ and $v_{0r} = 0.98$, $c_r = 0.0036$. Being $c/a < c_r$, we let $R(v) = (v - v_{0r})^2 + c_r$ and we proceed as before in order to adjust the second degree polynomial $a\lfloor(v - v_0)^2 + c/a\rfloor$. With $v^*(\alpha^*) = v_m - \sqrt{P_m/(a\alpha^*)}$, we obtain a twin project with cash flow stream

$$\mathbf{a}' = (46.9764, -230.6885, 420.814, -337.9395, 100.8751).$$

The IRR of this project (i.e., the quasi-IRR) is $r^*_{\alpha^*} = 43.8767\%$, as before.[7] The NPV of the twin project is 0.0399 and project P's return function is $y(x) = 0.1 + 0.04387/x$, whence $x^* = 0.12949$. The project is, again, an investment, and may be interpreted as a one-period investment of 0.12949 dollars yielding a 43.8767% internal rate of return. The interim capitals are found by $-47.477 + C_1/1.1 + C_2/1.1^2 + C_3/1.1^3 = 0.12949$. Unlike the two-period example, there is not a unique solution. Precisely, there are two degrees of freedom, that is infinite solutions. It may be convenient to choose $r_1^* = r_2^* = 0.438767$ (the quasi-IRR of the project), which implies $C_1 = 163.18$ and $C_2 = -185.269$. This in turn implies $C_3 = 69.71$ whence $r_3^* = 0.4373$, $r_4^* = 0.4345$. It is then easy to check that the mean of the rates weighted by its corresponding capitals is just the quasi-IRR. Geometrically, the present value of the twin project as a function of the discount factor v has the graph as in Figure 3, where the dashed curve is the NPV of the original project and the

[7] We only use four decimals after the comma, to avoid notational awkwardness. A better approximation of the IRR is found with

(4.873071949177,-14.022433915853, 10.008751024893)
(two-period twin project) and

(46.976413590066,-230.688473152691,
420.814023042214, -337.939540037641, 100.8751025)
(four-period twin project). Of course, this notational practice (four decimals only) has been applied to all the results in the example.

other one is the NPV of the twin project as a function of v .

7 - Concluding remarks

Several classes of projects may have no IRR. While this may occur rarely, the consequences are serious. The NPV can be still used for assessing wealth creation, but an important information is missed, that is, the amount of value created per unit of invested capital. In real life, industrial and engineering projects may occasionally meet this difficulty. Magni (2010) introduces a new approach to rate of return, based on the finding that any project is not associated with a rate of return but with a return function, which exists and is unique, and which maps aggregate capitals to rates of return, each of which is called *Average Internal Rate of Return* (AIRR), being a weighted average of one-period internal rates of return. He also shows that any IRR and its associated capital lies on such a return function (i.e., it is a particular case of AIRR). As any AIRR captures the project's economic profitability, there is the need of singling out the appropriate capital, which expresses the economic resources that are actually employed in the investment (i.e., one point on the graph of the return function). The choice should be made on the basis of sound economic reasoning and empirical evidence (Magni, 2013). Notwithstanding the flaws of the IRR, many investors are very familiar with IRR and privilege it as an intuitive relative measure of worth. In this paper, we show that the AIRR approach can come to the rescue of the IRR even when the IRR does not exist. To require an IRR whenever no IRR exists seems just to be (contradictory) wishful thinking. The paradox is overcome by minimizing an appropriate distance between the cash flow stream of the original project and the cash flows stream of a twin project which has a unique IRR and has the same NPV of the original project. Hence, using the return function of the original project, we show that the IRR of the twin project is an AIRR of the original project, associated with a well-determined capital. In other words, the IRR of the twin project captures the economic profitability of the original project, in the sense that it is NPV-consistent. The project may be interpreted as a one-period investment whose traditional IRR is just the IRR of the twin project. To sum up, the IRR of the twin project is an AIRR with two compelling features: (i) it is the IRR of the project whose cash flow stream is the closest one to the original project's; (ii) it is the IRR of a one-period project with initial outlay coinciding with the aggregate capital invested in the project. While not exactly the IRR of the project, the IRR of the twin project is *almost* the IRR of the project: it is the project's 'quasi-IRR'. Such a metric is NPV-consistent and overcomes the classical IRR problems in accept/reject decisions and choice between competing projects (to this end, incremental quasi-IRR can be computed).

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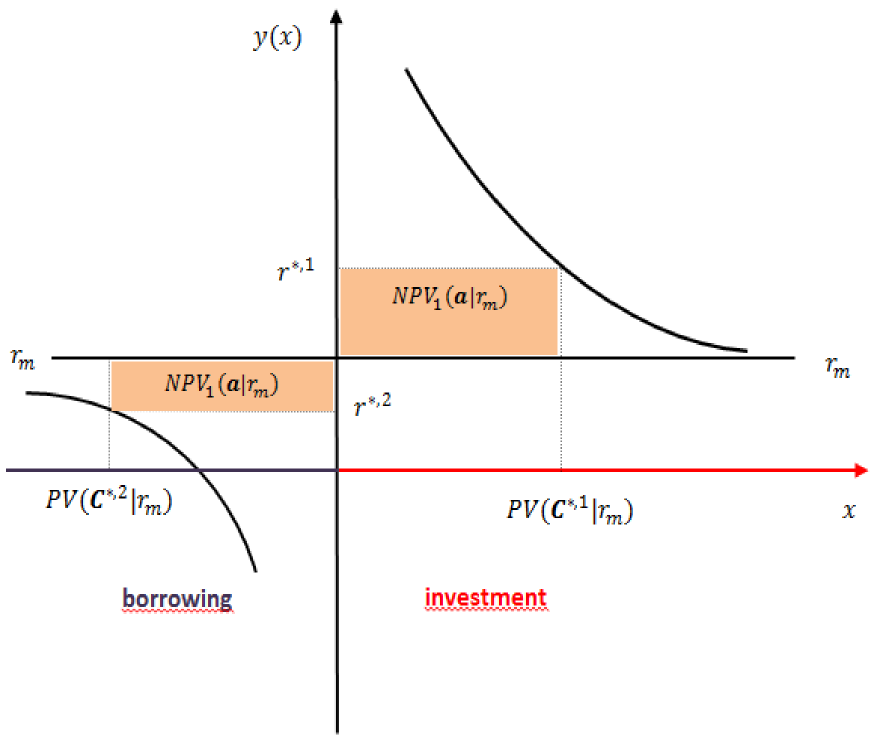

Figure 1. The return function for a positive-NPV projects with two IRRs ($r^{*,1}$ and $r^{*,2}$). Any value taken on by the function, labeled "AIRR", is a rate of return which correctly captures the project's economic profitability. Any one IRR is a particular case of AIRR.

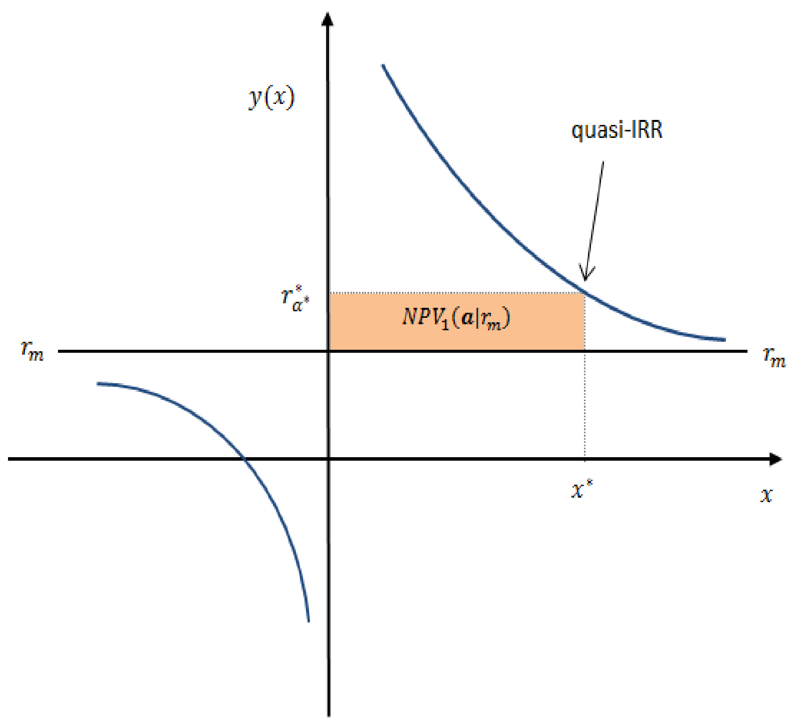

Figure 2. The quasi-IRR is the AIRR associated with an aggregate capital equal to $x^* = PV_n(C'^*|r_m)$.

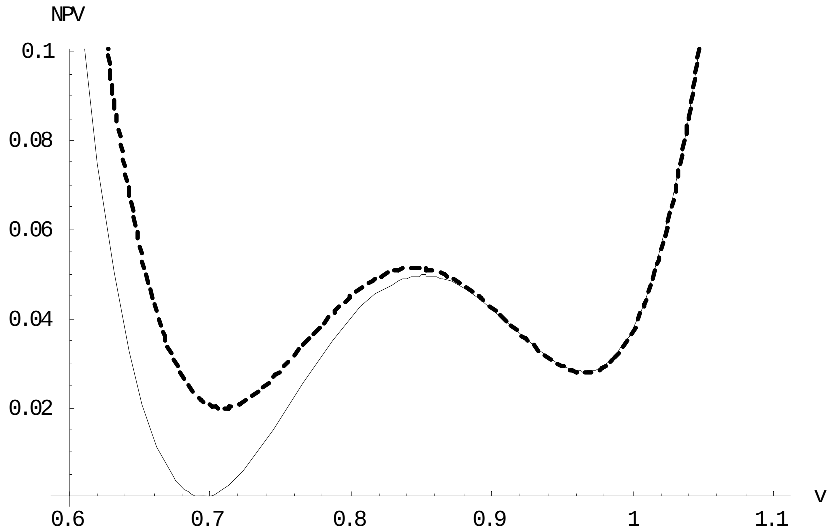

Figure 3. The project and the twin project: the four-period case