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The non-existence of stable Schottky forms

G. Codogni and N. I. Shepherd-Barron

Abstract

We show that there is no stable Siegel modular form that vanishes on every moduli space of curves.

1. Introduction

Denote by M_g and A_g the coarse moduli spaces of genus g curves and principally polarized abelian g-folds, respectively, and by $j_g : M_g \to A_g$ the Jacobian morphism, which sends every curve to its Jacobian. This separates geometric points, according to (a crude version of) the Torelli theorem, so that we can be somewhat careless in distinguishing between M_g and the Jacobian locus (the image of M_g under j_g).

The Satake compactification A_g^S of A_g is the minimal complete normal variety that contains A_g as an open subvariety and is also characterized by the property that the $\mathbb{Q}$-line bundle $\mathcal{L}$ (sometimes denoted by ω, but this notation conflicts with the use of ω to denote the canonical bundle) of weight 1 (Siegel) modular forms on A_g extends to an ample $\mathbb{Q}$-line bundle on A_g^S. There is a stratification $A_g^S = \bigcup_{0\leqslant h\leqslant g} A_h$, so that the boundary $\partial A_g^S = A_g^S - A_g$ of A_g^S is $\partial A_g^S = \bigcup_{0\leqslant h\leqslant g-1} A_h$. It follows that A_{g-1}^S is then the normalization of ∂A_g^S; in fact, $A_{g-1}^S = \partial A_g^S$, from the surjectivity of the Siegel operator Φ (some of whose basic properties recalled below) for modular forms of sufficiently high weight. In this way we can regard A_g^S as a subvariety of A_{g+m}^S for every positive g, m.

Define the Satake compactification M_g^S of M_g to be the closure of the Jacobian locus in A_g^S. Hence (see [Hoy63, Mum64]) the geometric points of A_g^S correspond to principally polarized abelian varieties of dimension at most g and those of M_g^S correspond to products of Jacobians, where the genera of the curves in question sum to at most g. Therefore, the intersection $M_{g+m}^S \cap A_g^S$, taken inside A_{g+m}^S is exactly M_g^S, as sets. The main result of this paper (which was inspired by a recent result of Grushevsky and Salvati Manni [GS11] that we recall below) is that this intersection is far from being transverse, however.

Theorem 1.1. *$M_{g+m}^S \cap A_g^S$ contains the mth order infinitesimal neighbourhood of M_g^S inside A_g^S.*

By definition, if an integral subscheme X of a scheme Y is defined by an ideal I, then the mth order infinitesimal neighbourhood of X in Y is the subscheme of Y defined by the $(m+1)$th symbolic power $I^{[m+1]}$ of I. When X and Y are both regular, then $I^{[m+1]} = I^{m+1}$, the ordinary $(m+1)$th power. If, moreover, X and Y are smooth over a field of characteristic zero, if $x_1, \ldots, x_n$ are holomorphic (or formal, or étale) co-ordinates on Y such that I is generated by $x_1, \ldots, x_r$,

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then I^m consists of those functions f on Y such that f and all its partial derivatives with respect to $x_1, \ldots, x_r$ up to and including those of order $m-1$ vanish along X.

Corollary 1.2. *If g is fixed, then $\bigcup_{m\geqslant 0}(M^S_{g+m} \cap A^S_g)$ equals the formal completion $(A^S_g)\widehat{\ }$ of A^S_g along M^S_g.*

Recall that the Schottky problem, in its general form, is the problem of distinguishing Jacobians from other principally polarized abelian varieties. One classical approach is to seek *Schottky forms*, that is, scalar-valued Siegel modular forms on A_g that vanish on the Jacobian locus, or, equivalently, forms on A^S_g that vanish on M^S_g.

The normalization $\nu : A^S_g \to \partial A^S_{g+1}$ gives a restriction map Φ, which coincides with the *Siegel operator*, from the vector space $[\Gamma_{g+1}, k] = H^0(A^S_{g+1}, \mathcal{L}^{\otimes k})$ of weight k forms on A_{g+1} (equivalently, on A^S_{g+1}) to $[\Gamma_g, k]$. This is surjective if k is even and $k > 2g$ [Fre83, p. 64] so that, since $\mathcal{L}$ is ample on the Satake compactification, ν is an isomorphism.

In terms of holomorphic functions on the Siegel upper half-planes $\mathfrak{H}_{g+1}$ and $\mathfrak{H}_g$ of degrees $g+1$ and g respectively, Φ is defined by

$$\Phi(F)(\tau) = \lim_{t\to+\infty} F(\tau \oplus it),$$

where the direct sum of two square matrices has its obvious meaning. In terms of a Fourier expansion

$$F(T) = \sum_S a(S) \exp \pi i \operatorname{tr}(ST),$$

where $T \in \mathfrak{H}_{g+1}$ and S runs over the positive semi-definite symmetric integral matrices with even diagonal, Φ is given by

$$\Phi(F)(\tau) = \sum_{S_1} a(S_1 \oplus 0) \exp \pi i \operatorname{tr}(S_1 \tau).$$

Or Φ can be calculated by first restricting to a copy of $\mathfrak{H}_g \times \mathfrak{H}_1$ in $\mathfrak{H}_{g+1}$ that arises as a cover of the image of $A_g \times A_1$ in A_{g+1} and then letting $\tau \in \mathfrak{H}_1$ tend to $i\infty$.

In genus 4, Schottky found one such Schottky form explicitly [Sch88]; later Igusa [Igu81a] showed that, if

$$F_g = \theta_{E_8^2, g} - \theta_{D_{16}^+, g},$$

the difference of the theta series in genus g associated to the two distinct positive even unimodular quadratic forms E_8^2 and D_{16}^+ of rank 16, then the form discovered by Schottky is an explicit rational multiple of F_4. He also [Igu81b] showed that it is reduced and irreducible, and so cuts out exactly the Jacobian locus in A_4.

(Recall that if a positive even unimodular quadratic form Q of rank k is regarded as a lattice Λ_Q in Euclidean space $\mathbb{R}^k$, then the theta series $\theta_{Q,g}$ is defined by

$$\theta_{Q,g}(\tau) = \sum_{x_1,\ldots,x_g \in \Lambda_Q} \exp \pi i \sum_{p,q=1}^{g} Q(x_p, x_q)\tau_{pq}.$$

This lies in the space $[\Gamma_g, k/2]$ and satisfies the formula $\Phi(\theta_{Q,g+1}) = \theta_{Q,g}$.)

Grushevsky and Salvati Manni [GS11] have shown that the genus 5 Schottky form F_5 does not vanish along M_5. (They go on to prove that F_5 cuts out exactly the trigonal locus in M_5.) They did this by proving that if F_5 did vanish along M_5, then F_4 would vanish with multiplicity at least 2 along M_4 (that is, F_4 and all its first partial derivatives would vanish), which would contradict Igusa's result on reducedness. Theorem 1.1 is a generalization of this.

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To begin with, say that a scalar-valued Siegel modular form F on A_g vanishes with multiplicity at least m along M_g if F and all its partial derivatives with respect to the co-ordinates τ_{pq} on $\mathfrak{H}_g$ of order at most $m-1$ vanish along M_g (rather, along the inverse image of M_g in $\mathfrak{H}_g$).

(At this point there is a slight conflict between the language of stacks and that of varieties. On the one hand the Satake compactification, which is fundamental here, is a coarse object, and on the other the variables τ_{pq} are local co-ordinates on the stack $\mathcal{A}_g$ but not on the coarse space A_g.)

THEOREM 1.3. *Suppose that $F = F_{g+1}$ is a scalar-valued Siegel modular form on A_{g+1} that vanishes with multiplicity at least $m \geqslant 1$ along the Jacobian locus M_{g+1} in A_{g+1}. Then $F_g = \Phi(F_{g+1})$ vanishes with multiplicity at least $m+1$ along M_g.*

Note that Theorem 1.1 follows at once from Theorem 1.3 and a lemma in commutative algebra, Lemma 3.5 below.

Finally, we restate this in terms of Freitag's description [Fre77] of the ring of stable Siegel modular forms. He showed that, for a fixed even integer k, the Siegel map $\Phi : [\Gamma_{g+1}, k] \to [\Gamma_g, k]$ is an isomorphism for all $g > 2k$. That is, the vector spaces $[\Gamma_g, k]$ stabilize to a vector space $[\Gamma_\infty, k]$ as g increases, and is the space of sections of a $\mathbb{Q}$-line bundle $\mathcal{L}^{\otimes k}$ on A_∞^S. Let $\mathcal{N}$ denote the restriction of $\mathcal{L}$ to M_g^S.

Put $A(\Gamma_g) = \oplus_k [\Gamma_g, k]$, the graded ring of Siegel modular forms on the moduli space A_g or on the Satake compactification A_g^S. Then $A = \oplus_k [\Gamma_\infty, k]$ is is an inverse limit, in the category of graded rings:

$$A = \lim_{\overleftarrow{g}} A(\Gamma_g).$$

Freitag proved also that A is the polynomial ring over $\mathbb{C}$ on the set of theta series θ_Q, where Q runs over the set of equivalence classes of indecomposable even, positive and unimodular quadratic forms over $\mathbb{Z}$.

Define the stable Satake compactification A_∞^S by

$$A_\infty^S = \bigcup_g A_g^S = \lim_{\overrightarrow{g}} A_g^S.$$

Define $M_\infty^S = \bigcup_g M_g^S$, similarly.

COROLLARY 1.4. *The homomorphism from A to the graded ring $\oplus_k H^0(M_\infty^S, \mathcal{N}^{\otimes k})$ that is induced by the inclusion $M_\infty^S \hookrightarrow A_\infty^S$ is injective. That is, there are no stable Schottky forms.*

Proof. An element F of the kernel would restrict, in each genus g, to a scalar-valued modular form F_g on A_g that vanished to arbitrarily high order along M_g. Then $F_g = 0$ for all g, and then $F = 0$. $\square$

COROLLARY 1.5. *If P, Q are positive even unimodular quadratic forms that are not equivalent, then there exists a curve C whose period matrix distinguishes between them.*

If P, Q have rank g, then there is a period matrix τ in $\mathfrak{H}_g$ such that $\theta_{P,k}(\tau) \neq \theta_{Q,k}(\tau)$. However, it is not clear how to find the genus of the curve whose existence is given by Corollary 1.5, let alone how to identify a particular such curve. However, more recently we have [Cod13, She13] shown that C can be taken to be a general point of the trigonal locus (of some genus, as yet undetermined), but not the hyperelliptic locus. That is, there exist stable modular forms vanishing on every hyperelliptic locus, but there are no such forms that vanish on every trigonal locus.

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2. Fay's degenerating families

We make a slight extension of a construction by Fay [Fay73, pp. 50–54], of 1-parameter families of genus $g+1$ curves that degenerate to an irreducible nodal curve of geometric genus g. His construction includes a calculation of the period matrix of the general member of the family, modulo the square of the parameter. The formula that he gives is, however, mistaken, as was pointed out by Yamada [Yam80], who gave the correct version. The reason for the extension is to permit a rescaling of local co-ordinates by non-zero parameters λ, μ; Fay's construction, with its original wording, only permits this rescaling when $|\lambda| = |\mu| = 1$.

Start with a curve C of genus g. Let $\mathcal{V}$ be the infinite-dimensional variety whose points are quadruples (a, b, z_a, z_b), where a, b are distinct points on C and z_a, z_b are local holomorphic co-ordinates on C at a, b respectively.

The 2-torus $\mathbb{G}_m^2$ acts on $\mathcal{V}$ by

$$(\lambda, \mu)(a, b, z_a, z_b) = (a, b, \lambda^{-1} z_a, \mu^{-1} z_b).$$

Fix a non-empty finite-dimensional (in order to avoid irrelevant difficulties) and smooth subvariety V of $\mathcal{V}$ that is preserved under this torus action and that maps onto the complement U of the diagonal in $C \times C$.

We want to construct a family of morphisms $\{f_v : \mathcal{C}_v \to \Delta\}_{v \in V}$ that is parametrized by V, where Δ is a complex disc centred at 0, each $\mathcal{C}_v$ is a smooth complex surface, each f_v is proper and each fibre over 0 is a nodal curve $C/(a \sim b)$, every other fibre is a smooth curve of genus $g+1$ and the parametrization is holomorphic in V.

It is clearer to run through the construction without referring to the parameter space V. So fix the data C, a, b, z_a, z_b and choose $\delta > 0$ such that there are disjoint neighbourhoods U^a of a and U^b of b such that $z_a : U^a \to \mathbb{C}$ and $z_b : U^b \to \mathbb{C}$ are each an isomorphism to some open set that contains a disc of radius δ centred at $z_a(a) = 0$ and $z_b(b) = 0$, respectively.

Let Δ_δ, D_{δ^2} denote complex discs of radius δ, δ^2, respectively.

Take $W = W_\delta$ to be the open subset of $C \times D_{\delta^2}$ obtained by deleting the two closed subsets

$$\{(p, t) \mid p \in U^a, 0 \leqslant \delta |z_a(p)| \leqslant |t| \leqslant \delta^2\},$$
$$\{(q, t) \mid q \in U^b, 0 \leqslant \delta |z_b(q)| \leqslant |t| \leqslant \delta^2\}.$$

Lemma 2.1. *If $\epsilon < \delta$, then $W_\epsilon \subset W_\delta$.*

In W_δ, define open subsets

$$W^a = W^a_\delta = \{(p, t) \mid p \in U^a,\ 0 < |z_a(p)| < \delta \text{ and } |t| < \delta |z_a(p)|\},$$
$$W^b = W^b_\delta = \{(q, t) \mid q \in U^b,\ 0 < |z_b(q)| < \delta \text{ and } |t| < \delta |z_b(q)|\}.$$

Consider the complex surface $S = S_\delta \subset (\Delta_\delta)^2 \times D_{\delta^2}$ defined by the equation $XY = t$, where X, Y are co-ordinates on the two copies of Δ_δ and t is a co-ordinate on D_{δ^2}. Then there are isomorphisms

$$W^a_\delta \to S - (X = 0) : (p, t) \mapsto (z_a(p), t/z_a(p), t),$$
$$W^b_\delta \to S - (Y = 0) : (q, t) \mapsto (t/z_b(q), z_b(q), t).$$

Together these define an étale morphism $j : W^a_\delta \cup W^b_\delta \to S$, where the union is the disjoint union, taken inside $C \times D_{\delta^2}$. Let $i : W^a_\delta \cup W^b_\delta \to W_\delta$ be the inclusion.

If Z is a subspace of a space X, then $\overline{Z}$ denotes the closure of Z in X.

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LEMMA 2.2. *$(i,j) : W_\delta^a \cup W_\delta^b \to W \times S$ is a closed embedding.*

Proof. It is enough to show that the image of W_δ^a in $\overline{W_\delta^a} \times S$ is closed. Now points in $\overline{W_\delta^a} \times S$ are of the form (p, t_1, X, Y, t_2) with

$$\delta \geqslant |z_a(p)| \geqslant t_1/\delta,\ t_2 = t_1,\ X = z_a(p),\ Y = t_2/z_a(p),$$
$$|X|, |Y| < \delta,\ |t_2| \leqslant \delta^2,\ XY = t_2.$$

However, these conditions force $\delta > |z_a(p)| = |t_2|/|Y| > |t_2|/\delta$, and we are done. □

Now define $\mathcal{C} = \mathcal{C}_\delta$ by gluing W_δ to S_δ by the inclusion i and the étale map j. By the lemma, $\mathcal{C}$ is Hausdorff,[1] and by construction there is a morphism $f : \mathcal{C} \to D_{\delta^2}$ whose fibre over 0 is the nodal curve $C/(a \sim b)$.

LEMMA 2.3. *The morphism f is proper.*

Proof. It is enough to show that, for any $r \in (0, \delta)$, the inverse image $Z_r = f^{-1}(\overline{D_{r^2}})$ is compact. By construction, Z_r is the union of the two compact spaces $\overline{W_\delta^1}$ and $\overline{S_r}$, where the subset $\overline{S_r}$ of S_δ is defined by $|X|, |Y| \leqslant r$. □

LEMMA 2.4. *The restriction of $f : \mathcal{C}_\delta \to D_{\delta^2}$ to the germ of the pair $(D_{\delta^2}, 0)$ is independent of δ.*

Proof. This follows from the facts that, by Lemma 2.1 above, $\mathcal{C}_\epsilon$ is open in $\mathcal{C}_\delta$, and that $C/(a \sim b)$ is proper. □

Note that, by construction, W is open in $C \times D_{\delta^2}$, the image of the projection $\mathrm{pr}_1 : W \to C$ is exactly $C - \{a, b\}$ and there is an étale morphism $\pi : W \to \mathcal{C}$.

Given cycles A_i, B_j on C that represent a symplectic basis of $H_1(C, \mathbb{Z})$ and are disjoint from $\{a, b\}$, we can then regard the A_i, B_j as cycles on $\mathcal{C}_t$ that represent part of a symplectic basis of $H_1(\mathcal{C}_t, \mathbb{Z})$ for $t \neq 0$ by taking $pr_1^{-1}(A_i) \cap pr_2^{-1}(t) = A_i \times \{t\}$ and the same thing for B_j. Define the cycle A_{g+1} on $\mathcal{C}_t$ by $A_{g+1} = \partial U^b \times \{t\}$; then $(A_1, \dots, A_{g+1}, B_1, \dots, B_g)$ can be extended to a symplectic basis of $H_1(\mathcal{C}_t, \mathbb{Z})$ where B_{g+1} projects to a cycle on the nodal curve $\mathcal{C}_0 = C/(a \sim b)$ that passes through the node.

We want to extend this construction of a single degenerating pencil $f : \mathcal{C} \to D$ of curves to the construction of a family of such pencils, where the parameter space is V and the pencil depends holomorphically on V. This is merely a matter of enhancing the notation that we have just used, and the details are omitted. The end result of the construction is a parameter space D that is an open neighbourhood of $V \times \{0\}$ in $V \times \mathbb{C}$ and a proper flat morphism $\mathcal{C} \to D$ from an $(n+1)$-dimensional complex manifold to a complex n-manifold that is smooth outside $V \times \{0\}$ and whose restriction to $V \times \{0\}$ is trivial, with fibre $C/(a \sim b)$.

Now we can follow Fay and Yamada.

We have already chosen 1-cycles $(A_i, B_j)_{i,j=1,\dots,g}$ that represent a symplectic basis of $H_1(C, \mathbb{Z})$. Take the corresponding normalized basis (ω_q) of $H^0(C, \Omega^1)$. ('Normalized' means that $\int_{A_p} \omega_q = \delta_{pq}$ rather than $2\pi i \delta_{pq}$, which latter is the sense in which Fay uses the word.) Denote by τ the resulting period matrix of C; that is, $\tau_{pq} = \int_{B_p} \omega_q$.

Also let $\omega_{g+1} = \omega_{b-a}$, the unique rational 1-form on C whose polar divisor is $a + b$, such that $\int_{A_p} \omega_{b-a} = 0$ for all p and $\mathrm{Res}_b\, \omega_{b-a} = -\mathrm{Res}_a\, \omega_{b-a} = 1/2\pi i$.

[1] See Bourbaki, Top. Gén. TG I.9, p. 57, Proposition 4 (1940 edition).

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Define scalars $v_p(a)$, etc., by $v_p(a) = (\omega_p/dz_a)(a)$; then for each p the map $(a, b, z_a, z_b) \mapsto (v_p(a), v_p(b))$ is a holomorphic function $V \to \mathbb{C}^2$. Also, take a co-ordinate t on $\mathbb{C}$, so that $V \times \{0\}$ is the divisor in D defined by $t = 0$.

When a curve varies in a holomorphic family, its period matrix is a holomorphic function of the parameters, and for the degenerating family just constructed Fay and Yamada make this explicit, as follows.

THEOREM 2.5 [Yam80, Corollary 6]. *After passing to a suitable infinite cyclic cover of $D - (V \times \{0\})$ there is a symplectic basis of the homology of a smooth fibre $\mathcal{C}_{v,t}$ with respect to which the period matrix $T = T(v, t)$ of $\mathcal{C}_{v,t}$ can be written in 2×2 block form*

$$T = \begin{bmatrix} \tau + t\sigma & AJ(b) - AJ(a) + ts \\ {}^t(AJ(b) - AJ(a) + ts) & \dfrac{1}{2\pi i}(\log t + c_1 + c_2 t) \end{bmatrix} + O(t^2).$$

Here, AJ is the Abel–Jacobi map from the curve C to its Jacobian, so that $AJ(b) - AJ(a)$ is the vector $(\int_a^b \omega_p)$; $s = (s_p)$ is some vector-valued holomorphic function on V whose explicit form we do not need; c_1, c_2 are holomorphic functions on V but independent of t; $O(t^2)$ is a holomorphic function on Δ that vanishes modulo t^2; and the $g \times g$ matrix $\sigma = (\sigma_{pq})$ is given by

$$\sigma_{pq} = -2\pi i(v_p(a)v_q(b) + v_q(a)v_p(b)).$$

Proof. This is only a matter of verifying that Yamada's calculation goes through in our, slightly more general, context. Note, however, that Fay uses the symbol v_p to denote the normalized holomorphic 1-form ω_p, while his expression $v_p(a)$ must be interpreted as $(\omega_p/dz_a)(a)$.

The calculation goes as follows. Fix a point in the parameter space V; then we have a degenerating family $\mathcal{C} \to D$ of curves, defined locally by an equation $XY = t$, where X, Y are co-ordinates on the smooth complex surface $\mathcal{C}$. By construction,

$$X = z_a(p_a) = t/z_b(p_b), \ Y = t/z_a(p_a) = z_b(p_b).$$

Let $h : C \to \mathcal{C}$ be the normalization of the degenerate fibre and $h_a : U^a \to \mathcal{C}$, $h_b : U^b \to \mathcal{C}$ be its restriction to the two given charts on C. Then h^a is defined by $Y = 0, 2x = X = z_a$ and h^b by $X = 0, 2x = z_b = Y$.

There are holomorphic 2-forms Ω_i, for $i = 1, \dots, g + 1$, such that if we define

$$u_i(\lambda) = \mathrm{Res}_{\mathcal{C}_\lambda} \frac{\Omega_i}{t - \lambda},$$

then $(u_i(\lambda))_{i=1,\dots,g+1}$ is a basis of the space of holomorphic 1-forms on $\mathcal{C}_\lambda$, normalized with respect to the cycles $A_1, \dots, A_{g+1}$ on $\mathcal{C}_\lambda$. For $i \leqslant g$, we have $h^* u_i(0) = \omega_i$, the normalized 1-form on C.

Define W_λ to be the Riemann surface defined in W by $t - \lambda = 0$; this equals $\pi^{-1}(\mathcal{C}_\lambda)$. Hence W_λ possesses an étale map $\pi : W_\lambda \to C_\lambda$ and an étale map $pr_1 : W_\lambda \to C$. So on W_λ there exist 1-forms $\tilde{\eta}_i(\lambda) = \pi^* u_i(\lambda) - pr_1^* \omega_i$. Switch notation from λ to t. Note that $\tilde{\eta}_i(0) = 0$. Now switch notation from λ to t and define η_i by

$$\eta_i = \lim_{t \to 0} \frac{\tilde{\eta}_i}{t}.$$

Then η_i is an intrinsic definition of $\partial u_i(t)/\partial t$.

We can expand Ω_i in terms of X, Y as

$$\Omega_i = -\phi(X, Y) dX \wedge dY = -\sum c_{m,n} X^m Y^n \, dX \wedge dY.$$

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Then

$$u_i(t) = \mathrm{Res}_{\mathcal{C}_t} \frac{\Omega_i}{XY - t} = \sum c_{n+p,n} X^{p-1} t^n \, dX = \sum c_{n+p,n} z_a^{p-1} t^n \, dz_a,$$

where the sum are over n, p with $n \geqslant 0$, $p \in \mathbb{Z}$ and $n + p \geqslant 0$. It follows that $\omega_i = u_i(0) = \sum c_{p,0} z_a^{p-1} \, dz_a$, so that $c_{0,0} = 0$ and $\omega_i = \sum_{p \geqslant 0} c_{p+1,0} z_a^p \, dz_a$ and then

$$v_i(a) = c_{1,0}.$$

By definition, η_i are then given by

$$\eta_i = \left(\sum n c_{n+p,n} z_a^{p-1} t^{n-1} \, dz_a \right) \bigg|_{t=0},$$

which gives

$$\eta_i = (c_{0,1} z_a^{-2} + c_{1,1} z_a^{-1} + \cdots) \, \mathrm{d}z_a.$$

Hence $(z_a^2 \eta_i / dz_a)(a) = c_{0,1}$ and $v_i(a) = c_{1,0}$.

An exactly similar calculation, after exchanging a with b and X with Y, gives $(z_b^2 \eta_i / dz_b)(b) = -c_{1,0}$ and $v_i(b) - c_{0,1}$. (The signs appear because computing residues in terms of Y instead of X changes the sign.) That is,

$$\frac{z_a^2 \eta_i}{dz_a}(a) = -v_i(b), \quad \frac{z_b^2 \eta_i}{dz_b}(b) = -v_i(a).$$

The formulae above show that η_i is meromorphic. Differentiating the identity $\int_{A_p} u_i = \delta_{ip}$ gives $\int_{A_p} \eta_i = 0$ for all $p \leqslant g$, and the residues of η_i vanish because the cycle A_{g+1} equals $\partial U^b \times \{t\}$, by construction, and the differentials u_q for $q \leqslant g$ are normalized, so that $\int_{A_{g+1}} u_q = 0$; differentiating this with respect to t, evaluating at $t = 0$ and then pulling back to C gives

$$2\pi i \, \mathrm{Res}_b \, \eta_q = \int_{\partial U^b} \eta_q = 0.$$

Then $\mathrm{Res}_a \, \eta_q = 0$ also, since the residues of a meromorphic 1-form sum to zero.

Now recall the bilinear relations between holomorphic forms and those of the second kind [Spr57, p. 260, Theorem 10-8]: if ϕ is a meromorphic 1-form with principal part $(\lambda_P / z_P^2) \, dz_P$ at each of its poles P (so that, in particular, ϕ has only double poles and all its residues vanish), where z_P is a local co-ordinate at P, and if ω is a holomorphic 1-form with $(\omega / dz_P)(P) = c_P$, then

$$\sum_{j=1}^{g} \left(\int_{A_j} \omega \int_{B_j} \phi - \int_{B_j} \omega \int_{A_j} \phi \right) = 2\pi i \sum_P \lambda_P c_P.$$

(Note that the left-hand side is exactly the cup product $[\omega] \cup [\phi]$ of the cohomology classes in $H^1(C, \mathbb{C})$ defined by these forms, so the bilinear relations give a formula for the cup product as a sum of local contributions.)

Take $\omega = \omega_k$ and $\phi = \eta_i$; we get

$$\int_{B_k} \eta_i = -2\pi i (v_i(a) v_k(b) + v_k(a) v_i(b)).$$

However, $\int_{B_k} \eta_i$ is exactly the entry σ_{ik} of the matrix σ appearing in the formula for $T(t)$.

Finally, the entry $T_{g+1,g+1}(t) = (1/2\pi i)(\log t + c_1 + c_2 t)$ for the reasons of monodromy that Fay gives. □

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3. The failure of transversality

Here we prove Theorem 1.3. Recall its statement.

Theorem 3.1 (the same as Theorem 1.3). *If F_{g+1} has multiplicity at least m along M_{g+1} then F_g has multiplicity at least $m+1$ along M_g.*

Proof. Suppose that $N_{g+1}(\{x_{ij}\})$ is a homogeneous polynomial of degree d in the entries x_{ij} of a symmetric $(g+1)\times(g+1)$ matrix X. Our hypothesis is that, for all $d \leqslant m-1$ and for all such N_{g+1}, the partial derivative

$$N_{g+1}(F_{g+1}) := N_{g+1}\left(\left\{\frac{\partial}{\partial T_{pq}}\right\}\right)(F_{g+1})$$

vanishes along M_{g+1} (rather, its inverse image in $\mathfrak{H}_{g+1}$) for $T=(T_{pq})\in\mathfrak{H}_{g+1}$.

Given such N_{g+1}, we let N_g denote the polynomial obtained from it by setting the bottom row and last column of X equal to zero. Our goal is to show that for every such N_g of degree m, the partial derivative $N_g(F_g)$ vanishes at every point τ in $\mathfrak{H}_g$ that comes from a curve of genus g.

For any positive integer n, let S_n denote the set of $n\times n$ integer matrices that are symmetric, positive semi-definite and whose diagonal entries are even. Then recall that every Siegel modular form $F=F_{g+1}(T)$ of degree $g+1$ over a ring R has a Fourier expansion

$$F(T) = \sum_{X\in S_{g+1}} a(X)\exp \pi i\,\mathrm{tr}(XT) = \sum_{X\in S_{g+1}} a(X)\exp\pi i\sum_{p,q=1}^{g+1} x_{pq}T_{pq}.$$

We write $X=(x_{pq})$ for $X\in S_{g+1}$. The Fourier coefficients $a(X)=a_F(X)$ lie in R. For us, $R=\mathbb{C}$.

Take T as above and take N to have degree $m-1$; then

$$\frac{1}{(\pi i)^{m-1}}N_{g+1}(F_{g+1})(T) = \sum_{X\in S_{g+1}} a(X)N_{g+1}(\{x_{pq}\})\exp\pi i\sum_{p,q=1}^{g+1} x_{pq}T_{pq}.$$

Our aim is to examine the coefficient of t in the expansion of this expression in powers of t, so calculate modulo t^2. Since $\exp 2\pi i T_{g+1,g+1} = \gamma_1\gamma_2^t t$ modulo t^2, where $\gamma_j=\exp c_j$, it follows that, modulo t^2, we can write

$$\frac{1}{(\pi i)^{m-1}}N_{g+1}(F_{g+1})(T) = \sum_{x_{g+1,g+1}=0} + \sum_{x_{g+1,g+1}=2},$$

where $\sum_{x_{g+1,g+1}=2r}$ denotes the sum over matrices $X=(x_{ij})\in S_{g+1}$ whose $(g+1,g+1)$ entry $x_{g+1,g+1}$ equals $2r$ (merely because all terms with $x_{g+1,g+1}\geqslant 4$ vanish modulo t^2). □

Lemma 3.2. *If $X\in S_{g+1}$ and $x_{g+1,g+1}=0$, then the right hand column and bottom row of X are both zero.*

Proof. This is an immediate consequence of semi-positivity. □

Therefore

$$\sum_{x_{g+1,g+1}=0} = \sum_{X\in S_g} a(X)N_g(\{x_{pq}\})\exp\pi i\sum_{p,q=1}^{g} x_{pq}(\tau_{pq}+t\sigma_{pq})$$

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and

$$\sum_{x_{g+1,g+1}=2} = t\gamma_1\gamma_2^t \sum_{X\in S_{g+1},x_{g+1,g+1}=2} a(X)N_{g+1}(\{x_{pq}\})$$
$$.\left(\exp 2\pi i \sum_{p=1}^{g} x_{p,g+1}\int_a^b \omega_p\right)\left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right)$$

since we are calculating modulo t^2. So the coefficient of t that we seek is $A+\gamma_1 B$, where

$$A = \sum_{x_{g+1,g+1}=0} a(X)N_g(\{x_{pq}\})\left(\pi i \sum_{p,q=1}^{g} x_{pq}\sigma_{pq}\right)\left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right)$$

and

$$B = \sum_{x_{g+1,g+1}=2} a(X)N_{g+1}(\{x_{pq}\})\left(\exp 2\pi i \sum_{p=1}^{g} x_{p,g+1}\int_a^b \omega_p\right)\left(\exp \pi i \sum_{p,q=1}^{g} x_{pq}\tau_{pq}\right).$$

The quantities A, B, γ_1 are holomorphic functions on V and, by assumption, $A+\gamma_1 B$ vanishes identically.

Now rescale the local co-ordinates z_a, z_b. That is, start with local co-ordinates ζ_a, ζ_b and then take $z_a = \lambda^{-1}\zeta_a$ and $z_b = \mu^{-1}\zeta_b$. Such a rescaling will produce a different family $\mathcal{C} \to \Delta$, but the quantity $A+\gamma_1 B$ will still vanish for the rescaled family. Moreover, B is invariant under this rescaling, as is revealed by a cursory inspection. Also c_1 is a holomorphic function of λ, μ because the entries of a period matrix are holomorphic functions of the parameters.

On the other hand, substituting

$$\sigma_{pq} = -2\pi i\lambda\mu\left(\frac{\omega_p}{d\zeta_a}(a)\frac{\omega_q}{d\zeta_b}(b) + \frac{\omega_q}{d\zeta_a}(a)\frac{\omega_p}{d\zeta_b}(b)\right)$$

into the expression above for A shows that A can be written as

$$A = D\lambda\mu,$$

where D are independent of λ, μ. Hence we have an identity

$$D\lambda\mu = -B\exp(c_1(\lambda,\mu))$$

of holomorphic functions on the 2-dimensional torus $\mathbb{G}_m^2 = \mathbf{Spec}\mathbb{C}[\lambda^{\pm},\mu^{\pm}]$, where we regard D, E as constants (constant as functions on $\mathbb{G}_m^2$, that is).

Lemma 3.3. *Suppose that f is a rational function on a complex algebraic variety X and that there is a holomorphic function h on some Zariski open subset U of X such that $f = \exp h$ on U. Then f is constant.*

Proof. It is enough to show that f is constant on a general curve in X. Therefore we can assume that $\dim X = 1$, and then that X is a compact Riemann surface. If f is not constant, then it has a zero, say at P, and in some neighbourhood U of P with a co-ordinate z we have $f = z^n f_1$ with f_1 holomorphic and invertible on U, and $n > 0$. Then $f_1 = \exp h_1$ with h_1 holomorphic on U, and h is holomorphic on $U - \{P\}$. Then z^n has a single-valued holomorphic logarithm on $U - \{P\}$, which is absurd. □

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Corollary 3.4. *A and B vanish identically.*

In fact, we do not exploit the vanishing of B, although it is a key step in the argument of [GS11] involving the linear system Γ_{00} of second order theta functions that vanish to order 4 at the origin and the heat equation.

Now A can also be written as

$$A = \frac{\partial}{\partial t}\bigg|_{t=0}\bigg(\sum_{X\in S_g} a(X)N_g(\{x_{pq}\})\exp \pi i \sum_{pq,=1}^{g} x_{pq}(\tau_{pq}+t\sigma_{pq})\bigg)$$
$$= \frac{\partial}{\partial t}\bigg|_{t=0} N_g(F_g(\tau+t\sigma)).$$

That is, σ lies in the Zariski tangent space H at the point τ to the divisor in $\mathfrak{H}_g$ defined by the function $N_g(F_g) = N_g(\{\partial/\partial\tau_{ij}\})(F_g)$. It is important to note that, from this description, H depends upon C but is independent of the points a, b, the local co-ordinates z_a, z_b and the scalars λ, μ.

We let M_g^0 denote the open subvariety of M_g corresponding to curves with no automorphisms and A_g^0 the open subvariety of A_g corresponding to principally polarized abelian varieties with no automorphisms except ± 1. Then M_g^0 lies in A_g^0 and both are smooth varieties, and, if C lies in M_g^0, there are natural identifications of tangent spaces given by

$$T_{[C]}M_g = H^0({\Omega^1_C}^{\otimes 2})^\vee,$$
$$T_{[C]}A_g = T_\tau\mathfrak{H}_g = \operatorname{Sym}^2 H^0(\Omega^1_C)^\vee$$

(this latter identification is also a consequence of the heat equation). The inclusion $T_{[C]}M_g \hookrightarrow T_{[C]}A_g$ is dual to the natural multiplication (which is surjective, by Max Noether's theorem) $\operatorname{Sym}^2 H^0(\Omega^1_C) \to H^0({\Omega^1_C}^{\otimes 2})$.

We are aiming to prove that H, when regarded as a Zariski tangent space, is the whole of the tangent space $T_\tau\mathfrak{H}_g = \operatorname{Sym}^2 H^0(C,\Omega^1)^\vee$. So assume otherwise; then H is a hyperplane. Projectivize; then $\sigma \in \mathbb{P}(H)$ and $\mathbb{P}(H)$ is a hyperplane in $\mathbb{P}(\operatorname{Sym}^2 H^0(C,\Omega^1)^\vee)$.

Now comes the point at which information about abelian integrals is transformed into projective geometry and thence moduli.

The symmetric square $\operatorname{Sym}^2 C$ is embedded in $\mathbb{P}(\operatorname{Sym}^2 H^0(C,\Omega^1_C)^\vee)$ via the identification $\operatorname{Sym}^2 H^0(C,\Omega^1_C) = H^0(\operatorname{Sym}^2 C, {\Omega^1_C}^{\boxtimes 2})$, where, by abuse of notation, ${\Omega^1_C}^{\boxtimes 2}$ denotes the line bundle on $\operatorname{Sym}^2 C$ obtained by symmetrizing the exterior tensor square ${\Omega^1_C}^{\boxtimes 2}$ on $C\times C$. The entries σ_{pq} of the matrix σ are obtained by taking a basis of $H^0(\operatorname{Sym}^2 C, {\Omega^1_C}^{\boxtimes 2})$ and evaluating at the point $\{a,b\}$ of $\operatorname{Sym}^2 C$. It follows, since H is independent of the points a and b, that the putative hyperplane $\mathbb{P}(H)$ contains the embedded $\operatorname{Sym}^2 C$. However, $\operatorname{Sym}^2 C$ is non-degenerate in $\mathbb{P}(\operatorname{Sym}^2 H^0(C,\Omega^1)^\vee)$ and therefore H does not exist.

Theorem 1.1 is an immediate corollary of this and the following lemma in commutative algebra.

Lemma 3.5. *Suppose that X is a closed subvariety of the variety Y defined by the ideal $I = I_{X/Y}$. Suppose that W is a smooth open subvariety of Y such that $W\cap X$ is smooth and non-empty and that J is an ideal of $\mathcal{O}_Y$ such that $J|_W = I^n|_W$. Then J is contained in $I^{[n]}$, the nth symbolic power of I.*

Proof. First, recall that if X and Y are smooth over a field of characteristic zero, then $I^n = I^{[n]}$ and consists of the functions f on Y all of whose derivatives, with respect to local co-ordinates on Y, of order up to and including the $(n-1)$st, vanish along X.

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We can assume that Y is affine, say $Y = \mathbf{Spec}\, A$, so that A is an integral domain and I is prime. For any ideal $\mathfrak{a}$ of A, write $V(\mathfrak{a}) = \mathbf{Spec}(A/\mathfrak{a})$.

We can increase J, provided that $J|_W$ is unchanged, so that in particular we can replace J by $J + I^{[n]}$. Then, without loss of generality, we can suppose that J contains $I^{[n]}$ and must prove that $J = I^{[n]}$. We have $V(J)_{\mathrm{red}} \subset V(I^{[n]})_{\mathrm{red}} = X$ and $V(J)_{\mathrm{red}} \cap W = X \cap W$, so that $V(J)_{\mathrm{red}} = X$, and therefore $\sqrt{J} = I$.

Recall that for any ideal $\mathfrak{a}$ with $\sqrt{\mathfrak{a}} = I$, there is a unique smallest I-primary ideal $\widetilde{\mathfrak{a}}$ containing $\mathfrak{a}$, given by the formula $\widetilde{\mathfrak{a}} = A \cap \mathfrak{a} \cdot A_I$, where A_I is the localization of A at the prime ideal I. As before, we can increase J, and so assume that $J = \widetilde{J}$, that is, that J is I-primary. The symbolic power $I^{[n]}$ is $I^{[n]} = \widetilde{I^n}$.

By assumption, the generic point ξ of X lies in W and $A_I = \mathcal{O}_{Y,\xi}$, so that $J \cdot A_I = I^n \cdot A_I$. Intersecting both sides of this equation with A gives $J = \widetilde{J} = I^{[n]}$. $\square$

Now regard the Satake compactifications A_g^S and M_{g+m}^S as closed subvarieties of A_{g+m}^S.

THEOREM 3.6 (the same as Theorem 1.1). *The intersection $A_g^S \cap M_{g+m}^S$ contains the mth order infinitesimal neighbourhood of M_g^S in A_g^S.*

Proof. The ideal defining M_{g+m}^S inside A_{g+m}^S is generated by those Siegel modular forms F_{g+m} that vanish along M_{g+m}^S. From Theorem 1.3 and induction on m it follows that F_g and all its partial derivatives with respect to the co-ordinates τ_{pq} on $\mathfrak{H}_g$ of orders at most m vanish along M_g, which is just the statement of the corollary. $\square$

Remark. For $m = 1$ this says that at a general point $[C]$ of M_g, the Zariski tangent space at $[C]$ to the $3g$-dimensional variety M_{g+1}^S contains the $g(g+1)/2$-dimensional tangent space $\mathrm{Sym}^2 H^0(C, \Omega_C^1)^\vee$ at $[C]$ to A_g, where these tangent spaces both lie in $T_{[C]} A_{g+1}^S$.

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