KAZHDAN–LUSZTIG POLYNOMIALS, TIGHT QUOTIENTS AND DYCK SUPERPARTITIONS

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Abstract. We give an explicit combinatorial product formula for the parabolic Kazhdan-Lusztig polynomials of the tight quotients of the symmetric group. This formula shows that these polynomials are always either zero or a monic power of q and implies the main result in [Pacific J. Math., 207 (2002), 257-286] on the parabolic Kazhdan-Lusztig polynomials of the maximal quotients. The formula depends on a new class of superpartitions.

1. Introduction

In 1979, Kazhdan and Lusztig [16] introduced a family of polynomials, indexed by pairs of elements in a Coxeter group W, which play an important role in various areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [1, p. 171] and the references cited there). These celebrated polynomials are now known as the Kazhdan–Lusztig polynomials of W (see, e.g., [1] or [13]).

In 1987, Deodhar [7] developed an analogous theory for the parabolic setup. Given any parabolic subgroup W_J in a Coxeter system (W, S), Deodhar introduced two Hecke algebra modules (one for each of the two roots q and -1 of the polynomial $x^2 - (q-1)x - q$) and two families of polynomials $\{P^{J,q}_{u,v}(q)\}_{u,v \in W^J}$ and $\{P^{J,-1}_{u,v}(q)\}_{u,v \in W^J}$ indexed by pairs of elements in the set of minimal coset representatives W^J. These polynomials are the parabolic analogues of the Kazhdan–Lusztig polynomials: while they are related to their ordinary counterparts in several ways (see, e.g., equation (1) and [7, Proposition 3.5]), they also play a direct role in several areas such as the geometry of partial flag manifolds [15], the theory of Macdonald polynomials [11], [12], tilting modules [25], [26], generalized Verma modules [4], canonical bases [10], [30], the representation theory of the Lie algebra $\mathfrak{gl}_n$ [18], and quantized Schur algebras [31].

The purpose of this work is to study the parabolic Kazhdan–Lusztig polynomials for the tight quotients of the symmetric group. The tight quotients have been introduced by Stembridge in [29] who classified them for finite Coxeter groups [29, Theorem 3.8]. For the symmetric group, the non-trivial tight quotients are obtained by taking either $J = [n-1] \setminus \{i\}$, $i \in [n-1]$ (*maximal quotients*), or $J = [n-1] \setminus \{i-1, i\}$, $i \in [2, n-1]$. The parabolic Kazhdan–Lusztig polynomials for the maximal quotients have been studied in [2]. In this paper we complete the

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study for all tight quotients giving an explicit closed combinatorial formula for the parabolic Kazhdan–Lusztig polynomials of the quotients W^J, with $J = [n-1] \setminus \{i-1, i\}$, $i \in [2, n-1]$. The formula, which implies that these polynomials are always either zero or a monic monomial, can be used to give another proof of the formula found in [2] for the maximal quotients, and involves a new class of (possibly skew) superpartitions, which we call Dyck. With every $v \in W^J$, $J = [n-1] \setminus \{i-1, i\}$, $i \in [2, n-1]$, we associate a superpartition (with fermionic degree equal to 1) and show that the parabolic Kazhdan–Lusztig polynomial associated to $u, v \in W^J$ is encoded in the pair of superpartitions associated to u and v. More precisely, the polynomial $P^{J,q}_{u,v}(q)$ is non-zero if and only if the two superpartitions form a Dyck skew superpartition (see section 4 for the definition) and, in this case, it is a power of q whose exponent is an explicit statistic of the Dyck skew superpartition.

Although superpartitions can be traced back to MacMahon diagrams [20], it is expecially in recent years that they attracted much attention, since they have been shown to arise in several contexts including mathematical physics, q-series, symmetric functions and combinatorics. Superpartitions (or strictly related concepts) have been extensively studied, sometimes under different names such as dotted partitions, joint partitions, colored partitions, jagged partitions, and overpartitions (see, for example, [5], [6], [9], [19] and references cited in these papers). This work provides a Lie theoretic application of the concept of superpartition (as asked for in [5]).

The organization of the paper is as follows. In §2, we recall some definitions, notation, and results that are used in the sequel. In §3, we explain the connection between the tight quotients of the symmetric groups and superpartitions with fermionic degree 1. In §4, we introduce and study the main new combinatorial concept of this work, namely Dyck superpartitions, which plays a fundamental role in the main result. In §5, using the results in the two previous sections, we prove our main result (Theorem 5.1) and derive some consequences of it including the formula for the maximal quotients found in [2] and new identities for the ordinary Kazhdan–Lusztig polynomials and for their leading terms.

2. Preliminaries

We let $\mathbf{P} = \{1, 2, \dots\}$ and $\mathbf{N} = \mathbf{P} \cup \{0\}$. Given $n, m \in \mathbf{N}$, with $n \leqslant m$, we let $[n, m] = \{n, n+1, \dots, m\}$ and for $n \in \mathbf{P}$ we let $[n] = [1, n]$. For a set T we let $S(T)$ be the set of all bijections $\pi \colon T \to T$, and $S_n = S([n])$. If $\sigma \in S_n$ then we denote σ by the word $\sigma(1)\sigma(2)\dots\sigma(n)$. If $\sigma \in S_n$ then we also write σ in disjoint cycle form omitting to write the 1-cycles of σ. For example, if $\sigma = 365492187$ then we also write $\sigma = (1, 3, 5, 9, 7)(2, 6)$. Given $\sigma, \tau \in S_n$ we let $\sigma\tau = \sigma \circ \tau$ (composition of functions) so that, for example, $(1, 2)(2, 3) = (1, 2, 3)$. Recall (see, e.g., [27, p. 21]) that, for $v \in S_n$, the *inversion table* of v is the sequence $(v_1, \dots, v_n)$ with

$$v_h = |\{k \in [n] : k > h, v^{-1}(k) < v^{-1}(h)\}|$$

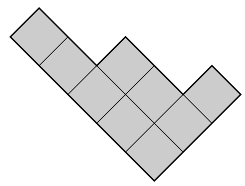
for all $h \in [n]$. The permutation v is uniquely determined by its inversion table: in fact it gives a bijection between S_n and $[0, n-1] \times [0, n-2] \times \cdots \times [0, 0]$.

We follow [27, Chapter 3] for poset notation and terminology.

By an (integer) *partition* we mean a sequence of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. We call the nonzero λ_i the *parts* of λ. We identify a partition λ with its *diagram*

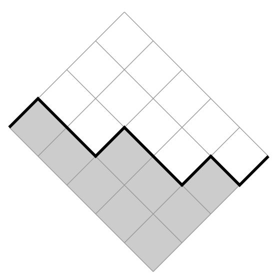
$$\{(i,j) \in \mathbf{P}^2 : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant \lambda_i\}.$$

We find convenient to draw the diagram of a partition λ rotated counterclockwise by 45 degrees with respect to the French convention (this is sometimes called the Russian convention). So, for example, the diagram of $(3,2,2,1,1)$ is the following:



We call the elements of $\mathbf{P}^2$, and hence of λ, *cells*. Expressions such as "to the left of", or "directly above", always refer to these rotated diagrams. The *level* of a cell $(i,j) \in \mathbf{P}^2$ is $\mathrm{lv}((i,j)) = i + j$. We denote by $\mathcal{P}$ the set of all integer partitions. It is well known, and not hard to see, that $\mathcal{P}$, partially ordered by set inclusion, is a lattice, usually called *Young's lattice* (see, e.g., [28, §7.2]).

Given a rectangular partition (n^m), we identify any partition $\lambda \subseteq (n^m)$ with the lattice path obtained by following the upper boundary of its diagram, which consists of n up $((1,1))$ steps and m down $((1,-1))$ steps. We denote such a path with a UD-word (that is, a word in the alphabet $\{\mathrm{U}, \mathrm{D}\}$) with $n+m$ letters, n of which are U. For example, given $(3,2,2,1,1) \subseteq (4^5)$, we have

$(3,2,2,1,1) =$  $=$ UDDUDDUDU.

Let $\lambda \subseteq (n^m)$ and let $a_1 a_2 \ldots a_{n+m}$ be the associated UD-word. We say that $j \in [n+m-1]$ is a *peak* (resp. *valley*) of λ if $(a_j, a_{j+1}) = (\mathrm{U}, \mathrm{D})$ (resp. $(a_j, a_{j+1}) = (\mathrm{D}, \mathrm{U})$). For any $j \in [n+m]$, we denote $d_\lambda(j) = |\{k \in [j] : a_k = \mathrm{U}\}|$.

Let λ, μ be partitions, with $\mu \subseteq \lambda$. Then $\lambda \setminus \mu$ is called a *skew partition*. Given a (skew) partition η, we denote by $|\eta|$ the number of its cells. The *conjugate* of η is $\eta' = \{(j,i) \in \mathbf{P}^2 : (i,j) \in \eta\}$. A (skew) partition η is *self-conjugate* if $\eta' = \eta$. A skew partition η is *connected* if it is "rookwise connected", so that, for instance, $(2,1) \setminus (1)$ is not connected. We say that a skew partition is a *border strip* if it contains no 2×2 square of cells. For brevity, we call a connected border strip a *cbs*. Given a skew partition $\lambda \setminus \mu$, the *outer border strip* θ of $\lambda \setminus \mu$ is the set of all cells of $\lambda \setminus \mu$ such that there is no cell of $\lambda \setminus \mu$ directly above it (see Figure 1: the outer border strip of the skew partition is shaded in dark grey). Given a skew partition $\lambda \setminus \mu \subseteq (n^m)$, for any $j \in [n+m]$, we denote $d_{\lambda \setminus \mu}(j) = d_\lambda(j) - d_\mu(j)$

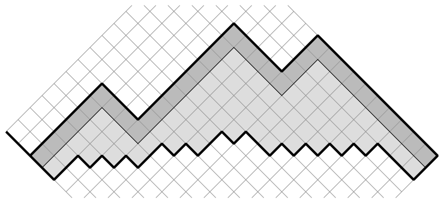


Figure 1. Outer border strip.

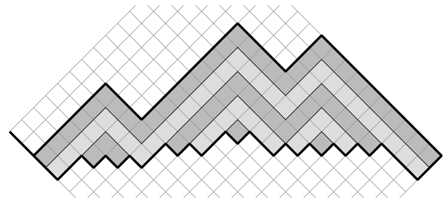


Figure 2. Dyck skew partition.

(the "thickness" of $\lambda \setminus \mu$ at j). Note that this definition of $d_{\lambda\setminus\mu}$ is equivalent to the one given in [2], right before Lemma 4.2. We follow [28, §7.2] for any undefined notation and terminology concerning partitions.

We now recall some notions introduced in [2]. A cbs θ is *Dyck* if it is a "Dyck path" (see, e.g., [28, p. 173]), which means that no cell of θ has level strictly less than that of the leftmost or the rightmost of its cells. In particular, in a Dyck cbs the leftmost and rightmost cells have the same level. A skew partition is defined to be *Dyck* in the following inductive way:

(1) if $\lambda\setminus\mu$ is not connected, then $\lambda\setminus\mu$ is *Dyck* if all of its connected components are Dyck;
(2) if $\lambda \setminus \mu$ is connected and non-empty, then $\lambda \setminus \mu$ is *Dyck* if the outer border strip θ of $\lambda \setminus \mu$ is a Dyck cbs and $(\lambda \setminus \mu) \setminus \theta$ is Dyck;
(3) the empty partition is *Dyck*.

If $\lambda \setminus \mu$ is a Dyck skew partition, the *depth* of $\lambda \setminus \mu$, denoted by $\mathrm{dp}(\lambda \setminus \mu)$, is defined in the following way:

(1) if $\lambda \setminus \mu$ is not connected and $\eta_1, \dots, \eta_k$ are its connected components, then
$$\mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}(\eta_1) + \cdots + \mathrm{dp}(\eta_k);$$
(2) if $\lambda \setminus \mu$ is connected and non-empty and θ is its outer border strip, then
$$\mathrm{dp}(\lambda \setminus \mu) = 1 + \mathrm{dp}((\lambda \setminus \mu) \setminus \theta);$$
(3) $\mathrm{dp}(\emptyset) = 0$.

For example, the skew partition $\lambda \setminus \mu$ in Figure 2 is Dyck, with $\mathrm{dp}(\lambda \setminus \mu) = 8$.

Let λ be a partition. If x is a peak or a valley of λ, we denote by $\hat{x}$ the cell immediately below x or above x, respectively. Then we set

$$\lambda^x = \begin{cases} \lambda \setminus \{\hat{x}\}, & \text{if } x \text{ is a peak of } \lambda, \\ \lambda \cup \{\hat{x}\}, & \text{if } x \text{ is a valley of } \lambda, \end{cases}$$

The operator $(\,\cdot\,)^x$ is clearly an involution.

We now recall two results of [2] that we will use in §3 (note that here x is a positive integer, whereas in [2] x and y are cells).

Proposition 2.1 ([2, Proposition 4.1]). *Let $\lambda \setminus \mu$ be a Dyck skew partition and let x be a peak of λ. Then x is either a peak or a valley of μ.*

Theorem 2.2 ([2, Theorem 4.3])**.** *Let $\lambda \setminus \mu$ be a skew partition and let x be a peak of both λ and μ. Then, the following are equivalent*:

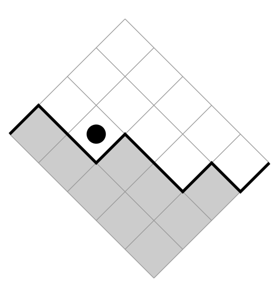
(1) $\lambda \setminus \mu$ *is Dyck*;
(2) $\lambda \setminus \mu^x$ *is Dyck*;
(3) *at least one of* $\lambda^x \setminus \mu$ *and* $\lambda^x \setminus \mu^x$ *is Dyck*;
(4) *exactly one of* $\lambda^x \setminus \mu$ *and* $\lambda^x \setminus \mu^x$ *is Dyck.*

Furthermore, if these conditions are satisfied, then

$$\begin{cases} \mathrm{dp}(\lambda \setminus \mu^x) - \mathrm{dp}(\lambda \setminus \mu) = 1, & \\ \mathrm{dp}(\lambda^x \setminus \mu) - \mathrm{dp}(\lambda \setminus \mu) = 1, & \text{if } \lambda^x \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}(\lambda^x \setminus \mu^x) - \mathrm{dp}(\lambda \setminus \mu) = 0, & \text{if } \lambda^x \setminus \mu^x \text{ is Dyck.} \end{cases}$$

A superpartition with fermionic degree f is a pair of partitions $(\tilde{\eta};\eta)$ such that $\tilde{\eta}$ is a partition with f distinct parts. With a superpartition, we can associate a partition with circled parts: circle the parts of $\tilde{\eta}$ and rearrange them together with the parts of η in order to obtain a single partition whose circled parts come first. For example, $((2,1);(3,2,2,1,1)) = (3,2,2,2,1,1,1)$. In the diagrammatic representation, we put a circle in correspondence of every circled part of the superpartition.

For brevity, we call *f-superpartitions* the superpartitions with fermionic degree equal to f, that is with exactly f circled parts. If $(\tilde{\eta};\eta)$ is an f-superpartition and $\mu \in \mathcal{P}$ then we say that $(\tilde{\eta};\eta)$ is *contained* in μ, and write $(\tilde{\eta};\eta) \subseteq \mu$, to mean that the partition λ obtained by rearranging the parts of η and $\tilde{\eta}$ is contained in μ. In this paper we will be concerned with 1-superpartitions $(\tilde{\eta};\eta)$ contained in a fixed rectangular partition (n^m). It will be convenient for us to think of the 1-superpartition $(\tilde{\eta};\eta) \subseteq (n^m)$ as the pair (λ, r), where r is the valley of λ corresponding to the circled part; we call r the *circle* of the 1-superpartition. Equivalently, we will consider the *dotted UD-word* associated with (λ, r), which is the UD-word associated with λ with an extra "$\bullet$" character inserted between the D and the U corresponding to r (that is, inserted after the r-th letter). For example, if $n = 4$, $m = 5$, $\tilde{\eta} = (1)$ and $\eta = (3,2,2,1)$, we identify $(\tilde{\eta};\eta) \subseteq (4^5)$ both with $((3,2,2,1,1),3)$ and UDD $\bullet$ UDDUDU.

$((1);(3,2,2,1)) =$  $= ((3,2,2,1,1),3) = \text{UDD} \bullet \text{UDDUDU}.$

Note that a 1-superpartitions contained in (n^m) can also be thought of as a cover relation in the interval $[\emptyset, (n^m)]$ of Young's lattice.

We follow [13] for general Coxeter group notation and terminology. In particular, given a Coxeter system (W, S) and $u \in W$ we denote by $\ell(u)$ the length of u in

W, with respect to S, and we let $D(u) = \{s \in S : \ \ell(us) < \ell(u)\}$ be the set of (right) descents of u. For $u, v \in W$ we let $\ell(u,v) = \ell(v) - \ell(u)$. We denote by e the identity of W, and we let $T = \{usu^{-1} : u \in W, s \in S\}$. Given $J \subseteq S$, we let W_J be the parabolic subgroup generated by J and

$$W^J = \{u \in W : \ell(su) > \ell(u) \text{ for all } s \in J\}$$

be the corresponding quotient. Note that $W^{\emptyset} = W$. If W_J is finite then we denote by $w_0(J)$ its longest element. The Coxeter group W is partially ordered by Bruhat order. Recall (see, e.g., [13, §5.9]) that this means that $x \leqslant y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \cdots t_1 \, x = y$ and $\ell(t_i \cdots t_1 \, x) > \ell(t_{i-1} \cdots t_1 x)$ for $i \in [r]$. The parabolic subgroup W_J and the quotient W^J have the induced order.

For $J \subseteq S$, $x \in \{-1, q\}$, and $u, v \in W^J$, we denote by $P^{J,x}_{u,v}(q)$ the *parabolic Kazhdan–Lusztig polynomials* of W^J of type x (we refer the reader to [7] for the definitions of these polynomials; see also below). It follows immediately from [7, §2 and §3] and from well-known facts (see, e.g., [13, §7.5] and [13, §§7.9–7.11]) that $P^{\emptyset,-1}_{u,v}(q) = P^{\emptyset,q}_{u,v}(q) = P_{u,v}(q)$, where $P_{u,v}(q)$ are the (*ordinary*) *Kazhdan–Lusztig polynomials* of W [16].

The following result is due to Deodhar, and we refer the reader to [7] for its proof. For $u, v \in W^J$ let $\mu_{J,q}(u,v)$ be the coefficient of $q^{(\ell(u,v)-1)/2}$ in $P^{J,q}_{u,v}(q)$ (so $\mu_{J,q}(u,v) = 0$ if $\ell(u,v)$ is even). We will often write $\mu(u,v)$ instead of $\mu_{J,q}(u,v)$ if there is no danger of confusion.

Proposition 2.3. *Let (W,S) be a Coxeter system, $J \subseteq S$, and $u, v \in W^J$, with $u \leqslant v$. Then, for each $s \in D(v)$, we have that*

$$P^{J,q}_{u,v}(q) = \widetilde{P}(q) - \sum_{\substack{w \in W^J \\ u \leqslant w < vs \\ ws < w}} \mu(w, vs) q^{\ell(w,v)/2} P^{J,q}_{u,w}(q),$$

where

$$\widetilde{P}(q) = \begin{cases} P^{J,q}_{us,vs}(q) + q P^{J,q}_{u,vs}(q), & \text{if } us < u, \\ q P^{J,q}_{us,vs}(q) + P^{J,q}_{u,vs}(q), & \text{if } u < us \in W^J, \\ 0, & \text{if } u < us \notin W^J. \end{cases}$$

Remark 2.4 ([23]). In the hypotheses of Proposition 2.3, if $u < us \notin W^J$, then all the $P^{J,q}_{u,w}(q)$ in the sum are zero, so $P^{J,q}_{u,v}(q) = 0$.

The next result follow easily from [7, Remark 3.8] and well-known properties of quotients of Coxeter groups (see, e.g., [1, Proposition 2.4.4]).

Proposition 2.5. *Let (W,S) be a Coxeter system, $I \subseteq J \subseteq S$ and $u, v \in W^J$, with $u \leqslant v$. Then*

$$P^{J,q}_{u,v}(q) = \sum_{w \in (W_J)^I} (-1)^{\ell(w)} P^{I,q}_{wu,v}(q).$$

Hence, knowledge of $\{P^{I,q}_{u,v}(q)\}_{u,v \in W^I}$ for some $I \subseteq S$ implies knowledge of $\{P^{J,q}_{u,v}(q)\}_{u,v \in W^J}$ for all $J \supseteq I$. Also, for $I = \emptyset$, we get

$$P^{J,q}_{u,v}(q) = \sum_{w \in W_J} (-1)^{\ell(w)} P_{wu,v}(q), \tag{1}$$

therefore parabolic Kazhdan–Lusztig polynomials of type q are alternating sums of ordinary Kazhdan–Lusztig polynomials. On the other hand, it is known (see Propositions 2.12 and 3.4, and Remark 3.8 in [7]) that if W_J is finite then

$$P^{J,-1}_{u,v}(q) = P_{w_0(J)u,w_0(J)v}(q). \tag{2}$$

Another relation between parabolic and ordinary Kazhdan–Lusztig polynomials is given in [7, Proposition 3.5].

It is well known that the symmetric group S_n is a Coxeter group with respect to the generating set $S = \{s_1, \dots, s_{n-1}\}$, where $s_i = (i, i+1)$ for all $i \in [n-1]$. If there is no danger of confusion we will denote s_i simply by i (so $S = [n-1]$). The following result is also well known (see, e.g., [1, Section 1.5]).

Proposition 2.6. *Let $v \in S_n$. Then*

$$\ell(v) = |\{(i,j) \in [n]^2 : i < j, v(i) > v(j)\}|$$

and

$$D(v) = \{i \in [n-1] : v(i) > v(i+1)\}.$$

Our purpose in this paper is to study the parabolic Kazhdan–Lusztig polynomials for the *tight quotients* of the symmetric group. These quotients have been introduced by Stembridge in [29] who classified them for finite Coxeter groups [29, Theorem 3.8]. For the symmetric group, the non-trivial tight quotients are obtained by taking either $J = [n-1] \setminus \{i\}$, $i \in [n-1]$ (*maximal quotients*), or $J = [n-1] \setminus \{i-1, i\}$, $i \in [2, n-1]$. The parabolic Kazhdan–Lusztig polynomials for the maximal quotients have been studied in [2]. In this paper we complete the study for all tight quotients, generalizing the results in [2].

3. Tight quotients and superpartitions

In this section, we explain the connection between the tight quotients of the symmetric groups and superpartitions with fermionic degree equal to 1.

Fix $n \in \mathbf{P}$ and $i \in [2, n-1]$ and note that, by Proposition 2.6, we have

$$S_n^{[n-1]\setminus\{i-1,i\}} = \{v \in S_n : v^{-1}(1) < \cdots < v^{-1}(i-1) \\ \text{and } v^{-1}(i+1) < \cdots < v^{-1}(n)\}.$$

Therefore $v \in S_n^{[n-1]\setminus\{i-1,i\}}$ if and only if its inversion table $(v_1, \dots, v_n)$ satisfies the following conditions:

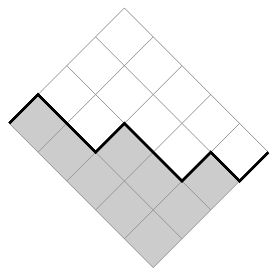
(1) $v_j = v^{-1}(j) - j$ for all $j \in [i-1]$;
(2) $v_j = 0$ for all $j \in [i+1, n]$.

Condition (1) can be replaced by the requirement that $(v_{i-1}, \dots, v_1)$ be a partition (note that, in this case, $((v_i); (v_{i-1}, \dots, v_1))$ is a 1-superpartition written as a pair of partitions: as said in the previous section, it is more convenient for us to use the other notation).

Given $v \in S_n^{[n-1]\setminus\{i-1,i\}}$, we set $\Lambda(v) = (v_{i-1}, \dots, v_k, v_i, v_{k-1}, \dots, v_1)$, where $k = \min\{j \in [n] : v_j > v_i\}$ (and $k = i$ if $\{j \in [n] : v_j > v_i\} = \emptyset$). In other words, $\Lambda(v)$ is the partition which is obtained from $(v_{i-1}, \dots, v_1)$ by inserting the term v_i in the leftmost suitable position (that is, the partition associated with the superpartition $((v_i);(v_{i-1},\dots,v_1))$). Note that $\Lambda(v) \subseteq ((n-i+1)^i)$.

For example, if $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$, we have $\Lambda(v) \subseteq (4^5)$ and

$\Lambda(v) = (3,2,2,1,1) =$



$= \mathrm{UDDUDDUDU}.$

It is easy to construct directly the UD-word associated with v.

Proposition 3.1. *Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then $\Lambda(v) = a_1a_2\dots a_n$, where*

$$a_h = \begin{cases} \mathrm{D}, & \text{if } v(h) < i, \\ \mathrm{U}, & \text{if } v(h) > i, \\ \mathrm{DU}, & \text{if } v(h) = i, \end{cases}$$

for all $h \in [n]$.

Proof. Let $\Lambda(v) = (v_{i-1},\dots,v_k,v_i,v_{k-1},\dots,v_1)$, as above. Then the UD-word associated with $\Lambda(v)$ is

$$\mathrm{U}^{v_1}\mathrm{D}\mathrm{U}^{v_2-v_1}\mathrm{D}\dots\mathrm{D}\mathrm{U}^{v_i-v_{k-1}}\mathrm{D}\mathrm{U}^{v_k-v_i}\mathrm{D}\dots\mathrm{D}\mathrm{U}^{v_{i-1}-v_{i-2}}\mathrm{D}\mathrm{U}^{n-i+1-v_{i-1}}.$$

On the other hand, note that $v_j - v_{j-1} = v^{-1}(j) - v^{-1}(j-1) - 1$, for all $j \in [i-1]$ (where we set $v_0 = v^{-1}(0) = 0$), and that $v_i - v_{k-1} = v^{-1}(i) - v^{-1}(k-1) - 1$ and $v_k - v_i = v^{-1}(k) - v^{-1}(i)$. The result follows. □

The following result establishes the link between the tight quotient $S_n^{[n-1]\setminus\{i-1,i\}}$ and the 1-superpartitions contained in $((n-i+1)^i)$.

Proposition 3.2. *The map $v \mapsto (\Lambda(v), v^{-1}(i))$ is a bijection between $S_n^{[n-1]\setminus\{i-1,i\}}$ and the set of 1-superpartitions contained in $((n-i+1)^i)$. Moreover, $\ell(v) = |\Lambda(v)|$.*

Proof. We have already observed that $\Lambda(v) \subseteq ((n-i+1)^i)$. Also, by Proposition 3.1, we have that $v^{-1}(i)$ is a valley of $\Lambda(v)$.

Conversely, given a 1-superpartition (λ, r), with $\lambda = (\lambda_i,\dots,\lambda_1) \subseteq ((n-i+1)^i)$, we construct a permutation $v \in S_n^{[n-1]\setminus\{i-1,i\}}$ as follows. Let λ_k be the circled part of λ (so that $r = \lambda_k + k$ and $\lambda_k < \lambda_{k+1}$). Then we set

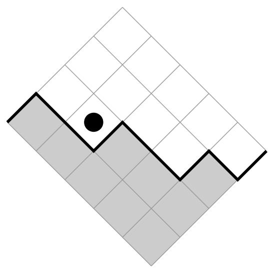
$$v^{-1}(j) = \begin{cases} \lambda_j + j, & \text{if } j \in [k-1], \\ \lambda_{j+1} + j, & \text{if } j \in [k, i-1], \\ r, & \text{if } j = i. \end{cases}$$

This uniquely determines $v \in S_n^{[n-1]\setminus\{i-1,i\}}$. It is easy to check that the two maps just defined are inverses of each other, so they are bijections. By Proposition 2.6, we have that $\ell(v) = |\Lambda(v)|$, since $v_j = 0$ for all $j \in [i+1,n]$. □

As a consequence of Proposition 3.2, we have that the number of 1-superpartitions contained in $((n-i+1)^i)$ is $i\binom{n}{i}$.

Given $v \in S_n^{[n-1]\setminus\{i-1,i\}}$, we let $\Lambda^\bullet(v) = (\Lambda(v), r(v))$ be the 1-superpartition contained in $((n-i+1)^i)$ associated with v, where $r(v) = v^{-1}(i)$ is its circle. For example, if $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$, then we have

$\Lambda^\bullet(v) = ((3,2,2,1,1),3) =$



$= \mathrm{UDD} \bullet \mathrm{UDDUDU}.$

The following result is a refinement of Proposition 3.1, and shows how to construct directly the dotted UD-word associated with v.

Proposition 3.3. *Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then $\Lambda^\bullet(v) = a_1 a_2 \dots a_n$, where*

$$a_h = \begin{cases} \mathrm{D}, & \text{if } v(h) < i, \\ \mathrm{U}, & \text{if } v(h) > i, \\ \mathrm{D}\bullet\mathrm{U}, & \text{if } v(h) = i, \end{cases}$$

for all $h \in [n]$.

Proof. Let $\Lambda(v) = (v_1, \dots, v_{k-1}, v_i, v_k, \dots, v_{i-1})$. Then, the dotted UD-word $\Lambda^\bullet(v)$ associated with v is

$$\mathrm{U}^{v_1}\mathrm{D}\mathrm{U}^{v_2-v_1}\mathrm{D}\dots\mathrm{D}\mathrm{U}^{v_i-v_{k-1}}\mathrm{D}\bullet\mathrm{U}^{v_k-v_i}\mathrm{D}\dots\mathrm{D}\mathrm{U}^{v_{i-1}-v_{i-2}}\mathrm{D}\mathrm{U}^{(n-i+1)-v_{i-1}}$$

and the result follows as in the proof of Proposition 3.1. □

The next result shows that the descents of v correspond to the peaks of $\Lambda(v)$, and describes the effect of multiplying v by a generator $s_j = (j, j+1)$. Given $v \in S_n^{[n-1]\setminus\{i-1,i\}}$ and $j \in [n-1]$, we set

$$p_v(j) = \begin{cases} j, & \text{if } j < r(v), \\ j+1, & \text{if } j \geqslant r(v). \end{cases} \tag{3}$$

Proposition 3.4. *Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$, $\Lambda^\bullet(v) = (\lambda, r)$, and $j \in [n-1]$. Then*

(1) *$vs_j < v$ if and only if $p_v(j)$ is a peak of λ; in this case*

$$\Lambda^\bullet(vs_j) = \begin{cases} (\lambda^{p_v(j)}, r), & \text{if } |p_v(j) - r| > 1, \\ (\lambda^{p_v(j)}, p_v(j)), & \text{if } |p_v(j) - r| = 1; \end{cases}$$

(2) *$v < vs_j \in S_n^{[n-1]\setminus\{i-1,i\}}$ if and only if $p_v(j)$ is a valley of λ^r; in this case*

$$\Lambda^\bullet(vs_j) = \begin{cases} (\lambda^{p_v(j)}, r), & \text{if } |p_v(j) - r| > 1, \\ (\lambda^r, p_v(j)), & \text{if } |p_v(j) - r| = 1. \end{cases}$$

Proof. This follows from Proposition 3.3, by distinguishing the eight possible cases depending on whether $v(j)$ and $v(j+1)$ are greater than, equal to, or smaller than i. For example, if $v(j) < i = v(j+1)$, then $v < vs_j \in S_n^{[n-1]\setminus\{i-1,i\}}$, $p_v(j)$ is a valley of λ^r and $\Lambda^\bullet(vs_j) = (\lambda^r, p_v(j))$. □

We illustrate Proposition 3.4 with an example. Let $v = 61523748 \in S_8^{[7]\setminus\{4,5\}}$ and $j = 3$. Then $\Lambda^\bullet(v) = ((3,2,2,1,1),3)$, $p_v(3) = 4$, $vs_3 = 61253748 \in S_8^{[7]\setminus\{4,5\}}$ and $\Lambda^\bullet(vs_3) = ((3,2,1,1,1),4)$.

We now give a description of the Bruhat order on $S_n^{[n-1]\setminus\{i-1,i\}}$ in terms of 1-superpartitions.

Let (λ, r) and (μ, t) be two 1-superpartitions. We say that (λ, r) is obtained from (μ, t) by an *elementary move* if $\lambda = \mu^x$, for some valley x of μ, and λ has no peaks strictly between r and t. In other words, if $x \neq t$ then $r = t$, while if $x = t$ then r is either the rightmost valley of λ to the left of x or the leftmost valley of λ to the right of x. We may think of an elementary move as changing a valley of μ into a peak and letting the circle of μ "slide down", as if it were subject to gravity.

The following result characterizes the cover relations of the Bruhat order on $S_n^{[n-1]\setminus\{i-1,i\}}$. We say that a cover relation $u \lhd v$ is *weak* if $v = us_j$ for some $s_j = (j, j+1)$.

Proposition 3.5. *Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then v covers u if and only if $\Lambda^\bullet(v)$ is obtained from $\Lambda^\bullet(u)$ by an elementary move. Moreover, the cover relation is weak if and only if $|r(v) - r(u)| \leqslant 1$.*

Proof. Let $\Lambda^\bullet(v) = (\lambda, r)$ and $\Lambda^\bullet(u) = (\mu, t)$. By [1, Lemma 2.1.4], we have that v covers u if and only if $v = u(j,k)$, with $j < k$, $u(j) < u(k)$, and there is no h with $j < h < k$ and $u(j) < u(h) < u(k)$. The cover relation is weak if and only if $k = j+1$. Suppose that v covers u. Since $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, the only possibilities are the following:

(1) $u(j) < i < u(k)$ and $k = j+1$;
(2) $u(j) = i < u(k)$ and $u(h) < i$ for all $j < h < k$;
(3) $u(j) < i = u(k)$ and $u(h) > i$ for all $j < h < k$.

If (1) holds, then, by Proposition 3.3, we have that $\lambda = \mu^x$, for some valley x of μ different from r, and $t = r$. In this case, the cover relation is weak.

If (2) (resp. (3)) holds, then, by Proposition 3.3, we have that $\lambda = \mu^t$ and r is the leftmost (resp. rightmost) valley of λ to the right (resp. left) of t. In this case, the cover relation is weak if and only if $|r - t| = 1$.

Conversely, if (λ, r) is obtained from (μ, t) by an elementary move, then, by Proposition 3.3, we are in one of the cases (1), (2) or (3), so the result follows. □

Given two 1-superpartitions (λ, r) and (μ, t), we say that (λ, r) *dominates* (μ, t) if $\mu \subseteq \lambda$ and, if $r < t$ (resp. $r > t$), there are no down steps (resp. up steps) of the paths λ and μ, strictly between the two circles, on which the two paths coincide. For example, $((3,2,2,1),2)$ dominates $((2,1,1,1),6)$, while $((2,2,2,1),2)$ does not dominate $((2,1,1,1),6)$. It is not hard to see that domination gives the set of 1-superpartitions a partial order structure.

Theorem 3.6. *Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then $u \leqslant v$ in the Bruhat order if and only if $\Lambda^\bullet(v)$ dominates $\Lambda^\bullet(u)$.*

Proof. By Proposition 3.5, we need to show that $\Lambda^\bullet(v)$ dominates $\Lambda^\bullet(u)$ if and only if $\Lambda^\bullet(v)$ is obtained from $\Lambda^\bullet(u)$ by a sequence of elementary moves.

Suppose that $\Lambda^\bullet(v)$ is obtained from $\Lambda^\bullet(u)$ by a sequence of elementary moves. By the transitivity of the dominance relation, we may assume that $\Lambda^\bullet(v)$ is obtained from $\Lambda^\bullet(u)$ by one elementary move. By our definitions, this implies that $\Lambda^\bullet(v)$ dominates $\Lambda^\bullet(u)$.

Conversely, assume that $\Lambda^\bullet(v)$ dominates $\Lambda^\bullet(u)$. Let $\Lambda^\bullet(v) = (\lambda, r)$ and $\Lambda^\bullet(u) = (\mu, t)$. We proceed by induction on $|\lambda \setminus \mu|$. If $|\lambda \setminus \mu| = 0$, then the fact that (λ, r) dominates (μ, t) implies that $r = t$, so $u = v$. Now let $|\lambda \setminus \mu| \geqslant 1$. By our induction hypothesis, we are done if we find a 1-superpartition (ν, s) which is obtained from (μ, t) by an elementary move and such that (λ, r) dominates (ν, s).

If $d_{\lambda\setminus\mu}(t) \geqslant 1$ then we let $\nu = \mu^t$ and s be the first valley of μ from t in the direction of r (any of the two possible choices works if $r = t$). Clearly, (ν, s) is still dominated by (λ, r).

If $d_{\lambda\setminus\mu}(t) = 0$, then, since (λ, r) dominates (μ, t), we have that $r = t$. In this case, since $\lambda \neq \mu$, there is a valley x of μ such that $d_{\lambda\setminus\mu}(x) \geqslant 1$. Then $(\nu, s) = (\mu^x, t)$ is dominated by (λ, r). □

4. Dyck superpartitions

In this section, we introduce and study the main new combinatorial concept of this work, namely Dyck superpartitions, which plays a fundamental role in the main result.

If (λ, r) and (μ, t) are two 1-superpartitions such that $\mu \subseteq \lambda$, we call $(\lambda \setminus \mu, r, t)$ a *skew 1-superpartition*. We will usually write $(\lambda, r) \setminus (\mu, t)$ rather than $(\lambda \setminus \mu, r, t)$.

Given $(i,j), (k,l) \in \mathbf{P}^2$ we say that (i,j) is *admissible* with respect to (k,l) if either $i \leqslant k+1$ and $j = l+1$, or $j \leqslant l+1$ and $i = k+1$. We can now define the main combinatorial concept of this work.

Let $(\lambda, r) \setminus (\mu, t)$ be a skew 1-superpartition, and θ be the outer border strip of $\lambda \setminus \mu$. We define $(\lambda, r) \setminus (\mu, t)$ to be *Dyck* in four steps:

1. If $\hat{r} = \hat{t}$ then $(\lambda, r) \setminus (\mu, t)$ is Dyck if and only if $\lambda \setminus \mu$ is Dyck.

We may therefore assume that $\hat{r} \neq \hat{t}$.

2. If λ has a peak strictly between r and t then $(\lambda, r) \setminus (\mu, t)$ is not Dyck.

If λ has no peaks strictly between r and t then $\hat{t} \in \lambda \setminus \mu$. Let $\lambda \setminus \mu^{(1)}, \ldots, \lambda \setminus \mu^{(k)}$ be the connected components of $\lambda \setminus \mu$ indexed so that $\hat{t} \in \lambda \setminus \mu^{(1)}$. Then

3. $(\lambda, r) \setminus (\mu, t)$ is Dyck if and only if:

i) $(\lambda, r) \setminus (\mu^{(1)}, t)$ is Dyck;
ii) $\lambda \setminus \mu^{(2)}, \ldots, \lambda \setminus \mu^{(k)}$ are Dyck.

Finally, if $\lambda \setminus \mu$ is connected, then

4. $(\lambda, r) \setminus (\mu, t)$ is Dyck if and only if either:

i): θ is Dyck and $(\lambda \setminus \theta, r') \setminus (\mu, t)$ is Dyck where

$$r' = \begin{cases} l_\theta, & \text{if } \hat{r} \text{ is to the left of } \theta, \\ r_\theta, & \text{if } \hat{r} \text{ is to the right of } \theta, \\ r, & \text{otherwise,} \end{cases} \tag{4}$$

and l_θ (resp, r_θ) is the valley of $\lambda \setminus \theta$ immediately below the leftmost (resp., rightmost) cell of θ;

or

ii): θ is not Dyck, $\hat{r}$ is admissible with respect to $\hat{t}$, and $\lambda \setminus \mu^t$ is Dyck.

If $\lambda \setminus \mu = \emptyset$, then $(\lambda, r) \setminus (\mu, t)$ is Dyck if and only if $\hat{r} = \hat{t}$.

Let $(\lambda, r) \setminus (\mu, t)$ be Dyck. We define the *depth* of $(\lambda, r) \setminus (\mu, t)$, denoted $dp((\lambda, r) \setminus (\mu, t))$, in the following way.

1. If $\hat{r} = \hat{t}$ then $dp((\lambda, r) \setminus (\mu, t)) = dp(\lambda \setminus \mu)$.

2. If $\hat{r} \neq \hat{t}$ and $\lambda \setminus \mu^{(1)}, \ldots, \lambda \setminus \mu^{(k)}$ are the connected components of $\lambda \setminus \mu$ indexed as above then

$$dp((\lambda, r) \setminus (\mu, t)) = dp((\lambda, r) \setminus (\mu^{(1)}, t)) + \sum_{i=2}^{k} dp(\lambda \setminus \mu^{(i)}).$$

3. If $\lambda \setminus \mu$ is connected and θ is its outer border strip then

$$dp((\lambda, r) \setminus (\mu, t)) = \begin{cases} dp((\lambda \setminus \theta, r') \setminus (\mu, t)) + 1, & \text{if } \theta \text{ is Dyck,} \\ dp(\lambda \setminus \mu^t) + 1, & \text{otherwise,} \end{cases}$$

where r' has the same meaning as in (4).

Four examples of Dyck skew 1-superpartitions are shown in Figure 3. For all of them, $\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 8$.

The following result characterizes Dyck skew 1-superpartitions in terms of Dyck skew partitions and will be used often in the sequel.

Theorem 4.1. *Let $(\lambda, r) \setminus (\mu, t)$ be a skew 1-superpartition. Then $(\lambda, r) \setminus (\mu, t)$ is Dyck if and only if λ has no peaks strictly between r and t and either $\lambda \setminus \mu$ or $\lambda \setminus \mu^t$ is Dyck. In this case,*

$$\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \begin{cases} \mathrm{dp}(\lambda \setminus \mu), & \text{if } \lambda \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}(\lambda \setminus \mu^t) + 1, & \text{if } \lambda \setminus \mu^t \text{ is Dyck.} \end{cases} \tag{5}$$

Proof. Suppose first that $(\lambda, r) \setminus (\mu, t)$ is Dyck. We proceed by induction on $|\lambda \setminus \mu|$, the result being clear if $|\lambda \setminus \mu| = 0$. So assume that $|\lambda \setminus \mu| \geqslant 1$. If $\hat{r} = \hat{t}$ then the result is clear so we may assume that $\hat{r} \neq \hat{t}$. Then, since $(\lambda, r) \setminus (\mu, t)$ is Dyck, λ has no peaks strictly between r and t. Hence $\hat{t} \in \lambda \setminus \mu$. Let $\lambda \setminus \mu^{(1)}, \ldots, \lambda \setminus \mu^{(k)}$

be the connected components of $\lambda \setminus \mu$ indexed so that $\hat{t} \in \lambda \setminus \mu^{(1)}$. Then, since $(\lambda, r) \setminus (\mu, t)$ is Dyck, $(\lambda, r) \setminus (\mu^{(1)}, t)$ and $\lambda \setminus \mu^{(2)}, \dots, \lambda \setminus \mu^{(k)}$ are Dyck and

$$\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \mathrm{dp}((\lambda, r) \setminus (\mu^{(1)}, t)) + \sum_{i=2}^{k} \mathrm{dp}(\lambda \setminus \mu^{(i)}).$$

If $k > 1$ then we conclude by induction that either $\lambda \setminus \mu^{(1)}$ or $\lambda \setminus (\mu^{(1)})^t$ are Dyck and

$$\mathrm{dp}((\lambda \setminus r) \setminus (\mu^{(1)}, t)) = \begin{cases} \mathrm{dp}(\lambda \setminus \mu^{(1)}), & \text{if } \lambda \setminus \mu^{(1)} \text{ is Dyck,} \\ \mathrm{dp}(\lambda \setminus (\mu^{(1)})^t) + 1, & \text{if } \lambda \setminus (\mu^{(1)})^t \text{ is Dyck,} \end{cases}$$

and the result follows. We may therefore assume that $\lambda \setminus \mu$ is connected. Let θ be the outer border strip of $\lambda \setminus \mu$. We have two cases to consider.

i): θ is Dyck and $(\lambda \setminus \theta, r') \setminus (\mu, t)$ is Dyck where r' has the same meaning as in (4).

Then, by our induction hypotheses, either $(\lambda \setminus \theta) \setminus \mu$ or $(\lambda \setminus \theta) \setminus \mu^t$ is Dyck and

$$\mathrm{dp}((\lambda \setminus \theta, r') \setminus (\mu, t)) = \begin{cases} \mathrm{dp}((\lambda \setminus \theta) \setminus \mu), & \text{if } (\lambda \setminus \theta) \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}((\lambda \setminus \theta) \setminus \mu^t) + 1, & \text{if } (\lambda \setminus \theta) \setminus \mu^t \text{ is Dyck.} \end{cases}$$

If $(\lambda \setminus \theta) \setminus \mu$ is Dyck then, since θ is Dyck, $\lambda \setminus \mu$ is Dyck and we are done. So suppose that $(\lambda \setminus \theta) \setminus \mu$ is not Dyck. Then $(\lambda \setminus \theta) \setminus \mu^t$ is Dyck and $(\lambda \setminus \theta) \setminus \mu^t \neq (\lambda \setminus \theta) \setminus \mu$. Hence $\hat{t} \notin \theta$. Therefore θ is also the outer border strip of $\lambda \setminus \mu^t$ so, since θ is Dyck, $\lambda \setminus \mu^t$ is Dyck and we are done.

ii): θ is not Dyck, $\hat{r}$ is admissible with respect to $\hat{t}$, and $\lambda \setminus \mu^t$ is Dyck.

Then $\lambda \setminus \mu^t$ is Dyck and we are done.

Conversely, suppose that λ has no peaks strictly between r and t and that either $\lambda \setminus \mu$ or $\lambda \setminus \mu^t$ is Dyck. We proceed by induction on $|\lambda \setminus \mu|$, the result being easy to check if $|\lambda \setminus \mu| = 0$. So assume $|\lambda \setminus \mu| \geqslant 1$. If $\hat{r} = \hat{t}$ then $\lambda \setminus \mu = \lambda \setminus \mu^t$ so $\lambda \setminus \mu$ is Dyck and we are done. We may therefore assume that $\hat{r} \neq \hat{t}$. Then , since λ has no peaks strictly between r and t, $\hat{t} \in \lambda \setminus \mu$. Let $\lambda \setminus \mu^{(1)}, \dots, \lambda \setminus \mu^{(k)}$ be the connected components of $\lambda \setminus \mu$ indexed so that $\hat{t} \in \lambda \setminus \mu^{(1)}$. Then $\lambda \setminus \mu^{(2)}, \dots, \lambda \setminus \mu^{(k)}$ are Dyck and either $\lambda \setminus \mu^{(1)}$ or $\lambda \setminus (\mu^{(1)})^t$ is Dyck. If $k > 1$ then by our induction hypothesis $(\lambda, r) \setminus (\mu^{(1)}, t)$ is Dyck and we are done. We may therefore assume that $\lambda \setminus \mu$ is connected. Let θ be the outer border strip of $\lambda \setminus \mu$. We then have two cases to consider.

i): either $\hat{t} \notin \theta$ or $\lambda \setminus \mu$ is Dyck.

Then θ is Dyck (for if $\lambda \setminus \mu$ is not Dyck then $\lambda \setminus \mu^t$ is and $\hat{t} \notin \theta$, so θ is also the outer border strip of $\lambda \setminus \mu^t$). We claim that there are no peaks of $\lambda \setminus \theta$ strictly between r' and t. In fact, note first that, since θ is Dyck, the only peaks of $\lambda \setminus \theta$ that are not peaks of λ are (possibly) $l_\theta - 1$ and $r_\theta + 1$. If $l_\theta \leqslant r \leqslant r_\theta$ then $l_\theta < r < r_\theta$ so $l_\theta < t < r_\theta$ and our claim follows since $r' = r$ in this case. If $r > r_\theta$ then, since λ has no peaks strictly between r and t, and $\hat{r} \neq \hat{t}$, all the steps of λ between r_θ and r are down and $l_\theta \leqslant t \leqslant r_\theta$ so our claim follows since $r' = r_\theta$ in this case. Similarly if $r < l_\theta$. This proves our claim. Hence, by our induction hypotheses, $(\lambda \setminus \theta, r') \setminus (\mu, t)$ is Dyck and the result follows.

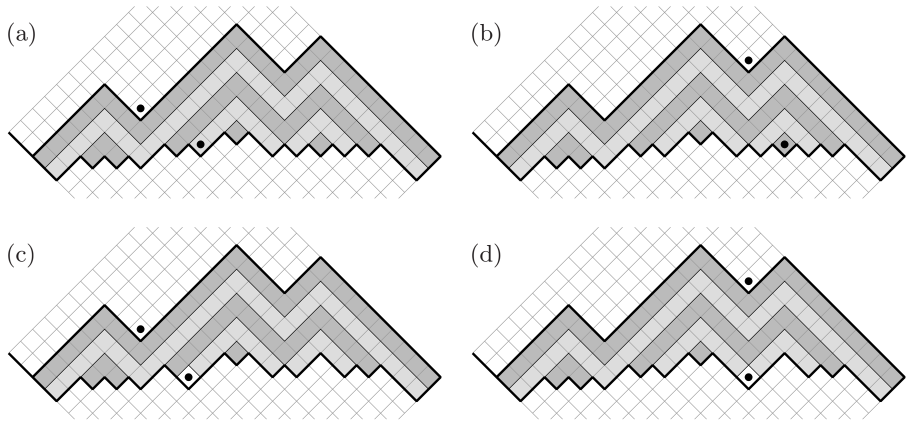


Figure 3. Four Dyck skew 1-superpartitions.

ii): $\hat{t} \in \theta$ and $\lambda \setminus \mu$ is not Dyck.

Then $\lambda \setminus \mu^t$ is Dyck and the outer border strip of $\lambda \setminus \mu^t$ is $\theta \setminus \{\hat{t}\}$. Let θ_L and θ_R be the two connected components of $\theta \setminus \{\hat{t}\}$ (possibly one of them is empty). Then $\theta \setminus \{\hat{t}\}$ is Dyck so θ_L and θ_R are Dyck cbs's. Therefore $lv(l_\theta) = lv(r_{\theta_L}) = lv(\hat{t})+1 = lv(l_{\theta_R}) = lv(r_\theta)$, where r_{θ_L} and l_{θ_R} denote the rightmost (resp., leftmost) cell of θ_L (resp., θ_R). Hence θ is not Dyck. Furthermore, since λ has no peaks strictly between r and t, we have that all the steps of λ between r and t are up (respectively, down) if $r \leqslant t$ (respectively, $r \geqslant t$). Hence $\hat{r}$ is admissible with respect to $\hat{t}$ and the result follows. □

It is useful to know, in the statement of Theorem 4.1, when both $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are Dyck. The following result answers this question, and is used in the proof of the main result of this section.

Proposition 4.2. *Let $\lambda \setminus \mu$ be a skew partition and let t be a valley of μ such that $d_{\lambda\setminus\mu}(t) \geqslant 1$. Suppose that at least one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck. Then $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are both Dyck if and only if t is a peak of λ.*

Proof. If t is a peak of λ, then, by equivalence (1) $\Leftrightarrow$ (2) in Theorem 2.2, $\lambda \setminus \mu$ is Dyck if and only if $\lambda \setminus \mu^t$ is Dyck.

Conversely, assume that $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are both Dyck. We prove that t is a peak of λ by induction on $d_{\lambda\setminus\mu}(t)$. Let θ be the outer border strip of $\lambda \setminus \mu$. If $d_{\lambda\setminus\mu}(t) = 1$, then the outer border strip of $\lambda \setminus \mu^t$ is $\theta \setminus \{\hat{t}\}$. Since both the outer border strips of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are Dyck, the only possibility is $\theta = \{\hat{t}\}$. Thus t is a peak of λ. If $d_{\lambda\setminus\mu}(t) \geqslant 2$, then θ is also the outer border strip of $\lambda \setminus \mu^t$. Let $\lambda^{(1)} = \lambda \setminus \theta$. Then $\lambda^{(1)} \setminus \mu$ and $\lambda^{(1)} \setminus \mu^t$ are both Dyck, so, by the induction hypothesis, t is a peak of $\lambda^{(1)}$. But this implies that t is also a peak of λ. □

Proposition 4.3. *Let $(\lambda, r) \setminus (\mu, t)$ be Dyck and x be a peak of λ. Then, either x is a peak or a valley of μ, or $|x - t| = 1$ and x is a valley of μ^t.*

Proof. Since $(\lambda, r) \setminus (\mu, t)$ is Dyck, $\lambda \setminus \mu$ or $\lambda \setminus \mu^t$ is Dyck. In the first case, by Proposition 2.1, x is a peak or a valley of μ. In the second case, x is a peak or a valley of μ^t. If $|x - t| > 1$, then x is a peak or a valley of μ, while if $|x - t| = 1$, then x is necessarily a valley of μ^t. □

The following is the main result of this section.

Theorem 4.4. *Let $(\lambda, r) \setminus (\mu, t)$ be a skew 1-superpartition and let x be a peak of λ, with $d_{\lambda\setminus\mu}(x) \geqslant 1$ and $|x - r| > 1$. Assume that y is a peak of μ, where*

$$y = \begin{cases} x, & \text{if } x < r, t \text{ or } x > r, t, \\ x + 1, & \text{if } t \leqslant x < r, \\ x - 1, & \text{if } r < x \leqslant t, \end{cases}$$

and set

$$t' = \begin{cases} y, & \text{if } |y - t| = 1, \\ t, & \text{if } |y - t| > 1. \end{cases}$$

Then, the following are equivalent:

1. *$(\lambda, r) \setminus (\mu, t)$ is Dyck*;
2. *$(\lambda, r) \setminus (\mu^y, t')$ is Dyck*;
3. *at least one of $(\lambda^x, r) \setminus (\mu, t)$ and $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck*;
4. *exactly one of $(\lambda^x, r) \setminus (\mu, t)$ and $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck.*

Furthermore, if these conditions are satisfied, then

$$\begin{cases} \mathrm{dp}((\lambda, r) \setminus (\mu^y, t')) - \mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 1, & \\ \mathrm{dp}((\lambda^x, r) \setminus (\mu, t)) - \mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 1, & \text{if } (\lambda^x, r) \setminus (\mu, t) \text{ is Dyck,} \\ \mathrm{dp}((\lambda^x, r) \setminus (\mu^y, t')) - \mathrm{dp}((\lambda, r) \setminus (\mu, t)) = 0, & \text{if } (\lambda^x, r) \setminus (\mu^y, t') \text{ is Dyck.} \end{cases}$$

Proof. If x is strictly between r and t, then $|y - t| > 1$, so $t' = t$. In this case, by Theorem 4.1, none of the four skew 1-superpartitions mentioned in the theorem is Dyck (both λ and λ^x have a peak strictly between r and t), and we are done. Thus we only have to consider two cases: $x < r, t$ (the case $x > r, t$ being similar) and $t = x < r$ (the case $r < x = t$ being similar).

Moreover, if λ has a peak different from x strictly between r and t, again none of the four skew 1-superpartitions is Dyck, and we are done. So we may assume that there are no peaks of λ strictly between r and t, which means that the path associated with λ is decreasing (resp. increasing) between the two circles if $t < r$ (resp. $r < t$).

We will prove that (1) implies (2), (4), and the identities involving the depth and then, conversely, that either (2) or (3) implies (1).

We start with the case $x < r, t$, for which $y = x$ is a peak of both λ and μ.

Suppose that $(\lambda, r) \setminus (\mu, t)$ is Dyck. Then, by Theorem 4.1, either $\lambda \setminus \mu$ or $\lambda \setminus \mu^t$ is Dyck. In both cases, λ has no peaks strictly between r and t', since either $t' = t$ or $t' = t - 1$.

If $\lambda \setminus \mu$ is Dyck, then, by Theorem 2.2, $\lambda \setminus \mu^x$ is Dyck, and $\mathrm{dp}(\lambda \setminus \mu^x) = \mathrm{dp}(\lambda \setminus \mu) + 1$, so by Theorem 4.1 we conclude that $(\lambda, r) \setminus (\mu^x, t')$ is Dyck and $\mathrm{dp}((\lambda, r) \setminus (\mu^x, t')) = \mathrm{dp}(\lambda \setminus \mu^x) = \mathrm{dp}(\lambda \setminus \mu) + 1 = \mathrm{dp}((\lambda, r) \setminus (\mu, t)) + 1$, as desired.

If $\lambda \setminus \mu^t$ is Dyck then it cannot be $t = x+1$, because in this case the peak x of λ is neither a peak nor a valley of μ^t, contradicting Proposition 2.1. Then $t > x+1$, so $t' = t$. By Theorem 2.2, $\lambda \setminus (\mu^t)^x$ is Dyck and $\mathrm{dp}(\lambda \setminus (\mu^t)^x) = \mathrm{dp}(\lambda \setminus \mu^t) + 1$. But $\lambda \setminus (\mu^t)^x = \lambda \setminus (\mu^x)^{t'}$. So $\lambda \setminus (\mu^x)^{t'}$ is Dyck. Therefore, by Theorem 4.1, $(\lambda, r) \setminus (\mu^x, t')$ is Dyck and $\mathrm{dp}((\lambda, r) \setminus (\mu^x, t')) = \mathrm{dp}(\lambda \setminus (\mu^x)^{t'}) + 1 = \mathrm{dp}(\lambda \setminus \mu^t) + 2 = \mathrm{dp}((\lambda, r) \setminus (\mu, t)) + 1$, as desired.

Suppose now that t is a peak of λ. Then $t > x+1$, so $t' = t$, and, by Proposition 4.2, both $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are Dyck. Hence, by Theorem 2.2 (applied to $\lambda \setminus \mu$ and x), exactly one of $\lambda^x \setminus \mu$ and $\lambda^x \setminus \mu^x$ is Dyck and

$$\mathrm{dp}(\lambda \setminus \mu) = \begin{cases} \mathrm{dp}(\lambda^x \setminus \mu) - 1, & \text{if } \lambda^x \setminus \mu \text{ is Dyck,} \\ \mathrm{dp}(\lambda^x \setminus \mu^x), & \text{if } \lambda^x \setminus \mu^x \text{ is Dyck.} \end{cases} \tag{6}$$

Since t is a peak of λ^x, by Proposition 4.2, we have that $\lambda^x \setminus \mu$ is Dyck if and only if $\lambda^x \setminus \mu^t$ is Dyck and $\lambda^x \setminus \mu^x$ is Dyck if and only if $\lambda^x \setminus (\mu^x)^t = \lambda^x \setminus (\mu^t)^x$ is Dyck. Also, since $t' = t > x+1$, we have that λ^x has no peaks strictly between r and t. Therefore, by Theorem 4.1, $(\lambda^x, r) \setminus (\mu, t)$ is Dyck if and only if $\lambda^x \setminus \mu$ is Dyck and $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck if and only if $\lambda^x \setminus \mu^x$ is Dyck and $\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}(\lambda^x \setminus \mu) - 1 = \mathrm{dp}((\lambda^x, r) \setminus (\mu, t)) - 1$, if $(\lambda^x, r) \setminus (\mu, t)$ is Dyck, while $\mathrm{dp}((\lambda, r) \setminus (\mu, t)) = \mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}(\lambda^x \setminus \mu^x) = \mathrm{dp}((\lambda^x, r)) \setminus (\mu^x, t'))$, if $(\lambda^x, r) \setminus (\mu^x, t')$ is Dyck, as desired.

Suppose now that t is not a peak of λ. Then, by Proposition 4.2, exactly one of $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ is Dyck.

Suppose that $\lambda \setminus \mu$ is Dyck and $\lambda \setminus \mu^t$ is not. By Theorem 2.2, exactly one of $\lambda^x \setminus \mu$ and $\lambda^x \setminus \mu^x$ is Dyck and (6) holds. If $t = x+1$, then $t' = x$. Since $|x - r| > 1$ by hypothesis, we have that $t = x+1 < r$. Therefore, all the steps of λ between t and r are down, so $x+1$ is a peak λ^x. On the other hand, $x+1$ is neither a peak nor a valley of μ^x, so, by Proposition 2.1, $\lambda^x \setminus \mu^x$ cannot be Dyck. Thus $\lambda^x \setminus \mu$ is Dyck. Also, λ^x has no peaks strictly between r and t, so by Theorem 4.1, $(\lambda^x, r) \setminus (\mu, t)$ is Dyck and $\mathrm{dp}((\lambda^x, r) \setminus (\mu, t)) = \mathrm{dp}(\lambda^x \setminus \mu) = \mathrm{dp}(\lambda \setminus \mu) + 1 = \mathrm{dp}((\lambda, r) \setminus (\mu, t)) + 1$. On the other hand, $(\lambda^x, r) \setminus (\mu^x, t')$ is not Dyck, since the peak $x+1$ of λ^x is strictly between $t' = x$ and r. If $t > x+1$, then $t' = t$ and λ^x has no peaks strictly between r and $t' = t$. Also, since $\lambda \setminus \mu^t$ is not Dyck, neither $\lambda^x \setminus \mu^t$ nor $\lambda^x \setminus (\mu^t)^x = \lambda^x \setminus (\mu^x)^t$ are Dyck by Theorem 2.2. Hence, by Theorem 4.1, $(\lambda^x, r) \setminus (\mu, t)$ is Dyck if and only if $\lambda \setminus \mu$ is Dyck, $(\lambda^x, r) \setminus (\mu^x, t')$ is Dyck if and only if $\lambda^x \setminus \mu^x$ is Dyck, and $\mathrm{dp}((\lambda^x, r) \setminus (\mu, t)) = \mathrm{dp}(\lambda^x \setminus \mu)$ if $\lambda^x \setminus \mu$ is Dyck while $\mathrm{dp}((\lambda^x, r) \setminus (\mu^x, t')) = \mathrm{dp}(\lambda^x \setminus \mu^x)$ if $\lambda^x \setminus \mu^x$ is Dyck so the result follows from (6).

Now suppose that $\lambda \setminus \mu^t$ is Dyck and $\lambda \setminus \mu$ is not. We already showed, that in these hypotheses, by Proposition 2.1, $t \neq x+1$. So $t > x+1$, and $t' = t$. By Theorem 2.2, exactly one of $\lambda^x \setminus \mu^t$ and $\lambda^x \setminus (\mu^t)^x = \lambda^x \setminus (\mu^x)^t$ is Dyck and

$$\mathrm{dp}(\lambda \setminus \mu^t) = \begin{cases} \mathrm{dp}(\lambda^x \setminus \mu^t) - 1, & \text{if } \lambda^x \setminus \mu^t \text{ is Dyck,} \\ \mathrm{dp}(\lambda^x \setminus (\mu^x)^t), & \text{if } \lambda^x \setminus (\mu^x)^t \text{ is Dyck.} \end{cases}$$

Also, since $\lambda \setminus \mu$ is not Dyck, by Theorem 2.2, neither $\lambda^x \setminus \mu$ nor $\lambda^x \setminus \mu^x$ are Dyck. Finally, λ^x has no peaks strictly between r and $t' = t$. Hence, by Theorem 4.1 exactly one of $(\lambda^x, r) \setminus (\mu, t)$ and $(\lambda^x, r) \setminus (\mu^x, t')$ is Dyck and we conclude as above.

Conversely, suppose that $(\lambda, r) \setminus (\mu^x, t')$ is Dyck. Then, by Theorem 4.1, either $\lambda \setminus \mu^x$ or $\lambda \setminus (\mu^x)^{t'}$ is Dyck. Moreover, we are assuming that λ has no peaks strictly between r and t. If $\lambda \setminus \mu^x$ is Dyck, then, by Theorem 2.2, $\lambda \setminus \mu$ is Dyck so, by Theorem 4.1, $(\lambda, r) \setminus (\mu, t)$ is Dyck, as desired. So assume that $\lambda \setminus (\mu^x)^{t'}$ is Dyck. If $t = x + 1$, then $t' = x$ and $\lambda \setminus (\mu^x)^{t'} = \lambda \setminus \mu$. So $\lambda \setminus \mu$ is Dyck and we conclude as above. If $t > x + 1$, then $t' = t$, so $\lambda \setminus (\mu^t)^x = \lambda \setminus (\mu^x)^{t'}$. Then, by Theorem 2.2 (applied to $\lambda \setminus \mu^t$ and x), $\lambda \setminus \mu^t$ is Dyck. Hence by Theorem 4.1 $(\lambda, r) \setminus (\mu, t)$ is Dyck, as desired.

Finally, suppose that either $(\lambda^x, r) \setminus (\mu, t)$ or $(\lambda^x, r) \setminus (\mu^x, t')$ is Dyck. Then, by Theorem 4.1, at least one of $\lambda^x \setminus \mu, \lambda^x \setminus \mu^t, \lambda^x \setminus \mu^x$ or $\lambda^x \setminus (\mu^x)^{t'}$ is Dyck. If $\lambda^x \setminus \mu$ or $\lambda^x \setminus \mu^x$ is Dyck, then, by Theorem 2.2, $\lambda \setminus \mu$ is Dyck. Now let $\lambda^x \setminus \mu^t$ be Dyck. If $t = x+1$, then, since $|x - r| > 1$ and we are assuming that λ is decreasing between t and r, we have that t is a peak of λ^x. So, by Theorem 2.2 (applied to $\lambda^x \setminus \mu^t$ and t), $\lambda^x \setminus \mu = \lambda^x \setminus (\mu^t)^t$ is Dyck, and this implies, as above, that $\lambda \setminus \mu$ is Dyck. If $t > x + 1$, then x is a peak of μ^t, so, by Theorem 2.2 (applied to $\lambda \setminus \mu^t$ and x), $\lambda \setminus \mu^t$ is Dyck. Finally, let $\lambda^x \setminus (\mu^x)^{t'}$ be Dyck. If $t = x + 1$, then $t' = x$, so $\lambda^x \setminus \mu = \lambda^x \setminus (\mu^x)^{t'}$ is Dyck, and this implies, as above, that $\lambda \setminus \mu$ is Dyck. If $t > x + 1$, then $t' = t$, so $\lambda^x \setminus (\mu^t)^x = \lambda^x \setminus (\mu^x)^{t'}$ is Dyck. Then, by Theorem 2.2 (applied to $\lambda \setminus \mu^t$ and x), $\lambda \setminus \mu^t$ is Dyck. So, in all cases, either $\lambda \setminus \mu$ or $\lambda \setminus \mu^t$ is Dyck. Moreover, we are assuming that λ has no peaks strictly between r and t, so, by Theorem 4.1, $(\lambda, r) \setminus (\mu, t)$ is Dyck.

We now consider the (easier) case $x = t < r$. Then $y = x + 1 = t'$.

Let $(\lambda, r) \setminus (\mu, t)$ be Dyck. Since $t = x$ is a peak of λ, by Proposition 4.2 we have that both $\lambda \setminus \mu$ and $\lambda \setminus \mu^t$ are Dyck. In this case $\lambda \setminus (\mu^y)^{t'} = \lambda \setminus \mu$ is Dyck. Also, λ has no peaks strictly between r and t'. Thus, by Theorem 4.1, $(\lambda, r) \setminus (\mu^y, t')$ is Dyck and $\mathrm{dp}((\lambda, r) \setminus (\mu^y, t')) = \mathrm{dp}(\lambda \setminus \mu) + 1 = \mathrm{dp}((\lambda, r) \setminus (\mu, t)) + 1$. Also, by Theorem 2.2 (applied to $\lambda \setminus \mu^t$ and x) exactly one of $\lambda^x \setminus \mu$ and $\lambda^x \setminus \mu^x$ is Dyck. But, by Proposition 2.1, $\lambda^x \setminus \mu^x$ is not Dyck, since $x+1$ is a peak of λ^x, but neither a peak nor a valley of μ^x. Thus $\lambda^x \setminus \mu$ is Dyck and $\mathrm{dp}(\lambda^x \setminus \mu) = \mathrm{dp}(\lambda \setminus \mu) - 1$. By Theorem 2.2 (applied to $\lambda^x \setminus \mu$ and y), we have that $\lambda^x \setminus \mu^y$ is Dyck and $\mathrm{dp}(\lambda^x \setminus \mu^y) = \mathrm{dp}(\lambda^x \setminus \mu) + 1$. Now note that the peak $x + 1$ of λ^x is strictly between $t = x$ and r, so $(\lambda^x, r) \setminus (\mu, t)$ is not Dyck, whereas λ^x has no peaks strictly between $t' = x + 1$ and r, so, by Theorem 4.1, $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck and $\mathrm{dp}((\lambda^x, r) \setminus (\mu^y, t')) = \mathrm{dp}(\lambda^x \setminus \mu^y) = \mathrm{dp}(\lambda \setminus \mu) = \mathrm{dp}((\lambda, r) \setminus (\mu, t))$, as desired.

Conversely, suppose that $(\lambda, r) \setminus (\mu^y, t')$ is Dyck. Then, by Theorem 4.1, one of $\lambda \setminus \mu^y$ and $\lambda \setminus (\mu^y)^{t'} = \lambda \setminus \mu$ is Dyck. But, by Proposition 2.1, $\lambda \setminus \mu^y$ is not Dyck, since x is a peak of λ, but neither a peak nor a valley of μ^y. So $\lambda \setminus \mu$ is Dyck. Also, we are assuming that λ has no peaks strictly between r and t, so, by Theorem 4.1, $(\lambda, r) \setminus (\mu, t)$ is Dyck.

Finally, suppose that either $(\lambda^x, r) \setminus (\mu, t)$ or $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck. Note that $(\lambda^x, r) \setminus (\mu, t)$ is not Dyck, since the peak $x + 1$ of λ^x is strictly between r and t, so $(\lambda^x, r) \setminus (\mu^y, t')$ is Dyck. Then, by Theorem 4.1, one of $\lambda^x \setminus \mu^y$ and $\lambda^x \setminus (\mu^y)^{t'} = \lambda^x \setminus \mu$ is Dyck. If $\lambda^x \setminus \mu$ is Dyck, then, by Theorem 2.2 (applied to $\lambda \setminus \mu^x$ and x), we have that $\lambda \setminus \mu$ is Dyck. On the other hand, if $\lambda^x \setminus \mu^y$ is Dyck, then, by Theorem 2.2 (now applied to $\lambda^x \setminus \mu$ and y), $\lambda^x \setminus \mu$ is Dyck. As before, this implies that $\lambda \setminus \mu$ is

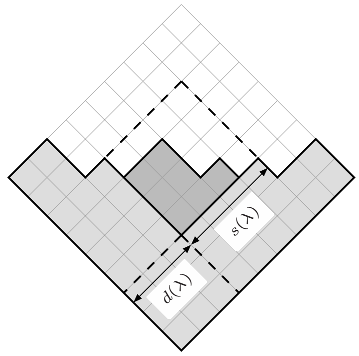


Figure 4. $(9,9,7,3,3,3,3,2,2)\angle(3,2,2) = (9,9,7,6,5,5,3,2,2)$.

Dyck. Also, we are assuming that λ has no peaks strictly between r and t, so, by Theorem 4.1, $(\lambda, r) \setminus (\mu, t)$ is Dyck. □

We conclude this section with some technical results that we will use in the proof of the main theorem. We first introduce some notation.

Given a partition $\lambda \subseteq (n^m)$, we denote by $d(\lambda)$ the length of the Durfee square of λ (largest square partition contained in λ). Note that $d(\lambda) = d_\lambda(m)$ (the "thickness" of λ at m). If $\lambda = (\lambda_1, \lambda_2, \dots) \subseteq (n^n)$ is self-conjugate, we set

$$s(\lambda) = \begin{cases} \lambda_{d(\lambda)} - d(\lambda), & \text{if } \lambda \neq \emptyset, \\ n, & \text{if } \lambda = \emptyset. \end{cases}$$

Let $d = d(\lambda)$ and $s = s(\lambda)$. Given a partition $\eta = (\eta_1, \eta_2, \dots, \eta_k) \subseteq (s^s)$, we denote

$$\lambda\angle\eta = (\lambda_1, \dots, \lambda_d, \lambda_{d+1} + \eta_1, \lambda_{d+2} + \eta_2, \dots, \lambda_{d+k} + \eta_k, \lambda_{d+k+1}, \dots).$$

Figure 4 illustrates an example: $\lambda = (9,9,7,3,3,3,3,2,2) \subseteq (9^9)$, with $d(\lambda) = 3$ and $s(\lambda) = 4$, $\eta = (3,2,2) \subseteq (4^4)$ and $\lambda\angle\mu = (9,9,7,6,5,5,3,2,2)$.

Consider the rectangular partition (n^m). If $\lambda \subseteq (\min\{n,m\})^{\min\{n,m\}}$, then we set

$$\lambda\!\downarrow_{(n^m)} = \begin{cases} \lambda, & \text{if } n = m, \\ (n^{m-n}, \lambda), & \text{if } n < m, \\ (n-m)^m + \lambda, & \text{if } n > m, \end{cases}$$

where the sum is componentwise.

Proposition 4.5. *Let (μ, r) be a 1-superpartition contained in $(n^{m-1}, n-1)$. Then $((n^{m-1}, n-1), n) \setminus (\mu, r)$ is Dyck if and only if there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 2$, such that*

$$\mu = (\lambda\angle(a,b))\!\downarrow_{(n^m)}, \tag{7}$$

where (a,b) is such that

$$((a,b), r-n+2) \in \{((2,1),2), ((2,0),1), ((1,1),3), ((1,0),1), ((1,0),3), ((0,0),2)\}.$$

In this case,

$$\mathrm{dp}(((n^{m-1}, n-1), n) \setminus (\mu, r)) = n + 1 - d(\lambda) - a - b. \tag{8}$$

There are $3 \cdot 2^{\min\{n,m\}-1}$ such 1-superpartitions (μ, r).

Proof. It is not hard to check, by induction on $n \geqslant 2$, that if $r \in \{n-1, n, n+1\}$ and μ is of the form (7) then $((n^{m-1}, n-1), n) \setminus (\mu, r)$ is Dyck and (8) holds.

Let (μ, r) be such that $((n^{m-1}, n-1), n) \setminus (\mu, r)$ is Dyck. Then, by Theorem 4.1, $r \in \{n-1, n, n+1\}$ and either $(n^{m-1}, n-1) \setminus \mu$ or $(n^{m-1}, n-1) \setminus \mu^r$ is Dyck.

Suppose first that $(n^{m-1}, n-1) \setminus \mu$ is Dyck. Then, by [3, Theorem 4.1], there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 2$, such that $\mu = (\lambda \angle (a,b)) \downarrow_{(n^m)}$ where $(a,b) \in \{(2,1), (2,0), (1,1), (1,0)\}$. Adding a circle in all possible ways yields

$$((a,b), r-n+2) \in \{((2,1),2), ((2,0),1), ((1,1),3), ((1,0),1), ((1,0),3)\},$$

as desired. Assume now that $(n^{m-1}, n-1) \setminus \mu$ is not Dyck; then $(n^{m-1}, n-1) \setminus \mu^r$ is Dyck and, by Proposition 4.2, $r = n$. Hence, by [3, Theorem 4.1], there exists a self-conjugate partition $\lambda \subseteq (n^n)$, with $s(\lambda) \geqslant 2$, such that $\mu^r = (\lambda \angle (a,b)) \downarrow_{(n^m)}$ where $(a,b) \in \{(2,1), (2,0), (1,1), (1,0)\}$. Since r is a peak of μ^r, we conclude that $(a,b) = (1,0)$, so $\mu = \lambda \downarrow_{(n^m)}$ and the result again follows. $\square$

Proposition 4.6. *Let (μ, r) be a 1-superpartition contained in $(n^{m-1}, n-2)$. Then $((n^{m-1}, n-2), n-1) \setminus (\mu, r)$ is Dyck if and only if there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 3$, such that $\mu = (\lambda \angle (a,b,c)) \downarrow_{(n^m)}$, where (a,b,c) is such that*

$$\begin{aligned}((a,b,c), r-n+3) \in \{&((3,3,1),2), ((3,3,0),1), ((3,2,1),2), ((3,2,1),4),\\ &((3,2,0),1), ((3,2,0),4), ((3,1,1),3), ((3,1,0),3),\\ &((2,1,1),3), ((2,1,0),1), ((2,1,0),3), ((2,0,0),2),\\ &((1,1,1),4), ((1,1,0),1), ((1,1,0),4), ((1,0,0),2)\}.\end{aligned}$$

In this case,

$$\begin{aligned}\mathrm{dp}(((n^{m-1}, n-2), n-1) \setminus (\mu, r)) = n - 3 - d(\lambda)&\\ + \mathrm{dp}(((3,3,1),2) \setminus ((a,b,c), r-n+3)).&\end{aligned}$$

In particular, $\mathrm{dp}(((n^{m-1}, n-2), n-1) \setminus (\mu, r)) = 1$ *if and only if either* $d(\lambda) = n-4$ *and* $((a,b,c), r-n+3) = ((3,3,1),2)$*, or* $d(\lambda) = n-3$ *and*

$$((a,b,c), r-n+3) \in \{((3,3,0),1), (3,2,1),2), ((3,2,1),4), ((2,1,1),3)\}.$$

There are $2^{\min\{n,m\}+1}$ such 1-superpartitions (μ, r).

Proof. The proof is similar to that of Proposition 4.5.

Let (μ, r) be such that $((n^{m-1}, n-2), n-1) \setminus (\mu, r)$ is Dyck. Then $r \in \{n-2, n-1, n, n+1\}$ and either $(n^{m-1}, n-2) \setminus \mu$ or $(n^{m-1}, n-2) \setminus \mu^r$ is Dyck. Suppose first that $(n^{m-1}, n-2) \setminus \mu$ is Dyck. Then, by [3, Theorem 4.1], there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 3$, such that $\mu = (\lambda \angle (a,b,c)) \downarrow_{(n^m)}$, where $(a,b,c) \in \{(3,3,1), (3,3,0), (3,2,1), (3,2,0), (2,1,1), (2,1,0), (1,1,1), (1,1,0)\}$. Adding a circle in all possible ways, with $r-n+3 \in \{1,2,3,4\}$, yields

$$\begin{aligned}((a,b,c), r-n+3) \in \{&((3,3,1),2), ((3,3,0),1), ((3,2,1),2), ((3,2,1),4),\\ &((3,2,0),1), ((3,2,0),4), ((2,1,1),3), ((2,1,0),1),\\ &((2,1,0),3), ((1,1,1),4), ((1,1,0),1), ((1,1,0),4)\},\end{aligned}$$

as desired. If $(n^{m-1}, n-2) \setminus \mu$ is not Dyck then $(n^{m-1}, n-2) \setminus \mu^r$ is Dyck and, by Proposition 4.2, $r \in \{n-1, n\}$. Hence, by [3, Theorem 4.1], there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 3$, such that $\mu^r = (\lambda \angle (a,b,c))\downarrow_{(n^m)}$, where

$$(a,b,c) \in \begin{cases} \{(3,2,1),(3,2,0)\}, & \text{if } r = n, \\ \{(2,1,0),(1,1,0)\}, & \text{if } r = n-1. \end{cases}$$

In this case, $\mu = (\lambda \angle (a,b,c))\downarrow_{(n^m)}$, where $((a,b,c), r-n+3) \in \{((3,1,1),3),((3,1,0),3),((2,0,0),2),((1,0,0),2)\}$, and the result again follows.

Furthermore, since $s(\lambda) \geqslant 3$, we have that $d(\lambda) \leqslant n-3$. Therefore, $\mathrm{dp}(((n^{m-1}, n-2), n-1) \setminus (\mu, r)) = 1$ if and only if either $d(\lambda) = n-3$ and $\mathrm{dp}(((3,3,1),2) \setminus ((a,b,c), r-n+3)) = 1$, or $d(\lambda) = n-4$ and $\mathrm{dp}(((3,3,1),2) \setminus ((a,b,c), r-n+3)) = 0$. Then, the result follows from the first part. □

Proposition 4.7. *Let (μ, r) be a 1-superpartition contained in $(n^{m-1}, n-1, n-2)$. Then $((n^{m-2}, n-1, n-2), n-1) \setminus (\mu, r)$ is Dyck if and only if there exists a self-conjugate partition $\lambda \subseteq (\min\{n,m\}^{\min\{n,m\}})$, with $s(\lambda) \geqslant 3$, such that $\mu = (\lambda \angle (a,b,c))\downarrow_{(n^m)}$, where (a,b,c) is such that*

$$\begin{aligned} ((a,b,c), r-n+3) \in \{&((3,2,1),2),((3,2,0),1),((3,1,1),3),((3,1,0),1), \\ &((3,1,0),3),((3,0,0),2),((2,2,1),2),((2,2,0),1), \\ &((2,1,1),3),((2,1,0),1),((2,1,0),3),((1,0,0),2)\}. \end{aligned}$$

In this case,

$$\begin{aligned} &\mathrm{dp}(((n^{m-2}, n-1, n-2), n-1) \setminus (\mu, r)) \\ &\qquad = n-3-d(\lambda) + \mathrm{dp}(((3,2,1),2) \setminus ((a,b,c), r-n+3)). \end{aligned}$$

There are $3 \cdot 2^{\min\{n,m\}-1}$ such 1-superpartitions (μ, r).

Proof. The proof follows the same line as that of Proposition 4.6, except that $(a,b,c) \in \{(3,2,1),(3,2,0),(3,1,1),(3,1,0),(2,2,1),(2,2,0),(2,1,1),(2,1,0)\}$ and $r = n-1$ if $(n^{m-2}, n-1, n-2) \setminus \mu$ is not Dyck. We omit the details. □

Proposition 4.8. *Let (λ, r) be a 1-superpartition contained in (n^m). Then $(\lambda, r) \setminus (\emptyset, m)$ is Dyck if and only if (λ, r) satisfies one of the following three conditions:*

(i) $\lambda = (k^k)$ *for some* $k \leqslant \min\{n,m\}$;
(ii) $(\lambda, r) = ((k^{k-1}, k-1), m)$ *for some* $k \leqslant \min\{n,m\}$;
(iii) $(\lambda, r) \in \{((2), m-1), ((1,1), m+1)\}$.

Proof. It is easy to see that, if (λ, r) satisfies one of the conditions (i), (ii) and (iii), then $(\lambda, r) \setminus (\emptyset, m)$ is Dyck.

Conversely, suppose that $(\lambda, r) \setminus (\emptyset, m)$ is Dyck. Then either λ or $\lambda \setminus (1)$ is Dyck. If λ is Dyck then, by [2, Theorem 5.1 and Corollary 5.2], λ is a square, thus (i) holds. Now assume that $\lambda \setminus (1)$ is Dyck. We proceed by induction on $d(\lambda)$. If $d(\lambda) \leqslant 3$ then it is easy to check that one of conditions (i), (ii) and (iii) holds. So assume $d(\lambda) \geqslant 4$. Since $\lambda \setminus (1)$ is Dyck and $d(\lambda) \geqslant 4$, the outer border strip θ of $\lambda \setminus (1)$ is Dyck and $\lambda \setminus \theta \supseteq (1)$. Hence, by induction, either $\lambda \setminus \theta = (k^k)$, thus

$\lambda = ((k+1)^{k+1})$, or $\lambda \setminus \theta = (k^{k-1}, k-1)$, thus $\lambda = ((k+1)^k, k)$. In the second case $m-1$ and $m+1$ are both peaks of λ, so necessarily $r = m$. $\square$

5. Parabolic Kazhdan–Lusztig polynomials

In this section, using the results in the two previous ones, we prove our main result (Theorem 5.1) and derive some consequences of it, including the formula for the maximal quotients found in [2] and new identities for the ordinary Kazhdan–Lusztig polynomials and for their leading terms.

Theorem 5.1. *Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with $u \leqslant v$. Then*

$$P_{u,v}^{[n-1]\setminus\{i-1,i\},q}(q) = \begin{cases} q^{(|\Lambda(v)\setminus\Lambda(u)| - \mathrm{dp}(\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)))/2}, & \text{if } \Lambda^\bullet(v)\setminus\Lambda^\bullet(u) \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We proceed by induction on $\ell(v) \geqslant 0$, the result being clear if $v = e$.

Let $\ell(v) \geqslant 1$, and $r = r(v)$ (recall that $r(v) = v^{-1}(i)$ is the circle of $\Lambda^\bullet(v)$).

Suppose first that $D(v) \not\subseteq \{s_{r-1}, s_r\}$. Let $s = s_j \in D(v) \setminus \{s_{r-1}, s_r\}$. Then $|p_v(j) - r| > 1$ and $r(vs) = r$. By Proposition 3.4, this implies $\Lambda(vs) = \Lambda(v)^{p_v(j)}$ and that $\Lambda(vs)$ has a valley at $p_{vs}(j)$.

Let $w \in S_n^{[n-1]\setminus\{i-1,i\}}$ be such that $\mu(w, vs) \neq 0$ and $ws < w$. By Proposition 3.4 and by our induction hypothesis, we have that $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(w)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(w)) = 1$. By Theorem 4.1, this implies that there are no peaks of $\Lambda(vs)$ strictly betweeen $r(vs)$ and $r(w)$ and either $\Lambda(vs) \setminus \Lambda(w)$ is Dyck and $\mathrm{dp}(\Lambda(vs) \setminus \Lambda(w)) = 1$ or $\Lambda(vs) \setminus \Lambda(w)^{r(w)}$ is Dyck and $\mathrm{dp}(\Lambda(vs) \setminus \Lambda(w)^{r(w)}) = 0$. In the first case $\Lambda(vs) \setminus \Lambda(w)$ is a Dyck cbs. In the second one, we conclude that $\Lambda(vs) \setminus \Lambda(w)^{r(w)} = \emptyset$ and therefore $\Lambda(w) \subseteq \Lambda(vs) \subseteq \Lambda(w)^{r(w)}$. Hence, since $\ell(w) < \ell(vs)$, $\Lambda(vs) = \Lambda(w)^{r(w)}$ and so $\Lambda(vs) \setminus \Lambda(w)$ is a Dyck cbs. Thus, in all cases, we conclude that $\Lambda(vs) \setminus \Lambda(w)$ is a Dyck cbs. Since $\Lambda(vs)$ has a valley at $p_{vs}(j)$ and $\Lambda(w)$ has a peak at $p_w(j)$, this implies that $p_{vs}(j) \neq p_w(j)$. Say $p_{vs}(j) = p_w(j) - 1$. Then $r(w) \leqslant p_w(j) - 1 = p_{vs}(j) < r(vs)$, so, since $p_{vs}(j)$ and $r(vs)$ are both valleys of $\Lambda(vs)$, $\Lambda(vs)$ has a peak strictly between $r(w)$ and $r(vs)$. This yields a contradiction. Similarly if $p_{vs}(j) = p_w(j) + 1$.

This shows that there are no $w \in S_n^{[n-1]\setminus\{i-1,i\}}$ such that $\mu(w, vs) \neq 0$ and $ws < w$. Therefore, by Theorem 2.3 we conclude that

$$P_{u,v}(q) = \begin{cases} P_{us,vs}(q) + qP_{u,vs}(q), & \text{if } us < u, \\ qP_{us,vs}(q) + P_{u,vs}(q), & \text{if } u < us \in S_n^{[n-1]\setminus\{i-1,i\}}, \\ 0, & \text{if } u < us \notin S_n^{[n-1]\setminus\{i-1,i\}}. \end{cases} \tag{9}$$

(For simplicity, in this proof we omit to write the superscripts "$[n-1] \setminus \{i-1, i\}$" and "$q$" on the polynomials.)

Note that, by (3),

$$p_u(j) = \begin{cases} p_v(j), & \text{if } p_v(j) < r(v), r(u) \text{ or } p_v(j) > r(v), r(u), \\ p_v(j) + 1, & \text{if } r(u) \leqslant p_v(j) < r(v), \\ p_v(j) - 1, & \text{if } r(v) < p_v(j) \leqslant r(u). \end{cases} \tag{10}$$

We have now three cases to consider.

(i) $u < us \notin S_n^{[n-1]\setminus\{i-1,i\}}$.

By Proposition 3.4, $p_u(j)$ is neither a peak of $\Lambda(u)$ nor a valley of $\Lambda(u)^{r(u)}$. Hence, by (10), $r(u) \neq p_v(j)$. If $r(v) < p_v(j) < r(u)$ or $r(u) < p_v(j) < r(v)$ then $\Lambda(v)$ has a peak strictly between $r(v)$ and $r(u)$ so $\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)$ is not Dyck and the result follows from (9). If $r(v), r(u) < p_v(j)$ or $p_v(j) < r(v), r(u)$ then, by (10), $p_v(j)$ is neither a peak of $\Lambda(u)$ nor a valley of $\Lambda(u)^{r(u)}$. So, by Proposition 4.3, $\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)$ is not Dyck and the result again follows from (9).

(ii) $us < u$.

By Proposition 3.4, $\Lambda(u)$ has a peak at $p_u(j)$ and the result follows from Theorem 4.4 (applied to $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ and $p_v(j)$), (9) and our induction hypothesis since, by Proposition 3.4, $\Lambda^\bullet(vs) = (\Lambda(v)^{p_v(j)}, r(v))$ and $\Lambda^\bullet(us) = (\Lambda(u)^{p_u(j)}, r(u)')$, where

$$r(u)' = \begin{cases} r(u), & \text{if } |p_u(j) - r(u)| > 1, \\ p_u(j), & \text{otherwise.} \end{cases}$$

(iii) $u < us \in S_n^{[n-1]\setminus\{i-1,i\}}$.

By Proposition 3.4, $p_u(j)$ is a valley of $\Lambda(u)^{r(u)}$.

Assume first that $|p_u(j) - r(u)| > 1$. Then, by Proposition 3.4, $\Lambda^\bullet(us) = (\Lambda(u)^{p_u(j)}, r(u))$ and the result follows immediately from (9), our induction hypothesis, and Theorem 4.4 (applied to $\Lambda^\bullet(v) \setminus \Lambda^\bullet(us)$). (Note that $\Lambda(u)^{p_u(j)} \subseteq \Lambda(v)$ since $us \leqslant v$.)

Assume now that $|p_u(j) - r(u)| = 1$, and so $\Lambda^\bullet(us) = (\Lambda(u)^{r(u)}, p_u(j))$. Suppose that $p_v(j) < r(v), p_u(j)$. Then, by (10), $p_u(j) = p_v(j) + 1$ and $r(u) \leqslant p_v(j)$. Hence $r(u) < p_u(j)$, so $r(u) = p_u(j) - 1 = p_v(j)$. Similarly, we conclude that $r(u) = p_v(j)$ if $r(v), p_u(j) < p_v(j)$, while $r(u) = p_v(j) - 1$ (resp. $p_v(j) + 1$) if $r(v) < p_v(j) \leqslant p_u(j)$ (resp. $p_u(j) \leqslant p_v(j) < r(v)$). The result then follows from (9), our induction hypothesis and Theorem 4.4 applied to $\Lambda^\bullet(v) \setminus \Lambda^\bullet(us)$.

Suppose now that $D(v) \subseteq \{s_{r-1}, s_r\}$. So v is the permutation

$$12 \ldots (r-k)\,(i+1) \ldots (i+k-1)\, i\, (r-k+1) \ldots (i-1)\,(i+k) \ldots n,$$

where $k = 1 + \max\{v(r-1) - i, 0\}$. If we set $h = i + k - r$, then $\Lambda(v) = (k^{h-1}, k-1)$.

Assume first that $D(v) = \{s_{r-1}, s_r\}$. Let $s = s_{r-1}$ and $t = s_r$. Then $p_v(r-1) = r-1$ and $r(vs) = r-1$. Let $w \in S_n^{[n-1]\setminus\{i-1,i\}}$ be such that $\mu(w, vs) \neq 0$ and $ws < w$. Then, by our induction hypothesis, $\Lambda^\bullet(vs)\setminus\Lambda^\bullet(w)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(vs)\setminus\Lambda^\bullet(w)) = 1$. By Proposition 4.6, this implies that there exists a self-conjugate partition $\lambda \subseteq (\min\{k,h\}^{\min\{k,h\}})$, with $s(\lambda) \geqslant 3$, such that $\Lambda(w) = (\lambda\angle(a,b,c))\!\downarrow_{(k^h)}$, where either $d(\lambda) = k-4$ and $((a,b,c), r(w) - r + 3) = ((3,3,1),2)$, or $d(\lambda) = k-3$ and $((a,b,c), r(w) - r + 3) \in \{((3,3,0),1), ((3,2,1),2), ((3,2,1),4), ((2,1,1),3)\}$. But since $ws < w$, by Proposition 3.4, we have that $p_w(r-1)$ is a peak of $\Lambda(w)$, so $d(\lambda) = k-3$ and $((a,b,c), r(w) - r + 3) = ((3,2,1),2)$.

This shows that the only $w \in S_n^{[n-1]\setminus\{i-1,i\}}$ such that $\mu(w, vs) \neq 0$ and $ws < w$ is $w = vst$. Hence, from Theorem 2.3 and Remark 2.4, we have that

$$(11)\quad P_{u,v}(q) = \begin{cases} P_{us,vs}(q) + qP_{u,vs}(q) - qP_{u,vst}(q), & \text{if } us < u, \\ qP_{us,vs}(q) + P_{u,vs}(q) - qP_{u,vst}(q), & \text{if } u < us \in S_n^{[n-1]\setminus\{i-1,i\}}, \\ 0, & \text{if } u < us \notin S_n^{[n-1]\setminus\{i-1,i\}}. \end{cases}$$

We have two cases to distinguish.

(a) $\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)$ is Dyck.

By Proposition 4.5, there exists a self-conjugate partition $\lambda \subseteq (\min\{k,h\}^{\min\{k,h\}})$, with $s(\lambda) \geqslant 3$, such that $\Lambda(u) = (\lambda \angle (a,b,c))\downarrow_{(k^h)}$, where

$$\begin{aligned} ((a,b,c), r(u) - r + 3) \in \{&((3,3,2),3), ((3,3,1),2), ((3,2,2),4), ((3,2,1),2), \\ &((3,2,1),4), ((3,1,1),3), ((2,1,0),3), ((2,0,0),2), \\ &((1,1,0),4), ((1,0,0),2), ((1,0,0),4), ((0,0,0),3)\}. \end{aligned}$$

Since these cases are all analogous, we treat only three of them.

Suppose that $((a,b,c), r(u) - r + 3) = ((1,0,0),2)$. Then $p_u(r-1) = r$ so, by Proposition 3.4, $us < u$ and $\Lambda^\bullet(us) = (\lambda\downarrow_{(k^h)}, r)$. Hence, by Propositions 4.5, 4.6 and 4.7, $\mathrm{dp}(\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)) = k - d(\lambda)$, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ is Dyck, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ are not Dyck, and $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)) = k + 1 - d(\lambda)$. Hence, by (11) and our induction hypothesis,

$$\begin{aligned} P_{u,v} &= P_{us,vs} + qP_{u,vs} - qP_{u,vst} = qP_{u,vs} \\ &= q \cdot q^{((\ell(u,v)-1)-(k+1-d(\lambda)))/2} = q^{(\ell(u,v)-(k-d(\lambda)))/2} \end{aligned}$$

and the result follows.

Suppose now that $((a,b,c), r(u) - r + 3) = ((2,1,0),3)$. Then $p_u(r-1) = r-1$ so, by Proposition 3.4, $us < u$ and $\Lambda^\bullet(us) = ((\lambda \angle (2,0,0))\downarrow_{(k^h)}, r-1)$. Hence, by Propositions 4.6 and 4.7, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ are all Dyck, and $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)) = k - d(\lambda)$, $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)) = k - 1 - d(\lambda)$, $\mathrm{dp}(\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)) = k - d(\lambda)$. Therefore, by (11) and our induction hypothesis, we have

$$\begin{aligned} P_{u,v} &= P_{us,vs} + qP_{u,vs} - qP_{u,vst} \\ &= q^{(\ell(u,v)-(k-d(\lambda)))/2} + q \cdot q^{((\ell(u,v)-1)-(k-1-d(\lambda)))/2} - q \cdot q^{((\ell(u,v)-2)-(k-d(\lambda)))/2} \\ &= q^{(\ell(u,v)-(k-2+d(\lambda)))/2}, \end{aligned}$$

and the result follows since $\mathrm{dp}(\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)) = k - 2 - d(\lambda)$.

Suppose now that $((a,b,c), r(u) - r + 3) = ((3,3,1),2)$. Then, $p_u(r-1) = r$ so, by Proposition 3.4, $u < us \in S_n^{[n-1]\setminus\{i-1,i\}}$ and $\Lambda^\bullet(us) = ((\lambda \angle (3,3,2))\downarrow_{(k^h)}, r)$. Hence, by Propositions 4.5, 4.6, and 4.7, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ is Dyck, whereas $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ are not, and $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)) = k - 3 - d(\lambda)$. Therefore, by (11) and our induction hypotesis,

$$\begin{aligned} P_{u,v} &= qP_{us,vs} + P_{u,vs} - qP_{u,vst} = P_{u,vs} \\ &= q^{((\ell(u,v)-1)-(k-3-d(\lambda)))/2} = q^{((\ell(u,v))-(k-2-d(\lambda)))/2}, \end{aligned}$$

and the result follows since $\mathrm{dp}(\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)) = k - 2 - d(\lambda)$.

(b) $\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)$ is not Dyck.

If $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ are not Dyck, then the result follows immediately from (11) and our induction hypothesis.

If $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ is Dyck, then, by Propositions 4.5 and 4.6, there exists a self-conjugate partition $\lambda \subseteq (\min\{k,h\}^{\min\{k,h\}})$, with $s(\lambda) \geqslant 3$, such that $\Lambda(u) = (\lambda \angle (a,b,c)) \downarrow_{(k^h)}$, where

$$((a,b,c), r(u) - r + 3) \in \{((3,3,0),1), ((3,2,0),1), ((3,2,0),4), ((3,1,0),3), ((2,1,1),3), ((2,1,0),1), ((1,1,1),4), ((1,1,0),1)\}.$$

We treat only four of these cases. If $((a,b,c), r(u)-r+3) \in \{((3,2,0),4), ((1,1,1),4)\}$ then, by Proposition 3.4, $us \notin S_n^{[n-1]\setminus\{i-1,i\}}$ and the result follows by (11) and our induction hypothesis. If $(a,b,c) = (2,1,1)$ then, by Proposition 3.4, $u < us \in S_n^{[n-1]\setminus\{i-1,i\}}$ and, by Proposition 4.7, $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)) = \mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)) - 1$. Also, $\Lambda(us) = (\lambda \angle (2,2,1)) \downarrow_{(k^h)}$, so, by Proposition 4.6, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ is not Dyck. Therefore, by (11) and our induction hypothesis, we have that

$$P_{u,v} = 0 + q^{(\ell(u,vs) - \mathrm{dp}(\Lambda^\bullet(vs)\setminus\Lambda^\bullet(u)))/2} - q \cdot q^{(\ell(u,vst) - \mathrm{dp}(\Lambda^\bullet(vst)\setminus\Lambda^\bullet(u)))/2} = 0.$$

If $((a,b,c), r(u) - r + 3) = ((3,2,0),1)$ then, by Proposition 3.4, $u > us$ and, by Proposition 4.7, $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)) = \mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)) - 1$. Also $\Lambda(us) = (\lambda \angle (3,1,0)) \downarrow_{(k^h)}$, so, by Proposition 4.6, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ is not Dyck. Therefore, by (11) and our induction hypothesis, we have

$$P_{u,v} = 0 + q \cdot q^{(\ell(u,vs) - \mathrm{dp}(\Lambda^\bullet(vs)\setminus\Lambda^\bullet(u)))/2} - q^{(\ell(u,vst) - \mathrm{dp}(\Lambda^\bullet(vst)\setminus\Lambda^\bullet(u)))/2} = 0.$$

If $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ is not Dyck and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ is Dyck, then, by Propositions 4.6 and 4.7, there exists a self-conjugate partition $\lambda \subseteq (\min\{k,h\}^{\min\{k,h\}})$, with $s(\lambda) \geqslant 3$, such that $\Lambda(u) = (\lambda \angle (a,b,c)) \downarrow_{(k^h)}$, where

$$((a,b,c), r(u) - r + 3) \in \{((3,1,0),1), ((3,0,0),2), ((2,2,1),2), ((2,2,0),1)\}.$$

We treat only one of these cases. For instance, if $(a,b,c) = (2,2,1)$ then, by Proposition 3.4, $us < u$ and $\Lambda(us) = (\lambda \angle (2,1,1)) \downarrow_{(k^h)}$, so, by Proposition 4.6, $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ is Dyck, $\mathrm{dp}(\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)) = \mathrm{dp}(\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u))$, and we conclude as above, by (11) and our induction hypothesis.

Finally suppose that $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(u)$ and $\Lambda^\bullet(vst) \setminus \Lambda^\bullet(u)$ are not Dyck. We claim that $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ is not Dyck. To prove that, assume by contradiction that $\Lambda^\bullet(vs) \setminus \Lambda^\bullet(us)$ is Dyck. Then, since $u, us \in S_n^{[n-1]\setminus\{i-1,i\}}$, we have from Proposition 3.4 that $p_{us}(r-1)$ is either a peak of $\Lambda(us)$ or a valley of $\Lambda(us)^{r(us)}$. Hence, by Proposition 4.6, there exists a self-conjugate partition $\lambda \subseteq (\min\{k,h\}^{\min\{k,h\}})$, with $s(\lambda) \geqslant 3$, such that $\Lambda(us) = (\lambda \angle (a,b,c)) \downarrow_{(h^k)}$, where

$$((a,b,c), r(us) - r + 3) \in \{((3,3,1),2), ((3,2,1),2), ((3,2,1),4), ((3,2,0),1), ((3,1,1),3), ((3,1,0),3), ((2,1,1),3), ((2,1,0),1), ((2,1,0),3), ((2,0,0),2), ((1,1,0),4), ((1,0,0),2)\},$$

(4 of the 16 configurations listed in Proposition 4.6 do not occur here, because in those cases $p_{us}(r-1)$ would be neither a peak of $\Lambda(us)$ nor a valley of $\Lambda(us)^{r(us)}$). Therefore, by Proposition 3.4, $\Lambda(u) = (\lambda\angle(a,b,c))\downarrow_{(k^h)}$, where

$$\begin{aligned}((a,b,c), r(u)-r+3) \in \{&((3,3,2),3),((3,1,1),3),((3,2,2),4),((3,1,0),1),\\ &((3,2,1),2),((3,0,0),2),((2,2,1),2),((2,2,0),1),\\ &((2,0,0),2),((2,1,0),3),((1,0,0),4),((0,0,0),3)\},\end{aligned}$$

By Propositions 4.5 and 4.7, this implies that at least one of $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ and $\Lambda^\bullet(vst)\setminus\Lambda^\bullet(u)$ is Dyck, contradicting our hypotheses.

Suppose now that $D(v)=\{s_{r-1}\}$. Then $\Lambda^\bullet(v)=((r-i),r)$ and $\Lambda^\bullet(u)=((r(u)-i),r(u))$, with $r(u)\leqslant r$. If $r(u)=r$, then $u=v$ and we are done. If $r(u)=r-1$, then $us_{r-1}=v\in S_n^{[n-1]\setminus\{i-1,i\}}$, $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ is Dyck, $\mathrm{dp}(\Lambda^\bullet(v)\setminus\Lambda^\bullet(u))=1$ and $P_{u,v}(q)=1$. If $r(u)<r-1$, then $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ is not Dyck and $P_{u,v}(q)=0$ since $us_{r-1}\notin S_n^{[n-1]\setminus\{i-1,i\}}$.

Finally, suppose that $D(v)=\{s_r\}$. Then $\Lambda^\bullet(v)=((1^{i-r}),r)$ and $\Lambda^\bullet(u)=((1^{i-r(u)}),r(u))$, with $r(u)\geqslant r$. We are in the mirror configuration of the preceding case and we may conclude in the same way. □

We illustrate the preceding theorem with an example. Let $n=9$, $i=5$, $u=125637849$ and $v=651728934$. Then $\Lambda^\bullet(u)=((4,2),3)$ and $\Lambda^\bullet(v)=((5,5,3,2,1),2)$ so $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ is not Dyck and $P_{u,v}^{[8]\setminus\{4,5\}}(q)=0$. On the other hand, if $n=7$, $i=4$, $u=1245367$ and $v=5647123$ then $\Lambda^\bullet(u)=((2),3)$ and $\Lambda^\bullet(v)=((4,4,4,2),3)$ so $\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(v)\setminus\Lambda^\bullet(u))=4$ so, by Theorem 5.1, $P_{u,v}^{[6]\setminus\{3,4\}}(q)=q^{\frac{1}{2}(12-4)}=q^4$.

For simplicity, we denote by $w_0(n,i)$ the longest element of $S_n^{[n-1]\setminus\{i-1,i\}}$. Note that, from (2), we have that

$$P_{u,v}^{[n-1]\setminus\{i-1,i\},-1}(q)=P_{w_0(n,i)u,w_0(n,i)v}(q)$$

for all $u,v\in S_n^{[n-1]\setminus\{i-1,i\}}$. On the other hand, the polynomials $P_{w_0(n,i)u,w_0(n,i)v}(q)$ have been computed (for $u,v\in S_n^{[n-1]\setminus\{i-1,i\}}$) in [17] (see also [1, Chap.5, Ex.39]). Thus Theorem 5.1 completes the computation of the parabolic Kazhdan–Lusztig polynomials of the tight quotients of the symmetric groups.

In view of the geometric interpretation given in [15] of $P_{u,v}^{J,q}(q)$ ($u,v\in W^J$, W a Weyl group), it would be interesting to have a geometric proof of Theorem 5.1.

We note the following simple consequence of Theorem 5.1 which seems to be difficult to prove directly.

Corollary 5.2. *Let $u,v\in S_n^{[n-1]\setminus\{i-1,i\}}$, $u<v$, and $(k,k+1)\in D(v)$ be such that either $v^{-1}(i)<k+1<u^{-1}(i)$ or $u^{-1}(i)<k<v^{-1}(i)$. Then*

$$P_{u,v}^{[n-1]\setminus\{i-1,i\},q}(q)=0.$$

Proof. Since $(k,k+1)$ is a descent of v, it follows from Proposition 3.4 that $\Lambda(v)$ has a peak at $k+1$ (respectively, k) if $v^{-1}(i)<k+1<u^{-1}(i)$ (respectively,

$u^{-1}(i) < k < v^{-1}(i)$). Hence $\Lambda(v)$ has a peak strictly between the circles of $\Lambda^\bullet(v)$ and $\Lambda^\bullet(u)$ so the result follows from Theorems 4.1 and 5.1. $\square$

For certain intervals, Theorem 5.1 becomes even more explicit. For simplicity, we call a 1-superpartition (λ, r) satisfying condition (ii) in Proposition 4.8 a *dented square*.

Corollary 5.3. *Let $v \in S_n^{[n-1]\setminus\{i-1,i\}}$, with $\ell(v) \geqslant 3$. Then*

$$P_{e,v}^{[n-1]\setminus\{i-1,i\},q}(q) = \begin{cases} q^{(|\Lambda(v)|-d(\Lambda(v)))/2}, & \text{if } \Lambda(v) \text{ is a square,} \\ q^{(|\Lambda(v)|-d(\Lambda(v)))/2-1}, & \text{if } \Lambda^\bullet(v) \text{ is a dented square,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. This follows immediately from Theorem 5.1, Proposition 4.8 and the definition of depth for Dyck 1-superpartitions. $\square$

Given a 1-superpartition (μ, r) contained in $(n^{m-1}, n-1)$, we say that $((n^{m-1}, n-1), n) \setminus (\mu, r)$ is *almost self-conjugate* (in short, *asc*) if it satisfies the conditions of Proposition 4.5.

Corollary 5.4. *Let $u \in S_n^{[n-1]\setminus\{i-1,i\}}$. Then*

$$P_{u,w_0(n,i)}^{[n-1]\setminus\{i-1,i\},q}(q) = \begin{cases} q^{((m+1)(m-2)-|\lambda|+d(\lambda))/2}, & \text{if } \Lambda^\bullet(w_0(n,i)) \setminus \Lambda^\bullet(u) \text{ is asc,} \\ 0, & \text{otherwise,} \end{cases}$$

where $m = \min(n-i+1, i)$ and λ has the same meaning as in (7).

Proof. This follows immediately from Theorem 5.1 and Proposition 4.5. $\square$

We now show that the main result of [2] follows from Theorem 5.1. Note that, if $u \in S_n^{[n-1]\setminus\{i\}}$, then the partition $\Lambda(u)$ defined in [2] (see (2)) coincides with the partition $\Lambda(u)$ defined in §3.

Corollary 5.5. *Let $u, v \in S_n^{[n-1]\setminus\{i\}}$, with $u \leqslant v$. Then*

$$P_{u,v}^{[n-1]\setminus\{i\},q} = \begin{cases} q^{(\ell(u,v)-\mathrm{dp}(\Lambda(v)\setminus\Lambda(u)))/2}, & \text{if } \Lambda(v) \setminus \Lambda(u) \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since $((S_n)_{[n-1]\setminus\{i\}})^{[n-1]\setminus\{i-1,i\}} = \{\omega_j : 0 \leqslant j \leqslant i-1\}$, where ω_j is the cycle $(i, i-1, \ldots, i-j)$, for $j = 0, \ldots, i-1$, by Proposition 2.5, we have that

$$P_{u,v}^{[n-1]\setminus\{i\},q}(q) = \sum_{j=0}^{i-1} (-1)^j P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q}(q).$$

Let $\Lambda(u) = \mu = (\mu_1, \ldots, \mu_i)$ and $\Lambda(v) = \lambda = (\lambda_1, \ldots, \lambda_i)$. Note that, since $u, v \in S_n^{[n-1]\setminus\{i\}}$, $r(v) = \lambda_1 + i$ (the rightmost valley of λ) and, for $0 \leqslant j \leqslant i-1$, we have $\Lambda(\omega_j u) = (\mu_1 + 1, \ldots, \mu_j + 1, \mu_{j+1}, \ldots, \mu_i)$ and $r(\omega_j u) = \mu_{j+1} + i - j$ (that is, the circle of $\Lambda^\bullet(\omega_j u)$ is in the valley between parts j and $j+1$ of $\Lambda(\omega_j u)$.)

Let x be the rightmost peak of λ. By Theorem 4.1, if $r(\omega_j u) < x$ then $\Lambda^\bullet(v) \setminus \Lambda^\bullet(\omega_j u)$ is not Dyck, and so $P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q} = 0$. Since $r(\omega_0 u) > r(\omega_1 u) > \cdots >$

$r(\omega_{i-1}u)$, if we set $j_0 = \max\{j : r(\omega_j u) \geqslant x\}$, we have that

$$P_{u,v}^{[n-1]\setminus\{i\},q}(q) = \sum_{j=0}^{j_0} (-1)^j P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q}(q).$$

Now note that, if $r(\omega_j u) > x$, then, by Proposition 4.2, $\Lambda(v) \setminus \Lambda(\omega_j u)$ and $\Lambda(v) \setminus \Lambda(\omega_j u)^{r(\omega_j u)}$ cannot both be Dyck. But $\Lambda(v) \setminus \Lambda(\omega_j u)^{r(\omega_j u)} = \Lambda(v) \setminus \Lambda(\omega_{j+1}u)$, hence, if $\Lambda(v) \setminus \Lambda(\omega_j u)$ is Dyck, then $\Lambda(v) \setminus \Lambda(\omega_{j+1}u)$ is not Dyck. Therefore, if we set

$$\begin{aligned} J_D &= \{j \in [0, j_0] : \Lambda(v) \setminus \Lambda(\omega_j u) \text{ is Dyck}\}, \\ J_D^\bullet &= \{j \in [0, j_0] : \Lambda^\bullet(v) \setminus \Lambda^\bullet(\omega_j u) \text{ is Dyck}\}, \end{aligned}$$

then J_D and $J_D \setminus \{0\} - 1 = \{j - 1 : j \in J_D \setminus \{0\}\}$ are disjoint and $J_D^\bullet = J_D \cup (J_D \setminus \{0\} - 1)$. Hence, if $j \in J_D \setminus \{0\}$, then $j, j-1 \in J_D^\bullet$ and $|\Lambda(v) \setminus \Lambda(\omega_j u)| = |\Lambda(v) \setminus \Lambda(\omega_{j-1}u)| - 1$, $\mathrm{dp}(\Lambda(v) \setminus \Lambda(\omega_j u)) = \mathrm{dp}(\Lambda(v) \setminus \Lambda(\omega_{j-1}u)) - 1$, so

$$P_{\omega_j u,v}^{[n-1]\setminus\{i,i-1\},q}(q) = P_{\omega_{j-1} u,v}^{[n-1]\setminus\{i,i-1\},q}(q).$$

We therefore conclude that

$$\begin{aligned} P_{u,v}^{[n-1]\setminus\{i\},q}(q) &= \sum_{j \in J_D^\bullet} (-1)^j P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q}(q) \\ &= \sum_{j \in J_D} (-1)^j P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q}(q) + \sum_{j \in J_D\setminus\{0\}-1} (-1)^j P_{\omega_j u,v}^{[n-1]\setminus\{i-1,i\},q}(q) \\ &= \begin{cases} P_{u,v}^{[n-1]\setminus\{i,i-1\},q}(q), & \text{if } 0 \in J_D, \\ 0, & \text{if } 0 \notin J_D. \end{cases} \end{aligned}$$

But if $0 \in J_D$ then, by definition, $\Lambda(v) \setminus \Lambda(u)$ is Dyck. Hence, by Proposition 2.1, x is either a peak or a valley of $\Lambda(u)$. Since $r(u)$ and $r(v)$ are the rightmost valleys of $\Lambda(u)$ and $\Lambda(v)$, this implies that $x \leqslant r(u)$ and hence that there are no peaks of $\Lambda(v)$ strictly between $r(u)$ and $r(v)$. Therefore $\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)$ is Dyck and $\mathrm{dp}(\Lambda^\bullet(v) \setminus \Lambda^\bullet(u)) = \mathrm{dp}(\Lambda(v) \setminus \Lambda(u))$ so the result follows from Theorem 5.1. $\square$

We conclude by deriving two consequences of our main result for the ordinary Kazhdan–Lusztig polynomials.

The next result follows immediately from (1) and Theorem 5.1 but we feel that it should be stated explicitly.

Corollary 5.6. *Let $u, v \in S_n^{[n-1]\setminus\{i-1,i\}}$, $u \leqslant v$. Then*

$$\begin{aligned} &\sum_{w \in (S_n)_{[n-1]\setminus\{i-1,i\}}} (-1)^{\ell(w)} P_{wu,v} \\ &\qquad = \begin{cases} q^{(|\Lambda(v)\setminus\Lambda(u)| - \mathrm{dp}(\Lambda^\bullet(v)\setminus\Lambda^\bullet(u)))/2}, & \text{if } \Lambda^\bullet(v) \setminus \Lambda^\bullet(u) \text{ is Dyck,} \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It is of interest to know which Kazhdan–Lusztig polynomials attain the maximum possible degree and, in that case, if the leading coefficient equals 1 (see, e.g., [1, Chap. 6], [14], [21], [22], [24]). Theorem 5.1 enables us to answer this question explicitly for permutations $u, v \in S_n$ such that $D(u) \cup D(v) \subseteq \{i-1, i\}$ for some $2 \leqslant i \leqslant n-1$.

Corollary 5.7. *Let $u, v \in S_n$, $u < v$ be such that $D(u), D(v) \subseteq \{i-1, i\}$ for some $2 \leqslant i \leqslant n-1$. Then the following are equivalent:*

(1) *the coefficient of $q^{(\ell(u,v)-1)/2}$ in $P_{u,v}$ is non-zero;*
(2) *$\Lambda(v^{-1}) \setminus \Lambda(u^{-1})$ is a Dyck cbs and $\Lambda(v^{-1})$ has no peaks strictly between $u(i)$ and $v(i)$;*
(3) *the coefficient of $q^{(\ell(u,v)-1)/2}$ in $P_{u,v}$ is 1.*

Proof. It is well known (see, e.g., [1, Chap.5, Ex.12]) that $P_{u,v} = P_{u^{-1},v^{-1}}$. Since $D(u), D(v) \subseteq \{i-1, i\}$, $u^{-1}, v^{-1} \in S_n^{[n-1]\setminus\{i-1,i\}}$ so it follows from (1), Proposition 3.1 of [7], and Proposition 2.4.4 of [1] that the coefficient of $q^{(\ell(u,v)-1)/2}$ in $P_{u^{-1},v^{-1}}$ equals the coefficient of $q^{(\ell(u,v)-1)/2}$ in $P^{[n-1]\setminus\{i-1,i\},q}_{u^{-1},v^{-1}}$ (see also [23]). The result then follows from Theorems 5.1, 4.1 and the definitions of Dyck and depth of a Dyck skew partition. □

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References

[1] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, Graduate Texts in Mathematics, 231, Springer-Verlag, New York, 2005.
[2] F. Brenti, Kazhdan-Lusztig and R-polynomials, Young's lattice, and Dyck partitions, Pacific J. Math., 202 (2002), 257–286.
[3] F. Brenti, Enumerative and combinatorial properties of Dyck partitions, J. Combin. Theory Ser. A, 99 (2002), 51–74.
[4] L. Casian, D. Collingwood, The Kazhdan–Lusztig conjecture for generalized Verma modules, Math. Z., 195 (1987), 581–600.
[5] S. Corteel, J. Lovejoy, Overpartitions, Trans. Amer. Math. Soc., 356 (2004), 1623–1635.
[6] S. Corteel, O. Mallet, Overpartitions, lattice paths, and Rogers–Ramanujan identities, J. Combin. Theory Ser. A, 114 (2007), 1407–1437.
[7] V. Deodhar, On some geometric aspects of Bruhat orderings. II. The parabolic analogue of Kazhdan–Lusztig polynomials, J. Algebra, 111 (1987), 483–506.
[8] V. Deodhar, Duality in parabolic setup for questions in Kazhdan–Lusztig theory, J. Algebra, 142 (1991), 201–209.
[9] P. Desrosiers, L. Lapointe, P. Mathieu, Jack superpolynomials, superpartition ordering and determinantal formulas, Comm. Math. Phys., 233, 383–402 (2003)
[10] I. Frenkel, M. Khovanov, A. Kirillov, Kazhdan–Lusztig polynomials and canonical basis, Transform. Groups, 3 (1998), 321–336.
[11] J. Haglund, M. Haiman, N. Loehr, A combinatorial formula for Macdonald polynomials, J. Amer. Math. Soc., 18 (2005), 735–761.
[12] J. Haglund, M. Haiman, N. Loehr, J. Remmel, A. Ulyanov, A combinatorial formula for the character of the diagonal coinvariants, Duke Math. J., 126 (2005), 195–232.
[13] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, 29, Cambridge Univ. Press, Cambridge, 1990.
[14] B. Jones, Leading coefficients of Kazhdan–Lusztig polynomials for Deodhar elements, J. Algebraic Combin., 29 (2009), 229–260.

[15] M. Kashiwara, T. Tanisaki, Parabolic Kazhdan–Lusztig polynomials and Schubert varieties, J. Algebra, 249 (2002), 306–325.
[16] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math., 53 (1979), 165–184.
[17] A. Lascoux, Polynômes de Kazhdan–Lusztig pour les variétés de Schubert vexillaires, C. R. Acad. Sci. Paris Sr. I Math., 321 (1995), 667–670.
[18] B. Leclerc, J.-Y. Thibon, Littlewood-Richardson coefficients and Kazhdan–Lusztig polynomials, Adv. Stud. Pure Math., 28 (2000), 155–220.
[19] J. Lovejoy, O. Mallet, Overpartition pairs and two classes of basic hypergeometric series, Adv. Math., 217 (2008), 386–418.
[20] P.A. MacMahon, Combinatory Analysis, Chelsea Publishing Co., New York, 1960.
[21] T. McLarnan, G. Warrington, Counterexamples to the 0-1 conjecture, Represent. Theory, 7 (2003), 181–195.
[22] M. Marietti, Boolean elements in Kazhdan–Lusztig theory, J. Algebra, 295 (2006), 1–26.
[23] M. Marietti, Parabolic Kazhdan–Lusztig and R-polynomials for Boolean elements in the symmetric group, European J. Combin., 31 (2010), 908–924.
[24] J. Shi, Joint relations on elements of the symmetric group, J. Algebra, 319 (2008), 3197–3221.
[25] W. Soergel, Kazhdan–Lusztig polynomials and a combinatoric for tilting modules, Represent. Theory, 1 (1997), 83–114.
[26] W. Soergel, Character formulas for tilting modules over Kac-Moody algebras, Represent. Theory, 1 (1997), 115–132.
[27] R. P. Stanley, Enumerative Combinatorics, Vol.1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
[28] R. P. Stanley, Enumerative Combinatorics, Vol.2, Cambridge Studies in Advanced Mathematics, 62, Cambridge Univ. Press, Cambridge, 1999.
[29] J. Stembridge, Tight quotients and double quotients in the Bruhat order, Electron. J. Combin., 11 (2005), R14, 41 pp.
[30] D. Uglov, Canonical bases of higher level q-deformed Fock spaces and Kazhdan–Lusztig polynomials, in: Physical Combinatorics, Progress in Math., 191, Birkhäuser, Boston, MA, 2000, pp. 249–299.
[31] M. Varagnolo, E. Vasserot, On the decomposition matrices of the quantized Schur algebra, Duke Math. J., 100 (1999), 267–297.

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