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A NEKHOROSHEV THEOREM FOR SOME INFINITE–DIMENSIONAL SYSTEMS

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Abstract. We study the persistence for long times of the solutions of some infinite–dimensional discrete hamiltonian systems with *formal hamiltonian* $\sum_{i=1}^{\infty} h(A_i) + V(\varphi)$, $(A,\varphi) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$. $V(\varphi)$ is not needed small and the problem is perturbative being the kinetic energy unbounded. All the initial data $(A_i(0), \varphi_i(0))$, $i \in \mathbb{N}$ in the phase–space $\mathbb{R}^{\mathbb{N}} \times \mathbb{T}^{\mathbb{N}}$, give rise to solutions with $|A_i(t) - A_i(0)|$ close to zero for exponentially–long times provided that $A_i(0)$ is large enough for $|i|$ large. We need $\frac{\partial h}{\partial A_i}(A_i(0))$ unbounded for $i \to +\infty$ making φ_i a *fast variable*; the greater is i, the faster is the angle φ_i (avoiding the resonances). The estimates are obtained in the spirit of the averaging theory reminding the analytic part of Nekhoroshev–theorem.

1. Introduction. In the study of hamiltonian ordinary differential equations, two of the main problems are: 1) to prove the existence of the solutions for a time as long as possible 2) to understand the qualitative properties of the solutions found. As a model problem let's consider the hamiltonian $H(A,\varphi) = h(A) + \varepsilon V(\varphi)$, $\varphi \in \mathbb{T}^N$, $A \in U = \overset{o}{U} \subset \mathbb{R}^N$, ($N \geq 2$ integer), h, V analytic functions, ε real. The canonical equations are of course $\dot{A} = -\varepsilon V_{\varphi}$, $\dot{\varphi} = h_A$ whose solution for $\varepsilon = 0$ is

$$A(t) \equiv A^o, \qquad \varphi(t) = \varphi^o + t\omega(A^o) \qquad \omega(A^o) \doteq \frac{\partial h}{\partial A}(A^o) \in \mathbb{R}^N \tag{1.1}$$

As well known by the theory of quasi–periodic motions, $\overline{\{\varphi(t)\}_{t\in\mathbb{R}}} = \mathbb{T}^N$ if and only if the components of $\omega(A^o)$ are non–resonant over $\mathbb{Z}^N$ i.e. $\sum_{i=1}^N \omega_i(A^o)\nu_i \neq 0$ for any $\nu \in \mathbb{Z}^N$ and $\sum_{i=1}^N |\nu_i| \neq 0$. Otherwise we have $\overline{\{\varphi(t)\}_{t\in\mathbb{R}}} = \mathbb{T}^{N-k}$, $1 \leq k \leq N-1$ (for $k = N-1$ the solution is periodic). By the celebrated KAM theorem, under some conditions which are essentially: *i)* the determinant of the matrix $\frac{\partial^2 h}{\partial A^2}$ different from zero, *ii)* $\omega(A^o)$ non–resonant over $\mathbb{Z}^N$: $\left|\sum_{i=1}^N \omega_i(A^o)\nu_i\right|^{-1} \leq C\,|\nu|^N$ for all $\sum_{i=1}^N |\nu_i| \neq 0$, $\nu_i \in \mathbb{Z}$, C suitable ($\omega(A^o)$ is said Diophantine), *iii)* $|\varepsilon| \leq \varepsilon_0$ small enough ,(1.1) can be continued into

$$A(t) = A^o + \alpha_\varepsilon(t\omega(A^o)), \quad \varphi(t) = \varphi^o + t\omega(A^o) + \beta_\varepsilon(t\omega(A^o)), \quad t \in \mathbb{R},$$

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(α_ε and β_ε are analytic functions of $t\omega(A^o)$ and ε such that $|\alpha_\varepsilon|+|\beta_\varepsilon|\xrightarrow[\varepsilon\to 0]{}0$; hence $(A(t),\varphi(t))$ are defined for all times in spite of $\varepsilon\neq 0$ (see [6] for a proof).[1]
The set of vectors ω satisfying *ii)* has full Lebesgue measure but is nowhere dense and the theorem cannot avoid its presence even weakening the non–resonance condition, see [14], [7], [13].

For including all the vectors ω and then all initial data by the diffeomorphism $\omega(A)=\frac{\partial h}{\partial A}$, one is forced to give up the solutions globally defined in time. This is essentially the content of the *Nekhoroshev Theorem* (see the pioneering work [11], the papers (plenty) of the "Milan group" of Bambusi, Benettin, Galgani, Giorgilli etc. and see also [3], [1] with the references therein). Let's consider again the hamiltonian $h(A)+\varepsilon V(\varphi)$ with the conditions *i)* and *iii)*. Roughly speaking, *all the solutions* are shown to exist for $|t|\le T\exp(\frac{1}{\varepsilon^a})$ and $|A(t)-A(0)|\le A_o\varepsilon^b$. No subset of the phase–space is excluded; a, b are positive constants depending on N such that $a\xrightarrow[N\to+\infty]{}0$. This is the finite–dimensional situation.

As far as we know, there are few papers on the extensions of the stability results for infinite–dimensional discrete systems (not originating from PDE's). The first one is [2] where an array of coupled *harmonic oscillators* over $\mathbb{Z}^d$ is considered. The hamiltonian is $H_1(p,q)=K+V$, $K=\frac{1}{2}\sum_{j\in\mathbb{Z}^d}\omega_j(p_j^2+q_j^2)$, $V=\sum_{i,j\in\mathbb{Z}^d}\sum_{k=0}^3 V_{(i,j)k}q_i^k q_j^{3-k}$; $(p_j,q_j)\in\mathbb{R}^2$. The coefficients $V_{(i,j)k}$ satisfy $\sum_{k=0}^3|V_{(i,j)k}|\le Ue^{-\alpha(1+dist(i,j))^\delta}$; ($U$, α, constants, $\delta\le 1$). According to our definitions V is of long–range type; see (2.2). Roughly speaking they prove that if the energy of the initial datum is of order ε (small) and concentrated in one point (say the origin), the variables $(p_i(t),q_i(t))_{i\in\mathbb{Z}^d}$ remain close to their initial value as long as $|t|\le\exp\frac{a(\ln\varepsilon^{-1})^2}{\ln\ln\varepsilon^{-1}}$ (faster than any power of ε but slower than an exponential). Each ω_j is a gaussian random variable with the same variance σ and the measure of the set of $\omega=\{\omega_j\}_{j\in\mathbb{Z}^d}$ excluded is $O(\frac{\alpha}{\sigma})$. In [3] the authors consider an infinite–dimensional hamiltonian system like (1.2) with $h(A)=\frac{1}{2}A^2$, $V_1(\varphi)=\varepsilon\sum_{i,j\in\mathbb{Z},i\neq j}\frac{1}{|i-j|^\alpha}(1-\cos(\varphi_i-\varphi_j))$ $\alpha>1$, ε a small parameter. When $\varepsilon\neq 0$ and small, the exponential stability for those quasi–periodic solutions whose vector–frequency ω has an arbitrary, finite number of components is proved. In [1] the author shows the exponential stability for the so called *breathers*, i.e. time–periodic, spatially localized solutions of perturbed systems whose hamiltonian is $H_2=\sum_{k\in\mathbb{Z}}(\frac{1}{2}p_k^2+V(q_k))+\frac{\varepsilon}{4}\sum_{i,j\in\mathbb{Z},i\neq j}\frac{1}{|i-j|^\alpha}(q_i-q_j)^2$, $V'(0)=0$, $V''(0)>0$, $\alpha>1$, ε small as usual. In all these models the kinetic and the potential energy are finite and the thermodynamic limit does not follow. Loosely speaking, a common feature of the previous results is the fact that "most of the energy is contributed" by few variables (hence the variables placed far away carry a small amount of energy).

Here we generalize the conclusions of the Nekhoroshev theorem too but in our model most of the energy is contributed by the variables far from the origin. We consider a class of infinitely many ODE's

$$\dot{A}_i=f_i(\varphi)\qquad\dot{\varphi}_i=h_{A_i}(A_i),\qquad i\in\mathbb{N},\quad A_i\in\mathbb{R},\ \varphi_i\in\mathbb{T},\quad\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}\quad(1.2)$$

For instance if $h(x)=\frac{1}{2}x^2$ and $f_i(\varphi)=-\frac{\partial V}{\partial\varphi_i}$ for a suitable function V, (1.2) would be the canonical equations of the *formal* hamiltonian $\sum_{i=1}^\infty\frac{1}{2}A_i^2+V(\varphi)$,

[1] ε_0 goes to zero when C and/or N goes to infinity. Up to some technicalities, it can be stated an analogue theorem for systems such that $\det(\frac{\partial^2 h}{\partial A^2})$ is equal to zero (for example harmonic oscillators where $h(A)\doteq\sum_{i=1}^N\omega_i A_1$ or celestial mechanics systems where some of the A_i's are not present in $h(A)$).

$\varphi = (\varphi_1, \varphi_2, \varphi_3, \dots) \in \mathbb{T}^{\mathbb{N}2}$. A Nekhoroshev–like theorem is proved. More specifically if we suppose in (1.2) that $0 < a' \leq |h_{A_i A_i}| \leq a < +\infty$, and $\frac{\partial h}{\partial A_i}(A_i(0))$ sufficiently large for $i \geq l-1 \geq 1$, we prove the existence of an increasing sequence of *time–scales* $\{t_{l+k}\}_{k\geq 0}$ $(t_{l+k+1} \geq t_{l+k})$ such that the action–variable $A_{l+k}(t)$ remains very close to its initial value $A_{l+k}(0)$ as long as $|t| \leq t_{l+k}$. The larger $A_{l+k}(0)$ is, the closer $A_{l+k}(t)$ remains to it. It follows that for $|t| \leq t_l$ the variables $A_l, A_{l+1}, A_{l+2}, \dots$ and $\varphi_l, \varphi_{l+1}, \varphi_{l+2}, \dots$ "do not affect" the motion of the system whose *effective hamiltonian* is: $\sum_{i=1}^{l-1} h(A_i) + V^{(l-1)}(\varphi_1, \dots, \varphi_{l-1})$; $\frac{\partial}{\partial \varphi_i} V^{(l-1)}(\varphi_1, \dots, \varphi_{l-1}) = -\int d\mu_J f_i(\varphi)$ and $J = \mathbb{N} \backslash \{1, 2, \dots, l-1\}$ (the possibility of doing the average respect to the infinite set of variables φ_i $i \in J$, is due to the weak topology introduced in the configuration space $\mathbb{T}^{\mathbb{N}}$; see section 2). If $\frac{\partial h}{\partial A_{l+k}} \doteq \omega^o_{l+k}$, we have $|A_{l+k}(t) - A_{l+k}(0)| = O(|\omega^o_{l+k}|^{-1})$ for $|t| \leq t_{l+k} \sim \exp\{O((\omega^o_{l+k})^{1/2})\}$ and $\varphi_{l+k}(t) \sim \varphi^o_{l+k} + \omega^o_{l+k} t$. The frequency ω^o_{l+k+1} is bigger than ω^o_{l+k} in such a way to determine a strong non–resonance condition. Actually we find $\omega^o_{l+k+1} \sim \exp\{O((\omega^o_{l+k})^{1/2})\}$.

Without doing any hypotheses on the size of $(\omega_1, \dots, \omega_{l-1})$, all we can say about the variables $A_i(t)$ $i = 1, 2, \dots, l-1$, is $|A_i(t) - A_i(0)| \leq C\rho$, where C is a constant and ρ is the size of the analyticity of the domain of the function $h(A)$. If $A_i(0)$ is not great for $|i| \to +\infty$ we cannot say that $|A_i(t) - A_i(0)|$ is small for large t.

This result is in agreement with [13] (see also [5], [12], [4]) where the same system of equations is considered and proved that if: 1) $|\mu|$ is different from 0 and small enough, 2) ω^o_k large enough and the vector $(\omega^o_1, \omega^o_2, \dots, \omega^o_j)$, $j \geq 1$ suitably non–resonant, then the solution of (1.2) for any t is $\big(\omega^o t = (\frac{\omega^o_1 t}{\mu}, \frac{\omega^o_2 t}{\mu}, \dots, \frac{\omega^o_{l-1} t}{\mu}, \omega^o_l t,$ $, \omega^o_{l+1} t, \dots,)\big)$

$$\varphi_i(t) = \varphi^o_i + \omega^o_i t + \alpha_i(\omega^o t), \quad A_i(t) = A^o_i + \beta_i(\omega^o t),$$

i.e. almost–periodic. Hence our theorem can be viewed also as a result about the persistence for long times of almost–periodic motions.

(1.2) can be viewed as a model of crystal lattice although we cannot perform a *thermodynamic limit* yet due to the very high energy per degree of freedom of the kinetic part (needed for applying the perturbative and averaging methods).

The paper is organized as follows. In section 2 we give the setup and some definitions. Section 3 contains the main results while the intermediate results and all the proofs, sometime sketched, are in Section 4.

2. Setup–Definitions.

Metrics ([9], [8]) The distance on $\mathbb{T}^{\mathbb{N}}$ is $\rho_w(\varphi, \varphi') \equiv \sum_{i\in\mathbb{N}} \rho(\varphi_i, \varphi'_i)\ w_i \qquad w_i > 0,\ \ \sum_{i\in\mathbb{N}} w_i < +\infty$, $(\mathbb{T}^{\mathbb{N}}, \rho_w) \doteq \mathcal{T}_w$ is a compact space. ρ is the standard (flat) metric on $\mathbb{T} \equiv \mathbb{T}_i$: $\rho([a], [b]) \equiv \inf_{n\in\mathbb{Z}} |a - b + 2\pi n|$ $a, b \in \mathbb{R}$ and $[\cdot]$ denotes equivalence (mod. 2π) class.

The distance on $\mathbb{R}^{\mathbb{N}}$ is $\lambda_w(A, A') \equiv \sum_{i\in\mathbb{N}} w_i \arctan|A_i - A'_i|$. $(\mathbb{R}^{\mathbb{N}}, \lambda_w) \doteq \mathcal{R}_w$ is a complete Banach space.

With the metrics given, the convergence is equivalent to the *weak convergence* ("component by component"): $\varphi^{(n)} \xrightarrow[n\to+\infty]{} \varphi$ means that $\forall$ i $\varphi^{(n)}_i \xrightarrow[n\to+\infty]{} \varphi_i$ (no uniformity in the components) and the same occurs for the space $\mathcal{R}_w$.

[2] We could have considered a system defined over $\mathbb{Z}^d$ and work on $\mathbb{N}$ after applying a bijection of $\mathbb{Z}^d$ onto $\mathbb{N}$.

Forces–Perturbations. We consider two examples of maps $\{f_i\}$ (the force). The first one is so called *short range*; fix $L \geq 1$

$$f_i \equiv \sum_{\|j-i\| \leq L} \partial_{\varphi_i} g_j \tag{2.1}$$

$g_j \equiv g_j(\varphi^{(L)})$, $\varphi^{(L)} \equiv \{\varphi_k\}_{k \in B_j(L)}$, $B_j(L) \equiv \{k : \|k-j\| \leq L\}$, $\|i-j\|$ is the Euclidean distance on $\mathbb{N}$. g_j are *real–analytic* functions from $\mathbb{T}^{|B_j(L)|} \to \mathbb{R}$ and for some positive M we have $\sup_{j,\varphi^{(L)} \in \mathbb{T}^{|B_j(L)|}} |g_j(\varphi^{(L)})| \leq M$.

The system (1.2) with such f_i is called a finite range system of infinitely many coupled variables. A particular case, often considered, is given in $d=1$ by $L=1$, $g_j = \cos(\varphi_j - \varphi_{j-1}) - \cos(\varphi_{j+1} - \varphi_j)$. Note that each variable is coupled only with a finite number of different variables

The second example is so called *long range* as each variable is coupled with any other variable. In $d=1$ it is given by

$$f_i \equiv \cos\varphi_i \sum_{j \in \mathbb{N}} a_j \prod_{k \neq 0} (1 + a_{j+k} \sin\varphi_{i+k}) \ , \quad \sum_{j \in \mathbb{N}} |a_j| < \infty \tag{2.2}$$

We point out that we don't need the existence of a function $V\colon \mathbb{T}^{\mathbb{N}} \to \mathbb{R}$ such that $-\frac{\partial V}{\partial \varphi_i} = f_i(\varphi)$. For instance we could take $V(\varphi) = -\sum_{i=1}^{\infty} mg(1-\cos\varphi_i) + \sum_{i=1}^{\infty} \kappa(1-\cos(\varphi_{i+1}-\varphi_i))$ for the short range case and $V(\varphi) = \sum_{i,j=1}^{\infty} e^{-|i-j|}(1-\cos(\varphi_i - \varphi_j))$ for the long range. In this sense (2.2) would be the derivative of the "function" $\sum_{n=1}^{\infty} \prod_{m=1}^{\infty} (1 + a_m \cos\varphi_{n+m})$. What we need well defined are certain averages described here

Averages ([Ha] section 38, [8]). For a measurable function $g\colon \mathcal{T}_w \to \mathbb{R}$, let's define the functions $g^{[I]}$ by means of

$$g^{[I]}\colon \mathbb{T}^{|I|} \to \mathbb{R}, \quad g^{[I]} \doteq \int g(\varphi) d\mu_J, \quad I \subset \mathbb{N}, \quad J = \mathbb{N} \backslash I, \quad d\mu_J = \bigotimes_{i \in J} d\mu_i$$

In $\mathcal{T}_w$ there exists a unique *probability measure* defined over the σ–algebra, $\mathcal{R}$, generated by the cylinders

$$\mathcal{R}_I = \bigotimes_{i \in I \subset \mathbb{N}} U_i \bigotimes_{j \notin I} \mathbb{T}_i, \quad U_i = \overset{o}{U_i} \subset \mathbb{T}_i, \quad |I| < \infty, \quad \mu(\mathcal{R}_I) = \prod_{i \in I} \mu_i(U_i)$$

where μ_i is the normalized "Lebesgue measure" on $\mathbb{T}_i$. If $|I| < \infty$, $g^{[I]}$ is a measurable function on $\mathbb{T}^{|I|}$ and $g^{[I]} \to g$ *a.e.* on $\mathcal{T}$ as $|I| \to \mathbb{N}$. For the examples in (2.1) and (2.2) the convergence is uniform

For the infinite–dimensional vector $\{f_i\}$ we shall suppose that for any finite $I \subset \mathbb{N}$ there exists a $C^1(\mathbb{T}^{|I|}; \mathbb{R})$ function, $V^{(I)}(\varphi)$, such that

$$f_i^{[I]}(\varphi) = -\partial_{\varphi_i} V^{(I)}(\varphi), \quad \forall\ i \in I \ , \qquad \forall\ \varphi \in \mathbb{T}^{|I|}.$$

We shall speak of *g–gradients.*

Definition. *A g-gradient f is said* uniformly weakly real-analytic *if there exists a real number $\sigma > 0$ such that for any finite set $I \subset \mathbb{Z}$, $V^{(I)}(\varphi)$ is real-analytic on $\mathbb{T}^{|I|}$*

and can be continued: analytically to the set $\{z \in \mathbb{C}^{|I|} : \ Re\, z_i \in \mathbb{T},\ |Im\, z_i| < \sigma\}$, *continuously on the closure*

Function–spaces. We shall work in the analytic class. Let $f\colon V \times \mathbb{T}^l \to \mathbb{R}$, $V = \overset{o}{V} \subset \mathbb{R}^l$ be an analytic function. $f = f(A, \varphi)$ can be extended to an holomorphic function on the complex domain $D \times \Delta_\xi \subset \mathbb{C}^l \times \mathbb{C}^l$ where $D = \overset{o}{D} = \cup_{x\in V}\{z \in \mathbb{C}^l : |z - x| < \rho\} \supset V$, $\Delta_\xi = \{z \in \mathbb{C}^l\colon\ Re\, z_i \in \mathbb{T},\ |Im\, z_i| < \xi\}$, $0 < \xi < 1$. The extension (which is called f too) is continuous on the closure $\overline{D} \times \overline{\Delta}_\xi$. This class of functions is denoted by $C^\omega(D \times \Delta_\xi; \mathbb{C}) \cap C(\overline{D} \times \overline{\Delta}_\xi; \mathbb{C})$ and its elements can be decomposed as

$$f(A, \varphi) = \sum_{\nu\in\mathbb{N}^l} e^{i\nu\cdot\varphi} f_{\nu,k}(A) = \sum_{\nu\in\mathbb{N}^l} e^{i\nu\cdot\varphi} \frac{1}{(2\pi)^l} \int_{\mathbb{T}^l} d\varphi e^{-i\nu\cdot\varphi} f(A, \varphi)$$

$\nu \cdot \varphi = \sum_{i=1}^l \nu_i \varphi_i$, $\quad \|f\|_{\rho,\xi} \doteq \sum_{\nu\in\mathbb{N}^l} e^{|\nu|\xi} \sup_{A\in D(A^o;\rho)} |f_\nu(A)| \doteq \sum_{\nu\in\mathbb{N}^l} e^{|\nu|\xi} \|f_\nu\|_\rho$, $|\nu| = \sum_{i=1}^l |\nu_i|$

$$\|f_\varphi\|_{\rho,\xi-\delta} \le \frac{1}{e\delta}\|f\|_{\rho,\xi}, \quad \|f_{A_j}\|_{\rho-r,\xi} \le \frac{1}{r}\|f\|_{\rho,\xi}, \quad \|f_A\|_{\rho-r,\xi} \le \frac{l}{r}\|f\|_{\rho,\xi}$$

$$\|f_{A,\varphi}\|_{\rho-r,\xi-\delta} \le \frac{1}{\delta}\frac{l}{r}\|f\|_{\rho,\xi} \qquad \|f\, g\|_{\rho,\xi} \le \|f\|_{\rho,\xi}\|g\|_{\rho,\xi}$$

For a vector valued function whose components are functions in $C^\omega(D \times \Delta_\xi; \mathbb{C}) \cap C(\overline{D} \times \overline{\Delta}_\xi; \mathbb{C})$, the norm is the sum of the norm of their components. For a matrix valued function $\{M(x, y)\}_{i,j=1}^k, \colon D \times \Delta_\xi \to \mathbb{C}^{2k}$ we set (only for the tensorial components)

$$\|M\| \doteq \sup_{v\in\mathbb{R}^k;|v|=1} |M(x, y)v| = \sup_{v\in\mathbb{R}^k;|v|=1} \sum_{i=1}^k |\sum_{j=1}^k M_{ij}(x, y)v_j|$$

For a generic square matrix $M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$ (each block is a $p \times p$ matrix), thinking of it as a linear operator over $\mathbb{R}^p \times \mathbb{R}^p$ and acting over the vectors $v = (v_1, v_2)$, $v_{1,2} \in \mathbb{R}^p$ with metric $|v| = |v_1| + \rho|v_2|$, we have $\|M\| = \sup_{v\neq 0} \frac{\|Mv\|}{|v|} = \frac{|M_1v_1+M_2v_2+\rho M_3v_1+\rho M_4v_2|}{|v_1|+\rho|v_2|} \le \|M_1\| + \frac{1}{\rho}\|M_2\| + \rho\|M_3\| + \|M_4\|$.

We shall make use also of the following notations: $v = (\hat{v}, v_d)$ where $\hat{v} \in \mathbb{R}^{d-1}$ or $v = (\check{v}, v_{d-1}, v_d)$ with $\check{v} \in \mathbb{R}^{d-2}$.

The idea of the proof in Theorem 3.1 is of “breaking” (1.2) in a sequence of finite–dimensional systems and then work in a finite–dimensional setting. Nevertheless, for obtaining the solution of (1.2), we have to make certain limits in suitable infinite–dimensional function–spaces which we are going to describe.

Let be : 1) $\tilde{D}^{\mathbb{N}}_\rho(A^o) = \otimes_{i\in\mathbb{N}} B(\rho; A^o_i) \subset \mathcal{R}_w$, $B(\rho; A^o_i) \subset \mathbb{R}$ is an interval centered in A^o_i of length 2ρ 2) $\tilde{D}^{(k)}_\rho(A^o_{(k)}) = \otimes_{i=1}^k B(\rho; A^o_i) \subset \mathbb{R}^k$ 3)$f\colon \tilde{D}^{\mathbb{N}}_\rho(A^o) \times \mathcal{T}_w \to \mathbb{R}$, $f = f(A, \varphi)$ a function integrable respect to measure over $\mathcal{T}_w$ and continuous on $\tilde{D}^{\mathbb{N}}_\rho(A^o)$,

Let’s call $\mathcal{A}$ the vector–space of functions defined in 3). $f \in \mathcal{A}$ can be given the Fourier series $f \sim \sum^*_{\nu\in\mathbb{Z}^{\mathbb{N}}} f_\nu(A)e^{i\nu\cdot\varphi}$ where the $\sum^*$ means that $f_\nu(A)$ are zero unless $\nu \cdot \varphi = \sum_{j=1}^k \nu_i\varphi_i$ for some $k \in \mathbb{N}$.

The Fourier coefficients f_ν determine f almost–everywhere and viceversa when f is integrable (everywhere when f is continuous)

$f^{(k)}(A,\varphi^{(k)}) \doteq \int f(A,\varphi)d\mu_J$ $J=\mathbb{N}\backslash\{1,2,\dots,k\}$ is well defined on $\tilde{D}^{\mathbb{N}}_\rho(A^o)\times\mathbb{T}^k$. By what said before, $f^{(k)}(A,\varphi^{(k)})$ can be extended to an holomorphic function of the variables $(\varphi_1,\varphi_2,\dots,\varphi_k)$ on the domain $\tilde{D}^{\mathbb{N}}_\rho(A^o)\times\Delta_\xi$ $(\Delta_\xi\subset\mathbb{C}^k)$. If $f^{(k)}\in\mathcal{A}$ depends only on a finite number of $A_i's$, $(A_1,\dots,A_k)$ for instance, $f^{(k)}$ can be extended to an holomorphic function also respect to these variables.

The space $\mathcal{A}$ can be endowed with the norm $\|f\|_\xi=\sum^*_{\nu\in\mathbb{Z}^{\mathbb{N}}}\sup_{A\in\tilde{D}^{\mathbb{N}}_\rho(A^o)}|f_\nu(A)|e^{|\nu|\xi}$ which makes it a Banach space

For a *g–gradient* $\{f_i\}$ we define $\|V^{(I)}\|_\sigma\le\mathcal{V}^{(|I|)}$ and let's suppose that $\mathcal{V}^{(|I|)}\le\mathcal{V}^{(|I|+1)}$ (otherwise $\mathcal{V}^{(|I|+1)}=\max\{\mathcal{V}^{(|I|)},\|V^{(I+1)}\|_\sigma\}$).

Great denominators. $\{h_{A_i}\}_{i=1,\dots,l}\doteq(h_{\hat{A}},h_{A_l})\colon\mathbb{R}^l\mapsto\mathbb{R}^l$, $\quad h_{\hat{A}}\doteq\{h_{A_i}\}_{i=1,\dots,l-1}$ For $A^o\in\mathbb{R}^l$ we will write $h_{A^o}\doteq h_A(A^o)\doteq\{h_{A_i}(A^o_i)\}\doteq\omega^o$.

The initial data of the system (1.2) are $(A^o,\varphi^o)\in\mathbb{R}^{\mathbb{N}}\times\mathbb{T}^{\mathbb{N}}$ and A^o is such that for any $k\ge l-1\ge1$, the d–dimensional vector $(h_{A^o_1},h_{A^o_2},\dots,h_{A^o_k})\doteq(\omega^o_1,\omega^o_2,\dots,\omega^o_k)$ verifies the relation

$$|\hat{\omega}^o\cdot\hat{\nu}+\omega^o_k\nu_k|^{-1}\le\frac{\gamma}{|\omega^o_k|}\quad|\nu|\le K(\omega^o),\quad\nu_k\ne0,\quad\gamma>1\quad K=O(|\omega^o_k|^{1/2})\tag{2.3}$$

3. Results. Let be $l\ge2$, $l_k=l+k$, $\|V^{(l_k)}\|_\xi\le\mathcal{V}^{(l_k)}$ for $V^{(l_k)}(\varphi_1,\dots,\varphi_{l_k})$

Theorem 3.1. *Let's consider the system (1.2) and let $A^o\in\mathbb{R}^{\mathbb{N}}$ be such that the vector $\omega^o_i=h_{A^o_i}$ $i=(l,l+1,l+2,\dots)$ satisfies ($k\ge0$ integer, C and C' universal constants)*

$$\frac{C}{\xi}\frac{\mathcal{V}^{(l_k)}}{\rho|\omega^o_{l_k}|}(k+1)^3\ln^6(k+2)+\frac{C}{\xi}\frac{a\rho}{|\omega^o_{l_k}|}<1,\qquad C\frac{\|\omega^o_{l_{k-1}}\|\rho}{\mathcal{V}^{(l_k)}}\Big(\frac{\mathcal{V}^{(l_k)}}{|\omega^o_{l_k}|\rho}\Big)^{1/2}\xi^{-1/2}\le1\tag{3.1}$$

$$k\ge1,\quad|\omega^o_{l_k}|\ge C\frac{(\mathcal{V}^{(l_k)})^2}{(\mathcal{V}^{(l_{k-1})})^2}|\omega^o_{l_{k-1}}|\exp\Big\{\frac{C}{k\ln^2(k+1)}\sqrt{\frac{\rho\xi}{\mathcal{V}^{(l_0)}}|\omega^o_{l_{k-1}}|}\Big\}\tag{3.2}$$

There exists a transformation $\mathcal{R}_\infty\colon D^{\mathbb{N}}_{\frac{\rho}{2}}(A^o)\times\Delta^{\mathbb{N}}_{\frac{\xi}{2}}\to D^{\mathbb{N}}_\rho(A^o)\times\Delta^{\mathbb{N}}_\xi$ such that in the new variables (v,u) defined by $(A,\varphi)=\mathcal{R}_\infty(v,u)$, (1.2) becomes

$$\frac{d}{dt}u_i=h_{v_i}+\frac{\partial}{\partial v_i}\sum_{j=0}^{\infty}G^{(l_j)}_{n_{l_j}}\tag{3.3}$$

$$\frac{d}{dt}v_i=-\frac{\partial}{\partial u_i}\Big(V^{(l-1)}+\sum_{j=0}^{\infty}G^{(l_j)}_{n_{l_j}}\Big)\qquad i\le l-1\tag{3.4}$$

$$\frac{d}{dt}v_{l_m}=-\frac{\partial}{\partial u_{l_m}}\sum_{j=m+1}^{\infty}G^{(l_j)}_{n_{l_j}}\qquad i=l-1+m,\quad m>0\tag{3.5}$$

$$\begin{aligned}&j\ge1\quad\|G^{(l_j)}_{n_{l_j}}\|_{D^{\mathbb{N}}_{\frac{\rho}{2}}\times\Delta^{\mathbb{N}}_{\frac{\xi}{2}}}\le Cj^2(\ln^4 j)\frac{(\mathcal{V}^{(l_{j-1})})^2}{\xi\rho|\omega^o_{l_{j-1}}|}\exp\Big\{\frac{-C'}{j\ln^2(j+1)}\sqrt{\frac{\xi\rho|\omega^o_{l_{j-1}}|}{\mathcal{V}^{(l_{j-1})}}}\Big\}\\&j=0\qquad\|G^{(l_0)}_{n_{l_0}}\|_{D^{\mathbb{N}}_{\frac{\rho}{2}}\times\Delta^{\mathbb{N}}_{\frac{\xi}{2}}}\le C\frac{(\mathcal{V}^{(l_0)})^2}{\xi\rho|\omega^o_{l_0}|}\end{aligned}\tag{3.6}$$

The functions $G^{(l_j)}_{n_{l_j}}$ *depend on the variables* $(v_1, v_2, \dots, v_{l_j}, u_1, u_2, \dots, u_{l_{j-1}})$ *and* $\sum_{j=0}^{\infty} \|G^{(l_j)}_{n_{l_j}}\|_{D^{\mathbb{N}}_{\frac{\rho}{2}} \times \Delta^{\mathbb{N}}_{\frac{\xi}{2}}}$ *converges.* $\mathcal{R}_\infty = \lim_{N\to+\infty} \tilde{\mathcal{C}}^{(n_{l_0})} \circ \dots \circ \tilde{\mathcal{C}}^{(n_{l_{N-1}})} \circ \tilde{\mathcal{C}}^{(n_{l_N})}$. $\tilde{\mathcal{C}}^{(l_j)}$ *is canonical of infinitely many canonical variables but it is the identity when acts on the variables* (A_{l_k}, φ_{l_k}) $k > j$.

Corollary 3.2. *There exists a sequence of time–scales* $\{t_{l_k}\}$, $k \geq 0$,

$$t_{l_k} = \frac{C\rho}{\mathcal{V}^{(l_0)}} \frac{1}{(k+2)\ln^2(k+3)} \exp\{\frac{C}{(k+1)\ln^2(k+2)} \sqrt{\frac{|\omega^o_{l_k}|\rho\xi}{\mathcal{V}^{(l_k)}}}\}$$

such that $|A_i(t) - A_i(0)| \leq C\rho, \quad 1 \leq i \leq l-1, \quad |A_{l_k}(t) - A_{l_k}(0)| \leq C(2k+4)^2(\ln^4(2k+4))\frac{\mathcal{V}^{(l_k)}}{|\omega^o_{l_k}|\xi}$ $k \geq 0$ *for* $|t| \leq t_{l_k}$

Remarks i) If $l = 1$ all the variables φ_i are fast (not only those one with index $i \geq l$) ii) Theorem 3.1 needs the condition $\omega^o_{l_k}$ large and then $A_{l_k}(0)$ large enough with $k \to +\infty$. Otherwise it would lack the perturbative character of the problem.

To prove Theorem 3.1 we make some steps. In Theorem 4.1 we start with the hamiltonian $H_0(A, \varphi) = \sum_{i=1}^{l} h(A_i) + V^{(l)}(\varphi)$, and end with the hamiltonian given by $H_1(A', \varphi') = \sum_{i=1}^{l} h(A'_i) + V^{(l-1)}(\hat{\varphi}') + R^{(l)}(A', \varphi')$, where the important point is that the fast variable φ_l has been confined in $R^{(l)}(A', \varphi')$ which is of order $|\omega^o_l|^{-1}$

With the hamiltonian $H^{(l)}_n$ of Theorem 4.2, the separation of φ_l has been pushed to $O(|\omega^o_l|^{-1} e^{-\sqrt{C|\omega^o_l|}})$. This is achieved with n (integer) canonical transformations and n increases with $|\omega^o_l|$ (see (4.4)). In Corollary 4.3 we give an estimate of the size of the canonical transformation constructed between Theorems 4.1 and 4.2. We emphasize that $\|\tilde{C}^{(n)} - Id\| \xrightarrow[n\to+\infty]{} 0$ and this fact, crucial in Theorem 4.8, is achieved because the analyticity loss in the first transformation of $\tilde{C}^{(n)}$ (Theorem 4.1) is large if compared with the analyticity looses in the other transformations of $\tilde{C}^{(n)}$ (a trick already used by A. Neishtadt in [10]).

In the next step $\tilde{C}^{(n)}$ is brought inside the hamiltonian with one more d.o.f. $H^{(l_1)}_0$ ((4.7)) and the separation of the fast variable φ_{l+1} is repeated (Theorem 4.6 and Corollary 4.7) (exactly as for φ_l).

Now we can continue adding more and more d.o.f. and obtain each time a canonical transformation $\tilde{C}^{(n_{l_k})}$. Finally in Theorem 4.8 we show essentially that under some hypotheses on the frequencies (see Corollary 3.2) the composition of all the $(\tilde{C}^{(n_{l_k})})'s$ admits the limit defining the solution of our infinite–dimensional system

4. Intermediate Theorems, Corollaries and Proofs.

Theorem 4.1. *Let's consider the hamiltonian* $H_0(A, \varphi) = \sum_{i=1}^{l} h(A_i) + V^{(l)}(\varphi)$, $(A, \varphi) \in \mathbb{C}^l \times \Delta_\xi$ *and let* $A_o \in \mathbb{R}^l$ *be a point such that the vector* $h_{A^o} = \omega^o = (\hat{\omega}^o, \omega^o_l) \in \mathbb{R}^l$ *satisfies (2.3) with* $k = l$ *and* $K \geq \frac{1}{2\delta} \ln \frac{\delta e r |\omega^o_l|}{2\gamma \mathcal{V}^{(l)}}$, $0 < \rho \leq \sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$, $r < \frac{\rho}{3}$. *If*

$$|\omega^o_l| \geq \frac{4\gamma}{e\delta}(ar + \frac{\mathcal{V}^{(l)}}{r}) \tag{4.1}$$

then via a suitable, canonical transformation $(A, \varphi) = \mathcal{C}^{(0)}(A', \varphi')$, $(H_0 \circ \mathcal{C})(A', \varphi') \doteq H_1(A', \varphi') = \sum_{i=1}^{l} h(A'_i) + V^{(l-1)}(\hat{\varphi}') + R^{(l)}(A', \varphi')$, $(A', \varphi') \in D_{\rho_1}(A^o) \times \Delta_{\xi_1}$, $\|R^{(l)}\|_{\rho_1, \xi_1} \leq \frac{3\gamma}{e\delta r |\omega^o_l|}(\mathcal{V}^{(l)})^2$, *(*$\rho_1 = \rho - 3r > 0$, $\xi_1 = \xi - 3\delta > 0$*)*

Remarks i) The size of ω_l^o makes φ_l a *fast variable* and $(\hat{\omega}^o, \omega_l^o)$ non–resonant up to order K ii) we write $D_{\rho_1}(A^o)$ instead of $D_{\rho_1}^{(l)}(A^o_{(l)})$ (see section 2 Function–spaces) because there is no ambiguity on the number of dimensions. iii) writing the equations of $H_1(A', \varphi')$, one can note that $\frac{d}{dt}A_i'(t) = -\frac{\partial}{\partial \varphi_i}V^{(l-1)}(\hat{\varphi}') + O(|\omega_l^o|^{-1})$ $i = 1, \ldots, l-1$ which means that $A_i(t) - A_i(0) = O(1)$ because $(\varphi_1, \ldots, \varphi_{l-1})$ are slow variables

Proof $V^{(l)}(\varphi) = (V^{(l)}(\varphi) - V^{(l-1)}(\hat{\varphi})) + V^{(l-1)}(\hat{\varphi})$ where $(V^{(l)}(\varphi) - V^{(l-1)}(\hat{\varphi})) = \sum_{\substack{|\nu|\le K \\ \nu_l \neq 0}} e^{i\nu\cdot\varphi}V_\nu^{(l)} + \sum_{\substack{|\nu|>K \\ \nu_l\neq 0}} e^{i\nu\cdot\varphi}V_\nu^{(l)}$,

Let's define the *generating function* $\tilde{S}(A', \varphi) = A' \cdot \varphi + S(A', \varphi)$, $S(A', \varphi) = \sum_{\substack{\nu\in\mathbb{N}^l \\ |\nu|\le K, \nu_l\neq 0}} e^{i\nu\cdot\varphi}S_\nu(A')$, $S_\nu(A') = \frac{V_\nu^{(l)}}{ih_{A'}\cdot\nu}$, that allows us to eliminate the harmonics $V_\nu^{(l)}$ of order $|\nu| \le K$. $|A' - A^o| \le \frac{|\omega_l^o|}{2a\gamma K}$ implies $|\omega(A')\cdot\nu| \ge |\omega(A^o)\cdot\nu| - |(\omega(A') - \omega(A^o))\cdot\nu| \ge \frac{|\omega_l^o|}{\gamma} - |(\omega(A') - \omega(A^o))\cdot\nu| \ge \frac{1}{2}\frac{|\omega_l^o|}{\gamma}$ for all $|\nu| \le K$ and $\nu_l \neq 0$. The condition $\rho \le \sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$ implies $\frac{|\omega_l^o|}{2a\rho\gamma} \ge \frac{1}{2\delta}\ln\frac{\delta e r|\omega_l^o|}{2\gamma\mathcal{V}^{(l)}}$ provided that $r < \frac{\rho}{3}$ (see the end of the proof). $r < \frac{\rho}{3}$ guarantees $\sqrt{\frac{6\mathcal{V}^{(l)}}{a}} \le \frac{|\omega_l^o|}{2a\gamma K}$ as well. By standard calculations (use $\max_{t\ge0} e^{-t\delta}t = \frac{1}{e\delta}$ and the exponential decay with $|\nu|$ of the coefficient f_ν of an analytic function on a complex strip; see [6] for instance) we have $\|S_\varphi\|_{\rho,\xi-\delta} \le \frac{2\gamma\mathcal{V}^{(l)}}{e\delta|\omega_l^o|}$, $\|S_{A'}\|_{\rho-r,\xi-\delta} \le \frac{2\gamma\mathcal{V}^{(l)}}{r|\omega_l^o|}$, $\|S_{A'\varphi}\|_{\rho-r,\xi-\delta} \le \frac{2\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|}$. The condition

$$\frac{4\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|} \le 1 \tag{4.2}$$

is guaranteed by (4.1) and allows us to define the canonical transformation (use the analytic–implicit function theorem (see [6])) $(A, \varphi) \doteq \mathcal{C}(A', \varphi') = (A' + \Xi(A', \varphi'), \varphi' + \Delta(A', \varphi'))$ and $(\Xi, \Delta)\colon D_{\rho-r}(A^o) \times \Delta_{\xi-2\delta} \mapsto D_\rho(A^o) \times \Delta_{\xi-\delta}$. The inverse transformation is $(A', \varphi') \doteq \overline{\mathcal{C}}(A, \varphi) = (A + \overline{\Xi}(A, \varphi), \varphi + \overline{\Delta}(A, \varphi))$, $(\overline{\Xi}, \overline{\Delta})\colon D_{\rho-2r}(A^o) \times \Delta_{\xi-\delta} \mapsto D_{\rho-r}(A^o) \times \Delta_\xi$

$$\|\Xi\|_{\rho-r,\xi-2\delta} \le \|S_\varphi\|_{\rho,\xi-\delta}, \qquad \|\Delta\|_{\rho-r,\xi-2\delta} \le \|S_{A'}\|_{\rho-r,\xi-\delta}. \tag{4.3}$$

$\mathcal{C} \circ \overline{\mathcal{C}} = \overline{\mathcal{C}} \circ \mathcal{C} = Identity$ on the domain $D_{\rho-3r}(A^o) \times \Delta_{\xi-3\delta}$
Putting $(A, \varphi) = \mathcal{C}(A', \varphi')$ into $H_0(A, \varphi)$ we obtain $H_1(A', \varphi')$ with $R^{(l)} = f_1 + f_2 + f_3$ and

$$\begin{aligned}
f_1 &= (\sum_{i=1}^{l}(h(A_i' + \Xi_i) - h(A_i') - h_{A_i'}\cdot\Xi_i),\\
f_2 &= V^{(l-1)}(\hat{\varphi}' + \hat{\Delta}) - V^{(l-1)}(\hat{\varphi}'),\\
f_3 &= \sum_{\substack{|\nu|>K \\ \nu_l\neq0}} e^{i\nu\cdot(\varphi'+\Delta)}V_\nu^{(l)} \|f_1\|_{\rho-3r,\xi-3\delta} \le a\sum_{i=1}^{l}\|\Xi_i\|^2_{\rho-3r,\xi-3\delta}\\
&\le a\sum_{i=1}^{l}\|S_{\varphi_j}\|^2_{\rho-3r,\xi-\delta} \le a(\frac{2\gamma}{e\delta|\omega_l^o|}\mathcal{V}^{(l)})^2\\
\|f_2\|_{\rho-3r,\xi-3\delta} &\le \|\frac{\partial V^{(l-1)}}{\partial\hat{\varphi}}\|_{\xi-2\delta}\|\hat{\Delta}\|_{\rho-r,\xi-2\delta} \le \frac{2\gamma}{r|\omega_l^o|}\mathcal{V}^{(l)}\frac{1}{2\delta}\mathcal{V}^{(l-1)}
\end{aligned}$$

$$\|f_3\|_{\rho-3r,\xi-3\delta} \le$$

(4.3) has been used. If $\frac{4a\gamma r}{e\delta|\omega_l^o|} \le 1$ (guaranteed by (4.1)) and $K \ge \frac{1}{2\delta}\ln\frac{\delta er|\omega_l^o|}{2\gamma\mathcal{V}^{(l)}}$ we have $\|R^{(l)}\|_{\rho-3r,\xi-3\delta} \le \frac{3\gamma}{e\delta r|\omega_l^o|}(\mathcal{V}^{(l)})^2$ The relation $\frac{|\omega_l^o|}{2a\rho\gamma} \ge \frac{1}{2\delta}\ln\frac{e\delta r|\omega_l^o|}{2\gamma\mathcal{V}^{(l)}}$ is equivalent to $f(x) = x\frac{\mathcal{V}^{(l)}}{r}\frac{1}{2a\gamma\rho} - \frac{1}{2}\ln x\frac{e}{2\gamma} \ge 0$ where $x = \frac{\delta r|\omega_l^o|}{\mathcal{V}^{(l)}}$. The function $f(x)$ has a minimum at $x = \frac{a\gamma\rho r}{\mathcal{V}^{(l)}}$ and the value of $f(x)$ is $1 - \ln\frac{e}{2}\frac{a\rho r}{\mathcal{V}^{(l)}}$ which is positive for $\rho \le \sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$ provided that $0 < r < \frac{\rho}{3}$ ∎

In Corollary 4.3 we will need the following estimates on the quantities $\Delta_{\varphi'} = -(Id+S_{A'\varphi})^{-1}S_{A'\varphi}$, $\Delta_{A'} = -(Id+S_{A'\varphi})^{-1}S_{A'A'}$, $\Xi_{A'} = S_{A'\varphi} + S_{\varphi\varphi}\Delta_{A'}$, $\Xi_{\varphi'} = S_{\varphi\varphi}(Id+\Delta_{\varphi'})$, (4.2) implies $\|(Id+S_{A'\varphi})^{-1}\|_{\rho-r,\xi-\delta} \le 2$, $\|S_{A'A'}\|_{\rho-r,\xi-\delta} \le \frac{4\gamma\mathcal{V}^{(l)}}{r^2|\omega_l^o|}$, $\|S_{\varphi'\varphi'}\|_{\rho,\xi-\delta} \le \frac{8\gamma\mathcal{V}^{(l)}}{e^2\delta^2|\omega_l^o|}$, (use $\max_{t\ge0} t^2e^{-t\delta} = \frac{4}{e^2\delta^2}$ and $r = \frac{\Delta\rho}{3}$, $\delta = \frac{\Delta\xi}{3}$).

$$\|\Delta_{\varphi'}\|_{\rho-r,\xi-2\delta} \le \frac{4\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|} = \frac{36\gamma\mathcal{V}^{(l)}}{e\Delta\rho\Delta\xi|\omega_l^o|} \le \frac{36}{81}\frac{1}{2(n-1)^2} = \frac{2}{9}\frac{1}{(n-1)^2}$$

$$\|\Delta_{A'}\|_{\rho-r,\xi-2\delta} \le \frac{8\gamma\mathcal{V}^{(l)}}{r^2|\omega_l^o|} = \frac{8\gamma\mathcal{V}^{(l)}}{er\delta|\omega_l^o|}\frac{\delta}{r}e \le \frac{4e}{9}\frac{\Delta\xi}{\Delta\rho}\frac{1}{(n-1)^2}$$

$$\|\Xi_{\varphi'}\|_{\rho-r,\xi-2\delta} \le \frac{16\gamma\mathcal{V}^{(l)}}{e^2\delta^2|\omega_l^o|} \le \frac{8\gamma\mathcal{V}^{(l)}}{er\delta|\omega_l^o|}\frac{r}{\delta e} \le \frac{8}{9e}\frac{\Delta\rho}{\Delta\xi}\frac{1}{(n-1)^2}$$

$$\|\Xi_{A'}\|_{\rho-r,\xi-2\delta} \le \frac{2\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|} + \frac{8\gamma\mathcal{V}^{(l)}}{e^2\delta^2|\omega_l^o|}\frac{4\gamma\mathcal{V}^{(l)}}{r^2|\omega_l^o|} \le \frac{1}{9}\frac{1}{(n-1)^2} + (\frac{2}{9})^2\frac{1}{(n-1)^4} = \frac{1}{3}\frac{1}{(n-1)^2}$$

Let's define $\partial\mathcal{C} = \begin{pmatrix} Id+\Xi_{A'} & \Xi_{\varphi'} \\ \Delta_{A'} & Id+\Delta_{\varphi'} \end{pmatrix}$. The metric in the phase–space $\mathbb{R}^l \times \mathbb{T}^l$, is $|A' - \overline{A}'| + \rho|\varphi' - \overline{\varphi}'|$. Then we have

$$\|\partial\mathcal{C}\|_{\rho_1,\xi_1} \le \|\Xi_{A'}\|_{\rho_1,\xi_1} + \frac{1}{\rho}\|\Xi_{\varphi'}\|_{\rho_1,\xi_1} + \rho\|\Delta_{A'}\|_{\rho_1,\xi_1} + \|\Delta_{\varphi'}\|_{\rho_1,\xi_1}$$

$$\|\partial(Id-\mathcal{C})\|_{\rho_1,\xi_1} \le \frac{1}{(n-1)^2}(1 + 3\rho\frac{\Delta\xi}{\Delta\rho} + \frac{\Delta\rho}{\Delta\xi}\frac{1}{\rho})$$

$$H_1(A',\varphi') = \sum_{i=1}^{l} h(A'_i) + V^{(l-1)}(\hat{\varphi}') + <R^{(l)}(A',\varphi')>_l + (R^{(l)}(A',\varphi') - <R^{(l)}(A',\varphi')>_l)$$

Starting with $H_1(A',\varphi')$, we perform a *finite number* n of canonical transformation and further reduce the perturbation to order $O(|\omega_l^o|^{-n})$. As usual n depends on $|\omega_l^o|$ in such a way that the perturbation is exponentially small respect to some power of $|\omega_l^o|$, ($\frac{1}{2}$ in our case)

Let be:

$\rho_{j+1} = \rho_j - 3r_j$, $r_j = r_{j+1}$ $j = 0,\dots n$, $\rho_0 \doteq \rho$, $\rho - \rho_1 = \rho_1 - \rho_n \doteq \Delta\rho$,

$\xi_{j+1} = \xi_j - 3\delta_j$, $\delta_j = \delta_{j+1}$ $j = 0,\dots n$, $\xi_0 \doteq \xi$, $\xi - \xi_1 = \xi_1 - \xi_n \doteq \Delta\xi$,

The canonical transformations $\mathcal{C}$ and $\overline{\mathcal{C}}$ are recalled $\mathcal{C}^{(0)}$ and $\tilde{\mathcal{C}}^{(0)}$.

Theorem 4.2. *Let's consider the hamiltonian* $H_1(A',\varphi')$, $(A',\varphi') \in D_{\rho_1}(A^o) \times \Delta_{\xi_1}$. *If*

$$a\gamma\frac{\Delta\rho}{\Delta\xi}|\omega_l^o|^{-1} \le 1, \quad \frac{1}{4} \le \frac{81\gamma}{e\Delta\xi\Delta\rho}\mathcal{V}^{(l)}\frac{(n-1)^2}{|\omega_l^o|} < \frac{1}{2} \tag{4.4}$$

there exist $n-1$ *canonical transformation* $(A',\varphi') = \mathcal{C}^{(1)}\circ\ldots\circ\mathcal{C}^{(n-1)}(A^{(n)},\varphi^{(n)})$, $(A^{(n)},\varphi^{(n)})\in D_{\rho_n}(A^o)\times\Delta_{\xi_n}$ *such that*

$$(H_1\circ\mathcal{C}^{(1)}\circ\ldots\circ\mathcal{C}^{(n-1)})(A^{(n)},\varphi^{(n)}) \doteq H_n^{(l)}(A^{(n)},\varphi^{(n)}) = \sum_{i=1}^l h(A_i^{(n)}) + V^{(l-1)}(\hat\varphi^{(n)})$$
$$+G_n^{(l)}(A^{(n)},\hat\varphi^{(n)}) + (R_n^{(l)}(A^{(n)},\varphi^{(n)}) - < R_n^{(l)}(A^{(n)},\hat\varphi^{(n)}) >_l),$$
$$\|G_n^{(l)}\|_{\rho_n,\xi_n} \le \frac{6\gamma}{\delta er}\frac{\mathcal{V}^{(l)}}{|\omega_l^o|},\quad \|R_n^{(l)}\|_{\rho_n,\xi_n} \le \frac{3\gamma}{e\delta r}\frac{(\mathcal{V}^{(l)})^2}{|\omega_l^o|}\exp\{-\ln 2\sqrt{|\omega_l^o|\frac{e\Delta\xi\Delta\rho}{(162)\gamma\mathcal{V}^{(l)}}}\}$$

Remarks i) In the spirit of *Nekhoroshev theorem* $R_n^{(l)}$, which depends on the fast variable $\varphi_l^{(n)}$, is exponentially small in $|\omega_l^o|^{1/2}$ ii) Another feature of the Nekhoroshev theorem is the fact that n depends on $\sqrt{|\omega_l^o|}$ and the bigger is $|\omega_l^o|$, the bigger is n.

Proof The calculations are analogous to those employed for $H_0(A,\varphi)$. We apply n times the same procedure, each time reducing the size of one order respect to $|\omega_l^o|^{-1}$. The variables (A'',φ'') play the role of the variables (A',φ'). The generating function is $\tilde S(A'',\varphi') = A''\cdot\varphi' + S(A'',\varphi')$ where

$$S(A'',\varphi') = \sum_{\substack{\nu\in\mathbb{N}^l,\nu_l\ne 0\\ |\nu|\le K_1}} e^{i\nu\cdot\varphi'}\frac{R_\nu^{(l)}(A'')}{i\omega(A'')\cdot\nu} \tag{4.5}$$

$|\omega(A'')\cdot\nu|^{-1}\le\frac{\gamma}{|\omega_l^o|}$ if $|A''-A^o|\le\frac{|\omega_l^o|}{2\gamma a K_1}$, $K_1\ge\frac{1}{2\delta_1}\ln\frac{\delta_1 e r_1|\omega_l^o|}{2\gamma\mathcal{V}^{(l)}}$, (see at the end of the proof that $\frac{|\omega_l^o|}{2a\gamma K_1}\ge\sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$ so that $|A''-A^o|\le\rho_1$).

$$\rho\to\rho_1,\quad \rho_1\to\rho_2, \xi\to\xi_1,\quad \xi_1\to\xi_2,\quad K\to K_1\quad \mathcal{C}^{(0)}\to\mathcal{C}^{(1)}$$
$$\overline{\mathcal{C}}^{(0)}\to\overline{\mathcal{C}}^{(1)},\quad \Xi\to\Xi',\quad \Delta\to\Delta'\quad \overline{\Xi}\to\overline{\Xi}',\quad \overline{\Delta}\to\overline{\Delta}'$$

$$\|\Delta'\|_{\rho_2,\xi_2}\le\|S_{A''}\|_{\rho_1-r_1,\xi_1}\le\frac{2\gamma}{r_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le\delta_1$$

$$\|\Xi'\|_{\rho_2,\xi_2}\le\|S_{\varphi'}\|_{\rho_1,\xi_1-\delta_1}\le\frac{2\gamma}{e\delta_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le r_1$$

$$\|S_{A''\varphi'}\|_{\rho_1-r_1,\xi_1-\delta_1}\le\frac{2\gamma}{er_1\delta_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le\frac12$$

$$\|\Delta'_{\varphi''}\|_{\rho_1-r_1,\xi_1-2\delta_1}\le\frac{4\gamma}{e\delta_1 r_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le\frac{4\mathcal{V}^{(l)}\gamma}{e\delta_1 r_1|\omega_l^o|}\frac{3\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|}\le\frac{4}{19}\frac34\le\frac16$$

$$\|\Delta'_{A''}\|_{\rho_1-r_1,\xi_1-2\delta_1}\le\frac{4\gamma}{r_1^2|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le\frac{4\gamma\mathcal{V}^{(l)}}{r_1^2|\omega_l^o|}\frac{3\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|}\le\frac{4}{19}\frac{e\delta_1}{r_1}\frac34\le\frac12\frac{\delta_1}{r_1}$$

$$\|\Xi'_{\varphi''}\|_{\rho_1-r_1,\xi_1-2\delta_1}\le\frac{4\gamma}{e^2\delta_1^2|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\le\frac{4\gamma\mathcal{V}^{(l)}}{e^2\delta_1^2|\omega_l^o|}\frac{3\gamma\mathcal{V}^{(l)}}{e\delta r|\omega_l^o|}(1+\frac16)\le\frac{4r_1}{19e\delta_1}\frac34\frac76\le\frac{1}{10}\frac{r_1}{\delta_1}$$

$$\|\Xi'_{A''}\|_{\rho_1-r_1,\xi_1-2\delta_1}\le\frac{2\gamma}{e\delta_1 r_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}+\frac{4\gamma}{e^2\delta_1^2|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}\frac{4\gamma}{r_1^2|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}$$
$$\le\frac12\frac34+\frac{1}{10}\frac{r_1}{\delta_1}\frac12\frac{\delta_1}{r_1}\le\frac12$$

$$\|\partial(Id-\mathcal{C})\|_{\rho_1,\xi_1}\le\frac{17}{40}+\frac{1}{\rho_2}\frac{r_1}{10\delta_1}+\rho_2\frac{9}{19}\frac{\delta_1}{r_1}+\frac{3}{19}\le\frac23+\frac{1}{\rho_2}\frac{r_1}{10\delta_1}+\rho_2\frac{9}{19}\frac{\delta_1}{r_1}$$

The new hamiltonian is

$$H_2^{(l)}(A'',\varphi'') \doteq H_1^{(l)} \circ \mathcal{C}^{(1)}(A'',\varphi'') = \sum_{i=1}^{l} h(A'') + V^{(l-1)}(\hat\varphi'') + < R^{(l)}(A'',\hat\varphi'') >_l$$
$$+ < R_1^{(l)}(A'',\hat\varphi'') >_l + (R_1^{(l)}(A'',\varphi'') - < R_1^{(l)}(A'',\hat\varphi'') >_l)$$

$$R_1^{(l)}(A'',\varphi'') = \sum_{i=1}^{5} f_i(A'',\varphi'')$$

$$f_1 = (V^{(l-1)}(\hat\varphi'' + \hat\Delta') - V^{(l-1)}(\hat\varphi'')),$$

$$\|f_1\|_{\rho_2,\xi_2} \le \|\frac{\partial V^{(l-1)}}{\partial \hat\varphi''}\|_{\xi_1-2\delta_1}\|\hat\Delta'\|_{\rho_2,\xi_2} \le \frac{\mathcal{V}^{(l-1)}}{2\delta_1 e}\frac{2\gamma}{r_1|\omega_l^o|}\|R^{(l)}\|_{\rho_2,\xi_2}$$

$$f_2 = < R^{(l)}(A''+\Xi',\hat\varphi''+\hat\Delta') >_l - < R^{(l)}(A'',\hat\varphi'') >_l$$

$$\|f_2\|_{\rho_2,\xi_2} \le \|\frac{\partial < R^{(l)} >_l}{\partial A''}\|_{\rho_1-2r_1,\xi_2}\|\Xi'\|_{\rho_2,\xi_2} + \|\frac{\partial < R^{(l)} >_l}{\partial \hat\varphi''}\|_{\rho_1-2r_1,\xi_2}\|\Delta'\|_{\rho_2,\xi_2}$$
$$\le \frac{2\gamma}{e\delta_1 r_1|\omega_l^o|}\|R^{(l)}\|^2_{\rho_1,\xi_1}$$

$$f_3 = \sum_{\substack{\nu:\nu_l\neq 0\\ |\nu|\le K_1}} e^{i\nu(\varphi''+\Delta')}(R_\nu^{(l)}(A''+\Xi') - R_\nu^{(l)}(A'')),$$

$$\|f_3\|_{\rho_2,\xi_2} \le \frac{\gamma}{e\delta_1 r_1|\omega_l^o|}\|R^{(l)}\|^2_{\rho_1,\xi_1}$$

$$f_4 = \sum_{\substack{\nu:\nu_l\neq 0\\ |\nu|> K_1}} e^{i\nu(\varphi''+\Delta')}R_\nu^{(l)}(A''+\Xi'), \qquad \|f_4\|_{\rho_2,\xi_2} \le e^{-2\delta_1 K_1}\|R^{(l)}\|^2_{\rho_1,\xi_1}$$

$$f_5 = (\sum_{i=1}^{l}(h(A_i''+\Xi') - h(A_i'') - h_{A_i''}\Xi_i'), \qquad \|f_5\|_{\rho_2,\xi_2} \le a(\frac{2\gamma}{e\delta_1|\omega_l^o|})^2\|R^{(l)}\|^2_{\rho_1,\xi_1}$$

The conditions $\frac{a\gamma r_1}{e\delta_1|\omega_l^o|} \le 1$ (see the analogous one in the previous theorem) and $K_1 \ge \frac{1}{2\delta_1}\ln\frac{e\delta_1 r_1|\omega_l^o|}{8\gamma\mathcal{V}^{(l)}}$ give $\|R_1^{(l)}\|_{\rho_2,\xi_2} \le \frac{9\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1}$ while $\frac{18\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1|\omega_l^o|} \le 1$ implies $\| < R^{(l)} + R_1^{(l)} >_l \|_{\rho_2,\xi_2} \le \frac{3\gamma}{e\delta r|\omega_l^o|}(\mathcal{V}^{(l)})^2$.

All the conditions on $|\omega_l^o|$ and (4.1) are implied by $\frac{4\gamma}{e\delta_1 r_1}\frac{1}{|\omega_l^o|}\|R^{(l)}\|_{\rho_1,\xi_1} + \frac{a\gamma r_1}{e\delta_1|\omega_l^o|} + \frac{18\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1 e|\omega_l^o|} \le 1$ which in turn is implied by $\frac{2}{3}\frac{\gamma}{e\delta_1 r_1|\omega_l^o|}\mathcal{V}^{(l)} + \frac{a\gamma r_1}{e\delta_1|\omega_l^o|} + \frac{18\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1 e|\omega_l^o|} \le 1$ ($\frac{18\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1|\omega_l^o|} \le 1$ and $\delta_1 r_1 \le \delta r$ have been taken into account). Thus all the lower bounds on $|\omega_l^o|$ are implied by

$$19\frac{\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1|\omega_l^o|} + \frac{ar\gamma}{e\delta}\frac{1}{|\omega_l^o|} \le 1. \tag{4.6}$$

Suppose to have a finite family of hamiltonians $H_j^{(l)}(A^{(j)},\varphi^{(j)}) = \sum_{i=1}^{l} h(A_i^{(j)}) + V^{(l-1)}(\hat\varphi^{(j)}) + G_j^{(l)}(A^{(j)},\hat\varphi^{(j)}) + (R_j^{(l)}(A^{(j)},\varphi^{(j)}) - < R_j^{(l)}(A^{(j)},\hat\varphi^{(j)}) >_l)$ and a set of canonical transformations $\mathcal{C}^{(j)}$ such that $(A^{(j)},\varphi^{(j)}) = \mathcal{C}^{(j)}(A^{(j+1)},\varphi^{(j+1)}),\quad 1 \le j \le n,\ \mathcal{C}^{(j-1)}(A^{(j)},\varphi^{(j)}) = (A^{(j)} + \Xi^{(j-1)},\varphi^{(j)} + \Delta^{(j-1)}).\qquad H_1 \circ \mathcal{C}^{(1)} \circ \ldots \circ$

$\mathcal{C}^{(n-1)})(A^{(j)},\varphi^{(j)}) = H_j^{(l)}(A^{(j)},\varphi^{(j)})$, $G_0^{(l)} \equiv 0$, $\quad G_j^{(l)} = G_{j-1}^{(l)} + < R_j^{(l)} >_l$, $\quad R_0^{(l)} \equiv 0 \quad 1 \le j \le n$.

If $S^{(k-1)}(A^{(k)},\varphi^{(k)})$ is the function involved in the construction of the canonical transformation (see (4.5)), we have the estimates

$$\|S^{(j-1)}_{A^{(j)}\varphi^{(j-1)}}\|_{\rho_{j-1}-r_{j-1},\xi_{j-1}-\delta_{j-1}} \le \frac{2\gamma}{e\delta_{j-1}r_{j-1}|\omega_l^o|}\|R_{j-1}^{(l)}\|$$

$$\|\Delta^{(j-1)}\|_{\rho_j,\xi_j} \le \frac{2\gamma}{r_{j-1}|\omega_l^o|}\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}},$$

$$\|\Xi^{(j-1)}\|_{\rho_j,\xi_j} \le \frac{2\gamma}{e\delta_{j-1}|\omega_l^o|}\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}}$$

Taking $\delta_j = \frac{\Delta\xi}{3(n-1)}$, $r_j = \frac{\Delta\rho}{3(n-1)}$, for any $j > 0$, it follows

$$\|R_n^{(l)}\|_{\rho_n,\xi_n} \le \frac{3\gamma}{\delta re}\frac{(\mathcal{V}^{(l)})^2}{|\omega_l^o|}\Big(\frac{81\gamma\mathcal{V}^{(l)}(n-1)^2}{e(\xi_1-\xi_n)(\rho_1-\rho_n)|\omega_l^o|}\Big)^{n-1}$$

By the second of (4.4) $\frac{|\omega_l^o|e\Delta\xi\Delta\rho}{324\gamma\mathcal{V}^{(l)}} \le (n-1)^2 < \frac{|\omega_l^o|e\Delta\xi\Delta\rho}{162\gamma\mathcal{V}^{(l)}}$ we have

$$\|R_n^{(l)}\|_{\rho_n,\xi_n} \le \frac{3\gamma}{e\delta r}\frac{(\mathcal{V}^{(l)})^2}{|\omega_l^o|}\exp\Big\{-\ln 2\Big(|\omega_l^o|\frac{e\Delta\xi\Delta\rho}{(162)\gamma\mathcal{V}^{(l)}}\Big)^{1/2}\Big\}$$

$a\gamma\frac{\Delta\rho}{\Delta\xi}|\omega_l^o|^{-1} \le 1$ implies (4.6) when the second of (4.4) is used to replace the product $\delta_1 r_1$ in terms of $|\omega_l^o|$. Moreover the second of (4.4) implies

$$\begin{aligned}
\|R_n^{(l)}\|_{\rho_n,\xi_n} &\le \frac{3}{18}\mathcal{V}^{(l)}.\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}} \le \frac{3\gamma}{\delta er}(\mathcal{V}^{(l)})^2\frac{1}{|\omega_l^o|}\Big(\mathcal{V}^{(l)}\frac{9\gamma}{e|\omega_l^o|}\Big)^{j-2}\Pi_{i=1}^{j-2}(\delta_i r_i)^{-1}\\
&\le \frac{6\gamma(\mathcal{V}^{(l)})^2}{e\delta r|\omega_l^o|}2^{-j+1} \le \frac{2}{3}\frac{\mathcal{V}^{(l)}}{(n-1)^2}2^{-j}\\
\|\Delta^{(j-1)}_{\varphi^{(j)}}\|_{\rho_j,\xi_j} &\le \frac{2\gamma}{e\delta_{j-1}r_{j-1}|\omega_l^o|}\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}} \le 4\Big(\frac{9(n-1)^2\mathcal{V}^{(l)}\gamma}{e\Delta\rho\Delta\xi|\omega_l^o|}\Big)\frac{2}{3}\frac{2^{-j}}{(n-1)^2}\\
&\le \frac{4}{18}\frac{2}{3}\frac{2^{-j}}{(n-1)^2} = \frac{4}{27}\frac{2^{-j}}{(n-1)^2}\\
\|\Delta^{(j-1)}_{A^{(j)}}\|_{\rho_j,\xi_j} &\le \frac{8\gamma}{r_{j-1}^2|\omega_l^o|}\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}} \le \frac{72e}{2\cdot 81}\frac{\delta_{j-1}}{r_{j-1}}\frac{2}{3}\frac{2^{-j}}{(n-1)^2} \le \frac{2^{-j}}{(n-1)^2}\\
\|\Xi^{(j-1)}_{\varphi^{(j)}}\|_{\rho_j,\xi_j} &\le \frac{16\gamma}{e^2\delta_{j-1}^2|\omega_l^o|}\|R_{j-1}^{(l)}\|_{\rho_{j-1},\xi_{j-1}} \le \frac{r_{j-1}}{e\delta_{j-1}}\frac{16\cdot 9}{2\cdot 81e}\frac{2}{3}\frac{2^{-j}}{(n-1)^2} = \frac{16}{27}\frac{r_{j-1}}{e\delta_{j-1}}\frac{2^{-j}}{(n-1)^2}\\
\|\Xi^{(j-1)}_{A^{(j)}}\|_{\rho_j,\xi_j} &\le \frac{2}{27}\frac{2^{-j}}{(n-1)^2} + \frac{16}{27}\frac{r_{j-1}}{e\delta_{j-1}}\frac{2^{-j}}{(n-1)^2}\frac{2^{-j}}{(n-1)^2}\frac{\delta_{j-1}}{r_{j-1}} \le \frac{2}{3}\frac{2^{-j}}{(n-1)^2}\\
\|\partial(Id-\mathcal{C}^{(j-1)})\|_{\rho_j,\xi_j} &\le \frac{1}{(n-1)^2 2^j}\Big(1+\frac{1}{\rho}\frac{r_{j-1}}{\delta_{j-1}}+\rho\frac{\delta_{j-1}}{r_{j-1}}\Big)
\end{aligned}$$

The presence of $(n-1)^2$ at denominator will be essential (see Theorem 4.8) and is due to the fact that δ_0, r_0, is much greater than respectively δ_j and r_j for $j > 0$ (in fact they are n independent)

$$\|G^{(l)}_{j+1}\|_{\rho_j,\xi_j} \le \frac{3\gamma}{\delta er}(\mathcal{V}^{(l)})^2\frac{1}{|\omega^o_l|}\sum_{k=0}^{j}\big(\frac{9\gamma}{e}\frac{\mathcal{V}^{(l)}}{\delta_1 r_1|\omega^o_l|}\big)^k$$

$$\|G^{(l)}_{j+1}\|_{\rho_j,\xi_j} \le \frac{3\gamma}{\delta er}(\mathcal{V}^{(l)})^2\frac{1}{|\omega^o_l|}\sum_{k=0}^{j}(\frac{1}{2})^k \le \frac{6\gamma}{\delta er}(\mathcal{V}^{(l)})^2\frac{1}{|\omega^o_l|} \le \frac{6}{18}\mathcal{V}^{(l)}$$

(using the second of (4.4))

$$\|R^{(l)}_{j+1}\|_{\rho_j,\xi_j} \le \frac{3\gamma}{\delta er}(\mathcal{V}^{(l)})^2\frac{1}{|\omega^o_l|}(\mathcal{V}^{(l)}\frac{9\gamma}{e|\omega^o_l|})^j\Pi_{k=1}^{j}(\delta_k r_k)^{-1}.$$

Now we prove that $\frac{|\omega^o_l|}{2a\gamma K_1} \le \frac{|\omega^o_l|}{2a\gamma K}$. It is equivalent to $K_1 \ge K$ and then $(n-1)\ln(\frac{e\Delta\xi\Delta\rho}{9\gamma\mathcal{V}^{(l)}(n-1)^2}|\omega^o_l|) \ge \ln(\frac{e\Delta\xi\Delta\rho|\omega^o_l|}{9\gamma\mathcal{V}^{(l)}})$. Let's call $B = \frac{e\Delta\xi\Delta\rho|\omega^o_l|}{9\gamma\mathcal{V}^{(l)}(n-1)^2}$, $18 < B \le 36$ and the inequality becomes $(n-2)\ln B \ge 2\ln(n-1)$ which is true if $n \ge 2$.

The last proof is $\frac{|\omega^o_l|}{2a\gamma K_1} \ge \sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$ which is equivalent to $r_1 \le \sqrt{\frac{2}{3}\frac{\mathcal{V}^{(l)}}{a}}$. The following chain $r_1 < r < \frac{\rho}{3} < \frac{1}{3}\sqrt{\frac{6\mathcal{V}^{(l)}}{a}}$ ∎

Corollary 4.3. *Let's consider the canonical transformation* $\tilde{\mathcal{C}}^{(n)}\colon D_{\rho_n}(A^o)\times\Delta_{\xi_n} \to D_\rho(A^o)\times\Delta_\xi$ *of the Theorem 4.2. We have* $\|\partial\tilde{\mathcal{C}}^{(n)}\|_{\rho_n,\xi_n} \le e^{\frac{2T_{l_0}}{(n-1)^2}}$ *(T_{l_0} is a constant depending only on l_0, ρ, ξ).*

Proof We make use of: 1) $\|\Xi^{(0)}\|_{\rho_1,\xi_1} \le \frac{2\gamma\mathcal{V}^{(l)}}{e\delta|\omega^o_l|}$ and 2)

$$\sum_{k=1}^{n-1}\frac{2\gamma}{e\delta_k|\omega^o_l|}\|R^{(l)}_k\|_{\rho_k,\xi_k} \le \sum_{k=1}^{n-1}\frac{2\gamma}{e\delta_k|\omega^o_l|}\frac{3\gamma}{\delta er}\frac{(\mathcal{V}^{(l)})^2}{|\omega^o_l|}\big(\frac{9\gamma\mathcal{V}^{(l)}}{e|\omega^o_l|}\big)^k\Pi_{h=1}^{k}(\delta_h r_h)^{-1}$$

$$= \sum_{k=1}^{n-1}\frac{6\gamma^2(\mathcal{V}^{(l)})^2}{e^2\delta r|\omega^o_l|^2}\big[\frac{9\gamma\mathcal{V}^{(l)}}{e|\omega^o_l|}\frac{9n^2}{\Delta\xi\Delta\rho}\big]^{k+1}\frac{\Delta\rho}{3n}\frac{e|\omega^o_l|}{9\gamma\mathcal{V}^{(l)}} \le \frac{(\Delta\rho)\gamma\mathcal{V}^{(l)}}{9ne\delta r|\omega^o_l|}\sum_{k=1}^{n-1}2^{-k} = \frac{(\Delta\rho)\gamma\mathcal{V}^{(l)}}{9ne\delta r|\omega^o_l|}$$

$$\le \frac{(\Delta\rho)\gamma\mathcal{V}^{(l)}}{9e\delta r|\omega^o_l|} = \frac{\gamma\mathcal{V}^{(l)}}{3e\delta|\omega^o_l|}$$

$$\|\partial\mathcal{C}^{(0)}\|_{\rho_1,\xi_1} \le 1 + \frac{1}{(n-1)^2}(1 + 3\rho\frac{e\Delta\xi}{\Delta\rho} + \frac{\Delta\rho}{e\Delta\xi}\frac{1}{\rho})(see after Theorem 4.1)$$

$$\|\partial\mathcal{C}^{(j)}\|_{\rho_{j+1},\xi_{j+1}} \le 1 + \frac{1}{(n-1)^2 2^j}(1 + \frac{1}{\rho}\frac{r_j}{\delta_j} + \rho\frac{\delta_j}{r_j}) = 1 + \frac{1}{(n-1)^2 2^j}(1 + \frac{1}{\rho}\frac{\Delta\rho}{\Delta\xi} + \rho\frac{\Delta\xi}{\Delta\rho})$$

$$\|\partial\tilde{\mathcal{C}}^{(n)}\|_{\rho_n,\xi_n} \le \Pi_{j=0}^{n-1}\|\partial\mathcal{C}^{(j)}\|_{\rho_{j+1},\xi_{j+1}} \le e^{\frac{2T_{l_0}}{(n-1)^2}}, \quad T_{l_0} = (1 + 3e\frac{1}{\rho}\frac{\Delta\rho}{\Delta\xi} + \rho\frac{\Delta\xi}{e\Delta\rho}).$$

Now we must pass from the hamiltonian with l d.o.f. to the hamiltonian with $l+1$ d.o.f.; then to $l+2$ d.o.f. and so on. Let be $l_k = l + k$. Change variables

$$(\hat{A}, \hat{\varphi}, A_{l_1}, \varphi_{l_1}) \to (A^{(n)}, \varphi^{(n)}, A_{l_1}, \varphi_{l_1});$$
$$(\hat{A}, \hat{\varphi}) = \mathcal{C}^{(0)} \circ \tilde{\mathcal{C}}^{(n)}(A^{(n)}, \varphi^{(n)}), (A^{(n)}, \varphi^{(n)}) \in D_{\rho_n}(A^o) \times \Delta_{\xi_n}$$

are those of the hamiltonian $H^{(l)}_n$ in Theorem 4.2. $A_{l_1} \in \mathbb{R}$, $\varphi_{l_1} \in \Delta_{\xi_n}$;

The hamiltonian $\tilde{H}_0^{(l_1)}(A,\varphi) \doteq \sum_{i=1}^{l_1} h(A_i) + V^{(l_0)}(\hat{\varphi}) + (V^{(l_1)}(\varphi) - V^{(l_0)}(\hat{\varphi}))$, becomes

$$\sum_{i=1}^{l_0} h(A_i^{(n)}) + h(A_{l_1}) + V^{(l-1)}(\hat{\varphi}^{(n)}) + G_n^{(l_0)}(A^{(n)},\hat{\varphi}^{(n)}) + (R_n^{(l_0)}(A^{(n)},\varphi^{(n)}) - \\ - < R_n^{(l_0)}(A^{(n)},\hat{\varphi}^{(n)}) >_{l_0}) + (V^{(l_1)}(\hat{\varphi}(A^{(n)},\varphi^{(n)}),\varphi_{l_1}) - V^{(l_0)}(\hat{\varphi}(A^{(n)},\varphi^{(n)})) \tag{4.7}$$

Let's recall $(A^{(n)},A_{l_1}) \doteq (A_1,\dots,A_{l_1})$, $(\varphi^{(n)},\varphi_{l_1}) \doteq (\varphi_1,\dots,\varphi_{l_1})$; we rewrite (4.7) as

$$H_0^{(l_1)}(A,\varphi) = \sum_{i=1}^{l_1} h(A_i) + V^{(l_0-1)}(\check{\varphi}) + G_n^{(l_0)}(\hat{A},\check{\varphi}) + (R_n^{(l_0)}(\hat{A},\hat{\varphi}) - \\ - < R_n^{(l_0)}(\hat{A},\check{\varphi}) >_{l_0}) + (\tilde{V}^{(l_1)}(\hat{A},\varphi) - \tilde{V}^{(l_0)}(\hat{A},\hat{\varphi})) \tag{4.8}$$

The following theorem is analogous to Theorem 4.1 but with one more degree of freedom.

Theorem 4.4. *Let's consider the hamiltonian (4.8) with* $(A,\varphi) \in D_{\rho_n}(A^o) \times \Delta_{\xi_n}$. *Let* $A^o \in \mathbb{R}^{l_1}$ *be a point such that the vector* $h_{A^o} = (\hat{\omega}^o, \omega_{l_1}^o)$ *satisfies* $|\omega^o \cdot \nu|^{-1} \leq \frac{\gamma}{|\omega_{l_1}^o|}$ *for* $|\nu| \leq K'$, $\nu_{l_1} \neq 0$, $K' \geq \frac{1}{2\overline{\delta}} \ln \frac{r\overline{\delta}\overline{r}e|\omega_{l_1}^o|}{2\gamma \mathcal{V}^{(l_1)}}$. *If*

$$|\omega_{l_1}^o| \geq \frac{4\gamma}{e\overline{\delta}}(a\overline{r} + \frac{\mathcal{V}^{(l_1)}}{\overline{r}}) \tag{4.9}$$

then via a suitable canonical transformation $(A,\varphi) = \mathcal{C}^{(0)}(A',\varphi')$, $(A',\varphi') \in D_{\overline{\rho}_1}(A^o) \times \Delta_{\overline{\xi}_1}$, $\overline{\rho} = \rho_n$, $\overline{\xi} = \xi_n$, $\overline{\rho}_1 = \overline{\rho} - 3\overline{r}$, $\overline{\xi}_1 = \overline{\xi} - 3\overline{\delta}$

$$H_1^{(l_0)} \circ \mathcal{C}^{(0)}(A',\varphi') = H_1^{(l_1)}(A',\varphi') = \sum_{i=1}^{l_1} h(A_i') + V^{(l-1)}(\check{\varphi}') + G_1^{(l_1)}(A',\hat{\varphi}') + \\ + R_1^{(l_1)}(A',\varphi') - < R_1^{(l_1)}(A',\hat{\varphi}') >_l \tag{4.10}$$

$$G_1^{(l_1)}(A',\hat{\varphi}') = G_n^{(l_0)}(\hat{A}',\check{\varphi}') + R_n^{(l_0)}(\hat{A}',\hat{\varphi}') - < R_n^{(l_0)}(\hat{A}',\check{\varphi}') >_{l_0} + < R_1^{(l_1)}(A',\hat{\varphi}') >_{l_1}$$

$$R_1^{(l_1)} = \sum_{i=1}^{10} f_i and f_1 = (V^{(l_0-1)}(\check{\varphi}' + \check{\Delta}) - V^{(l_0-1)}(\check{\varphi}'),$$

$$f_2 = (G_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}' + \check{\Delta}) - G_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}'))$$

$$f_3 = (G_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}') - G_n^{(l_0)}(\hat{A}', \check{\varphi}'))$$

$$f_4 = (R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \hat{\varphi}' + \hat{\Delta}) - R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \hat{\varphi}'))$$

$$f_5 = (< R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}' + \check{\Delta}) >_{l_0} - < R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}') >_{l_0})$$

$$f_6 = (R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \hat{\varphi}') - R_n^{(l_0)}(\hat{A}', \hat{\varphi}'))$$

$$f_7 = (< R_n^{(l_0)}(\hat{A}' + \hat{\Xi}, \check{\varphi}') >_{l_0} - < R_n^{(l_0)}(\hat{A}', \check{\varphi}') >_{l_0})$$

$$f_8 = \sum_{\substack{\nu \in \mathbb{N}^{l_1} \\ \nu_{l_1} \neq 0 \; |\nu| > K'}} e^{i\nu\cdot(\varphi' + \Delta)} \tilde{V}_\nu^{(l_1)}(\hat{A}' + \hat{\Xi})$$

$$f_9 = \sum_{\substack{\nu \in \mathbb{N}^{l_1} \\ \nu_{l_1} \neq 0 \, |\nu| \leq K'}} e^{i\nu\cdot(\varphi'+\Delta)}(\tilde{V}_\nu^{(l_1)}(\hat{A}' + \hat{\Xi}) - \tilde{V}_\nu^{(l_1)}(\hat{A}'))$$

$$f_{10} = \sum_{i=1}^{l_1}(h(A'_i + \Xi_i) - h(A'_i) - h_{A'_i}\Xi_i)$$

$$\|R_1^{(l_1)}\|_{\overline{\rho}_1, \overline{\xi}_1} = \frac{6\gamma}{e\overline{\delta}\overline{r}|\omega_{l_1}^o|}(\mathcal{V}^{(l_1)})^2$$

Remarks The variable φ'_l is present in $R_n^{(l_0)}(\hat{A}', \check{\varphi}')$ and $< R_1^{(l_1)}(A', \hat{\varphi}') >_{l_1}$. But the first is exponentially small in $\sqrt{|\omega_l^o|}$ while the second is $O(|\omega_{l_1}^o|^{-1})$. This forces us to take $|\omega_{l_1}^o|$ exponentially small respect to $|\omega_l^o|$ in order to get $\varphi_l(t) \sim \varphi_l^o + \omega_l^o t$ for a time exponentially–long.

Proof The proof is similar to that of Theorem 4.1 so we omit it. ∎

We rewrite the hamiltonian (4.10) as

$$H_1^{(l_1)}(A', \varphi') = \sum_{i=1}^{l+1} h(A'_i) + V^{(l-1)}(\check{\varphi}') + \big(G_n^{(l_0)}(\hat{A}', \check{\varphi}') + R_n^{(l_0)}(\hat{A}', \hat{\varphi}')$$
$$- < R_n^{(l_0)}(\hat{A}', \check{\varphi}') >_l + < R_1^{(l_1)}(A', \hat{\varphi}') >_{l_1}\big) + R_1^{(l_1)}(A', \varphi') - < R_1^{(l_1)}(A', \hat{\varphi}') >_{l_1}$$

$$H_1^{(l_1)}(A', \varphi') = \sum_{i=1}^{l+1} h(A'_i) + V^{(l-1)}(\check{\varphi}') + G_n^{(l_0)}(\hat{A}', \check{\varphi}') + G_1^{(l_1)}(A', \hat{\varphi}') + \hat{R}_1^{(l_1)}(A', \varphi') \tag{4.11}$$

For a generic function $G(A, \varphi)$ we set $\hat{G}(A, \varphi) \doteq G(A, \varphi) - < G(A, \varphi) >_{l_m}$ $\varphi \doteq (\varphi_1, \ldots, \varphi_m)$ m integer.$\|R_1^{(l_1)}\|_{\overline{\rho}_1, \overline{\xi}_1} \leq \frac{6\gamma}{e\overline{\delta}\overline{r}} \frac{(\mathcal{V}^{(l_1)})^2}{|\omega_{l_1}^o|} \leq \frac{3}{2}\mathcal{V}^{(l_1)}$ using (4.9)

$$\|G_n^{(l_0)}\|_{\rho_n, \xi_n} \leq \frac{6\gamma}{\delta e r}(\mathcal{V}^{(l)})^2 \frac{1}{|\omega_l^o|}$$
$$\|G_1^{(l_1)}\|_{\overline{\rho}_1, \overline{\xi}_1} \leq 2\|R_n^{(l_0)}\|_{\rho_n, \xi_n} + \|R_1^{(l_1)}\|_{\overline{\rho}_1, \overline{\xi}_1}$$
$$\leq \frac{2\gamma}{e\delta r} \frac{(\mathcal{V}^{(l)})^2}{|\omega_l^o|} \exp\{-\ln 2 \sqrt{|\omega_l^o| \frac{e\Delta\xi\Delta\rho}{(162)\gamma\mathcal{V}^{(l)}}}\} + \frac{6\gamma}{e\overline{\delta}\overline{r}} \frac{(\mathcal{V}^{(l_1)})^2}{|\omega_{l_1}^o|}$$
$$\doteq P^{(l_1)} + \frac{6\gamma}{e\overline{\delta}\overline{r}} \frac{(\mathcal{V}^{(l_1)})^2}{|\omega_{l_1}^o|} P^{(l_0)} \equiv 0.$$

Theorem 4.5. *Let's consider the hamiltonian (4.11). If* $\frac{6\gamma a \overline{r}_1 c}{e|\omega_l^o|\overline{\delta}_1} \leq 1$ *and* $\frac{9\mathcal{V}^{(l_1)}\gamma}{e\overline{\delta}_1\overline{r}_1|\omega_{l_1}^o|} \leq 1$, *there exists a canonical transformation*

$$(A', \varphi') = (A'' + \Xi', \varphi'' + \Delta') \doteq \mathcal{C}^{(1)}(A'', \varphi'')(A'', \varphi'') \in D_{\overline{\rho}_2}(A^o) \times \Delta_{\overline{\xi}_2}$$

such that

$$H_1^{(l_1)} \circ \mathcal{C}^{(1)}(A'', \varphi'') \doteq H_2^{(l_1)}(A'', \varphi'') = \sum_{i=1}^{l+1} h(A''_i) + V^{(l-1)}(\check{\varphi}'') + G_n^{(l_0)}(\hat{A}'', \check{\varphi}'')$$
$$+ G_2^{(l_1)}(A'', \hat{\varphi}'') + \hat{R}_2^{(l_1)}(A'', \varphi'')$$

where $G_2^{(l_1)} = G_1^{(l_1)} + < \hat{R}_2^{(l_1)} >_{l_1}$, $R_2^{(l_1)}(A'', \varphi'') = \sum_{i=1}^{12} f_i$ *and*

$$
\begin{aligned}
f_1 &= V^{(l-1)}(\check{\varphi}'' + \check{\Delta}') - V^{(l-1)}(\check{\varphi}''),\\
f_2 &= G_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'' + \check{\Delta}') - G_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'')\\
f_3 &= G_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'') - G_n^{(l_0)}(\hat{A}'', \check{\varphi}'')\\
f_4 &= R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \hat{\varphi}'' + \hat{\Delta}') - R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \hat{\varphi}'')\\
f_5 &= R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \hat{\varphi}'') - R_n^{(l_0)}(\hat{A}'', \hat{\varphi}'')\\
f_6 &=< R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'' + \check{\Delta}') - R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'') >_l\\
f_7 &=< R_n^{(l_0)}(\hat{A}'' + \hat{\Xi}', \check{\varphi}'') - R_n^{(l_0)}(\hat{A}'', \check{\varphi}'') >_l\\
f_8 &=< R_1^{(l_1)}(A'' + \Xi', \hat{\varphi}'' + \hat{\Delta}') >_{l_1} - < R_1^{(l_1)}(A'' + \Xi', \hat{\varphi}'') >_{l_1}\\
f_9 &=< R_1^{(l_1)}(A'' + \Xi', \hat{\varphi}'') >_{l_1} - < R_1^{(l_1)}(A'', \hat{\varphi}'') >_{l_1}\\
f_{10} &= \sum_{\substack{\nu \in \mathbb{N}^{l_1} \\ \nu_{l_1} \neq 0\,, |\nu| > K_1'}} e^{i\nu\cdot(\varphi'' + \Delta')} R_1^{(l_1)}(A'' + \Xi')\\
f_{11} &= \sum_{\substack{\nu \in \mathbb{N}^{l_1} \\ \nu_{l_1} \neq 0\,, |\nu| \leq K_1'}} e^{i\nu\cdot(\varphi'' + \Delta')} (R_{1,\nu}^{(l_1)}(A'' + \Xi') - R_{1,\nu}^{(l_1)}(A''))\\
f_{12} &= \sum_{i=1}^{l_1} (h(A_i'' + \Xi_i') - h(A_i'') - h_{A_i''}\Xi_i')
\end{aligned}
$$

Proof It is the same as that one of Theorem 4.2 ∎

Theorem 4.6. *Let's consider the hamiltonian* $\tilde{H}_0^{(l_1)}$, *together with (4.9).If*

$$
\frac{1}{4} \leq \frac{81\gamma \mathcal{V}^{(l_1)}}{e\overline{\Delta\xi}\,\overline{\Delta\rho}} \frac{(n_{l_1} - 1)^2}{|\omega_{l_1}^o|} < \frac{1}{2} \qquad a\gamma \frac{\overline{\Delta}\rho}{\overline{\overline{\Delta}}\xi} |\omega_{l_1}^o|^{-1} \leq 1
$$

there exists the canonical transformation

$$
(A, \varphi) = \widetilde{\mathcal{C}}^{(n_{l_1})}(A^{(n_{l_1})}, \varphi^{(n_{l_1})}), (A^{(n_{l_1})}, \varphi^{(n_{l_1})}) \in D_{\overline{\rho}_{n_{l_1}}}(A^o) \times \Delta_{\overline{\xi}_{n_{l_1}}}
$$

such that

$$
(\tilde{H}_0^{(l_1)} \circ \widetilde{\mathcal{C}}^{(n_{l_1})})(A^{(n_{l_1})}, \varphi^{(n_{l_1})}) \doteq H_{n_{l_1}}^{(l_1)}(A^{(n_{l_1})}, \varphi^{(n_{l_1})}) = \sum_{j=1}^{l+1} h(A_j^{(n_{l_1})})
$$

$$
+ V^{(l-1)}(\check{\varphi}^{(n_{l_1})}) + G_{n_{l_0}}^{(l_0)}(\hat{A}^{(n_{l_1})}, \check{\varphi}^{(n_{l_1})}) + G_{n_{l_1}}^{(l_1)}(A^{(n_{l_1})}, \hat{\varphi}^{(n_{l_1})}) + R_{n_{l_1}}^{(l_1)}(A^{(n_{l_1})}, \varphi^{(n_{l_1})})
$$

$$
\|G_{n_{l_1}}^{(l_1)}\|_{\overline{\rho}_{n_{l_1}}, \overline{\xi}_{n_{l_1}}} \leq P^{(l_1)} + \frac{18\gamma}{e\overline{\delta}\overline{r}} \frac{(\mathcal{V}^{(l_1)})^2}{|\omega_{l_1}^o|}
$$

$$
\|R_{n_{l_1}}^{(l_1)}\|_{\overline{\rho}_{n_{l_1}}, \overline{\xi}_{n_{l_1}}} \leq \frac{6\mathcal{V}^{(l_1)}\gamma}{e\overline{\delta}\overline{r}|\omega_{l_1}^o|} \Big(\frac{9\mathcal{V}^{(l_1)}\gamma}{e|\omega_{l_1}^o|}\Big)^{n_{l_1}} \Pi_{h=1}^{n_{l_1}-1} (\overline{\delta}_h \overline{r}_h)^{-1}
$$

$$
\|R_{n_{l_1}}^{(l_1)}\|_{\overline{\rho}_{n_{l_1}}, \overline{\xi}_{n_{l_1}}} \leq \frac{6\gamma}{e\overline{\delta}\overline{r}} \frac{(\mathcal{V}^{(l_1)})^2}{|\omega_{l_1}^o|} \exp\Big\{- \ln 2 \sqrt{|\omega_{l_1}^o| \frac{e\overline{\Delta}\xi\overline{\Delta}\rho}{(216)\gamma \mathcal{V}^{(l_1)}}}\Big\}
$$

Proof Apply enough times the Theorem 4.5 ∎

Corollary 4.7. *Let's consider the transformation* $\tilde{\mathcal{C}}^{(n_{l_1})}: D_{\overline{\rho}_{n_{l_1}}}(A^o) \times \Delta_{\overline{\xi}_{n_{l_1}}} \to D_{\overline{\rho}}(A^o) \times \Delta_{\overline{\xi}}$ *of Theorem 4.6. Then we have* $\|\partial\tilde{\mathcal{C}}^{(n_{l_1})}\|_{\overline{\rho}_{n_{l_1}},\overline{\xi}_{n_{l_1}}} \le e^{\frac{2T_{l_1}}{(n_{l_1}-1)^2}}$ *(*$T_{l_1} = (1 + 6e\frac{1}{\rho}\frac{\overline{\Delta}\rho}{\overline{\Delta}\xi} + \rho\frac{\overline{\Delta}\xi}{e\overline{\Delta}\rho})$.

Remarks We impose $\overline{\rho}_{n_{l_1}} \ge \frac{\rho}{2}$ and this explains the 6 in place of 3 (see Corollary 4.3).

Proof Same as in Corollary 4.3. It changes slightly only T_{l_1} respect to T_{l_0} ∎

Let's define some quantities we are going to use.
1) $l_k \doteq l+k, l_0 = l$, 2) $\rho_i^{(l_k)}$ $0 \le i \le n_{l_k}$, 3) $\xi_i^{(l_k)}$ $0 \le i \le n_{l_k}$, 4) $\tilde{H}_0^{(l_k)}(A,\varphi) \doteq \sum_{i=1}^{l_k} h(A_i) + V^{(l_k)}(\varphi)$, $\varphi \in \mathbb{T}^{l_k}$, 5) $n_{l_k} \in \mathbb{N}$ $n_{l_0} \doteq n$ (the n of Theorem 4.2), 6) $K_i^{(l_k)}$ $0 \le i \le n_{l_k}$ ($K_0^{(l_0)} \doteq K$ of Theorem 4.1, $K_1^{(l_0)} \doteq K_1$ of Theorem 4.2), 7) $\tilde{\mathcal{C}}^{(n_{l_k})} \doteq \mathcal{C}_{l_k}^{(0)} \circ \mathcal{C}_{l_k}^{(1)} \circ \mathcal{C}_{l_k}^{(2)} \ldots \circ \mathcal{C}_{l_k}^{(n_{l_k}-1)}$ ($\mathcal{C}_{l_0}^{(0)}$ is the transformation of Theorem 4.1, $\mathcal{C}_{l_0}^{(1)}$ is one of the transformations of Theorem 4.2). $T_{l_k} \le T_{l_1}$ of Corollary 4.7.

Let's define for $k \ge 0$,

$$S = \sum_{k=0}^{\infty} \frac{1}{(k+1)\ln^2(k+2)},$$

$$\xi_{n_{l_k}}^{(l_k)} - \xi_{n_{l_{k+1}}}^{(l_{k+1})} = \frac{1}{(k+1)\ln^2(k+2)}\frac{1}{2}\frac{\xi^{(l_0)}}{S} = 2\Delta\xi^{(l_k)},$$

$$\xi^{(l_k)} - \xi_1^{(l_k)} = \frac{1}{4}\frac{\xi^{(l_0)}}{S}\frac{1}{(k+1)\ln^2(k+2)} = \Delta\xi^{(l_k)},$$

$$\delta_j^{(l_k)} = \frac{\Delta\xi^{(l_k)}}{3(n_{l_k}-1)}, 1 \le j \le n_{l_k-1},$$

$$\xi_j^{(l_k)} \ge \frac{1}{2}\xi^{(l_0)}, \quad \delta_0^{(l_k)} = \frac{1}{3}\Delta\xi^{(l_k)},$$

$$\rho_{n_{l_k}}^{(l_k)} - \rho_{n_{l_{k+1}}}^{(l_{k+1})} = \frac{1}{(k+1)\ln^2(k+2)}\frac{1}{2}\frac{\rho^{(l_0)}}{S} = 2\Delta\rho^{(l_k)},$$

$$\rho^{(l_k)} - \rho_1^{(l_k)} = \frac{1}{4}\frac{\rho^{(l_0)}}{S}\frac{1}{(k+1)\ln^2(k+2)} = \Delta\rho^{(l_k)},$$

$$r_j^{(l_k)} = \frac{\Delta\rho^{(l_k)}}{3(n_{l_k}-1)}, 1 \le j \le n_{l_k-1},$$

$$\rho_j^{(l_k)} \ge \frac{1}{2}\rho^{(l_0)}, \quad \rho_0^{(l_k)} = \frac{1}{3}\Delta\rho^{(l_k)},$$

For example $\rho = \rho_0 = \rho_0^{(l_0)}$, $\overline{\rho} = \rho_0^{(l_1)}$, $\xi = \xi_0 = \xi_0^{(l_0)}$, $\overline{\xi} = \xi_0^{(l_1)}$, $\overline{\Delta}\xi = \xi^{(l_1)} - \xi_1^{(l_1)}$, $\overline{\Delta}\rho = \rho^{(l_1)} - \rho_1^{(l_1)}$., $K_j^{(l_k)} = \frac{1}{\delta_j^{(l_k)}} \ln \frac{e\delta_j^{(l_k)} r_j^{(l_k)} \mathcal{V}^{(l_k)}}{2\gamma|\omega_{l_k}^o|}$

For each $k \ge 0$ we have the relations $\frac{1}{4} \le \frac{81\gamma\mathcal{V}^{(l_k)}(n_{l_l}-1)^2}{e\Delta\xi^{(l_k)}\Delta\rho^{(l_k)}|\omega_{l_k}^o|} < \frac{1}{2}$, $a\gamma\frac{\Delta\rho^{(l_k)}}{\Delta\xi^{(l_k)}}|\omega_{l_k}^o|^{-1} \le$

1, $|\omega^o_{l_k}| \geq \frac{4\gamma}{e\delta_0^{(l_k)}}(ar_0^{(l_k)} + \frac{\mathcal{V}^{(l_k)}}{r_0^{(l_k)}})$, $\qquad \|\partial\tilde{\mathcal{C}}^{(n_{l_k})}\|_{\overline{\rho}_{n_{l_k}},\overline{\xi}_{n_{l_k}}} \leq e^{\frac{2T_{l_k}}{(n_{l_k}-1)^2}}$, $T_{l_k} = T_{l_1}$ for any $k \geq 1$.

It follows that there exists a universal constant B_0 such that (being $n_{l_k} \geq 2$)

$$B_0 \frac{\mathcal{V}^{(l_k)}}{\xi_0^{(l_0)}\rho_0^{(l_0)}} \frac{(k+1)^3 \ln^6(k+2)}{|\omega^o_{l_k}|} + B_0 a \frac{\rho_0^{(l_0)}}{\xi_0^{(l_0)}} \frac{1}{|\omega^o_{l_k}|} < 1$$

and

$$n_{l_k} \geq 1 + \left[\frac{(B_1|\omega^o_{l_k}|\xi_0^{(l_0)}\rho_0^{(l_0)})^{1/2}}{(k+1)\ln^2(k+2)} \right] \geq [B_2(k+1)^{1/2}\ln(k+2)] \tag{4.12}$$

(2.3) sets a strong restriction on how small the frequencies $\omega^o_{l_k}$ could be.

$$\begin{aligned} H^{(l_k)}_{n_{l_k}}(A^{(n_{l_k})},\varphi^{(n_{l_k})}) &= \sum_{j=1}^{l+k} h(A_j^{(n_{l_k})}) + V^{(l-1)}(\varphi_1^{(n_{l_k})},\dots,\varphi_{l-1}^{(n_{l_k})}) + \\ &+ \sum_{j=0}^{k} G^{(l_j)}_{n_{l_j}}(A_1^{(n_{l_k})},\dots,A_{l_j}^{(n_{l_k})},\varphi_1^{(n_{l_k})},\dots,\varphi_{l_j-1}^{(n_{l_k})}) + R^{(l_k)}_{n_{l_k}}(A^{(n_{l_k})},\varphi^{(n_{l_k})}) \end{aligned} \tag{4.13}$$

$$\begin{aligned} &\|G^{(l_j)}_{n_{l_j}}\|_{\rho^{(l_j)}_{n_{l_j}},\xi^{(l_j)}_{n_{l_j}}} \leq P^{(l_j)} + \frac{18\gamma}{e\delta_0^{(l_j)}r_0^{(l_j)}}\frac{(\mathcal{V}^{(l_j)})^2}{|\omega^o_{l_j}|} \\ &P^{(l_{j+1})} = \frac{2\gamma(\mathcal{V}^{(l_j)})^2}{e\delta_0^{(l_j)}r_0^{(l_j)}|\omega^o_{l_j}|}\exp\{-\ln 2\sqrt{|\omega^o_{l_j}|\frac{e\Delta\xi^{(l_j)}\Delta\rho^{(l_j)}}{(162)\gamma\mathcal{V}^{(l_j)}}}\} \\ &\|R^{(l_k)}_{n_{l_k}}\|_{\overline{\rho}_{n_{l_k}},\overline{\xi}_{n_{l_k}}} \leq \frac{6\gamma}{e\delta_0^{(l_k)}r_0^{(l_k)}}\frac{(\mathcal{V}^{(l_k)})^2}{|\omega^o_{l_k}|}\exp\{-\ln 2\sqrt{|\omega^o_{l_k}|\frac{e\Delta\xi^{(l_k)}\Delta\rho^{(l_k)}}{(216)\gamma\mathcal{V}^{(l_k)}}}\} \end{aligned}$$

In the variables $(A^{(l_k)},\varphi^{(l_k)})$ the system is

$$\frac{d}{dt}\varphi_i^{(l_k)} = h_{A_i^{(l_k)}} + \frac{\partial}{\partial A_i^{(l_k)}}\sum_{j=0}^{k} G^{(l_j)}_{n_{l_j}} + \frac{\partial}{\partial A_i^{(l_k)}} R^{(l_k)}_{n_{l_k}} \tag{4.14}$$

If $i \leq l-1$ we have

$$\frac{d}{dt}{A_i}^{(l_k)} = -\frac{\partial}{\partial\varphi_i^{(l_k)}}(V^{(l-1)} + \sum_{j=0}^{k} G^{(l_j)}_{n_{l_j}} + R^{(l_k)}_{n_{l_k}}) \tag{4.15}$$

if $i > l-1$, $i = l-1+m$, $m \leq k$ we have

$$\frac{d}{dt}{A_i}^{(l_k)} = -\frac{\partial}{\partial\varphi_i^{(l_k)}}(\sum_{j=m+1}^{k} G^{(l_j)}_{n_{l_j}} + R^{(l_k)}_{n_{l_k}}) \tag{4.16}$$

Let's consider the sequence of transformations $\{\widetilde{\mathcal{C}}^{(n_{l_k})}\}$, $k = 0, 1, \dots$ whose domain is $D^{\mathbb{N}}_{\frac{\rho}{2}}(A^o) \times \Delta^{\mathbb{N}}_{\frac{\xi}{2}} \subset \mathbb{C}^{\mathbb{N}} \times \mathbb{C}^{\mathbb{N}}$. Define $\mathcal{R}_N \colon D^{\mathbb{N}}_{\frac{\rho}{2}}(A^o) \times \Delta^{\mathbb{N}}_{\frac{\xi}{2}} \to \mathbb{C}^{l_N} \times \mathbb{C}^{l_N}$, $\mathcal{R}_N \doteq$

$\widetilde{\mathcal{C}}^{(n_{l_0})} \circ \ldots \circ \widetilde{\mathcal{C}}^{(n_{l_{N-1}})} \circ \widetilde{\mathcal{C}}^{(n_{l_N})}$, $\mathcal{R}_N = (\mathcal{R}_N^{(A)}, \mathcal{R}_N^{(\varphi)})$, $\mathcal{R}_N^{(A)} = \{\mathcal{R}_N^{(A)}\}_i$, $\mathcal{R}_N^{(\varphi)} = \{\mathcal{R}_N^{(\varphi)}\}_i$ $i = 1, \ldots, l_N$. Note that the functions $(\mathcal{R}_N^{(\varphi)})_i$ and $(\mathcal{R}_N^{(A)})_i$ are analytic on the larger domain $D_{\rho_{l_N}}(A^o_{(l_N)}) \times \Delta_{\xi_{l_N}}$ and continuous on $\overline{D}_{\rho_{l_N}}(A^o_{(l_N)}) \times \overline{\Delta}_{\xi_{l_N}}$, $A^o_{(l_N)} = (A^o_1, A^o_2, \ldots, A^o_{l_N})$.

Theorem 4.8 *For each $i \in \mathbb{N}$ the following four limits are defined uniformly in $D^{\mathbb{N}}_{\frac{\varrho}{2}}(A^o) \times \Delta^{\mathbb{N}}_{\frac{\xi}{2}}$ for $N \to +\infty$: 1) $(\mathcal{R}_N^{(A)})_i$ 2) $(\mathcal{R}_N^{(\varphi)})_i$ 3) $\frac{d}{dt}(\mathcal{R}_N^{(A)})_i$ 4) $\frac{d}{dt}(\mathcal{R}_N^{(\varphi)})_i$.*

Remarks i) Observe that $\mathcal{R}_\infty \doteq \lim_{N\to+\infty}\{(\mathcal{R}_N^{(A)})_i, (\mathcal{R}_N^{(\varphi)})_i\}_{i\in\mathbb{N}} \doteq (A, \varphi) \in (\mathbb{C}^{\mathbb{N}}, \mathbb{C}^{\mathbb{N}})$ defines the action–angle variables of the equations (1.2); $(A, \varphi) = \mathcal{R}_\infty \doteq (\mathcal{P}(v, u), \mathcal{Q}(v, u))$. There is no uniformity respect to i ii) $u_i \doteq \lim_{N\to\infty} \varphi_i^{(l_N)}$ and $v_i \doteq \lim_{N\to\infty} A_i^{(l_N)}$ iii) In (4.17) becomes apparent that without the factor $(n_{l_k} - 1)^{-2}$ the limits do not exist and the presence of the factor is due to the different choice of the first analyticity loss: $\delta_i^{(l_k)}$ and $r_i^{(l_k)}$ $i \geq 1$, much smaller respectively than $\delta_0^{(l_k)}$ and $r_0^{(l_k)}$ (see after Corollary 4.7)

Proof We show that $(\mathcal{R}_N^{(A)})_i$ and $(\mathcal{R}_N^{(\varphi)})_i$ are Cauchy sequences defined in $D^{\mathbb{N}}_{\frac{\varrho}{2}}(A^o) \times \Delta^{\mathbb{N}}_{\frac{\xi}{2}}$ and then define $(\mathcal{P}_i, \mathcal{Q}_i)$ being in the space $\mathcal{A}$ which is complete

$$\begin{aligned}
&\|(\mathcal{R}_{N+1}^{(A)})_i - (\mathcal{R}_N^{(A)})_i\|_{\frac{\varrho}{2},\frac{\xi}{2}} \leq \|(\mathcal{R}_{N+1}^{(A)})_i - (\mathcal{R}_N^{(A)})_i\|_{\rho_{l_{N+1}},\xi_{l_{N+1}}} \\
&\leq \Pi_{k=0}^N \|\partial\widetilde{\mathcal{C}}^{(n_{l_k})}\|_{\rho_{l_k},\xi_{l_k}} |(\widetilde{\mathcal{C}}^{(n_{l_{N+1}})})_i^{(A)} - A_i^{(n_{l_N})}| \\
&\leq (\Pi_{k=0}^\infty \exp\{\frac{2T_{l_k}}{(n_{l_k}-1)^2}\})(\exp\{\frac{2T_{l_{N+1}}}{(n_{l_{N+1}}-1)^2}\} - 1) \\
&\leq B_3(\exp\{\frac{2T_{l_{N+1}}}{(n_{l_{N+1}}-1)^2}\} - 1)
\end{aligned} \tag{4.17}$$

and the same occurs for $\|(\mathcal{R}_{N+1}^{(\varphi)})_i - (\mathcal{R}_N^{(\varphi)})_i\|_{\frac{\varrho}{2},\frac{\xi}{2}}$. The second $\leq$ is due to Theorem 4.2. (4.12) implies that $\{\mathcal{R}_N^{(A)}\}_i$ and $\{\mathcal{R}_N^{(\varphi)}\}_i$ are Cauchy sequences in the space $\mathcal{A}$ which is complete (see section 2)

Moreover we have $\frac{d}{dt}(\mathcal{R}_N^{(\varphi)})_i = h_{A_i}((\mathcal{R}_N^{(A)})_i)$ and by Lagrange theorem, using that $|h_{A_iA_i}| \leq a$, we can do the limit $N \to +\infty$ at left. The uniform convergence respect to time allows us to interchange the limits $N \to +\infty$ with the derivative so that

$$\begin{aligned}
&\lim_{N\to+\infty} \frac{d}{dt}(\mathcal{R}_N^{(\varphi)})_i = \frac{d}{dt} \lim_{N\to+\infty} (\mathcal{R}_N^{(\varphi)})_i = \frac{d}{dt}\mathcal{Q}_i = h_{A_i}(\mathcal{P}_i). \\
&\frac{d}{dt}(\mathcal{R}_N^{(A)})_i = -V^{(l_N)}_{\varphi_i^{(l_N)}}(\mathcal{R}_N^{(\varphi)}) = f_i^{(l_N)}(\mathcal{R}_N^{(\varphi)}) = (f_i^{(l_N)}(\mathcal{R}_N^{(\varphi)}) - f_i(\mathcal{Q})) + f_i(\mathcal{Q}) \\
&\qquad = [f_i^{(l_N)}(\mathcal{R}_N^{(\varphi)}) - f_i^{(l_N)}(\mathcal{Q}^{(l_N)})] + [f_i^{(l_N)}(\mathcal{Q}^{(l_N)}) - f_i(\mathcal{Q})] + f_i(\mathcal{Q}) \\
&\mathcal{Q}^{(l_N)} = (\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \ldots, \mathcal{Q}_{l_N}).
\end{aligned}$$

The first difference goes to zero because of the regularity properties of the function $f_i^{(l_N)}$ and $R_N^{(\varphi)} - \mathcal{Q}^{(l_N)} \xrightarrow[N\to+\infty]{} 0$. The second difference goes to zero by the fact that the functions $f_i^{|I|}: \mathcal{T}_w \to \mathbb{R}$ converge uniformly to f_i for $|I| \to \mathbb{N}$ (see Section 2, Averages). Actually $f_i^{|I|}$ would be defined over $\mathbb{T}^{|I|}$ but it does not matter because $f_i^{|I|}$ does not depend on the variables φ_i for $i \notin I$. It follows that $\lim_{N\to+\infty} \frac{d}{dt}(\mathcal{R}_N^{(A)})_i = f_i(\mathcal{Q})$ and the same interchange as before of the limit $N \to +\infty$ with the derivative can be performed here. Then we get what expected namely $\frac{d}{dt}\mathcal{P}_i = f_i(\mathcal{Q})$ ∎

Proof of Theorem 3.1 Let's consider the hamiltonian (4.13).

$$\|G^{(l_j)}_{n_{l_j}}\|_{\rho^{(l_j)}_{n_{l_j}},\xi^{(l_j)}_{n_{l_j}}} \leq P^{(l_j)} + \frac{18\gamma}{e\delta_0^{(l_j)} r_0^{(l_j)}} \frac{(\mathcal{V}^{(l_j)})^2}{|\omega^o_{l_j}|}$$

$$P^{(l_j)} = \frac{2\gamma(\mathcal{V}^{(l_{j-1})})^2}{e\delta_0^{(l_{j-1})} r_0^{(l_{j-1})}|\omega^o_{l_{j-1}}|} \exp\{-\ln 2\sqrt{|\omega^o_{l_{j-1}}|\frac{e\Delta\xi^{(l_{j-1})}\Delta\rho^{(l_{j-1})}}{(162)\gamma\mathcal{V}^{(l_{j-1})}}}\}$$

Remember that $P^{(l_j)}$ is the estimate of a term containing the angle–variables $(\varphi_1^{(n_{l_k})},\dots,\varphi_{l_{j-1}}^{(n_{l_k})})$ (see after Corollary 4.7)

$$B_3 = \sup_{k\geq 0} \frac{\delta_0^{(l_k)} r_0^{(l_k)}}{\delta_0^{(l_{k+1})} r_0^{(l_{k+1})}} > 1, \quad B_4 = \sqrt{\frac{e}{2\gamma}}\frac{\ln 2}{36S}, \quad S = \sum_{k=0}^{\infty} \frac{1}{(k+1)\ln^2(k+2)},$$

If $|\omega^o_{l_j}| \geq B_3 \frac{(\mathcal{V}^{(l_j)})^2}{(\mathcal{V}^{(l_{j-1})})^2}|\omega^o_{l_{j-1}}| \exp\{\frac{B_4}{j\ln^2(j+1)}\sqrt{\frac{\rho^{(l_0)}\xi^{(l_0)}}{\mathcal{V}^{(l_0)}}|\omega^o_{l_{j-1}}|}\}$ then $\frac{18\gamma}{e\delta_0^{(l_j)} r_0^{(l_j)}}\frac{(\mathcal{V}^{(l_j)})^2}{|\omega^o_{l_j}|} \leq P^{(l_j)}$ and by virtue of (3.1) (in particular the power 6 of the logarithm) the series $\sum_{j=0}^{\infty} P^{(l_j)}$ is convergent. It follows that the series $\sum_{j=0}^{\infty} \|G^{(l_j)}_{n_{l_j}}\|_{\rho^{(l_j)}_{n_{l_j}},\xi^{(l_j)}_{n_{l_j}}}$ is convergent too while $R^{(l_k)}_{n_{l_k}}(A^{(n_{l_k})},\varphi^{(n_{l_k})})$ goes to zero when k goes to infinity. Then the equations (4.14)–(4.16) admit the limit for $k \to +\infty$ and (3.3)–(3.6) follow ∎

Proof of Corollary 3.2 Iterating the procedure of Corollary 3.1, (3.2) is implied by $|\omega^o_{l_k}| \geq (B_3)^k \frac{(\mathcal{V}^{(l_k)})^2}{(\mathcal{V}^{(l_0)})^2}|\omega^o_l| \exp\{\frac{kB_4}{\ln^2(2)}\sqrt{\frac{\rho^{(l_0)}\xi^{(l_0)}|\omega^o_l|}{\mathcal{V}^{(l_0)}}}\}$; $\sum_{j=2}^{\infty} \|G^{(l_j)}_{n_{l_j}}\|_{\frac{\rho}{2},\frac{\xi}{2}} \leq \|G^{(l_1)}_{n_{l_1}}\|_{\frac{\rho}{2},\frac{\xi}{2}}$ provided that $\frac{B_4}{\ln^2(2)}\sqrt{\frac{\rho^{(l_0)}\xi^{(l_0)}|\omega^o_l|}{\mathcal{V}^{(l_0)}}} \geq \ln(1+B_5)$ where $B_5 = \sup_{j\geq 2} \frac{(j\ln^2(j+1))}{\ln^2 2(B_3)^j}$. By the same condition on $|\omega^o_l|$ we get $\sum_{j=j_0+1}^{\infty} \|G^{(l_j)}_{n_{l_j}}\|_{\frac{\rho}{2},\frac{\xi}{2}} \leq \|G^{(l_{j_0})}_{n_{l_{j_0}}}\|_{\frac{\rho}{2},\frac{\xi}{2}}$ being $\sup_{j\geq j_0+1} \frac{(j\ln^2(j+1))}{(j_0\ln^2(j_0+1))(B_3)^j} \leq B_5$. In particular

$$\|\sum_{j=p+2}^{\infty} G^{(l_j)}_{n_{l_j}}\|_{\frac{\rho}{2},\frac{\xi}{2}} \leq 2\|G^{(l_{p+1})}_{n_{l_{p+1}}}\|_{\frac{\rho}{2},\frac{\xi}{2}} \leq 2\|G^{(l_{p+1})}_{n_{l_{p+1}}}\|_{\rho^{(l_{p+1})}_{n_{l_{p+1}}},\xi^{(l_{p+1})}_{n_{l_{p+1}}}}$$
$$\leq \frac{2\gamma(\mathcal{V}^{(l_p)})^2}{e\delta_0^{(l_p)} r_0^{(l_p)}|\omega^o_{l_p}|} \exp\{-\ln 2\sqrt{|\omega^o_p|\frac{e\Delta\xi^{(l_p)}\Delta\rho^{(l_p)}}{(162)\gamma\mathcal{V}^{(l_p)}}}\}$$

and then

$$|v_{l_p}(t) - v_{l_p}(0)| \leq |t|e(\xi^{(l_{p+1})}_{n_{l_{p+1}}} - \frac{\xi}{2})^{-1}\|G^{(l_{p+1})}_{n_{l_{p+1}}}\|_{\rho^{(l_{p+1})}_{n_{l_{p+1}}},\xi^{(l_{p+1})}_{n_{l_{p+1}}}}$$
$$\leq |t|e(\xi^{(l_{p+1})}_{n_{l_{p+1}}} - \xi^{(l_{p+2})}_{n_{l_{p+2}}})^{-1}2P^{(l_{p+1})} \leq |t|\frac{(\mathcal{V}^{(l_p)})^2}{\rho^{(l_0)}(\xi^{(l_0)})^2|\omega^o_{l_p}|}1152S^2(p+2)^3\ln^6(p+3)$$
$$\cdot \exp\{-\frac{B_4}{(p+1)\ln^2(p+2)}\sqrt{\frac{|\omega^o_{l_p}|\rho^{(l_0)}\xi^{(l_0)}}{\mathcal{V}^{(l_p)}}}\}$$

being

$$v_{l_p}(t) - v_{l_p}(0) = -\int_0^t \frac{\partial}{\partial u_{l_p}} \sum_{j=p+1}^{\infty} G_{n_{l_j}}^{(l_j)}$$
$$\varphi_{l_p} - u_{l_p} = \mathcal{Q}_{l_p}(v,u) - u_{l_p} = \big((\widetilde{\mathcal{C}}^{(l_p)} \circ \widetilde{\mathcal{C}}^{(l_{p+1})} \circ \widetilde{\mathcal{C}}^{(l_{p+2})} \circ \dots)(v,u)\big)_{l_p}^{(\varphi)} - u_{l_p}$$
$$A_{l_p} - v_{l_p} = \mathcal{P}_{l_p}(v,u) - v_{l_p} = \big((\widetilde{\mathcal{C}}^{(l_p)} \circ \widetilde{\mathcal{C}}^{(l_{p+1})} \circ \widetilde{\mathcal{C}}^{(l_{p+2})} \circ \dots)(v,u)\big)_{l_p}^{(A)} - v_{l_p}$$

For what written after Corollary 4.7, we have

$$\|\big((\widetilde{\mathcal{C}}^{(l_p)} \circ \widetilde{\mathcal{C}}^{(l_{p+1})} \circ \widetilde{\mathcal{C}}^{(l_{p+2})} \circ \dots)(v,u)\big)_{l_p}^{(\varphi)} - u_{l_p}\|_{\frac{\varrho}{2},\frac{\xi}{2}}$$
$$\leq \Pi_{k=p}^{\infty} \exp\{\frac{2T_{l_k}}{(n_{l_k}-1)^2}\} - 1 = \exp\{\sum_{k=p}^{\infty} \frac{2T_{l_k}}{(n_{l_k}-1)^2}\} - 1.$$

By

$$|\omega_{l_k}| \geq B_3^{k-p} (\frac{\mathcal{V}^{(l_k)}}{\mathcal{V}^{(l_p)}})^2 \exp\{(k-p)B_4 \frac{\sqrt{Q|\omega_{l_p}^o|}}{(p+1)\ln^2(p+2)}\}|\omega_{l_p}^o|,$$
$$Q = \frac{\rho^{(l_0)}\xi^{(l_0)}}{\mathcal{V}^{(l_p)}}, \quad T_{l_k} \leq T, \quad B_3 \exp\{B_4 \frac{\sqrt{Q|\omega_{l_p}^o|}}{(p+1)\ln^2(p+2)}\} \doteq B_6 \geq 2,$$

the following holds

$$\exp\Big\{\sum_{k=p}^{\infty} \frac{2T_{l_k}}{(n_{l_k}-1)^2}\Big\} \leq \exp\Big\{2T \frac{324}{e} \frac{(\mathcal{V}^{(p)})^2}{|\omega_{l_p}^o|} \frac{(4S)^2}{\xi^{(l_0)}\rho^{(l_0)}} (\sum_{t=0}^{p+2} \frac{(2p+4)^2 \ln^4(2p+4)}{\mathcal{V}^{(l_p)}} B_6^{-t}$$
$$+ \sum_{t=p+2}^{\infty} \frac{(2t)^2 \ln^4(2t)}{\mathcal{V}^{(l_p)}} B_6^{-t})\Big\}$$

The sums are bounded by

$$\frac{2}{\mathcal{V}^{(l_p)}}(2p+4)^2 \ln^4(2p+4) + \frac{B_7}{\mathcal{V}^{(l_p)}},$$
$$B_7 = \sum_{t=1}^{\infty} (2t)^2 \ln^4(2t)(B_6)^{-t}$$
$$\exp\{\sum_{k=p}^{\infty} \frac{2T_{l_k}}{(n_{l_k}-1)^2}\} - 1 \leq \exp\{T \frac{648}{e} \frac{(4S)^2}{|\omega_{l_p}^o|} \frac{\mathcal{V}^{(l_p)}}{\xi^{(l_0)}\rho^{(l_0)}} (B_7 + 2(2p+4)^2 \ln^4(2p+4))\} - 1$$
$$\leq T B_8 (2p+4)^2 \ln^4(2p+4)) \frac{\mathcal{V}^{(l_p)}}{|\omega_{l_p}^o| \xi^{(l_0)} \rho^{(l_0)}}$$

Then $\varphi_{l_p} - u_{l_p}$ goes to zero when $|\omega_{l_p}^o|$ goes to $+\infty$ and the behavior depends on the choice of $\{|\omega_{l_k}^o|\}$ with $k \geq p$. The same happens for $A_{l_p} - v_{l_p}$.

As a consequence, being $A_{l_p}(t) - A_{l_p}(0) = (A_{l_p}(t) - v_{l_p}(t)) + (v_{l_p}(t) - v_{l_p}(0)) + (A_{l_p}(0) - v_{l_p}(0))$, if

$$|t| \leq B_9 \frac{\rho^{(l_0)}}{\mathcal{V}^{(l_0)}} \frac{1}{(p+2)\ln^2(p+3)} \exp\Big\{ \frac{B_4}{(p+1)\ln^2(p+2)} \sqrt{\frac{|\omega^o_{l_p}|\rho^{(l_0)}\xi^{(l_0)}}{\mathcal{V}^{(l_p)}}} \Big\}$$

we have $|A_{l_p}(t) - A_{l_p}(0)| \leq 3TB_8(2p+4)^2 \ln^4(2p+4)) \frac{\mathcal{V}^{(l_p)}}{|\omega^o_{l_p}|\xi^{(l_0)}}$. For the variables $A_i(t)$, $i = 1, \ldots, l-1$ it is valid what written in the remarks of Theorem 4.1 namely $|A_i(t) - A_i(0)| \leq C\rho$. The constant C is the greatest of the $B_i's$ $i = 0, \ldots, 8$. ∎

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