*Acta Applicandae Mathematicae* **63:** 3–25, 2000.



# A White-Noise Approach to Stochastic Calculus

LUIGI ACCARDI[1], YUN GANG LU[1,2] and IGOR V. VOLOVICH[1,3]
[1]*Centro Matematico Vito Volterra, Università di Roma, Rome, 00133, Italy.*
*e-mail: accardi@volterra.mat.uniroma2.it*
[2]*Dipartimento di Matematica, Università di Bari, Bari, 70125, Italy*
[3]*Steklov Mathematical Institute, Russian Academy of Sciences, Vavilov St. 42, 117966, Moscow, Russia*

(Received: 8 August 2000)

**Abstract.** During the past 15 years a new technique, called *the stochastic limit of quantum theory*, has been applied to deduce new, unexpected results in a variety of traditional problems of quantum physics, such as quantum electrodynamics, bosonization in higher dimensions, the emergence of the noncrossing diagrams in the Anderson model, and in the large-$N$-limit in QCD, interacting commutation relations, new photon statistics in strong magnetic fields, etc. These achievements required the development of a new approach to classical and quantum stochastic calculus based on white noise which has suggested a natural nonlinear extension of this calculus. The natural theoretical framework of this new approach is the white-noise calculus initiated by T. Hida as a theory of infinite-dimensional generalized functions. In this paper, we describe the main ideas of the white-noise approach to stochastic calculus and we show that, even if we limit ourselves to the first-order case (i.e. neglecting the recent developments concerning higher powers of white noise and renormalization), some nontrivial extensions of known results in classical and quantum stochastic calculus can be obtained.

**Mathematics Subject Classification (2000):** 60H40.

**Key words:** quantum probability, white noise, stochastic limit.

## 1. The Main Idea of the Stochastic Limit of Quantum Theory

Quantum stochastic differential equations are now widely used to construct phenomenological models of physical systems, for example in quantum optics, in quantum measurement theory, etc. However, the fundamental equation of quantum theory is not a stochastic equation but a usual Schrödinger equation. Therefore, the problem of understanding the physical meaning of these phenomenological models naturally arose.

The stochastic limit of quantum theory was developed to solve this problem and its main result can be concisely formulated as follows: *stochastic equations are limits, in an appropriate sense, of the usual Hamiltonian equations of quantum physics*.

Thus, the stochastic limit provides a derivation of the phenomenological stochastic equations from the fundamental quantum laws. In particular, this gives a microscopic interpretation of the coefficients of these equations and proves that the most important examples of quantum Markov flows arise in physics from the stochastic limit of Hamiltonian models.

From the mathematical point of view, the stochastic limit suggested a new interpretation of the usual stochastic equations, both classical and quantum, as *normally ordered Hamiltonian white noise equations*. In this section, we give a short illustration of the basic ideas of the stochastic limit and show how this naturally leads to the identification of normally ordered first-order white-noise Hamiltonian equations with stochastic differential equations in the sense of Hudson and Parthasarathy.

The starting point of the stochastic limit is not a stochastic equation but a usual Schrödinger equation in interaction representation, depending on a parameter $\lambda$

$$\partial U_t^{(\lambda)} = -i\lambda(DA_t^+(S_t g) - D^+ A_t(S_t g))U_t^{(\lambda)} \tag{1.1}$$

describing a system $S$ with state space $\mathcal{H}_S$ interacting with a field with creation and annihilation operators $A_t^+(g)$, $A_t(g)$, and $D$, $D^+$ are operators on a Hilbert space $\mathcal{H}_S$. One rescales the time parameter according to the law $t \to t/\lambda^2$. This rescaling is motivated both by mathematics (central limit theorem) and by physics (Friedrichs–van Hove rescaling). After the rescaling, one arrives to an equation of the form

$$\partial U_{t/\lambda^2}^{(\lambda)} = (Da_t^{(\lambda)+} - D^+ a_t^{(\lambda)})U_{t/\lambda^2}^{(\lambda)}, \tag{1.2}$$

where

$$a_t^{(\lambda)} = \lambda \int_0^{t/\lambda^2} \mathrm{d}s A(S_s g).$$

It was proved in [1] that, as $\lambda \to 0$, the iterated series solution of this equation converges, in a sense which is the natural generalization of the notion of quantum convergence in law, to the solution of the QSDE

$$\mathrm{d}U_t = \left(D\,\mathrm{d}B_t^+ - D^+\,\mathrm{d}B_t + \left(-\frac{\gamma}{2}D^+D + i\alpha D^+D\right)\mathrm{d}t\right)U_t, \tag{1.3}$$

where $B_t^+$, $B_t$ is the Fock Brownian motion with variance $\gamma$ acting on the Boson Fock space $L^2(\mathbf{R})\otimes\mathcal{K}$, $H = \kappa D^+D$ and $\kappa, \gamma > 0$, are real numbers, $\mathcal{K}$ is a Hilbert space, whose explicit structure is described in terms of the original Hamiltonian model.

In [3, 4] it was proved that the iterated series solution of this equation converges term by term, in the same limit, and in the same sense as above, to the iterated series solution of the distribution equation

$$\partial_t U_t = (Db_t^+ - D^+ b_t)U_t, \tag{1.4}$$

where $b_t^+$, $b_t$ are the annihilation and creation operators of the Boson Fock white noise with variance $\gamma$ ($> 0$) which is characterized, up to unitary equivalence, by the algebraic relations

$$[b_t, b_s^+] = \gamma\delta(s-t), \quad t, s \in \mathbf{R}, \tag{1.5}$$

$$b_t\Phi = 0, \tag{1.6}$$

where $\Phi$ is the Fock vacuum. It is therefore natural to conjecture that Equations (1.3) and (1.4) are just two different ways of writing the same equation. To prove this conjecture we have to develop a purely analytical white noise approach to the standard, classical, and quantum Itô calculus and, in particular, a white-noise formulation of the Itô table, based on the general white-noise theory initiated by Hida [10] and developed in [12, 13, 16].

## 2. Notations on Fock Spaces

We begin by describing a concrete representation of the Fock space which, being well suited for explicit calculations, is most often used in the physical literature. Such a representation can be used whenever the 1-particle space is concretely realized as an $L^2$-space over some measure space (finite or $\sigma$-finite) $(S, \mu)$. In this case, the $n$-particle space can be realized as the space $L^2_{\mathrm{sym}}(S^n, \otimes^n\mu)$ of all the symmetric, square integrable functions on the product space

$$S^n := S \times S \times \cdots \times S \quad (n\text{-times})$$

with the measure $\otimes^n\mu$, which is the product of $n$ copies of the measure $\mu$. In the following we shall fix the choice

$$S = \mathbf{R}^d; \qquad \mu = \text{Lebesgue measure}.$$

Let $\mathcal{F}_1 = L^2(\mathbf{R}^d)$ be the Hilbert space of functions on $\mathbf{R}^d$ with the inner product

$$(f, g) = \int_{\mathbf{R}^d} \overline{f}(s)g(s)\,\mathrm{d}s, \quad f, g \in \mathcal{F}_1 \tag{2.1}$$

and $\mathcal{F}_n = L^2_{\mathrm{sym}}(\mathbf{R}^{nd})$, $n = 1, 2, \ldots$ be the Hilbert space of square integrable functions of $n$-variables in $\mathbf{R}^d$, symmetric under the permutation of their arguments. The elements of $\mathcal{F}_n$ are called *n-particle vectors*. For an element $\psi_n \in \mathcal{F}_n$ we write $\psi_n = \psi_n(s_1, \ldots, s_n)$, $s_i \in \mathbf{R}^d$ and one has $\psi_n(s_1, \ldots, s_n) = \psi_n(s_{\pi(1)}, \ldots, s_{\pi(n)})$ for any permutation $\pi$.

DEFINITION. The *symmetric representation* of the scalar Boson Fock space $\mathcal{F}$ is the direct sum of the Hilbert spaces $\mathcal{F}_n$

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_{\mathrm{sym}}(\mathbf{R}^{dn}) = \mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n. \tag{2.2}$$

Here we set $\mathcal{F}_0 = \mathbf{C}$. So an element of the Boson Fock space $\mathcal{F}$ is a sequence of functions

$$\Psi = \{\psi_0, \psi_1, \psi_2, \ldots\},$$

where $\psi_0 \in \mathbf{C}$, $\psi_n \in \mathcal{F}_n$, $n = 1, 2, \ldots$ and

$$\|\psi\|^2 = \sum_{n=0}^{\infty} \|\psi^{(n)}\|^2_{L^2(\mathbf{R}^{dn})} < \infty. \tag{2.3}$$

More explicitly

$$\|\psi\|^2 = |\psi^{(0)}|^2 + \sum_{n=1}^{\infty} \int_{\mathbf{R}^{dn}} |\psi^{(n)}(s_1, \ldots, s_n)|^2 \, \mathrm{d}s_1 \ldots \mathrm{d}s_n. \tag{2.4}$$

The inner product of elements $\Psi = \{\psi_n\}_{n=0}^{\infty}$ and $\Phi = \{\phi_n\}_{n=0}^{\infty}$ from $\mathcal{F}$ is given by

$$\begin{aligned} (\Psi, \Phi) &= \sum_{n=0}^{\infty} (\psi_n, \phi_n) \\ &= \overline{\psi}_0 \phi_0 + \sum_{n=1}^{\infty} \int_{\mathbf{R}^{nd}} \overline{\psi_n(s_1, \ldots, s_n)} \phi_n(s_1, \ldots, s_n) \, \mathrm{d}s_1 \ldots \mathrm{d}s_n. \end{aligned} \tag{2.5}$$

The vector $\Psi_0 = (1, 0, 0, \ldots)$ is called the *vacuum vector*. It describes the state of a system in which no particle is present.

## 3. Annihilator and Creator Densities

Define

$$\mathcal{D}_{\mathcal{S}} := \{\psi \in \mathcal{F} \mid \psi^{(n)} \in \mathcal{S}(\mathbf{R}^{dn})\}. \tag{3.1}$$

In the remaining of this section, unless otherwise specified, all the $n$-particle vectors shall belong to $\mathcal{D}_{\mathcal{S}}$. Define, moreover,

$$\mathcal{D}^o_{\mathcal{S}} := \{\psi \in \mathcal{D}_{\mathcal{S}} \mid \psi^{(n)} = 0 \text{ for almost all } n \in \mathbf{N}\}, \tag{3.2}$$

$$\mathcal{D}(a) := \left\{ \psi \in \mathcal{D}_{\mathcal{S}} \ : \ \sum_{n=1}^{\infty} n \|\psi^{(n)}\|^2 < \infty \right\} \tag{3.3}$$

and notice that $\mathcal{D}(a)$ is a vector space containing both the number and the exponential vectors with test functions in $\mathcal{S}$. Define the *annihilation density*[(1)] $a_s$

$$(a_s \psi)^{(n)}(s_1, \ldots, s_n) = \sqrt{n+1} \psi^{(n+1)}(s, s_1, \ldots, s_n); \quad s \in \mathbf{R}^d, \ n \in \mathbf{N}. \tag{3.4}$$

The right-hand side of (3.4) is well defined whenever it makes sense to speak of the values $\psi^{(n)}$ on any point, for example when $\psi^{(n)}$ is in the $L^2$-equivalence class of a continuous function for each $n$, and the sequence of functions $\{(a_s\psi)^{(n)}\}$ defines an element of $\mathcal{F}$. This is surely the case if $\psi$ is in $\mathcal{D}(a)$. Thus, for any $t \in \mathbf{R}^d$, the annihilator $a_t$ is a densely defined operator which maps $\mathcal{D}(a)$ into $\mathcal{F}$.

From (8) it follows that the map $a_s$ is weakly measurable and therefore, for any square integrable function $g$, the integral

$$A(g) = \int_{\mathbf{R}^d} \mathrm{d}s\ \overline{g}(s)a_s \tag{3.5}$$

called the *annihilation operator* is well defined as a Bochner integral on $\mathcal{D}(a)$. Proposition 1 below shows that it is a preclosed operator. The explicit action of $A(g)$ on vectors in $\mathcal{D}(a)$ is deduced from (8) to be, for $n \in \mathbf{N}$,

$$\begin{aligned}(A(g)\psi)^{(n)} &= \int_{\mathbf{R}^d} \mathrm{d}s\ \overline{g}(s)(a_s\psi)^{(n)}(s_1, \dots, s_n)\\ &= \sqrt{n+1}\int_{\mathbf{R}^d} \mathrm{d}s\ \overline{g}(s)\psi^{(n+1)}(s, s_1, \dots, s_n).\end{aligned} \tag{3.6}$$

For example, the explicit action of $A(g)$ on the exponential vectors is also deduced from (3.4):

$$A(g)\psi_f = \int_{\mathbf{R}^d} \mathrm{d}s\ \overline{g}(s)a_s\psi_f = \int_{\mathbf{R}^d} \mathrm{d}s\ \overline{g}(s)f(s)\psi_f = \langle g, f\rangle\psi_f. \tag{3.7}$$

The *creation density* $a_s^+$ is defined for $\psi \in \mathcal{D}_{\mathcal{S}}$ by

$$(a_s^+\psi)^{(n)}(s_1 \dots s_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \delta(s - s_i)\psi^{(n-1)}(s_1 \dots \hat{s}_i \dots s_n). \tag{3.8}$$

The $\delta$-function on the right-hand side of (3.8) shows that the creation density $a^+(t)$ is not an operator but a sesquilinear form on the number vectors.

PROPOSITION 1. *For any square integrable function $g$ there exists a preclosed operator $A^+(g)$, defined on the $n$-particle vectors, represented by continuous functions, by the relation*

$$(A^+(g)\psi)^{(n)}(s_1, \dots, s_n) := \frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(s_i)\psi^{(n-1)}(s_1, \dots, \hat{s}_i, \dots, s_n). \tag{3.9}$$

*Moreover, on the $n$-particle space, $A^+(g)$ is bounded with norm less or equal to $n^{1/2}\|g\|$ ($L^2$-norm of $g$) and, on $\mathcal{D}(a)$, $A^+(g)$ satisfies the relation*

$$\langle A^+(g)\psi, \psi'\rangle = \langle\psi, A(g)\psi'\rangle. \tag{3.10}$$

*Proof.* Let $\psi$ be as in the statement. Then, using the definition (3.9) of $A^+(g)$:

$$\|(A^+(g)\psi)^{(n)}\|^2 = \frac{1}{n}\sum_{i,j=1}^{n} \langle g_i\psi_i^{(n-1)}, g_j\psi_j^{(n-1)}\rangle,$$

where

$$g_i(s_1, \ldots, s_i, \ldots, s_n) := g(s_i);$$
$$\psi_i^{(n-1)}(s_1, \ldots, s_i, \ldots, s_n) := \psi^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n).$$

Since $\psi^{(n-1)}$ is a symmetric function, it follows that

$$|\langle g_i \psi_i^{(n-1)}, g_j \psi_j^{(n-1)} \rangle| \leqslant \|g_i \psi_i^{(n-1)}\| \|g_j \psi_j^{(n-1)}\| = \|g\|^2 \|\psi^{(n-1)}\|^2$$

and therefore

$$\|(A^+(g)\psi)^{(n)}\|^2 \leqslant n\|g\|^2 \|\psi^{(n-1)}\|^2.$$

This shows that $A^+(g)$ is a well defined operator on the domain $\mathcal{D}(a)$, bounded on each $n$-particle space. To prove (3.10) we compute

$$\begin{aligned}
&\langle \psi, A(g)\psi' \rangle \\
&= \sum_n \langle \psi^{(n)}, (A_g \psi')^n \rangle = \sum_n \sqrt{n+1} \int \mathrm{d}s \overline{g}(s) \langle \psi^{(n)}, \psi'^{(n+1)}(s, \cdot) \rangle \\
&= \sum_n \sqrt{n+1} \int \mathrm{d}s \overline{g}'_s \int \overline{\psi}^{(n)}(s_1, \ldots, s_n) \psi'^{(n+1)}(s, s_1, \ldots, s_n) \\
&= \sum_n \sqrt{n+1} \int \mathrm{d}s \overline{g}_s \int \overline{\psi}^{(n)}(s, \ldots, s_n) \psi'^{(n+1)}(s, s_1, \ldots, s_n) \\
&= \sum_n \int \mathrm{d}s \int \mathrm{d}s_1 \ldots \int \mathrm{d}s_n \left[ \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} g_{s_i} \psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_{n+1}) \right] \times \\
&\quad \times \psi'^{(n+1)}(s_1, s_2, \ldots, s_{n+1}) \\
&= \langle A^+(g)\psi, \psi' \rangle. \qquad \square
\end{aligned}$$

LEMMA 2. *The following formulae hold on* $\mathcal{D}_a$*:*

$$\begin{aligned}
&(a(t_1)a^+(t_2)\psi)^{(n)}(s_1, \ldots, s_n) \qquad (3.11)\\
&= \sum_{i=1}^n \delta(t_2 - s_i)\psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_n, t_1) + \delta(t_2 - t_1)\psi^{(n)}(s_1, \ldots, s_n) \times \\
&\quad \times (a^+(t_1)a(t_2)\psi)^{(n)}(s_1, \ldots, s_n) \\
&= \sum_{i=1}^n \delta(t_2 - s_i)\psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_n, t_1). \qquad (3.12)
\end{aligned}$$

*Proof.* Define

$$\begin{aligned}
&\phi_{t_2}^{(n+1)}(s_1, \ldots, s_{n+1}) \\
&:= (a^+(t_2)\psi)^{(n+1)}(s_1, \ldots, s_{n+1}) \\
&= \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \delta(t_2 - s_i)\psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_{n+1}). \qquad (3.13)
\end{aligned}$$

$\phi^{(n+1)})_{t_2}$ is a distribution with values in $\mathcal{F}_{n+1}$, i.e., for any $g \in \mathcal{S}$,

$$\left[\int \mathrm{d}t_2 g(t_2)\phi_{t_2}\right]^{(n+1)} = \frac{1}{\sqrt{n+1}} \sum_i g(s_i)\psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_{n+1}).$$

Now, applying $a_{t_1}$, as defined by (3.8), we find

$$\begin{aligned}
&\left[a_{t_1}\int \mathrm{d}t_2 g(t_2)\phi_{t_2}\right]^{(n)}(s_1, \ldots, s_n) \\
&\quad = g(t_1)\psi^{(n)}(s_1, \ldots, s_n) + \sum_{i=1}^{n} g(s_i)\psi^{(n)}(t_1, \ldots, \hat{s}_i, \ldots, s_n) \\
&\quad = \int \mathrm{d}t_2 g(t_2)\delta(t_2 - t_1)\psi^{(n)}(s_1, \ldots, s_n) + \\
&\qquad + \sum_{i=1}^{n} \delta(t_2 - s_i)g(t_2)\psi^{(n)}(t_1, \ldots, \hat{s}_i, \ldots, s_n) \\
&\quad = \int \mathrm{d}t_2 g(t_2)\sqrt{n+1}\phi_{t_2}^{(n+1)}(t_1, s_1, \ldots, s_n).
\end{aligned}$$

Therefore, in the sense of distributions,

$$(a(t_1)\phi_{t_2})^{(n)}(s_1, \ldots, s_n) = \sqrt{n+1}\phi_{t_2}^{(n+1)}(t_1, s_1, \ldots, s_n) \tag{3.14}$$

and from (3.7) we get that (3.12), (3.12) are proved in a similar way. □

*Remark.* Comparing (3.12) and (3.12), one deduces the *Boson commutation relations* for a scalar Boson white noise

$$(t_1)a^+(t_2) - a^+(t_2)a(t_1) = \delta(t_2 - t_1).$$

## 4. Stochastic Integrals with Respect to the Boson Fock White Noises

In this section we shall discuss white noises and stochastic integrals in $\mathbf{R}^d$ rather than in $\mathbf{R}$ because exactly the same formulae are valid in the 1- and in the $d$-dimensional case.

We have defined the operators

$$A(F) = \langle F, A\rangle = \int_{\mathbf{R}^d} \mathrm{d}s F_s a_s; \qquad A^+(F) = \langle F, A^+\rangle = \int_{\mathbf{R}^d} \mathrm{d}s F_s a_s^+, \tag{4.1}$$

when $F$ is a complex-valued function on $\mathbf{R}$. The generalization of these integrals to the case when $F$ is an operator valued function are called *right stochastic integrals* with respect to $a_s$ (resp. $a_s^+$). One has also to define the *left stochastic integrals*

$$\langle A, F\rangle = \int_{\mathbf{R}^d} \mathrm{d}s a_s F_s; \qquad \langle A^+, F\rangle = \int_{\mathbf{R}^d} \mathrm{d}s a_s^+ F_s \tag{4.2}$$

and the *two-sided stochastic integrals*

$$\int_{\mathbf{R}^d} \mathrm{d}s F_s a_s^{\pm} G_s; \qquad \int_{\mathbf{R}^d} \mathrm{d}s a_s^{+} F_s a_s. \tag{4.3}$$

Let $\mathcal{D}_S^0$ be as in (5.6) and let $\mathcal{L}$ be a space of maps from $\mathbf{R}^d$ to linear operators from a dense subspace of $\mathcal{D}_{\mathcal{S}}^o$ to $\mathcal{F}$ with the property that the maps

$$s \mapsto \langle \psi, F_s \varphi \rangle; \qquad s \mapsto \| F_s \psi \|^2; \quad \varphi, \psi \in \mathcal{D}_{\mathcal{S}}^0,$$

are locally integrable. Clearly $s \mapsto a_s$, then $a \in \mathcal{L}$, while $s \mapsto a_s^+$ is not in $\mathcal{L}$.

If $P_n$ denotes the projection onto the $n$-particle space of the Fock space, then for any $t$, we can write

$$F_t = \sum_{n,k} P_n F_t P_k =: \sum_{n,k} F_t^{(n,k)}.$$

*Remark.* By inspection from formulae (4.2) and (4.3), one can guess that even if the *integrand* $F_s$ is bounded, in general the stochastic integrals will not be bounded operators. So a precise definition of the notion of stochastic integral should always specify the domain of vectors where this inegral is defined. The general scheme we shall adopt to define stochastic integrals is the following. If $G_s$ denotes any of the integrands in formulae (4.1) or (4.2) or (4.3), $I$ denotes the corresponding stochastic integral and $\psi$ an arbitrary vector, then $I$ will be characterized by the following two properties:

(i) The $n$-particle component of $I\psi$ is the Bochner integral of the $n$-particle component of $G_s\psi$:

$$\left( \int_{\mathbf{R}^d} \mathrm{d}s G_s \psi \right)^n := \int_{\mathbf{R}^d} \mathrm{d}s (G_s \psi)^n.$$

(ii) The $n$-particle component of $G_s\psi$ is explicitly computed using the rules of the previous section.

In the following sections we shall show how these general principles work in concrete applications.

## 5. Right Annihilator Integrals

Let $\mathcal{D}_S^0 =: \mathcal{D}$ and $\mathcal{L} := \mathcal{L}(\mathcal{D})$ be as in Equation (3.6).

DEFINITION 3. The right annihilator stochastic integral of $F \in \mathcal{L}$ is the operator

$$\psi = \int F_s a_s \psi \, \mathrm{d}s = \langle F^*, A \rangle \psi := \int F_s A_s \psi \, \mathrm{d}s, \tag{5.1}$$

where the integral is meant as a Bochner integral in the Fock space. It is defined for each $\psi \in \mathcal{D}_{\mathcal{S}}^o$ such that $a_s\psi$ is in the domain of $F_s$ for each $s$ and the vector-valued function $s \in \mathbf{R}^d \mapsto F_s a_s \psi$ is Bochner integrable.

The explicit form of the right annihilator stochastic integral on the $n$-particle vectors can be easily obtained by using the same technique as in Section 3. In fact, because of definition (3.8), one has that

$$(a_s\psi)^{(n)} = \sqrt{n+1}\psi^{(n+1)}(s,\cdot), \tag{5.2}$$

where $\psi^{(n+1)}(s,\cdot)$ is the function

$$(s_1,\dots,s_n) \in \mathbf{R}^{dn} \mapsto \psi^{(n+1)}(s,s_1,\dots,s_n). \tag{5.3}$$

Therefore, (5.1) is equivalent to

$$\int F_s a_s \psi \, \mathrm{d}s = \sum_{n\geq 0} \sqrt{n+1} \int \mathrm{d}s F_s \psi^{(n+1)}(s,\cdot). \tag{5.4}$$

In particular, on the exponential vectors this explicit form is

$$\int F_t a_t \, \mathrm{d}t \psi_f = \int \mathrm{d}t F_t f(t) \psi_f, \tag{5.5}$$

where the right-hand side of (5.1) is a usual Bochner integral. The right-hand side of (5.5) is defined on the set of the exponential vectors $\psi_f$ with test function in $\mathcal{H}_1$ such that the vector valued function $s \mapsto f(s)F(s)\psi_f$ is Bochner integrable. From definition (5.1) we have that

$$\langle F^*, A\rangle := \int F a_s \, \mathrm{d}s. \tag{5.6}$$

In the case where $S = \mathbf{R}$ and $F = \chi_I F$ with $I = [0,t]$ we shall simply write

$$\langle F, A_t\rangle := \int_0^t F_s a_s \, \mathrm{d}s. \tag{5.7}$$

Thus the right annihilator integral maps functions $F\colon \mathbf{R}^d \to \mathcal{L}(\mathcal{D})$ into elements of $\mathcal{L}(\mathcal{D})$.

From (5.4) and (5.5) we deduce the estimate

$$\begin{aligned} \left\| \int F_s a_s \psi \, \mathrm{d}s \right\| &\leqslant \sum_{n\geqslant 0} \sqrt{n+1} \int \mathrm{d}s \| F_s \psi^{(n+1)}(s,\cdot)\| \\ &= \int \mathrm{d}s \| F_s (N+1)^{1/2} \psi^{(n+1)}_{(s,\cdot)} \| (s,\cdot) \| \end{aligned} \tag{5.8}$$

and $A(\chi_{I_j})$.

The definition of exponential vector implies that

$$(N+1)^{1/2}\psi_f^{(n+1)}(s,\cdot) = f(s)\psi_f^{(n)}, \tag{5.9}$$

therefore (5.5) implies that for any exponential vector $\psi_f$ one has

$$\left\| \int F_s a_s \, \mathrm{d}s \psi_f \right\| \leqslant \int |f(s)| \cdot \|F_s \psi_f\| \, \mathrm{d}s. \tag{5.10}$$

A sufficient condition for the finiteness of the right-hand side of (5.10) is that the vector-valued function $s \mapsto f(s)F(s)\psi(s)$ is Bochner integrable.

## 6. The Left Creator Stochastic Integral

DEFINITION 1. The definition of left creator stochastic integrals is the natural extension of formula (3.13) for the scalar case

$$\left(\int a_t^+ F_t \psi \,\mathrm{d}t\right)^{(n)}(s_1, \ldots, s_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (F(s_i)\psi)^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n). \tag{6.1}$$

Definition 1 has a meaning for any measurable function $F_s$ and, given such an $F$, the natural domain of its left creator stochastic integral is

$$\mathcal{D}\left(\int a_t^+ F_t \,\mathrm{d}t\right) = \left\{\psi \,\middle|\, \sum_{n=1}^{\infty} \left\| \left(\int a_t^+ F_t \psi \,\mathrm{d}t\right)^{(n)} \right\|^2 < \infty\right\} \tag{6.2}$$

or, more explicitly, a vector $\psi$ is in $\mathcal{D}(\int a_t^+ F_t \,\mathrm{d}t)$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_{\mathbf{R}^{dn}} \left| \sum_{i=1}^{n} (F(s_i)\psi)^{n-1}(s_1, \ldots, \hat{s}_i, \ldots, s_n) \right|^2 \mathrm{d}s_1 \ldots \mathrm{d}s_n < \infty. \tag{6.3}$$

We want now to obtain an estimate on the norm of $\int a_s^+ F_s \,\mathrm{d}s \psi$ which guarantees that the stochastic integral exists. This is given by the following lemma:

LEMMA 1. *Let $\psi^{(n-1)}$ belong to $\mathcal{D}(F_s)$ for all $s \in \mathbf{R}^d$. Then one has, for each $n \in \mathbf{N}$,*

$$\left\| P_n\left(\int \mathrm{d}s a_s^+ F_s \psi\right) \right\|^2 \leqslant n \int \mathrm{d}s \| P_{n-1}(F_s \psi)\|^2 = \int \mathrm{d}s \left\| \sqrt{(N+1)} (F_s \psi)^{(n-1)} \right\|^2. \tag{6.4}$$

*In particular*

$$\left\| \int \mathrm{d}s a_s^+ F_s \psi \right\|^2 \leqslant \int \mathrm{d}s \| (\sqrt{N+1}) F_s \psi \|^2. \tag{6.5}$$

*Proof.* The norm square of (6.1) is

$$\int \ldots \int \mathrm{d}s_1 \ldots \mathrm{d}s_n \frac{1}{n} \sum_{i,j} \langle (F_{s_i} \psi)^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n), (F_{s_j} \psi)^{(n-1)}(s_1, \ldots, \hat{s}_j, \ldots, s_n) \rangle$$

$$\leqslant \frac{1}{n}\sum_{ij=1}^{n}\int \mathrm{d}s_1\ldots \mathrm{d}s_n \|(F_{s_i}\psi)^{(n-1)}(s_1,\ldots,\hat{s}_i,\ldots,s_n)\|\times$$
$$\times \|(F_{s_j}\psi)^{(n-1)}(s_1,\ldots,\hat{s}_j,\ldots,s_n)\|$$
$$= \frac{n^2}{n}\int_{\mathbf{R}^d}\mathrm{d}s\int_{\mathbf{R}^{d(n-1)}}\mathrm{d}s_2\ldots \mathrm{d}s_n \|(F_s\psi)^{(n-1)}(s_2,\ldots,s_{n-1})\|^2$$
$$= n\int_{\mathbf{R}^d}\mathrm{d}s\|(F_s\psi)^{(n-1)}\|^2. \square$$

COROLLARY 2. *Let $L_s$ be a function with values in $\mathcal{B}(\mathcal{F})$ such that*

(i) *for any* $0 < T < +\infty \sup_{s\in[0,T]}\|L_s\|_\infty < \sqrt{C_T}$,
(ii) $L_s$ *and* $L_s^+$ *commute with every* $a_t$, $a_t^+$, $P_k$ $t \in \mathbf{R}$, $k \in \mathbf{N}$.

*Then*

$$\left\|P_n\int \mathrm{d}s a_s^+ L_s F_s\psi\right\|^2 \leqslant C_T n\int \mathrm{d}s\|P_{n-1}F_s\psi\|^2. \tag{6.6}$$

*Proof.* From Lemma 1 the left-hand side of (6.6) is less than or equal to

$$C_T n\int \mathrm{d}s\|P_{n-1}L_sF_s\psi\|^2$$

and, using (ii) and (i), the thesis follows. $\square$

## 7. The Normally Ordered Two-Sided Integral

DEFINITION 1. The two-sided (normally ordered) integral $\int \mathrm{d}s b_s^+ F_s b_s$ is defined, weakly on the exponential or number vectors by

$$\langle\xi, \int \mathrm{d}s b_s^+ F_s b_s\eta\rangle = \int \mathrm{d}s\langle b_s\xi, F_s b_s\eta\rangle. \tag{7.1}$$

In particular, on exponential vectors one has

$$\langle\psi_f, \int \mathrm{d}s b_s^+ F_s b_s\psi_g\rangle = \int \mathrm{d}s\overline{f}(s)g(s)\langle\psi_f, F_s\psi_g\rangle. \tag{7.2}$$

LEMMA 2. *For any $n \in \mathbf{N}$ and for any exponential vector $\psi_f$, one has the estimate*

$$\left\|\left(\int \mathrm{d}s b_s^+ F_s b_s\psi_f\right)^{(n)}\right\|^2 \leqslant n\int \mathrm{d}s|f(s)|^2\|(F_s\psi_f)\|^2. \tag{7.3}$$

*In particular,*

$$\left\|\int \mathrm{d}s b_s^+ F_s b_s\psi_f\right\|^2 \leqslant \int \mathrm{d}s|f(s)|^2\|(N+1)^{1/2}\psi_f\|^2. \tag{7.4}$$

*Proof.* Using $b_s\psi_f = f(s)\psi_f$, the left-hand side of (7.3) becomes

$$\left\| \left( \int \mathrm{d}s b_s^+ f(s) F_s \psi_f \right)^{(n)} \right\|$$

and, because of (8.4), this is

$$\leqslant n \int \mathrm{d}s \|(F_s\psi_f)^{(n)}\|^2 |f(s)|^2,$$

i.e. (1). (2) is obtained from (1) by summing over all $n$. □

## 8. Differential Calculus

Usually in stochastic calculus one considers the differentials only as symbolic expressions for the corresponding integrals. We want to develop a differential calculus directly in analogy with classical analysis.

Let us first consider the differentiability properties, with respect to $t$, of the Brownian motion operators $B_t$, $B_t^+$.

THEOREM 1. *Let $\psi \in \mathcal{D}_S^o$ be such that, for each $n$, $\psi^{(n)}$ is continuous with compact support. Then, with $\Delta B_t$ defined by* (6.5) *below, one has the following:*

(i) $$\lim_{\Delta t\to 0} \left\| \left( \frac{\Delta B_t}{\Delta t} - b(t) \right)\psi \right\| = 0, \tag{8.1}$$

*where the operator $b(t)$ is defined in* (6.5).

(ii) *The strong limit, as $\Delta t \to 0$, of $\Delta B_t^+/\Delta t - b^+(t)$ does not exist on the number vectors. However, the weak limit of this expression on $\mathcal{D}_S^o$ does exist, i.e.* $\forall \psi_1, \psi_2 \in \mathcal{D}_S^o$,

$$\begin{aligned} \lim_{\Delta t\to 0} \left( \psi_1, \frac{\Delta B_t^+}{\Delta t} \psi_2 \right) &= (\psi_1, b^+(t)\psi_2) \qquad (8.2)\\ &\equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \int_{\mathbf{R}^{n-1}} \mathrm{d}s_1 \ldots \mathrm{d}\hat{s}_i \ldots \mathrm{d}s_n \times \\ &\quad \times \psi_1^{(n)}(s_1, \ldots, t, s_n) \psi_2^{(n-1)}(s_1, \ldots, \hat{s}_i, \ldots, s_n). \end{aligned}$$

(iii) $$\lim_{\Delta t\to 0} \left\| \left( \frac{\Delta B_t \cdot \Delta B_t^+}{\Delta t} - 1 \right)\psi \right\| = 0. \tag{8.3}$$

(iv) $$\lim_{\Delta t\to 0} \left\| \frac{\Delta B_t^+}{\Delta t} \Delta B_t \psi \right\| = 0. \tag{8.4}$$

*Proof.* One has

$$(\Delta B_t\psi)^{(n)}(s_1,\dots,s_n)=\sqrt{n+1}\int_t^{t+\Delta t}\psi^{(n+1)}(s,s_1,\dots,s_n)\,\mathrm{d}s, \tag{8.5}$$

$$(\Delta B_t^+\psi)^{(n)}(s_1,\dots,s_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\chi_{[t,t+\Delta t]}(s_i)\psi^{(n-1)}(s_1,\dots,\hat{s}_i,\dots,s_n) \tag{8.6}$$

and

$$(\Delta B_t\Delta B_t^+\psi)^{(n)}(s_1,\dots,s_n) = \psi^{(n)}(s_1,\dots,s_n)\cdot\Delta t + + \sum_{i=1}^{n}\chi_{[t,t+\Delta t]}(s_i)\int_t^{t+\Delta t}\psi^{(n)}(s_1,\dots,\hat{s}_i,\dots,s_n,t_1)\,\mathrm{d}t_1. \tag{8.7}$$

From (8.2), we have

$$\left\|\left(\frac{\Delta B_t}{\Delta t}-b(t)\right)\psi^{(n)}\right\|^2 = (n+1)\int_{\mathbf{R}^{dn}}\mathrm{d}s_1\dots\mathrm{d}s_n\times \times\left|\frac{1}{\Delta t}\int_t^{t+\Delta t}\psi^{(n+1)}(s,s_1,\dots,s_n)\,\mathrm{d}s-\psi^{(n+1)}(t,s_1,\dots,s_n)\right|^2. \tag{8.8}$$

Because $\psi^{(n+1)}$ are continuous functions with compact support, one can go to the limit $\Delta t\to 0$ under the integral over $\mathrm{d}s_1,\dots,\mathrm{d}s_n$ and we get (1).

To see that the limit in (ii) does not exist let us take $\psi^{(0)}=\Phi$. Then one has

$$(\Delta B_t^+\psi)^{(1)}(s_1)=\chi_{[t,t+\Delta t]}(s_1) \tag{8.9}$$

and

$$\left\|\frac{(\Delta B_t^+\psi)^{(1)}}{\Delta t}\right\|^2=\frac{1}{(\Delta t)^2}\int\chi_{[t,t+\Delta t]}(s_1)\,\mathrm{d}s_1=\frac{1}{\Delta t}. \tag{8.10}$$

From (8.10) it is clear that the limit when $\Delta t\to 0$ does not exist. However, there exist the limit of the bilinear form (8.2). Now, from (8.7) one has

$$\begin{aligned}&\left\|\left(\frac{\Delta B_t\cdot\Delta B_t^+}{\Delta t}-1\right)\psi^{(n)}\right\|^2\\ &=\int_{\mathbf{R}^{dn}}\mathrm{d}s_1\dots\mathrm{d}s_n\left|\frac{1}{\Delta t}\sum_{i=1}^{n}\chi_{[t,t+\Delta t]}(s_i)\int_t^{t+\Delta t}\psi^{(n)}(s_1,\dots,\hat{s}_i,\dots,s_n,t_1)\,\mathrm{d}t\right|^2\\ &=\sum_{i,j=1}^{n}\frac{1}{(\Delta t)^2}\int_t^{t+\Delta t}\mathrm{d}s_i\int_t^{t+\Delta t}\mathrm{d}t_1\int_t^{t+\Delta t}\mathrm{d}t_1'\int_{\mathbf{R}^{n-1}}\mathrm{d}s_1\dots\mathrm{d}\hat{s}_i\dots\mathrm{d}s_n\times\\ &\quad\times\overline{\psi^{(n)}}(s_1,\dots,\hat{s}_i,\dots,s_n,t_1)\psi^{(n)}(s_1,\dots,\hat{s}_j,\dots,s_n,t_1')\end{aligned} \tag{8.11}$$

which tends to zero when $\Delta t \to 0$. This ends the proof of the theorem. □

*Remark 1.* One can symbolically write the relation (8.1) in the form

$$\mathrm{d}B_t = b(t)\,\mathrm{d}t \tag{8.12}$$

and the relation for bilinear form (8.2) as

$$\mathrm{d}B_t^+ = b^+(t)\,\mathrm{d}t. \tag{8.13}$$

Formulas (8.3), (8.4) look like the Itô rules

$$\mathrm{d}B_t\,\mathrm{d}B_t^+ = \mathrm{d}t, \tag{8.14}$$

$$\mathrm{d}B_t^+\mathrm{d}B_t = 0. \tag{8.15}$$

However, we emphasize that we get them as differential relations (8.3), (8.4) and not as integral relations like in the Itô calculus.

## 9. Mutual Quadratic Variation

In this section we prove that the limit relation (8.3) is true in a topology much stronger than the one given by strong convergence in a dense subspace of $\mathcal{F}$.

For a real number $\Delta t > 0$, we shall denote

$$\Delta B_t^{\pm} := B^{\pm}_{(0,t+\Delta t)} - B^{\pm}_{(0,t)}. \tag{9.1}$$

From (8.4), (8.5) we deduce that

$$(\Delta B_t\psi)^{(n)}(s_1,\ldots,s_n) = \sqrt{n+1}\int_t^{t+\Delta t}\psi^{(n+1)}(s,s_1,\ldots,s_n)\,\mathrm{d}s, \tag{9.2}$$

$$(\Delta B_t^+\psi)^{(n)}(s_1,\ldots,s_n) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\chi_{(t,t+\Delta t]}(s_i)\cdot\psi^{(n-1)}(s_1,\ldots,\hat{s}_i,\ldots,s_n). \tag{9.3}$$

From (8.6), one deduces the identity, for any real number $\Delta t$,

$$(\Delta B_t\Delta B_t^+\psi)^{(n)}(s_1,\ldots,s_n) = \sum_{i=1}^{n}\chi_{(t,t+\Delta t]}(s_i)\int_t^{t+\Delta t}\psi^{(n)}(s_1,\ldots,\hat{s}_i,\ldots,s_n,t_1)\,\mathrm{d}t_1 + \Delta t\cdot\psi^{(n)}(s_1,\ldots,s_n). \tag{9.4}$$

The Itô multiplication table is obtained from (9.4) when $\Delta t \to 0$ and has the form

$$(\Delta B_t\cdot\Delta B_t^+\psi)^{(n)}(s_1,\ldots,s_n) = \psi^{(n)}(s_1,\ldots,s_n)\Delta t + \mathrm{o}(\Delta t), \tag{9.5}$$

where $o(\Delta t)$ means something that, when summed over all the intervals $(t, t+\Delta t)$ of a partition of a fixed interval $(S, T)$, ten ds to zero as $\Delta t \to 0$. In order to make this statement precise, one has to choose a topology and this can be done in a multitude of ways. In the following, we prove some estimates which show that some topologies arise quite naturally in our context. For example, if the function $\psi^{(n)}$ is measurable and bounded, then one has the estimate

$$|(\Delta B_t \cdot \Delta B_t^+ \psi)^{(n)}(s_1, \ldots, s_n) - \psi^{(n)}(s_1, \ldots, s_n)\Delta t| \leqslant \|\psi^{(n)}\|_\infty \Delta t \sum_{i=1}^{n} \chi_{(t,t+\Delta t]}(s_i). \tag{9.6}$$

LEMMA 1. *Assume that $\psi^{(n)}$ is bounded, fix a bounded interval $(S, T)$ and consider the partition of $(S, T)$ into intervals of equal width $\Delta t$. Then, if $\sum_t$ denotes summation over the intervals of the partition, one has*

$$\left|\sum_t (\Delta B_t \Delta B_t^+ \psi)^{(n)}(s_1, \ldots, s_n) - (T-S)\cdot\psi^{(n)}(s_1, \ldots, s_n)\right| \leqslant n\Delta t \cdot \|\psi^{(n)}\|_\infty. \tag{9.7}$$

*In particular, the limit*

$$\lim_{\Delta t \to 0} \sum_t (\Delta B_t \Delta B_t^+ \psi)^{(n)}(s_1, \ldots, s_n) = (T-S)\cdot\psi^{(n)}(s_1, \ldots, s_n) \tag{9.8}$$

*holds uniformly in $s_1, \ldots, s_n$.*

*Proof.* Summing the identity (9.4) over all the intervals of the partition, we obtain

$$\sum_t (\Delta B_t \Delta B_t^+ \psi)^{(n)}(s_1, \ldots, s_n) = \sum_{i=1}^{n}\sum_t \chi_{(t,t+\Delta t]}(s_i)\int_t^{t+\Delta t} \psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_n, t_1)\, dt_1 + (T-S)\cdot\psi^{(n)}(s_1, \ldots, s_n).$$

But

$$\left|\sum_{i=1}^{n}\sum_t \chi_{(t,t+\Delta t]}(s_i)\int_t^{t+\Delta t} \psi^{(n)}(s_1, \ldots, \hat{s}_i, \ldots, s_n, t_1)\, dt_1\right| \leqslant \Delta t \cdot \|\psi^{(n)}\|_\infty \sum_{i=1}^{n}\sum_t \chi_{(t,t+\Delta t]}(s_i) = \Delta t \cdot \|\psi^{(n)}\|_\infty,$$

which tends to zero, as $\Delta t \to 0$, uniformly in $s_1, \ldots, s_n$. □

## 10. Normally Ordered White Noise Equations in $R^d$

Given a notion of stochastic integral, one can study the problem of the meaning, existence, uniqueness, and unitarity of the corresponding integral equations. We will study integral equations of the form

$$\begin{aligned} Y_t \;=\; & Y_0 + \int_{\mathbf{R}^d} L_{01}(s,t) Y_s a_s \,\mathrm{d}s + \int_{\mathbf{R}^d} L_{10}(s,t) a_s^+ Y_s \,\mathrm{d}s + \\ & + \int_{\mathbf{R}^d} L_{11}(s,t) a_s^+ Y_s a_s \,\mathrm{d}s + \int_{\mathbf{R}^d} L_{00}(s,t) Y_s \,\mathrm{d}s, \end{aligned} \tag{10.1}$$

where the coefficients $L_{\varepsilon,\varepsilon'}(s,t)$ $(\varepsilon, \varepsilon' = 0, 1)$ are linear operators acting on $\mathcal{H}_S$ such that,

(i) for any $(\varepsilon, \varepsilon' = 0, 1)$ and $s, t \in \mathbf{R}^d$, the operator $L_{\varepsilon,\varepsilon'}(s,t)$ is bounded;

(ii) defining

$$\max_{\varepsilon,\varepsilon'=0,1} \|L_{\varepsilon,\varepsilon'}(s,t)\| =: l(s,t), \tag{10.2}$$

then for any bounded set $B \subseteq \mathbf{R}^d$, the functions

$$s \in \mathbf{R}^d \mapsto l(s,t) \tag{10.3}$$

are integrable for each $t \in B$ and the set of integrals, as a function of $t \in B$, is bounded;

(iii) for any bounded set $B \subseteq \mathbf{R}^d$, there exists a constant $L \geqslant 0$ such that, for any natural integer $k$ one has

$$\int_{\mathbf{R}^d} \ldots \int_{\mathbf{R}^d} \mathrm{d}s_1 \ldots \mathrm{d}s_k l(s_1,t) l(s_2,s_1) \ldots l(s_k,s_{k-1}) \leqslant \frac{L^k}{k!} \tag{10.4}$$

uniformly in $t \in B$. (In Section 12, we shall give examples of coefficients $L_{\varepsilon,\varepsilon'}(s,t)$ which satisfy this condition.)

We shall write Equation (10.1) in the notation

$$Y_t = Y_0 + \int_{\mathbf{R}^d} L_{\varepsilon,\varepsilon'}(s,t) \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s) Y_s, \tag{10.5}$$

where summation is understood in the indices $\varepsilon$, $\varepsilon' \in \{0, 1\}$.

In this notation we define the $k$th iterated approximation solution of Equation (10.5) by

$$Y_t^{(0)} = Y_0, \tag{10.6}$$

$$Y_t^{(k+1)} = \int_{\mathbf{R}^d} L_{\varepsilon\varepsilon'}(s,t) \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s) Y_s^{(k)}. \tag{10.7}$$

The iterated series, associated to Equation (10.1) is

$$\sum_{k=0}^{\infty} Y_t^{(k)}. \tag{10.8}$$

In this section we shall fix the set

$$\mathcal{S}_0 = \{f \in L^2(\mathbf{R}^d) : \max\{\|f\|_\infty, \|f\|_2\} \leqslant 1\} \tag{10.9}$$

and we denote $\mathcal{E}(\mathcal{S}_0)$ as the corresponding set of exponential vectors. It is known that $\mathcal{S}_0$ is a total set in $\mathcal{F}$.

THEOREM 1. *Suppose that the coefficients of Equation* (10.1) *satisfy conditions* (i), (ii), (iii) *and, moreover,*

$$\|L\| < \frac{1}{16e}. \tag{10.10}$$

*Then the iterated series* (10.8) *converges, strongly in norm on* $\mathcal{E}(\mathcal{S}_0)$ *to a solution of this equation uniformly in bounded subsets of* $\mathbf{R}^d$.

For the proof of Theorem 1 we shall use several lemmata.

LEMMA 2. *Let* $L_{\varepsilon,\varepsilon'}(s,t)$ $(\varepsilon, \varepsilon' = 0, 1)$ *be linear operators on* $\mathcal{H}_S$ *satisfying the conditions* (i), (ii) *and* (iii), *then for any* $t \in B \subseteq \mathbf{R}^d$, *a bounded set*, $n \in \mathbf{N}$ *and* $f \in \mathcal{S}_0$ *one has*

$$\begin{aligned} &\left\| P_n \int L_{\varepsilon\varepsilon'}(s,t) Y_s \, \mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s) \psi_f \right\|^2 \\ &\quad \leqslant 8n \int l(s,t) \, \mathrm{d}s (\|P_{n-1} Y_s \psi_f\|^2 + \|P_n Y_s \psi_f\|^2). \end{aligned} \tag{10.11}$$

*In particular,* $Y_t^{(k)}$ *defined by* (10.7) *verifies that*

$$\begin{aligned} &\|P_n Y_t^{(k+1)} \psi_f\|^2 \\ &\quad \leqslant 8n \int \mathrm{d}s \; l(s,t) (\|P_{n-1} Y_s^{(k)} \psi_f\|^2 + \|P_n Y_s^{(k)} \psi_f\|^2). \end{aligned} \tag{10.12}$$

*Proof.* First of all,

$$\begin{aligned} &\left\| P_n \int L_{\varepsilon,\varepsilon'}(s,t) Y_s \, \mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s) \psi_f \right\|^2 \\ &\quad \leqslant \left( \sum_{\varepsilon,\varepsilon'=0,1} \left\| P_n \int_{\mathbf{R}^d} L_{\varepsilon,\varepsilon'}(s,t) \, \mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s) Y_s \psi_f \right\| \right)^2. \end{aligned}$$

The Schwarz inequality

$$\left(\sum_{j=1}^{M} a_j\right)^2 \leqslant M \sum_{j=1}^{M} a_j^2$$

($M$ is an integer and $M$ and $a_j$ are real numbers), implies that, for any $n \in \mathbf{N}$,

$$\begin{aligned}&\left\| P_n \int L_{\varepsilon,\varepsilon'}(s,t) Y_s \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 \\ &\quad \leqslant 4 \sum_{\varepsilon,\varepsilon'=0,1} \left\| P_n \int_{\mathbf{R}^d} L_{\varepsilon,\varepsilon'} Y_s \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\|^2 . \end{aligned} \tag{10.13}$$

Now we investigate the quantity in the right-hand side of (10.13) term by term according to the values of $\varepsilon, \varepsilon'$.

By letting $a_s$ act on the exponential vector, we deduce

$$\begin{aligned}&\left\| P_n \int L_{01}(s,t) Y_s a_s \psi_f \,\mathrm{d}s \right\|^2 \\ &\quad \leqslant \|f\|_2^2 \int \|P_n L_{01}(s,t) Y_s \psi_f\|^2 \,\mathrm{d}s \\ &\quad \leqslant \|f\|_2^2 \int \mathrm{d}s l(s,t) \|P_n Y_s \psi_f\|^2 \,\mathrm{d}s. \end{aligned} \tag{10.14}$$

From formula (6.4), one has

$$\left\| P_n \int \mathrm{d}s\; L_{10}(s,t) a_s^+ Y_s \psi_f \right\|^2 \leqslant n \int \mathrm{d}s\; l(s,t) \|P_{n-1} Y_s \psi_f\|^2 \tag{10.15}$$

and from formula (7.3),

$$\begin{aligned}&\left\| P_n \int \mathrm{d}s\; L_{11}(s,t) a_s^+ Y_s a_s \psi_f \right\|^2 \\ &\quad \leqslant \|f\|_\infty^2 n \int \mathrm{d}s\; l(s,t) \|P_{n-1} Y_s \psi_f\|^2 . \end{aligned} \tag{10.16}$$

Finally, the usual properties of Bochner’s integral imply that

$$\left\| P_n \int \mathrm{d}s L_{00}(s,t) Y_s \psi_f \right\|^2 \leqslant \int \mathrm{d}s\; l(s,t) \|P_n Y_s \psi_f\|^2 . \tag{10.17}$$

Because of our assumption (10.9) on $f$, the sum of the left-hand sides of (10.14), (10.15), (10.16), (10.17) is less than or equal to

$$\begin{aligned}&2 \int \mathrm{d}s\; l(s,t) \|P_n Y_s \psi_f\|^2 + 2n \int \mathrm{d}s\; l(s,t) \|P_{n-1} Y_s \psi_f\|^2 \\ &\quad \leqslant 2n \int \mathrm{d}s\; l(s,t) (\|P_{n-1} Y_s \psi_f\|^2 + \|P_n Y_s \psi_f\|^2) \end{aligned}$$

and this is (10.11). To deduce (10.12), one simply applies (10.11) to the definition of $Y_t^{(k+1)}$. □

LEMMA 3. *If the series*

$$\sum_{k=0}^{\infty} \|Y_t^{(k)}\psi_f\| \tag{10.18}$$

*converges uniformly on a bounded set $B$ in $\mathbf{R}^d$, then for each $t \in B$ there exists a unique operator $Y_t$ on $\mathcal{H}_S \otimes \mathcal{E}(\mathcal{S}_0)$ such that*

$$\sum_{k=0}^{\infty} Y_t^{(k)} = Y_t \tag{10.19}$$

*and the series on the left-hand side of* (10.19) *converges strongly in norm on $\mathcal{E}(\mathcal{S}_0)$, uniformly for $t \in B$. Moreover, the function $\mapsto Y_t$ is a solution of Equation* (10.1).

*Proof.* From Lemma 2 we know that there exists an operator $Y_t$ on $\mathcal{H}_S \otimes \mathcal{E}(\mathcal{S}_0)$ such that (10.19) hol ds. And the convergence estimates also imply that the stochastic integrals of $Y_t$ for the basic integrators exist. To prove that $Y_t$ satisfies Equation (10.1) it will be sufficient to prove that, for each $n \in \mathbf{N}$, $P_n Y_t$ satisfies Equation (10.1). To show this, we use the estimate of Lemma 2 to deduce that

$$\begin{aligned} &\left\| P_n \int_{\mathbf{R}^d} L_{\varepsilon\varepsilon'}(s,t) Y_s \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f - P_n \int_{\mathbf{R}^d} \sum_{k=1}^{N} L_{\varepsilon,\varepsilon'}(s,t) Y_s^{(k)} \,\mathrm{d}\Lambda_\varepsilon^{\varepsilon'}(s)\psi_f \right\| \\ &\quad \leqslant 8n \int_{\mathbf{R}^d} \sum_{k=N+1}^{\infty} \|P_n Y_s^{(k)}\psi_f\| l(s,t)\,\mathrm{d}s. \end{aligned} \tag{10.20}$$

By assumption, for each $t \in B$, the function $s \mapsto l(s,t)$ is integrable. Therefore, the right-hand side of (10.20) tends to zero by dominated convergence as $N \to \infty$.

Letting $N \to \infty$ in (10.20), we see that $Y$ satisfies Equation (10.1) and this completes the proof. □

LEMMA 4. *Let $I_{n,k}$ $(n, k \in \mathbf{N})$ be positive numbers satisfying the inequality*

$$I_{n,k+1} \leqslant cn(I_{n,k} + I_{n-1,k}), \tag{10.21}$$

*where $c > 0$ is a constant, then*

$$I_{n,k+1} \leqslant (2cn)^k \sum_{m=n-k}^{n} I_{m,0}. \tag{10.22}$$

*Proof.* By iterating the inequality (10.21) we see that the right-hand side is equal to

$$\begin{aligned} &cn(cnI_{n,k-1} + cnI_{n-1,k-1} + c(n-1)I_{n-1,k-1} + c(n-1)I_{n-2,k-1}) \\ &\quad \leqslant (cn)^2(I_{n,k-1} + 2I_{n-1,k-1} + I_{n-2,k-1}) \\ &\quad \leqslant (cn)^3(I_{n,k-2} + 3I_{n-1,k-2} + 3I_{n-2,k-2} + I_{n-3,k-2}) \dots \\ &\quad \leqslant (cn)^k(I_{n,0} + h_1 I_{n-1,0} + h_2 I_{n-2,0} + \cdots + h_k I_{n-k,0}), \end{aligned}$$

where the coefficients $h_\alpha$ satisfy $h_\alpha \leqslant 2^k$ and (10.22) immediately follows from this. □

*Proof of Theorem 1.* Introducing the notation

$$I_{n,k+1}(s) := \| P_n Y_t^{(k+1)} \psi_f \|^2,$$

we have from Lemma 2

$$I_{n,k+1}(t) \leqslant \int \mathrm{d}s\; l(s,t) 8n(I_{n,k}(s) + I_{n-1,k}(s)),$$

therefore, arguing as in Lemma 4

$$\begin{aligned} I_{n,k+1}(t) \;\leqslant\; 16^k n^k \sum_{m=n-k}^{n} I_{m,0}(s_k) \int \dots \int \mathrm{d}s_1 \dots \mathrm{d}s_k l(s_1,t) \times \\ \times\, l(s_2,s_1) \dots l(s_k,s_{k-1}). \end{aligned} \tag{10.23}$$

But for any $s_k \in \mathbf{R}^d$

$$I_{m,0}(s_k) = \| P_m Y_0 \psi_f \|^2 = \| Y_0 \|^2 \frac{\| f \|^{2m}}{m!}$$

and, without loss of generality, we can assume that

$$\| Y_0 \| = 1. \tag{10.24}$$

Moreover, according to assumption (iii), the multiple integral in (10.23) is dominated by $L^k/k!$. In conclusion

$$\| P_n Y_t^{(k+1)} \psi_f \|^2 \leqslant \frac{(16L)^k}{k!} n^k \sum_{m=n-k}^{n} \frac{\| f \|^{2m}}{m!}. \tag{10.25}$$

Since for large $m$ the sequence $\| f \|^{2m}/m!$ is decreasing the sum in (24) is majorized by

$$k \frac{\| f \|^{2(n-k)}}{(n-k)!}.$$

Therefore

$$\| P_n Y_t^{(k+1)} \psi_f \|^2 \leqslant \frac{(16L)^k}{(k-1)!} \frac{n^k \| f \|^{2(n-k)}}{(n-k)!}.$$

So in order to estimate

$$\| Y_t^{(k+1)} \psi_f \|^2,$$

we are lead to estimate the series

$$\begin{aligned} \sum_{n \geqslant k} \frac{n^k \| f \|^{2(n-k)}}{(n-k)!} &= \left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \right|_{t=0} \sum_{n \geqslant k} \mathrm{e}^{tn} \frac{\| f \|^{2(n-k)}}{(n-k)!} \\ &= \left. \frac{\mathrm{d}^k}{\mathrm{d}t^k} \right|_{t=0} \mathrm{e}^{tk} \mathrm{e}^{\| f \|^2 \mathrm{e}^t}. \end{aligned} \tag{10.26}$$

Moreover, because of our assumption (10.9) on the test functions $f$, we can restrict our attention to the case in which $\| f \| = 1$ in (10.26). (We could have put $\| f \| = 1$ directly in (10.25), but it is convenient to leave it to show the opportunity of introducing *Bell numbers depending on a parameter*.) In this case by Leibnitz rule the expression (10.26) is

$$\sum_{h=0}^{k} \binom{k}{h} k^h B_2(k-h), \tag{10.27}$$

where $B_2(k-h)$ are the Bell numbers of order 2 as defined in [9].

Under this assumption denoting

$$c := 16L, \tag{10.28}$$

we have

$$\begin{aligned} \| P_n Y_t^{(k+1)} \psi_f \|^2 &\leqslant \frac{c^k}{(k-1)!} \sum_{h=0}^{k} \binom{k}{h} k^h B_2(k-h) \\ &= (kc^k) \sum_{h=0}^{k} \frac{k^h}{h!} \frac{B_2(k-h)}{(k-h)!}. \end{aligned} \tag{10.29}$$

Now, since all the terms involved are positive, clearly

$$\sum_{h=0}^{k} \frac{k^h}{h!} \frac{B_2(k-h)}{(k-h)!} \leqslant \left( \sum_{h=0}^{k} \frac{k^h}{h!} \right) \left( \sum_{h'=0}^{k} \frac{B_2(k-h)}{(k-h)!} \right)$$

and, from [9] we know that this is

$$\leqslant \mathrm{e}^k G_2(1)/2,$$

where $G_2$ is an analytic function. Therefore

$$\|Y_t^{(k+1)}\psi_f\|^2 \leqslant G_2(1)k(ce)^k/2. \tag{10.30}$$

But if $ce < 1$ or, equivalently due to (10.28), if

$$L < \frac{1}{16e}$$

the series on the right-hand side of (10.30) is convergent. □

## 11. An Example

In this section we produce an example of coefficients which satisfy condition (iii) of Equation (10.11). Let, for $s, t \in \mathbf{R}^d$

$$L_{\varepsilon,\varepsilon'}(s,t) = L_{\varepsilon,\varepsilon'}\psi(|s|)\chi_{[0,|t|)}(|s|)\varphi(\hat{s},\hat{t}), \tag{11.1}$$

where $L_{\varepsilon,\varepsilon'} \in \mathcal{B}(\mathcal{H}_S)$ $(\varepsilon, \varepsilon' = 0, 1)$,

$$\chi_I(x) = \begin{cases} 0, & \text{if } x \notin I \subseteq \mathbf{R}, \\ 1, & \text{if } x \in I. \end{cases} \tag{11.2}$$

$\psi\colon \mathbf{R}_+ \to \mathbf{C}$ and $\varphi\colon S^{(d)} \times S^{(d)} \to \mathbf{C}$ are continuous functions ($S^{(d)}$ is the unit sphere in $\mathbf{R}^d$) and

$$t = |t|\hat{t} \in \mathbf{R}^d; \quad |t| \in \mathbf{R}_+; \ \hat{t} \in S^{(d)} \quad \text{(unit sphere in } \mathbf{R}^d) \tag{11.3}$$

is the polar decomposition of $t \in \mathbf{R}^d$. Then

$$\begin{aligned}
&\int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \mathrm{d}s_1 \ldots \mathrm{d}s_k l(s_1,t)l(s_1,s_2)\ldots l(s_k,s_{k-1}) \\
&\quad = \int \cdots \int \rho_1^{d-1}\,\mathrm{d}\rho_1\,\mathrm{d}\hat{s}_1 \ldots \rho_k^{d-1}\,\mathrm{d}\rho_k\,\mathrm{d}\hat{s}_k\,\chi_{[0,t]}(\rho_1)\chi_{[0,\rho_1)}(\rho_2)\ldots\chi_{[0,\rho_{n-1})}(\rho_n)\times \\
&\qquad \times \varphi(\hat{s}_1,\hat{t})\varphi(\hat{s}_2,\hat{s}_1)\ldots\varphi(s_k,s_{k-1})\psi(\rho_1)\ldots\psi(\rho_k) \\
&\quad \leqslant (|t|^{d-1})^k\|\varphi\|_\infty^k\sigma_d^k \cdot\cdot \int_0^{|t|}\mathrm{d}\rho_1\int_0^{\rho_1}\mathrm{d}\rho_2\ldots\int_0^{\rho_{k-1}}\mathrm{d}\rho_k\psi(\rho_1)\ldots\psi(\rho_k) \\
&\quad = (|t|^{d-1})^k\|\varphi\|_\infty^k\sigma_d^k\frac{(\int_0^{|t|}\psi(s)\,\mathrm{d}\rho)^k}{k!}.
\end{aligned}$$

Therefore, if $B \subseteq \mathbf{R}^d$ is a bounded set and $t \in B$, condition (iii) is satisfied.

## References

1. Accardi, L., Frigerio, A. and Lu, Y. G.: On the weak coupling limit problem, In: *Quantum Probability and Applications IV*, Lecture Notes in Math. 1396, Springer, New York, 1987, pp. 20–58.

2. Accardi, L., Frigerio, A. and Lu, Y. G.: The weak coupling limit as a quantum functional central limit, *Comm. Math. Phys.* **131** (1990), 537–570.
3. Accardi, L., Lu, Y. G. and Volovich, I. V.: The stochastic sector of quantum field theory, Volterra Preprint 138, 1993; *Mat. Zam.* (1994).
4. Accardi, L., Lu, Y. G. and Volovich, I. V.: Non-commutative (quantum) probability, master fields and stochastic bosonization, Volterra Preprint CVV-198-94, hep-th/9412241.
5. Accardi, L. and Mohari, A.: Stochastic flows and imprimitivity systems, In: *Quantum Probability and Related Topics*, World Scientific, Singapore, 1994, pp. 43–65. Volterra Preprint, 1994.
6. Accardi, L., Lu, Y. G. and Volovich, I. V.: Nonlinear extensions of classical and quantum stochastic calculus and essentially infinite dimensional analysis, In: L. Accardi and Chris Heyde (eds), *Probability Towards 2000*, Lecture Notes in Statist. 128, Springer, New York, 1998, pp. 1–33.
7. Accardi, L. and Volovich, I. V.: Quantum white noise with singular non-linear interaction, Volterra Preprint 278, 1997.
8. Accardi, L., Lu, Y. G. and Volovich, I. V.: *A White Noise Approach to Classical and Quantum Stochastic Calculus*, Volterra Preprint 375, Rome, July 1999, World Scientific, Singapore, 2000.
9. Cochran, W. G, Kuo, H.-H. and Sengupta, A.: A new class of white noise generalized functions, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **1** (1998), 43–67.
10. Hida, T.: *Analysis of Brownian Functionals*, Carleton Mathematical Lecture Notes 13, 1975.
11. Hida, T.: The impact of classical functional analysis on white noise calculus, Volterra Preprints 90, 1992.
12. Hida, T., Kuo, H.-H., Potthoff, J. and Streit L.: *White Noise. An Infinite Dimensional Calculus*, Kluwer Acad. Publ., Dordrecht, 1993, pp. 185–231.
13. Hida, T., Potthoff, J. and Streit, L.: *White Noise Analysis and Applications*, Kluwer Acad. Publ., Dordrecht, 1994.
14. Hudson, R. L. and Parthasarathy, K. R.: Construction of quantum diffusions, In: L. Accardi, A. Frigerio and V. Gorini (eds), *Quantum Probability and Applications to the Quantum Theory of Irreversible Processes*, Lecture Notes in Math. 1055, Springer, New York, 1984.
15. Hudson, R. L. and Parthasarathy, K. R.: Quantum Itô's formula and stochastic evolutions, *Comm. Math. Phys.* **93** (1984), 301–323.
16. Kuo, H.-H.: *White Noise Distribution Theory*, CRC Press, Boca Raton, 1996.
17. Parthasarathy, K. R.: Quantum stochastic calculus, Preprint, 1995.
18. Obata, N.: Coherent state representation and unitarity condition in white noise calculus, *J. Korean Math. Soc.* (2000), to appear; White noise operator theory: fundamental concepts and developing applications, to appear in *Proc. Volterra Internat. School*, World Scientific, Singapore, 2000.