THE QUANTUM BLACK-SCHOLES EQUATION

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ABSTRACT. Motivated by the work of Segal and Segal in [16] on the Black-Scholes pricing formula in the quantum context, we study a quantum extension of the Black-Scholes equation within the context of Hudson-Parthasarathy quantum stochastic calculus,. Our model includes stock markets described by quantum Brownian motion and Poisson process.

1. The Merton-Black-Scholes Option Pricing Model

An option is a ticket which is bought at time t = 0 and which allows the buyer at (in the case of European call options) or until (in the case of American call options) time t = T (the time of maturity of the option) to buy a share of stock at a fixed exercise price K. In what follows we restrict to European call options. The question is: how much should one be willing to pay to buy such an option? Let X_T be a reasonable price. According to the definition given by Merton, Black, and Scholes (M-B-S) an investment of this reasonable price in a mixed *portfolio* (i.e part is invested in stock and part in bond) at time t = 0, should allow the investor through a self-financing strategy (i.e. one where the only change in the investor's wealth comes from changes of the prices of the stock and bond) to end up at time t = T with an amount of $(X_T - K)^+ := \max(0, X_T - K)$ which is the same as the payoff, had the option been purchased (cf. [12]). Moreover, such a reasonable price allows for no arbitrage i.e., it does not allow for risk free unbounded profits. We assume that there are no transaction costs and that the portfolio is not made smaller by consumption. If $(a_t, b_t), t \in [0, T]$ is a self-financing trading strategy (i.e an amount a_t is invested in stock at time t and an amount b_t is invested in bond at the same time) then the value of the portfolio at time t is given by $V_t = a_t X_t + b_t \beta_t$ where, by the self-financing assumption, $dV_t = a_t dX_t + b_t d\beta_t$. Here X_t and β_t denote, respectively, the price of the stock and bond at time t. We assume that $dX_t = c X_t dt + \sigma X_t dB_t$ and $d\beta_t = \beta_t r dt$ where B_t is classical Brownian motion, r > 0 is the constant interest rate of the bond, c > 0 is the mean rate of return, and $\sigma > 0$ is the volatility of the stock. The assets a_t and b_t are in general stochastic processes. Letting $V_t = u(T - t, X_t)$ where $V_T = u(0, X_T) = (X_T - K)^+$ it can be shown (cf. [12]) that u(t, x) is the solution of the Black-Scholes equation

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$$\frac{\partial}{\partial t}u(t,x) = (0.5\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + r x \frac{\partial}{\partial x} - r) u(t,x)$$
$$u(0,x) = (X_T - K)^+, \quad x > 0, \ t \in [0,T]$$

and it is explicitly given by

$$u(t,x) = x \Phi(g(t,x)) - K e^{-rt} \Phi(h(t,x))$$

where

$$g(t,x) = (\ln(x/K) + (r+0.5\,\sigma^2)\,t)(\sigma\,\sqrt{t})^{-1}, \quad h(t,x) = g(t,x) - \sigma\sqrt{t}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n+1}}{2n+1}.$$

Thus a reasonable (in the sense described above) price for a European call option is

$$V_0 = u(T, X_0) = X_0 \Phi(g(T, X_0)) - K e^{-rT} \Phi(h(T, X_0))$$

and the self-financing strategy $(a_t, b_t), t \in [0, T]$ is given by

$$a_t = \frac{\partial}{\partial x}u(T-t, X_t), \quad b_t = \frac{u(T-t, X_t) - a_t X_t}{\beta_t}.$$

2. Quantum Extension of the M-B-S Model

In recent years the fields of Quantum Economics and Quantum Finance have appeared in order to interpret erratic stock market behavior with the use of quantum mechanical concepts (cf. [3], [4],[6]-[9], [11], and [14]-[16]). While no approach has yet been proved prevalent, in [16] Segal and Segal introduced quantum effects into the Merton-Black-Scholes model in order to incorporate market features such as the impossibility of simultaneous measurement of prices and their instantaneous derivatives. They did that by adding to the Brownian motion B_t used to represent the evolution of public information affecting the market, a process Y_t which represents the influence of factors not simultaneously measurable with those involved in B_t . They then sketched a calculus for dealing with such processes. Segal and Segal concluded that the combined process $a B_t + b Y_t$ may be represented as (in their notation) $\Phi\left((a + ib) \chi_{[0,t]}\right)$ where for a Hilbert space element f, $e^{i \Phi(f)}$ is the corresponding Weyl operator, and $\chi_{[0,t]}$ is the characteristic function of the interval [0, t]. In the context of the Hudson-Parthasarathy quantum stochastic calculus of [10] and [13] (see Theorem 20.10 of [13]) simple linear combinations of $\Phi(f)$ and $\Phi(i f)$ define the Boson Fock space annihilator

and creator operators A_f and A_f^{\dagger} . Segal and Segal used $\Phi(\chi_{[0,t]})$ as the basic integrator process with integrands restricted to a special class of exponential processes. In view of the above reduction of Φ to A and A^{\dagger} , it makes sense to study option pricing using as integrators the annihilator and creator processes of Hudson-Parthasarathy quantum stochastic calculus, thus exploiting its much larger class of integrable processes than the one considered in [16]. The Hudson-Parthasarathy calculus has a wide range of applications. For applications to, for example, control theory we refer to [2], [5] and the references therein. Quantum stochastic calculus was designed to describe the dynamics of quantum processes and we propose that we use it to study the non commutative Merton-Black-Scholes model in the following formulation (notice that our model includes also the Poisson process): We replace (see [1] for details on quantization) the stock process $\{X_t / t \ge 0\}$ of the classical Black-Scholes theory by the quantum mechanical process $j_t(X) = U_t^* X \otimes 1 U_t$ where , for each $t \ge 0$, U_t is a unitary operator defined on the tensor product $\mathcal{H} \otimes \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$ of a system Hilbert space \mathcal{H} and the noise Boson Fock space $\Gamma = \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$ satisfying

(2.1)
$$dU_t = -\left(\left(iH + \frac{1}{2}L^*L\right)dt + L^*S dA_t - L dA_t^{\dagger} + (1-S) d\Lambda_t\right)U_t, \ U_0 = 1$$

where X > 0, H, L, S are in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on \mathcal{H} , with S unitary and X, H self-adjoint. We identify time-independent, bounded, system space operators x with their ampliation $x \otimes 1$ to $\mathcal{H} \otimes \Gamma(L^2(\mathbf{R}_+, \mathcal{C}))$. The value process V_t is defined for $t \in [0, T]$ by $V_t = a_t j_t(X) + b_t \beta_t$ with terminal condition $V_T = (j_T(X) - K)^+ = \max(0, j_T(X) - K)$, where K > 0 is a bounded self-adjoint system operator corresponding to the strike price of the quantum option, a_t is a real-valued function, b_t is in general an observable quantum stochastic processes (i.e. b_t is a self-adjoint operator for each $t \geq 0$) and $\beta_t = \beta_0 e^{tr}$ where β_0 and r are positive real numbers. Therefore $b_t = (V_t - a_t j_t(X)) \beta_t^{-1}$. We interpret the above in the sense of expectation i.e. given $u \otimes \psi(f)$ in the exponential domain of $\mathcal{H} \otimes \Gamma$, where we will always assume $u \neq 0$ so that $||u \otimes \psi(f)|| \neq 0$,

$$< u \otimes \psi(f), V_t u \otimes \psi(f) > = a_t < u \otimes \psi(f), j_t(X) u \otimes \psi(f) > + < u \otimes \psi(f), b_t u \otimes \psi(f) > \beta_t$$

(i.e the value process is always in reference to a particular quantum mechanical state, so we can eventually reduce to real numbers) and

$$< u \otimes \psi(f), V_T u \otimes \psi(f) > = < u \otimes \psi(f), (j_T(X) - K)^+ u \otimes \psi(f) >$$
$$= \max(0, < u \otimes \psi(f), (j_T(X) - K) u \otimes \psi(f) >).$$

As in the classical case we assume that the portfolio $(a_t, b_t), t \in [0, T]$ is self-financing i.e

$$dV_t = a_t \, dj_t(X) + b_t \, d\beta_t$$

or equivalently

$$da_t \cdot j_t(X) + da_t \cdot dj_t(X) + db_t \cdot \beta_t + db_t \cdot d\beta_t = 0.$$

Remark 1.

The fact that the value process (and all other operator processes X_t appearing in this paper) is always in reference to a particular quantum mechanical state, allows for a direct translation of all classical financial concepts described in Section 1 to the quantum case by considering the expectation (or matrix element) $\langle u \otimes \psi(f), X_t u \otimes$ $\psi(f) >$ of the process at each time t. If the process is classical (i.e., if $X_t \in \mathbb{R}$) then we may divide out $||u \otimes \psi(f)||^2$ and everything is reduced to the classical case described in Section 1.

Lemma 1. Let $j_t(X) = U_t^* X \otimes 1 U_t$ where $\{U_t / t \ge 0\}$ is the solution of (2.1). If

$$\alpha = [L^*, X] S, \ \alpha^{\dagger} = S^* [X, L], \ \lambda = S^* X S - X,$$

and

$$\theta = i [H, X] - \frac{1}{2} \{ L^* L X + X L^* L - 2 L^* X L \}$$

then

(2.2)
$$dj_t(X) = j_t(\alpha^{\dagger}) dA_t^{\dagger} + j_t(\lambda) d\Lambda_t + j_t(\alpha) dA_t + j_t(\theta) dt$$

and for $k \geq 2$

$$(2.3) \ (dj_t(X))^k = j_t(\lambda^{k-1} \alpha^{\dagger}) \, dA_t^{\dagger} + j_t(\lambda^k) \, d\Lambda_t + j_t(\alpha \lambda^{k-1}) \, dA_t + j_t(\alpha \lambda^{k-2} \alpha^{\dagger}) \, dt$$

Proof. Equation (2.2) is a standard result of quantum flows theory (cf. [13]). To prove (2.3) we notice that for k = 2, using (2.2), the Itô table

	dA_t^{\dagger}		dA_t	dt
dA_t^{\dagger}	$egin{array}{c} 0 \\ dA_t^\dagger \\ dt \\ 0 \end{array}$	0	0	0
$d\Lambda_t$	dA_t^{\dagger}	$d\Lambda_t$	0	0
dA_t	dt	dA_t	0	0
dt	0	0	0	0

and the homomorhism property $j_t(x y) = j_t(x) j_t(y)$, we obtain

 $(dj_t(X))^2 = dj_t(X) dj_t(X) = j_t(\lambda \alpha^{\dagger}) dA_t^{\dagger} + j_t(\lambda^2) d\Lambda_t + j_t(\alpha \lambda) dA_t + j_t(\alpha \alpha^{\dagger}) dt$ so (2.3) is true for k = 2. Assuming (2.3) to be true for k we have

$$\begin{aligned} (dj_t(X))^{k+1} &= dj_t(X) \ (dj_t(X))^k \\ &= dj_t(X) \ \left(j_t(\lambda^{k-1} \alpha^{\dagger}) \ dA_t^{\dagger} + j_t(\lambda^k) \ d\Lambda_t + j_t(\alpha \lambda^{k-1}) \ dA_t + j_t(\alpha \ \lambda^{k-2} \ \alpha^{\dagger}) \ dt \right) \\ &= j_t(\lambda^k \ \alpha^{\dagger}) \ dA_t^{\dagger} + j_t(\lambda^{k+1}) \ d\Lambda_t + j_t(\alpha \lambda^k) \ dA_t + j_t(\alpha \ \lambda^{k-1} \ \alpha^{\dagger}) \ dt \end{aligned}$$

Thus (2.3) is true for k + 1 also.

3. Derivation of the Quantum Black-Scholes Equation

In the spirit of the previous section, let $V_t := F(t, j_t(X))$ where $F : [0, T] \times \mathcal{B}(\mathcal{H} \otimes \Gamma) \longrightarrow \mathcal{B}(\mathcal{H} \otimes \Gamma)$ is the extension to self-adjoint operators $x = j_t(X)$ of the analytic function $F(t, x) = \sum_{n,k=0}^{+\infty} a_{n,k}(t_0, x_0) (t - t_0)^n (x - x_0)^k$, where x and $a_{n,k}(t_0, x_0)$ are in **C**, and for $\lambda, \mu \in \{0, 1, ...\}$

$$F_{\lambda\mu}(t,x) := \frac{\partial^{\lambda+\mu}F}{\partial t^{\lambda} \partial x^{\mu}}(t,x)$$
$$= \sum_{n=\lambda,k=\mu}^{+\infty} \frac{n!}{(n-\lambda)!} \frac{k!}{(k-\mu)!} a_{n,k}(t_0,x_0) (t-t_0)^{n-\lambda} (x-x_0)^{k-\mu}$$

and so, if 1 denotes the identity operator then

$$a_{n,k}(t_0, x_0) = a_{n,k}(t_0, x_0) \ 1 = \frac{1}{n! \, k!} F_{n\,k}(t_0, x_0).$$

Notice that for $(t_0, x_0) = (0, 0)$ we have

$$V_t = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0) t^n j_t(X)^k = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0) t^n j_t(X^k).$$

Proposition 1. (Quantum Black-Scholes Equation)

$$\begin{aligned} a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) \, j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) \, j_t(\alpha \, \lambda^{k-2} \, \alpha^{\dagger}) &= \\ a_t \, j_t(\theta) + V_t \, r - a_t \, j_t(X) \, r \end{aligned}$$

(this is the quantum analogue of the classical Black-Scholes equation) and

$$a_{0,1}(t, j_t(X)) j_t(\alpha^{\dagger}) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^{k-1} \alpha^{\dagger}) = a_t j_t(\alpha^{\dagger})$$
$$a_{0,1}(t, j_t(X)) j_t(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \lambda^{k-1}) = a_t j_t(\alpha)$$
$$\sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\lambda^k) = a_t j_t(\lambda).$$

Proof. By Lemma 2.1 and the Itô table for quantum stochastic differentials

$$\begin{aligned} dV_t &= dF(t, j_t(X)) = F(t + dt, j_{t+dt}(X)) - F(t, j_t(X)) \\ &= F(t + dt, j_t(X) + dj_t(X)) - F(t, j_t(X)) \\ &= \sum_{\substack{n,k=0\\n+k>0}}^{+\infty} a_{n,k}(t, j_t(X)) (dt)^n (dj_t(X))^k \\ &= a_{1,0}(t, j_t(X)) dt + \sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) (dj_t(X))^k \\ &= a_{1,0}(t, j_t(X)) dt + a_{0,1}(t, j_t(X)) dj_t(X) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) \{j_t(\lambda^{k-1} \alpha^{\dagger}) dA_t^{\dagger} \\ &+ j_t(\lambda^k) d\Lambda_t + j_t(\alpha \lambda^{k-1}) dA_t + j_t(\alpha \lambda^{k-2} \alpha^{\dagger}) dt \} \end{aligned}$$

where $\alpha, \alpha^{\dagger}, \lambda$ are as in Lemma 2.1. Thus

$$dV_{t} = \left(a_{1,0}(t, j_{t}(X)) + a_{0,1}(t, j_{t}(X)) j_{t}(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_{t}(X)) j_{t}(\alpha \lambda^{k-2} \alpha^{\dagger})\right) dt + \left(a_{0,1}(t, j_{t}(X)) j_{t}(\alpha^{\dagger}) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_{t}(X)) j_{t}(\lambda^{k-1} \alpha^{\dagger})\right) dA_{t}^{\dagger} + \left(a_{0,1}(t, j_{t}(X)) j_{t}(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_{t}(X)) j_{t}(\alpha \lambda^{k-1})\right) dA_{t} (3.1) + \sum_{k=1}^{+\infty} a_{0,k}(t, j_{t}(X)) j_{t}(\lambda^{k}) d\Lambda_{t}$$

where θ is as in Lemma 2.1. We can obtain another expression for dV_t with the use of the self-financing property. We have

$$dV_t = a_t dj_t(X) + b_t d\beta_t = a_t dj_t(X) + b_t \beta_t r dt$$

$$= a_t dj_t(X) + (V_t - a_t j_t(X)) \beta_t^{-1} \beta_t r dt$$

$$= a_t dj_t(X) + (V_t - a_t j_t(X)) r dt$$

$$= a_t \left(j_t(\alpha^{\dagger}) dA_t^{\dagger} + j_t(\lambda) d\Lambda_t + j_t(\alpha) dA_t + j_t(\theta) dt \right) + (V_t - a_t j_t(X)) r dt$$

which can be written as

$$dV_t = (a_t j_t(\theta) + V_t r - a_t j_t(X) r) dt + a_t j_t(\alpha^{\dagger}) dA_t^{\dagger} + a_t j_t(\alpha) dA_t$$
(3.2)
$$+ a_t j_t(\lambda) d\Lambda_t$$

Equating the coefficients of dt and the quantum stochastic differentials in (3.1) and (3.2) we obtain the desired equations.

4. The case S = 1: Quantum Brownian motion

Proposition 2. Let F be as in the previous section. If S = 1 then the equations of Proposition 3.1 combine into

$$u_{10}(t,x) = \frac{1}{2} u_{02}(t,x) g(x) + u_{01}(t,x) h(x) - u(t,x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where $u(t, x) = F(T - t, x), g(x) = [y^*, x] [x, y], h(x) = x r$ and $x, y \in \mathcal{B}(\mathcal{H} \otimes \Gamma)$

Proof. If S = 1 then, in the notation of Lemma 2.1, $\alpha = [L^*, X]$, $\alpha^{\dagger} = [X, L]$, $\lambda = 0$, and $\theta = i [H, X] - \frac{1}{2} \{L^* L X + X L^* L - 2 L^* X L\}$ and the equations of Proposition 3.1 reduce to

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + a_{0,2}(t, j_t(X)) j_t(\alpha \alpha^{\dagger}) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and

$$a_{0,1}(t, j_t(X)) j_t(\alpha^{\dagger}) = a_t j_t(\alpha^{\dagger})$$

$$a_{0,1}(t, j_t(X)) j_t(\alpha) = a_t j_t(\alpha)$$

which are condensed into

 $a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + a_{0,2}(t, j_t(X)) j_t(\alpha \alpha^{\dagger}) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$ and

$$a_{0,1}(t, j_t(X)) = a_t.$$

Upon substituting the second of the last two equations into the first one and simplifying we obtain

 $a_{1,0}(t, j_t(X)) + a_{0,2}(t, j_t(X)) j_t([L^*, X] [X, L]) + a_{0,1}(t, j_t(X)) j_t(X) r - V_t r = 0$

which can be written as

$$F_{10}(t, j_t(X)) + \frac{1}{2} F_{02}(t, j_t(X)) j_t([L^*, X] [X, L]) + F_{01}(t, j_t(X)) j_t(X) r = F(t, j_t(X)) r$$

with terminal condition $F(T, j_T(X)) = (j_T(X) - K)^+$. Letting $x = j_t(X), y = j_t(L)$ be arbitrary elements in $\mathcal{B}(\mathcal{H} \otimes \Gamma)$ and letting $g(x) = [y^*, x][x, y], h(x) = xr$, we obtain

$$F_{10}(t,x) + \frac{1}{2} F_{02}(t,x) g(x) + F_{01}(t,x) h(x) = F(t,x) r.$$

Letting $u(t,x) := F(T-t,x), u_{10}(t,x) = -F_{10}(T-t,x), u_{02}(t,x) = F_{02}(T-t,x)$ and $u_{01}(t,x) = F_{01}(T-t,x)$ we obtain

$$u_{10}(t,x) = \frac{1}{2} u_{02}(t,x) g(x) + u_{01}(t,x) h(x) - u(t,x) r$$
$$u(0, j_T(X)) = (j_T(X) - K)^+.$$

5. The case $S \neq 1$: Quantum Poisson Process

In this section we examine the equations of Proposition 3.1 under the assumption $S \neq 1$.

Proposition 3. Let F be as in Section 3. If [X, S] = S then the equations of Proposition 3.1 combine into

$$u_{10}(t,x) = \sum_{k=2}^{+\infty} \frac{1}{k!} u_{0k}(t,x) g(x) + u_{01}(t,x) h(x) - u(t,x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where $u(t, x) = F(t - T, x), g(x) = [y^*, x] [x, y] - i [z, x] + \frac{1}{2} \{y^* y x + x y^* y - 2 y^* x y\}, h(x) = x r \text{ and } x, y, z \in \mathcal{B}(\mathcal{H} \otimes \Gamma)$

Proof. Since X is self-adjoint and S is unitary, assuming that [X, S] = S is equivalent to assuming that $\lambda = S^* X S - X = 1$ and the equations of Proposition 3.1 take the form

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \alpha^{\dagger}) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

with

and

$$a_{0,1}(t, j_t(X)) j_t(\alpha^{\dagger}) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha^{\dagger}) = a_t j_t(\alpha^{\dagger})$$
$$a_{0,1}(t, j_t(X)) j_t(\alpha) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha) = a_t j_t(\alpha)$$
$$\sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X)) = a_t$$

which are satisfied if

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(\theta) + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) j_t(\alpha \alpha^{\dagger}) = a_t j_t(\theta) + V_t r - a_t j_t(X) r$$

and $a_t = \sum_{k=1}^{+\infty} a_{0,k}(t, j_t(X))$ which, if substituted in the previous one, yields

$$a_{1,0}(t, j_t(X)) + a_{0,1}(t, j_t(X)) j_t(X) r + \sum_{k=2}^{+\infty} a_{0,k}(t, j_t(X)) \left(j_t(\alpha \, \alpha^{\dagger} - \theta) + j_t(X) \, r \right) = V_t \, r.$$

But

$$j_t(\alpha \,\alpha^{\dagger} - \theta) = [j_t(L)^*, j_t(X)] [j_t(X), j_t(L)] - i [j_t(H), j_t(X)] + \frac{1}{2} \{ j_t(L)^* j_t(L) j_t(X) + j_t(X) j_t(L)^* j_t(L) - 2 j_t(L)^* j_t(X) j_t(L) \}$$

Letting $x = j_t(X)$, $y = j_t(L)$, $z = j_t(H)$, h(x) = x r and

$$g(x) = [y^*, x] [x, y] - i [z, x] + \frac{1}{2} \{y^* y x + x y^* y - 2 y^* x y\}$$

using the notation of the previous section we obtain the Black-Scholes equation for the case $S \neq 1$ as stated in the Proposition.

6. Solution of the Quantum Brownian Motion Black-Scholes Equation

To solve the Quantum Brownian motion Black-Scholes equation we assume that $j_t(X^2) = j_t([L^*, X] [X, L])$ which is the same as $X^2 = [L^*, X] [X, L]$. Since $X = X^*$, it follows that $[L^*, X] = [X, L]^*$ and so letting $\phi(X) = [X, L]$ we find $X^2 = \phi(X)^* \phi(X)$ i.e $\phi(X) = WX$ which implies that [X, L] = WX and $[L^*, X] = XW^*$, where W is

an arbitrary unitary operator acting on the system space. In this case equation (2.2) takes the form

$$dj_t(X) = j_t \left(i \left[H, X \right] + \frac{1}{2} \left(L^* W X + X W^* L \right) \right) dt + j_t(X W) dA_t^{\dagger} + j_t(W^* X) dA_t.$$

Lemma 2. If H > 0 is a bounded self-adjoint operator on a Hilbert space \mathcal{H} then there exists a bounded self-adjoint operator A on \mathcal{H} such that $H = e^A$.

Proof. Let $H = \int_a^b \lambda \, dE_\lambda$ where $[a, b] \subset (0, +\infty)$ and $a \leq ||H|| \leq b$. Letting $\lambda = e^\mu$ we obtain $H = \int_{\ln a}^{\ln b} e^\mu \, dF(\mu)$ where $F(\mu) = E(e^\mu)$. Thus $H = e^A$ where $A = \int_{\ln a}^{\ln b} \mu \, dF(\mu)$ with $||A|| \leq \max(|\ln a|, |\ln b|)$. To show that the family $\{F(\mu)/\ln a \leq \mu \leq \ln b\}$ is a resolution of the identity we notice that for $h \in \mathcal{H}$ and $\lambda, \mu \in [\ln a, \ln b]$ we have:

(i)
$$F(\lambda) F(\mu) = E(e^{\lambda}) E(e^{\mu}) = E(e^{\lambda} \wedge e^{\mu}) = F(\lambda \wedge \mu),$$

(ii) $\lim_{\lambda \to \mu^{-}} F(\lambda) h = \lim_{e^{\lambda} \to e^{\mu^{-}}} E(e^{\lambda}) h = E(e^{\mu}) h = F(\mu) h,$
(iii) $\lambda < \mu \Rightarrow e^{\lambda} < e^{\mu} \Rightarrow E(e^{\lambda}) < E(e^{\mu}) \Rightarrow F(\lambda) < F(\mu),$
(iv) $\lambda < \ln a \Rightarrow e^{\lambda} < a \Rightarrow E(e^{\lambda}) = 0 \Rightarrow F(\lambda) = 0,$
(v) $\lambda > \ln b \Rightarrow e^{\lambda} > b \Rightarrow E(e^{\lambda}) = 1 \Rightarrow F(\lambda) = 1.$

and the proof is complete.

The equation in Proposition 4.1 now has the form

$$u_{10}(t,x) = \frac{1}{2} u_{02}(t,x) x^2 + u_{01}(t,x) x r - u(t,x) r$$

with initial condition $u(0, j_T(X)) = (j_T(X) - K)^+$ where we may assume that x is a bounded self-adjoint operator. Since

$$u(t,x) = F(T-t,x) = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0) (T-t)^n x^k$$

and $x = j_t(X) > 0$, and K are invertible, we may let $x = K e^z$ where z is a bounded self-adjoint operator commuting with K, and obtain

$$\begin{split} \omega(t,z) &:= u(t,Ke^z) = \sum_{n,k=0}^{+\infty} a_{n,k}(0,0) \, (T-t)^n \, (Ke^z)^k \\ \omega_{01}(t,z) &= \sum_{n,k=0}^{+\infty} a_{n,k}(0,0) \, (T-t)^n \, k \, (Ke^z)^k = \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) \, (T-t)^n \, k \, x^k \\ &= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) \, (T-t)^n \, k \, x^{k-1} \, x = u_{01}(t,x) \, x \end{split}$$

Similarly

$$\begin{split} \omega_{02}(t,z) &= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) (T-t)^n k^2 (K e^z)^k = \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) (T-t)^n k^2 x^k \\ &= \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) (T-t)^n (k (k-1) + k) x^k \\ &= \sum_{n=0,k=2}^{+\infty} a_{n,k}(0,0) (T-t)^n k (k-1) x^{k-2} x^2 \\ &+ \sum_{n=0,k=1}^{+\infty} a_{n,k}(0,0) (T-t)^n k x^{k-1} x \\ &= u_{02}(t,x) x^2 + u_{01}(t,x) x \end{split}$$

and so

$$\omega_{02}(t,z) - \omega_{01}(t,z) = u_{02}(t,x) \ x^2.$$

Finally

$$\omega_{10}(t,z) = -\sum_{n=1,k=0}^{+\infty} a_{n,k}(0,0) n (T-t)^{n-1} (K e^z)^k$$
$$= -\sum_{n=1,k=0}^{+\infty} a_{n,k}(0,0) n (T-t)^{n-1} x^k = u_{10}(t,x)$$

and so

(6.1)
$$\omega_{10}(t,z) = \frac{1}{2}\,\omega_{02}(t,z) + \omega_{01}(t,z)\,\left(r - \frac{1}{2}\right) - \omega(t,z)\,r$$

with initial condition $\omega(0, z_T) = (j_T(X) - K)^+$ where z_T is defined by $K e^{z_T} = j_T(X)$.

Theorem 1. In analogy with the classical case presented in section 1, the solution of (6.1) is given by

$$\omega(t,z) = K e^z \Phi(g(t, K e^z)) - K \Phi(h(t, K e^z)) e^{-rt}$$

where

$$g(t, K e^{z}) = z t^{-1/2} + (r + 0.5) t^{1/2}$$

$$h(t, K e^{z}) = z t^{-1/2} + (r - 0.5) t^{1/2}$$

and

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{x^{2n+1}}{2n+1}$$

Proof. We have

$$\omega_{10}(t,z) = K e^{z} \ (\Phi \circ g)_{10}(t, K e^{z}) - K \ (\Phi \circ h)_{10}(t, K e^{z}) \ e^{-rt} + K \ (\Phi \circ h)(t, K e^{z}) \ r \ e^{-rt},$$

 $\omega_{01}(t,z) = K e^{z} \ (\Phi \circ g)(t, K e^{z}) + K e^{z} \ (\Phi \circ g)_{01}(t, K e^{z}) - K \ (\Phi \circ h)_{01}(t, K e^{z}) e^{-rt},$ and

$$\omega_{02}(t,z) = K e^{z} (\Phi \circ g)(t, K e^{z}) + 2 K e^{z} (\Phi \circ g)_{01}(t, K e^{z}) + K (\Phi \circ g)_{02}(t, K e^{z}) - K (\Phi \circ h)_{02}(t, K e^{z}) e^{-rt}$$

where

$$\begin{aligned} (\Phi \circ h)(t, K e^{z}) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^{n}}{2^{n} n!} \frac{(z t^{-1/2} + (r - 0.5) t^{1/2})^{2n+1}}{2n+1} \\ (\Phi \circ g)(t, K e^{z}) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^{n}}{2^{n} n!} \frac{(z t^{-1/2} + (r + 0.5) t^{1/2})^{2n+1}}{2n+1} \end{aligned}$$

Thus

$$\omega_{10}(t,z) - \frac{1}{2}\,\omega_{02}(t,z) - \omega_{01}(t,z)\,(r - \frac{1}{2}) + \omega(t,z)\,r = K\,\left(A\,e^{-rt} + e^{z}\,B\right)$$

where

$$A = -(\Phi \circ h)_{10}(t, K e^{z}) + \frac{1}{2} (\Phi \circ h)_{02}(t, K e^{z}) + (\Phi \circ h)_{01}(t, K e^{z}) (r - \frac{1}{2})$$
$$B = (\Phi \circ g)_{10}(t, K e^{z}) - \frac{1}{2} (\Phi \circ g)_{02}(t, K e^{z}) - (\Phi \circ g)_{01}(t, K e^{z}) (r + \frac{1}{2})$$

It follows that A = B = 0 thus proving (6.1). Moreover, in order to prove that the initial condition is satisfied, we have

$$\begin{aligned} \omega(0,z_T) &= K e^{z_T} \Phi(g(0,K e^{z_T})) - K \Phi(h(0,K e^{z_T})) \\ &= (K e^{z_T} - K) \Phi(g(0,K e^{z_T})) + K \left(\Phi(g(0,K e^{z_T})) - \Phi(h(0,K e^{z_T})) \right). \end{aligned}$$

But

$$g(0, Ke^{z_T}) - h(0, Ke^{z_T}) = \lim_{t \to 0^+} \left(\frac{z}{\sqrt{t}} + (r+0.5)\sqrt{t} - \frac{z}{\sqrt{t}} - (r-0.5)\sqrt{t}\right) = 0$$

and so $\Phi(g(0, Ke^{z_T})) - \Phi(h(0, Ke^{z_T})) = 0$. Thus, it suffices to show that

$$\Phi(g(0, K e^{z_T})) = \begin{cases} 1 & \text{if } K e^{z_T} \ge K \\ 0 & \text{if } K e^{z_T} < K \end{cases}$$

We have

$$\begin{split} \Phi(g(0, K e^{z_T})) &= \lim_{t \to 0^+} \left(\Phi \circ g \right)(t, K e^{z_T})) \\ &= \frac{1}{2} + \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{1}{t^{n+1/2}} \frac{z_T^{2n+1}}{2n+1} \end{split}$$

Suppose that $K e^{z_T} \ge K$. Then $z_T \ge 0$ and by the spectral resolution theorem $z_T^{2n+1} = \int_0^{+\infty} \lambda^{2n+1} dE_{\lambda}$. So

$$\begin{split} \Phi(g(0, K e^{z_T})) &= \frac{1}{2} + \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{+\infty} \frac{(-1)^n}{2^n n!} \frac{1}{t^{n+1/2}} \int_0^{+\infty} \frac{\lambda^{2n+1}}{2n+1} dE_\lambda \\ &= \frac{1}{2} + \lim_{t \to 0^+} \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} \int_0^{\frac{\lambda}{\sqrt{t}}} e^{-\frac{s^2}{2}} ds \ dE_\lambda \\ &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} \int_0^{+\infty} \frac{\sqrt{2\pi}}{2} \ dE_\lambda = 1 \end{split}$$

Similarly, if $K e^{z_T} < K$ then $z_T < 0$ and if we let $z_T = -w_T$ where $w_T = \int_0^{+\infty} \lambda \, dE_\lambda > 0$, then

$$z_T^{2n+1} = (-1)^{2n+1} \int_0^{+\infty} \lambda^{2n+1} dE_{\lambda} = -\int_0^{+\infty} \lambda^{2n+1} dE_{\lambda}$$

and so, as before, $\Phi(g(0, K e^{z_T})) = \frac{1}{2} - \frac{1}{2} \cdot 1 = 0.$

Corollary 1. The reasonable price for a quantum option is $\omega(T, z_0)$ where ω is as in Theorem 6.1 and z_0 is defined by $X = K e^{z_0}$. The associated quantum portfolio (a_t, b_t) is given by

$$a_t = \omega_{01}(t - T, z_t)$$

$$b_t = (\omega(T - t, z_t) - a_t j_t(X)) e^{-tr} \beta_0^{-1}$$

where z_t is defined by $j_t(X) = K e^{z_t}$. (As in the classical case described in Section 1, a reasonable price is defined as one which when invested at time t = 0 in a mixed portfolio, allows the investor through a self-financing strategy to end up at time t = T with an amount of

$$< u \otimes \psi(f), V_T u \otimes \psi(f) > = < u \otimes \psi(f), (j_T(X) - K)^+ u \otimes \psi(f) >$$
$$= \max(0, < u \otimes \psi(f), (j_T(X) - K) u \otimes \psi(f) >)$$

which is the same as the payoff, had the option been purchased. Here, $u \otimes \psi(f)$ is any vector in the exponential domain of $\mathcal{H} \otimes \Gamma$).

Proof. By Theorem 6.1, the reasonable price for a quantum option is $V_0 = F(0, j_0(X)) = F(0, X) = u(T, X) = \omega(T, z_0)$. The formulas for a_t and b_t follow from the definition of the portfolio, given in Section 2.

References

- L. Accardi, Quantum Probability: an introduction to some basic ideas and trends, Course given at the VI Symposium de probabilidad y Procesos Estocasticas, CIMAT, guanajuato, 24–27 May 2000, Mexico
- [2] L. Accardi, A. Boukas, Quadratic control of quantum processes, Russian Journal of Mathematical Physics, vol.9, no. 4, pp. 381-400, 2002, MR1966015.
- B. E. Baaquie, Price of coupon bond options in a quantum field theory of forward interest rates, Phys. A 370 (2006),no. 1, 98–103, MR2263766
- [4] B. E. Baaquie, C. Coriano, M. Srikant, Quantum mechanics, path integrals and option pricing: reducing the complexity of finance, Nonlinear physics: theory and experiment, II (Gallipoli, 2002), 33-339, World Sci. Publ., River Edge, NJ, 2003. MR2028802.
- [5] A. Boukas, Application of Quantum Stochastic Calculus to Feedback Control, Global Journal of Pure and Applied Mathematics, Vol. 1, no.1, (2005).

- [6] O. A. Choustova, Pilot wave quantum model for the stock market, Quantum theory: reconsideration of foundations, Proceedings of the International Conference in Vaxjo, June 2001. Ed. A. Khrennikov. Series Math. Mod. in Phys., Eng., Cogn. Sc., 2, 41-58 (Vaxjo University Press:Vxj).
- [7] E. Haven, A Black-Scholes Schrodinger Option Price: bit versus qubit, Physica A 324, (1-2), 201-206 (2003).
- [8] E. Haven, A Discussion on Embedding the Black-Scholes Option Pricing Model in a Quantum Physics Setting, Physica A 304 (3-4), 507-524 (2002).
- [9] E. G. Hidalgo, *Quantum Econophysics*, arXiv: physics/0609245, v1, 28 Sep 2006.
- [10] R. L. Hudson, K. R. Parthasarathy, Quantum Ito's formula and stochastic evolutions, Comm. Math. Phys. 93 (1984), 301-323.
- [11] V. P. Maslov, Quantum Economics, Russ. J. Math. Phys 12 (2005), no. 2, 219–231. MR2199003.
- [12] T. Mikosch, Elementary Stochastic Calculus with Finance in View, World Scientific 1998.
- [13] K. R. Parthasarathy, An introduction to quantum stochastic calculus, Birkhauser Boston Inc., 1992.
- [14] P. W. Piotrowski, M. Schroeder, A. Zambrzycka, Quantum extension of European option pricing based on the Ornstein-Uhlenbeck process, arXiv: quant-ph/0510121, v1, 16 Oct 2005.
- [15] P. W. Piotrowski, J. Sladkowski, Quantum market games, Phys. A 312 (2002), no.1–2, 208–216. MR1926828.
- [16] W. Segal, I. E. Segal, The Black-Scholes pricing formula in the quantum context, Proc. Natl. Acad. Sci. USA, vol. 95, pp. 4072-4075, March 1998, Economic Sciences.

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