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A UNIVERSAL, NON-COMMUTATIVE C^*-ALGEBRA ASSOCIATED TO THE HECKE ALGEBRA OF DOUBLE COSETS

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Dedicated to the memory of Mihaly Bakonyi

Abstract. Let G be a discrete group and Γ an almost normal subgroup. The operation of cosets concatenation, extended by linearity, gives rise to an operator system that is embeddable in a natural C* algebra. The Hecke algebra embeds diagonally in the tensor product of this C* algebra with its opposite. When represented on the l^2 space of the group Γ, by left and right convolution operators, this representation gives rise to abstract Hecke operators that in the modular group case, are unitarily equivalent to the classical operators on Maass wave forms

Let G be a discrete group, Γ an almost normal subgroup. In this work, we introduce a formalism for the free algebra having as free algebra generators the cosets of Γ in G, subject to the relations that define the Hecke algebra and its standard action on the space of cosets.

Let $\mathbb{C}(\Gamma\backslash G)$, respectively $\mathbb{C}(G\backslash\Gamma)$ be the $\mathbb{C}$-linear space having as a linear basis the left and respectively right cosets. Let $\mathbb{C}(\Gamma\backslash G/\Gamma)$ be the $\mathbb{C}$-linear space having as a linear basis the double cosets of Γ in G. Since Γ is almost normal, it follows that every double coset $[\Gamma\sigma\Gamma]$ is a finite union (of which we think as of a sum in the linear space $\mathbb{C}(\Gamma\backslash G)$, respectively $\mathbb{C}(G\backslash\Gamma)$) of left and respectively right cosets of Γ in G.

For σ in G, let $\Gamma_\sigma = \sigma\Gamma\sigma^{-1}\cap\Gamma$. This subgroup is, by hypothesis, of finite index in G. Let $\mathcal{S}$ be the lattice of subgroups generated by the above subgroups. Note that all the subgroups in $\mathcal{S}$ have finite index.

We endow the Hilbert space $\ell^2(\Gamma\backslash G)$ with the canonical Hilbert space structure such that the cosets $[\Gamma\sigma]$, $\sigma\in G$, where $\sigma\in G$ runs over a system representatives of right cosets of Γ in G, is an orthonormal basis (and similarly for $\ell^2(G\backslash\Gamma)$).

There is a canonical prehilbert space structure on $\bigcup_{\Gamma_0\in\mathcal{S}}\ell^2(\Gamma_0/G)$ that is uniquely determined by the requirement that $[\sigma_1\Gamma]=\sum[\sigma_1 s_i\Gamma_0]$, for all $\sigma_1\in G$, where s_i are a system of representatives for right cosets of Γ_0 in $\mathcal{S}$, and by the requirement that all the inclusions $\ell^2(\Gamma/G)$ in $\ell^2(\Gamma_0/G)$, for all $\Gamma_0\in\mathcal{S}$, are isometric.

The space $\mathbb{C}(\Gamma\backslash G/\Gamma)$ has a natural action on left and respectively right cosets. The action of double coset $[\Gamma\sigma_1\Gamma]$ on $[\Gamma\sigma_2]$ is simply the projection of $[\sigma_1\Gamma\sigma_2]$, (which belongs to the reunion $\bigcup_{\Gamma_0\in\mathcal{S}}\ell^2(\Gamma_0\backslash G)$), onto $\ell^2(\Gamma/G)$. Naturally, we have a similar left action of $\mathbb{C}(\Gamma\backslash G/\Gamma)$ on $\ell^2(G/\Gamma)$. This actions define canonically a $\mathbb{C}-$ algebra with involution structure on $\mathbb{C}(\Gamma\backslash G/\Gamma)$. With this structure the space $\mathbb{C}(\Gamma\backslash G/\Gamma)$ is the Hecke algebra associated with the inclusion Γ in G.

The relations defining the Hecke algebra are thus summarized by the requirement that

$$\sum_i [\sigma_1^i\Gamma][\Gamma\sigma_2^i] = \sum_j [\theta_1^j\Gamma][\Gamma\theta_2^j] \tag{$*$}$$

if the unions $\bigcup_i \sigma_1^i\Gamma\sigma_2^i$, $\bigcup_j \theta_1^j\Gamma\theta_2^j$ are disjoint and equal. In particular the coset $[\Gamma]$ corresponding to the identity is the unit of this algebra.

This allows, naturally, to define an abstract $*$-algebra, whose free generators are the cosets (left or right) of Γ in G, subject to the relations $(*)$. By analogy with the Jones's algebra of higher relative commutants ([Jo], [Bi]) (or the algebra of bimodules over Γ in $\ell^2(G)$), we define a larger algebra $\mathcal{B}(\Gamma, G)$ which in addition allows to have a consistent definition for the localization of the support of the cosets (the support of a coset in the algebra $\mathcal{B}(\Gamma, G)$, is the coset itself viewed as a characteristic function). This is done by adding to the algebra of cosets the algebra of characteristic functions of cosets of groups in $\mathcal{S}$.

Our main result is that, under the assumption that there exists a (projective) unitary representation π of G on $\ell^2(\Gamma)$ that extends the left regular representation (with the cocycle induced from π) on $\ell^2(\Gamma)$, the algebra $\mathcal{B}(\Gamma, G)$ admits a unital C^*-representation, which is faithful on $\mathbb{C}(\Gamma\backslash G/\Gamma)$ and is consistent with the action of the Hecke algebra on $\mathbb{C}(G\backslash\Gamma)$. (Examples of such representations π are found in [GHJ]). Here the * operation is obtained as a linear extension of the relation $[\Gamma\sigma]^* = [\sigma^{-1}\Gamma]$, σ in G.

The construction of the C^*-representation of the algebra $\mathcal{B}(\Gamma, G)$ is based on the analysis ([Ra]) of the properties of the positive definite function on G

$$\varphi_I(g) = \langle \pi(g)I, I\rangle, g \in G$$

where $I \in \ell^2(\Gamma)$ is the vector corresponding to the identity of Γ in $\ell^2(\Gamma)$.

We let $\boldsymbol{X}_\Gamma$ be the (commutative) subalgebra of $\ell^\infty(G)$ generated by characteristic functions of cosets of groups in $\mathcal{S}$.

Let $\chi_\Gamma \in \ell^\infty(G)$ be the characteristic function of the group Γ. The unit of the algebra $\mathcal{B}(\Gamma, G)$ will be χ_Γ. We will prove that there exists a canonical $*$-representation of the Hecke algebra $\mathbb{C}(\Gamma\backslash G/\Gamma)$ into $\mathcal{B}(\Gamma, G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma, G)^{\mathrm{op}}$, defined by the mapping

$$\mathbb{C}(\Gamma\backslash G/\Gamma) \ni [\Gamma\sigma\Gamma] \mapsto \chi_\Gamma([\Gamma\sigma\Gamma] \otimes [\Gamma\sigma\Gamma]^*)\chi_\Gamma, \sigma \in G.$$

This map is, in the canonical representation of $\mathcal{B}(\Gamma, G) \otimes_{\chi_\Gamma} \mathcal{B}(\Gamma, G)$ by left and right convolution operators on $\ell^2(\Gamma)$, unitarily equivalent to classical representation of the Hecke algebra into Hecke operators on Maass forms ([Ra]).

We extend this representation to a representation of $\mathbb{C}(\Gamma\backslash G)$ into the algebra $\mathcal{B}(\Gamma\backslash G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma/G)^{\mathrm{op}}$, that is compatible with the action of the Hecke algebra $\mathbb{C}(\Gamma\backslash G/\Gamma)$ on $\mathbb{C}(\Gamma/G)$.

We remark that this construction gives a canonical operator system structure on the linear space $\{\mathbb{C}([\sigma_1\Gamma\sigma_2]) \mid \sigma_1, \sigma_2 \in G\}$, having as basis the sets $[\sigma_1\Gamma\sigma_2]) \mid \sigma_1, \sigma_2 \in G$. This space is is isomorphic to $\mathbb{C}(G\backslash\Gamma) \underset{\mathbb{C}(\Gamma\backslash G/\Gamma)}{\otimes} \mathbb{C}(\Gamma/G)$.

This last space is canonically included in the similar spaces, constructed by using subgroups of the form Γ_σ instead of Γ. It is conceivable that one could obtain such an operator system structure, canonically associated to the Hecke algebra, simultaneously for all levels (Γ_σ in $\mathcal{S}$).

We start with the definition of the free algebra of cosets of G in Γ.

Definition 1. Let Γ be an almost normal subgroup of a discrete group G. Let $\mathbb{C}(\Gamma\backslash G)$, $\mathbb{C}(G\backslash\Gamma)$, $\mathbb{C}(\Gamma\backslash G/\Gamma)$ be the linear vector space having as basis, respectively the left, right, and double cosets of Γ in G. For σ in G we denote such a coset (respectively a double coset) by $[\Gamma\sigma]$, $[\sigma\Gamma]$ and by $[\Gamma\sigma\Gamma]$ respectively.

Let $I(G,\Gamma)$ be the free $\mathbb{C}$-algebra whose (algebra) generators are $[\Gamma\sigma]$, $[\sigma\Gamma]$, for σ in G. We define a natural $*$-operation by requiring that the $*$ operation verifies the equality $[\Gamma\sigma]^* = [\sigma^{-1}\Gamma]$, for σ in G.

Let $J(G,\Gamma)$ be the double sided ideal generated by the differences corresponding to all the relations of the form

$$\sum_i [\sigma_1^i\Gamma][\Gamma\sigma_2^i] = \sum_j [\theta_1^j\Gamma][\Gamma\theta_2^j], \tag{1}$$

whenever σ_α^i, θ_β^j, $\alpha = 1,2$, $\beta = 1,2$, $i = 1,2,\dots,N$, $j = 1,2,\dots,M$ are elements in G such that the unions $\bigcup_i[\sigma_1^i\Gamma\sigma_2^i]$, $\bigcup_j[\theta_1^j\Gamma\theta_2^j]$ are disjoint, and equal. We also assume that the ideal $J(G,\Gamma)$ contains the elements corresponding to the equalities corresponding to the fact that $[\Gamma]$ is the identity of the algebra.

Then the universal $*$-algebra associated to Γ, G will be

$$\mathcal{A}(\Gamma, G) = I(\Gamma, G)/J(\Gamma, G).$$

Then the Hecke algebra $\mathbb{C}(\Gamma\backslash G/\Gamma)$ is a subalgebra of $\mathcal{A}(\Gamma, G)$, and the embedding is compatible with the left (and right) action of $\mathbb{C}(\Gamma\backslash G/\Gamma)$ on $\mathbb{C}(\Gamma/G)$ (and respectively $\mathbb{C}(G\backslash\Gamma)$).

Proof. Indeed the relations defining the action of the Hecke algebra, on the space of left and right cosets, and its multiplication are of the form of the relation in (1).

For example, the action of a double coset $[\Gamma\sigma_1\Gamma]$ on a coset $[\Gamma\sigma_2]$ is

$$[\Gamma\sigma_1\Gamma][\Gamma\sigma_2] = \sum_j [\Gamma][\Gamma\sigma_1 r_j\sigma_2],$$

where r_j are cosets representatives for $\Gamma_{\sigma_1} = \sigma_1\Gamma\sigma_1^{-1}\cap\Gamma$ (that is $[\Gamma\sigma_1\Gamma] = \bigcup[\Gamma\sigma_1 r_j]$).

Hence, if s_j are left coset representatives, (that is $[\Gamma\sigma\Gamma] = \sum[s_i\sigma\Gamma]$), then we get

$$\sum_i [s_i\sigma\Gamma][\Gamma\sigma_2] = \sum_j [\Gamma][\Gamma\sigma_1 r_j\sigma_2]. \qquad \square$$

We will prove in the sequel that the algebra $\mathcal{A}(\Gamma, G)$ admits a $*$-representation into a C^*-algebra. Hence we obtain that there exists a maximal C^*-algebra associated with $\mathcal{A}(\Gamma, G)$.

Remark 2. Let θ be an automorphism of G preserving Γ. Then θ obviously extends to an automorphism of $\mathcal{A}(\Gamma, G)$.

We also have a simple method to algebraically describe the localization (as characteristic functions) of the supports of the cosets $[\Gamma\sigma]$, $[\sigma\Gamma]$, $\sigma \in G$, by considering a larger algebra containing the $*$-algebra $\mathcal{A}(\Gamma, G)$.

Let $\boldsymbol{X}_\Gamma$ be the $\mathbb{C}$-subalgebra of $L^\infty(G)$ consisting of characteristic functions of left and right cosets in G of subgroups in $\mathcal{S}$ (with $\mathcal{S}$ as above).

Definition 3. The localized $*$-free $\mathbb{C}$-algebra of cosets of Γ in G. We consider this time the free $\mathbb{C}$-algebra $I_1(\Gamma, G)$ whose generators are cosets $[s_1\Gamma_{\sigma_1}]$, $[\Gamma_{\sigma_2}s_2]$, where $\sigma_1, \sigma_2 \in G$, s_1, s_2 in G and the algebra $\boldsymbol{X}_\Gamma$. Let $J_1(\Gamma, G)$ be the bilateral ideal in $I_1(\Gamma, G)$ corresponding to the following relations.

(0) If C is a coset of some modular subgroup in $\mathcal{S}$, which in turn is a disjoint reunion $\bigcup_j D_j$ of cosets then

$$[C] = \sum [D_j].$$

(1) Fix two cosets C_1, C_2 (left or right) of subgroups in $\mathcal{S}$ (that is, $C_i = [\sigma_i\Gamma_{\theta_i}]$ or $[\Gamma_{\theta_i}\sigma_i]$ for some σ_i, θ_i in G, $i = 1, 2$).

By considering cosets of smaller subgroups, there exists partitions (into cosets) $C_1 = \bigcup_{i\in I} A_i$, $C_1C_2 = \bigcup_{j\in J} D_j$ and a map $\pi : I \to J$ such that if $c \in A_i$ then $cC_2 \subseteq D_{\pi(i)}$.

Then we require that

$$[C_1]\chi_{C_2} = \sum_i \chi_{D_{\pi(i)}}[A_i].$$

For example $[\sigma\Gamma]\chi_\Gamma = \chi_{[\sigma\Gamma]}[\sigma\Gamma]$ and $[\Gamma\sigma]\chi_\Gamma = \chi_{[\sigma\Gamma]}[\Gamma\sigma]$. This corresponds to the fact that, when the group algebra of G acts on $L^2(G)$, then for every subset A of G, we have $g\chi_A g^{-1} = \chi_{gA}$, for all g in G.

This property (1) will correspond to the fact that the quotient algebra over $J_1(\Gamma, G)$ will be, as a linear space, the linear span Sp $(A(\Gamma, G)\boldsymbol{X}_\Gamma)$, which will also be equal to the linear span Sp $(\boldsymbol{X}_\Gamma\mathcal{A}(\Gamma, G))$.

The relation (1) from the previous definition becomes:

(2) for all σ_1^i, σ_2^i in G, if $\bigcup[\sigma_1^i\Gamma\sigma_2^i]$ is a disjoint union and its further equal to the disjoint union $\bigcup C_i$ of cosets then

$$\sum [\sigma_1^i\Gamma][\Gamma\sigma_2^i] = \sum_j [C_j].$$

In addition to this property we will add some properties which hold true in our model and are due to the fact that our algebra is related to the Jones's construction ([Jo]) for the inclusion $\Gamma \subset G$, and its Pimsner-Popa basis ([PP])

(3) Let $C_1 = [\sigma_1\Gamma]$, $C_2 = [\sigma_2\Gamma]$ be two cosets of Γ in G. Then $\chi_\Gamma[\sigma_1\Gamma][\Gamma\sigma_2]\chi_\Gamma = \delta_{C_1,C_2}\chi_\Gamma$, where δ is the Kronecker symbol.

(4) For all σ in G, if $[\Gamma\sigma\Gamma]$ is the finite disjoint reunion of $[s_i\sigma\Gamma]$ then

$$\sum_i [\Gamma\sigma s_i]\chi_\Gamma[s_i\sigma\Gamma] = \chi_{[\Gamma\sigma\Gamma]}.$$

(5) In the same conditions as in (4) we have that

$$\sum_i [\Gamma \sigma s_i][s_i \sigma \Gamma] = [\Gamma : \Gamma_\sigma]\mathrm{Id}.$$

We define then the localized free algebra of cosets as the algebra $\mathcal{B}(\Gamma, G) = I_1(\Gamma, G)/J_1(\Gamma, G)$. Then the *-algebra $\mathcal{B}(\Gamma, G)$ is an algebra having as generators the cosets in G of subgroups in $\mathcal{S}$ and their characteristic functions. We will use the convention to denote by $[C]$, a coset C, viewed as an element of the algebra $\mathcal{A}(\Gamma, G) \subseteq \mathcal{B}(\Gamma, G)$.

Our main result is that the $*$-algebra $\mathcal{B}(\Gamma, G)$ admits a $*$ unital representation into a C^*-algebra, (more precisely into a C^*-subalgebra of $\mathcal{L}(G)$, the II_1 von Neumann algebra associated to the discrete group G).

Theorem 4. *Let $\mathcal{L}(G)$ be the finite von Neumann algebra associated to the countable discrete group G. Assume that there exists a unitary (eventually projective, with 2-cocycle ε) representation π of G on $\ell^2(\Gamma)$, that extends the left regular unitary representation of Γ on $\ell^2(\Gamma)$ (eventually with cocycle ε). Then there exists an embedding of $\mathcal{B}(\Gamma, G)$ into $\mathcal{L}(G)$, such that every product $[\sigma_1\Gamma][\Gamma\sigma_2] = [\sigma_1\Gamma\sigma_2]$ in $\mathcal{B}(\Gamma, G)$, $\sigma_1, \sigma_2 \in G$, is supported in $\mathcal{L}(G) \cap \ell^2(\sigma_1\Gamma\sigma_2)$.*

(Examples of such representations π can be found in ([GHJ], [Jo])*).*

Proof. This is based on the results proven in [Ra]. Let u be a unitary in $\mathcal{L}(\Gamma)$ (viewed as trace vector) and let φ_J^u be the positive definite functional on G, defined as the matrix coefficient

$$\varphi_J^u(g) = \overline{\langle \pi(g)u, u\rangle}_{\ell^2(\Gamma)}, \quad g \in G.$$

For simplicity when u is the identity 1 we denote

$$t(g) = \varphi_J^1(g), \quad g \in G.$$

We proved in [Ra] that the matrix coefficients t have the property that

$$t(\theta_1\theta_2) = \varepsilon(\theta_1, \theta_2) \sum_{\gamma \in \Gamma} t(\theta_1\gamma^{-1})t(\gamma\theta_2), \quad \theta_1, \theta_2 \in G.$$

We denote for A a subset of G,

$$t^A = \sum_{\theta \in A} t(\theta)\theta \in \mathcal{L}(G).$$

Then, as proven in ([Ra]), we have that

$$t^{\sigma_1\Gamma} t^{\Gamma\sigma_2} = t^{\sigma_1\Gamma\sigma_2}, \quad \sigma_1, \sigma_2 \in \Gamma.$$

We also proved in [Ra], that by a change coordinates we may assume that all $t^{\Gamma\sigma}$ belong to $\mathcal{L}(G)$. (Note that $t^{\Gamma\sigma}$ has in fact the precise formula $(\pi(\sigma)1)^*\sigma$, $\sigma \in G$).

To define the representation of $\mathcal{B}(\Gamma, G)$, we associate to each coset $[C] \in \mathcal{A}(\Gamma, G)$ the element t^C of $\mathcal{L}(G)$, and to each characteristic function of a coset χ_C we associate the multiplication operator by the characteristic function χ_C, viewed as a bounded linear operator on $\ell^2(\Gamma)$.

The properties (0), (1), (2) are obvious. It remains to show the properties (3), (4), (5). The properties are proved by using the properties of Jones's basic construction for the subfactor $\mathcal{L}(\Gamma) \subseteq L(G)$ ([GHJ]).

Property (3) is equivalent to the fact that

$$E^{\mathcal{L}(G)}_{\mathcal{L}(\Gamma)}((t^{\Gamma\sigma_1})^*(t^{\Gamma\sigma_2})) = \delta_{\Gamma\sigma_1,\Gamma\sigma_2}\mathrm{Id}, \sigma_1, \sigma_2 \in G.$$

But $(t^{\Gamma\sigma_1})^*(t^{\Gamma\sigma_2}) = t^{\sigma_1^{-1}\Gamma\sigma_2}$, $\sigma_1, \sigma_2 \in G$ and $t^{\sigma_1^{-1}\Gamma\sigma_2}$ has a nonzero coefficient in Γ if and only if $\sigma_1 = \sigma_2$ and in this case the coefficient is exactly the identity.

It then follows that $(t^{\Gamma\sigma})\chi_\Gamma$ is a partial isometry, and if $[\Gamma\sigma\Gamma] = \sum[\Gamma\sigma s_i]$, where s_i are coset representatives, then $(t^{\Gamma\sigma s_i})\chi_\Gamma$ are orthogonal partial isometries with initial space $\ell^2(\Gamma)$ and range contained in $\ell^2(\Gamma\sigma\Gamma)$. Because there are $[\Gamma : \Gamma_\sigma]$ such isometries, and the $\mathcal{L}(\Gamma)$ bimodule $\ell^2(\Gamma\sigma\Gamma)$ has multiplicity $[\Gamma : \Gamma_\sigma]$ it follows that the sum of the orthogonal ranges of the partial isometries $(t^{\Gamma\sigma s_i})\chi_\Gamma$ is exactly $\ell^2(\Gamma\sigma\Gamma)$.

This proves property (4) and property (5) is proved as in [PP]. Note that property (5) also proves the boundedness of $t^{\Gamma\sigma}$. □

In fact, our representation of the algebra $\mathcal{B}(\Gamma, G)$ proves that the algebra $\mathcal{L}(G)$ plays the role of an algebra of coefficients. Thus we have the following additional property (which is valid in our particular representation):

(6) All the products $[\Gamma\sigma]\sigma^{-1}$ belong to $\mathcal{L}(\Gamma)$, $\sigma \in G$.

We will now prove that the algebra $\mathcal{A}(\Gamma, G)$ has a special diagonal representation into $\mathcal{B}(\Gamma, G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma, G)^{\mathrm{op}}$. Here the tensor product is with amalgamation over χ_Γ (with $\mathcal{B}(\Gamma, G)^{\mathrm{op}}$ acting from the right).

For computational purposes we consider a larger algebra, containing $\mathcal{B}(\Gamma, G)$. In this way we take into account the action of Γ.

Definition 5. The localized, Γ-free algebra $\mathcal{C}(\Gamma, G)$ of cosets of modular subgroups of Γ in G, is the $*$-algebra generated by $\mathcal{B}(\Gamma, G)$ and the group $*$-algebra of Γ, where the action of Γ on $\boldsymbol{X}_\Gamma$ is the canonical action on characteristic functions of cosets (belonging to $\boldsymbol{X}_\Gamma$), that is $\gamma\chi_A\gamma^{-1} = \chi_{\gamma A}$, $\gamma \in \Gamma$, A subset of G.

We denote for a coset C of a modular subgroup in $\mathcal{S}$, by $[C]_\gamma$, $\gamma \in \Gamma$, the sum

$$[C]_\gamma = \sum_i \gamma[D_i]\gamma^{-1},$$

if, as a disjoint reunion of sets, we have that

$$C = \bigcup_i \gamma D_i \gamma^{-1},$$

where D_i is a finite family of cosets of subgroups in $\mathcal{S}$. Then the algebra $\mathcal{C}(\Gamma, G)$ is generated by the elements of the form $[C]_\gamma$, C a coset, $\gamma \in \Gamma$, as above and by the group algebra of Γ.

As a consequence of the fact that Ad γ is an inner automorphism of $\mathcal{B}(\Gamma, G)$, $\gamma \in \Gamma$, we have with the notations of property (2) in the preceding definition that

$$\sum_i [\sigma_1^i\Gamma]_\gamma[\Gamma\sigma_2^i]_\gamma = \sum_j [C_j]_\gamma.$$

The product in the algebra $\mathcal{C}(\Gamma, G)$ is determined by the equalities above.

We now prove a diagonal representation of the Hecke algebra, which we extend to the action of the Hecke algebra $\mathbb{C}(\Gamma\backslash G/\Gamma)$ on $\mathbb{C}(\Gamma\backslash G)$.

Consider the algebra $\mathcal{B}(\Gamma, G)^{\mathrm{op}}$ where multiplication of $b_1 \odot b_2$, $b_1, b_2 \in \mathcal{B}(\Gamma, G)^{\mathrm{op}}$ is by definition $b_2 b_1$. With this we have the following

Theorem 6. *Define the following map*

$$\Phi : \mathbb{C}(\Gamma\backslash G) \to \chi_\Gamma(\mathcal{B}(\Gamma, G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma, G)^{\mathrm{op}})\chi_\Gamma$$

by

$$\Phi([\Gamma\sigma]) = \chi_\Gamma[\Gamma\sigma] \otimes [\Gamma\sigma\Gamma]^*\chi_\Gamma, \quad \sigma \in G.$$

Then Φ is a $\mathbb{C}(\Gamma\backslash G/\Gamma)$-linear map, that is for all σ_1, σ_2 in G,

$$\Phi([\Gamma\sigma_1\Gamma])\Phi([\Gamma\sigma_2]) = \Phi([\Gamma\sigma_1\Gamma][\Gamma\sigma_2]).$$

Note that in particular this proves that the map

$$[\Gamma\sigma\Gamma] \to \chi_\Gamma([\Gamma\sigma\Gamma] \otimes [\Gamma\sigma\Gamma]^*)\chi_\Gamma$$

is a $*$-algebra morphism from the Hecke algebra into

$$\chi_\Gamma(\mathcal{B}(\Gamma, G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma, G)^{\mathrm{op}})\chi_\Gamma.$$

As proved in [Ra]*, these maps, when representing $\mathcal{B}(\Gamma, G) \otimes_{\boldsymbol{X}_\Gamma} \mathcal{B}(\Gamma, G)^{\mathrm{op}}$ on $\ell^2(\Gamma)$ are unitarily equivalent to classical Hecke operators.*

Proof. Denote by $\mathcal{B} = \mathcal{B}(\Gamma, G)$. To make the calculations, we write elements X in $\mathcal{B} \bigotimes_{\boldsymbol{X}_\Gamma} \mathcal{B}^{\mathrm{op}}$ in the form $X = \sum [C_i] \otimes [D_i] f_i$, where $[C_i]$, $[D_i]$ are cosets and f_i functions in $\boldsymbol{X}_\Gamma$. For γ in Γ we denote by

$$X(\gamma) = \sum_i [C_i] \otimes [D_i] f_i(\gamma).$$

Working in the algebra $\mathcal{C} = \mathcal{C}(\Gamma, G)$ instead of $\mathcal{B} = \mathcal{B}(\Gamma, G)$, we have that, for γ in Γ, and for $[A], [B]$ cosets in $\mathcal{C}(\Gamma, G)$, and e the identity of the group Γ, the following equality holds true:

$$([A] \otimes [B])f(\gamma) = ([A] \otimes \gamma[B])f(e).$$

This holds true since $[B]$ acts from the right on $\boldsymbol{X}_\Gamma$.

This formula has a meaning also if we have another function h in front of $[A] \otimes [B]$. By $h([A] \otimes [B])(\gamma)$ we will denote the result of the above expression evaluated at γ, when h is moved to the left.

Then $([A] \otimes \gamma[B])f(e)$ is further equal to

$$([A] \otimes \gamma \odot (\gamma[B]\gamma^{-1}))f(e) = ([A] \otimes (\gamma \odot [B]_\gamma))f(e).$$

Note that the following identity holds for all cosets $[A], [B], [M], [N] \in \mathcal{C}(\Gamma, G)$ and all $\gamma \in \Gamma$:

$$([M]\otimes[N])([A]\otimes(\gamma\odot[B]_\gamma)) = [M][A]\otimes(\gamma\odot[N]_\gamma\odot[B]_\gamma) = (1\otimes\gamma)([M][A]\otimes([N]_\gamma\odot[B]_\gamma)). \quad (2)$$

Then, if $\sigma \in G$, and $[\Gamma\sigma\Gamma] = \sum[\Gamma\sigma s_i]$ with s_i coset representatives, then

$$\begin{aligned}\Phi([\Gamma\sigma])(\gamma) &= \chi_{[\Gamma]}([\Gamma\sigma]\otimes([\Gamma\sigma\Gamma])^*)(\gamma) = \chi_{[\Gamma]}\sum_i[\Gamma\sigma]\otimes([\Gamma\sigma s_i])^*(\gamma)\\ &= \chi_{[\Gamma]}\sum_i[\Gamma\sigma]\otimes(\gamma\odot[\Gamma\sigma s_i]_\gamma)(e) = \chi_{[\Gamma]}(1\otimes\gamma)\sum_i[\Gamma\sigma]\otimes([\Gamma\sigma s_i]_\gamma)(e)\\ &= (1\otimes\gamma)\chi_{[\Gamma]}\sum_i[\Gamma\sigma]\otimes([\Gamma\sigma s_i]_\gamma)(e).\end{aligned}$$

Passing $\chi_{[\Gamma]}$ to the right hand side, this is further equal to

$$\sum_i[\Gamma\sigma]\otimes(\gamma\odot[\Gamma\sigma s_i]^*_\gamma)\chi_{\sigma^{-1}\Gamma(\sigma s_i)^{-1}}(e).$$

The only non zero terms that remains in the sum is

$$\Phi([\Gamma\sigma])(\gamma) = [\Gamma\sigma]\otimes\gamma\odot[\Gamma\sigma]^*_\gamma. \tag{3}$$

Assume σ_1, σ_2 are elements of G and that $[\Gamma\sigma_2\Gamma] = \bigcup[\Gamma\sigma_2 r_j]$. Then from the Hecke algebra structure $[\Gamma\sigma_2\Gamma][\Gamma\sigma_1] = \sum_i[\Gamma\sigma_2 r_j\sigma_1]$.

Hence by using the identity (2)

$$\begin{aligned}\Phi([\Gamma\sigma_1\Gamma])\Phi([\Gamma\sigma_2])(\gamma) &= \chi_\Gamma([\Gamma\sigma_1\Gamma]\otimes[\Gamma\sigma_2\Gamma]^*)([\Gamma\sigma_1]\otimes\gamma\odot[\Gamma\sigma_2]^*_\gamma)(e)\\ &= \chi_\Gamma\sum_{j,k}[\Gamma\sigma_1 r_j\sigma_2]\otimes(\gamma\odot[\Gamma\sigma_1 r_k\sigma_2]^*_\gamma)(e).\end{aligned}$$

Passing χ_Γ to the right hand side, we obtain

$$\sum_{j,k}[\Gamma\sigma_1 r_j\sigma_2]\otimes(\gamma\odot[\Gamma\sigma_1 r_k\sigma_2]^*_\gamma)\cdot\chi_{\sigma_1 r_j\sigma_2\Gamma\sigma_2^{-1}r_k^{-1}\sigma_1^{-1}\cap\Gamma}(e).$$

Since in the expansion $[\Gamma\sigma_2\Gamma][\Gamma\sigma_1] = \sum_j[\Gamma\sigma_2 r_j\sigma_1]$, the cosets $[\Gamma\sigma_2 r_j\sigma_1]$ do not repeat themselves, the last sum is further equal to

$$\sum_j[\Gamma\sigma_2 r_j\sigma_1]\otimes(\gamma\odot[\Gamma\sigma_1 r_j\sigma_2]^*_\gamma)$$

which is clearly equal by formula (3) to

$$\Phi([\Gamma\sigma_2\Gamma][\Gamma\sigma_1])(\gamma) = \sum_j\Phi([\Gamma\sigma_2 r_j\sigma_1])(\gamma).$$ □

Remark 7. The representation in the previous theorem is extended with the same method to $\mathrm{Sp}\{[\sigma_1\Gamma\sigma_2] \mid \sigma_1, \sigma_2 \in G\}$ by defining for σ_1, σ_2 in G

$$\Phi([\sigma_1\Gamma\sigma_2]) = \chi_\Gamma\left([\sigma_1\Gamma\sigma_2]\otimes\sum_{[\Gamma z]\subseteq\Gamma\sigma_1\Gamma\sigma_2\Gamma}[\Gamma z]\right)\chi_\Gamma$$

and

$$\Phi([\sigma_1\Gamma]) = \chi_\Gamma([\sigma_1\Gamma]\otimes[\Gamma\sigma_1\Gamma])\chi_\Gamma$$

and verifying

$$\Phi([\sigma_1\Gamma])\Phi([\Gamma\sigma_2]) = \Phi([\sigma_1\Gamma\sigma_2]).$$

We conclude by noting that in particular we obtained a canonical operator system structure on $\mathrm{Sp}\{[\sigma_1\Gamma][\Gamma\sigma_2] \mid \sigma_1, \sigma_2 \in G\}$ with a trace

$$\langle[\sigma_1\Gamma][\Gamma\sigma_2], [\sigma_3\Gamma][\Gamma\sigma_4]\rangle = \tau\langle[\sigma_1\Gamma][\Gamma\sigma_2]([\sigma_3\Gamma][\Gamma\sigma_4])^*\rangle.$$

Remark 8. Assume $G = \mathrm{PSL}_2(\mathbb{Z}[\frac{1}{p}])$, $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. Then there exists a canonical isometry $\Psi : \ell^2(\Gamma/G) \to \ell^2(F_N)$ where $N = \frac{p+1}{2}$, that is a $\mathbb{C}(\Gamma\backslash G/\Gamma) - A_{\mathrm{red}}$ bimodule map (where A_{red} is the algebra generated in $\mathcal{L}(F_N)$ by

$$\sum_{i=1}^{N} u_i + \sum_{i=1}^{N} u_i^{-1},$$

u_i the generators of F_N [Py]).

Then, since

$$\mathrm{Sp}[\sigma_1\Gamma][\Gamma\sigma_2] \cong \mathbb{C}(G/\Gamma) \underset{\mathbb{C}(\Gamma\backslash G/\Gamma)}{\otimes} \mathbb{C}(\Gamma/G)$$

it follows that the operator system structure corresponds to an operator system structure on $\ell^2(F_N) \otimes_A \ell^2(F_N)$.

Proof. Indeed to define Ψ one divides the cosets $[\Gamma\sigma_p s_i]_{i=1}^{p+1}$, $\sigma_p = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ into two sets, one set which is mapped into $u_1, \dots, u_N$ the other set is mapped into $u_1^{-1}, \dots, u_N^{-1}$. □

Remark 9. Note that there exists a canonical embedding

$$\ell^2(G/\Gamma_\sigma) \underset{\mathbb{C}(\Gamma_\sigma\backslash G/\Gamma_\sigma)}{\otimes} \ell^2(\Gamma_\sigma\backslash G)$$

into $\ell^2(G/\Gamma_{\sigma_1}) \underset{\mathbb{C}(\Gamma_{\sigma_1}\backslash G/\Gamma_{\sigma_1})}{\otimes} \ell^2(\Gamma_{\sigma_1}\backslash G)$ if $\Gamma_{\sigma_1} \subseteq \Gamma_{\sigma_0}$.

Indeed if the identity (1) is satisfied for $\sigma_1^i, \sigma_2^i, \theta_1^j, \theta_2^j$ with Γ_{σ_1} instead of Γ, it will be satisfied also for Γ_{σ_0} instead of Γ_{σ_1} (this is proven by splitting Γ_{σ_0} into Γ_{σ_1} cosets). Hence one can define (in analogy with the Hecke modular algebra from ([CoMo]) an operator system structure on the inductive limit

$$\bigcup_{\Gamma_\sigma \in \mathcal{S}} \mathbb{C}(G\backslash\Gamma_\sigma) \underset{\mathbb{C}(\Gamma_\sigma\backslash G/\Gamma_\sigma)}{\otimes} \mathbb{C}(\Gamma_\sigma\backslash G).$$

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