Parabolic Kazhdan-Lusztig R-polynomials for tight quotients of the symmetric groups [1]

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Abstract

We give explicit closed combinatorial formulas for the parabolic Kazhdan-Lusztig R-polynomials of the tight quotients of the symmetric groups. We give two formulations of our result, one in terms of permutations and one in terms of Motzkin paths. As an application of our results we obtain explicit closed combinatorial formulas for certain sums and alternating sums of ordinary Kazhdan-Lusztig R-polynomials.

1 Introduction

In their fundamental paper [11] Kazhdan and Lusztig defined, for any Coxeter group W, a family of polynomials, indexed by pairs of elements of W, which have become known as the Kazhdan-Lusztig polynomials of W (see, e.g., [9, Chap.7] or [2, Chap.5]). These polynomials play an important role in several areas of mathematics, including the algebraic geometry and topology of Schubert varieties and representation theory (see, e.g., [2, Chap.5], and the references cited there). In order to prove the existence of these polynomials Kazhdan and Lusztig introduced another family of polynomials, usually called the R-polynomials, whose knowledge is equivalent to that of the Kazhdan-Lusztig polynomials.

In 1987 Deodhar ([5]) introduced parabolic analogues of all these polynomials. These parabolic Kazhdan-Lusztig and R-polynomials reduce to the ordinary ones for the trivial parabolic subgroup of W and are also related to them in other ways

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(see, e.g., Proposition 2.2 below). Besides these connections the parabolic polynomials also play a direct role in several areas including the theories of generalized Verma modules ([4]), tilting modules ([13], [14]), quantized Schur algebras ([17]), Macdonald polynomials ([8], [7]), Schubert varieties in partial flag manifolds ([10]), and in the representation theory of the Lie algebra gl_n ([12]).

The purpose of this work is to study the parabolic Kazhdan-Lusztig R-polynomials for the tight quotients of the symmetric groups. These quotients were first introduced and studied by Stembridge in [16], who classified them for the finite Coxeter groups. For the symmetric groups S_n, the tight quotients are the ones obtained by deleting either a single node (maximal quotients) or two adjacent nodes in the Dynkin diagram of S_n. The parabolic Kazhdan-Lusztig R-polynomials for the maximal quotients of the symmetric groups have been computed in [3], here we complete the computation of the parabolic R-polynomials of the tight quotients of the symmetric groups by dealing with the other ones. More precisely, we obtain explicit combinatorial product formulas for these polynomials. We give two formulations of our result, one in terms of permutations and one in terms of Motzkin paths. As an application of our results, we obtain combinatorial closed product formulas for certain sums and alternating sums of ordinary Kazhdan-Lusztig R-polynomials.

The organization of the paper is as follows. In the next section we recall definitions, notation and results that are used in the rest of this work. In §3 we prove our main result, and derive some consequences of it.

2 Preliminaries

In this section we collect some definitions, notation and results that are used in the rest of this paper. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$ and $\mathbf{N} \stackrel{\text{def}}{=} \mathbf{P} \cup \{0\}$. For $m, n \in \mathbf{N}$, $m \geq n$, we let $[n, m] \stackrel{\text{def}}{=} \{n, n+1, \ldots, m-1, m\}$ and $[n] \stackrel{\text{def}}{=} [1, n]$ (where $[0] \stackrel{\text{def}}{=} \emptyset$). The cardinality of a set A will be denoted by $|A|$. For $S \subseteq \mathbf{N}$ we write $S = \{s_1, \ldots, s_k\}_<$ to mean that $S = \{s_1, \ldots, s_k\}$ and $s_1 < s_2 < \cdots < s_k$. If P is a statement then we let $\chi(P) \stackrel{\text{def}}{=} 1$ if P is true and $\chi(P) \stackrel{\text{def}}{=} 0$ if P is false. For $i, j \in \mathbf{N}$ we let $\delta_{i,j}$ be the Kronecker delta.

Given a set T we let $S(T)$ be the set of all bijections $\pi : T \to T$, and $S_n \stackrel{\text{def}}{=} S([n])$. If $\sigma \in S_n$ then we write $\sigma = a_1 \cdots a_n$ to mean that $\sigma(i) = a_i$ for all $i \in [n]$. We also write σ in *disjoint cycle form* (see, e.g., [15], p.17) and we usually omit writing the 1-cycles of σ. So, for example, if $\sigma = (9,7,1,3,5)(2,6)$ then $\sigma(1) = 3$, $\sigma(2) = 6$, $\sigma(3) = 5$, $\sigma(4) = 4$, etc... Given $\sigma, \tau \in S_n$ we let $\sigma\tau \stackrel{\text{def}}{=} \sigma \circ \tau$ (composition of functions) so that, for example, $(1,2)(2,3) = (1,2,3)$.

Let $n \in \mathbf{P}$. By a *Motzkin path* of length n we mean a function $M : [0,n] \to \mathbf{Z}$ such that $M(0) = 0$ and $M(j) - M(j-1) \in \{0, 1, -1\}$ for all $j \in [n]$. If $j \in [n]$ then we call $M(j) - M(j-1)$ the j-th *step* of M and say that such a step is *up* (respectively, *horizontal*, *down*) if $M(j) - M(j-1) = 1$ (respectively, 0, -1). We will usually depict a Motzkin path by its diagram. So, for example, the Motzkin path depicted in Figure 1 is the Motzkin path $M : [0,9] \to \mathbf{Z}$ such that $(M(1), \ldots, M(9)) = (1, 0, -1, -1, 0, 0, -1, -2, -1)$.

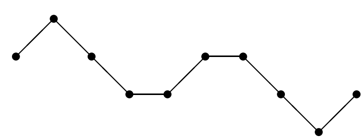

Figure 1.

We follow [9] and [2] for general Coxeter groups notation and terminology. In particular, given a Coxeter system (W,S) and $u \in W$ we denote by $\ell(u)$ the length of u in W, with respect to S, and we let $D(u) \stackrel{\text{def}}{=} \{s \in S : \ \ell(us) < \ell(u)\}$ and $\varepsilon_u \stackrel{\text{def}}{=} (-1)^{\ell(u)}$. For $u, v \in W$ we let $\ell(u,v) \stackrel{\text{def}}{=} \ell(v) - \ell(u)$. We denote by e the identity of W, and we let $T \stackrel{\text{def}}{=} \{usu^{-1} : u \in W, \ s \in S\}$ be the set of reflections of W. Given $J \subseteq S$ we let W_J be the parabolic subgroup generated by J and

$$W^J \stackrel{\text{def}}{=} \{u \in W : \ell(su) > \ell(u) \text{ for all } s \in J\}. \tag{1}$$

Note that $W^{\emptyset} = W$. We always assume that W^J is partially ordered by *Bruhat order*. Recall that this means that $x \leq y$ if and only if there exist $r \in \mathbf{N}$ and $t_1, \ldots, t_r \in T$ such that $t_r \cdots t_1 \, x = y$ and $\ell(t_i \cdots t_1 \, x) > \ell(t_{i-1} \cdots t_1 x)$ for $i = 1, \ldots, r$.

The following result is due to Deodhar, and we refer the reader to [5, §§2-3] for its proof.

Theorem 2.1 *Let (W,S) be a Coxeter system, and $J \subseteq S$. Then, for each $x \in \{-1, q\}$, there is a unique family of polynomials $\{R^{J,x}_{u,v}(q)\}_{u,v \in W^J} \subseteq \mathbf{Z}[q]$ such that, for all $u, v \in W^J$:*

i) $R^{J,x}_{u,v}(q) = 0$ *if* $u \not\leq v$;

ii) $R^{J,x}_{u,u}(q) = 1$;

iii) *if* $u < v$ *and* $s \in D(v)$ *then*

$$R_{u,v}^{J,x}(q) = \begin{cases} R_{us,vs}^{J,x}(q), & \text{if } us < u, \\ (q-1)R_{u,vs}^{J,x}(q) + qR_{us,vs}^{J,x}(q), & \text{if } u < us \in W^J, \\ (q-1-x)R_{u,vs}^{J,x}(q), & \text{if } u < us \notin W^J. \end{cases}$$

The polynomials $R_{u,v}^{J,x}(q)$, whose existence is guaranteed by the previous theorem, are called the *parabolic* R*-polynomials* of W^J of type x. It follows immediately from Theorem 2.1 and from well known facts (see, e.g., [9, §7.5]) that $R_{u,v}^{\emptyset,-1}(q)$ $(= R_{u,v}^{\emptyset,q}(q))$ are the (ordinary) R*-polynomials* of W which we will denote simply by $R_{u,v}(q)$, as customary. The parabolic R-polynomials can then be used to define and compute the parabolic Kazhdan-Lusztig polynomials of W^J of type x (see [5, Proposition 3.1]).

The parabolic R-polynomials are related to their ordinary counterparts also in the following way.

Proposition 2.2 *Let* (W, S) *be a Coxeter system,* $J \subseteq S$*, and* $u, v \in W^J$*. Then*

$$R_{u,v}^{J,x}(q) = \sum_{w \in W_J} (-x)^{\ell(w)} R_{wu,v}(q),$$

for all $x \in \{-1, q\}$*.*

A proof of this result can be found in [5, Proposition 2.12].

There is one more property of the parabolic R-polynomials that we will use and that we recall for the reader's convenience. A proof of it can be found in [6, Corollary 2.2].

Proposition 2.3 *Let* (W, S) *be a Coxeter system, and* $J \subseteq S$*. Then*

$$q^{\ell(u,v)} R_{u,v}^{J,q}\left(\frac{1}{q}\right) = (-1)^{\ell(u,v)} R_{u,v}^{J,-1}(q)$$

for all $u, v \in W^J$*.*

It is well known (see, e.g., [2, Chap. 1]) that the symmetric group S_n is a Coxeter group with respect to the generating set $S = \{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$ for all $i \in [n-1]$. The following result is also well known (see, e.g., [2, §1.5]).

Proposition 2.4 *Let $v \in S_n$. Then*

$$\ell(v) = |\{(i,j) \in [n]^2 : i < j, v(i) > v(j)\}|$$

and

$$D(v) = \{(i,i+1) \in S : v(i) > v(i+1)\}.$$

For $k \in [n]$ and $U, T \subseteq [n]$ such that $|U| = |T| = k$ let $U \preceq T$ if and only if $u_i \leq t_i$ for all $i \in [k]$ where $\{u_1, \ldots, u_k\}_< \stackrel{\text{def}}{=} U$ and $\{t_1, \ldots, t_k\}_< \stackrel{\text{def}}{=} T$. Note that $U \preceq T$ if and only if

$$|\{j \geq r \,:\, j \in T\}| \geq |\{j \geq r \,:\, j \in U\}| \tag{2}$$

for all $r \in [n]$. The following result is well known (see, e.g., [2, Theorem 2.6.3]).

Theorem 2.5 *Let $u, v \in S_n$. Then the following are equivalent:*

i) $u \leq v$;

ii) $u([j]) \preceq v([j])$ *for all* $j \in [n-1]$;

iii) $u([j]) \preceq v([j])$ *for all* j *such that* $s_j \in D(u)$.

Our purpose in this work is to study the parabolic R-polynomials of the tight quotients of the symmetric groups. These quotients were first introduced and studied by Stembridge in [16], who classified them for the finite Coxeter groups. For the symmetric groups S_n, the tight quotients are the ones obtained by deleting either a single node (maximal quotients) or two adjacent nodes in the Dynkin diagram of S_n. The parabolic R-polynomials for the maximal quotients have been computed in [3], in this work we complete the computation of the parabolic R-polynomials of the tight quotients by dealing with the other ones.

Let $n \in \mathbf{P}$ and $2 \leq i \leq n-1$. For simplicity, we let $S_n^{(i)} \stackrel{\text{def}}{=} (S_n)^{J_i}$ where $J_i \stackrel{\text{def}}{=} \{s_1, s_2, \ldots, s_{i-2}, s_{i+1}, \ldots, s_{n-1}\}$. It follows immediately from (1), Proposition 2.4 and well known facts (see, e.g., [2, Proposition 1.4.2]) that

$$S_n^{(i)} = \{v \in S_n : v^{-1}(1) < \cdots < v^{-1}(i-1),\, v^{-1}(i+1) < \cdots < v^{-1}(n)\}. \tag{3}$$

For $u, v \in S_n^{(i)}$ and $r \in [n]$ we let

$$a_r(u,v) \stackrel{\text{def}}{=} |\{j \geq r : \; j \in v^{-1}([i-1])\}| - |\{j \geq r : \; j \in u^{-1}([i-1])\}| \tag{4}$$

and

$$\tilde{a}_r(u,v) \stackrel{\text{def}}{=} |\{j \geq r:\ j \in v^{-1}([i])\}| - |\{j \geq r:\ \ j \in u^{-1}([i])\}|. \tag{5}$$

So, for example, if $n = 9$, $i = 5$, $u = 1\,6\,2\,3\,5\,7\,4\,8\,9$ and $v = 6\,5\,7\,1\,8\,2\,3\,9\,4$ then $(a_1(u,v), \ldots, a_9(u,v)) = (0,1,1,2,2,2,1,1,1)$ and $(\tilde{a}_1(u,v), \ldots, \tilde{a}_9(u,v)) = (0,1,0,1,1,2,1,1,1)$. Note that

$$\tilde{a}_k(u,v) = \begin{cases} a_k(u,v)+1, & \text{if } u^{-1}(i) < k \leq v^{-1}(i), \\ a_k(u,v)-1, & \text{if } v^{-1}(i) < k \leq u^{-1}(i), \\ a_k(u,v), & \text{otherwise,} \end{cases} \tag{6}$$

for all $k \in [n]$.

Proposition 2.6 *Let $u, v \in S_n^{(i)}$. Then $u \leq v$ if and only if $a_r(u,v) \geq 0$ and $\tilde{a}_r(u,v) \geq 0$ for all $r \in [n]$.*

Proof. It is well known (see, e.g., [2, Cor. 2.2.5]) that $u \leq v$ if and only if $u^{-1} \leq v^{-1}$. Therefore we conclude from (3) and Theorem 2.5 that $u \leq v$ if and only if $u^{-1}([i-1]) \preceq v^{-1}([i-1])$ and $u^{-1}([i]) \preceq v^{-1}([i])$. The result then follows from (2), (4), and (5). $\square$

3 Main result

In this section we prove our main result and derive some consequences of it. More precisely, we obtain explicit combinatorial product formulas for the parabolic R-polynomials of $S_n^{(i)}$. As an application of our results, we derive explicit combinatorial formulas for certain sums and alternating sums of ordinary Kazhdan-Lusztig R-polynomials.

For $u, v \in S_n^{(i)}$ we define

$$D(u,v) \stackrel{\text{def}}{=} v^{-1}([i-1]) \setminus u^{-1}([i-1]),$$

and

$$\tilde{D}(u,v) \stackrel{\text{def}}{=} v^{-1}([i]) \setminus u^{-1}([i]).$$

So, for example, if $n = 9$, $i = 5$, $v = 657182394$, and $u = 162375489$, then $D(u,v) = \{6,9\}$ and $\tilde{D}(u,v) = \{2,9\}$. Note that $D(u,v) = \tilde{D}(u,v)$ if $u^{-1}(i) = v^{-1}(i)$.

Theorem 3.1 *Let $u, v \in S_n^{(i)}$, $u \leq v$. Then*

$$R_{u,v}^{J_i,q} = \begin{cases} \varepsilon_u \varepsilon_v \left(1 - q + c\, q^{1+a_{u^{-1}(i)}(u,v)}\right) \prod_{k \in D(u,v)} (1 - q^{a_k(u,v)}), & \text{if } u^{-1}(i) \leq v^{-1}(i), \\ \varepsilon_u \varepsilon_v \left(1 - q + c\, q^{1+\tilde{a}_{u^{-1}(i)}(u,v)}\right) \prod_{k \in \tilde{D}(u,v)} (1 - q^{\tilde{a}_k(u,v)}), & \text{if } u^{-1}(i) \geq v^{-1}(i), \end{cases}$$

where $c \stackrel{\text{def}}{=} \delta_{u^{-1}(i),v^{-1}(i)}$.

Proof. We proceed by induction on $\ell(v) \geq 0$, the result being easy to check if $\ell(v) = 0$. So assume $\ell(v) > 0$. Let $s \in D(v)$, say $s = (j, j+1)$. Then $v(j) > v(j+1)$ and hence, since $v \in S_n^{(i)}$, $v(j) \geq i \geq v(j+1)$. Note that it follows immediately from our definitions that

$$a_k(u, v) = a_k(u, vs) = a_k(us, vs) \tag{7}$$

and

$$\tilde{a}_k(u, v) = \tilde{a}_k(u, vs) = \tilde{a}_k(us, vs) \tag{8}$$

for all $k \in [n] \setminus \{j+1\}$. We will use these facts throughout the proof often without explicit mention. For simplicity, we write "$R_{w,z}$" rather than "$R_{w,z}^{J_i,q}$" for all $w, z \in S_n^{(i)}$.

Assume first that $u^{-1}(i) \leq v^{-1}(i)$. There are two main cases to consider.

i) $v(j+1) < i$.

There are then six cases to consider.

a) $u(j) \geq i > u(j+1)$.

Then $u > us$, $(us)^{-1}(i) \leq (vs)^{-1}(i)$, $D(u, v) = D(us, vs) \not\ni j+1$, and $\delta_{(us)^{-1}(i),(vs)^{-1}(i)} = \delta_{u^{-1}(i),v^{-1}(i)}$ so we have from Theorem 2.1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= R_{us,vs} \\ &= \varepsilon_{us}\varepsilon_{vs}(1 - q + \delta_{(us)^{-1}(i),(vs)^{-1}(i)}\, q^{a_{(us)^{-1}(i)}(us,vs)+1}) \prod_{k \in D(us,vs)} (1 - q^{a_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v \left(1 - q + \delta_{u^{-1}(i),v^{-1}(i)}\, q^{a_{(us)^{-1}(i)}(us,vs)+1}\right) \prod_{k \in D(u,v)} (1 - q^{a_k(u,v)}), \end{aligned}$$

and the result follows since $a_{u^{-1}(i)}(u, v) = a_{(us)^{-1}(i)}(us, vs)$ if $u^{-1}(i) = v^{-1}(i)$.

b) $u(j) > i = u(j+1)$.

Then $v(j) > i$, $u > us$, $(us)^{-1}(i) < (vs)^{-1}(i)$, $D(us, vs) = (D(u,v) \setminus \{j+1\}) \cup \{j\}$, so we have from Theorem 2.1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= R_{us,vs} \\ &= \varepsilon_{us}\varepsilon_{vs}\,(1-q) \prod_{k\in D_(us,vs)} (1-q^{a_k(us,vs)}) \\ &= \varepsilon_u\varepsilon_v\,(1-q)\,(1-q^{a_j(us,vs)}) \prod_{k\in D(u,v)\setminus\{j+1\}} (1-q^{a_k(u,v)}), \end{aligned}$$

and the result follows since $a_j(us, vs) = a_{j+1}(u, v)$.

c) $i < u(j) < u(j+1)$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) \leq (vs)^{-1}(i)$, $D(u, vs) = (D(u,v) \setminus \{j+1\}) \cup \{j\}$, $\delta_{u^{-1}(i),(vs)^{-1}(i)} = \delta_{u^{-1}(i),v^{-1}(i)}$, so we have from Theorem 2.1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= -R_{u,vs} \\ &= -\varepsilon_u\varepsilon_{vs}\,(1-q+\delta_{u^{-1}(i),(vs)^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,vs)+1}) \prod_{k\in D(u,vs)} (1-q^{a_k(u,vs)}) \\ &= \varepsilon_u\varepsilon_v\,(1-q+\delta_{u^{-1}(i),v^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,vs)+1})\,(1-q^{a_j(u,vs)}) \\ &\quad \prod_{k\in D(u,v)\setminus\{j+1\}} (1-q^{a_k(u,v)}), \end{aligned}$$

and the result follows since $a_j(u, vs) = a_{j+1}(u, v)$ and $u^{-1}(i) \neq j+1$.

d) $u(j) = i < u(j+1)$

Then $u < us \in S_n^{(i)}$, $u^{-1}(i)+1 = (us)^{-1}(i) \leq (vs)^{-1}(i)$, $D(u, vs) = (D(u,v) \setminus \{j+1\}) \cup \{j\} = D(us, vs)$, $\delta_{u^{-1}(i),v^{-1}(i)} = \delta_{(us)^{-1}(i),(vs)^{-1}(i)}$. Furthermore, $a_{j+1}(us, vs) = a_{j+2}(u, v)$ so, by (6), (7), (8) and Proposition 2.6, $us \leq vs$. Hence we have from Theorem 2.4 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= qR_{us,vs} + (q-1)R_{u,vs} \\ &= (1-q)^2(-\varepsilon_u\varepsilon_{vs}) \prod_{k\in D(u,vs)} (1-q^{a_k(u,vs)}) \\ &+ q\,\varepsilon_{us}\varepsilon_{vs}\,(1-q+\delta_{(us)^{-1}(i),(vs)^{-1}(i)}\,q^{a_{(us)^{-1}(i)}(us,vs)+1}) \\ &\quad \prod_{k\in D(us,vs)} (1-q^{a_k(us,vs)}) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon_u\varepsilon_v\,((1-q)^2+q(1-q+\delta_{u^{-1}(i),v^{-1}(i)}\,q^{a_{j+1}(us,vs)+1}) \\
&\quad (1-q^{a_j(u,v)}) \prod_{k\in D(u,v)\setminus\{j+1\}} (1-q^{a_k(u,v)}),
\end{aligned}
$$

and the result follows since $a_{j+1}(us,vs)+1 = a_j(u,v) = a_{j+1}(u,v)$.

e) $u(j) < u(j+1) < i$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) \le (vs)^{-1}(i)$, $D(u,vs) = D(u,v) \not\ni j+1$, $\delta_{u^{-1}(i),v^{-1}(i)} = \delta_{u^{-1}(i),(vs)^{-1}(i)}$ and we have from Theorem 2.1 and our induction hypotheses that

$$
\begin{aligned}
R_{u,v} &= -R_{u,vs} \\
&= -\varepsilon_u\varepsilon_{vs}\,(1-q+\delta_{u^{-1}(i),(vs)^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,vs)+1}) \prod_{k\in D(u,vs)} (1-q^{a_k(u,vs)}) \\
&= \varepsilon_u\varepsilon_v\,(1-q+\delta_{u^{-1}(i),v^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,v)+1}) \prod_{k\in D(u,v)} (1-q^{a_k(u,v)}),
\end{aligned}
$$

as desired.

f) $u(j) < i \le u(j+1)$.

Then $u < us \in S_n^{(i)}$, $(us)^{-1}(i) \le u^{-1}(i) \le v^{-1}(i) \le (vs)^{-1}(i)$, $D(u,vs) = D(u,v) \setminus \{j+1\}$, $D(us,vs) = (D(u,v)\setminus\{j+1\}) \cup \{j\}$, and $\delta_{u^{-1}(i),v^{-1}(i)} = \delta_{(us)^{-1}(i),(vs)^{-1}(i)} = \delta_{u^{-1}(i),(vs)^{-1}(i)}$. Then from Theorem 2.1 and our induction hypotheses we have that

$$
\begin{aligned}
R_{u,v} &= q\,R_{us,vs} + (q-1)R_{u,vs} \\
&= (q-1)\varepsilon_u\varepsilon_{vs}(1-q+\delta_{u^{-1}(i),(vs)^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,vs)+1}) \\
&\quad \prod_{k\in D(u,vs)} (1-q^{a_k(u,vs)}) + \chi(us\le vs)\,q\varepsilon_{us}\varepsilon_{vs} \\
&\quad (1-q+\delta_{(us)^{-1}(i),(vs)^{-1}(i)}\,q^{a_{(us)^{-1}(i)}(us,vs)+1}) \prod_{k\in D(us,vs)} (1-q^{a_k(us,vs)}) \\
&= \varepsilon_u\varepsilon_v\,((1-q)+\chi(us\le vs)\,q(1-q^{a_j(us,vs)})) \\
&\quad (1-q+\delta_{u^{-1}(i),v^{-1}(i)}\,q^{a_{u^{-1}(i)}(u,v)+1}) \prod_{k\in D(u,v)\setminus\{j+1\}} (1-q^{a_k(u,v)})
\end{aligned}
$$

where we have used the fact that, if $\delta_{u^{-1}(i),v^{-1}(i)} = 1$, then $u^{-1}(i) = (us)^{-1}(i) \ne j+1$, and the result follows if $us \le vs$ since $a_j(us,vs)+1 = a_{j+1}(u,v)$. If $us \not\le vs$ then, by (6) and Proposition 2.6, $a_{j+1}(us,vs) < 0$. But $a_{j+1}(us,vs)+1 = a_{j+1}(u,v)-1 = a_{j+2}(u,v)$ so $a_{j+1}(u,v) = 1$ and the result again follows.

ii) $v(j+1) = i$.

There are then seven cases to consider.

a) $u(j) > i \geq u(j+1)$.

Then we conclude as in case i)a) above.

b) $u(j) = i > u(j+1)$

Then $u > us$, $(us)^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(us, vs) = D(u, v) \not\ni j+1$ so we have from Theorem 2.1 that

$$\begin{aligned} R_{u,v} = R_{us,vs} &= \varepsilon_{us}\varepsilon_{vs}(1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v \, (1-q) \prod_{k \in D(u,v)} (1 - q^{a_k(u,v)}), \end{aligned}$$

as desired.

c) $i < u(j) < u(j+1)$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) < (vs)^{-1}(i)$, $D(u, vs) = D(u, v) \not\ni j+1$, and the result follows from Theorem 2.1 and our induction hypothesis.

d) $u(j) = i < u(j+1)$.

Then $u < us \in S_n^{(i)}$, $u^{-1}(i) = (vs)^{-1}(i) < (us)^{-1}(i)$, $D(u, vs) = D(u, v) \not\ni j+1$, $\tilde{D}(us, vs) = D(u, v) \cup \{j\}$. Then by Theorem 2.1 and our induction hypothesis we have that

$$\begin{aligned} R_{u,v} &= q\, R_{us,vs} + (q-1) R_{u,vs} \\ &= (q-1)\, \varepsilon_u \varepsilon_{vs} (1 - q + q^{a_j(u,vs)+1}) \prod_{k \in D(u,vs)} (1 - q^{a_k(u,vs)}) \\ &\quad + q\, \chi(us \leq vs) \varepsilon_{us}\varepsilon_{vs} \, (1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v \, (1-q) \prod_{k \in D(u,v)} (1 - q^{a_k(u,v)}) \\ &\quad (1 - q + q^{a_j(u,vs)+1} + q\, \chi(us \leq vs)(1 - q^{\tilde{a}_j(us,vs)})), \end{aligned}$$

and the result follows if $us \leq vs$ since $a_j(u, vs) = \tilde{a}_j(us, vs)$. If $us \not\leq vs$ then, by (6) and Proposition 2.6, $\tilde{a}_{j+1}(us, vs) < 0$. But $\tilde{a}_{j+1}(us, vs) = a_j(u, vs) - 1$ so, by Proposition 2.6, $a_j(u, vs) = 0$ and the result again follows.

e) $u(j) < u(j+1) < i$

Then we conclude as in case i)e) above, except that $\delta_{u^{-1}(i),v^{-1}(i)} = \delta_{u^{-1}(i),(vs)^{-1}(i)} = 0$ in this case.

f) $u(j) < i < u(j+1)$.

Then $u < us \in S_n^{(i)}$, $(us)^{-1}(i) = u^{-1}(i) < (vs)^{-1}(i)$, $D(u,vs) = D(u,v) = D(us,vs) \not\ni j+1$. Furthermore, $\tilde{a}_{j+1}(us,vs) = a_{j+1}(us,vs) = a_j(u,v)$ so, by Proposition 2.6, $us \leq vs$ and we have from Theorem 2.1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= q\,R_{us,vs} + (q-1)\,R_{u,vs} \\ &= (-\varepsilon_u\varepsilon_{vs})(1-q)^2 \prod_{k\in D(u,vs)} (1-q^{a_k(u,vs)}) \\ &\quad +\varepsilon_{us}\varepsilon_{vs}\; q\;(1-q) \prod_{k\in D(us,vs)} (1-q^{a_k(us,vs)}) \\ &= \varepsilon_u\varepsilon_v\;(1-q) \prod_{k\in D(u,v)} (1-q^{a_k(u,v)}), \end{aligned}$$

as desired.

g) $u(j) < i = u(j+1)$.

Then $u < us \in S_n^{(i)}$, $(us)^{-1}(i) = (vs)^{-1}(i) < u^{-1}(i)$, $D(us,vs) = D(u,v) \not\ni j+1$, $\tilde{D}(u,vs) = D(u,v)$. Furthermore, $\tilde{a}_{j+1}(us,vs) = a_{j+1}(us,vs) = a_j(u,v)$ so, by Proposition 2.6, $us \leq vs$ and we have from Theorem 2.1 and our induction hypotheses that

$$\begin{aligned} R_{u,v} &= qR_{us,vs} + (q-1)R_{u,vs} \\ &= (-\varepsilon_u\,\varepsilon_{vs})(1-q)^2 \prod_{k\in\tilde{D}(u,vs)} (1-q^{\tilde{a}_k(u,vs)}) \\ &\quad +q\,\varepsilon_{us}\varepsilon_{vs}(1-q+q^{a_j(us,vs)+1}) \prod_{k\in D(us,vs)} (1-q^{a_k(us,vs)}) \\ &= \varepsilon_u\varepsilon_v((1-q)^2 + q(1-q+q^{a_j(us,vs)+1})) \prod_{k\in D(u,v)} (1-q^{a_k(u,v)}), \end{aligned}$$

and the result follows since $a_j(us,vs)+1 = a_{j+1}(u,v)$.

Assume now that $u^{-1}(i) > v^{-1}(i)$. There are again two main cases to consider.

i) $v(j) > i$.

There are then five cases to consider.

a) $u(j) > u(j+1)$.

Then $u > us$, $u(j+1) \leq i$, $(us)^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(us, vs) = \tilde{D}(u, v) \not\ni j+1$, so by Theorem 2.1 and our induction hypotheses we have that

$$R_{u,v} = R_{us,vs} = \varepsilon_{us}\varepsilon_{vs}\,(1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}).$$

and the result follows.

b) $i < u(j) < u(j+1)$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(u, vs) = (\tilde{D}(u,v) \setminus \{j+1\}) \cup \{j\}$, so by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} &= -R_{u,vs} = -\varepsilon_u\varepsilon_{vs}\,(1-q) \prod_{k \in \tilde{D}(u,vs)} (1 - q^{\tilde{a}_k(u,vs)}) \\ &= \varepsilon_u\varepsilon_v\,(1-q)\,(1 - q^{\tilde{a}_j(u,vs)}) \prod_{k \in \tilde{D}(u,v)\setminus\{j+1\}} (1 - q^{\tilde{a}_k(u,v)}) \end{aligned}$$

and the result follows since $\tilde{a}_j(u, vs) = \tilde{a}_{j+1}(u, v)$.

c) $u(j) \leq i < u(j+1)$.

Then $u < us \in S_n^{(i)}$, $(us)^{-1}(i) \geq u^{-1}(i) > v^{-1}(i) \geq (vs)^{-1}(i)$, $\tilde{D}(u, vs) = \tilde{D}(u,v) \setminus \{j+1\}$, $\tilde{D}(us, vs) = (\tilde{D}(u,v) \setminus \{j+1\}) \cup \{j\}$. Hence, by induction and Theorem 2.1 we conclude that

$$\begin{aligned} R_{u,v} &= (q-1)R_{u,vs} + q\,R_{us,vs} \\ &= (-\varepsilon_u\varepsilon_{vs})(1-q)^2 \prod_{k \in \tilde{D}(u,vs)} (1 - q^{\tilde{a}_k(u,vs)}) \\ &\quad +\varepsilon_{us}\varepsilon_{vs}\chi(us \leq vs)(1-q)\,q \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u\varepsilon_v(1-q) \prod_{k \in \tilde{D}(u,v)\setminus\{j+1\}} (1 - q^{\tilde{a}_k(u,v)}) \\ &\quad (1 - q + q\,\chi(us \leq vs)(1 - q^{\tilde{a}_j(us,vs)})) \end{aligned}$$

and the result follows if $us \leq vs$ since $\tilde{a}_j(us, vs) + 1 = \tilde{a}_{j+1}(u, v)$. If $us \not\leq vs$ then, by (6) and Proposition 2.6, $\tilde{a}_{j+1}(us, vs) < 0$. But $\tilde{a}_{j+1}(us, vs) = \tilde{a}_{j+1}(u, v) - 2 = \tilde{a}_{j+1}(u, vs) - 1$, so we conclude from Proposition 2.6 that $\tilde{a}_{j+1}(u, v) = 1$ and the result again follows.

d) $u(j) < u(j+1) < i$

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(u, vs) = \tilde{D}(u, v) \not\ni j+1$, so the result follows from Theorem 2.1 and our induction hypotheses.

e) $u(j) < u(j+1) = i$

Then $u < us \in S_n^{(i)}$, $v(j+1) < i$, $u^{-1}(i) > (us)^{-1}(i) > v^{-1}(i) = (vs)^{-1}(i)$, $\tilde{D}(u, vs) = \tilde{D}(u, v) = \tilde{D}(us, vs) \not\ni j+1$. Furthermore, $a_{j+1}(us, vs) = \tilde{a}_{j+1}(us, vs) = \tilde{a}_j(u, v)$ so, by Proposition 2.6, $us \leq vs$. Hence by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} &= q\, R_{us,vs} + (q-1)\, R_{u,vs} \\ &= (-\varepsilon_u \varepsilon_{vs})\,(1-q)^2 \prod_{k \in \tilde{D}(u,vs)} (1 - q^{\tilde{a}_k(u,vs)}) \\ &\quad + \varepsilon_{us}\varepsilon_{vs} q(1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v\,(1-q) \prod_{k \in \tilde{D}(u,v)} (1 - q^{\tilde{a}_k(u,v)}) \end{aligned}$$

and the result follows.

ii) $v(j) = i$.

Then $u(j) \neq i$ and there are six cases to consider.

a) $u(j) > i > u(j+1)$.

Then $u > us$, $(us)^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(us, vs) = (\tilde{D}(u, v) \setminus \{j\}) \cup \{j+1\}$, so by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} &= R_{us,vs} = \varepsilon_{us}\varepsilon_{vs}\,(1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v\,(1-q)(1 - q^{\tilde{a}_{j+1}(us,vs)}) \prod_{k \in \tilde{D}(u,v)\setminus\{j\}} (1 - q^{\tilde{a}_k(u,v)}), \end{aligned}$$

and the result follows since $\tilde{a}_{j+1}(us, vs) = \tilde{a}_j(u, v)$.

b) $u(j) > u(j+1) = i$.

Then $u > us$, $(us)^{-1}(i) < (vs)^{-1}(i)$, $D(us,vs) = \tilde{D}(u,v) \not\ni j+1$, so by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} = R_{us,vs} &= \varepsilon_{us}\varepsilon_{vs}\,(1-q) \prod_{k \in D(us,vs)} (1-q^{a_k(us,vs)}) \\ &= \varepsilon_{u}\varepsilon_{v}\,(1-q) \prod_{k \in \tilde{D}(u,v)} (1-q^{\tilde{a}_k(u,v)}) \end{aligned}$$

as desired.

c) $i < u(j) < u(j+1)$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(u,vs) = \tilde{D}(u,v) \ni j+1$, so by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} = -R_{u,vs} &= -\varepsilon_{u}\varepsilon_{vs}\,(1-q) \prod_{k \in \tilde{D}(u,vs)} (1-q^{\tilde{a}_k(u,vs)}) \\ &= \varepsilon_{us}\varepsilon_{v}\,(1-q)\,(1-q^{\tilde{a}_{j+1}(u,vs)}) \prod_{k \in \tilde{D}(u,v)\setminus\{j+1\}} (1-q^{\tilde{a}_k(u,v)}), \end{aligned}$$

and the result follows since $\tilde{a}_{j+1}(u,vs) = \tilde{a}_{j+1}(u,v)$.

d) $u(j) < u(j+1) < i$.

Then $u < us \notin S_n^{(i)}$, $u^{-1}(i) > (vs)^{-1}(i)$, $\tilde{D}(u,vs) = \tilde{D}(u,v) \not\ni j+1$, so by Theorem 2.1 and our induction hypotheses we have that

$$R_{u,v} = -R_{u,vs} = -\varepsilon_{u}\varepsilon_{vs}\,(1-q) \prod_{k \in \tilde{D}(u,vs)} (1-q^{\tilde{a}_k(u,vs)})$$

and the result follows.

e) $u(j) < u(j+1) = i$.

Then $u < us \in S_n^{(i)}$, $(us)^{-1}(i) = v^{-1}(i) < (vs)^{-1}(i) = u^{-1}(i)$, $D(u,vs) = \tilde{D}(u,v) \not\ni j+1$, $D(us,vs) = \tilde{D}(u,v) \cup \{j\}$ so by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} &= q\,R_{us,vs} + (q-1)\,R_{u,vs} \\ &= -\varepsilon_{u}\varepsilon_{vs}\,(1-q)\,(q^{a_{j+1}(u,vs)+1} - q + 1) \prod_{k \in D(u,vs)} (1-q^{a_k(u,vs)}) \\ &\quad + q\,\chi(us \le vs)\varepsilon_{us}\varepsilon_{vs}\,(1-q) \prod_{k \in D(us,vs)} (1-q^{a_k(us,vs)}) \\ &= \varepsilon_{u}\varepsilon_{v}\,(1-q) \prod_{k \in \tilde{D}(u,v)} (1-q^{\tilde{a}_k(u,v)}) \\ &\quad (q^{a_{j+1}(u,vs)+1} - q + 1 + q\,\chi(us \le vs)(1-q^{a_j(us,vs)})) \end{aligned}$$

and the result follows if $us \leq vs$ since $a_j(us, vs) = a_{j+1}(u, vs)$. If $us \not\leq vs$ then, by (6) and Proposition 2.6, $a_{j+1}(us, vs) < 0$. But $a_{j+1}(us, vs) = a_{j+1}(u, vs) - 1$, so we conclude from Proposition 2.6 that $a_{j+1}(u, vs) = 0$ and the result again follows.

f) $u(j) < i < u(j+1)$

Then $u < us \in S_n^{(i)}$, $(vs)^{-1}(i) < u^{-1}(i) = (us)^{-1}(i)$, $\tilde{D}(u, vs) = \tilde{D}(u, v) \ni j+1$, $\tilde{D}(us, vs) = (\tilde{D}(u, v) \setminus \{j+1\}) \cup \{j\}$. Furthermore, $\tilde{a}_{j+1}(us, vs) = \tilde{a}_{j+2}(u, v)$ so, by (6) and Proposition 2.6, $us \leq vs$. Then by Theorem 2.1 and our induction hypotheses we have that

$$\begin{aligned} R_{u,v} &= (q-1)\, R_{u,vs} + q\, R_{us,vs} \\ &= -\varepsilon_u \varepsilon_{vs}\, (1-q)^2 \prod_{k \in \tilde{D}(u,vs)} (1 - q^{\tilde{a}_k(u,vs)}) \\ &\quad + q \varepsilon_{us} \varepsilon_{vs}\, (1-q) \prod_{k \in \tilde{D}(us,vs)} (1 - q^{\tilde{a}_k(us,vs)}) \\ &= \varepsilon_u \varepsilon_v\, (1-q) \prod_{k \in \tilde{D}(u,v) \setminus \{j+1\}} (1 - q^{\tilde{a}_k(u,v)}) \\ &\quad ((1-q)\,(1 - q^{\tilde{a}_{j+1}(u,vs)}) + q\,(1 - q^{\tilde{a}_j(us,vs)})) \end{aligned}$$

and the result follows since $\tilde{a}_{j+1}(u, vs) = \tilde{a}_j(us, vs) = \tilde{a}_{j+1}(u, v)$.

This concludes the induction step and hence the proof. □

We illustrate the preceding theorem with two examples. Suppose $n = 9$, $i = 5$, $u = 162578349$ and $v = 657819234$. Then $u^{-1}(5) = 4 > 2 = v^{-1}(5)$, $\tilde{D}(u, v) = \{2, 5, 9\}$, $(\tilde{a}_2(u, v), \tilde{a}_5(u, v), \tilde{a}_9(u, v)) = (1, 2, 1)$ so by Theorem 3.1 we have that $R_{u,v}^{J_5,q} = (1-q)^3(1-q^2)$. On the other hand, if $u = 123567489$ and $v = 617582394$ then $u^{-1}(5) = 4 = v^{-1}(5)$, $D(u, v) = \{6, 9\}$, $(a_4(u, v),\, a_6(u, v), a_9(u, v)) = (2, 2, 1)$ so by Theorem 3.1 $R_{u,v}^{J_5,q} = -(1 - q + q^3)(1-q)(1-q^2)$.

From Proposition 2.3 we obtain the following "dual" version of Theorem 3.1.

Corollary 3.2 *Let $u, v \in S_n^{(i)}$, $u \leq v$. Then*

$$R_{u,v}^{J_i,-1} = \begin{cases} q^{\ell(u,v)}\, (1 - q^{-1} + c q^{-a_{u^{-1}(i)}(u,v) - 1}) \prod_{k \in D(u,v)} (1 - q^{-a_k(u,v)}), & \text{if } u^{-1}(i) \leq v^{-1}(i), \\ q^{\ell(u,v)}\, (1 - q^{-1} + c q^{-\tilde{a}_{u^{-1}(i)}(u,v) - 1}) \prod_{k \in \tilde{D}(u,v)} (1 - q^{-\tilde{a}_k(u,v)}), & \text{if } u^{-1}(i) \geq v^{-1}(i), \end{cases}$$

where $c \stackrel{\text{def}}{=} \delta_{u^{-1}(i), v^{-1}(i)}$. □

If the first element is the identity, only one of the integers appearing in Theorem 3.1 is sufficient to determine the corresponding polynomial.

Corollary 3.3 *Let $v \in S_n^{(i)}$. Then*

$$R_{e,v}^{J_i,q} = \varepsilon_v (1 - q + \delta_{v^{-1}(i),i}\, q^{k+1}) \prod_{j=1}^{k} (1 - q^j)$$

and

$$R_{e,v}^{J_i,-1} = q^{\ell(v)} (1 - q^{-1} + \delta_{v^{-1}(i),i}\, q^{-k-1}) \prod_{j=1}^{k} (1 - q^{-j}),$$

where $k \stackrel{\text{def}}{=} a_i(e,v)$ if $i \leq v^{-1}(i)$ and $k \stackrel{\text{def}}{=} \tilde{a}_{i+1}(e,v)$ if $i \geq v^{-1}(i)$.

Proof. We have that $D(e,v) = \{j \geq i : \ v(j) < i\}$ and hence that $|D(e,v)| = a_i(e,v)$. But, for all $r \in D(e,v)$, $a_r(e,v) = |\{j \geq r : \ v(j) < i\}| = |D(e,v) \cap [r,n]|$, so the result follows from Theorem 3.1 if $i \leq v^{-1}(i)$. Similarly, $\tilde{D}(e,v) = \{j > i : v(j) \leq i\}$ so $|\tilde{D}(e,v)| = \tilde{a}_{i+1}(e,v)$. But, for all $r \in \tilde{D}(e,v)$, $\tilde{a}_r(e,v) = |\{j \geq r : v(j) \leq i\}| = |\tilde{D}(e,v) \cap [r,n]|$, and the result again follows since $\tilde{a}_{i+1}(e,v) = \tilde{a}_i(e,v)$ if $i = v^{-1}(i)$. $\square$

It is an open problem, in the theory of the (ordinary) R-polynomials, to know if given $u, v \in W$ there exists $w \in W$ such that $R_{u,v}(q) = R_{e,w}(q)$ ([1]). The last three results (and simple examples) show that, in general, this is false for the parabolic R-polynomials of $S_n^{(i)}$.

As a further consequence of our main result we obtain combinatorial closed product formulas for certain sums and alternating sums of ordinary R-polynomials.

Corollary 3.4 *Let $u, v \in S_n^{(i)}$, $u < v$, and $x \in \{-1, q\}$. Then*

$$\sum_{w \in (S_n)_{J_i}} (-x)^{\ell(w)} R_{wu,v} =$$

$$\begin{cases} (q-x-1)^{\ell(u,v)} \left(1 - \frac{x^2}{q} + c\left(\frac{x^2}{q}\right)^{a_{u^{-1}(i)}(u,v)+1}\right) \prod\limits_{r \in D(u,v)} \left(1 - \left(\frac{x^2}{q}\right)^{a_r(u,v)}\right), & \text{if } u^{-1}(i) \leq v^{-1}(i), \\ (q-x-1)^{\ell(u,v)} \left(1 - \frac{x^2}{q} + c\left(\frac{x^2}{q}\right)^{\tilde{a}_{u^{-1}(i)}(u,v)+1}\right) \prod\limits_{r \in \tilde{D}(u,v)} \left(1 - \left(\frac{x^2}{q}\right)^{\tilde{a}_r(u,v)}\right), & \text{if } u^{-1}(i) \geq v^{-1}(i), \end{cases}$$

where $c \stackrel{\text{def}}{=} \delta_{u^{-1}(i),v^{-1}(i)}$.

Proof. This follows immediately from Theorem 3.1, Corollary 3.2 and Proposition 2.2. $\square$

We conclude by giving a geometric interpretation of our main result. Given $u \in S_n^{(i)}$ we associate to u a Motzkin path, which we denote by M_u, with $n-i$ up, $i-1$ down, and 1 horizontal steps, in the following way. For $1 \leq j \leq n$ the j-th step of M_u is down (respectively, horizontal, up) if and only if $u(j) < i$ (resp., $= i$, $> i$). So, for example, if $n = 9$, $i = 5$, $u = 123657489$ then M_u is the Motzkin path depicted in Figure 2. Note that, if $u, v \in S_n^{(i)}$, then $D(u,v)$ (resp., $\tilde{D}(u,v)$) is the set of all $j \in [n]$ such that the j-th step of M_v (resp., M_u) is down (resp., up) and the j-th step of M_u (resp., M_v) is not.

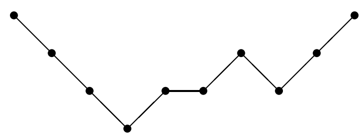

Figure 2.

Proposition 3.5 *Let $u, v \in S_n^{(i)}$. Then*

$$\left\lfloor \frac{M_v(j-1) - M_u(j-1)}{2} \right\rfloor = \begin{cases} a_j(u,v), & \text{if } u^{-1}(i) \leq v^{-1}(i), \\ \tilde{a}_j(u,v), & \text{if } u^{-1}(i) \geq v^{-1}(i), \end{cases} \tag{9}$$

for $j \in [n]$. In particular, $u \leq v$ if and only if $M_v(j) \geq M_u(j)$ for all $j \in [n]$.

Proof. Let $j \in [n]$. Clearly, $M_v(j-1)$ equals the difference between the number of up steps and down steps among the first $j-1$ steps of M_v. Therefore, by our definitions

$$\begin{aligned} M_v(j-1) &= |\{r \in [j-1] : \ v(r) > i\}| - |\{r \in [j-1] : \ v(r) < i\}| \\ &= n - 2i + 1 - |\{r \in [j,n] : \ v(r) > i\}| + |\{r \in [j,n] : \ v(r) < i\}|) \\ &= j - 2i + 2 + |\{r \in [j,n] : \ v(r) < i\}| + |\{r \in [j,n] : \ v(r) \leq i\}|. \end{aligned}$$

Hence, by (4), (5), and (6)

$$\begin{aligned} M_v(j-1) - M_u(j-1) &= a_j(u,v) + \tilde{a}_j(u,v) \\ &= \begin{cases} 2\, a_j(u,v) + 1, & \text{if } u^{-1}(i) < j \leq v^{-1}(i), \\ 2\, \tilde{a}_j(u,v) + 1, & \text{if } v^{-1}(i) < j \leq u^{-1}(i), \\ \tilde{a}_j(u,v) + a_j(u,v), & \text{otherwise,} \end{cases} \end{aligned}$$

and (9) follows since $a_j(u,v) = \tilde{a}_j(u,v)$ if either $j \leq u^{-1}(i), v^{-1}(i)$ or $u^{-1}(i), v^{-1}(i) < j$. The second statement follows immediately from (9), (6), and Proposition 2.6. □

We can now give the following geometric reformulation of Theorem 3.1.

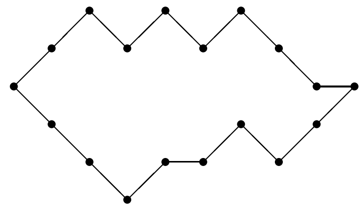

Figure 3.

Corollary 3.6 *Let $u, v \in S_n^{(i)}$, $u \leq v$. Then*

$$R_{u,v}^{J_i,q} = \varepsilon_u \varepsilon_v \left(1 - q + cq^{\left\lfloor \frac{M_v(v^{-1}(i)-1)-M_u(u^{-1}(i)-1)}{2} \right\rfloor + 1}\right) \prod_{j \in \mathcal{D}(u,v)} \left(1 - q^{\left\lfloor \frac{M_v(j-1)-M_u(j-1)}{2} \right\rfloor}\right),$$

where $c \stackrel{\text{def}}{=} \delta_{u^{-1}(i),v^{-1}(i)}$, $\mathcal{D}(u,v) \stackrel{\text{def}}{=} D(u,v)$ if $u^{-1}(i) \leq v^{-1}(i)$ and $\mathcal{D}(u,v) \stackrel{\text{def}}{=} \tilde{D}(u,v)$ if $u^{-1}(i) \geq v^{-1}(i)$.

Proof. This follows immediately from Theorem 3.1 and Proposition 3.5. □

So, for example, if $n = 9$, $i = 5$, $u = 123657489$ and $v = 671829345$ then the two Motzkin paths M_u and M_v are depicted in Figure 3, $u^{-1}(5) < v^{-1}(5)$, $D(u,v) = \{5, 8\}$, $M_v(4) - M_u(4) = 4$ and $M_v(7) - M_u(7) = 3$ so we have from Corollary 3.6 that $R_{u,v}^{J_5,q} = -(1-q)^2(1-q^2)$.

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