# Simplifying the Virtual Safety Stock formula

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**Abstract:** The paper deepen the analysis into the Virtual Safety Stock theory, which is an approach intended to drastically reduce safety inventory levels exploiting the eventual time lag between the moment when a product is ordered and the time the product needs to be available, while preserving the same performance as a production system that operates with physical safety stock. The original virtual safety stock definition embeds two major problems: a double Gaussian integral in the formulation together with the heritage of the unrealistic assumptions already included in the earliest Hadley and Whitin's safety stock conception. This paper describes an alternative approach in which the virtual safety stock is defined with a closed-form expression much easier to compute and use in operations management practice.

Keywords: Safety Stock; Virtual Safety Stock; Inventory Management; Service Level.

## 1. Introduction

Over the years, several authors have focussed their research on emphasizing the crucial role of inventory control in production planning, aiming at finding out the right trade-off between the benefits and the drawback connected with each material management method: from the Western traditional approach to the Lean philosophy and Just-in-Time technique, through the Theory of Constraints and up to the modern solutions of stock virtualization. This latter concept dates back in 1985, when Chang introduced the opportunity of a valuable alternative to reduce the stock levels in specific industrial contexts; the concept was then analysed by other authors which concentrated on the trade-off between stockholding and preserving available productive capacity through the exploitation of safety lead times (Glasserman, Wang 1998; Wijngaard, 2002). Safety stock is necessary to deal with the production scenario uncertainties, due both to demand variability and to supplier procurement delays (Hadley and Whitin, 1963), in order to meet the expected service level ensuring the customer satisfaction and retention. The trade-off between stock and time is, indeed, the fundamental principle on which alternative solutions to reduce inventory level are founded, such as: bringing forward the production runs while leaving service level unchanged (Whybark and Williams, 1976), retrieving unproductive time, scheduling backlog or combining an amount of safety stock with the opportunity of delayed deliveries.

## 2. The Virtual Safety Stock theory

### *2.1 The main pillar*

The traditional safety stock (SS) theory is based on the assumption of immediate delivery of ordered products, that is a condition often missing either in B2C or, mainly in B2B context: before being delivered to final customers, ordered products may wait in warehouses even one or two days, for several reasons. So there may be a lag between the moment when a product is ordered and the time the product needs to be available, that is defined as DST, Delivery Slack Time (Nenni, Schiraldi, Van de Velde, 2005).

The possibility to regain a portion of time, between the time an order is received and the time the product is delivered to the customer, plays a key role in defining safety stock, since this time delay deeply affects service level. Removing the immediate delivery constraint and allowing a small DST, a stock out event would not necessarily happen when an order is received and the stock is not available. Indeed, an order could be on hand before the DST is elapsed. In this case, the demand will be met without incurring any backlog, and thus the DST would entirely replace the physical safety stock. This logic encouraged some authors to formalize the Virtual Safety Stock theory, which is based on the reduction - or elimination - of the physical safety stock through the exploitation of DST. However, the resulting formulation of service level in VSS theory still seems too complex due to the presence of two nested Gaussian integrals (Nenni, Schiraldi, Van de Velde, 2005). Thus this paper presents the efforts of the authors in trying to obtain an handier expression to be more easily adopted in industrial context.

### *2.2 Service Level formulation in the VSS theory*

The computation of service level in the original VSS theory is based on the assumption that a customer order, which is received on time $t^*$, will not be satisfied if these two conditions are both true:

1. the stock has run out on time $t$, where $t < t^*$ ;

2. not any replenishment will arrive before time $t^*$+ DST,

where DST, as previously introduced, represents the time interval in which the customer cannot complain for the absence of the requested item, thanks to his expectance/acceptance of a delayed delivery. Now, if $t^*$ is the moment in which the customer's order is received, Nenni et alii (2005) define:

- $P_{EX}(t)$ as the probability that all the stock run out within time $t$ and depends on demand variability and on stock level;
- $P_{REP}(t)$ as the probability that within time $t$ no replenishment lot has arrived and depends only on the supplier's delivery process; thus it is not directly controllable.

Thus, the probability $P_{NSAT}(t^*)$, that an order received in time $t^*$ is not satisfied can be determined as follows:

$$P_{NSAT}(t^*) = P_{EX}(t^*)\cdot P_{REP}(t^*+DST) \qquad (1)$$

According to traditional SS theory hypotheses (Hadley & Whitin, 1963) in a Re-Order Cycle (ROC) inventory management context:

- customer demand for each time bucket (e.g. day) is a stochastic variable ($d$) and follows a Normal distribution $N(\bar{d},\sigma_d)$;
- the supplier delivery time is a stochastic variable ($DT$) and follows a Normal distribution $N(\overline{DT},\sigma_{DT})$;
- both $d$ and $DT$ are independent and identically distributed (i.i.d.).

So if

$$P_{EX}(t^*) = \frac{1}{\sqrt{2\pi}} \int_{k(t^*-\varepsilon)}^{k(t^*+\varepsilon)} e^{-\frac{z^2}{2}}\, dz$$

$$P_{REP}(t^*) = 1 - \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\hat{k}(t^*)} e^{\frac{z^2}{2}}\, dz$$

we have that

$$k(t) = (\bar{d}\cdot\overline{DT} - \bar{d}\cdot t)\,\frac{\sqrt{\overline{DT}}}{\sigma_d\cdot t} \quad \text{and} \quad \hat{k}(t) = \frac{t - \overline{DT}}{\sigma_{DT}}$$

And, thus, the service level results to be (Nenni, Schiraldi, Van de Velde, 2005):

$$SL = 1 - \int_0^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{\hat{k}(t^*)}^{\infty}\exp\left(-\frac{z^2}{2}\right)dz\right]\cdot$$

$$\cdot\left[-\frac{1}{\sqrt{2\pi}}\int_{k(t^*-\varepsilon)}^{k(t^*+\varepsilon)}\exp\left(-\frac{z^2}{2}\right)dz\right]d(t^*)$$

It is clear that this formulation is not easy manageable and this is the reason for the proposal which follows.

## 3. A proposal for simplifying the virtual safety stock formula

### *3.1 An alternative stochastic model to determine stock-out probability*

This paragraph introduces an alternative modeling approach to estimate the stock out probability in order to simplify the SL formulation in VSS theory. Assume that after the replenishment order is launched to the supplier at time $t_{ord}$ – i.e. when the inventory level has dropped below the re-order level (RL) - the replenishment lot is received at time $t_R$.

If we define

$t^* = t_R - t_{ord}$

and if $D(t_R)$ represents the cumulative demand at time $t_R$, that is when $t^*$ time buckets have passed since the launch of the order, we have that

$$D(t_R) = \bar{d}\cdot t^* \quad \text{and} \quad \sigma_D^2(t_R) = \sigma_d^2\cdot t^* \qquad (2)$$

Thus, defining:

- $P\{D(t_R)>RL\}$ the probability that in time $t_R$ , the cumulative demand is greater than $RL$, where $RL$ includes the safety stock $SS$;
- $P\{t=t_R\}$ the probability that the replenishment will arrive in time $t_R$,

we have that the stock-out probability may be also expressed as:

$$P_{SO} = \int_t P\{D(t_R) > RL\}\cdot P\{t = t_R\}\,dt \qquad (3)$$

### *3.2 The stochastic demand model: triangular approximation*

We proceed now to show how to approximate the demand distribution – which was assumed to follow a Normal distribution since the first contribution of Hadley & Whitin in 1963 - through the triangular distribution. Scherer, Pomroy and Fuller (2003) have proposed an heuristic approach to estimate triangular distribution parameters, in order to fit a Normal distribution N(σ, μ):

$a = \mu - w\cdot\sigma$

$b = \mu + w\cdot\sigma$

$c = \mu$

where $w \approx 2{,}436$.

Therefore, starting from the distribution parameters defined by equations (2) we find:

$$a(t) = \bar{d}\cdot t - w\cdot\sigma_d\cdot\sqrt{t} \qquad (4)$$

$$b(t) = \bar{d}\cdot t + w\cdot\sigma_d\cdot\sqrt{t} \qquad (5)$$

$$c(t) = \bar{d}\cdot t \qquad (6)$$

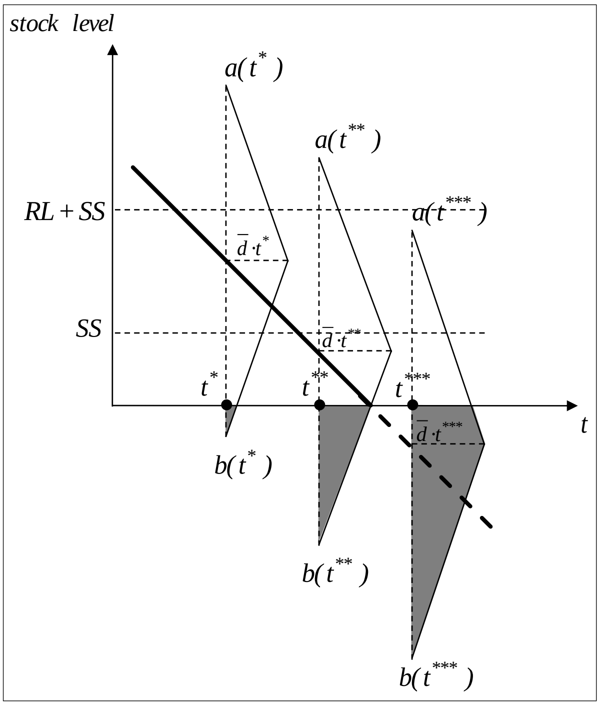


Figure 1. Stock out probability due to demand variability

We now define, the probabilities $P_{D1}$ and $P_{D2}$. As show in figure 1, these correspond, at each time *t*, to the area between *L* (under the stock level - on the time axis) and *b(t)*, respectively in $L<\bar{d}\cdot t$ and $L>\bar{d}\cdot t$ case. Therefore we have:

$$P_{D1}=1-F^{1}_{D(t)}=1-\frac{(w\cdot\sigma_{d}\cdot\sqrt{t}-\bar{d}\cdot t+L)^{2}}{2\cdot w^{2}\cdot\sigma_{d}{}^{2}\cdot t}\text{, with } L<\bar{d}\cdot t \quad (7)$$

$$P_{D2}=1-F^{2}_{D(t)}=\frac{(w\cdot\sigma_{d}\cdot\sqrt{t}+\bar{d}\cdot t-L)^{2}}{2\cdot w^{2}\cdot\sigma_{d}{}^{2}\cdot t}\text{, with } L>\bar{d}\cdot t \quad (8)$$

*Proof.* As shown in figure 1, in $L<\bar{d}\cdot t$ case we have $P\{D(t)>L\}=1-F^{1}_{D(t)}(L)$, that is the complementary cumulative distribution of *D(t)*, and then equal to the area between *L* and *b(t)*, that is the total area (equal to 1) minus the area of the triangle having base of length *n* and height of length *m*, as shown in figure 2, where:

$$n=[L-a(t)]\;;\; m=\frac{h}{q}\cdot n\;;\; h=\frac{2}{b(t)-a(t)}\;;\; q=\bar{d}\cdot t-a(t)$$

Therefore, we find:

$$P\{D(t)>L\}=1-F^{1}_{D(t)}(L)=1-\frac{m\cdot n}{2} \quad (9)$$

Replacing equations (4), (5), (6) in the (9) we have:

$$1-F^{1}_{D(t)}=1-\frac{(w\cdot\sigma_{d}\cdot\sqrt{t}-\bar{d}\cdot t+L)^{2}}{2\cdot w^{2}\cdot\sigma_{d}{}^{2}\cdot t}$$

Similarly, for $L>\bar{d}\cdot t$ case we have:

$$P\{D(t)>L\}=1-F^{2}_{D(t)}(L)=\frac{m'\cdot n'}{2} \quad (10)$$

where: $n'=[b(t)-L]$; $m'=\frac{h}{q'}\cdot n'$; $q'=b(t)-\bar{d}\cdot t$

Replacing equations (4), (5), (6) in the (10) we have:

$$1-F^{2}_{D(t)}=\frac{(w\cdot\sigma_{d}\cdot\sqrt{t}+\bar{d}\cdot t-L)^{2}}{2\cdot w^{2}\cdot\sigma_{d}{}^{2}\cdot t}$$

*q.e.d.*

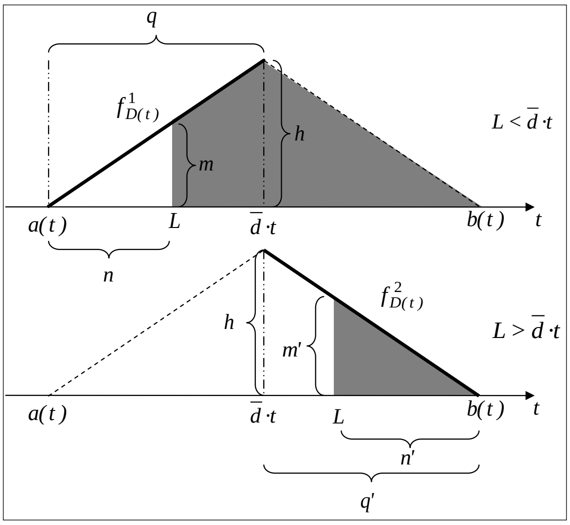


Figure 2. The computation of stock out probability

As discussed, the probability $P\{D(t)>L\}$ is given by $P_{D1}$ for *L* within the interval [*a(t)*, *c(t)*], otherwise, it is give by $P_{D2}$ for *L* within the interval [*c(t)*, *b(t)*]. Thus, the probability that the demand *D(t)* is greater than *RL* is given by:

$$P_{D1}=1-F^{1}_{D(t)} \quad for \quad t>\overline{DT}+\frac{SS}{\bar{d}} \quad (11)$$

$$P_{D2}=1-F^{2}_{D(t)} \quad for \quad t<\overline{DT}+\frac{SS}{\bar{d}} \quad (12)$$

*Proof.* Taking into account the interval [*a(t)*; *c(t)*], we refer to density function $f^{1}_{D(t)}$ if $L<\bar{d}\cdot t$, where $L=RL+SS$ and $RL=\bar{d}\cdot\overline{DT}$. Therefore the condition became $RL+SS<\bar{d}\cdot t$ that is $t>\overline{DT}+\frac{SS}{\bar{d}}$. Whereas, we refer to density function $f^{2}_{D(t)}$ if $L>\bar{d}\cdot t$ that is $t<\overline{DT}+\frac{SS}{\bar{d}}$.

*q.e.d.*

### *3.2 The stochastic model for the demand: uniform approximation*

In order to reduce model complexity, we assume that supplier delivery time follows an uniform distribution defined by the following parameters:

- $t_E$ is the minimum expected time for supply delivery (latest delivery).
- $t_L$ is the maximum expected time for supply delivery (earliest delivery).

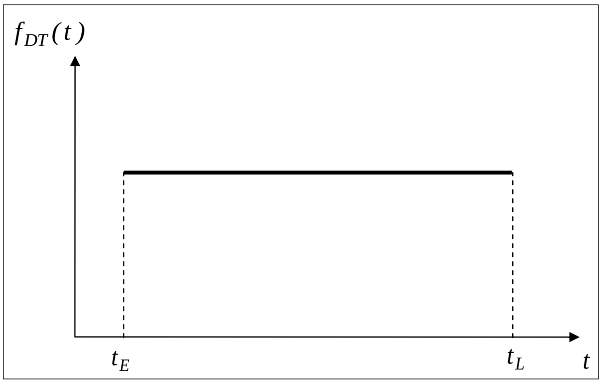


Figure 3. Uniform distribution model for supplier's delivery time

Therefore the probability that a replenishment lot arrives exactly in time $t$ is given by:

$$P\{t = t_R\} = f_{DT}(t_R) = \frac{1}{t_L - t_E} \tag{13}$$

Analogously to previously discussed triangular model, we can estimate the parameters of the distribution, starting from the Normal $N(\overline{DT}, \sigma_{DT})$ adopted in Hadley & Whitin's model. So, through the uniform approximation of the Gaussian distribution, proposed by Scherer et al (Scherer, Pomroy, Fuller, 2003) we have:

$$t_L = \overline{DT} + \sqrt{3} \cdot \sigma_{DT}$$
$$t_E = \overline{DT} - \sqrt{3} \cdot \sigma_{DT}$$

### *3.3 The approximation of stock-out probability*

Once we have defined the demand and delivery lead time distribution, we can determine stock out probability staring from (3). First, we define the interval in which the (3) will be integrated with respect to $t$. The maximum integration interval is given by the time interval [$t_L$, $t_E$] in which the delivery is expected. This assumption is necessary because outside this interval the expression (13) is not defined, and this is reasonably acceptable in the case of $P\{D(t_E) > L\} \approx 0$. In order to define the integration interval, we should take into account two other stock out possible occurrences, due to the demand variability:

- $t_{NSO}$ is the time before which the stock out probability is equal to 0;
- $t_{SSO}$ is the time after which the stock out probability is equal to 1.

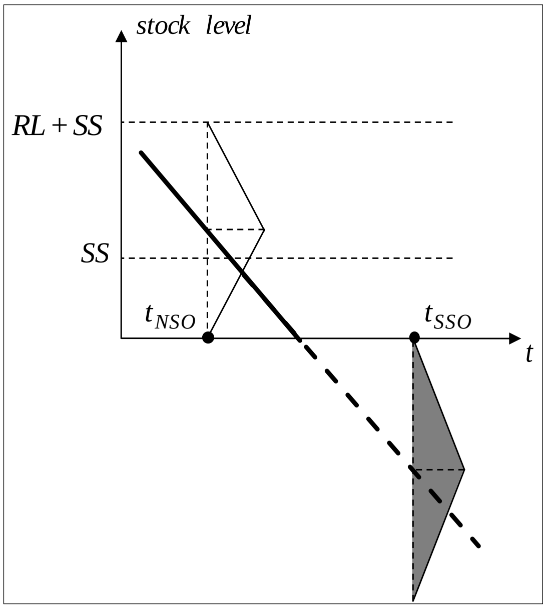


Figure 4. The stock out probability before the instant $t_{NSO}$ and after the instant $t_{SSO}$

Therefore, the integration interval is given by [$T_m$, $T_M$] where:

$T_m = \max(t_E; t_{NSO})$

$T_M = \min(t_L; t_{SSO})$

The instant $t_{NSO}$ is given setting the follow condition:

$$b(t) = L$$

And then, replacing the (5) the condition turns into:

$$\bar{d} \cdot t + w \cdot \sigma_d \cdot \sqrt{t} = L$$

finding:

$$t_{NSO} = \frac{L}{\bar{d}} + \frac{w^2 \cdot \sigma_d{}^2}{2 \cdot \bar{d}^2} - \frac{w \cdot \sigma_d}{2 \cdot \bar{d}^2} \sqrt{w^2 \cdot \sigma_d{}^2 + 4 \cdot L\bar{d}} \tag{14}$$

Whereas, $t_{SSO}$ is given setting the condition:

$$a(t) = L$$

And thus, replacing the (4), the above condition turns into:

$$-\bar{d} \cdot t + w \cdot \sigma_d \cdot \sqrt{t} = L$$

finding:

$$t_{SSO} = \frac{L}{\bar{d}} + \frac{w^2 \cdot \sigma_d{}^2}{2 \cdot \bar{d}^2} + \frac{w \cdot \sigma_d}{2 \cdot \bar{d}^2} \sqrt{w^2 \cdot \sigma_d{}^2 + 4 \cdot L\bar{d}} \tag{15}$$

After the integration boundaries, $T_m$ and $T_M$, have been found, we can proceed to compute the stock out probability:

$$P_{SO} = \frac{1}{t_L - t_E} \left\{ P_{D2}^T + P_{D1}^T \right\} + \frac{t_L - T_M}{t_L - t_E} =$$

$$= \frac{1}{t_L - t_E} \left\{ \int_{T_m}^{\overline{DT}+\frac{SS}{\overline{d}}} P_{D2}\ dt + \int_{\overline{DT}+\frac{SS}{\overline{d}}}^{T_M} P_{D1}\ dt \right\} + \frac{t_L - T_M}{t_L - t_E} \qquad (16)$$

We point out that stock out probability is 0 within the interval $[t_{NSO}, t_E]$ (if $t_{NSO}>t_E$) given that $P\{D(t_{NSO})>L\}=0$, whereas, is equal to $\frac{t_L - T_M}{t_L - t_E}$ within the interval $[t_{SSO}, t_L]$ (if $t_{SSO}<t_L$) given that $P\{D(t_{SSO})>L\}=1$.

Replacing (7) and (8) in (16) we can solve the two integrals:

$$P_{D2}^T = \left| \alpha \cdot t^{3/2} + \beta \cdot t^2 + \gamma t - \varphi \cdot \sqrt{t} + \theta \cdot log(t) \right|_{T_m}^{\overline{DT}+\frac{SS}{\overline{d}}} \qquad (17)$$

$$P_{D1}^T = \left| \alpha \cdot t^{3/2} - \beta \cdot t^2 - \gamma t - \varphi \cdot \sqrt{t} - \theta \cdot log(t) + t \right|_{\overline{DT}+\frac{SS}{\overline{d}}}^{T_M} \qquad (18)$$

with:

$$\alpha = \frac{2 \cdot \overline{d}}{3 w \cdot \sigma_d}\ ;\ \beta = \frac{\overline{d}^2}{4 w^2 \cdot \sigma_d^2}\ ;\ \gamma = \frac{w^2 \cdot \sigma_d^2 - 2 \cdot L \cdot \overline{d}}{2 \cdot w^2 \cdot \sigma_d^2}$$

$$\varphi = \frac{2L}{w \cdot \sigma_d}\ ;\ \theta = \frac{L^2}{2 \cdot w^2 \cdot \sigma_d^2}$$

*3.2 The formulation with Delivery Slack Time*

Let's include the DST in the above formulation The stock out probability, given by (3), turns into:

$$P'_{SO} = \int_t P\{D(t_R - DST) > L\} \cdot P\{t = t_R\}\ dt$$

Thus the probabilities (7) and (8) can be respectively rewritten as:

$$P'_{D1} = 1 - F_{D(t)}^1 = 1 - \frac{(w \cdot \sigma_d \cdot \sqrt{t - DST} - \overline{d} \cdot (t - DST) + L)^2}{2 \cdot w^2 \cdot \sigma_d^2 \cdot (t - DST)}$$

if $t > \overline{DT} + \frac{SS}{\overline{d}} + DST$

$$P'_{D2} = 1 - F_{D(t)}^2 = \frac{(w \cdot \sigma_d \cdot \sqrt{t - DST} + \overline{d} \cdot (t - DST) - L)^2}{2 \cdot w^2 \cdot \sigma_d^2 \cdot (t - DST)}$$

if $t < \overline{DT} + \frac{SS}{\overline{d}} + DST$

Furthermore, we modify the instants $t_{NSO}$ and $t_{SSO}$, solving the equations (14) and (15) with respect to *(t-DST)*:

$$t_{NSO} = DST + \frac{L}{\overline{d}} + \frac{w^2 \cdot \sigma_d^2}{2 \cdot \overline{d}^2} - \frac{w \cdot \sigma_d}{2 \cdot \overline{d}^2} \sqrt{w^2 \cdot \sigma_d^2 + 4 \cdot L\overline{d}}$$

$$t_{SSO} = DST + \frac{L}{\overline{d}} + \frac{w^2 \cdot \sigma_d^2}{2 \cdot \overline{d}^2} + \frac{w \cdot \sigma_d}{2 \cdot \overline{d}^2} \sqrt{w^2 \cdot \sigma_d^2 + 4 \cdot L\overline{d}}$$

Finally, the stock out probability turns into:

$$P'_{SO} = \frac{1}{t_L - t_E} \left\{ P'^T_{D2} + P'^T_{D1} \right\} + \frac{t_L - T_M}{t_L - t_E}$$

where:

$$P'^T_{D2} = \Big| \ \alpha \cdot (t - DST)^{3/2} + \beta \cdot (t - DST)^2 + $$
$$+ \gamma t - \varphi \cdot \sqrt{t - DST} + \theta \cdot log(t - DST)\ \Big|_{T_m}^{\overline{DT}+\frac{SS}{\overline{d}}+DST}$$

$$P'^T_{D1} = \Big| \ \alpha \cdot (t - DST)^{3/2} - \beta \cdot (t - DST)^2 + $$
$$- \gamma t - \varphi \cdot \sqrt{t - DST} - \theta \cdot log(t - DST) + t\ \Big|_{\overline{DT}+\frac{SS}{\overline{d}}+DST}^{T_M}$$

Despite its length, differently from the VSS original formula, this expression is computable and solvable in a closed form. Starting from $P'_{SO}$ is then possible to determine the service level as a function of SS and DST:

$$SL = 1 - P'_{SO}$$

## 4. Conclusions

In this paper a method to simplify the Virtual Safety Stock formula is presented. The triangular distribution is known to be a good approximation for fitting Gaussian model and then to represent the stochastic demand behaviour over time. For the sake of simplicity, we have assumed that the supplier's delivery time follows an uniform distribution. This hypothesis is the limit of our approach and further research could aim to finding the way of adopting a triangular distribution for the delivery time as well. For sure, a further contribution is necessary to numerically validate the proposed model in order to verify how it performs compared to Hadley & Whitin model, and the authors are already working on it. Furthermore, other studies should be addressed in order to identify more realistic stochastic models for both the two main critical variables and, if possible, easy to be adopted in operations management practice.

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