Quantum stochastic equation for the low density limit

L Accardi†, A N Pechen†§ and I V Volovich‡

† Centro Vito Volterra, Universita di Roma Tor Vergata 00133,Italia
‡ Steklov Mathematical Institute of Russian Academy of Sciences, Gubkin St.8, GSP-1, 117966, Moscow, Russia

E-mail: accardi@volterra.mat.uniroma2.it, pechen@mi.ras.ru, volovich@mi.ras.ru

Abstract. A new derivation of quantum stochastic differential equation for the evolution operator in the low density limit is presented. We use the distribution approach and derive a new algebra for quadratic master fields in the low density limit by using the energy representation. We formulate the stochastic golden rule in the low density limit case for a system coupling with Bose field via quadratic interaction. In particular the vacuum expectation value of the evolution operator is computed and its exponential decay is shown.

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§ permanent address: Steklov Mathematical Institute of Russian Academy of Sciences, Gubkin St.8, GSP-1, 117966, Moscow, Russia

1. Introduction

There are many studies of the large time behavior in quantum theory. One of the powerful methods is the stochastic limit. Many important physical models have been investigated by using this method (see [1, 2, 3] for more discussions). This method, however, is restricted to the studying of the long time behavior of the models in the weak coupling case, i.e. when coupling constant is a small parameter, and it can not be applied directly to important class of models which contain terms in the interaction without small coupling constant.

This last class of models includes models in which the small parameter is the density. Such models naturally arise in the low density limit (LDL). The LDL for a classical Lorentz gas is the Boltzmann-Grad limit. For classical systems there has been a considerable progress in the rigorous derivation of the Boltzmann equation. Lanford [4] using ideas of Grad [5] proved the convergence of the hierarchy of correlation functions for a hard sphere gas in the Boltzmann–Grad limit for sufficiently short times. This proof was extended by King [6] to positive potentials of finite range. The limiting evolution of the one particle distribution is governed by the non-linear Boltzmann equation.

The test particle problem was studied by many authors(see a review of Spohn [7]). One considers the motion of a single particle through an environment of randomly placed, infinitely heavy scatterers (Lorentz gas). In the Boltzmann–Grad limit successive collisions become independent and the averaged over the positions of the scatterers the position and velocity distribution of the particle converges to the solution of the linear Boltzmann equation.

To describe a quantum physical model to which LDL can be applied let us consider an N-level atom immersed in a free gas whose molecules can collide with the atom; the gas is supposed to be very dilute. Then the reduced time evolution for the atom will be Markovian, since the characteristic time t_S for appreciable action of the surroundings on the atom (time between collisions) is much larger than the characteristic time t_R for relaxation of correlations in the surroundings. Rigorous results substantiating this idea have been obtained in [8].

It is known [9] that the dynamics of the N-level atom interacting with the free gas converges, in the low density limit, to the solution of a quantum stochastic differential equation driven by quantum Poisson noise. Indeed, from a semiclassical point of view, collision times, being times of occurrence of rare events, will tend to become Poisson distributed, whereas the effect of each collision will be described by the (quantum-mechanical) scattering operator of the atom with one gas particle (see the description of the quantum Poisson process in [10]).

In this paper a derivation of the quantum stochastic differential equation for the evolution operator in the low density limit is presented. The equation obtained is equivalent to the stochastic equation which has been derived in [9] but we use a new method. We use the distribution approach [2, 3] and derive a new algebra for quadratic master fields in the low density limit by using the energy representation. An advantage of this method is the simplicity of derivation of quantum stochastic equations and computation of correlation functions. We formulate the stochastic golden rule in the low density limit case for a system coupling with Bose field via quadratic interaction. In particular the vacuum expectation value of the evolution operator is computed and its exponential decay is shown.

Main results of the paper are quantum stochastic differential equation (19), the new algebra of commutation relations for the master field (Theorem 1) and the expression for the expectation value of the evolution operator (28).

In this paper we obtain unitary evolution which is given by the solution of the quantum stochastic differential equation. Using this equation one can obtain corresponding quantum Langevin and master equations.

An important problem in theory of open quantum systems is the rigorous derivation of quantum Boltzmann equation from microscopic dynamics. Aa approach to the derivation of classical and quantum Boltzmann equations, based on BBGKY-hierarchy, has been presented in the work of Bogoliubov [11]. The low density limit for the model under consideration, with completely different methods, based on quantum BBGKY hierarchy has been investigated by Dümcke [8].

Let us explain our notations. We consider a quantum model of a test particle interacting with a reservoir (heat bath). We shall restrict ourselves to the case of a Boson reservoir in this paper. Let $\mathcal{H}_{\rm S}$ be the Hilbert space of the system (test particle) with the system Hamiltonian $H_{\rm S}$. The system Hilbert space for the N-level atom $\mathcal{H}_{\rm S}=\mathbb{C}^N$. The reservoir is described by the Boson Fock space $\Gamma(\mathcal{H}_1)$ over the one particle Hilbert space $\mathcal{H}_1=L^2(\mathbb{R}^d)$, where $d=3$ in physical case. Moreover, the Hamiltonian of the reservoir is given by $H_{\rm R}:={\rm d}\Gamma(H_1)$ (the second quantization of the one particle Hamiltonian H_1) and the total Hamiltonian of the compound system is given by a self–adjoint operator on the total Hilbert space $\mathcal{H}_{\rm S}\otimes\Gamma(\mathcal{H}_1)$, which has the form

$$H_{\rm tot}:=H_{\rm S}\otimes 1+1\otimes H_{\rm R}+H_{\rm int}=:H_{\rm free}+H_{\rm int}.$$

Here $H_{\rm int}$ is the interaction Hamiltonian between the system and the reservoir. The evolution operator at time t is given by:

$$U_t:={\rm e}^{{\rm i}tH_{\rm free}}\cdot{\rm e}^{-{\rm i}tH_{\rm tot}}.$$

Obviously, it satisfies the differential equation

$$\partial_t U_t=-{\rm i}H_{\rm int}(t)U_t$$

where the quantity $H_{\rm int}(t)$ will be called the evolved interaction and defined as

$$H_{\rm int}(t)={\rm e}^{{\rm i}tH_{\rm free}}H_{\rm int}{\rm e}^{-{\rm i}tH_{\rm free}}.$$

The interaction Hamiltonian will be assumed to have the following form:

$$H_{\rm int}:=D\otimes A^+(g_0)A(g_1)+D^+\otimes A^+(g_1)A(g_0)$$

where D is a bounded operator in $\mathcal{H}_S$, $D\in\mathbf{B}(\mathcal{H}_S)$, A and A^+ are annihilation and creation operators and $g_0,g_1\in\mathcal{H}_1$ are form-factors describing the interaction of the system with the reservoir. This Hamiltonian preserves the particle number of the reservoir, and therefore the particles of the reservoir are only scattered and not created or destroyed. This model was considered by Davies [12] in the analysis of the weak coupling limit. The development of the method to the Bose gas is a subject of further works.

With the notion

$$S_t^0 := \mathrm{e}^{\mathrm{i}tH_1} \ ; \qquad D(t) := \mathrm{e}^{\mathrm{i}tH_\mathrm{S}} D \mathrm{e}^{-\mathrm{i}tH_\mathrm{S}}$$

the evolved interaction can be written in the form

$$H_{\mathrm{int}}(t) := D(t) \otimes A^+(S_t^0 g_0) A(S_t^0 g_1) + D^+(t) \otimes A^+(S_t^0 g_1) A(S_t^0 g_0) \tag{1}$$

The initial state of the compound system is supposed to be of the form

$$\rho = \rho_\mathrm{S} \otimes \varphi^{(\xi)}.$$

Here ρ_S is arbitrary density matrix of the system and the initial state of the reservoir is the Gibbs state, at inverse temperature β, of the free evolution, i.e. the gauge invariant quasi-free state $\varphi^{(\xi)}$, characterized by

$$\varphi^{(\xi)}(W(f)) = \exp\Big(-\frac{1}{2} < f, (1 + \xi \mathrm{e}^{-\beta H_1})(1 - \xi \mathrm{e}^{-\beta H_1})^{-1} f >\Big) \tag{2}$$

for each $f \in \mathcal{H}_1$. Here $W(f)$ is the Weyl operator, β the inverse temperature of the reservoir, $\xi = \mathrm{e}^{\beta\mu}$ the fugacity, μ the chemical potential. We suppose that the temperature $\beta^{-1} > 0$. Therefore for sufficiently low density one is above the transition temperature, and no condensate is present. The generalization to the case then the condensate is present, is a subject of further investigations.

We will study the dynamics, generated by the Hamiltonian (1) and the initial state of the reservoir (2) in the low density regime: $n \to 0$, $t \sim 1/n$ (n is the density of particles of the reservoir). In the low density limit the fugacity ξ and the density of particles of the reservoir n have the same asymptotic, i.e.

$$\lim_{n\to 0} \frac{\xi(n)}{n} = 1$$

Therefore the limit $n \to 0$ is equivalent to the limit $\xi \to 0$.

The low density limit for this model, with completely different methods, based on quantum BBGKY hierarchy has been considered by Dümcke [8].

Throughout the paper, for simplicity, the following technical condition is assumed: the two test functions in the interaction Hamiltonian have disjoint supports in the energy representation. Thus the disjointness is invariant under the action of any function of H. This assumption means that the two test function g_0, g_1 in the interaction Hamiltonian satisfy:

$$< g_0, S_t^0 \mathrm{e}^{-\beta H} g_1 >= 0 \qquad \forall t \in \mathbb{R}.$$

This means that, even if the particles of the reservoir have generically a continuous energy spectrum, they behave like a 2–level system as far as their interaction with the system is concerned: if P_0 and P_1 project onto disjoint intervals (energy bands) I_0 and I_1, these energy bands act as the counterpart of the energy levels ϵ_0, ϵ_1 of the system.

Also the rotating wave approximation condition will be assumed. This condition means that

$$e^{itH_S} D e^{-itH_S} = e^{-it\omega_0} D$$

where ω_0 is a real number. This is a familiar assumption, satisfied by all the Hamiltonians commonly used in quantum optics. This assumption is satisfied if $D = |\epsilon_0 >< \epsilon_1|$, where $|\epsilon_0 >$ and $|\epsilon_1 >$ are eigenvectors of the free system Hamiltonian with eigenvalues ϵ_0 and ϵ_1 so that $\omega = \epsilon_1 - \epsilon_0$.

We will fix a projection operator P_0 in $\mathcal{H}_1$ commuting with H_1 and H and such that

$$P_0 g_0 = g_0 \quad \text{and} \quad P_0 g_1 = 0.$$

Using this projection let us define the group $\{S_t;\, t \in \mathbb{R}\}$ of unitary operators on $\mathcal{H}_1$ by

$$S_t = S_t^0 e^{-it\omega_0 P_0} = e^{it(H_1 - \omega_0 P_0)}.$$

The infinitesimal generator H_1' of S_t is given by

$$H_1' = H_1 - \omega_0 P_0.$$

Following Palmer [13] we realize the representation space as the tensor product of a Fock and anti-Fock representations. Then the expectation values with respect to the state $\varphi^{(\xi)}$ for the model with the interaction Hamiltonian (1) can be conveniently represented as the vacuum expectation values in the Fock-anti-Fock representation for the modified Hamiltonian.

Denote by $\mathcal{H}_1^\iota$ the conjugate of $\mathcal{H}_1$, i.e.

$$\iota :\ \mathcal{H}_1 \longrightarrow \mathcal{H}_1\ , \quad \iota(\lambda f) := \bar{\lambda}\iota(f)$$
$$< \iota(f), \iota(g) >_\iota \ := \ < g, f >$$

then, $\mathcal{H}_1^\iota$ is a Hilbert space. The corresponding Fock space $\Gamma(\mathcal{H}_1^\iota)$ is called in this context the anti-Fock space.

It was shown in [9] that with notations $D_0 = D$, $D_1 = D^+$ the modified Hamiltonian acting in $\Gamma(\mathcal{H}_1) \otimes \Gamma(\mathcal{H}_1^\iota)$ has the form

$$H_\lambda(t) = \sum_{\varepsilon=0,1} D_\varepsilon \otimes (A^+(S_t g_\varepsilon) A(S_t g_{1-\varepsilon}) \otimes 1$$
$$+\lambda(A(S_t g_{1-\varepsilon}) \otimes A(S_t L g_\varepsilon) + A^+(S_t g_\varepsilon) \otimes A^+(S_t L g_{1-\varepsilon})).$$

Here A and A^+ are Bose annihilation and creation operators acting in the Fock spaces $\Gamma(\mathcal{H}_1)$ and $\Gamma(\mathcal{H}_1^\iota)$ and $L := e^{-\beta H/2}$.

Moreover, it will be assumed that there exists a subset $\mathcal{K}$ (which includes g_0, g_1) of the one particle Hilbert space $\mathcal{H}_1$, such that

$$\int_{\mathbb{R}} | < f, S_t g > | dt < \infty \qquad \forall f, g \in \mathcal{K}.$$

The interaction Hamiltonian determines the evolution operator $U_t^{(\lambda)}$ which is the solution of the Schrödinger equation in interaction representation:

$$\partial_t U_t^{(\lambda)} = -\mathrm{i} H_\lambda(t) U_t^{(\lambda)}$$

with initial condition

$$U_0^{(\lambda)} = 1.$$

One has the following integral equation for the evolution operator.

$$U_t^{(\lambda)} = 1 - \mathrm{i} \int_0^t \mathrm{d}t' H_\lambda(t') U_{t'}^{(\lambda)}.$$

2. Energy representation

We will investigate the limit of the evolution operator when $\xi \to +0$ after the time rescaling $t \to t/\xi$, where $\xi = \lambda^2$. After this time rescaling the equation for the evolution operator becomes

$$\partial_t U_{t/\lambda^2}^{(\lambda)} = -\mathrm{i} \sum_{\varepsilon=0,1} D_\varepsilon \otimes (N_{\varepsilon,1-\varepsilon,\lambda}(t) + B_{1-\varepsilon,\varepsilon,\lambda}(t) + B^+_{\varepsilon,1-\varepsilon,\lambda}(t)) U_{t/\lambda^2}^{(\lambda)}$$

where we introduced the notations:

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(t) = \frac{1}{\lambda^2} A^+(S_{t/\lambda^2} g_{\varepsilon_1}) A(S_{t/\lambda^2} g_{\varepsilon_2}) \otimes 1$$
$$B^+_{\varepsilon_1,\varepsilon_2,\lambda}(t) = \frac{1}{\lambda} A^+(S_{t/\lambda^2} g_{\varepsilon_1}) \otimes A^+(S_{t/\lambda^2} L g_{\varepsilon_2}).$$

Let us introduce the energy representation for the creation and annihilation operators by the formulae

$$A^+_{\mathrm{E}}(g) = A^+(P_{\mathrm{E}} g) = \int \mathrm{d}k (P_E g)(k) a^+(k) = \int \mathrm{d}k \delta(H_1' - E) g(k) a^+(k) \tag{3}$$

$$A_E(g) = A(P_E g)$$

Here

$$P_E = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}t S_t \mathrm{e}^{-\mathrm{i}tE} = \delta(H_1' - E).$$

It has the properties

$$S_t = \int \mathrm{d}E P_E \mathrm{e}^{\mathrm{i}tE}$$
$$P_E P_{E'} = \delta(E - E') P_E$$
$$P_E^* = P_E$$

In the case when $\mathcal{H}_1 = L^2(\mathbb{R}^d)$ the one-particle Hamiltonian is the multiplication operator to the function $\omega(k)$ and acts on any element $f \in L^2(\mathbb{R}^d)$ as $H_1 f(k) = \omega(k) f(k)$.

It is easy to check that

$$[A_E(f), A^+_{E'}(g)] = \delta(E-E') < f, P_E g > .$$

Here $< \cdot, \cdot >$ means the scalar product in $\mathcal{H}_1$.

Using the energy representation one gets

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(t) = \int \mathrm{d}E_1 \mathrm{d}E_2 N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t)$$

$$B_{\varepsilon_1,\varepsilon_2,\lambda}(t) = \int \mathrm{d}E_1 \mathrm{d}E_2 B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t)$$

where

$$N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t) := \frac{\mathrm{e}^{\mathrm{i}t(E_1-E_2)/\lambda^2}}{\lambda^2} A^+_{E_1}(g_{\varepsilon_1}) A_{E_2}(g_{\varepsilon_2}) \otimes 1$$

$$B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t) := \frac{\mathrm{e}^{\mathrm{i}t(E_2-E_1)/\lambda^2}}{\lambda} A_{E_1}(g_{\varepsilon_1}) \otimes A_{E_2}(Lg_{\varepsilon_2}).$$

Let us also denote

$$\gamma_\varepsilon(E) := \int_{-\infty}^{0} \mathrm{d}t < g_\varepsilon, S_t g_\varepsilon > \mathrm{e}^{-\mathrm{i}tE}.$$

3. The limiting commutation relations

Besides the operators $B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t)$ and $N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t)$ defined above let us consider the following operators:

$$N^\iota_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t) = \frac{\mathrm{e}^{\mathrm{i}t(E_2-E_1)/\lambda^2}}{\lambda^2}\, 1 \otimes A^+_{E_1}(Lg_{\varepsilon_1}) A_{E_2}(Lg_{\varepsilon_2})$$

with $A^+_E(Lg_\varepsilon)$ has been defined in (3). The commutators of these operators are:

$$\begin{aligned}
&[B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t), B^+_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,t')] \\
&= \frac{\mathrm{e}^{\mathrm{i}(t'-t)(E_1-E_2)/\lambda^2}}{\lambda^2}\Big(\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\delta(E_1-E_3)\delta(E_2-E_4) < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} >< g_{\varepsilon_2}, P_{E_2} L^2 g_{\varepsilon_2} > \\
&\qquad + \lambda^2 \delta_{\varepsilon_1,\varepsilon_3}\delta(E_1-E_3) < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > N^\iota_{\varepsilon_4,\varepsilon_2,\lambda}(E_4,E_2,t') \\
&\qquad + \lambda^2 \delta_{\varepsilon_2,\varepsilon_4}\delta(E_2-E_4) < g_{\varepsilon_2}, P_{E_2} L^2 g_{\varepsilon_2} > N_{\varepsilon_3,\varepsilon_1,\lambda}(E_3,E_1,t')\Big) \qquad (4)
\end{aligned}$$

$$\begin{aligned}
&[B_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t), N_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,t')] \\
&\qquad = \delta_{\varepsilon_1,\varepsilon_3}\frac{\mathrm{e}^{\mathrm{i}(t'-t)(E_1-E_2)/\lambda^2}}{\lambda^2}\delta(E_1-E_3) \\
&\qquad \times < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > B_{\varepsilon_4,\varepsilon_2,\lambda}(E_4,E_2,t') \qquad (5)
\end{aligned}$$

$$\begin{aligned}
&[N_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t), N_{\varepsilon_3,\varepsilon_4,\lambda}(E_3,E_4,t')] \\
&\qquad = \frac{\mathrm{e}^{\mathrm{i}(t'-t)(E_3-E_1)/\lambda^2}}{\lambda^2}\Big(\delta_{\varepsilon_2,\varepsilon_3}\delta(E_2-E_3) < g_{\varepsilon_2}, P_{E_2} g_{\varepsilon_2} > N_{\varepsilon_1,\varepsilon_4,\lambda}(E_1,E_4,t') \\
&\qquad - \delta_{\varepsilon_1,\varepsilon_4}\delta(E_1-E_4) < g_{\varepsilon_1}, P_{E_1} g_{\varepsilon_1} > N_{\varepsilon_3,\varepsilon_2,\lambda}(E_3,E_2,t)\Big). \qquad (6)
\end{aligned}$$

Notice that in the sense of distributions one has the limit

$$\lim_{\lambda\to 0}\frac{\mathrm{e}^{\mathrm{i}(t'-t)(E_1-E_2)/\lambda^2}}{\lambda^2}=2\pi\delta(t'-t)\delta(E_1-E_2) \tag{7}$$

and, in the sense of distributions over the standard simplex (cf. [3]) one has the limit

$$\lim_{\lambda\to 0}\frac{\mathrm{e}^{\mathrm{i}(t'-t)(E_1-E_2)/\lambda^2}}{\lambda^2}=\delta_+(t'-t)\frac{1}{\mathrm{i}(E_1-E_2-\mathrm{i}0)}. \tag{8}$$

The following theorem describes the algebra of commutation relations for the master field in the LDL.

Theorem 1 *The limits*

$$X_{\varepsilon_1,\varepsilon_2}(E_1,E_2,t):=\lim_{\lambda\to 0}X_{\varepsilon_1,\varepsilon_2,\lambda}(E_1,E_2,t)\qquad (X=B,N)$$

exist in the sense of convergence of correlators and satisfy the (causal) commutation relations

$$\begin{aligned}&[B_{\varepsilon_1,\varepsilon_2}(E_1,E_2,t),B^+_{\varepsilon_3,\varepsilon_4}(E_3,E_4,t')]\\ &\quad=\delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\delta_+(t'-t)\delta(E_1-E_3)\delta(E_2-E_4)\\ &\quad\times\frac{<g_{\varepsilon_1},P_{E_1}g_{\varepsilon_1}>}{\mathrm{i}(E_1-E_2-\mathrm{i}0)}<g_{\varepsilon_2},P_{E_2}L^2g_{\varepsilon_2}>\end{aligned} \tag{9}$$

$$\begin{aligned}&[B_{\varepsilon_1,\varepsilon_2}(E_1,E_2,t),N_{\varepsilon_3,\varepsilon_4}(E_3,E_4,t')]\\ &\quad=\delta_{\varepsilon_1,\varepsilon_3}\delta_+(t'-t)\delta(E_1-E_3)\frac{<g_{\varepsilon_1},P_{E_1}g_{\varepsilon_1}>}{\mathrm{i}(E_1-E_2-\mathrm{i}0)}B_{\varepsilon_4,\varepsilon_2}(E_4,E_2,t')\end{aligned} \tag{10}$$

$$\begin{aligned}&[N_{\varepsilon_1,\varepsilon_2}(E_1,E_2,t),N_{\varepsilon_3,\varepsilon_4}(E_3,E_4,t')]=\delta_+(t'-t)\frac{1}{\mathrm{i}(E_3-E_1-\mathrm{i}0)}\\ &\quad\times\Big(\delta_{\varepsilon_2,\varepsilon_3}\delta(E_2-E_3)<g_{\varepsilon_2},P_{E_2}g_{\varepsilon_2}>N_{\varepsilon_1,\varepsilon_4}(E_1,E_4,t')\\ &\quad-\delta_{\varepsilon_1,\varepsilon_4}\delta(E_1-E_4)<g_{\varepsilon_1},P_{E_1}g_{\varepsilon_1}>N_{\varepsilon_3,\varepsilon_2}(E_3,E_2,t)\Big)\end{aligned} \tag{11}$$

The commutation relations of the master field are obtained by (9),(10),(11) replacing the factor $\delta_+(t'-t)$ *by* $\delta(t'-t)$ *and* $(\mathrm{i}(E_1-E_2-\mathrm{i}0))^{-1}$ *by* $2\pi\delta(E_1-E_2)$.

Proof. The proof of the theorem follows by induction from the commutation relations (4)-(6) using the limits (7) and (8) and standard methods of the stochastic limit.

4. The master space and the associated white noise

Let $\mathcal{H}_{0,1}$ denote the closed subspace of $\mathcal{H}_1=L^2(\mathbb{R}^d)$ spanned by the vectors

$$S_tg_\varepsilon,\quad \varepsilon\in\{0,1\},\quad t\in\mathbb{R}.$$

Let K be a non zero subspace of $\mathcal{H}_{0,1}$ such that $g_\varepsilon\in K$ $(\varepsilon=0,1)$ and

$$\int_{-\infty}^{\infty}|<f,S_tg>|\mathrm{d}t<\infty\qquad\forall f,g\in K.$$

This assumption implies that the sesquilinear form $(\cdot|\cdot) : K \times K \longrightarrow C$ defined by

$$(f|g) = \int_{-\infty}^{\infty} < f, S_t g > \mathrm{d}t, \qquad f, g \in K$$

is well defined. Moreover it defines a pre-scalar product on K. We denote $\{K, (\cdot|\cdot)\}$ or simply K, the completion of the quotient of K by the zero $(\cdot|\cdot)$-norm elements.

Define then Hilbert space $K_{0,1}$ which will be denoted also as $K \otimes_\beta K$ as the completion of the algebraic tensor product $K \otimes K^*$ with respect to the scalar product

$$\begin{aligned}(f_0 \otimes_\beta f_1 | f_0' \otimes_\beta f_1') &:= \int_{-\infty}^{\infty} < f_0, S_t f_0' > \overline{< f_1, S_t L^2 f_1' >} \mathrm{d}t \\ &= \int_{-\infty}^{\infty} < f_0, S_t f_0' >< f_1', S_{-t} L^2 f_1 > \mathrm{d}t.\end{aligned}$$

Bounded operators acts naturally on $K_{0,1}$ by

$$(A \otimes_\beta B)(f_0 \otimes_\beta f_1) = A f_0 \otimes_\beta B f_1 \qquad \forall A, B \in \mathbf{B}(K).$$

The limit reservoir (or master) space is the space

$$\mathcal{F}(L^2(\mathbb{R}) \otimes K_{0,1}).$$

The (non-causal) commutation relations (9),...,(11) mean that operators $B_{\varepsilon_1,\varepsilon_2}(E_1, E_2, t)$ are the white noise operators $b_t(\cdot)$ in $\mathcal{F}(L^2(\mathbb{R}) \otimes K_{0,1})$:

$$B_{\varepsilon_1,\varepsilon_2}(E_1, E_2, t) =: b_t(P_{E_1} g_{\varepsilon_1} \otimes_\beta P_{E_2} g_{\varepsilon_2}).$$

The number operator is

$$\begin{aligned}N_{\varepsilon_1,\varepsilon_2}(E_1, E_2, t) &= \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E_1) b_t^+(g_{\varepsilon_1} \otimes_\beta P_{E_1} g_{\varepsilon'}) b_t(P_{E_2} g_{\varepsilon_2} \otimes_\beta P_{E_1} g_{\varepsilon'}) \\ &= \sum_{\varepsilon'=0,1} \int \mathrm{d}E n_{\varepsilon'}(E_1) B^+_{\varepsilon_1,\varepsilon'}(E, E_1, t) B_{\varepsilon_2,\varepsilon'}(E_2, E_1, t) \qquad (12)\end{aligned}$$

where we denoted

$$n_\varepsilon(E) := \frac{1}{< g_\varepsilon, P_E L^2 g_\varepsilon >}.$$

By identifying the element of the algebraic tensor product $f \otimes g \in K \otimes K^*$ with the operator

$$|f >< g| : \xi \in \mathcal{H}_1 \to < g, \xi > f \in \mathcal{H}_1$$

so that

$$(|f >< g|)^* = |g >< f|$$

and introducing the scalar product of such operators X, Y (notice that $L^2 = e^{-\beta H}$) by

$$< Y, X > := Tr \int \mathrm{e}^{-\beta H} Y^* S_t X S_t^* \mathrm{d}t = 2\pi Tr \int \mathrm{d}E \mathrm{e}^{-\beta H} Y^* P_E X P_E$$

one can rewrite white noise $b_t(g\otimes_\beta f)$ as $b_t(|g><f|)$ with commutation relations defined by

$$[b_t(Y), b^+_{t'}(X)] = \delta(t'-t) < Y, X > .$$

Let us introduce for simplicity

$$B_{\varepsilon_1,\varepsilon_2}(E,t) := \int \mathrm{d}E' B_{\varepsilon_1,\varepsilon_2}(E',E,t)$$
$$N_{\varepsilon_1,\varepsilon_2}(E,t) := \int \mathrm{d}E' N_{\varepsilon_1,\varepsilon_2}(E,E',t)$$

with (causal) commutation relations:

$$[B_{\varepsilon_1,\varepsilon_2}(E,t), B^+_{\varepsilon_3,\varepsilon_4}(E',t')] = \delta_{\varepsilon_1,\varepsilon_3}\delta_{\varepsilon_2,\varepsilon_4}\delta_+(t'-t)\delta(E-E')\gamma_{\varepsilon_1}(E) < g_{\varepsilon_2}, P_E L^2 g_{\varepsilon_2} >$$
$$[B_{\varepsilon_1,\varepsilon_2}(E,t), N_{\varepsilon_3,\varepsilon_4}(E',t')] = \delta_{\varepsilon_1,\varepsilon_3}\delta_+(t'-t)\frac{< g_{\varepsilon_1}, P_{E'} g_{\varepsilon_1} >}{\mathrm{i}(E'-E-\mathrm{i}0)} B_{\varepsilon_4,\varepsilon_2}(E,t').$$

In these notations the limiting Hamiltonian acts on $\mathcal{H}_S\otimes\mathcal{F}(L^2(\mathbb{R})\otimes K_{0,1})$ as

$$H(t) = \int \mathrm{d}E \sum_{\varepsilon=0,1} D_\varepsilon \otimes (N_{\varepsilon,1-\varepsilon}(E,t) + B_{1-\varepsilon,\varepsilon}(E,t) + B^+_{\varepsilon,1-\varepsilon}(E,t)).$$

5. Emergence of the drift term and annihilation process

The results of the preceding section allow us to write the equation for the evolution operator in the stochastic limit

$$\partial_t U_t = -\mathrm{i}H(t)U_t = -\mathrm{i}\sum_{\varepsilon=0,1} D_\varepsilon\otimes(N_{\varepsilon,1-\varepsilon}(t) + B_{1-\varepsilon,\varepsilon}(t) + B^+_{\varepsilon,1-\varepsilon}(t))U_t \tag{13}$$

In order to bring it to the normally ordered form one needs to compute the commutator

$$-\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(t), U_t] = -\mathrm{i}D_\varepsilon\int \mathrm{d}E[B_{1-\varepsilon,\varepsilon}(E,t), U_t].$$

Notice that $D_\varepsilon D_{1-\varepsilon}$ is a positive self-adjoint operator. Therefore one can assume that for each $E\in\mathbb{R}$, the inverse operator

$$T_\varepsilon(E) := (1 + (\gamma_\varepsilon\gamma_{1-\varepsilon})(E)D_\varepsilon D_{1-\varepsilon})^{-1}$$

always exists. Notice also that, since $D_\varepsilon D_{1-\varepsilon}$ commutes with $T_\varepsilon(E)$, one has

$$1 - D_\varepsilon D_{1-\varepsilon}(\gamma_\varepsilon\gamma_{1-\varepsilon})(E)T_\varepsilon(E) = T_\varepsilon(E).$$

Therefore

$$\mathrm{i}D_\varepsilon(1 - D_{1-\varepsilon}(\gamma_\varepsilon\gamma_{1-\varepsilon})(E)T_\varepsilon(E)D_\varepsilon) = \mathrm{i}T_\varepsilon(E)D_\varepsilon \tag{14}$$

Theorem 2 *For the model described above one has*

$$-\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(t), U_t] = -\int \mathrm{d}E\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)$$
$$\times\Big(< g_\varepsilon, P_E L^2 g_\varepsilon > U_t - \mathrm{i}D_\varepsilon\gamma_\varepsilon(E)U_t B_{1-\varepsilon,\varepsilon}(E,t) + U_t B_{\varepsilon,\varepsilon}(E,t)\Big) \tag{15}$$

Proof. Using the integral equation for the evolution operator and the commutation relations (9),(10), one gets

$$-\,\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(E,t),U_t]$$

$$= -\sum_{\varepsilon'=0,1} D_\varepsilon D_{\varepsilon'}\int \mathrm{d}E'\int_0^t \mathrm{d}t_1[B_{1-\varepsilon,\varepsilon}(E,t),N_{\varepsilon',1-\varepsilon'}(E',t_1)+B^+_{\varepsilon',1-\varepsilon'}(E',t_1)]U_{t_1}$$

$$= -D_\varepsilon D_{1-\varepsilon}\gamma_{1-\varepsilon}(E)\Big(< g_\varepsilon, P_E L^2 g_\varepsilon > + B_{\varepsilon,\varepsilon}(E,t)\Big)U_t \qquad (16)$$

Notice that the first equality in (16) holds because, due to the time consecutive principle

$$[B_{\varepsilon,\varepsilon'}(E,t),U_{t_1}]=0.$$

Similarly one computes the commutator

$$[B_{\varepsilon,\varepsilon}(E,t),U_t] = -\mathrm{i}\sum_{\varepsilon'} D_{\varepsilon'}\int \mathrm{d}E'\int_0^t \mathrm{d}t_1[B_{\varepsilon,\varepsilon}(E,t),N_{\varepsilon',1-\varepsilon'}(E',t_1)]U_{t_1}$$

$$= -\mathrm{i}D_\varepsilon\gamma_\varepsilon(E)B_{1-\varepsilon,\varepsilon}(E,t)U_t \qquad (17)$$

After substitution of this commutator into (16) one gets

$$-\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(E,t),U_t] = -D_\varepsilon D_{1-\varepsilon}\gamma_{1-\varepsilon}(E)(< g_\varepsilon, P_E L^2 g_\varepsilon > U_t - \mathrm{i}D_\varepsilon\gamma_\varepsilon(E)$$
$$\times([B_{1-\varepsilon,\varepsilon}(E,t),U_t]+U_tB_{1-\varepsilon,\varepsilon}(E,t))+U_tB_{\varepsilon,\varepsilon}(E,t))$$

Then for

$$f_\varepsilon(E,t) := -\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(E,t),U_t]$$

one has

$$(1+(\gamma_\varepsilon\gamma_{1-\varepsilon})(E)D_\varepsilon D_{1-\varepsilon})f_\varepsilon(E,t) = -\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}(< g_\varepsilon, P_E L^2 g_\varepsilon > U_t$$
$$-\mathrm{i}D_\varepsilon\gamma_\varepsilon(E)U_tB_{1-\varepsilon,\varepsilon}(E,t)+U_tB_{\varepsilon,\varepsilon}(E,t)).$$

Since the inverse operator $(1+(\gamma_\varepsilon\gamma_{1-\varepsilon})(E)D_\varepsilon D_{1-\varepsilon})^{-1}$ exists we can solve the equation above for $f_\varepsilon(E,t)$. Using this solution we find

$$-\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon}(t),U_t] = \int \mathrm{d}E f_\varepsilon(E,t) = -\int \mathrm{d}E\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)$$
$$\times\Big(< g_\varepsilon, P_E L^2 g_\varepsilon > U_t - \mathrm{i}D_\varepsilon\gamma_\varepsilon(E)U_tB_{1-\varepsilon,\varepsilon}(E,t)+U_tB_{\varepsilon,\varepsilon}(E,t)\Big).$$

6. Emergence of the number and creation processes

In order to bring equation (13) to the normally ordered form one needs also to move the annihilation operators in $N_{\varepsilon,1-\varepsilon}(t)$ to the right of the evolution operator. Using (12) this leads to

$$-\,\mathrm{i}D_\varepsilon N_{\varepsilon,1-\varepsilon}(t)U_t = -\mathrm{i}D_\varepsilon\int \mathrm{d}E\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)B_{1-\varepsilon,\varepsilon'}(E,t)U_t$$

$$= -\mathrm{i}D_\varepsilon\int \mathrm{d}E\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)\Big([B_{1-\varepsilon,\varepsilon'}(E,t),U_t]+U_tB_{1-\varepsilon,\varepsilon'}(E,t)\Big) \qquad (18)$$

and the commutator is evaluated using (16) and (17).

Theorem 3 *For the model described above one has*

$$-\mathrm{i}D_\varepsilon N_{\varepsilon,1-\varepsilon}(t)U_t = -\int \mathrm{d}E\Big(\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)$$
$$\times\Big(-\mathrm{i}D_\varepsilon\gamma_\varepsilon(E)B^+_{\varepsilon,1-\varepsilon}(E,t)+B^+_{\varepsilon,\varepsilon}(E,t)\Big)U_t$$
$$+\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)\Big(\mathrm{i}T_\varepsilon(E)D_\varepsilon B^+_{\varepsilon,\varepsilon'}(E,t)U_t B_{1-\varepsilon,\varepsilon'}(E,t)$$
$$+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)B^+_{\varepsilon,\varepsilon'}(E,t)U_t B_{\varepsilon,\varepsilon'}(E,t)\Big)\Big)$$

Proof. From (16) and (17) it follows that

$$-\mathrm{i}D_\varepsilon[B_{1-\varepsilon,\varepsilon'}(E,t),U_t] = -\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)\Big(n^{-1}_{\varepsilon'}(E)(\delta_{\varepsilon,\varepsilon'}-\delta_{1-\varepsilon,\varepsilon'}\mathrm{i}D_\varepsilon\gamma_\varepsilon)U_t$$
$$-\mathrm{i}D_\varepsilon\gamma_\varepsilon(E)U_t B_{1-\varepsilon,\varepsilon'}(E,t)+U_t B_{\varepsilon,\varepsilon'}(E,t)\Big).$$

After substitution of these commutators in (18) and using (14) one finishes the proof of the theorem.

7. The normally ordered equation

Theorems (2) and (3) allow us to obtain immediately the normally ordered equation for the evolution operator in the LDL. This procedure of deduction of quantum stochastic differential equation is being called a stochastic golden rule. The normally ordered equation has the form

$$\partial_t U_t = -\sum_{\varepsilon=0,1}\int \mathrm{d}E\Big\{\mathrm{i}T_\varepsilon(E)D_\varepsilon \sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)U_t B_{1-\varepsilon,\varepsilon'}(E,t)$$
$$+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)U_t B_{\varepsilon,\varepsilon'}(E,t)$$
$$+\mathrm{i}T_\varepsilon(E)D_\varepsilon B^+_{\varepsilon,1-\varepsilon}(E,t)U_t+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)B^+_{\varepsilon,\varepsilon}(E,t)U_t$$
$$+\mathrm{i}T_\varepsilon(E)D_\varepsilon U_t B_{1-\varepsilon,\varepsilon}(E,t)+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)U_t B_{\varepsilon,\varepsilon}(E,t)$$
$$+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E) < g_\varepsilon, P_E L^2 g_\varepsilon > U_t\Big\} \qquad (19)$$

We will represent equation (19) in the form of quantum stochastic differential equation [14].

Theorem 4 *Equation (19) is equivalent to the quantum stochastic differential equation*

$$\mathrm{d}U_t = -\sum_{\varepsilon=0,1}\int \mathrm{d}E\Big\{\mathrm{i}T_\varepsilon(E)D_\varepsilon \mathrm{d}N_t(2\pi|g_\varepsilon><g_{1-\varepsilon}|P_E\otimes_\beta P_E)$$
$$+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)\mathrm{d}N_t(2\pi|g_\varepsilon><g_\varepsilon|P_E\otimes_\beta P_E)$$

$$+\mathrm{i}T_\varepsilon(E)D_\varepsilon \mathrm{d}B_t^+(g_\varepsilon\otimes_\beta P_E g_{1-\varepsilon})+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)\mathrm{d}B_t^+(g_\varepsilon\otimes_\beta P_E g_\varepsilon)$$

$$+\mathrm{i}T_\varepsilon(E)D_\varepsilon \mathrm{d}B_t(g_{1-\varepsilon}\otimes_\beta P_E g_\varepsilon)+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)\mathrm{d}B_t(g_\varepsilon\otimes_\beta P_E g_\varepsilon)$$

$$+\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E) < g_\varepsilon, P_E L^2 g_\varepsilon > \mathrm{d}t\Big\}U_t \tag{20}$$

Proof. Let us consider the following term in (19)

$$\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)U_t B_{1-\varepsilon,\varepsilon'}(E,t)=n_{1-\varepsilon}(E)b_t^+(g_\varepsilon\otimes_\beta P_E g_{1-\varepsilon})U_t b_t(g_{1-\varepsilon}\otimes_\beta P_E g_{1-\varepsilon})$$
$$+n_\varepsilon(E)b_t^+(g_\varepsilon\otimes_\beta P_E g_\varepsilon)U_t b_t(g_{1-\varepsilon}\otimes_\beta P_E g_\varepsilon) \tag{21}$$

The matrix element of this expression on the exponential vectors $\psi(f)$, $\psi(f')$ is (we use Dirac's notations also for bra- and ket-vectors from $K_{0,1}$)

$$(n_{1-\varepsilon}(E) < f|g_\varepsilon\otimes_\beta P_E g_{1-\varepsilon} >< g_{1-\varepsilon}\otimes_\beta P_E g_{1-\varepsilon}|f' >$$
$$+n_\varepsilon(E) < f|g_\varepsilon\otimes_\beta P_E g_\varepsilon >< g_{1-\varepsilon}\otimes_\beta P_E g_\varepsilon|f' >) < \psi(f)|U_t|\psi(f') >$$
$$=< f|T_{\varepsilon,1-\varepsilon}(E)|f' >< \psi(f)|U_t|\psi(f') > \tag{22}$$

which is the time derivative of the matrix element $< \psi(f)|\mathrm{d}N_t(T_{\varepsilon,1-\varepsilon}(E))|\psi(f') >$ where $N_t(T_{\varepsilon,1-\varepsilon}(E))$ is the number process with intensity

$$T_{\varepsilon,1-\varepsilon}(E)=n_{1-\varepsilon}(E)|g_\varepsilon\otimes_\beta P_E g_{1-\varepsilon} >< g_{1-\varepsilon}\otimes_\beta P_E g_{1-\varepsilon}|$$
$$+n_\varepsilon(E)|g_\varepsilon\otimes_\beta P_E g_\varepsilon >< g_{1-\varepsilon}\otimes_\beta P_E g_\varepsilon|$$

Let us now prove that

$$T_{\varepsilon,1-\varepsilon}(E)=2\pi|g_\varepsilon >< g_{1-\varepsilon}|P_E\otimes_\beta P_E \tag{23}$$

To this goal let us consider the action of the $T_{\varepsilon,1-\varepsilon}(E)$ on vectors of the form

$$|f >=|P_{E_1}g_{\varepsilon_1}\otimes_\beta P_{E_2}g_{\varepsilon_2} > .$$

One has

$$T_{\varepsilon,1-\varepsilon}(E)|f >$$
$$=2\pi\delta_{\varepsilon,1-\varepsilon_1}\delta(E_1-E)\delta(E_2-E) < g_{1-\varepsilon}, P_E g_{1-\varepsilon} >$$
$$(\delta_{\varepsilon,1-\varepsilon_2}|g_\varepsilon\otimes_\beta P_E g_{1-\varepsilon} > +\delta_{\varepsilon,\varepsilon_2}|g_\varepsilon\otimes_\beta P_E g_\varepsilon >)$$
$$=2\pi\delta_{\varepsilon,1-\varepsilon_1} < g_{1-\varepsilon}, P_E g_{1-\varepsilon} > \delta(E_1-E)\delta(E_2-E)$$
$$(\delta_{\varepsilon,1-\varepsilon_2}|g_\varepsilon\otimes_\beta P_E g_{\varepsilon_2} > +\delta_{\varepsilon,\varepsilon_2}|g_\varepsilon\otimes_\beta P_E g_{\varepsilon_2} >)$$
$$=2\pi\delta_{\varepsilon,1-\varepsilon_1} < g_{1-\varepsilon}, P_E g_{1-\varepsilon} > \delta(E_1-E)\delta(E_2-E)|g_\varepsilon\otimes_\beta P_E g_{\varepsilon_2} >$$
$$=2\pi|g_\varepsilon >< g_{1-\varepsilon}|P_E\otimes_\beta P_E|f > \tag{24}$$

Therefore (23) holds and the term (21) corresponds to the number process

$$\mathrm{d}N_t(2\pi|g_\varepsilon >< g_{1-\varepsilon}|P_E\otimes_\beta P_E)$$

Computing the same matrix element for the term

$$\sum_{\varepsilon'=0,1} n_{\varepsilon'}(E)B^+_{\varepsilon,\varepsilon'}(E,t)U_tB_{\varepsilon,\varepsilon'}(E,t) = n_{1-\varepsilon}(E)b^+_t(g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon})U_t b_t(g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon})$$

$$+n_\varepsilon(E)b^+_t(g_\varepsilon \otimes_\beta P_E g_\varepsilon)U_t b_t(g_\varepsilon \otimes_\beta P_E g_\varepsilon) \qquad (25)$$

one finds an expression like (22) with $T_{\varepsilon,1-\varepsilon}$ replaced by the operator

$$T_{\varepsilon,\varepsilon}(E) = n_{1-\varepsilon}(E)|g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon} >< g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon}| + n_\varepsilon(E)|g_\varepsilon \otimes_\beta P_E g_\varepsilon >< g_\varepsilon \otimes_\beta P_E g_\varepsilon|$$

and a calculation similar to the one done in (24) leads to the conclusion that

$$T_{\varepsilon,\varepsilon}(E) = 2\pi|g_\varepsilon >< g_\varepsilon|P_E \otimes_\beta P_E.$$

Therefore the term (25) corresponds to the number process

$$\mathrm{d}N_t(2\pi|g_\varepsilon >< g_\varepsilon|P_E \otimes_\beta P_E).$$

This finishes the proof of the theorem.

Let us introduce the notations:

$$R_{\varepsilon,\varepsilon}(E) = -\gamma_{1-\varepsilon}(E)D_\varepsilon D_{1-\varepsilon}T_\varepsilon(E)$$
$$R_{\varepsilon,1-\varepsilon}(E) = -\mathrm{i}T_\varepsilon(E)D_\varepsilon.$$

In these notations the quantum stochastic differential equation for the evolution operator can be rewritten as

$$\mathrm{d}U_t = \sum_{\varepsilon=0,1}\int \mathrm{d}E\Big\{R_{\varepsilon,1-\varepsilon}(E)\mathrm{d}N_t(2\pi|g_\varepsilon >< g_{1-\varepsilon}|P_E \otimes_\beta P_E)$$

$$+R_{\varepsilon,\varepsilon}(E)\mathrm{d}N_t(2\pi|g_\varepsilon >< g_\varepsilon|P_E \otimes_\beta P_E)$$

$$+R_{\varepsilon,1-\varepsilon}(E)\mathrm{d}B^+_t(g_\varepsilon \otimes_\beta P_E g_{1-\varepsilon}) + R_{\varepsilon,\varepsilon}(E)\mathrm{d}B^+_t(g_\varepsilon \otimes_\beta P_E g_\varepsilon)$$

$$+R_{\varepsilon,1-\varepsilon}(E)\mathrm{d}B_t(g_{1-\varepsilon} \otimes_\beta P_E g_\varepsilon) + R_{\varepsilon,\varepsilon}(E)\mathrm{d}B_t(g_\varepsilon \otimes_\beta P_E g_\varepsilon)$$

$$+R_{\varepsilon,\varepsilon}(E) < g_\varepsilon, P_E L^2 g_\varepsilon > \mathrm{d}t\Big\}U_t.$$

Notice that the quantum stochastic differential equation (20) can be written also in the Frigerio-Maassen form [15]. In order to prove this recall that for any pair of Hilbert spaces $\mathcal{X}_0, \mathcal{X}_1$ if N, A denote the number and annihilation processes on the Fock space $\mathcal{F}(\mathcal{X}_1)$ then for $X_0 \in B(\mathcal{X}_0)$, $X_1 \in B(\mathcal{X}_1)$, $x \in \mathcal{X}_1$, Frigerio and Maassen [15] introduced the notation:

$$N(X_0 \otimes X_1) := X_0 \otimes N(X_1)$$
$$A(X_0 \otimes X_1 x) := X_0 \otimes A(X_1 x)$$
$$< x, X_0 \otimes X_1 x >:= X_0 \otimes 1 < x, X_1 x >$$

Let us also introduce an operator $T_3(E)$ acting on the triple $\mathcal{H}_S \otimes K \otimes_\beta K$ (this is the reason for introducing index 3) as

$$T_3(E) := 2\pi \sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon,\varepsilon'}(E) \otimes |g_\varepsilon >< g_{\varepsilon'}| P_E \otimes_\beta P_E$$

and the vector $\xi(E) \in K \otimes_\beta K$

$$\xi(E) := \frac{1}{2\pi} \sum_{\varepsilon=0,1} \frac{1}{< g_\varepsilon, P_E g_\varepsilon >} g_\varepsilon \otimes_\beta g_\varepsilon.$$

In these notations equation (20) can be written as

$$\mathrm{d}U_t = \int \mathrm{d}E \Big(\mathrm{d}N_t(T_3(E)) + \mathrm{d}B_t^+(T_3(E)\xi(E)) \\ + \mathrm{d}B_t(T_3^*(E)\xi(E)) + < \xi(E), T_3(E)\xi(E) > \mathrm{d}t \Big) U_t.$$

8. Connection with scattering theory

Here we consider relation between the evolution operator and scattering theory. Because of the number conservation, the closed subspace of $\mathcal{H}_S \otimes \mathcal{F}$ generated by vectors of the form $u \otimes A^+(f)\Phi$ $(u \in \mathcal{H}_\mathrm{S},\ f \in \mathcal{H}_1 = L^2(\mathbb{R}^d))$ which is naturally isomorphic to $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_1$, is globally invariant under the time evolution operator $\exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H_\mathrm{R} + V)t]$. Therefore the restriction of the time evolution operator to this subspace corresponds to an evolution operator on $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_1$ given by

$$\exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H_1 + V_1)t]$$

where

$$V_1 = \sum_{\varepsilon=0,1} D_\varepsilon \otimes |g_\varepsilon >< g_{1-\varepsilon}| \tag{26}$$

The 1-particle Møller wave operators are defined as

$$\Omega_\pm = s - \lim_{t\to\pm\infty} \exp[\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H + V_1)t] \exp[-\mathrm{i}(H_\mathrm{S} \otimes 1 + 1 \otimes H)t]$$

and the 1-particle T-operator is defined as

$$T = V_1 \Omega_+ \tag{27}$$

From (26) it follows that

$$\Omega_\pm = s - \lim_{t\to\pm\infty} U_t^{(1)}$$

where $U_t^{(1)}$ is the solution of

$$\partial_t U_t^{(1)} = -\mathrm{i} \left(\sum_{\varepsilon=0,1} D_\varepsilon \otimes |S_t g_\varepsilon >< S_t g_{1-\varepsilon}| \right) U_t^{(1)} \qquad U_0^{(1)} = 1.$$

In order to make a connection between the stochastic process U_t and scattering theory notice that the operator $T_3(E)$ can be written as

$$T_3(E) = 2\pi T(E) \otimes_\beta P_E$$

where operator $T(E)$ acts on $\mathcal{H}_{\mathrm{S}} \otimes K$ as

$$T(E) = \sum_{\varepsilon,\varepsilon'=0,1} R_{\varepsilon,\varepsilon'}(E) \otimes |g_\varepsilon >< g_{\varepsilon'}| P_E.$$

In [9] it was proved that T-operator defined by (27) connected with $T(E)$ by the following formula

$$T = \int \mathrm{d}E T(E).$$

9. Vacuum expectation value

For the vacuum matrix element of the evolution operator from (19) one immediately gets

$$< U(t) >_{vac} = \mathrm{e}^{-\Gamma t}. \tag{28}$$

The operator Γ acts in $\mathcal{H}_{\mathrm{S}}$ as

$$\Gamma = \sum_{\varepsilon=0,1} \int \mathrm{d}E \gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) < g_\varepsilon, P_E L^2 g_\varepsilon > .$$

Theorem 5 *The operator* Γ *has a non-negative real part (i.e. this operator describes the damping).*

Proof. From the definition of $T_\varepsilon(E)$ we know that

$$\gamma_{1-\varepsilon}(E) D_\varepsilon D_{1-\varepsilon} T_\varepsilon(E) = \frac{D_\varepsilon D_{1-\varepsilon}}{\gamma_{1-\varepsilon}^{-1}(E) + \gamma_\varepsilon(E) D_\varepsilon D_{1-\varepsilon}}$$

But γ_ε and $\gamma_{1-\varepsilon}$ (hence also $1/\gamma_{1-\varepsilon}$) have positive real part and $D_\varepsilon D_{1-\varepsilon}$ is positive self–adjoint.

Hence the above expression has a positive real part because it is of the form:

$$\frac{H}{z_1 + z_2 H} = \frac{H(\mathrm{Re}\ z_1 + H\ \mathrm{Re}\ z_2)}{|z_1 + z_2 H|^2} - \mathrm{i}\frac{H(\mathrm{Im}\ z_1 + H\ \mathrm{Im}\ z_2)}{|z_1 + z_2 H|^2}$$

where H is positive self–adjoint and z_1, z_2 have a positive real part. Since $\langle g_\varepsilon, P_E L^2 g_\varepsilon \rangle \geq 0$ the thesis follows.

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