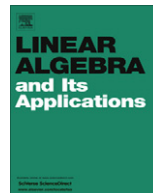




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Optimal rank matrix algebras preconditioners

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ARTICLE INFO

Article history:

Received 23 April 2012

Accepted 20 July 2012

Available online 19 September 2012

Submitted by R.A. Brualdi

AMS classification:

65F05

65F10

65F15

65F50

65T50

Keywords:

Preconditioning

Matrix algebras

Toeplitz

Hankel

Clustering

Fast discrete transforms

ABSTRACT

When a linear system $Ax = y$ is solved by means of iterative methods (mainly CG and GMRES) and the convergence rate is slow, one may consider a preconditioner P and move to the preconditioned system $P^{-1}Ax = P^{-1}y$. The use of such preconditioner changes the spectrum of the matrix defining the system and could result into a great acceleration of the convergence rate. The construction of *optimal rank* preconditioners is strongly related to the possibility of splitting A as $A = P + R + E$, where E is a small perturbation and R is of low rank (Tyrtyshnikov, 1996) [1]. In the present work we extend the black-dot algorithm for the computation of such splitting for P circulant (see Oseledets and Tyrtyshnikov, 2006 [2]), to the case where P is in \mathcal{A} , for several known low-complexity matrix algebras \mathcal{A} . The algorithm so obtained is particularly efficient when A is Toeplitz plus Hankel like. We finally discuss in detail the existence and the properties of the decomposition $A = P + R + E$ when A is Toeplitz, also extending to the φ -circulant and Hartley-type cases some results previously known for P circulant.

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1. Introduction

In this work we consider a new approach to the construction of preconditioning matrices P for the solution of linear systems $Ax = y$. We call this kind of preconditioners *optimal rank* because they are produced trying to force the rank of $A - P$ to be as small as possible. Optimal rank circulant preconditioners P were initially proposed for Toeplitz systems by Tyrtyshnikov et al. in [2,3]. Here we

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basically extend them to several known low complexity matrix algebras [4–7] and to Toeplitz plus Hankel like matrices.

1.1. Notations

We use some standard notations, which are briefly described here.

With $\mathbf{M}(n)$ we denote the Hilbert space of $n \times n$ matrices whose entries are in the complex field \mathbb{C} , with $\mathbf{U}(n)$ the group of unitary $n \times n$ matrices and with $\mathbf{M}(n)_+$ the cone of positive semi-definite $n \times n$ matrices. We use also the notation $A \geq 0$ for an element of $\mathbf{M}(n)_+$ and $A > 0$ for those matrices which are strictly positive definite. Given $A \in \mathbf{M}(n)$, $\sigma(A)$ and $\lambda_i(A)$ are the spectrum and the i th eigenvalue of A , respectively.

The square bracket $[\cdot, \cdot] : \mathbf{M}(n) \times \mathbf{M}(n) \rightarrow \mathbf{M}(n)$ denotes the commutator

$$A, B \mapsto [A, B] = AB - BA$$

also denoted by $\nabla_A(B) = [B, A]$. The round bracket (\cdot, \cdot) denotes the standard scalar product on \mathbb{C}^n (sometimes it can denote the scalar product on different Hilbert spaces; it will be clear from the context).

The symbol e_i is used for the i th canonical vector $(e_i)_k = 1$ if $k = i$ and $(e_i)_k = 0$ otherwise.

Finally given a matrix $W \in \mathbf{M}(n)$ we shall use the symbol $\mathcal{A}(W)$ for the algebra generated by W , namely the closed set

$$\mathcal{A}(W) = \{p(W) \mid p \text{ polynomials}\}$$

2. Low complexity matrix algebras

Suppose we are given a unitary matrix $U \in \mathbf{U}(n)$. Then it can be naturally defined the algebra of normal matrices

$$\mathcal{A} = \text{sd } U = \{U \text{diag}(\theta_1, \dots, \theta_n)U^* \mid \theta_i \in \mathbb{C}\}$$

also called *algebra of matrices simultaneously diagonalized by a unitary transform* or briefly *sd U algebra*. For an element $A \in \mathcal{A}$, the complexity of the products $A \times \text{vector}$ and $A^{-1} \times \text{vector}$ depends only on the complexity of $U \times \text{vector}$ and $U^* \times \text{vector}$. Therefore we say that a *sd U algebra \mathcal{A}* is of low complexity if both these products are computable with less than $O(n^2)$ operations, in particular if they can be performed with $O(n \log n)$ floats.

Notice that, whenever $W \in \mathbf{M}(n)$ is diagonalized by $M \in \mathbf{M}(n)$, for any $A \in \mathcal{A}(W)$ we have $A = M \text{diag}(\lambda_1(A), \dots, \lambda_n(A))M^{-1}$. Moreover if (A_λ) is a family of mutually commuting matrices, it is known that (A_λ) admits a common Schur basis. These two facts together imply that the algebra $\mathcal{A}(N)$ generated by a normal $n \times n$ matrix N , must satisfy $\mathcal{A}(N) \subset \text{sd } U$ for a unitary matrix $U \in \mathbf{U}(n)$. The inclusion can be proper, and precisely it is an identity if and only if N is *non-derogatory*³ [4]. Therefore the non-derogatoricity hypothesis on the matrix W leads to the following further characterization

$$\mathcal{A}(W) = \ker \nabla_W$$

and if W is normal we also have $\mathcal{A}(W) = \text{sd } U$, for an $U \in \mathbf{U}(n)$.

The best known low complexity *sd U algebras* are commonly divided into three classes: φ -circulants, Trigonometric and Hartley-type. Some specific choices in such classes have been used successfully to solve linear algebra problems involving Toeplitz matrices or, more generally, structured matrices related to shift invariance of the mathematical model considered (see f.i. [8] and references therein).

³ A matrix $A \in \mathbf{M}(n)$ is said to be non-derogatory if $\deg(p) \geq n$, for any polynomial p such that $p(A) = 0$, or, equivalently, if the geometric multiplicity of any eigenvalue of A is one. We make often use of matrices that are both normal and non-derogatory, which therefore are those normal matrices that have pairwise different eigenvalues.

Anyway, depending on the problem, any algebra among the three families could find a potential application as a preconditioner.

2.1. φ -Circulant algebras

Let us consider the matrix

$$\Pi_\varphi = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \varphi & & & \end{pmatrix}, \quad \varphi \in \mathbb{C}, \tag{1}$$

which is the basis for the definition of the family of φ -circulant algebras

Definition 2.1. Given $\varphi \in \mathbb{C}$, the algebra $\mathcal{C}_\varphi = \mathcal{A}(\Pi_\varphi)$ generated by Π_φ is called φ -circulant algebra.

Note that $e_1^\top \Pi_\varphi^k = e_{k+1}^\top$ for $k = 0, 1, \dots, n - 1$, which implies that \mathcal{C}_φ is a 1-space [4] or, in other words, that any matrix $C \in \mathcal{C}_\varphi$ is uniquely defined by its first row. It is now natural to introduce the operator

$$C_\varphi : \mathbb{C}^n \longrightarrow \mathcal{C}_\varphi, \quad x \mapsto C_\varphi(x)$$

which maps $x \in \mathbb{C}^n$ into the matrix $C_\varphi(x) \in \mathcal{C}_\varphi$ whose first row is x^\top .

It is not difficult to observe that the matrix

$$F_\varphi = \frac{1}{\sqrt{n}} \left(\varphi^{\frac{i}{n}} \omega^{ij} \right)_{i,j=0,1,\dots,n-1}, \quad \omega = e^{-\frac{2\pi i}{n}} \tag{2}$$

diagonalizes the algebra \mathcal{C}_φ , namely that

$$\mathcal{C}_\varphi = \{ F_\varphi \text{diag}(\theta_1, \dots, \theta_n) F_\varphi^{-1} \mid \theta_i \in \mathbb{C} \}.$$

Moreover $F_\varphi \in \mathbf{U}(n)$ if and only if $|\varphi| = 1$ (cf. [9]). From now on we assume that φ , defining \mathcal{C}_φ , has modulus one, unless otherwise specified. Nevertheless we underline that several formulas that we obtain could be adapted to the case of a generic complex φ . The choice $\varphi = 1$ gives rise to the well known circulant algebra \mathcal{C} , diagonalized by the Fourier matrix

$$F = \frac{1}{\sqrt{n}} \left(e^{-\frac{2\pi k h i}{n}} \right)_{k,h=0,1,\dots,n-1} \tag{3}$$

which is naturally related to F_φ up to the diagonal scaling

$$F_\varphi = \text{diag} \left(1, \varphi^{\frac{1}{n}}, \dots, \varphi^{\frac{n-1}{n}} \right) F = \Delta_\varphi F.$$

We shall use simply the symbol Π for Π_1 . Another very popular choice is $\varphi = -1$ which defines the so called skew-circulant algebra \mathcal{C}_{-1} . Both \mathcal{C} and \mathcal{C}_{-1} and their applications has been widely studied, see for instance [10]. Since Π_φ is normal and non-derogatory we have $\mathcal{C}_\varphi = \ker \nabla_{\Pi_\varphi}$.

2.2. Trigonometric algebras

There are sixteen different trigonometric algebras presently known [7, 11]. Eight of them are diagonalized by a discrete sine-type transform, the other eight by a cosine-type one. We start again by introducing a family of matrices

Table 1
Sixteen choices for $\mu \in \mathbb{R}^4$ and respective trigonometric algebras.

	$\mu_3 = 2$ $\mu_4 = 0$	$\mu_3 = 1$ $\mu_4 = 0$	$\mu_3 = 1$ $\mu_4 = 1$	$\mu_3 = 1$ $\mu_4 = -1$
$\mu_1 = 0, \mu_2 = 2$	DCT_1	DCT_3	DCT_5	DCT_7
$\mu_1 = 0, \mu_2 = 1$	DST_3	DST_1	DST_7	DST_5
$\mu_1 = 1, \mu_2 = 1$	DCT_6	DCT_8	DCT_2	DCT_4
$\mu_1 = -1, \mu_2 = 1$	DST_8	DST_6	DST_4	DST_2

$$X_\mu = X_{(\mu_1, \mu_2, \mu_3, \mu_4)} = \begin{pmatrix} \mu_1 & \mu_2 & & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 0 & 1 \\ & & & & \mu_3 & \mu_4 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}, \tag{4}$$

and defining $\mathcal{T}_\mu = \mathcal{A}(X_\mu)$. Sixteen different choices for the vector $\mu \in \mathbb{R}^4$ give rise to the sixteen trigonometric algebras (see f.i. appendix 1 in [12]). We list in Table 1 such particular values for μ naming the corresponding trigonometric algebras \mathcal{T}_μ also DST or DCT so to recall that they are diagonalized by a discrete sine transform or a discrete cosine transform, respectively.

It is not difficult to observe that for such choices of μ the matrix X_μ is normal and non-derogatory, since $\mu_2 \neq 0$, so $\mathcal{T}_\mu = \ker \nabla_{X_\mu}$.

The well known *tau-algebra* is $DST_1 = \mathcal{T}_{(0,1,1,0)} = \mathcal{T}$ whose generating matrix will be denoted simply by X . Such algebra was considered in [13] where it is defined as the set of $n \times n$ matrices satisfying the *cross-sum* rule with null boundary conditions

$$\mathcal{T} = \{A \in \mathbf{M}(n) \mid a_{ij} \in \mathbb{C}, a_{i-1,j} + a_{i+1,j} = a_{i,j-1} + a_{i,j+1}, i, j = 1, \dots, n, a_{n+1,j} = a_{0,j} = a_{i,n+1} = a_{i,0} = 0\}.$$

This is a computationally useful definition and it clearly is nothing but the scalar form of our previous characterization of \mathcal{T} as the kernel of ∇_X . Other trigonometric algebras which attained particular attention are $DCT_2, DST_2, DST_3, DST_7$ mainly because of their applications to image processing [14], displacement decomposition [11,15], and preconditioning techniques (see [4,6] and the references therein).

As for the φ -circulants, also the trigonometric algebras are 1-spaces [4], namely each matrix in \mathcal{T}_μ is uniquely defined up to its first row. In [15] such property is explicitly shown for the more generic set of Hessenberg algebras which contains both φ -circulant and trigonometric algebras. As a special case of that result we derive the following characterization for \mathcal{T}_μ : consider $n - 1$ polynomials $\Phi_1, \dots, \Phi_{n-1}$, each Φ_k defined as the characteristic polynomial of the principal submatrix of X_μ of order k . Setting $\Phi_0 = 1$ and

$$\mathcal{X}_\mu^{(k)} = \mu_2^{-1} \Phi_{k-1}(X_\mu), \quad k = 1, \dots, n,$$

then $\mathcal{T}_\mu = \text{Span}(\mathcal{X}_\mu^{(1)}, \dots, \mathcal{X}_\mu^{(n)})$ and the 1-space property $e_1^\top \mathcal{X}_\mu^{(k)} = e_k^\top$ follows. So, as for the circulant case, we introduce in a natural way the operator

$$\tau_\mu : \mathbb{C}^n \rightarrow \mathcal{T}_\mu \quad x \mapsto \tau_\mu(x)$$

which maps $x \in \mathbb{C}^n$ into the matrix $\tau_\mu(x) \in \mathcal{T}_\mu$ whose first row is x^\top .

Since X_μ is non-derogatory and normal as well, there exists a matrix $U_\mu \in \mathbf{U}(n)$ such that $\mathcal{T}_\mu = \text{sd } U_\mu$. Using the symbol $S_\mu (C_\mu)$ for the matrix U_μ diagonalizing \mathcal{T}_μ when the choice of μ gives rise to a DST (DCT), we list S_μ, C_μ and the eigenvalues of X_μ in Tables 2, 3. We finally underline that all of them satisfy the low complexity property (see [16,17] and references therein).

Table 2
Discrete sine transform S_{μ} and the eigenvalues λ_k of X_{μ} , $k, h = 0, \dots, n - 1$.

	$(S_{\mu})_{kh}$	$\lambda_k(X_{\mu})$
DST_1	$\sin(k + 1)(h + 1) \frac{\pi}{n + 1}$	$\cos \frac{(k + 1)\pi}{n + 1}$
DST_2	$\sin(k + 1) \left(h + \frac{1}{2}\right) \frac{\pi}{n}$	$\cos \frac{(k + 1)\pi}{n}$
DST_3	$\sin \left(k + \frac{1}{2}\right) (h + 1) \frac{\pi}{n}$	$\cos \frac{(k + \frac{1}{2})\pi}{n}$
DST_4	$\sin \left(k + \frac{1}{2}\right) \left(h + \frac{1}{2}\right) \frac{\pi}{n}$	$\cos \frac{(k + \frac{1}{2})\pi}{n}$
DST_5	$\sin(k + 1)(h + 1) \frac{\pi}{n + \frac{1}{2}}$	$\cos \frac{(k + 1)\pi}{n + \frac{1}{2}}$
DST_6	$\sin(k + 1) \left(h + \frac{1}{2}\right) \frac{\pi}{n + \frac{1}{2}}$	$\cos \frac{(k + 1)\pi}{n + \frac{1}{2}}$
DST_7	$\sin \left(k + \frac{1}{2}\right) (h + 1) \frac{\pi}{n + \frac{1}{2}}$	$\cos \frac{(k + \frac{1}{2})\pi}{n + \frac{1}{2}}$
DST_8	$\sin \left(k + \frac{1}{2}\right) \left(h + \frac{1}{2}\right) \frac{\pi}{n - \frac{1}{2}}$	$\cos \frac{(k + \frac{1}{2})\pi}{n - \frac{1}{2}}$

Table 3
Discrete cosine transform C_{μ} and the eigenvalues λ_k of X_{μ} , $k, h = 0, \dots, n - 1$.

	$(C_{\mu})_{kh}$	$\lambda_k(X_{\mu})$
DCT_1	$\cos kh \frac{\pi}{n - 1}$	$\cos \frac{k\pi}{n - 1}$
DCT_2	$\cos k \left(h + \frac{1}{2}\right) \frac{\pi}{n}$	$\cos \frac{k\pi}{n}$
DCT_3	$\cos \left(k + \frac{1}{2}\right) h \frac{\pi}{n}$	$\cos \frac{(k + \frac{1}{2})\pi}{n}$
DCT_4	$\cos \left(k + \frac{1}{2}\right) \left(h + \frac{1}{2}\right) \frac{\pi}{n}$	$\cos \frac{(k + \frac{1}{2})\pi}{n}$
DCT_5	$\cos kh \frac{\pi}{n - \frac{1}{2}}$	$\cos \frac{k\pi}{n - \frac{1}{2}}$
DCT_6	$\cos k \left(h + \frac{1}{2}\right) \frac{\pi}{n - \frac{1}{2}}$	$\cos \frac{k\pi}{n - \frac{1}{2}}$
DCT_7	$\cos \left(k + \frac{1}{2}\right) h \frac{\pi}{n - \frac{1}{2}}$	$\cos \frac{(k + \frac{1}{2})\pi}{n - \frac{1}{2}}$
DCT_8	$\cos \left(k + \frac{1}{2}\right) \left(h + \frac{1}{2}\right) \frac{\pi}{n + \frac{1}{2}}$	$\cos \frac{(k + \frac{1}{2})\pi}{n + \frac{1}{2}}$

2.3. Hartley-type algebras

In [5] eight different unitary matrices H_i are introduced and eight Hartley-type algebras $\mathcal{H}_i = \text{sd } H_i$ are defined as the set of matrices simultaneously diagonalized by such H_i . The Hartley algebra \mathcal{H}_1 was introduced in [18] and the well known Hartley matrix H_1 , which diagonalizes it, is defined as follows

$$H_1 = \frac{1}{\sqrt{n}} \left[\cos \left(\frac{2\pi ij}{n} \right) + \sin \left(\frac{2\pi ij}{n} \right) \right]_{ij=0,1,\dots,n-1}.$$

The multiplication $H_1 \times \text{vector}$ can be performed with $O(n \log n)$ operations, nonetheless in [5] is shown that the same low complexity property holds for all the H_i , $i = 1, \dots, 8$. Let us introduce in detail the Hartley-type transforms and the corresponding algebras. Consider the matrices

Table 4
Definitions for the eight Hartley-type algebras \mathcal{H}_i .

$\mathcal{H}_1 = \mathcal{H} = \text{sd } H = \text{sd } H_1 = \mathbb{C}^s + J\Pi\mathbb{C}^{sk}$
$\mathcal{H}_2 = \mathcal{K} = \text{sd } K = \text{sd } H_2 = \mathbb{C}_{-1}^s + J\Pi_{-1}\mathbb{C}_{-1}^{sk}$
$\mathcal{H}_3 = \mathcal{G} = \text{sd } G = \text{sd } H_3 = \mathbb{C}_{-1}^s + J\mathbb{C}_{-1}^{sk}$
$\mathcal{H}_4 = \tilde{\mathcal{K}} = \text{sd } K^T = \text{sd } H_4 = \mathbb{C}^s + J\mathbb{C}^{sk}$
$\mathcal{H}_5 = \eta = \text{sd}(K^T E_1) = \text{sd } H_5 = \mathbb{C}^s + J\mathbb{C}^s$
$\mathcal{H}_6 = \mu = \text{sd}(G E_2) = \text{sd } H_6 = \mathbb{C}_{-1}^s + J\mathbb{C}_{-1}^s$
$\mathcal{H}_7 = \alpha = \text{sd}(H E_1^T) = \text{sd } H_7 = \mathbb{C}^s + J\Pi\mathbb{C}^s$
$\mathcal{H}_8 = \beta = \text{sd}(K E_2^T) = \text{sd } H_8 = \mathbb{C}_{-1}^s + J\Pi_{-1}\mathbb{C}_{-1}^s$

$$K = H_2 = \frac{1}{\sqrt{n}} \left[\text{cas} \left(\frac{2\pi i(2j+1)}{n} \right) \right]_{j=0,1,\dots,n-1},$$

$$G = H_3 = \frac{1}{\sqrt{n}} \left[\text{cas} \left(\frac{2\pi(2i+1)(2j+1)}{2n} \right) \right]_{j=0,1,\dots,n-1},$$

where $\text{cas } x = \cos x + \sin x$. Both of them are orthonormal matrices. Furthermore consider the two sparse $n \times n$ matrices

$$E_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & & & \\ & I & & J \\ & & \sqrt{2} & \\ & -J & & I \end{pmatrix}, \quad E_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -J \\ & \sqrt{2} \\ J & \\ & I \end{pmatrix}$$

where the presence of the central row and column depends on the oddness of n . Consider finally the four subspaces of ± 1 -circulant algebras

$$\mathbb{C}_{\pm 1}^s = \{C \in \mathbb{C}_{\pm 1} \mid C^T = C\}, \quad \mathbb{C}_{\pm 1}^{sk} = \{C \in \mathbb{C}_{\pm 1} \mid C^T = -C\},$$

the first two being a proper subalgebra of $\mathbb{C}_{\pm 1}$.

The eight Hartley-type transforms and algebras are defined by the identities in Table 4. Note that only the algebra $\mathcal{G} = \text{sd } G$ is not a 1-space, in fact its elements are not defined uniquely by the first row (for more details see [4,5]).

Unlike φ -circulant and trigonometric algebras, it is not so clear that Hartley-type algebras can be introduced as the algebras generated by matrices whose structure is predictable for all n . However, since all of them are algebras of normal matrices simultaneously diagonalized by a unitary transform, there exist non-derogatory matrices W_i such that $\mathcal{H}_i = \mathcal{A}(W_i) = \ker \nabla_{W_i}$. Nevertheless let us note that the $\mathbb{C}_{\pm 1}^s$ part of \mathcal{H}_i is the subalgebra $\mathcal{A}(Y_{\pm 1})$ generated by the following derogatory matrix

$$Y_\varphi = \Pi_\varphi + \Pi_\varphi^T = \begin{pmatrix} 0 & 1 & & \varphi \\ 1 & & \ddots & \\ & \ddots & & 1 \\ \varphi & & & 1 & 0 \end{pmatrix}. \tag{5}$$

Noting that

$$F_\varphi^* \Pi_\varphi F_\varphi = \Omega_\varphi = \varphi^{1/n} \Omega = \varphi^{1/n} \text{diag}(\omega^i \mid i = 0, \dots, n-1), \quad \omega = e^{-\frac{2\pi i}{n}}$$

and that $\Pi_{\pm 1} + \Pi_{\pm 1}^T = \Pi_{\pm 1} + \Pi_{\pm 1}^* = 2\Re \Pi_{\pm 1}$,⁴ we easily obtain an explicit formula for the eigenvalues of Y_1 and Y_{-1} , namely

⁴ The real part (Hermitian part) of a matrix X is the Hermitian matrix $\Re X = \frac{1}{2}(X + X^*)$

$$\lambda_k(Y_1) = 2 \cos\left(\frac{2k\pi}{n}\right), \quad \lambda_k(Y_{-1}) = 2 \cos\left(\frac{(2k+1)\pi}{n}\right) \tag{6}$$

for $k = 0, \dots, n - 1$.

3. Optimal rank preconditioning

Let us consider an $n \times n$ linear system

$$Ax = y, \quad A \in \mathbf{M}(n), \quad x, y \in \mathbb{C}^n \tag{7}$$

which should be solved by some Krylov subspace iterative methods (CG and GMRES are good examples). When the convergence rate of such methods is low, one may consider a suitable preconditioning matrix P , switch to the preconditioned linear system

$$P^{-1}Ax = P^{-1}y, \tag{8}$$

and apply to this new system the iterative methods. Clearly, except for trivial cases, the spectrum of the matrix defining the system (8) is different from the original one, while the solution x maintains unchanged. The introduction of the preconditioner P could lead to a substantial improvement of the convergence rate, provided that P satisfies some “good” properties, that we summarize into the following two

1. The condition number $\kappa(P^{-1}A)$ is uniformly bounded in n
2. The spectrum of $P^{-1}A$ has a cluster around 1.

When $A > 0$ (the general case needs some further hypothesis cf. [19]), the first property leads to the linear convergence of the methods, independently on the dimension n of the problem. The second one is related with the super-linear convergence of the methods and, fixed an $\varepsilon > 0$, it is the same as requiring that the following splitting for the matrix A in (7)

$$A = P + R + E$$

holds, being E a small perturbation, $\|E\| \leq \varepsilon$, and R a low-rank matrix, namely $\text{rank } R$ is $o(n)$ [1,2]. It is clear that the cluster of $\sigma(P^{-1}A)$ is a proper cluster whenever $\text{rank } R = r_\varepsilon(n)$ is uniformly bounded with respect n , in fact $r_\varepsilon(n)$ is exactly the number of eigenvalues of $P^{-1}A$ which are outside a ball of radius ε around 1. Moreover we can heuristically affirm that “the smaller is rank R , the smaller is the cluster of $\sigma(P^{-1}A)$ and the better is the preconditioner P ”.

In the following we consider low complexity algebras of matrices simultaneously diagonalized by a unitary fast transform U , or rather closed sets of the form

$$\mathcal{A} = \text{sd } U = \{U \text{diag}(\theta_1, \dots, \theta_n)U^* \mid \theta_i \in \mathbb{C}\},$$

where $U \times \text{vector}$ and $U^* \times \text{vector}$ require $O(n \log n)$ operations.

Our considerations about property 2 above suggest the definition of an optimal rank matrix algebra preconditioner for a given linear system (7).

Definition 3.1. Given an invertible $n \times n$ matrix A and a matrix algebra \mathcal{A} , we call optimal rank preconditioner in \mathcal{A} for A any matrix

$$\mathcal{A} = \arg \min\{\text{rank}(A - P - E) \mid P \in \mathcal{A}, \|E\| \leq \varepsilon\}.$$

In [2] it is stressed that in order to construct such preconditioner for A one should solve the following

Problem 1. Given $A \in \mathbf{M}(n)$, given an $\text{sd } U$ algebra \mathcal{A} and given $\varepsilon > 0$, find $\mathcal{A} = P + R$, with $P \in \mathcal{A}$ and $\|A - \mathcal{A}\| \leq \varepsilon$ such that $\text{rank } R$ is as small as possible.

Observe that for any given $M \in \text{sd } U$ there exists a diagonal matrix D such that $M = UDU^*$. Therefore if the norm considered in Problem 1 $\|\cdot\|$ is unitarily invariant

$$\|A - \mathcal{A}\| = \|U^*AU - D - \tilde{R}\|, \quad \text{rank } \tilde{R} = \text{rank } R.$$

In other words we can split problem 1 into the following problems 2 and 3 and calculate the minimum over the algebra of diagonal $n \times n$ matrices \mathcal{D}

Problem 2. Given $A \in \mathbf{M}(n)$ and $\varepsilon > 0$, find $\hat{A} = D + R$, with $D \in \mathcal{D}$ and $\|A - \hat{A}\| \leq \varepsilon$ such that $\text{rank } R$ is minimum, for a chosen unitarily invariant norm $\|\cdot\|$.

Problem 3. Given $A \in \mathbf{M}(n)$ and given an $\text{sd } U$ algebra $\mathcal{A} = \text{sd } U$, compute the image U^*AU .

If $\mathcal{A} = \text{sd } U$ is of low complexity, the computation of U^*AU requires an amount of $O(n^2 \log n)$ operations which is not acceptable. For this reason we have posed Problem 3. However, we shall see that in solving Problem 3 it is not necessary to compute all the entries of the image matrix U^*AU , it is instead enough to have an algorithm that computes any prescribed entry $(U^*AU)_{ij}$ in a fast way, by a number of operations independent of the matrix size.

Problem 2 can be approached as in the circulant-Toeplitz case [2], i.e., by means of the *black-dot algorithm*, an ad-hoc version of the incomplete cross algorithm [20–22]. Given an algebra $\mathcal{A} = \text{sd } U = \mathcal{A}(W)$ (W non-derogatory), in order to apply the black-dot algorithm we need to know some elements $(U^*AU)_{ij}$, where the pair (i, j) belongs to a certain set of indices Ω . Typical choices for Ω give rise to a method whose complexity can be estimated with $O(nr_\varepsilon(n)^2)$ [2]. When the coefficients matrix is positive definite we expect that the preconditioner \mathcal{A} computed by such algorithm is positive definite as well. Actually such property is not always ensured, in fact it may happen that some entry of the computed diagonal matrix D is negative. However, in typical cases – see [2] and the proposition here below – it is possible to ensure the positive definiteness of \mathcal{A} by applying a low-rank correction to the computed D , i.e., modifying a small number of its diagonal entries.

Proposition 3.2. Let $A = P + R + E$ with $\text{rank } R = r < n$ and $\|E\| < \varepsilon$. If $A > \varepsilon I$ and $R \geq 0$ then at least $n - r$ eigenvalues of P are positive.

Proof. Weyl’s inequalities for the Hermitian eigenvalues problem $Z = X + Y$ give us the following

$$\begin{aligned} \lambda_{i+j-1}(Z) &\leq \lambda_i(X) + \lambda_j(Y), & 1 \leq i + j - 1 \leq n, \\ \lambda_i(X) + \lambda_n(Y) &\leq \lambda_i(Z) \leq \lambda_i(X) + \lambda_1(Y), & 1 \leq i \leq n, \end{aligned}$$

where the eigenvalues of a Hermitian matrix are supposed in decreasing order (i.e., $\lambda_i \geq \lambda_{i+1}$). Since $r < n$, we have $\lambda_{r+k}(R) = 0$ for $k = 1, \dots, n - r$. Thus Weyl’s inequalities applied to $A = P + R + E$ give us

$$\lambda_{i+r}(A) = \max_{k=r, \dots, n-1} \lambda_{i+k}(A) \leq \lambda_i(P + E) \leq \lambda_i(A), \quad 1 \leq i \leq n - k.$$

Therefore we know that all the first $n - r$ eigenvalues of $P + E$ are greater or equal to ε , thus $n - r$ eigenvalues of P are positive, since $A > \varepsilon I$. \square

In the next section we discuss Problem 3 and propose a method for computing any element $(U^*AU)_{ij}$ at a very low cost, after a preprocessing phase of complexity $O(n \log n)$. Such method works well if the displacement rank of A , the matrix defining our system (7), with respect to the chosen algebra \mathcal{A} , is sufficiently small. More precisely let us introduce the following

Definition 3.3. Let $\mathcal{A} = \mathcal{A}(W)$ be a matrix algebra. We say that a matrix $A \in \mathbf{M}(n)$ almost-belongs to \mathcal{A} , in symbols $A \tilde{\in} \mathcal{A}$, if $\text{rank}([A, W])$ is uniformly bounded in n .

Note that by the fundamental theorem of homomorphism we have a canonical isomorphism $\dot{\nabla}_W$ between $\mathbf{M}(n)/\mathcal{A}$ and $\text{range } \nabla_W$ which implies that the pre-image of $A \in \text{range } \nabla_W$ is given by the closed set $\{\dot{\nabla}_W^{-1}(A) + \mathcal{A}\}_{A \in \mathcal{A}}$.

The method proposed in the next section for computing cheaply the elements $(U^*AU)_{ij}$, works if $A \tilde{\in} \mathcal{A}$ and $[A, W]$ is explicitly known. We underline since now that our method works well for all the low complexity $\text{sd } U$ algebras \mathcal{A} previously presented and for any Toeplitz or Hankel matrix A .

4. The computation of U^*AU

Let $\mathcal{A} \subset \mathbf{M}(n)$ be an algebra of normal matrices $\mathcal{A} = \text{sd } U, U \in \mathbf{U}(n)$. Let us consider an element $W \in \mathcal{A}$ which is non-derogatory. Clearly $\mathcal{A} = \mathcal{A}(W) = \ker \nabla_W$ and $U^*WU \in \mathcal{D}$. Set $D = U^*WU$. Given a matrix $A \in \mathbf{M}(n)$, we have

$$(U^*AU)D - D(U^*AU) = U^*[A, W]U$$

thus the off-diagonal elements of U^*AU satisfy the identity

$$(U^*AU)_{ij} = \frac{(U^*[A, W]U)_{ij}}{\lambda_j(W) - \lambda_i(W)}, \quad i \neq j. \tag{9}$$

Such equalities let us state the theorem below, whose detailed proof can be derived also by the observations which follow.

Theorem 4.1. *Let $\mathcal{A} = \mathcal{A}(W)$ be a low complexity $\text{sd } U$ algebra, $A \tilde{\in} \mathcal{A}$ and $\text{rank}[A, W] = \rho$ (thus ρ is uniformly bounded in n and $\lambda_i(W) = \lambda_k(W) \Leftrightarrow i = k$). Assume that $\sigma(W)$ is explicitly known. Then, after a preprocessing phase of complexity $O(n \log n)$ required for the computation of $U^*[A, W]U$ (see below), each off-diagonal element $(U^*AU)_{ij}$ can be calculated with $\rho + 1$ multiplications.*

The theorem above gives us a tool for approaching Problem 3, or rather for computing the image of a matrix A under a unitary fast transformation. In fact, we are requested to compute 2ρ transformations $\hat{x}_k = U^*x_k, \hat{y}_k = U^*y_k$, for $k = 1, \dots, \rho$, in a preprocessing phase, where x_k and y_k are the vectors defining a dyadic decomposition of $[A, W]$. Then each off-diagonal entry of U^*AU is known up to $\rho + 1$ multiplicative operations $(U^*AU)_{ij} = (\lambda_j(W) - \lambda_i(W))^{-1} \sum_{k=1}^{\rho} (\hat{x}_k)_i \overline{(\hat{y}_k)_j}$. Finally observe that in order to apply the black-dot algorithm we actually do not need the diagonal entries of U^*AU [2], thus Theorem 4.1 is not restrictive in our situation.

To summarize, in order to compute the optimal rank preconditioner A for A into the algebra $\mathcal{A} = \text{sd } U = \ker \nabla_W$, we can propose the following DR-scheme, where the black-dot algorithm is like a black box which sometime requires an entry $(U^*AU)_{ij}, i \neq j$.

DR-scheme

Assume $\text{sd } U = \ker \nabla_W$ and $[A, W] = \sum_{k=1}^{\rho} x_k y_k^*$.

1. (Preprocessing) Compute the 2ρ fast transforms $\hat{x}_k = U^*x_k, \hat{y}_k = U^*y_k, k = 1, \dots, \rho$.
2. Start the black-dot iterations
 - 2.1. **if $(U^*AU)_{ij}$ is required**, compute it via the identity

$$(U^*AU)_{ij} = \frac{\sum_{k=1}^{\rho} (\hat{x}_k)_i \overline{(\hat{y}_k)_j}}{\lambda_j(W) - \lambda_i(W)}$$

where \bar{z} denotes the complex conjugate of z .

3. Proceed with the iterations until convergence, passing through 2.1 if necessary
-

Note that the preprocessing phase requires $O(\rho n \log n)$ operations, whereas only $\rho + 1$ arithmetic multiplications are needed each time step 2.1 must be performed.

Despite the somewhat general formulation of the DR-scheme, in the following we consider some specific cases, in which A is a Toeplitz or Hankel matrix, and discuss how to compute U^*AU explicitly,

for the low complexity algebras presented in Section 2, underlining that for these choices of A , we actually have $A \tilde{\in} \mathcal{A}$.

4.1. φ -circulant algebras

Recall that the generic φ -circulant algebra is defined as

$$\mathcal{C}_\varphi = \mathcal{A}(\Pi_\varphi) = \text{sd } F_\varphi = \ker \nabla_{\Pi_\varphi},$$

where $F_\varphi = \Delta_\varphi F = \text{diag}(1, \varphi^{\frac{1}{n}}, \dots, \varphi^{\frac{n-1}{n}})F$, $|\varphi| = 1$, and F is the Fourier matrix (3). We will make use of the following matrix J , also called *reverse identity*

$$J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}.$$

Given $p \in \mathbb{Z}$, consider the equivalence class $\{p \pmod n\}$ and let $[p]_n$ denote its unique representative in $\{0, 1, \dots, n - 1\}$.

Proposition 4.2. *Let $T_n = (t_{i-j})_{ij}$ be a Toeplitz matrix and Π_φ the matrix (1) generating \mathcal{C}_φ . Then $T_n \tilde{\in} \mathcal{C}_\varphi$ and*

$$[T_n, \Pi_\varphi] = x_\varphi e_1^\top + e_n y_\varphi^\top \tag{10}$$

being $x_\varphi = (\varphi t_{1-n} - t_1, \dots, \varphi t_{-1} - t_{n-1}, 0)^\top$ and $y_\varphi = -Jx_\varphi$.

Proof. Due to the definition of Π_φ we have the following equality

$$[T_n, \Pi_\varphi]_{ij} = \sum_{k=0}^{n-1} (t_{i-k}(\Pi_\varphi)_{kj} - (\Pi_\varphi)_{ik}t_{k-j}) = \varphi t_{i-[j+n-1]_n} - t_{[i+1]_n-j}.$$

Therefore (10) holds for $i \neq n - 1$ and $j \neq 0$. In fact for such choices of indexes we have $i - [j + n - 1]_n = i - (j - 1) = [i + 1]_n - j$ thus $[T_n, \Pi_\varphi]_{ij} = 0$ and $\text{rank}([T_n, \Pi_\varphi]) = 2$. It is not difficult to observe that (10) also holds for $i = n - 1$ and $j = 0$. \square

The above Proposition immediately implies that, when A has Toeplitz structure, the DR-scheme can be applied to the case $U = F_\varphi$, the unitary matrix diagonalizing \mathcal{C}_φ . The Hankel case needs some further observation since the rank of $[H_n, \Pi_\varphi]$ is not bounded in general and a direct use of the DR-scheme would be prohibitive. Given a Hankel matrix H_n call $T(H_n)$ the Toeplitz matrix JH_n . Here below we observe that when φ is 1 or -1 , the computation of $F_\varphi^* H_n F_\varphi$ can be brought back to the Toeplitz case $F_\varphi^* T(H_n) F_\varphi$, for which, instead, the DR-scheme works well.

Observe that

$$\Delta_\varphi^* J \Delta_\varphi = \text{diag} \left(\varphi^{\frac{n-2k-1}{n}} \mid k = 0, \dots, n - 1 \right) J,$$

therefore when $\varphi \in \{-1, 1\}$ we get the equality $F_\varphi^* J F_\varphi = \varphi^{\frac{n-1}{n}} F^* J F$. Moreover, since $F = J \Pi F^*$, $F^2 = J \Pi$ and $F^* J F = \Omega$, we also have

$$F^* J F = (F^* \Pi F)(F^*)^2 = \Omega J \Pi,$$

where $\Omega = \text{diag}(1, \omega, \dots, \omega^{n-1})$, $\omega = e^{-2\pi i/n}$. Now use the definition of $T(H_n)$ to write

$$[T_n + H_n, X] = \begin{pmatrix} * & \cdots & * \\ \vdots & O_{n-2} & \vdots \\ * & \cdots & * \end{pmatrix}$$

where $*$ are in general non-null entries and O_{n-2} is the null matrix of order $n - 2$. \square

Proposition 4.4. *Let $a, b, c, d \in \mathbb{C}^n$ with $a_1 = b_1$ and $c_n = d_1$. Then*

$$[T_n(a, b), X] = \Theta(a, b) - J\Theta(a, b)^T J, \tag{13}$$

$$[H_n(c, d), X] = \Theta(d, Jc)J - J\Theta(d, Jc)^T \tag{14}$$

where $\Theta(a, b)$ is the rank two matrix

$$\Theta(a, b) = e_1(\Pi b)^T - (\Pi a)e_1^T. \tag{15}$$

Proof. Displacement formula (14) for $H_n(c, d)$ clearly follows from the previous one (13), since $H_n(c, d) = T_n(d, Jc)J$ and

$$[H_n(c, d), X] = [T_n(d, Jc), X]J = \Theta(d, Jc)J - J\Theta(d, Jc)^T.$$

By Proposition 4.3 only the border rows and columns of $[T_n, X]$ are non-null, for a Toeplitz matrix T_n . Thus we just need to check (13) for such four vectors. For instance, noting that $X = \Pi_0 + \Pi_0^T$, we have

$$[T_n(a, b), X]e_1 = T_n(a, b)e_2 - Xa = \Pi_0^T a + b_2e_1 - Xa = b_2e_1 - \Pi_0 a$$

and analogously

$$e_1^T [T_n(a, b), X] = b^T X - e_2^T T_n(a, b) = (\Pi_0 b)^T - a_2 e_1^T.$$

Therefore (13) holds for the first row and column, if we define $\Theta(a, b)$ with Π_0 in place of Π . It is not difficult to observe that the same can be said also for the last row and column. Moreover, thanks to arithmetic cancellations for the corner positions $(1, n)$ and $(n, 1)$, identity (13) holds for our definition of $\Theta(a, b)$, given in terms of Π . \square

The above proposition gives us an explicit formula for the displacement rank of a Toeplitz or Hankel matrix into the tau-algebra \mathcal{T} . Then, if $S = S_{(0, 1, 1, 0)}$ is the sine transform which diagonalizes \mathcal{T} , we can apply the DR-scheme to the case where $U = S$ and $A = T_n + H_n$ is the sum of any two Toeplitz and Hankel matrices.

Anyway, by Proposition 4.3, the displacement rank of a Toeplitz or Hankel matrix into any trigonometric algebra \mathcal{T}_μ does not exceed 8. Let us first derive an explicit formula for the dyadic decomposition of $[T_n, X_\mu]$. Since

$$M_\mu = e_1(\mu_1 e_1 + (\mu_2 - 1)e_2)^T + e_n((\mu_3 - 1)e_{n-1} + \mu_4 e_n)^T,$$

we have

$$\begin{aligned} [T_n(a, b), M_\mu] &= c_1(a, b)(\mu_1 e_1 + (\mu_2 - 1)e_2)^T + c_n(a, b)((\mu_3 - 1)e_{n-1} + \mu_4 e_n)^T \\ &\quad - e_1(\mu_1 c_1(a, b) + (\mu_2 - 1)c_2(a, b))^T \\ &\quad - e_n((\mu_3 - 1)c_{n-1}(a, b) + \mu_4 c_n(a, b))^T \end{aligned}$$

where $c_k(a, b)$ is the k th column of $T_n(a, b)$. The required formula is obtained by summing the latter one and (13). A similar computation provides a formula for $[H_n, X_\mu]$ and thus the DR-scheme can be applied when $U = S_\mu$. C_μ is any trigonometric transform and $A = T_n + H_n$.

4.3. Hartley-type algebras

Let $\mathcal{H}_k = \text{sd } H_k$ denote a generic Hartley-type algebra, we can characterize \mathcal{H}_k as the set [23]

$$\mathcal{H}_k = \{A \mid [A, Y_\varphi] = [A, M_k] = 0\} \tag{16}$$

where $\varphi \in \{1, -1\}$ and $M_k \in \mathbf{M}(n)$ depend on the Hartley-type algebra. We are considering. Despite it is obviously possible to define \mathcal{H}_k as the set $\ker \nabla_W$ for some non derogatory matrix W , it is not always easy to find a “simple” W with such property. Therefore here we make use of (16) and derive an easy variant of the DR-scheme.

Theorem 4.5. *Let X and Y be two distinct normal matrices which commute. Let $\lambda_i(X), \lambda_i(Y)$ be the eigenvalues of X, Y corresponding to the same common eigenvector. Consider the algebra $\mathcal{A} = \{A \mid [A, X] = [A, Y] = 0\}$. Then \mathcal{A} is n dimensional if and only if $\lambda_i(X) = \lambda_j(X)$ implies $\lambda_i(Y) \neq \lambda_j(Y)$, for any pair of distinct indices (i, j) .*

Proof. Denote with $\lambda_1, \dots, \lambda_n$ and μ_1, \dots, μ_n the eigenvalues of X and Y , respectively. Since $[X, Y] = 0$ we have $X = UD_\lambda U^*$ and $Y = UD_\mu U^*$, where D_λ and D_μ are the diagonal matrices such that $(D_\lambda)_{ii} = \lambda_i$ and $(D_\mu)_{ii} = \mu_i$.

First let us prove the implication (\Leftarrow). Take $A \in \mathcal{A}$, then $U^*AU D_\lambda = D_\lambda U^*AU$ and $U^*AU D_\mu = D_\mu U^*AU$. Let $(U^*AU)_{ij} = \hat{a}_{ij}$. By writing the first relation entrywise we get $\hat{a}_{ij}(\lambda_j - \lambda_i) = 0$, therefore \hat{a}_{ij} must be zero if $\lambda_i \neq \lambda_j$. When $\lambda_i = \lambda_j$ we use the second relation obtaining $\hat{a}_{ij}(\mu_j - \mu_i) = 0$, which gives us $\hat{a}_{ij} = 0$ due to our hypothesis. As a consequence we have $\hat{a}_{ij} = 0 \ \forall i \neq j$, hence $U^*AU \in \mathcal{D}$ and $\dim \mathcal{A} = \dim \text{sd } U = n$.

Viceversa, since any $A \in \text{sd } U$ is an element of \mathcal{A} and $\dim \mathcal{A} = n$, we have $\mathcal{A} = \text{sd } U$. Now proceed by absurd and assume the claim to be false. Without loss of generality, suppose $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$, and consider the matrix

$$A = U \begin{pmatrix} d_1 & b & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} U^* = UBU^*$$

where $b \neq 0$. We have $U^*[A, X]U = BD_\lambda - D_\lambda B = 0$ and $U^*[A, Y]U = BD_\mu - D_\mu B = 0$. Therefore A commutes with both X and Y , namely $A \in \mathcal{A}$. This is impossible since $A \notin \text{sd } U$. \square

By the above theorem it is clear how to adapt the DR-scheme for Hartley-type algebras

Assume $[A, Y_\varphi] = \sum_{s=1}^\rho x_s y_s^*$ and $[A, M_k] = \sum_{s=1}^\tau w_s z_s^*$.

1. (Preprocessing) Compute the $2(\rho + \tau)$ fast transforms $\hat{x}_s = H_k^* x_s, \hat{y}_s = H_k^* y_s, \hat{w}_t = H_k^* w_t, \hat{z}_t = H_k^* z_t, s = 1, \dots, \rho, t = 1, \dots, \tau$.
2. Start the black-dot iterations
 - 2.1. **if $(H_k^* A H_k)_{ij}$ is required**, compute it via the identity

$$(H_k^* A H_k)_{ij} = \frac{\sum_{s=1}^\rho (\hat{x}_s)_i \overline{(\hat{y}_s)_j}}{\lambda_j(Y_\varphi) - \lambda_i(Y_\varphi)}$$

if $\lambda_i(Y_\varphi) \neq \lambda_j(Y_\varphi)$, or

$$(H_k^* A H_k)_{ij} = \frac{\sum_{s=1}^\tau (\hat{w}_s)_i \overline{(\hat{z}_s)_j}}{\lambda_j(M_k) - \lambda_i(M_k)}$$

otherwise

3. Proceed with the iterations until convergence, passing through 2.1 if necessary

Note that in this modified version the DR-scheme requires one more assumption, since the matrix A we are considering should have small displacement rank with respect to two different matrices Y_φ and M_k , rather than only one. Of course there are many matrices which satisfy such assumption, however it is not obvious if among them there are also Toeplitz or Hankel matrices. The rest of the section is devoted to observe that this is true in particular cases.

First of all, note that the matrices Y_φ are derogatory and only $\dim \mathcal{C}_\varphi^s$ of their eigenvalues are distinct, in fact Y_φ is the generator of the family of algebras

$$\mathcal{C}_\varphi^s = \{C \in \mathcal{C}_\varphi \mid C = C^T\}.$$

Note that in (6) we derived an explicit formula for the spectrum of $Y_{\pm 1}$.

Moreover it is clear from Propositions 4.2 that for any Toeplitz matrix A we have $\text{rank}[A, Y_\varphi] \leq 4$. The same conclusion holds for $[A, Y_\varphi]$ with A Hankel. Namely

Proposition 4.6. *Assume $|\varphi| = 1$. Let $a, b \in \mathbb{C}^n$ such that $a_1 = b_1$. Then*

$$[T_n(a, b), Y_\varphi] = \mathcal{K}_\varphi(a, b) - \mathcal{K}_\varphi(b, a)^T + J(\mathcal{K}_\varphi(b, a) - \mathcal{K}_\varphi(a, b)^T)J,$$

where $\mathcal{K}_\varphi(x, y)$ is the rank one matrix in (12).

Let $c, d \in \mathbb{C}^n$ with $c_n = d_1$. Then

$$[H_n(c, d), Y_\varphi] = e_1 d^T (J\Pi - \varphi I) - (J\Pi - \varphi I) d e_1^T + e_n c^T (\Pi J - \varphi I) - (\Pi J - \varphi I) c e_n^T.$$

Proof. The formula for $T_n(a, b)$ immediately follows from (11), noting that $T_n(a, b)^T = T_n(b, a)$ and that $[A, B^T] = -[A^T, B]^T, \forall A, B \in \mathbf{M}(n)$. The second one, for $H_n(c, d)$, follows from (14) and the decomposition $Y_\varphi = X + R_\varphi$, being $R_\varphi = \varphi(e_n e_1^T + e_1 e_n^T)$. In fact $H_n(c, d)e_n = d, H_n(c, d)e_1 = c, e_1^T H_n(c, d) = c^T$ and $e_n^T H_n(c, d) = d^T$. \square

Clearly we can also represent $[T_n(a, b), Y_\varphi]$ by means of $\Theta(a, b)$, in fact

$$[T_n(a, b), Y_\varphi] = \Theta(a, b) - J\Theta(a, b)^T J + \varphi J(b e_1^T - e_1 b^T) + \varphi(a e_1^T - e_1 a^T)J$$

which easily comes from (13).

Now, concerning M_k , let us fix four choices among the eight possible indices k , precisely $k = 1, 2, 5, 6$, and consider the corresponding Hartley-type algebras $\mathcal{H} = \mathcal{H}_1, \mathcal{K} = \mathcal{H}_2, \eta = \mathcal{H}_5$ and $\mu = \mathcal{H}_6$. For such choices we can characterize M_k somehow explicitly, in fact in [4] it is shown that

$$M_k = J + \begin{pmatrix} 0 & 0^T \\ 0 & \tau(z_k) \end{pmatrix}, \tag{17}$$

where $\tau(z_k) = \tau_{(0,1,1,0)}(z_k) \in \mathbf{M}(n - 1)$ and the vectors $z_k \in \mathbb{R}^{n-1}$ are, respectively,

$$z_1 = \frac{1}{2}(e_2 - e_{n-1}), \quad z_2 = -\frac{1}{2}(e_2 + e_{n-1}), \quad z_5 = z_6 = 0.$$

As a consequence we get $\tau(z_5) = \tau(z_6) = O_{n-1}$, and

$$\tau(z_1) = \frac{1}{2}X(I - J), \quad \tau(z_2) = -\frac{1}{2}X(I + J)$$

where $X = X_{(0,1,1,0)} \in \mathbf{M}(n - 1)$ is defined in (4). Also note that for the algebras $\mathcal{H}_5 = \eta$ and $\mathcal{H}_6 = \mu$ a more elegant characterization does hold. We state it by means of the following

Proposition 4.7. *Let η and μ be the Hartley-type algebras defined in Table 4. Then*

$$\eta = \ker \nabla_{Y_1+J} \quad \mu = \ker \nabla_{Y_{-1}+J}.$$

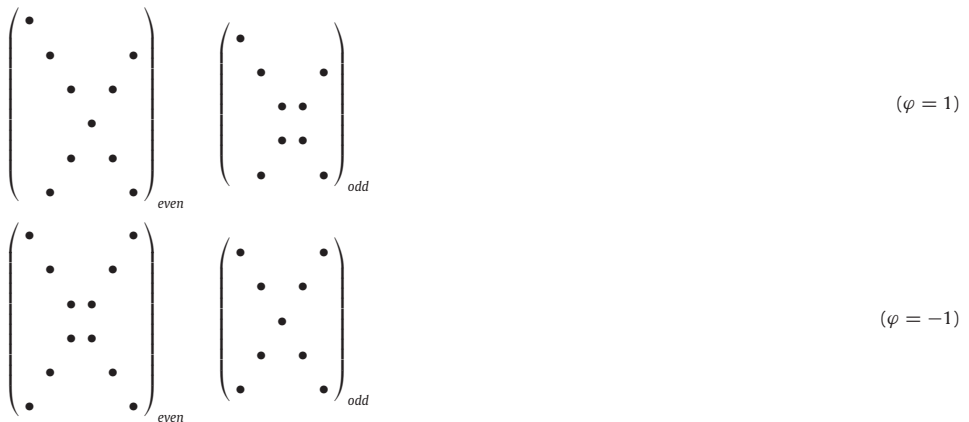
Proof. Clearly $\eta \subset \ker \nabla_{Y_1+J}, \mu \subset \ker \nabla_{Y_{-1}+J}$ and the equalities hold if and only if $Y_1 + J$ and $Y_{-1} + J$ are non-derogatory. If H_5 is the unitary matrix diagonalizing η , it is not difficult to observe that

$$H_5^* J H_5 = \begin{pmatrix} I_m & \\ & -I_{n-m} \end{pmatrix}$$

where $m = n/2$ if n is even and $m = (n + 1)/2$ otherwise. This remark and (6) imply that the eigenvalues $\lambda_i(Y_1 + J) = \lambda_i(Y_1) + \lambda_i(J)$ are all distinct; thus $Y_1 + J$ is non derogatory, and the thesis follows for η . In the same way one proves the thesis also for μ . \square

It is now clear that we can apply the modified DR-scheme to a quite general class of Toeplitz and Hankel matrices. In fact for any symmetric Toeplitz matrix T_n and any persymmetric Hankel matrix H_n we have $[T_n, J] = [H_n, J] = 0$, therefore $[T_n, XJ] = [T_n, X]J$ and $[H_n, XJ] = [H_n, X]J$. Using (13) and (14), it is now straightforward to derive the formulas for the commutator of T_n and H_n with M_k , for $k = 1, 2, 5, 6$, taking into account that the matrix X which appears into (17) has order $n - 1$. This eventually allows us to apply the modified DR-scheme to the case $A = \text{Toeplitz symmetric} + \text{Hankel persymmetric}$.

We conclude this section by noting that using only Y_φ and formula (9) we can compute at least $n(n - 2)$ entries of $H_i^* A H_i$, for any $i = 1, \dots, 8$. The elements \bullet of $H_i^* A H_i$ that we can not compute this way, are in the positions shown in the figure below (we represent them for $n = 5, 6$).



5. Algebra-plus-low-rank approximation of a matrix

As we underlined the previous section, an optimal rank preconditioner P should realize the splitting

$$A = P + R + E \tag{18}$$

where $\|E\| \leq \varepsilon$ and $\text{rank } R$ is minimum. When A is Toeplitz, it can be shown that optimal or Strang-type preconditioners P , chosen inside suitable $sd U$ algebras, realize an analogous decomposition where in general $\text{rank } R = O(\varepsilon^{-p})O(n)$, for a $p > 0$ [1,4,8]. In this section we show that for particular classes of matrices A and algebras \mathscr{A} there exists $P \in \mathscr{A}$ that realizes the splitting (18) with $\|E\| \leq \varepsilon$ and $\text{rank } R = o(n)$.

If \mathbb{T} is the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, let us denote with $T_n, H_n : L^\infty(\mathbb{T}, \mathbb{C}) \rightarrow \mathbf{M}(n)$ the Toeplitz and Hankel operators, respectively, which map $f \in L^\infty(\mathbb{T}, \mathbb{C})$ into the $n \times n$ Toeplitz or Hankel matrices $T_n(f), H_n(f)$. Finally call $\mathcal{L}(\mathbb{T})$ the subset of $L^\infty(\mathbb{T}, \mathbb{C})$ of all piecewise holomorphic functions with logarithmic singularities, i.e., functions given by an holomorphic function plus a function with logarithmic singularities. A generic $f \in \mathcal{L}(\mathbb{T})$ has the form

$$f(z) = g(z) + \sum_{k=0}^p \sum_{h=0}^q \alpha_{kh} \cdot (z - z_h)^k \log(z - z_h), \quad z \in \mathbb{T},$$

with g holomorphic over a set containing \mathbb{T} and $z_h \in \mathbb{T}, h = 0, \dots, q$.

In [3] it is shown that for an $f \in \mathcal{L}(\mathbb{T})$ the Toeplitz matrix $T_n(f)$ admits the decomposition

$$T_n(f) = P + R + E, \quad \text{rank}(R) = O\left(\log \frac{1}{\varepsilon} \left(\log \frac{1}{\varepsilon} + \log n\right)\right),$$

where P is a circulant matrix and E is a small perturbation as usual. The space $\mathcal{L}(\mathbb{T})$ is a special class of symbol functions that, however, covers all examples considered in literature on superlinear preconditioners [2].

In this section we will show that a splitting analogous to the one in [3] also holds for $H_n(f)$ and P chosen inside a generic φ -circulant algebra, $\varphi \in \mathbb{T}$. We will explicitly describe such matrix P in several cases, and we will discuss also the case of Hartley-type algebras.

Let us consider a $\lambda \in \mathbb{C}$ and define the vector

$$p(\lambda) = \left(1 \ \lambda \ \lambda^2 \ \dots \ \lambda^{n-1}\right)^\top.$$

It is not difficult to observe that the Toeplitz matrix

$$T_n(p(\lambda), \lambda^{1-n} J p(\lambda)) = p(\lambda) p(\lambda^{-1})^\top$$

is a rank one matrix. Moreover, by requiring a Toeplitz matrix to be of rank one, we observe in fact that

$$\text{rank } T_n(a, b) = 1 \iff \exists \lambda \mid a = p(\lambda) \text{ and } b = \lambda^{1-n} J p(\lambda).$$

Given a real number λ set

$$Z_n(\lambda) = T_n(p(\lambda), e_1) = \begin{pmatrix} 1 & & & & \\ \lambda & 1 & & & \\ \vdots & \ddots & \ddots & & \\ \lambda^{n-1} & \dots & \lambda & 1 & \end{pmatrix}.$$

This is the Toeplitz matrix generated by the symbol

$$\zeta_\lambda(\theta) = \frac{1}{1 - \lambda e^{i\theta}}, \quad \lambda, \theta \in \mathbb{R}.$$

By noting that $p(\lambda)$ satisfies

$$\Pi_\varphi p(\lambda) = \lambda p(\lambda) + (\varphi - \lambda^n) e_n$$

we obtain the following identities

$$\begin{aligned} \mathcal{H}_\varphi(p(\lambda), e_1) &= \varphi e_n e_1^\top - \Pi_\varphi p(\lambda) e_1^\top = (\lambda^n e_n - \lambda p(\lambda)) e_1^\top, \\ \mathcal{H}_\varphi(p(\lambda), \lambda^{1-n} J p(\lambda)) &= ((\varphi \lambda^{1-n} - \lambda) p(\lambda) - (\varphi - \lambda^n) e_n) e_1^\top \end{aligned}$$

which lead to

Lemma 5.1. For any $\lambda \in \mathbb{R}$ there exists $P_\varphi \in \mathcal{C}_\varphi$ such that the triangular Toeplitz matrix $Z_n(\lambda)$ splits into $Z_n(\lambda) = P_\varphi + R$, with $\text{rank } R = 1$.

Proof. Using (11) and the formulas above we have

$$[T_n(p(\lambda), e_1), \Pi_\varphi] = \left[\frac{\lambda^n}{\lambda^n - \varphi} T_n(p(\lambda), \lambda^{1-n} J p(\lambda)), \Pi_\varphi \right].$$

Notice, in fact, that, after the subtraction between \mathcal{H}_φ and $J\mathcal{H}_\varphi^T J$, the term $\varphi e_n e_1^T$ is canceled. \square

Observe that we can explicitly write the φ -circulant matrix P_φ in Lemma 5.1, in fact it is given by the difference

$$P_\varphi = T_n(p(\lambda), e_1) - \frac{\lambda^n}{\lambda^n - \varphi} T_n(p(\lambda), \lambda^{1-n} J p(\lambda)). \tag{19}$$

Hence, using the operator $C_\varphi : \mathbb{C}^n \rightarrow \mathbb{C}_\varphi$ defined in Section 2.1, we have the equality

$$Z_n(\lambda) = C_\varphi(x_\varphi(\lambda)) + R, \quad x_\varphi(\lambda) = \frac{1}{\varphi - \lambda^n} J \Pi_\varphi p(\lambda).$$

As a consequence, the following characterization holds

Proposition 5.2. *The eigenvalues of $C_\varphi(x_\varphi(\lambda))$ are*

$$\zeta_{\frac{\lambda}{\varphi^{1/n}}} \left(\frac{2\pi k}{n} \right), \quad k = 0, \dots, n - 1.$$

Proof. Observe that for a generic v the following formulas hold

$$\Pi_\varphi p(v) = v p(v) + (\varphi - v^n) e_n, \quad J p(v) = v^{n-1} p(v^{-1}).$$

Then note that

$$\begin{aligned} \sqrt{n} F_\varphi^T x_\varphi(\lambda) &= \frac{\sqrt{n}}{\varphi - \lambda^n} F \Delta_\varphi J \Pi_\varphi p(\lambda) \\ &= \frac{\sqrt{n}}{\varphi - \lambda^n} F \Delta_\varphi (\lambda J p(\lambda) + (\varphi - \lambda^n) e_1) = \frac{\lambda^n \sqrt{n}}{\varphi - \lambda^n} F \Delta_\varphi p(\lambda^{-1}) + e \\ &= \frac{\lambda^n}{\varphi - \lambda^n} \begin{pmatrix} \vdots \\ \frac{1 - \varphi \lambda^{-n}}{1 - \lambda^{-1} \varphi^{1/n} \omega^k} \\ \vdots \end{pmatrix} + e = \begin{pmatrix} \vdots \\ \frac{1}{1 - \lambda \varphi^{-1/n} \omega^{-k}} \\ \vdots \end{pmatrix}. \end{aligned}$$

The thesis follows by recalling that, given $y \in \mathbb{C}^n$, the eigenvalues of $C_\varphi(y)$ are the entries of the vector $\sqrt{n} F_\varphi^T y$. \square

Consider now the matrix

$$K_n(\lambda) = Z_n(\lambda) + Z_n(\lambda)^T - I,$$

i.e., the well known Kac–Murdock–Szego (KMS) matrix generated by the symbol

$$\kappa_\lambda(\theta) = 2\Re \zeta_\lambda(\theta) - 1 = \sum_{n \in \mathbb{Z}} \lambda^{|n|} e^{in\theta} = \frac{1 - \lambda^2}{1 - 2\lambda \cos \theta + \lambda^2}. \tag{20}$$

Observe that when λ is real, we have the identity, $K_n(\lambda) = 2\Re Z_n(\lambda) - I$. It is also easy to show that $\kappa_\lambda(\theta) > 0 \iff |\lambda| < 1$ or, equivalently, $K_n(\lambda) > 0 \iff |\lambda| < 1$. Therefore

Proposition 5.3. For a given $\lambda \in \mathbb{R}$, let $K_n(\lambda)$ be the corresponding KMS matrix. Then

1. For any $\varphi \in \mathbb{T}$ there exist $Q_\varphi \in \mathcal{C}_\varphi$ and R of rank 2 such that $K_n(\lambda) = Q_\varphi + R$
2. Let $\xi_\varphi(\lambda) = \left(\frac{1}{\varphi - \lambda^n} J \Pi_\varphi + \frac{\bar{\varphi}}{\bar{\varphi} - \lambda^n}\right) p(\lambda) - 1$. Then $Q_\varphi = C_\varphi(\xi_\varphi(\lambda))$
3. The eigenvalues of Q_φ are $\kappa_{\frac{\lambda}{\varphi^{1/n}}} \left(\frac{2\pi k}{n}\right)$, $k = 0, \dots, n - 1$

Notice that we also have $K_n(\lambda) > 0 \iff Q_\varphi > 0$.

Proof. By Proposition 5.1 we have the equality $K_n(\lambda) = (2\Re C_\varphi(x_\varphi(\lambda)) - I) + R$ where R is a rank two Hermitian matrix. Note that the matrix $\Re C_\varphi(x_\varphi(\lambda))$ belongs to \mathcal{C}_φ , and this proves (1). Moreover its eigenvalues are the real part of the eigenvalues of $C_\varphi(x_\varphi(\lambda))$. Therefore from (20) we derive (3). Concerning (2) just observe that, for any $y \in \mathbb{C}^n$, $C_\varphi(y)^* = \bar{\varphi} C_\varphi(J \Pi_\varphi \bar{y})$, and use such remark to compute the first row of $2\Re C_\varphi(x_\varphi(\lambda)) - I$. \square

It is important to note that the φ -circulant matrix Q_φ in the previous proposition is indeed the optimal rank φ -circulant preconditioner for a KMS matrix. This fact can be easily proved by a direct calculation. Just try to impose that the difference between $K_n(\lambda)$ and a rank one matrix is φ -circulant to reach an absurd. Therefore we have an explicit formula for the optimal rank preconditioner of $K_n(\lambda)$. Note that it outperforms, from the clustering point of view, any other known preconditioner for a KMS matrix [8,24]. In fact the preconditioned matrix $Q_\varphi^{-1} K_n(\lambda)$ has only three distinct eigenvalues.

Theorem 5.4. Let p, q be two complex valued mutually prime polynomials defined on \mathbb{T} , such that $0 \notin q(\mathbb{T})$, $\deg p < \deg q$, and q has all distinct roots. Then, for all $\varphi \in \mathbb{T}$, the lower triangular Toeplitz matrix generated by p/q satisfies the identity

$$T_n(p/q) = P_\varphi + R, \quad P_\varphi \in \mathcal{C}_\varphi, \quad \text{rank } R \leq \deg p + 1.$$

Moreover, if the roots of q are real, then also the Hermitian Toeplitz matrix generated by $\Re(p/q)$ splits into

$$T_n(\Re(p/q)) = Q_\varphi + \tilde{R}, \quad Q_\varphi \in \mathcal{C}_\varphi, \quad \text{rank } \tilde{R} \leq 2 \text{rank } R.$$

Proof. By the fundamental theorem of algebra, any polynomial $f : \mathbb{T} \rightarrow \mathbb{C}$ admits the splitting $f(z) = \prod_{i=1}^{\deg f} (z - z_i)$. As a consequence, the rational function $\frac{p(z)}{q(z)}$ admits the simple fractions decomposition

$$\frac{p(z)}{q(z)} = \sum_{i=1}^{\deg q} \frac{\rho_i}{z - z_i}$$

where ρ_i is the residual given by $\rho_i = (z - z_i) \frac{p(z)}{q(z)} \Big|_{z=z_i}$ and z_i are the roots of q . Therefore, by the linearity of the Toeplitz operator $T_n : L^2(\mathbb{T}) \rightarrow \mathbf{M}(n)$ we have the identity

$$T_n(p/q) = - \sum_{k=1}^{\deg q} \frac{\rho_k}{z_k} Z_n(1/z_k).$$

Now the existence of P_φ and R follows from Proposition 5.1. Finally, if all the roots of q are real, then the residuals ρ_i are real, and

$$T_n(\mathfrak{R}(p/q)) = - \sum_{k=1}^{\deg q} \frac{\rho_k}{2z_k} (K_n(1/z_k) + I),$$

which together with Proposition 5.3 concludes the proof. \square

Notice that, if all the roots of q are known, then one can explicitly compute the φ -circulant matrices of Theorem 5.4, obtaining not exactly the optimal but a rank bounded preconditioner for an important class of Toeplitz matrices.

There follows one further lemma, whose proof can be found in [25] as underlined in [3].

Lemma 5.5. *Let $k \in \{1, \dots, n\}$ and $\alpha \in \mathbb{R}$. For any $\varepsilon > 0$ there exist a_i, b_i such that*

$$\left| k^{-\alpha} - \sum_{i=1}^{\rho} a_i (e^{-b_i})^k \right| \leq \varepsilon k^{-\alpha}$$

with $\rho \leq \log \varepsilon^{-1} (\beta_0 + \beta_1 \log \varepsilon^{-1} + \beta_2 \log n)$, and the coefficients β_i depend only on α .

By using the result stated in Lemma 5.1, we can reformulate the above Lemma 5.5. Let $\|\cdot\|_C$ denotes the Chebyshev norm on $\mathbf{M}(n)$,

$$A = (a_{ij})_{ij} \in \mathbf{M}(n), \quad \|A\|_C = \max_{ij} |a_{ij}|.$$

Consider the lower triangular Toeplitz matrix $T_n = [(i - j)^{-\alpha}]_{i \geq j}$. Then, for any $\varepsilon > 0$, there exist $P_\varphi^{(k)} \in \mathcal{C}_\varphi$ and R_k of rank one, such that

$$\left\| T_n - \sum_{k=1}^{\rho} a_k z_n(e^{-b_k}) \right\|_C = \left\| T_n - \sum_{k=1}^{\rho} P_\varphi^{(k)} + R_k \right\|_C \leq \varepsilon \|T_n\|_C$$

where ρ is bounded as in Lemma 5.5. By taking the transpose of T_n and then subtracting by I , one immediately observes (via Proposition 5.3) that also the symmetric Toeplitz matrix $T_n^s = (|i - j|^{-\alpha})_{ij}$ admits the decomposition

$$\|T_n^s - C_\varphi - R\|_C \leq \varepsilon \|T_n^s\|_C, \quad C_\varphi \in \mathcal{C}_\varphi$$

for a matrix R whose rank is bounded by 2ρ .

From now on when referring to Lemma 5.5 we will always think at the latter two inequalities.

Observe that the results obtained at this stage are enough to say that for any polynomial f and any symmetrized polynomial $g(x) = f(|x|)$, the Toeplitz matrices $T_n(f)$ and $T_n(g)$ admit the decomposition

$$T_n = C_\varphi + R + E$$

where $C_\varphi \in \mathcal{C}_\varphi$, R has sufficiently small rank and $\|E\|_C \leq \varepsilon$. Notice furthermore that the same can be said for continuous symbol functions, since they can be approximated by polynomials (Weierstrass theorem).

An even better result can be obtained for the Toeplitz matrix whose entries are positive integer powers of the indexes, namely

Lemma 5.6. *Let $T_n = [(i - j)^p]_{i \geq j}$ and $T_n^s = (|i - j|^p)_{ij}$. Then for any $\varphi \in \mathbb{T}$, we have $T_n = P_\varphi + R$ and $T_n^s = Q_\varphi + \tilde{R}$ with $P_\varphi, Q_\varphi \in \mathcal{C}_\varphi$ and $\text{rank } \tilde{R} \leq 2 \text{rank } R \leq 2(p + 2)$.*

Proof. If we prove the decomposition for T_n , the thesis for T_n^s follows because of the identity $T_n^s = T_n + T_n^T$. Fix $p \in \mathbb{N}$ and consider the polynomial χ_φ such that $\deg(\chi_\varphi) \leq p + 1$ and

$$\chi_\varphi(k) - \varphi \chi_\varphi(k - n) = k^p, \quad k = 1, \dots, n - 1.$$

Call v_p the first column of T_n , i.e., $T_n = T_n(v_p, 0)$, then set

$$a(\chi_\varphi) = \begin{pmatrix} \chi_\varphi(0) \\ \chi_\varphi(1) \\ \vdots \\ \chi_\varphi(n-1) \end{pmatrix}, \quad b(\chi_\varphi) = \begin{pmatrix} \chi_\varphi(0) \\ \chi_\varphi(-1) \\ \vdots \\ \chi_\varphi(1-n) \end{pmatrix}.$$

It follows that $\varphi Jb(\chi_\varphi) - \Pi_\varphi a(\chi_\varphi) = -\Pi_\varphi v_p$, and therefore

$$\mathcal{K}_\varphi(a(\chi_\varphi), b(\chi_\varphi)) = \mathcal{K}_\varphi(v_p, 0), \quad [T_n(a(\chi_\varphi), b(\chi_\varphi)), \Pi_\varphi] = [T_n(v_p, 0), \Pi_\varphi].$$

The thesis for T_n follows by noting that $\text{rank } T_n(a(\chi_\varphi), b(\chi_\varphi)) \leq p + 2$. \square

Now consider a polynomial f of degree d . By the previous Lemma we can affirm that the Toeplitz matrix whose entries are $f(|i - j|)$ can be decomposed into the sum of a φ -circulant matrix and a matrix R whose rank is bounded by $\sum_{i=1}^d i + 2 = O(d^2)$. Such particular Toeplitz matrix is indeed a generalized KMS matrix. For the sake of completeness we recall that, given the matrix

$$K_n(f, \lambda) = \left(f(|i - j|) \lambda^{|i-j|} \right)_{ij}$$

a generalized KMS matrix is defined as

$$G_n = \sum_{k=1}^m \gamma_k K_n(f_k, \lambda_k)$$

where γ_k and λ_k are all real and f_k are polynomials of degree d_k .

It is clear that using both Proposition 5.3 and Lemma 5.5 we have

$$\|G_n - P_\varphi + R\|_C \leq \varepsilon \|G_n\|_C, \quad P_\varphi \in \mathcal{C}_\varphi,$$

with $\text{rank } R = O(\log \frac{1}{\varepsilon} (\log \frac{1}{\varepsilon} + \log n) \sum_{k=1}^m d_k)$.

Proposition 5.7. *Set $f(z) = \log(z - z_0)$, $z_0, z \in \mathbb{T}$. Then, for any $\varepsilon > 0$ there exist $P_\varphi \in \mathcal{C}_\varphi$ and R_ε with $\text{rank } R_\varepsilon \leq \log \varepsilon^{-1} (\beta_0 + \beta_1 \log \varepsilon^{-1} + \beta_2 \log n)$ such that*

$$\|T_n(f) - P_\varphi - R_\varepsilon\|_C \leq \varepsilon \|T_n(f)\|_C.$$

Proof. By the logarithmic singularity of f there follows the equality

$$f(z) = \log z_0 + \sum_{k \geq 1} \frac{z^k}{kz_0^k},$$

and thus for $i > j$, we have

$$T_n(f)_{ij} = \frac{1}{(i-j)z_0^{i-j}} = (i-j)^{-1} Z_n(z_0^{-1})_{ij}.$$

Note that the $\log z_0$ term gives rise to a multiple of the identity, thus to an element of \mathcal{C}_φ . Therefore we do not care about it. By Lemma 5.1, the matrix $Z_n(z_0)$ has the form $\tilde{P}_\varphi + R$ for a $\tilde{P}_\varphi \in \mathcal{C}_\varphi$ and a rank one matrix R . Therefore by Lemma 5.5, for any $\varepsilon > 0$ there exist $P_\varphi^{(k)}$ and a rank one R_k such that

$$\left\| T_n(f) - (\tilde{P}_\varphi + R) \sum_k (P_\varphi^{(k)} + R_k) \right\|_C \leq \varepsilon \|T_n(f)\|_C.$$

As a consequence we have the thesis, since $(\tilde{P}_\varphi + R) \sum_k (P_\varphi^{(k)} + R_k) = P_\varphi + R_\varepsilon$, with R_ε and P_φ as in the statement. \square

We can, finally, combine Theorem 5.4, Lemma 5.6 and Proposition 5.7, to obtain

Theorem 5.8. *Let $f \in \mathcal{L}(\mathbb{T})$. For any $\varepsilon > 0$ there exist $P_\varphi, Q_\varphi \in \mathcal{C}_\varphi$ such that*

$$\begin{aligned} \|T_n(f) - P_\varphi - R_\varepsilon\|_C &\leq \varepsilon \|T_n(f)\|_C = O(\varepsilon), \\ \|T_n(\mathfrak{H}f) - Q_\varphi - \tilde{R}_\varepsilon\|_C &\leq \varepsilon \|T_n(\mathfrak{H}f)\|_C = O(\varepsilon) \end{aligned}$$

with $\text{rank } \tilde{R}_\varepsilon \leq 2 \text{rank } R_\varepsilon \leq 2 \log \varepsilon^{-1} (a + b \log \varepsilon^{-1} + c \log n) + d$, and all the coefficients a, b, c, d do not depend on n neither on ε .

The Hankel case can be discussed analogously. It is not difficult to check that $H_n(a, b) = JT_n(Ja, b)$, for any $a, b \in \mathbb{C}^n$ with $a_1 = b_1$. As before we call $H(T_n(a, b))$ such matrix. Therefore we can reformulate the results obtained in this section simply multiplying them by J on the left, since this clearly does not affect the rank neither the arbitrariness of ε . Moreover, we can write the Hankel matrix generated by the symbols

$$\zeta_\mu(\theta) = \frac{1}{1 - \mu e^{i\theta}}, \quad g(z) = \log(z - z_0), \quad z, z_0 \in \mathbb{T},$$

in terms of $H(Z_n(\lambda))$. In fact, for instance, we have

$$H_n(\zeta_\mu) = \mu^{n-1} JZ_n(\mu^{-1}) = \mu^{n-1} H(Z_n(\mu^{-1})) \tag{21}$$

and a similar identity holds for $H_n(g)$.

We finally stress the fact that our initial problem is not well posed when f is a rational function and the linear system is defined by the Hankel matrix with symbol f . In fact such matrix $H_n(f)$ has in general a small rank which equals the number of poles of f (due to the Kronecker theorem, 1881 [26]) and therefore the linear system $H_n(f)x = y$, when n is large enough, could even be unsolvable.

Hartley-type algebras

Let us conclude with few observations concerning Hartley-type algebras. In studying this case the arbitrariness of $\varphi \in \mathbb{T}$ is crucial, in fact it allows us to use both circulant and skew-circulant matrices and thus to consider Hartley-type algebras. If \mathcal{H} is a generic Hartley-type algebra, recall that $\mathcal{C}_\varphi^S \subset \mathcal{H}$ with $\varphi \in \{-1, 1\}$.

Observe that by Proposition 5.3 we already know that the KMS matrix $K_n(\lambda)$ admits the splitting $K_n(\lambda) = H + R$ where $\text{rank } R = 2$ and H is an element of $\mathcal{C}_{\pm 1}^S \subset \mathcal{H}$. In fact we have $K_n(\lambda) = C_\varphi(\xi_\varphi(\lambda)) + R$ where the matrix $C_\varphi(\xi_\varphi(\lambda))$, $\varphi \in \{1, -1\}$, is circulant or skew-circulant symmetric, respectively. Thus by the definitions in Table 4, when $\varphi \in \{1, -1\}$, $C_\varphi(\xi_\varphi(\lambda)) \in \mathcal{H}$.

Let us summarize this remark into the following:

Lemma 5.9. *Let $\lambda \in \mathbb{R}$. For any Hartley-type algebra \mathcal{H} there exists $H \in \mathcal{H}$ and R of rank two, such that $K_n(\lambda) = H + R$.*

Proof. Specialize Proposition 5.3 for $\varphi = 1$ and $\varphi = -1$ and use the definitions in Table 4. \square

Nonetheless we would stress the fact that all the results we obtained in terms of symmetric φ -circulants may also be seen as involving Hartley-type algebras. One just need to specialize them for $\varphi = 1$ or $\varphi = -1$.

6. Conclusions

We have tried to produce a first step towards the generalization of the ideas presented in [2]. After a brief overview about matrix algebras of low complexity, their generators and main properties, we have proposed a way to extend the applicability of the black-dot algorithm, proposed in [2] for the construction of optimal rank circulant preconditioners for a Toeplitz system, to other types of linear systems (including Toeplitz plus Hankel like) and to preconditioners chosen in other low complexity matrix algebras. Then we have shown that, in fact, a suitable class of Toeplitz and Hankel matrices is indeed representable as the sum of a φ -circulant matrix and a small rank perturbation, for any φ of modulus one. Combining such representation for $\varphi = 1$ and $\varphi = -1$ we then derive an analogous decomposition involving matrices from a Hartley-type algebra and a low rank perturbation.

It is important to note that for a significant class of Toeplitz (and Hankel) matrices associated with a rational symbol, the optimal rank φ -circulant and Hartley-type preconditioners (as we called it) can be explicitly computed without the use of the black-dot method, provided that the symbol function and its poles are explicitly known.

Acknowledgements

This paper includes results from the master dissertation of the corresponding author. He thanks the researchers of INM for the support they gave him during his permanence in Moscow, and reserves a special and kind thank to his two supervisors.

References

- [1] E.E. Tyrtyshnikov, A unifying approach to some old and new theorems on distribution and clustering, *Linear Algebra Appl.* 232 (1996) 1–43.
- [2] I.V. Oseledets, E.E. Tyrtyshnikov, A unifying approach to the construction of circulant preconditioners, *Linear Algebra Appl.* 418 (2006) 435–449.
- [3] N.L. Zamarashkin, I.V. Oseledets, E.E. Tyrtyshnikov, Approximation of Toeplitz matrices by sums of circulants and small-rank matrices, *Doklady Math.* 73 (1) (2006) 100–101.
- [4] C. Di Fiore, P. Zellini, Matrix Algebras in Optimal Preconditioning, *Linear Algebra Appl.* 335 (2001) 1–54.
- [5] A. Bortolotti, C. Di Fiore, On a set of matrix algebras related to discrete Hartley-type transforms, *Linear Algebra Appl.* 366 (2003) 65–85.
- [6] C. Di Fiore, S. Fanelli, P. Zellini, On the best least squares fit to a matrix and its applications, in: Ched E. Stedman (Ed.), *Algebra and Algebraic Topology*, Nova Science Publishers, 2007, pp. 73–109.
- [7] F. Di Benedetto, Gram matrices of fast algebras have a rank structure, *SIAM J. Matrix Anal. Appl.* (31) (2009) 526–545.
- [8] R.H. Chan, M. Ng, Conjugate gradient methods for Toeplitz systems, *SIAM Rev.* 38 (1996) 427–482.
- [9] P. Davis, *Circulant Matrices*, John Wiley and Sons, New York, 1979.
- [10] T. Huckle, Circulant and skewcirculant matrices for solving Toeplitz matrix problems, *SIAM J. Matrix Anal. Appl.* 13 (1992) 767–777.
- [11] E. Bozzo, C. Di Fiore, On the use of certain matrix algebras associated with real discrete transforms in matrix displacement decomposition, *SIAM J. Matrix Anal. Appl.* (16) (1995) 312–326.
- [12] T. Ceccherini-Silberstein, F. Scarabotti, F. Tolli, *Harmonic Analysis on Finite Groups*, Cambridge Studies in Advanced Mathematics, 2008.
- [13] D. Bini, M. Capovani, Spectral and computational properties of band symmetric Toeplitz matrices, *Linear Algebra Appl.* 52/53 (1983) 99–126.
- [14] M. Ng, R.H. Chan, W.C. Tang, A fast algorithm for deblurring models with Neumann boundary conditions, *SIAM J. Sci. Comput.* 21 (1999) 851–866.
- [15] C. Di Fiore, P. Zellini, Matrix decompositions using displacement rank and classes of commutative matrix algebras, *Linear Algebra Appl.* (229) (1995) 49–99.
- [16] V. Sanchez, P. Garcia, A.M. Peinado, J.C. Segura, A.J. Rubio, Diagonalizing properties of the discrete cosine transforms, *IEEE Trans. Signal Process.* 43 (1995) 2631–2641.
- [17] V. Sanchez, A.M. Peinado, J.C. Segura, P. Garcia, A.J. Rubio, Generating matrices for the discrete sine transforms, *IEEE Trans. Signal Process.* 44 (1996) 2644–2646.
- [18] D. Bini, P. Favati, On a matrix algebra related to the discrete Hartley transform, *SIAM J. Matrix Anal. Appl.* 14 (1993) 500–507.
- [19] E.E. Tyrtyshnikov, N.L. Zamarashkin, A.Y. Yemin, Clusters, preconditioners, convergence, *Linear Algebra Appl.* 263 (1997) 25–48.
- [20] S.A. Goreinov, E.E. Tyrtyshnikov, N.L. Zamarashkin, A theory of pseudo-skeleton approximations, *Linear Algebra Appl.* (261) (1997) 1–21.
- [21] E.E. Tyrtyshnikov, Incomplete cross approximation in the mosaic-skeleton method, *Computing* 64 (4) (2000) 367–380.
- [22] M. Bebendorf, Approximation of boundary element matrices, *Numer. Math.* 86 (2000) 565–589.
- [23] C. Di Fiore, Matrix algebras and displacement decompositions, *SIAM J. Matrix Anal. Appl.* (21) (2000) 646–667.

- [24] G. Strang, A. Edelman, The Toeplitz-circulant eigenvalue problem $Ax = \lambda x$, in: L. Bragg, J. Dettman (Eds.), *Oakland Conference on PDE's*, Longman, London, 1987
- [25] N. Yarvin, V. Rokhlin, Generalized Gaussian quadratures and singular value decompositions of integral operators, *SIAM J. Sci. Comput.* 20 (2) (1999) 699–718.
- [26] L. Kronecker, Zur theorie der elimination einer variablen aus zwei algebraischen gleichungen, *Math. Königl. Preussischen. Akad. Wiss. Berlin*, 1881, pp. 535–600.