

# New results in the deformed $\mathcal{N} = 4$ SYM theory

G.C. Rossi<sup>a,b</sup>, E. Sokatchev<sup>c</sup>, Ya.S. Stanev<sup>a</sup>

<sup>a</sup> Dipartimento di Fisica, Università di Roma “*Tor Vergata*”  
and INFN, Sezione di Roma 2

Via della Ricerca Scientifica - 00133 Roma, Italy

<sup>b</sup> John von Neumann-Institut für Computing NIC  
Platanenallee 6, D-15738 Zeuthen, Germany

<sup>c</sup> Laboratoire d’Annecy-le-Vieux de Physique Théorique LAPTH,  
B.P. 110, F-74941 Annecy-le-Vieux, France\*

## Abstract

We investigate various perturbative properties of the deformed  $\mathcal{N} = 4$  SYM theory. We carry out a three-loops calculation of the chiral matter superfield propagator and derive the condition on the couplings for maintaining finiteness at this order. We compute the 2-, 3- and 4-point functions of composite operators of dimension 2 at two loops. We identify all the scalar operators (chiral and non-chiral) of bare dimension 4 with vanishing one-loop anomalous dimension. We compute some 2- and 3-point functions of these operators at two loops and argue that the observed finite corrections cannot be absorbed by a finite renormalization of the operators.

---

\*UMR 5108 associée à l’Université de Savoie

# 1 Introduction

Recently there has been a renewed interest [1]–[10] in the deformed  $\mathcal{N} = 4$  supersymmetric Yang-Mills (SYM) theory with superpotential<sup>1</sup>

$$\mathcal{W} = g \kappa \int d^4x d^2\theta \operatorname{tr} \left( \Phi^1 [\Phi^2, \Phi^3]_\omega \right) + \text{h.c.} \quad , \quad (1)$$

where  $g$  is the SYM coupling constant and the deformed commutator is defined as

$$[A, B]_\omega = \omega AB - BA. \quad (2)$$

The parameter  $\kappa$  can be taken real since its phase can be absorbed into a redefinition of the chiral superfields, while  $\omega$  is in general complex<sup>2</sup>.

The undeformed  $\mathcal{N} = 4$  SYM is recovered when  $\kappa = \omega = 1$ , where the action has a manifest  $SU(3) \times U_R(1)$  invariance. For generic values of  $\omega$ ,  $\mathcal{N} = 4$  supersymmetry is reduced to  $\mathcal{N} = 1$ , and the  $SU(3)$  symmetry is broken down to  $U(1) \times U(1)$  [3]. The latter can be chosen to act as follows

$$\begin{aligned} (\Phi^1, \Phi^2, \Phi^3) &\rightarrow (e^{2i\alpha_1} \Phi^1, e^{-i\alpha_1} \Phi^2, e^{-i\alpha_1} \Phi^3), \\ (\Phi^1, \Phi^2, \Phi^3) &\rightarrow (\Phi^1, e^{i\alpha_2} \Phi^2, e^{-i\alpha_2} \Phi^3). \end{aligned} \quad (3)$$

The key observation is that apparently many of the interesting properties of the  $\mathcal{N} = 4$  SYM theory are preserved by the deformation. First of all, it is believed that the deformed theory is finite, provided the couplings satisfy some condition which ensures the vanishing of all  $\beta$ -functions. A general argument to this effect has been given in [1]. This claim has been explicitly verified to order  $g^4$  in [4, 5], using results of [1, 11, 12]. In our notation, the condition reads

$$\kappa^2 = \frac{2 N^2}{(N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega})}. \quad (4)$$

Another feature retained by the deformed theory is the existence of composite operators with vanishing anomalous dimension. A list of such single trace chiral primary operators (CPO) has been proposed in [2]. These include  $\operatorname{tr}(\Phi^1\Phi^1)$ ,  $\operatorname{tr}(\Phi^1\Phi^1\Phi^1)$ , etc., which in the undeformed theory belong to short (1/2 BPS) supermultiplets. In [4] it has been shown that also CPOs of dimension two of the form  $\operatorname{tr}(\Phi^1\Phi^2)$  have vanishing order  $g^2$  anomalous

---

<sup>1</sup>The trace is over the colour indices of the fundamental representation of the  $SU(N)$  gauge group. The generators,  $T^a$ , of the fundamental representation are normalized according to  $\operatorname{tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$ .

<sup>2</sup>We use a slightly different notation than in [3, 4]. The correspondence is given by  $\omega = q^2 = e^{2i\pi\beta}$  (with  $\beta$  complex) and  $g\kappa = h/q$ . The reason for our choice is that in all the formulae only the quantities  $\kappa^2$  and  $\omega$  appear.

dimension. This is non-trivial since, while in the undeformed  $\mathcal{N} = 4$  theory  $\text{tr}(\Phi^1\Phi^2)$  and  $\text{tr}(\Phi^1\Phi^1)$  belong to the same supermultiplet, in the deformed theory the two operators are not related by supersymmetry. In [5] this result has been extended to order  $g^4$ . Neither multi-trace, nor non-chiral operators have been considered in the literature.

In this paper we investigate further the non-renormalization properties of the deformed theory. To simplify the calculations, we shall use the obvious observation that if some quantity is known in  $\mathcal{N} = 4$  SYM, then it is sufficient to compute the difference between the perturbative corrections in the deformed and undeformed cases.

The outline of the paper is as follows. In Section 2 we show that a new condition is necessary for the finiteness of the deformed theory at three loops (order  $g^6$ ). We give the explicit form of this condition and discuss its general solution. Section 3 is devoted to a systematic search for all operators of dimension  $\Delta \leq 4$  with vanishing order  $g^2$  anomalous dimensions. Among them we find non-chiral operators, both of dimension 2 and of dimension 4, as well as numerous mixtures of single and double trace operators of dimension 4. We also compute the perturbative corrections to some 2-, 3- and 4-point functions at order  $g^4$  and argue that the observed finite corrections cannot be absorbed by any finite renormalization of the operators. Some conclusions can be found in Section 4.

## 2 Perturbative corrections to the correlation functions

We shall write the action of the deformed theory in the form

$$S_\kappa = S_0 + S_v + \mathcal{W}_\kappa, \quad (5)$$

where  $S_0$  contains the kinetic terms, and  $S_v$  is the part of the standard  $\mathcal{N} = 4$  SYM action involving the couplings of the gauge superfield,  $V$ . Finally  $\mathcal{W}_\kappa$  is the deformed superpotential<sup>3</sup>

$$\begin{aligned} \mathcal{W}_\kappa = & g \kappa \int d^4x d^2\theta \left( \omega \text{tr}(\Phi^1\Phi^2\Phi^3) - \text{tr}(\Phi^1\Phi^3\Phi^2) \right) + \\ & + g \kappa \int d^4x d^2\bar{\theta} \left( \bar{\omega} \text{tr}(\Phi_1^\dagger\Phi_3^\dagger\Phi_2^\dagger) - \text{tr}(\Phi_1^\dagger\Phi_2^\dagger\Phi_3^\dagger) \right). \end{aligned} \quad (6)$$

As a matter of principle, both parameters,  $\kappa$  and  $\omega$ , could depend on  $g$  and the number of colours,  $N$ . We shall assume, however, that they have a Taylor series expansion in powers of  $g$  around  $g = 0$ .

<sup>3</sup>We shall use the shorthand notation  $\mathcal{W}_\kappa$  instead of  $\mathcal{W}_{\kappa,\omega}$ , keeping in mind that the superpotential also depends on  $\omega$ .

In our notations the action of the undeformed  $\mathcal{N} = 4$  SYM theory reads ( $\kappa = \omega = 1$ )

$$S_g = S_0 + S_v + \mathcal{W}_g, \quad (7)$$

with  $\mathcal{W}_g$  the  $\mathcal{N} = 4$  superpotential

$$\mathcal{W}_g = g \int d^4x d^2\theta \operatorname{tr} \left( \Phi^1 [\Phi^2, \Phi^3] \right) - g \int d^4x d^2\bar{\theta} \operatorname{tr} \left( \Phi_1^\dagger [\Phi_2^\dagger, \Phi_3^\dagger] \right). \quad (8)$$

## 2.1 General considerations

We want to compute the order  $g^{2n}$  correction in the deformed theory to a given  $\ell$ -point correlation function,  $\langle \mathcal{O}_1(x_1) \dots \mathcal{O}_\ell(x_\ell) \rangle$ , where  $\mathcal{O}_i(x_i)$  are some local (fundamental or composite) operators. To this end we have to evaluate the correlator

$$G_\kappa^{2n}(x_1, \dots, x_\ell) = \langle e^{S_v + \mathcal{W}_\kappa} \Big|_{g^{2n}} \mathcal{O}_1(x_1) \dots \mathcal{O}_\ell(x_\ell) \rangle, \quad (9)$$

where by  $e^{S_v + \mathcal{W}_\kappa} \Big|_{g^{2n}}$  we denote all terms of order  $g^{2n}$  in the expansion of the exponent. Since both  $S_v$  and  $\mathcal{W}_\kappa$  are non-linear in  $g$ , one gets in general a rather complicated expression.

The same computation in  $\mathcal{N} = 4$  SYM will give

$$G_g^{2n}(x_1, \dots, x_\ell) = \langle e^{S_v + \mathcal{W}_g} \Big|_{g^{2n}} \mathcal{O}_1(x_1) \dots \mathcal{O}_\ell(x_\ell) \rangle. \quad (10)$$

Both computations are very involved as soon as  $n$  gets large ( $n \geq 2$ ), since there are numerous diagrams which contribute. Most of them, however, come from  $S_v$  which contains all the non-polynomial interactions involving the gauge field. This suggests that it might be simpler to compute the difference

$$\delta G^{2n}(x_1, \dots, x_\ell) = G_\kappa^{2n}(x_1, \dots, x_\ell) - G_g^{2n}(x_1, \dots, x_\ell), \quad (11)$$

between the corrections to the correlator in the two theories.

In particular, if we are interested in the corrections to the 2-point function of an operator, say  $\mathcal{O}$ , which is known to be protected in  $\mathcal{N} = 4$  SYM (*i.e.* for which  $G_g^{2n} = 0$ ,  $n \geq 1$ ), then the vanishing of  $\delta G^{2n}$  implies in an obvious way  $G_\kappa^{2n} = 0$ . Hence, if  $\delta G^{2n} = 0$ , the operator  $\mathcal{O}$  will have vanishing anomalous dimension at order  $g^{2n}$  also in the deformed theory.

Inserting eqs. (9) and (10) in (11), one obtains

$$\delta G^{2n}(x_1, \dots, x_\ell) = \left\langle \left\{ e^{S_v + \mathcal{W}_\kappa} \Big|_{g^{2n}} - e^{S_v + \mathcal{W}_g} \Big|_{g^{2n}} \right\} \mathcal{O}_1(x_1) \dots \mathcal{O}_\ell(x_\ell) \right\rangle. \quad (12)$$

It is useful to rewrite the factor in the braces as

$$\begin{aligned}
e^{S_v+\mathcal{W}_\kappa}\Big|_{g^{2n}} - e^{S_v+\mathcal{W}_g}\Big|_{g^{2n}} &= \sum_{p=1}^{2n-1} e^{S_v}\Big|_{g^{2n-p}} \left( e^{\mathcal{W}_\kappa}\Big|_{g^p} - e^{\mathcal{W}_g}\Big|_{g^p} \right) + \\
&+ \left( e^{\mathcal{W}_\kappa}\Big|_{g^{2n}} - e^{\mathcal{W}_g}\Big|_{g^{2n}} \right). \tag{13}
\end{aligned}$$

As expected, the most complicated term corresponding to  $p=0$  has disappeared from the right hand side (r.h.s.). For future convenience we have separated from the rest the pure superpotential interaction. This equations can be used to simplify the computation of all correlation functions whose expressions in  $\mathcal{N} = 4$  SYM are explicitly known.

## 2.2 Correction to the propagators at order $g^6$

Let us start by considering the corrections to the chiral propagator

$$\langle \Phi_a^1(x_1, \theta_1) \Phi_1^\dagger{}_b(x_2, \bar{\theta}_2) \rangle \tag{14}$$

in the deformed theory. As we already recalled, the vanishing of the coefficient to the (logarithmically divergent) order  $g^2$  contribution requires the relation (4) between  $\kappa$  and  $\omega$  to hold. Then, if  $\kappa$  and  $\omega$  are assumed not to depend on  $g$ , the finiteness<sup>4</sup> of the order  $g^4$  correction follows without further constraints [11] (for an explicit check see also [5]). If one allows a  $g$ -dependence in  $\kappa$  and/or  $\omega$ , the finiteness of the order  $g^4$  correction to the propagator implies the absence of order  $g^2$  corrections to eq. (4). This means that the solution to the condition for the finiteness of the chiral propagator (*i.e.* vanishing of  $\gamma_\Phi$ ) at orders  $g^2$  and  $g^4$  can be written in the more general form

$$\kappa^2 = \frac{2 N^2}{(N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega})} + O(g^4). \tag{15}$$

Let us now consider the order  $g^6$  corrections to the chiral propagator (14). As explained before, to this end we shall make use of eqs. (12) and (13), and compute only the difference of the order  $g^6$  corrections to the propagator in the deformed and in the  $\mathcal{N} = 4$  SYM theory. At first sight this still looks like a very difficult task, since there are very many different superdiagrams. The crucial and rather surprising observation, however, is that through order  $g^6$ , if  $\kappa$  and  $\omega$  are related by eq. (15) then all contributions coming from the first

---

<sup>4</sup>The difference between the order  $g^4$  corrections in the deformed and undeformed theories is zero. The finite correction in  $\mathcal{N} = 4$  SYM has been computed in ref. [12].

line in the r.h.s. of eq. (13) have a vanishing colour factor<sup>5</sup>. Let us stress that the unspecified  $O(g^4)$  correction in eq. (15) does not modify this conclusion.

As a consequence of this analysis, we conclude that at order  $g^6$  we will only have to compute the correction to the chiral propagator in the deformed theory of the form

$$\langle (e^{\mathcal{W}_\kappa} - e^{\mathcal{W}_g}) \Big|_{g^6} \Phi_a^1(x_1, \theta_1) \Phi_{1\ b}^\dagger(x_2, \bar{\theta}_2) \rangle. \quad (16)$$

This is a drastic simplification, because one is effectively left with only the task of computing the propagator corrections coming exclusively from the superpotential. Moreover, since we already cancelled the propagator corrections at lower orders, we have to consider only primitive divergent superdiagrams (*i.e.* those which do not contain divergent subdiagrams). There

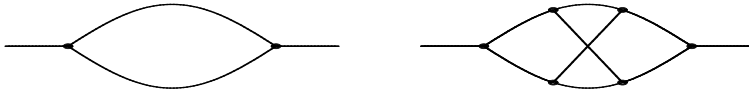


Figure 1: The 1-loop and the 3-loop diagrams.

are only two such contributions to order  $g^6$  (see figure), both logarithmically divergent [14]. One is given by the one-loop diagram, because at order  $g^6$  it will appear multiplied by the possibly non-vanishing  $O(g^4)$  term in eq. (15). The second is a genuine three-loop nonplanar diagram, which is present only in the deformed theory, since the corresponding colour factor in  $\mathcal{N} = 4$  SYM is zero. We note that, owing to the structure of the superpotential, there are no primitive divergent two-loop superdiagrams. After computing the colour factor of the three-loop diagram, we find that its contribution is proportional to

$$g^6 \kappa^6 (\omega - 1) (\bar{\omega} - 1) P(\omega, N) \frac{(N^2 - 4)}{N^3}, \quad (17)$$

where

$$P(\omega, N) = ((\omega^2 + \omega + 1)(\bar{\omega}^2 + \bar{\omega} + 1) - 9 \omega \bar{\omega}) N^2 + 5 (\omega - 1)^2 (\bar{\omega} - 1)^2. \quad (18)$$

In view of this result we may argue that there are three essentially different ways to achieve perturbative finiteness at three-loops (*i.e.* at order  $g^6$ ) if we allow  $\kappa$  or  $\omega$  to depend on  $g$ .

---

<sup>5</sup>To compute the complicated colour traces we use a Maple program to implement the split/join rules for  $SU(N)$  [13].

1) One may choose the  $O(g^4)$  term in eq. (15) so as to compensate the contribution of the three-loop diagram.

2) One can make the coefficient in (17) proportional to  $g^8$ , shifting in this way the problem to the next order. This in turn can be done either by choosing  $\omega = 1 + g \omega_1 + \dots$ , or by seeking for an  $\omega$  which solves the equation  $P(\omega, N) = O(g^2)$ .

3) One may insist that  $\kappa$  and  $\omega$  do not depend on  $g$ . In this case  $\omega$  has to satisfy the equation

$$((\omega^2 + \omega + 1)(\bar{\omega}^2 + \bar{\omega} + 1) - 9 \omega \bar{\omega}) N^2 + 5 (\omega - 1)^2 (\bar{\omega} - 1)^2 = 0. \quad (19)$$

As for the gauge field propagator, we find that eq. (15) alone is sufficient for its finiteness both at order  $g^4$  and order  $g^6$ , as expected on the basis of general renormalization arguments [11].

To summarize, we have proven by an explicit computation that the deformed theory can be made as finite as  $\mathcal{N} = 4$  SYM also at order  $g^6$ , and in various ways. A preliminary investigation shows that the different scenarios have rather different implications at the next perturbative order<sup>6</sup>. We end this Section by noting that, since the only relevant contribution at order  $g^6$  is nonplanar, as far as the planar limit is concerned, eq. (15) alone is sufficient for finiteness also at order  $g^6$ .

### 3 Composite operators with vanishing anomalous dimension

In this Section we present some results for the 2-, 3- and 4-point functions of the composite operators of naive dimension  $\Delta_0 \leq 4$  in the deformed theory at orders  $g^2$  and  $g^4$ . We shall assume that  $\kappa$  and  $\omega$  satisfy eq. (15), hence both the chiral and the gauge field propagator are finite up to order  $g^4$ . In practice this also means that there is only one parameter left in all the formulae, namely  $\omega$ .

To label operators we shall sometimes use a shorthand notation where

$$\text{tr}(121\bar{3}) \text{ denotes } \text{tr}(\Phi^1 \Phi^2 \Phi^1 \Phi_3^\dagger) \quad (20)$$

with all fields at the same point and in the indicated order under the colour trace. We list operators up to relabelling of the flavour indices. Hence (1112) is also meant to denote (1113), (2221), etc. As usual, we shall expand the anomalous dimension of the operator  $\mathcal{O}$  in power series of  $g^2$  by writing

$$\gamma_{\mathcal{O}}(g^2) = g^2 \gamma_{\mathcal{O}}^{(1)} + g^4 \gamma_{\mathcal{O}}^{(2)} + \dots \quad (21)$$

---

<sup>6</sup>We will not pursue this question any further in this paper [15].

### 3.1 Operators of $\Delta_0 = 2$ up to order $g^4$

The 21 different scalar operators of naive dimension  $\Delta_0 = 2$  can be organized as follows. There are 6 chiral and 6 antichiral operators, namely

$$\mathcal{C}^{IJ} = \text{tr}(\Phi^I \Phi^J), \quad \mathcal{C}_{IJ}^\dagger = \text{tr}(\Phi_I^\dagger \Phi_J^\dagger), \quad (22)$$

there are 3 non-chiral, mutually orthogonal, flavour singlet operators which we shall choose as

$$\begin{aligned} \mathcal{V}_X &= 2 \text{tr}(\Phi^1 \Phi_1^\dagger) - \text{tr}(\Phi^2 \Phi_2^\dagger) - \text{tr}(\Phi^3 \Phi_3^\dagger), \\ \mathcal{V}_Y &= \text{tr}(\Phi^2 \Phi_2^\dagger) - \text{tr}(\Phi^3 \Phi_3^\dagger), \\ \mathcal{K}_1 &= \text{tr}(\Phi^1 \Phi_1^\dagger) + \text{tr}(\Phi^2 \Phi_2^\dagger) + \text{tr}(\Phi^3 \Phi_3^\dagger), \end{aligned} \quad (23)$$

and 6 operators which are neither chiral nor flavour singlets

$$\mathcal{V}_J^I = \text{tr}(\Phi^I \Phi_J^\dagger) \quad \text{for } I \neq J. \quad (24)$$

In the undeformed  $\mathcal{N} = 4$  SYM theory the 20 operators  $\mathcal{C}^{IJ}$ ,  $\mathcal{C}_{IJ}^\dagger$ ,  $\mathcal{V}_J^I$ ,  $\mathcal{V}_X$  and  $\mathcal{V}_Y$  are the lowest components of the short stress-tensor supermultiplet, and hence all have protected dimension, while  $\mathcal{K}_1$  is the lowest component of the long Konishi supermultiplet and acquires anomalous dimension. The chiral primary operators (CPO)  $\mathcal{C}^{IJ}$  have been already considered in the literature. In [4] it was shown that their 2-point functions get neither infinite nor finite corrections at order  $g^2$ . The absence of anomalous dimension for these operators was confirmed at order  $g^4$  in [5], but at this order finite corrections appear. Our calculations agree with these references. We present here new results about the non-chiral operators (23) and (24).

Since all the 2-, 3- and 4-point functions of all the 21 scalar operators have already been computed in  $\mathcal{N} = 4$  SYM to order  $g^4$ , we shall compute only the difference of these correlators between the deformed and the undeformed theory. To this purpose the splitting in eq. (13) again turns out to be very useful.

Our results can be summarized as follows. If the deformation parameters  $\kappa$  and  $\omega$  satisfy eq. (15), then

- The 6 operators  $\mathcal{V}_J^I$  defined in eq. (24) have non-vanishing order  $g^2$  anomalous dimension given by

$$\gamma_{\mathcal{V}_J^I}^{(1)} = \frac{N}{(4\pi^2)} \frac{(N^2 - 4)(\omega - 1)(\bar{\omega} - 1)}{(N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega})}. \quad (25)$$

- The flavour singlets  $\mathcal{V}_X$  and  $\mathcal{V}_Y$ , defined in eq. (23), have vanishing anomalous dimensions both at order  $g^2$  and  $g^4$ , *i.e.*

$$\gamma_{\mathcal{V}_{X,Y}}^{(1)} = 0, \quad \gamma_{\mathcal{V}_{X,Y}}^{(2)} = 0. \quad (26)$$



Their 2- and 3-point functions do not receive any (not even finite) corrections up to order  $g^4$ . In fact, since these operators are the lowest components of the supermultiplets which contain also the generators of the two  $U(1)$  symmetries (see eq. (3)), we expect them to be protected to all orders.

- The order  $g^2$  and  $g^4$  anomalous dimensions of the Konishi scalar  $\mathcal{K}_1$  are the same as in the undeformed  $\mathcal{N} = 4$  SYM, *i.e.*

$$\gamma_{\mathcal{K}_1}^{(1,2)} \Big|_{\text{Deformed}} = \gamma_{\mathcal{K}_1}^{(1,2)} \Big|_{\mathcal{N}=4 \text{ SYM}} . \quad (27)$$

- All the 3- and 4-point functions of the flavour singlets  $\mathcal{O}_{\text{fs}} = \{\mathcal{V}_X, \mathcal{V}_Y, \mathcal{K}_1\}$  in any combination at orders  $g^2$  and  $g^4$  are *exactly equal* to the corresponding functions in  $\mathcal{N} = 4$  SYM. This implies that up to order  $g^4$  the same property will hold for all the  $n$ -point functions involving these operators. In formulae

$$\langle \mathcal{O}_{\text{fs}}(x_1) \dots \mathcal{O}_{\text{fs}}(x_n) \rangle \Big|_{\text{Deformed } g^2, g^4} = \langle \mathcal{O}_{\text{fs}}(x_1) \dots \mathcal{O}_{\text{fs}}(x_n) \rangle \Big|_{\mathcal{N}=4 \text{ } g^2, g^4} . \quad (28)$$

Let us note that this result, combined with the absence of protected dimension  $\Delta_0 = 4$  scalar operators in the flavour singlet sector (see below), poses severe constraints on the conformal Operator Product Expansion (OPE) interpretation of these correlators.

- The 3-point functions  $\langle \mathcal{C}^{IJ}(x_1) \mathcal{C}_{IJ}^\dagger(x_2) \mathcal{V}_{X,Y}(x_3) \rangle$  receive finite non vanishing corrections at order  $g^4$  (but not at order  $g^2$ ). These corrections affect only the normalizations of the functions and are proportional to the corrections of the 2-point functions  $\langle \mathcal{C}^{IJ}(x_1) \mathcal{C}_{IJ}^\dagger(x_2) \rangle$  at the same order (see eq. (54) below). In fact, since the operators  $\mathcal{V}_X$  and  $\mathcal{V}_Y$  are the lowest components of supermultiplets containing conserved currents, the above 3-point and 2-point functions are related by the Ward identities.

### 3.2 Operators of $\Delta_0 = 3$ up to order $g^4$

All the operators of  $\Delta_0 = 3$  with vanishing order  $g^2$  anomalous dimension are chiral (or antichiral) and were all found in [4]. Among them we find that (in the notation introduced in (20))

$$\mathcal{O}^{111} = \text{tr}(111) \quad (29)$$

has also vanishing order  $g^4$  anomalous dimension

$$\gamma_{\mathcal{O}^{111}}^{(2)} = 0 , \quad (30)$$

while its 2-point function receives finite corrections at the same order.

The only other operator with vanishing order  $g^2$  anomalous dimension is

$$\mathcal{O}^{123} = \text{tr}(123) + \frac{(N^2 - 2) \bar{\omega} + 2}{N^2 - 2 + 2 \bar{\omega}} \text{tr}(132). \quad (31)$$

This is the first example (we shall give more below) of an operator with coefficients explicitly depending on the parameter  $\omega$ . In this case we could not push our calculation to the next order, because  $\mathcal{O}^{123}$  (for  $\omega \neq 1$ ) is not protected in the undeformed theory. As a consequence, it is not sufficient to compute the difference of the perturbative corrections in the two theories.

### 3.3 Operators of $\Delta_0 = 4$ at order $g^2$ and $g^4$

We shall consider operators made of scalars only, since at order  $g^2$  they form a closed subspace. It should be noted that some of them (*e.g.* the non-chiral operators listed below) can have a fermionic contribution. However, since this piece will enter multiplied by an overall  $g$  factor, it will not affect the lowest-order logarithmic behaviour of the 2-point functions. Thus, the fermionic contribution is irrelevant as far as only the order  $g^2$  anomalous dimension of the scalar operators is considered and we will omit it.

To compute the anomalous dimensions of the operators at order  $g^2$  we use the following (standard) procedure (for the details see *e.g.* [16]).

- We write down all scalar operators  $\mathcal{O}_i$  with given flavour and bare dimension  $\Delta_0$ .
- We compute the matrix of the tree-level 2-point functions

$$F_{ij}^{(0)} = \langle \mathcal{O}_i(x_1) \mathcal{O}_j^\dagger(x_2) \rangle_{\text{tree}}, \quad (32)$$

and notice that the coordinate dependence in all the functions is the same, *i.e.*  $(x_{12}^2)^{-\Delta_0}$ .

- We compute at order  $g^2$  the matrix of the 2-point functions in the deformed theory

$$F_{ij}^{(1)} = \langle \mathcal{O}_i(x_1) \mathcal{O}_j^\dagger(x_2) \rangle_{g^2}. \quad (33)$$

To this order the only relation one needs is eq. (15), by which we can express  $\kappa$  in terms of  $\omega$  and  $\bar{\omega}$ . In this way  $F^{(1)}$  will depend only on  $N$ ,  $\omega$  and  $\bar{\omega}$ . The coordinate dependence of the logarithmically divergent one-loop contributions is identical for all diagrams and it is given by  $(x_{12}^2)^{-\Delta_0} \times B(\epsilon, x_{12})$ , where  $B(\epsilon, x_{12})$  is the (regularized) massless box integral. It is the latter that contains the logarithmic divergence responsible for the anomalous dimension of the operators. Hence in both  $F^{(0)}$  and  $F^{(1)}$  we can factorize out the common coordinate dependence, and consider them as matrices with numerical entries.

- The anomalous dimensions of the operators are finally given by the eigenvalues of the matrix

$$\left(F^{(0)}\right)^{-1} F^{(1)}. \quad (34)$$

In particular operators with vanishing order  $g^2$  anomalous dimension correspond to the zero eigenvalues of this matrix. Their explicit tree level form in the  $\mathcal{O}_i$  basis is given by the corresponding eigenvectors. In other words, to find all the operators of vanishing order  $g^2$  anomalous dimension, it is sufficient to find the kernel of the matrix (34).

We now list all the scalar operators of  $\Delta_0 = 4$  which have vanishing anomalous dimensions at order  $g^2$ . The list is exhaustive for generic  $\omega$ . For special values, in particular for  $\omega = -1$ , there may be more operators (see *e.g.* eq. (3.29) in [4]). If some flavour choice is not in the list, then there are no operators with vanishing order  $g^2$  anomalous dimension with that flavour. In particular, we found no operators with vanishing order  $g^2$  anomalous dimension among the 39 flavour singlets of the type  $(I\bar{I}J\bar{J})$ . In order to maximally simplify the coefficients of the various linear combinations we make reference to some suitably chosen (possibly non-orthogonal) basis of operators belonging to the kernel of the matrix (34).

### Chiral operators

- **flavour 1111** - There are 2 operators, one single and one double trace

$$\text{tr}(1111) \quad \text{and} \quad \text{tr}(11)\text{tr}(11). \quad (35)$$

Both have vanishing order  $g^2$  and  $g^4$  anomalous dimensions, *i.e.*

$$\gamma_{\mathcal{O}^{1111}}^{(1)} = \gamma_{\mathcal{O}^{1111}}^{(2)} = 0. \quad (36)$$

Their 2-point functions receive finite non-vanishing corrections at order  $g^4$ .

- **flavour 1112** - There are 2 operators, one single and one double trace

$$\text{tr}(1112) \quad \text{and} \quad \text{tr}(11)\text{tr}(12). \quad (37)$$

Only one of them, namely

$$\mathcal{O}^{1112} = \text{tr}(1112) - \frac{(N^2 - 6)}{2N} \text{tr}(11)\text{tr}(12) \quad (38)$$

has vanishing order  $g^2$  and  $g^4$  anomalous dimensions, *i.e.*

$$\gamma_{\mathcal{O}^{1112}}^{(1)} = \gamma_{\mathcal{O}^{1112}}^{(2)} = 0. \quad (39)$$

Its 2-point function receives finite non-vanishing corrections at order  $g^4$ .

- **flavour 1122** - There are 4 operators, two single and two double trace operators

$$\text{tr}(1122), \text{tr}(1212), \text{tr}(11)\text{tr}(22) \quad \text{and} \quad \text{tr}(12)\text{tr}(12). \quad (40)$$

Two of them have vanishing order  $g^2$  anomalous dimensions. Both are linear combinations of single and double trace. They can be taken as the following combinations

$$2 \text{tr}(1122) + \text{tr}(1212) - \frac{(N^2 - 6)}{2N} (2 \text{tr}(12)\text{tr}(12) + \text{tr}(11)\text{tr}(22)), \quad (41)$$

and

$$\text{tr}(1122) - \text{tr}(1212) - \frac{N}{4} \text{tr}(11)\text{tr}(22) + N \text{tr}(12)\text{tr}(12). \quad (42)$$

- **flavour 1123** - There are 5 operators, three single and two double trace operators

$$\begin{aligned} &\text{tr}(1123), \text{tr}(1132), \text{tr}(1213), \\ &\text{tr}(11)\text{tr}(23) \quad \text{and} \quad \text{tr}(12)\text{tr}(13). \end{aligned} \quad (43)$$

Two of them have vanishing order  $g^2$  anomalous dimensions. Both are linear combinations of single and double traces. They can be taken to be

$$\begin{aligned} &\text{tr}(1123) + \text{tr}(1132) + \text{tr}(1213) + \\ &-\frac{(N^2 - 6)}{2N} (2 \text{tr}(12)\text{tr}(13) + \text{tr}(11)\text{tr}(23)), \end{aligned} \quad (44)$$

and

$$\begin{aligned} &\text{tr}(1123) - \text{tr}(1132) + \frac{((\bar{\omega}^2 + 1)(N^2 - 2) + 4\bar{\omega})}{N^2(\bar{\omega}^2 - 1)} \text{tr}(1213) + \\ &-\frac{(N^2 - 2)((\bar{\omega}^2 + 1)(N^2 - 2) + 4\bar{\omega})}{2N^3(\bar{\omega}^2 - 1)} \text{tr}(11)\text{tr}(23) + \\ &-\frac{N^4(\bar{\omega} + 1)^2 - 2(\bar{\omega}^2 + 1)N^2 + 4(\bar{\omega} - 1)^2}{N^3(\bar{\omega}^2 - 1)} \text{tr}(12)\text{tr}(13). \end{aligned} \quad (45)$$

Let us note the similarity of the mixing coefficients in the operators (38), (41) and (44).

## Non-chiral operators

- **flavour**  $11J\bar{J}$  - There are 12 operators, seven single and five double trace operators. They are

$$\begin{aligned} & \text{tr}(111\bar{1}) , \text{tr}(11) \text{tr}(1\bar{1}) , \\ & \text{tr}(112\bar{2}) , \text{tr}(11\bar{2}2) , \text{tr}(121\bar{2}) , \text{tr}(11) \text{tr}(2\bar{2}) , \text{tr}(12) \text{tr}(1\bar{2}) , \\ & \text{tr}(113\bar{3}) , \text{tr}(11\bar{3}3) , \text{tr}(131\bar{3}) , \text{tr}(11) \text{tr}(3\bar{3}) , \text{tr}(13) \text{tr}(1\bar{3}) . \end{aligned} \quad (46)$$

Two operators have vanishing order  $g^2$  anomalous dimensions. The first one is pure double trace

$$\text{tr}(11) (2 \text{tr}(1\bar{1}) - \text{tr}(2\bar{2}) - \text{tr}(3\bar{3})) = \mathcal{C}^{11} \mathcal{V}_X \quad (47)$$

(see eqs. (22) and (23) for the notation). Let us stress that, surprisingly, the operator  $\mathcal{C}^{11} \mathcal{V}_Y$ , although very similar in structure to  $\mathcal{C}^{11} \mathcal{V}_X$  has a non-vanishing order  $g^2$  anomalous dimension.

The second one can be taken as the following linear combination of single and double trace terms

$$\begin{aligned} & 2(N^2 - 3) (\text{tr}(112\bar{2}) + \text{tr}(11\bar{2}2)) + 12 \text{tr}(121\bar{2}) + \\ & -N(N^2 - 7) \text{tr}(11) \text{tr}(2\bar{2}) - 4N \text{tr}(12) \text{tr}(1\bar{2}) + \\ & -(2 \leftrightarrow 3 , \bar{2} \leftrightarrow \bar{3}) . \end{aligned} \quad (48)$$

- **flavour**  $12J\bar{J}$ : There are 19 operators, twelve single and seven double trace ones

$$\begin{aligned} & \text{tr}(123\bar{3}) , \text{tr}(12\bar{3}3) , \text{tr}(132\bar{3}) , \text{tr}(13\bar{3}2) , \text{tr}(1\bar{3}23) , \text{tr}(1\bar{3}32) , \\ & \text{tr}(12) \text{tr}(3\bar{3}) , \text{tr}(13) \text{tr}(2\bar{3}) , \text{tr}(1\bar{3}) \text{tr}(23) , \\ & \text{tr}(112\bar{1}) , \text{tr}(11\bar{1}2) , \text{tr}(121\bar{1}) , \text{tr}(11) \text{tr}(2\bar{1}) , \text{tr}(12) \text{tr}(1\bar{1}) , \\ & \text{tr}(221\bar{2}) , \text{tr}(22\bar{2}1) , \text{tr}(212\bar{2}) , \text{tr}(22) \text{tr}(1\bar{2}) , \text{tr}(21) \text{tr}(2\bar{2}) . \end{aligned} \quad (49)$$

Again there are two operators with vanishing order  $g^2$  anomalous dimensions, both are linear combinations of single and double traces. The first one can be taken as

$$\begin{aligned} & \text{tr}(123\bar{3}) + \text{tr}(1\bar{3}32) + N \text{tr}(13) \text{tr}(2\bar{3}) + \\ & - \text{tr}(112\bar{1}) - \text{tr}(11\bar{1}2) - (N^2 - 2) \text{tr}(121\bar{1}) + N(N^2 - 3) \text{tr}(12) \text{tr}(1\bar{1}) + \\ & - (1 \leftrightarrow 2 , \bar{1} \leftrightarrow \bar{2}) , \end{aligned} \quad (50)$$

while the second has rather complicated mixing coefficients depending on both  $\omega$  and  $\bar{\omega}$ . One finds

$$\begin{aligned}
& N^2(N^2\bar{\omega}(3\omega\bar{\omega} + 1) - 2(\bar{\omega} - 1)(4\omega\bar{\omega} - 3\bar{\omega} + \omega - 2)) \times \\
& \times (-\omega\text{tr}(123\bar{3}) - \omega\text{tr}(12\bar{3}3) + 2\text{tr}(132\bar{3}) - \text{tr}(11\bar{1}2) - \text{tr}(221\bar{2})) + \\
+ & N^2(N^2(\omega\bar{\omega} + 3) - 2(\bar{\omega} - 1)(2\bar{\omega}\omega - \bar{\omega} + 3\omega - 4)) \times \\
& \times (-\text{tr}(13\bar{3}2) + 2\omega\text{tr}(1\bar{3}23) - \text{tr}(1\bar{3}32) - \omega\text{tr}(112\bar{1}) - \omega\text{tr}(22\bar{2}1)) + \\
+ & N(\omega - 1)((3\omega\bar{\omega}^2 - \bar{\omega}\omega + \bar{\omega} - 3)N^2 - 4(\bar{\omega} - 1)^2(\omega - 1)) \times \\
& \times (\text{tr}(13) \text{tr}(2\bar{3}) + \text{tr}(1\bar{3}) \text{tr}(23)) \\
+ & N^2((2\bar{\omega}\omega + 3 + 3\bar{\omega}^2\omega^2)N^2 + \\
& - 2(\bar{\omega} - 1)(4\omega^2\bar{\omega} + \omega^2 - \bar{\omega}\omega - \bar{\omega} + \omega - 4))(\text{tr}(121\bar{1}) + \text{tr}(212\bar{2})) + \\
- & \frac{1}{3}N(6(\bar{\omega}\omega + 1)^2N^4 + 28(\bar{\omega} - 1)^2(\omega - 1)^2 + \\
& - (\omega - 1)(25\omega\bar{\omega}^2 - 25 - 29\bar{\omega}\omega + 29\bar{\omega} + 2\bar{\omega}^2 - 2\omega)N^2) \times \\
& \times (\text{tr}(12) \text{tr}(1\bar{1}) + \text{tr}(21) \text{tr}(2\bar{2})) + \\
+ & \frac{4}{3}N(3(\bar{\omega}\omega + 1)^2N^4 - (\bar{\omega} - 1)(\omega - 1)(17(\bar{\omega}\omega + 1) + \omega + \bar{\omega})N^2 + \\
& + 20(\bar{\omega} - 1)^2(\omega - 1)^2) \text{tr}(12) \text{tr}(3\bar{3}). \tag{51}
\end{aligned}$$

### 3.4 Finite corrections to the invariant trilinear couplings at order $g^4$

As already noted, most of the operators with vanishing order  $g^2$  and  $g^4$  anomalous dimension receive finite corrections to their 2-point functions at order  $g^4$ . In principle the finite correction to the normalization of the 2-point function of an operator is not in conflict with conformal invariance or the vanishing of its anomalous dimension at next perturbative orders. Moreover, if these operators are not in the same supermultiplet with the generators of some symmetry, their normalization is not protected, so we can change it arbitrarily by a finite  $g$  dependent rescaling. There are, however, quantities which are invariant under such a rescaling, namely the ratio of the normalization of a 3-point function,  $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$ , to the square root of the product of the normalizations of the 2-point functions, *i.e.*

$$C_{\mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3} = \frac{\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle}{\sqrt{\langle \mathcal{O}_1 \mathcal{O}_1^\dagger \rangle \langle \mathcal{O}_2 \mathcal{O}_2^\dagger \rangle \langle \mathcal{O}_3 \mathcal{O}_3^\dagger \rangle}}. \tag{52}$$

Hence the real question is whether also these invariant trilinear couplings receive finite corrections or not. The explicit calculation to order  $g^4$  of the

3-point function

$$\langle \mathcal{O}^{1112}(x_1) \mathcal{C}_{12}^\dagger(x_2) \mathcal{C}_{11}^\dagger(x_3) \rangle \quad (53)$$

(see eqs. (22) and (38) for the definition of the operators) and of the associated 2-point functions, shows that such invariant trilinear couplings get indeed a finite correction to this order. Thus, the finite order  $g^4$  corrections cannot be absorbed by a redefinition of the operators.

For the reader's convenience we report below the ratios of the order  $g^4$  corrections to the tree-level expressions for the relevant 2- and 3- point functions and for the invariant trilinear coupling. For simplicity we have omitted a common numerical factor proportional to  $\zeta(3)$ , but we kept the complete dependence on  $\omega$  and  $N$ . One gets

$$\begin{aligned} \mathcal{C}^{11} \Big|_{2\text{pt}} &: \frac{N^2(N^2 - 4)(N^2((\omega\bar{\omega} + 1)^2 - 2(\omega^2 + \bar{\omega}^2)) + 2(\omega - 1)^2(\bar{\omega} - 1)^2)}{4((N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega}))^2}, \\ \mathcal{C}^{12} \Big|_{2\text{pt}} &: \frac{N^2(N^2 - 4)(N^2(\omega\bar{\omega} - 1)^2 + 2(\omega - 1)^2(\bar{\omega} - 1)^2)}{4((N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega}))^2}, \\ \mathcal{O}^{1112} \Big|_{2\text{pt}} &: \frac{(N^2 - 4)(\omega - 1)(\bar{\omega} - 1)}{2(N^2 - 8)((N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega}))^2} \times \\ &\quad \times [(N^6 - 4N^4 - 56N^2 + 48)(\omega + 1)(\bar{\omega} + 1) + \\ &\quad - 4(N^4 - 16N^2 + 24)(\omega + \bar{\omega})], \end{aligned} \quad (54)$$

and

$$\begin{aligned} \mathcal{O}^{1112} \mathcal{C}_{12}^\dagger \mathcal{C}_{11}^\dagger \Big|_{3\text{pt}} &: - \frac{(N^2 - 4)(\omega - 1)(\bar{\omega} - 1)}{((N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega}))^2} \times \\ &\quad \times ((N^4 + 6N^2 - 8)(\omega\bar{\omega} + 1) + (N^4 + 2N^2 + 8)(\omega + \bar{\omega})), \\ C_{\mathcal{O}^{1112} \mathcal{C}_{12}^\dagger \mathcal{C}_{11}^\dagger} &: - \frac{(N^2 - 4)(\omega - 1)(\bar{\omega} - 1)}{(N^2 - 8)((N^2 - 2)(\omega\bar{\omega} + 1) + 2(\omega + \bar{\omega}))^2} \times \\ &\quad \times (N^2(3N^2 - 20)(\omega + 1)(\bar{\omega} + 1) + 40(\omega - 1)(\bar{\omega} - 1)). \end{aligned} \quad (55)$$

The first two expressions in (54) for  $|\omega| = 1$  agree with the results of [5]. Note, however, that the conclusion that the correction to the 2-point function of  $\mathcal{C}_{12}$  is suppressed in the planar limit, while the correction to the 2-point function of  $\mathcal{C}_{11}$  is not, crucially depends on the choice  $|\omega| = 1$  made in [5]. For generic  $\omega$  both corrections are non-vanishing in the large  $N$  limit. The invariant trilinear couplings  $C_{\mathcal{O}^{1112} \mathcal{C}_{12}^\dagger \mathcal{C}_{11}^\dagger}$  of the two operators in (35) also receive finite corrections at order  $g^4$ .

## 4 Conclusions

Our explicit analysis suggests that although the deformed theory has only  $\mathcal{N} = 1$  supersymmetry, it inherits many of the interesting properties of  $\mathcal{N} = 4$  SYM. It can be made (at least up to order  $g^6$ ) perturbatively finite. It is endowed with a rich spectrum of composite operators with vanishing anomalous dimensions and a complicated operator mixing pattern. It also shows some surprising new features, like the finite perturbative corrections to the 2- and 3-point functions of the operators with vanishing anomalous dimension. Finally we observe that even in the large  $N$  limit, the deformed and the undeformed theories are significantly different.

A crucial but still open question is whether the deformed theory is exactly conformal.

## Acknowledgements

It is a pleasure to thank Dan Freedman for extensive discussion and for showing us the manuscript of ref. [4] before publication. The work of G.C.R. and Ya.S. was supported in part by INFN, MIUR-COFIN contract 2003-023852, EU contracts MRTN-CT-2004-503369 and MRTN-CT-2004-512194, INTAS contract 03-51-6346, and NATO grant PST.CLG.978785. E.S. is grateful to the organizers for the invitation to the conference “The Legacy of Supergravity”, Frascati, June 2005, where part of this work was done. G.C.R. thanks the Alexander von Humboldt Foundation for partial financial support.



## References

- [1] R. G. Leigh and M. J. Strassler, Nucl. Phys. B **447** (1995) 95 [arXiv:hep-th/9503121].
- [2] D. Berenstein and R. G. Leigh, JHEP **0001** (2000) 038 [arXiv:hep-th/0001055];  
D. Berenstein, V. Jejjala and R. G. Leigh, Nucl. Phys. B **589** (2000) 196 [arXiv:hep-th/0005087].
- [3] O. Lunin and J. Maldacena, JHEP **0505** (2005) 033 [arXiv:hep-th/0502086].
- [4] D. Z. Freedman and U. Gursoy, arXiv:hep-th/0506128.
- [5] S. Penati, A. Santambrogio and D. Zanon, arXiv:hep-th/0506150.
- [6] S. A. Frolov, R. Roiban and A. A. Tseytlin, arXiv:hep-th/0503192.
- [7] S. Frolov, JHEP **0505** (2005) 069 [arXiv:hep-th/0503201].
- [8] S. A. Frolov, R. Roiban and A. A. Tseytlin, arXiv:hep-th/0507021.
- [9] N. Beisert and R. Roiban, arXiv:hep-th/0505187.
- [10] R. de Mello Koch, J. Murugan, J. Smolic and M. Smolic, arXiv:hep-th/0505227.
- [11] I. Jack, D. R. T. Jones and C. G. North, Nucl. Phys. B **473** (1996) 308 [arXiv:hep-ph/9603386].
- [12] M. T. Grisaru, W. Siegel and M. Rocek, Nucl. Phys. B **159** (1979) 429.
- [13] P. Cvitanovic, Phys. Rev. D **14** (1976) 1536.
- [14] L. F. Abbott and M. T. Grisaru, Nucl. Phys. B **169** (1980) 415.
- [15] G.C. Rossi, E. Sokatchev and Ya.S. Stanev, in preparation.
- [16] M. Bianchi, B. Eden, G.C. Rossi and Ya. S. Stanev, Nucl. Phys. B **646**, 69 (2002) [arXiv:hep-th/0205321];  
M. Bianchi, G.C. Rossi and Ya. S. Stanev, Nucl. Phys. B **685** (2004) 65 [arXiv:hep-th/0312228].