

# Mixture Dynamics and Option Pricing: a Regime Switching Model

A. Ramponi

Dipartimento S.E.F.eME.Q., Università di Roma - Tor Vergata, Italy  
via Columbia, 2 - 00133 Roma, Italy  
e-mail: ramponi@economia.uniroma2.it

June 2008

The need of considering price dynamics alternative to the classical Black-Scholes model for derivatives pricing is widely known. The stochastic variability of market parameters and in particular the empirical evidence of non constant surfaces of implied volatility in real markets require more realistic models for the assets dynamics. Many approaches are available to obtain a better fitting of market data, each aiming to relax some of the restrictive assumption implied by the geometric Brownian motion dynamic of the Black - Scholes model. These modifications typically consider explicit models for the local volatility, the addition of jump components and/or the introduction of stochastic volatility in the underlying diffusive dynamic.

In a series of papers Brigo and Mercurio (2000, 2001) proposed a class of one-dimensional diffusion models characterized by a local volatility function which induces a mixture marginal distribution for the asset price. In particular, the case of log-normal mixtures has been deeply developed. The main advantages of their model are:

1. that analytical formulas for European derivatives are readily available together with the corresponding Greeks and
2. an improved flexibility for the calibration to real market data.

Furthermore, the explicit analytical form of the local volatility makes available Monte Carlo simulation to price path-dependent derivatives, through simple discretization schemes.

Our main contribution is to develop a dynamic model in a regime - switching framework in order to extend the class of mixture models proposed by Brigo and Mercurio. The main advantage of our approach is twofold: first it can be easily applied to multivariate processes, secondly in some very important models sample paths can be exactly simulated.

Regime switching process were introduced by Hamilton (1989, 1990) in a financial econometric context. The main idea consists in introducing a discrete and in general unobservable Markov chain which generates switches among a finite set of "regimes": each regime characterizes a particular parameter set for the dynamic model. Indeed, empirical studies demonstrated that markets may randomly switch between low-volatility/high growth and/or high-volatility/low growth regimes. Correspondingly, there is a practical interest to develop methods for pricing and hedging options for regime switching underlying models. There has been a considerable progress in the case of standard European and/or American style options: see e.g Naik (1993), Di Masi et al. (1994), Bollen (1998), Guo (2001), Hardy (2001), Duan et al. (2002), Buffington and Elliott (2002), Guo and Zhang (2004), Liu et al. (2006), Yao et al. (2006), Jobert and Rogers (2006). Few results are available for exotic options: see Boyle and Draviam (2007) and the recent Elliott et al. (2007) who considered the pricing of volatility swaps in a regime-switching version of the Heston stochastic volatility model. Finally, switching Lévy processes have been considered in Elliott and Osakwe (2006).

In many of these applications the Markov chain is independent from the other model factors and its generator is characterized by constant elements. In this work we develop an explicit mixture dynamics in a regime-switching framework. Mathematically the model we consider is described by the pair  $(X(t), Y(t)) \in \mathbb{R}^d \times \mathcal{S}$ ,  $\mathcal{S} = \{1, 2, \dots, N\}$ , such that

$$dX(t) = \mu(t, X(t), Y(t))dt + \sigma(t, X(t), Y(t))dW(t) \tag{1}$$

$$\mathbb{P}\{Y(t + \Delta t) = j | Y(t) = i, (X(s), Y(s)), s \leq t\} = q_{ij}(t, X(t))\Delta t + o(\Delta t), \quad i \neq j$$

where  $W(t)$  is a  $d$ -dimensional brownian motion,  $\mu : \mathbb{R}_+ \times \mathbb{R}^n \times \mathcal{S} \rightarrow \mathbb{R}^n$  and  $\sigma : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{S} \rightarrow \mathbb{R}^{n \times d}$ , with  $\sigma(t, x, i)\sigma(t, x, i)^T = a(x, i)$ . The first component represents a "controlled diffusion" process since its drift and diffusion coefficients depend on the state of the second discrete component  $Y(t)$ . On the other hand  $Y(t)$  is a "controlled Markov Chain" with finite state space and a generator depending on the continuous component  $X(t)$ . They appear naturally in many engineering applications where the system under study may show abrupt changes in its structure and parameters (see e.g. Ghosh et al. (1993, 1997)).

We present, under some conditions, a characterization of the Markov chain generator  $q_{ij}(t, x)$  to get a target mixture marginal distribution for the diffusive component  $X(t)$ . Furthermore, since the state dependent generator makes the regime-switching diffusion model difficult to work with computationally, a simulation scheme recently proposed for a general jump-diffusion process by Glasserman and Merener (2004), has been adapted to our model. Finally, as an example of application of our dynamic mixture model we extend two well-known option pricing formulas to our setting: the first one consider the Margrabe (or Exchange) Option and the second a plain vanilla under the Heston stochastic volatility model .

## References

- Bollen N. P. B. (1998), Valuing options in regime-switching models. *Journal of Derivatives* 6, pp. 38-49.
- Boyle P. and Draviam T. (2007), Pricing exotic options under regime switching, *Insurance: Mathematics and Economics* 40 , pp. 267-282.
- Brigo D. and Mercurio F. (2000), A Mixed-up Smile. *Risk*, September, pp. 123-126.
- Brigo D. and Mercurio F. (2002), Lognormal-mixture dynamics and calibration to market volatility smiles, *International Journal of Theoretical and Applied Finance*, Vol. 5, No. 4, pp. 427-446.
- Buffington J. and Elliott R. J. (2002), American options with regime switching, *International Journal of Theoretical and Applied Finance*, 5 , pp. 497-514.
- Di Masi, G.B., Kabanov, Y.M., Runggaldier, W.J. (1994), Mean-variance hedging of options on stocks with Markov volatility. *Theory of Probability and Its Applications* 39, pp. 173-181.
- Duan, J.C., Popova, I., Ritchken, P. (2002), Option pricing under regime switching, *Quantitative Finance* 2, pp. 1-17.
- Elliott R. J. and Osakwe C. J. U. (2006), Option pricing for pure jump processes with Markov switching compensators, *Finance and Stochastics*, 10, pp. 250-275.
- Elliott R. J. , Siu T. K. and Chan L. (2007), Pricing volatility swaps under Heston's stochastic volatility model with regime switching, *Applied Mathematical Finance*, Vol 14, 1, pp. 41-62.
- Ghosh M.K. , Arapostathis A. , Marcus S.I. (1993), Optimal control of switching diffusions with application to flexible manufacturing systems, *SIAM J. Control and Optimization* Vol.31, pp. 1183-1204.
- Glasserman P. , Merener N. (2004), Convergence of a discretization scheme for jump-diffusion process with state-dependent probability, *Journal of the Royal Statistical Society, Series* , pp.. 32, No.2, pp. 443-458.
- Guo X., (2001), Information and option pricing *Quantitative Finance* 1, pp. 38-44.
- Guo X. and Zhang Q. Z (2004), Closed-form solutions for perpetual American put options with regime switching, *SIAM Journal of Applied Mathematics*, 64 , pp. 2034-2049.

- Hamilton, J.D., (1989), A new approach to the economic analysis of non-stationary time series, *Econometrica* 57, pp. 357-384.
- Hamilton, J.D., (1990), Analysis of time series subject to changes in regime, *Journal of Econometrics* 45, pp. 39-70.
- Hardy, M.R., (2001), A regime switching model of a long term stock-returns. *North American Actuarial Journal* 3, pp. 185-211.
- Heston, S. L. (1993), A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Rev. Fin. Studies*, n.6, pp. 327-343.
- Jobert A. and Rogers L. C. G. (2006), Option pricing with Markov-Modulated dynamics, *SIAM Journal of Control and Optimization*, Vol. 44, No. 6, pp. 2063-2078.
- Liu R. H. , Zhang Q. , and Yin G. (2006), Option pricing in a regime-switching model using the fast Fourier transform. *J. Appl. Math. Stoch. Anal.*, Vol. 22, pp.1-22.
- Naik, V. (1993), Option valuation and hedging strategies with jumps in the volatility of asset returns. *Journal of Finance* 48, pp. 1969-1984.
- Yao D. D., Zhang Q. and Zhou X. Y. (2006), A regime-switching model for European options, *Stochastic Processes, Optimization, and Control Theory Applications in Financial Engineering, Queueing Networks, and Manufacturing Systems* (H. M. Yan, G. Yin, and Q. Zhang, eds.), Springer, New York, pp. 281-300.