DICII Technical report RR-13.02

Appendix to "Torque setpoint tracking for parallel hybrid electric vehicles using dynamic input allocation", published on IEEE Transactions on Control Systems Technology

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I. Input allocation with linear dynamic redundancy

A dynamic allocator is proposed in this section to generalize the strategy for input redundant plants introduced in [1] and extended in [2]. The approach of [1] applies to linear plants with multiple and redundant inputs. The theory is extended here to the case of multiple linear actuators, each of them with its own dynamics, acting on a nonlinear plant with strong input redundancy. In the HEV case the two redundant inputs are the ICE and EM torques and the two actuators with different dynamics are the two propulsion systems.

Referring to Figure 3, the "Plant" block corresponds to the following nonlinear system:

$$\begin{aligned} \dot{x}_p &= f_p(t, x_p, Bu_p, d) \\ y_p &= h_p(t, x_p, Du_p, d) \end{aligned} \tag{1}$$

where $u_p \in \mathbb{R}^{n_u}$ is the control input, $d \in \mathbb{R}^{n_d}$ is a disturbance input and $y_p \in \mathbb{R}^{n_y}$ is the plant output. The block "Actuators" is a diagonal and square linear time invariant (LTI) system representing the dynamics of the n_u linear actuators. These actuators establish the following relationship (2) at the control input of the plant, which is reported here for the easier reading:

$$u_p(s) = G(s)u(s) = \operatorname{diag}\{g_i(s)\}\, u(s) \tag{2}$$

where $g_i(s)$, $i = 1, ..., n_u$ are proper asymptotically stable transfer functions. The block "Controller" is the following a-priori given nonlinear controller:

$$\begin{aligned} \dot{x}_c &= f_c(t, x_c, y_p, r) \\ y_c &= h_c(t, x_c, y_p, r) \end{aligned} \tag{3}$$

where $r \in \mathbb{R}^{n_r}$ is the reference input and $y_c \in \mathbb{R}^{n_u}$ is the controller output. The following natural assumption is made on the closed-loop of Figure 3 in the absence of the "Allocator" block and on the actuators transfer functions in (2). Note that the assumption that $g_i(0) > 0$ can always be guaranteed as long as $g_i(0) \neq 0$ by possibly inverting the sign of the corresponding columns in $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$, namely $g_i(0) > 0$, $\forall i \in \{1, ..., n_u\}$.

Assumption 1. The closed loop system before allocation or *unallocated closed loop*, given by (1), (2), (3) with $u = y_c$ and $r = const$, is well posed and has an unique globally asymptotically stable equilibrium. Moreover, all the actuators transfer functions in (2) are asymptotically stable and are strictly positive at zero. ◦

Mimicking the approach of [1], it is assumed that the matrices B and D filtering the control input u_p of the nonlinear plant (1) characterize a strong input redundancy in the sense that $\ker(B) \cap \ker(D) \neq \emptyset$. Based on this property, a nonempty full column rank matrix $B_\perp$ can be defined satisfying:

$$\operatorname{Im}(B_\perp) = \ker(B) \cap \ker(D) \tag{4}$$

where clearly $\operatorname{rank}(B_\perp) = n_u - \operatorname{rank}(\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right])$.

Definition 1. Two constant vectors $y_c, u_0 \in \mathbb{R}^{n_u}$ are statically compatible with respect to the strongly input redundant plant (1) and the actuators (2) if they satisfy relation (8), which is reported here for the easier reading:

$$\begin{bmatrix} B \\ D \end{bmatrix} G(0)(y_c - u_0) = 0 \tag{5}$$

◦

The structure of the proposed dynamic allocator is shown in Figure 4 and corresponds to the linear dynamics (6), represented in the Laplace domain, which is reported here for the easier reading:

$$w = \frac{1}{s} \operatorname{sat}_M(K B_\perp' \bar{G}(0) W(s)(u - u_0)) \tag{6a}$$

$$\delta_u = -\bar{G}(s) B_\perp w \tag{6b}$$

$$u = y_c + \delta_u \tag{6c}$$

where $w \in \mathbb{R}^{n_a}$ is the state of the dynamic allocator and δ_u is the input variation imposed by the allocator. The function sat_M: $R^{n_a} \rightarrow \mathbb{R}^{n_a}$ is a symmetric decentralized vector saturation having saturation limits $M = \begin{bmatrix} m_1 & \dots & m_{n_a} \end{bmatrix}'$, which specify the maximum allocation rates. The diagonal matrix $K > 0$ on the other hand specifies the local exponential convergence rate of the allocator. Moreover, $\bar{G}(s)$ and $W(s)$ are diagonal and proper matrix transfer functions selected in such a way that there exists a scalar asymptotically stable transfer function $\kappa(s)$ satisfying (7) (here reported for the easier reading):

$$G(s)\bar{G}(s) = \bar{G}(s)G(s) = I\kappa(s) \tag{7a}$$
$$\kappa(0) \neq 0 \tag{7b}$$
$$W(0) = W_0 > 0, \tag{7c}$$

where equality (7a) easily follows from the diagonal structure of $G(s)$ and $\bar{G}(s)$.

Remark 1. From the first equation in (7), and the definition of $B_\perp$ in (4), if two vectors y_c and u_0 are statically compatible, then there exists a third vector w^* satisfying:

$$y_c - u_0 = \bar{G}(0)B_\perp w^*, \tag{8}$$

indeed from (5) there exists $\bar{w}$ such that $B_\perp \bar{w} = G(0)(y_c - u_0)$ and (8) holds also using (7b), with $w^* = \bar{w}/\kappa(0)$. ○

Remark 2. If $\bar{G}(s) = I$ and $W(s) = W_0 > 0$ diagonal, then the dynamic allocator (6) reduces to the structure proposed in [1]. ○

The following theorem formalizes two fundamental properties of the *allocated closed loop* (1), (2), (3), (6) and represented in Figure 3. These properties hold under the assumption that the closed loop in Figure 4 satisfies the following property. Later in Procedure 1, we will provide constructive technique for selecting $\bar{G}(s)$ and $W(s)$ to guarantee this property whenever the actuators dynamics is minimum phase.

Property 1. The parameters of allocator (6) satisfy (4) and (7) and are such that $\bar{G}(s)$ and $W(s)$ are proper[1]. Moreover for any constant selection of y_c and u_0, it has a globally asymptotically stable (GAS) and locally exponentially stable (LES) equilibrium point. ○

The first item of Theorem 1 ensures that the allocator does not enforce any modification at the plant state x_p and output y_p but only affects the actuator input u (and the plant input u_p). The second property ensures that the allocator is capable of asymptotically tracking constant actuator input references u_0 as long as they are statically compatible with the controller output y_c (in the sense of Definition 1) generated by the controller before allocation. The proof of the theorem is reported in Section II.

Theorem 1. Consider the allocated closed-loop (1), (2), (3), (6) satisfying Property 1 and the unallocated closed-loop (1), (2), (3), $u = y_c$ satisfying Assumption 1. Initializing the allocator dynamics (6) at zero, the following holds:

1) using the same plant-controller initial conditions and the same external signals r, d, the plant state and output responses (x_p, y_p) produced by the unallocated closed-loop and by the allocated closed-loop coincide;
2) given any constant u_0 and a steady-state controller output y_c generated by the unallocated closed-loop if (y_c, u_0) is a statically compatible pair (in the sense of Definition 1), then the plant input response u of the allocated closed-loop from the same initial conditions satisfies $\lim_{t\to\infty} u(t) = u_0$.

○

In the following procedure, under the assumption that all the diagonal elements in (2) are minimum phase, a procedure is proposed for the selection of the diagonal matrix transfer functions $\bar{G}(s)$ and $W(s)$ in (6), guaranteeing Property 1.

Procedure 1.

Inputs: $G(s) = \text{diag}\{g_i(s)\}$ diagonal and proper matrix transfer function where $g_i(s)$, $i = 1, ..., n_u$, have all poles and zeros with strictly negative real part; $B_\perp$ satisfying (4); $K > 0$ and $W_0 > 0$ diagonal matrices.

Step 1: (Least common multiple). Define $\bar{g}_i(s) = \prod_{j\neq i} g_j(s)$, $i = 1, ..., n_u$ and select $\bar{G}(s) := \text{diag}\{\bar{g}_i(s)\}$. If the allocator (6) satisfies Property 1 [2], then select $W(s) = W_0$ and terminate, otherwise repeat the following steps for $i = 1, ..., n_u$.

Step 2: (Relative degree). If the relative degree of $\bar{g}_i(s)\frac{1}{s}$ is equal to one[3] then select $w_{1,i}(s)$ as the $i-$th diagonal component of W_0. Else define $w_{1,i}(s)$ as the i-th diagonal component of W_0 times a number of stable zeros such that $w_{1,i}(s)\bar{g}_i(s)\frac{1}{s}$ has relative degree one and, without loss of generality $w_{1,i}(0) = 1$. Go to Step 3.

Step 3: (Positive realness). If $w_{1,i}(s)\bar{g}_i(s)\frac{1}{s}$ is positive real then $w_{2,i}(s) := 1$. Else define $w_{2,i}(s)$ as a cascade of biproper, stable and minimum phase lead and/or lag compensators such that $w_{2,i}(s)w_{1,i}(s)\bar{g}_i(s)\frac{1}{s}$ is positive real and without loss of generality $w_{2,i}(0) = 1$. Go to Step 4.

Step 4: (Proper $W(s)$). If $w_{1,i}(s)w_{2,i}(s)$ is proper then $w_{3,i}(s) := 1$. Else define $w_{3,i}(s)$ with a sufficient number of high - frequency stable poles such that $w_{3,i}(s)w_{2,i}(s)w_{1,i}(s)$ is proper.

Output: $\bar{G}(s)$, $W(s) := \text{diag}\{w_{3,i}(s)w_{2,i}(s)w_{1,i}(s)\}$ and $\kappa(s) = \prod_{i=1}^{n_u} g_i(s)$

⋆

The following theorem establishes that Procedure 1 always leads to an allocator satisfying Property 1. The proof is passivity-based and hinges upon the property that the construction in Procedure 1 ensures that the lower system in Figure 1 (which is an equivalent representation of the closed loop in Figure 4) is positive real. Then, from the passivity theorem, there exists a single GAS and LES equilibrium for its interconnection with the upper system (a saturation function). The proof of the theorem is reported in the Section II.

[1]The properness assumption on $\bar{G}(s)$ and $W(s)$ ensures that (6) is causal, hence physically realizable.

[2]As shown in the proof, $\bar{g}_i(s)\frac{1}{s}$ positive real $\forall i = 1, ..., n_u$ is a sufficient condition to have an allocator (6) respecting the Property 1.

[3]Note that, due to the selection of $\bar{g}_i(s)$ in Step 1, $\bar{g}_i(s)/s$ is always strictly proper.

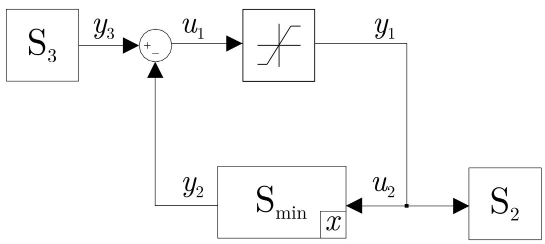

Fig. 1. Equivalent block diagram of the internal feedback of the proposed dynamic allocator for the case $y_c - u_0 = 0$.

Theorem 2. If each diagonal element of $G(s)$ in (2) is minimum phase and asymptotically stable, then Procedure 1 always leads to a set of parameters $\bar{G}(s)$, $W(s)$ and $\kappa(s)$ satisfying Property 1. ◦

Remark 3. Since the proof of Theorem 2 is based on the passivity of the interconnection in Figure 1, then, as long as the procedure did not terminate at Step 1 , the allocator gain $K = K' > 0$ can be selected arbitrarily large while preserving closed-loop stability. This property allows for a useful tuning of the dynamic allocator speed without needing to worry about stability. ◦

Remark 4. The main extension given here from the dynamic allocator proposed in [1] is the presence of the new dynamics $\bar{G}(s)$ in Figure 4. This transfer matrix is based on a model of the actuators dynamics and has the goal of making w completely invisible from $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] u_p$. This goal is obtained by the property in equation (7) which ensures that w is filtered by some common multiple $\kappa(s) = \prod_{i=1}^{n_u} g_i(s)$ of all the actuators dynamics. ◦

II. Proofs of the main results

Proof 1. *Item 1.* By linearity of the actuators and of the allocator (6), it is sufficient to show that the transfer function from w to $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right] u_p$ is zero. As a matter of fact, the allocator contribution at the plant input will consequently be zero, thus fully preserving the unallocated closed-loop response from the point of view of the plant state and output. This property follows from the following calculations arising from the second equation in (6), from (7), (4) and (2):

$$\begin{bmatrix} B \\ D \end{bmatrix} G(s)\bar{G}(s)B_\perp w = \kappa(s) \begin{bmatrix} B \\ D \end{bmatrix} B_\perp w = 0 \quad \forall w \in \mathbb{R}^{n_a} \tag{9}$$

Item 2. By Property 1, for each constant selection of y_c, u_0, all signals in (6) converge. Then also w converges to a steady-state value $\bar{w}$. In particular, using the fact that $\dot{w} = 0$, at the steady-state (namely replacing s by 0 in the transfer functions):

$$0 = \mathrm{sat}_M(KB_\perp^{'}\bar{G}(0)W(0)(y_c - u_0 - \bar{G}(0)B_\perp\bar{w})) \tag{10a}$$
$$= \mathrm{sat}_M(KB_\perp^{'}\bar{G}(0)W(0)\bar{G}(0)B_\perp(w^* - \bar{w})) \tag{10b}$$

where, according to Remark 1, w^* satisfies (8). Since $K > 0$, $\bar{G}(0)W(0)\bar{G}(0) > 0$ from (7), and $B_\perp$ is full column rank by assumption, then the matrix $KB_\perp^{'}\bar{G}(0)W(0)\bar{G}(0)B_\perp$ is non singular, which together with (10) implies $\bar{w} = w^*$. Finally, using (8), the steady-state value of u can be computed as $\bar{u} = y_c - \bar{\delta}_u = y_c - \bar{G}(0)B_\perp w^* = u_0$ which completes the proof. ◦

Proof 2. To prove the theorem, it will be first shown in Step A that the procedure always terminates with parameters satisfying (7), with proper, therefore causal, $\bar{G}(s)$ and $W(s)$. Then it is shown in Step B that for any constant exogenous signal, the dynamics (6) with $W(s) = \mathrm{diag}\{w_{2,i}(s)w_{1,i}(s)\}$ has a single GAS and LES equilibrium. Finally in Step C it will be shown that the same property holds with $W(s)$ coming from Procedure 1, thus completing the proof of Property 1.

Step A. If the procedure terminates at Step 1, then (7a) holds with $\kappa(s) = \prod_{i=1}^{n_u} g_i(s)$ by the definition of $\bar{G}(s)$, (7b) holds by Assumption 1, (7c) holds by definition of W_0 and the proof is completed because Property 1 holds. Otherwise, Step 2 of the procedure can be completed because all the functions $g_i(s)$ (therefore also $\bar{g}_i(s)\frac{1}{s}$) are proper. Step 3 can be completed because, by Assumption 1, all the functions $w_{1,i}(s)\bar{g}_i(s))$ are asymptotically stable and minimum phase and the residual of $w_{1,i}(s)\bar{g}_i(s)\frac{1}{s}$ relative to $s = 0$ is $w_{1,i}(0)\bar{g}_i(0) > 0$. Then it is always possible to correct the phase of $w_{1,i}(s)\bar{g}_i(s)\frac{1}{s}$ by suitable biproper stable minimum phase lead or lag filters and the relative degree property ensured at Step 2 together with the phase property ensured at Step 3, guarantees that $w_1(s)w_2(s)w_3(s)\bar{g}_i(s)\frac{1}{s}$ is positive real. Finally Step 4 can always be performed, and it is ensured that $W(s)$ and $\bar{G}(s)$ are proper in addition to stable. Note that (7) still holds since (7a) and (7b) only depend on $\bar{G}(s)$ and (7c) holds because $w_{j,i}(0) > 0$, $j = 1, 2, 3$.

Step B. Since the proof has already been completed in the case when the procedure stops at Step 1, in the following it is assumed that Steps 2-4 are performed. First note that, referring to Figure 4, given the response of (6) with $y_c - u_0$ constant and some fixed initial conditions, it is always possible to find another set of initial conditions for w, the block $-\bar{G}(s)B_\perp$ and $W(s)$ such that the state response of (6) with $y_c - u_0 = 0$ is the same. Then from this equivalence, in the following it will be assumed that $y_c - u_0 = 0$. Now consider the function $\hat{H}(s) := KB_\perp^{'}\bar{G}(0)\mathrm{diag}\{w_{2,i}(s)w_{1,i}(s)\}\bar{G}(s)B_\perp\frac{1}{s}$, define S_{min} as its minimal realization with state x and consider its negative interconnection with $\mathrm{sat}_M(\cdot)$ (which is shown in Figure 1 when $y_3 = 0$ and S_2, S_3 are not considered). Since $\hat{H}(s)$ is positive real by the construction at Step 3, then (see [3, Chapter 6]) there exists a matrix $P = P' > 0$ such that the function $V(x) := x'Px$ satisfies $\dot{V}(x) \leq 0$. This function is also a Lyapunov function proving stability and passivity of the negative feedback interconnection of $\hat{H}(s)$ and $\mathrm{sat}_M(\cdot)$. $\dot{V}$ is only negative semi-definite in x but the largest invariant set in $\left\{x : \dot{V}(x) = 0\right\}$ is the origin because $\hat{H}(s)$ consists in a pool of integrators times a nonsingular matrix transfer function. Then, from LaSalle's invariance principle [3, Corollary 4.2], the origin is GAS and LES for the negative feedback interconnection between $\mathrm{sat}_M(\cdot)$ and the minimal realization of $\hat{H}(s)$. Consider now the actual (non-necessarily minimal) realization of $\hat{H}(s)$ composed by S_{min} and S_2, S_3 which represent, respectively, the uncontrollable and unreachable exponentially stable (ES) parts. Since S_2, S_3 are ES so that their outputs converge to zero, GAS and LES of the cascade system can be proven by establishing global boundedness and applying [4]. To establish global boundedness, first note that the saturation output y_1 is bounded by $|y_1| \leq \sqrt{n_a}|M|_\infty$ therefore the trajectories of the exponentially stable system S_2 remain bounded. Regarding the remaining states, note that from passivity of S_{min}, there exists $U(x)$ positive definite and radially unbounded such that $\dot{U}(x) \leq u_2 y_2$. Moreover, since by the properties of $\mathrm{sat}_M(\cdot)$ it follows that $u_1 y_1 \geq 0$, then:

$$\dot{U} \leq u_2 y_2 + u_1 y_1 = y_1 y_3 \leq |y_1||y_3|. \tag{11}$$

Denote now by $x_{3,0}$ the initial conditions of S_3, it follows from ES of S_3 that, for some positive scalars k_3, λ_3, $|y_3| \leq k_3 \exp(-\lambda_3 t)|x_{3,0}|$. Therefore, integrating (11) it follows that $U(x(t)) \leq U(x_0) + (\sqrt{n_a}|M|_\infty k_3 |x_{3,0}|)/\lambda_3$, which implies global boundedness of x.

Step C. Similar to Step B, define $H(s) := KB_\perp^{'}\bar{G}(0)W(s)\bar{G}(s)B_\perp \frac{1}{s}$ and note that its negative interconnection with $\mathrm{sat}_M(\cdot)$ is equivalent to the scheme of Figure 4 with $y_c - u_0 = 0$. $H(s)$ differs from $\hat{H}(s)$ only by a number of high frequency asymptotically stable poles corresponding to $\mathrm{diag}\{w_{3,i}(s)\}$ constructed at Step 4. Then the closed-loop between $H(s)$ and $\mathrm{sat}_M(\cdot)$ is a perturbation of the asymptotically stable closed loop between $\hat{H}(s)$ and $\mathrm{sat}_M(\cdot)$ by way of a high frequency strictly proper filter. Then, since $H(s)$ is strictly proper (due to Step 2 of Procedure 1), so that there is no algebraic loop in the feedback interconnection, as long as the high-frequency poles of $\mathrm{diag}\{w_{3,i}(s)\}$ are sufficiently fast, asymptotic stability of the feedback interconnection is preserved by the results in [5, Section 4.7] (see also a similar discussion in [6]). ○

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