NON-COMMUTATIVE MARKOV PROCESSES IN FREE GROUPS FACTORS, RELATED TO BEREZIN'S QUANTIZATION AND AUTOMORPHIC FORMS

Florin Rădulescu

Abstract. In this paper we use the description of free group factors as the von Neumann algebras of Berezin's deformation of the upperhalf plane, modulo $\mathrm{PSL}(2,\mathbb{Z})$.

The derivative, in the deformation parameter, of the product in the corresponding algebras, is a positive — 2 Hochschild cocycle, defined on a dense subalgebra. By analyzing the structure of the cocycle we prove that there is a generator, $\mathcal{L}$, for a quantum dynamical semigroup, that implements the cocycle on a strongly dense subalgebra.

For x in the dense subalgebra, $\mathcal{L}(x)$ is the (diffusion) operator

$$\mathcal{L}(x) = \Lambda(x) - 1/2\{T, x\},$$

where Λ is the pointwise (Schurr) multiplication operator with a symbol function related to the logarithm of the automorphic form Δ. The operator T is positive and affiliated with the algebra $\mathcal{A}_t$ and T corresponds to $\Lambda(1)$, in a sense to be made precise in the paper. After a suitable normalization, corresponding to a principal value type method, adapted for II_1 factors, Λ becomes (completely) positive on a union of weakly dense subalgebras. Moreover the 2- cyclic cohomology cocycle associated to the deformation may be expressed in terms of Λ

2000 Mathematics Subject Classification: 46L09, secondary 11F03, 81R15

Keywords and phrases: Berzin quantization, Free group factors, Quantum Dynamics, Automorphic forms

1. Introduction

In this paper we analyze the structure of the positive Hochschild cocycle that determines the Berezin's deformation [4] of the upper halfplane $\mathbb{H}$, modulo $\mathrm{PSL}(2,\mathbb{Z})$.

As described in [27], the algebras $\mathcal{A}_{t,t>1}$, in the deformation are II_1 factors, (free group factors, by [15] and [27], based on [33]) whose elements are (reproducing) kernels k, that are functions on $\mathbb{H}\times\mathbb{H}$, analytic in the second variable and antianalytic in the first variable, diagonally $\mathrm{PSL}(2,\mathbb{Z})$ invariant and subject to boundedness conditions (see [27]).

The product $k *_t l$ of two such kernels is the convolution product

$$(k *_t l)(\overline{z},\xi) = c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t \mathrm{d}\nu_t(\eta), \quad z,\xi\in\mathbb{H}.$$

Here $[\overline{z},\eta,\overline{\eta},\xi]$ is the cross ratio $\dfrac{(\overline{z}-\xi)(\overline{\eta}-\eta)}{(\overline{z}-\eta)(\overline{\eta}-\xi)}$ while $\mathrm{d}\nu_t$ is the measure on the upper half plane, $\mathbb{H}$ defined by $\mathrm{d}\nu_t = (\mathrm{Im}\eta)^{t-2}\mathrm{d}\overline{\eta}\mathrm{d}\eta$, and c_t is a constant.

For k,l in a weakly dense subalgebra $\widehat{\mathcal{A}}_t$, that will be constructed later in the paper, the following 2-Hochschild cocycle is well defined:

$$\mathcal{C}_t(k,l) = \text{the derivative at } t, \text{ from above, of } s\to k *_s l$$

Clearly

$$\mathcal{C}_t(k,l) = \frac{c_t'}{c_t}(k *_t l) + c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t \ln[z,\eta,\overline{\eta},\xi]\mathrm{d}\nu_0(\eta).$$

In what follows we will prove that $\mathcal{C}_t$ is always a completely positive 2-Hochschild cocycle (for example in the sense introduced in [13]). More precisely, for all $k_1,k_2,\ldots,k_N$ in $\widehat{\mathcal{A}}_{t_0}$, $l_1,l_2,\ldots,l_N$ in $\mathcal{A}_t$, we have that

$$\sum_{i,j} \tau_{\mathcal{A}_t}(l_i^*\mathcal{C}_t(k_i^*,k_j)l_j)\leq 0.$$

This also holds true for more general, discrete, subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

In the case of $\mathrm{PSL}(2,\mathbb{Z})$, it turns out that $\mathcal{C}_t(k,l)$ behaves like the corresponding cocycle obtained from the generator of a quantum dynamical semigroup, that is there exists a (necessary completely diffusive, i.e. completely conditionally negative) $\mathcal{L}$ such that

$$\mathcal{C}_t(k,l) = \mathcal{L}_t(k *_t l) - k *_t \mathcal{L}_t(l) - \mathcal{L}_t(k) *_t l.$$

It turns out that $\mathcal{L}$ is defined on a unital, dense subalgebra $\mathcal{D}_t$ of $\mathcal{A}_t$, and that $\mathcal{L}(k)$ belongs to the algebra of unbounded operators affiliated with $\mathcal{A}_t$. Moreover, by a restricting to a smaller, dense, *but not unital* subalgebra $\mathcal{D}_t^0$, the completely positive part of $\mathcal{L}$ will take values in the predual $L^1(\mathcal{A}_t)$.

The construction of $\mathcal{L}_t$ is done by using automorphic forms. Let Δ be the unique (normalized) automorphic form for $\mathrm{PSL}(2,\mathbb{Z})$ in order 12. Then Δ is not vanishing in $\mathbb{H}$, so that the following expression

$$\ln \varphi(\overline{z}, \xi) = \ln \left(\overline{\Delta(z)} \Delta(\xi) [(\overline{z} - \xi)/(-2\mathrm{i})]^{12} \right)$$

$$= \overline{\ln \Delta(z)} + \ln \Delta(\xi) + 12 \ln[(\overline{z} - \xi)/(-2\mathrm{i})], \quad z, \xi \in \mathbb{H}$$

is well defined, and diagonally Γ-invariant, for a suitable choice of the logarithmic function.

Let Λ be the multiplication operator on $\mathcal{A}_t$, corresponding to pointwise (Schurr) multiplication of a symbol k by $\ln \varphi$. Then Λ is defined on a weakly dense subalgebra $\mathcal{D}_t$ of $\mathcal{A}_t$. If $\{a, b\}$ denotes the Jordan product $\{a, b\} = ab + ba$, then

$$\mathcal{L}(k) = \Lambda(k) - 1/2\{T, k\}$$

where T is related to $\Lambda(1)$ in a sense made explicit in 9. Moreover by adding a suitable constant, times the identity operator to the linear map $-\Lambda$, we get a completely positive map, defined on a weakly dense subalgebra.

By analogy with the Sauvageot's construction ([31]), the 2-Hochschild cocycle $\mathcal{C}_t$ corresponds to a construction of a cotangent bundle, associated with the deformation. Moreover there is a "real and imaginary part" of $\mathcal{C}_t$. Heuristically, this is analogous to the decomposition, of d, the exterior derivative, on a Kahler manifold, into δ and $\overline{\delta}$ (we owe this analogy to A. Connes).

The construction of the "real part" of $\mathcal{C}_t$ is done as follows. One considers the "Dirichlet form" $\mathcal{E}_t$ associated to $\mathcal{C}_t$, which is defined as follows:

$$\mathcal{E}_t(k, l) = \tau_{\mathcal{A}_t}(\mathcal{C}_t(k, l)),$$

defined for k, l in weakly dense, unital subalgebra $\widehat{\mathcal{A}}_t$. Out of this one constructs the operator Y_t defined by

$$\langle Y_t(k), l \rangle_{L^2(\mathcal{A}_t)} = \mathcal{E}_t(k, l), \quad k, l \in \widehat{\mathcal{A}}_t.$$

The imaginary part of $\mathcal{C}_t$ is rather defined as 2-cyclic cohomology cocycle. The formula for this cyclic ([27], [29]) cocycle is:

$$\Psi_t(k,l,m) = \tau_{\mathcal{A}_t}([\mathcal{C}_t(k,l) - (\nabla Y_t)(k,l)]m), \quad k,l,m \in \widehat{\mathcal{A}}_t,$$

with

$$(\nabla Y_t)(k,l) = Y_t(k,l) - kY_tl - Y_t(k)l.$$

This is a construction similar to one used in [13].

Let χ be the antisymmetric form defined on $\mathcal{D}_t^0$, a weakly dense subalgebra of $\mathcal{A}_t$, by the formula

$$\chi_t(k,l) = \frac{1}{2}[\langle \Lambda k, l\rangle - \langle k, \Lambda(l)\rangle].$$

Then there is a nonzero constant β, depending on t, such that

$$\Psi_t(k,l,m) + \beta\tau_{\mathcal{A}_t}(klm) = \chi_t(kl,m) - \chi_t(k,lm) + \chi_t(mk,l),$$

for k,l,m in $\mathcal{D}_t^0$.

We will show in the paper that $L^2(\mathcal{A}_t)$ can be identified with the Bargmann type Hilbert space of diagonally $\Gamma-$ invariant functions on $\mathbb{H}\times\mathbb{H}$, that are square summable on $F\times\mathbb{H}$, analyic in the second variable and analytic in the first variable. Here F is a fundamental domain for PSL $(2,\mathbb{Z})$ in $\mathbb{H}$, and on $F\times\mathbb{H}$ we consider the invariant measure

$$d(z,\eta)^{2t}\mathrm{d}\nu_0(z)\mathrm{d}\nu_0(w) = \left(\frac{(\operatorname{Im} z)^{1/2}(\operatorname{Im}\eta)^{1/2}}{|[(\overline{z}-\eta)/(-2\mathrm{i})]|}\right)^{2t}\mathrm{d}\nu_0(z)\mathrm{d}\nu_0(w).$$

With this identification, the "real part" of $\mathcal{C}_t$ is implemented is implemented (on $\widehat{\mathcal{A}_t}$) by the analytic Toeplitz operator, on $L^2(\mathcal{A}_t)$, (compression of multiplication) of symbol $\ln d$, The "imaginary part" of $\mathcal{C}_t$ is implemented (on the smaller algebra $\mathcal{D}_t^0$) by the Toeplitz operator, on $L^2(\mathcal{A}_t)$, of symbol $\frac{1}{12}\ln\varphi$.

The expression that we have obtain for $\mathcal{C}_t(k,l) = \mathcal{L}_t(k *_t l) - k *_t \mathcal{L}_t(l) - \mathcal{L}_t(k) *_t l$, $\mathcal{L}(k) = \Lambda(k) - 1/2\{T,k\}$, is in concordance with known results in quantum dynamics: Recall that in Christensen and Evans ([8]), by improving a result due to Lindblad [22] and [16], it is proved that for every uniformly normic continuos semigroup $(\Phi_t)_{t\geq 0}$, of completely positive maps on a von Neumann algebra $\mathcal{A}$, the generator $\mathcal{L} = \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t$ has the following form:

$$\mathcal{L}(x) = \Psi(x) - 1/2\{\Psi(1), x\} + \mathrm{i}[H, x],$$

where $\Psi : \mathcal{A} \to \mathcal{A}$ is a completely positive map and H is a bounded selfadjoint operator.

For a semigroup of completely positive maps that is only strongly uniformly continuous, the generator has a similar form, although $\mathcal{L}(x)$, for x in $\mathcal{A}$ is defined as quadratic form, affiliated to the von Neumann algebra $\mathcal{A}$.

Conversely, given $\mathcal{L}$, a minimal semigroup may be constructed under certain conditions (see e.g. [7], [20], [23], [17] [14]), although the semigroup might not be conservative (i.e unital) even if $\mathcal{L}(1) = 0$.

If $\mathcal{L}(x) = \Lambda(x) + (G^*x + xG)$, let $\widehat{\Lambda}_t x = \mathrm{e}^{-tG^*}x\mathrm{e}^{-tG}$. Then in the case of $\mathcal{A} = B(H)$, the corresponding semigroup Φ_t, verifying the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \Phi_t(x), \xi, n\rangle = \langle \mathcal{L}(\Phi_t(x))\xi, \eta\rangle$$

for ξ, η in a dense domain is constructed, by the Dyson expansion ([20])

$$\Phi_t(x) = \widehat{\Lambda}_t(x) + \sum_{n\geq 0} \int_{0\leq t_1\leq t_2\leq \ldots\leq t_n<t}\cdots\int \widehat{\Lambda}_{t_1}\circ\Lambda\circ\widehat{\Lambda}_{t_2-t_1}\circ\cdots\circ\Lambda\circ\widehat{\Lambda}_{t-t}\mathrm{d}t_1\mathrm{d}t_2\ldots\mathrm{d}t_n$$

which is proved to be convergent ([7], [23]).

It is not clear if a minimal conservative semigroup exists for the quantum dynamical generator $\mathcal{L}_t$ constructed in our paper. The quantum dynamical generators $\mathcal{L}_t$ constructed in this paper have the following formal property:

Assume that there exists families of completely positive maps, $(\Phi_{s,t})_{s\geq t}$, with $\Phi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ verifying the following variant of the master equation:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Psi_{s,t}(\Phi_{s,t}(X)))|_{s=s_0} = \mathcal{L}_{s_0}(\Psi_{s_0,t}(\Phi_{s_0,t}(X))). \tag{1.1}$$

Then $\Phi_{s,t}$ would verify the Chapmann Kolmogorov condition:

$$\Phi_{s,t}\Phi_{s,v} = \Phi_{s,v};\ s\geq t\geq v,\ \Phi_{ss} = \mathrm{I}d$$

Moreover

$$\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_{s,t}(X) *_s \Phi_{s,t}(Y))|_{s=s_0}$$

would be

$$\mathcal{C}_{s_0}(\Phi_{s_0,t}(X),\Phi_{s_0,t}(Y))+\mathcal{L}_{s_0}(\Phi_{s_0,t}(X))*s_0\Phi_{s_0,t}(Y)+\Phi_{s_0,t}(X)*_{s_0}\mathcal{L}_{s_0}(\Phi_{s_0,t}(Y)),$$

which by the cocycle property would be

$$\mathcal{L}_{s_0}(\Phi_{s_0,t}(X *_{s_0} Y)).$$

Thus $\frac{\mathrm{d}}{\mathrm{d}s}(\Phi_{s,t}(X) *_s \Phi_{s,t}(Y)) = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_{s,t}(X *_t Y)$. If unicity (conservativity) holds, it would follow that $\Psi_{s,t}\Phi_{s,t}(X)$ would be a (unital) multiplicative map from $\mathcal{A}_t$ into $\mathcal{A}_s$.

At present we do not know if this conservativity condition of the minimal solution and the subsequent considerations hold true.

Acknowledgement. *This work was initiated while the author was visiting the Erwin Schroedinger Institute in Wien. This work was completed while the author was visiting IHP and IHES to which the author is greatefull for the great conditions and warm receiving. The author acknowledges enlightening discussion with L. Beznea, P. Biane, A. Connes, P. Jorgensen, R. Nest, J.L. Sauvageot., L. Zsido*

2. Definitions

We recall first some notions associated with the Berezin's deformation ([4]) of the upper halfplane that were proved in [27] (see also [28]), in the Γ-equivariant context.

We consider the Hilbert space $H_t = H^2(\mathbb{H}, \mathrm{d}\nu_t)$, $t > 1$ of square summable analytic functions on the upperhalf plane $\mathbb{H}$, with respect to the measure $\mathrm{d}\nu_t = (\Im z)^{t-2}\mathrm{d}\overline{z}\mathrm{d}z$. $\mathrm{d}\nu_0$ is the $\mathrm{PSL}_2(\mathbb{R})$ invariant measure on $\mathbb{H}$. This spaces occur as the Hilbert spaces for the series of projective unitary irreducible representations π_t of $\mathrm{PSL}_2(\mathbb{R})$ on H_t, $t > 1$ ([30], [26]).

Recall that $\pi_t(g)$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PSL}_2(\mathbb{R})$ are defined by means of left translation (using the Möbius action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$) by the formula

$$(\pi_t(g)f)(z) = f(g^{-1}z)(cz+d)^{-t},\ z \in \mathbb{H},\ f \in H_t.$$

Here the factor $(cz+d)^{-t}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined by using a preselected branch of $\ln(cz+d)$ on $\mathbb{H}$, which is always possible ([30]). If $t = n$ is an integer, ≥ 2, then π_t is actually a representation of $\mathrm{PSL}_2(\mathbb{R})$, in the discrete series.

Let Γ be a discrete subgroup of finite covolume in $\mathrm{PSL}_2(\mathbb{R})$ and consider the von Neumann algebra $\mathcal{A}_t = \{\pi_t(\Gamma)\}' \subseteq B(H_t)\}$ consisting of all operators that commute with $\pi_t(\Gamma)$.

By generalizing a result of [3], [10], [11], [18], it was proved in [27] that $\{\pi_t(\Gamma)\}''$ (the enveloping von Neumann algebra of the image of Γ through π_t) is isomorphic to $\mathcal{L}(\Gamma, \sigma_t)$, which is the enveloping von Neumann algebra of the image of the left regular, cocycle representation of Γ into $B(l^2(\Gamma)))$. Thus $\mathcal{L}(\Gamma, \sigma_t)$ is a II_1, factor. Here σ_t is the cocycle coming from the projective, unitary representation π_t.

Therefore, H_t, as a left Hilbert module over $\{\pi_t(\Gamma)\}'' \simeq \mathcal{L}(\Gamma, \sigma_t)$ has Murray von Neumann dimension (see e.g. [GHJ]) equal to $\dfrac{t-1}{\pi}\mathrm{covol}(\Gamma)$ (this generalizes to projective, unitary representations, the formula in [3], [10]). The precise formula is

$$\dim_{\mathcal{L}(\Gamma,\sigma_t)} H_t = \dim_{\{\pi_t(\Gamma)\}''} H_t = \frac{t-1}{\pi}\mathrm{covol}(\Gamma).$$

Hence the commutant $\mathcal{A}_t$ is isomorphic to $\mathcal{L}(\Gamma, \sigma_t)_{\frac{t-1}{\pi}\mathrm{covol}(\Gamma)}$. We use the convention to denote by M_t, for a type II_1 factor M, the isomorphism class of eMe, with e an idempotent of trace t. If $t > 1$ then one has to replace M by $M \otimes M_N(\mathbb{C})$ (see [24]).

When $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ the class of the cocycle σ_t vanishes (although not in the bounded cohomology, see [6]). Consequently, since in this case [18]

$$\frac{t-1}{\pi}\mathrm{covol}(\Gamma) = \frac{t-1}{12},$$

it follows that when $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ we have

$$\mathcal{A}_t \simeq L(\mathrm{PSL}(2, \mathbb{Z}))_{(t-1)/12}.$$

We want to analyze the algebras $\mathcal{A}_t$ by means of the Berezin's deformation of $\mathbb{H}$. Recall that the Hilbert space H_t has reproducing vectors e_z^t, $z \in \mathbb{H}$, that are defined by the condition $\langle f, \mathrm{e}_z^t \rangle = f(z)$, for all f in H. The precise formula is,

$$\mathrm{e}_z^t(\xi) = \langle \mathrm{e}_\xi^t, \mathrm{e}_z^t \rangle = \frac{c_t}{[(\overline{z} - \xi)/(-2\mathrm{i})]^t}, \quad \xi \in \mathbb{H}, \; c_t = \frac{t-1}{4\pi}.$$

Each operator A in $B(H_t)$ has then a reproducing kernel $\widehat{A}(\overline{z}, \xi)$. To obtain the Berezin's symbol, one normalizes so that the symbol of $A = \mathrm{Id}$ is the identical function 1.

Thus the Berezin's symbol of A is a bivariable function on $\mathbb{H} \times \mathbb{H}$, anti-analytic in the first variable, analytic in the second and given by

$$\widehat{A}(\overline{z}, \xi) = \frac{\langle A\mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle}{\langle \mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle}, \; \overline{z}, \xi \in \mathbb{H}.$$

We have that $\langle A\mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle$ is a reproducing kernel for $A \in B(H_t)$ and hence the formula for the symbol $\widehat{AB}$ of the composition of two operators A, B in $B(H_t)$ is computed as

$$\widehat{AB}(\overline{z}, \xi) \cdot \langle \mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle \langle AB\mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle = \langle \mathrm{e}_z^t, \mathrm{e}_\xi^t \rangle \int_{\mathbb{H}} \langle A\mathrm{e}_z^t, \mathrm{e}_\eta^t \rangle \langle B\mathrm{e}_\eta^t, \mathrm{e}_\xi^t \rangle \mathrm{d}\nu_t(\eta).$$

Definition 1.1 *By making explicit the kernels involved in the product, one obtains the following formula: Let* $\widehat{A}(\overline{z}, \xi) = k(\overline{z}, \xi)$, $\widehat{B}(\overline{z}, \xi) = l(\overline{\eta}, \xi)$, *and let* $(k *_t l)(\overline{z}, \xi)$ *be the symbol of* AB *in* H_t. *Then*

$$(k *_t l)(\overline{z}, \xi) = c_t \int_{\mathbb{H}} (k(\overline{z}, \xi))(l(\overline{\eta}, \xi))[\overline{z}, \eta, \overline{\eta}, \xi]^t d\nu_0(\eta) \tag{2.1}$$

with $[\overline{z}, \eta, \overline{\eta}, \xi] = \dfrac{(\overline{z} - \xi)(\overline{\eta} - \eta)}{(\overline{z} - \eta)(\overline{\eta} - \xi)}$.

Here one uses the choice of the branch of $\ln(\overline{z} - \xi) \in [-\pi, \pi]$ *that appears in the definition of* e_z^t *(see [30]).*

The above definition can be extended, when the integrals are convergent, to an (associative) operation on the space of bivariant kernels, by the formula (2.1). One problem that remains open is to determine when a given bivariant function represents a bounded operator on H_t.

Let $d(\overline{z},\eta) = \left((\operatorname{Im} z)^{1/2}(\operatorname{Im}\eta)^{1/2}\right)/(|[(\overline{z}-\eta)/(-2\mathrm{i})]|)$ for z,η in $\mathbb{H}$. Then $d(\overline{z},\eta)^2$ is the hyperbolic cosine of the hyperbolic distance between z,η in $\mathbb{H}$. The following criteria was proven in [27]

Criterion 2.2 *Let h be a bivariant function on $\mathbb{H}\times\mathbb{H}$, antianalytic in the first variable, and analytic in the second variable. Consider the following norm: $\|h\|_t^{\widehat{}}$ is the maximum of the following two quantities*

$$\sup_{z\in\mathbb{H}} \int |h(\overline{z},\eta)|(\mathrm{d}(z,\eta))^t \mathrm{d}\nu_0(\eta),$$

$$\sup_{\eta\in\mathbb{H}} \int |h(\overline{z},\eta)|(\mathrm{d}(z,\eta))^t \mathrm{d}\nu_0(z).$$

Then $\|h\|_t^{\widehat{}}$ is a norm on $B(H_t)$, finer then the uniform norm, and the vector space of all elements in $B(H_t)$ whose kernel have finite $\|\cdot\|_t^{\widehat{}}$ norm, is an involutive weakly dense, unital, normal subalgebra of $B(H_t)$. We denote this algebra by $\widehat{B(H_t)}$.

In [27] we proved a much more precise statement about the algebra $\widehat{B(H_t)}$:

Proposition 2.3.([27]) *The algebra of symbols corresponding to $\widehat{B(H_t)}$ is closed under all the product operations $*_s$, for $s\geq t$. In particular $\widehat{B(H_t)}$ embeds continuously into $\widehat{B(H_t)}$ and its image is closed under the product in $\widehat{B(H_t)}$.*

Since this statement will play an essential role in proving that the domains of some linear maps in our paper, are algebras, we'll briefly recall the proof of this proposition:

Assume that k,l are kernels such that $\|k\|_t^{\widehat{}}, \|l\|_t^{\widehat{}} < \infty$. Consider the product of k,l in $\mathcal{A}_s$. We are estimating

$$\int |(k *_s l)(\overline{z},\xi)||d(z,\xi)|^t \mathrm{d}\nu_0(\xi).$$

This should be uniformly bounded in z.

The integrals are bounded by

$$\iint_{\mathbb{H}^2} |k(\overline{z},\eta)||l(\overline{\eta},\xi)||[\overline{z},\eta,\overline{\eta},\xi]|^s|d(z,\xi)|^t \mathrm{d}\nu_0(\eta)\mathrm{d}\nu_0(\zeta).$$

Since obviously

$$|[\overline{z},\eta,\overline{\eta},\xi]|^s = \left[\frac{\mathrm{d}(z,\eta)\mathrm{d}(\eta,\xi)}{\mathrm{d}(z,\xi)}\right]^s,$$

the integral is bounded by

$$\iint\limits_{\mathbb{H}^2} |k(\overline{z},\eta)||\mathrm{d}(\overline{z},\eta)|^t|l(\overline{\eta},\xi)|\mathrm{d}(\overline{\eta},\xi)^t|\cdot M(z,\eta,\xi)\mathrm{d}\nu_0(\eta,\xi).$$

If we can show that $M(z,\eta,\xi)$ is a bounded function on $\mathbb{H}\times\mathbb{H}\times\mathbb{H}$, then last integral will be bounded by $\|M\|_\infty\|k\|_t^{\widehat{}}\|l\|_t^{\widehat{}}$.

But it is easy to see that

$$M(\overline{z},\eta,\overline{\eta},\xi) = \left|\frac{d(\overline{z},\eta)d(\overline{\eta},\xi)}{d(\overline{z},\xi)}\right|^{s-t} = |[\overline{z},\eta,\overline{\eta},\xi]|^{s-t}.$$

This is a diagonally $\mathrm{PSL}(2,\mathbb{R})$-invariant function on $\mathbb{H}\times\mathbb{H}\times\mathbb{H}$. Since $d(z,\eta)$ is an intrinsic notion of the geometry on $\mathbb{H}$ we can replace $\mathbb{H}$ by D, the unit disk. Then the expression of $d(z',\xi')$ becomes: $\dfrac{(1-|z'|^2)^{1/2}(1-|\xi'|^2)^{1/2}}{|1-\overline{z'}\xi'|}$, $z',\xi'\in\mathbb{D}$. We thus consider M as a function of three variables $z',\eta',\xi'\in\mathbb{D}$. By $\mathrm{PSL}(2,\mathbb{R})$-invariance when computing the maximum we may let $\eta=0$ and we have

$$\begin{aligned} M(z',0,\xi') &= \left|\frac{d(\overline{z},0)d(0,\xi')}{d(z',\xi')}\right|^{s-t} \\ &= \left|\frac{(1-|z'|^2)^{1/2}(1-|\xi'|^2)^{1/2}}{d(z',\xi')}\right|^{s-t} \\ &= |(1-z'\xi')|^{s-t} \leq 2 \end{aligned}$$

since $t>1$. This completes the proof of Proposition 2.3.

In [27] we proved that there is a natural symbol map $\Psi_{s,t}: B(H_t)\to B(H_s)$ defined as follows:

Definition 2.4. *Let $\Psi_{s,t}: B(H_t)\to B(H_s)$ be the map that assigns to every operator A in $B(H_t)$, of Berezin's symbol $\widehat{A}(\overline{z},\eta)$, $\overline{z},\eta\in\mathbb{H}$, the operator*

$\Psi_{s,t}(A)$ on $B(H_s)$ whose Berezin's symbol (as operator on H_s) coincides with the symbol of A. Then $\Psi_{s,t}$ is continuous on $B(H_s)$.

A proof of this will be given in Section 4 and we will in fact prove even more, that is that $\Psi_{s,t}$ is a completely positive map.

Obviously one has

$$\begin{aligned} \Psi_{s,t}\Psi_{sv} &= \Psi_{sv} \quad \text{for } s \geq t \geq v > 1 \\ \Psi_{s,s} &= \mathrm{Id} \quad \text{for } s > 1. \end{aligned}$$

Assume k, l represent two symbols of bounded operators in $B(H_t)$. Then the product $h *_s l$ makes sense for all $s \geq t$. The following definition of differentiation of the product structure appears then naturally. In this way we get a canonical Hochschild 2-cocycle associated with the deformation.

DEFINITION-PROPOSITION 2.5. ([27]) *Fix $1 < t_0 < t$. Let k, l be operators in $\widehat{B(H_{t_0})}$. Consider $k *_s l$ for $s \geq t$, and differentiate pointwise the symbol of this expression at $s = t$. Denote the corresponding kernel by $\mathcal{C}_t(k, l) = k *'_t l$. Then $\mathcal{C}_t(k, l)$ corresponds to a bounded operator in $B(H_t)$. Moreover $\mathcal{C}_t(k, l)$ has the following expression*

$$\mathcal{C}_t(k, l) = \frac{\mathrm{d}}{\mathrm{d}s}(k *_s l)\bigg|_{s=t},$$

$$\mathcal{C}_t(k, l)(\overline{z}, \xi) = \frac{c_t{}'}{c_t}(k *_s l)(\overline{z}, \xi) + c_t \int_{\mathbb{H}} k(\overline{z}, \eta) l(\overline{\eta}, \xi) [\overline{z}, \eta, \overline{\eta}, \xi]^t \ln[\overline{z}, \eta, \overline{\eta}, \xi] \mathrm{d}\nu_0(\eta).$$

Moreover (by differentiation of the associativity property) it follows that $\mathcal{C}_t(k, l)$ defines a two Hochschild cocycle on the weakly dense subalgebra $\widehat{B(H_{t_0})}$ (viewed as a subalgebra of $B(H_t)$ through the symbol map).

We specialize now this construction for operators $A \in \mathcal{A}_t = \{\pi_t(\Gamma)\}'$, that is, operators that commute with the image of Γ in $B(H_t)$. We have the following lemma, which was proved in [27].

LEMMA 2.6. [27] *Let Γ be a discrete subgroup of finite covolume in $PSL(2, \mathbb{R})$. Assume F is a fundamental domain of Γ in $\mathbb{H}$ (of finite area $\nu_0(F)$ with respect to the $PSL(2, \mathbb{R})$ - invariant measure $d\nu_0$ on $\mathbb{H}$). Let $\mathcal{A}_t = \{\pi_t(\Gamma)\}'$, which is a type II_1 factor with trace τ. Then*

1) Any operator A in $\mathcal{A}_t$ has a diagonally Γ-equivariant kernel $k = k_A(\overline{z}, \xi)$, $z, \xi \in \mathbb{H}$. (that is $k(\overline{z}, \xi) = k(\overline{\gamma z}, \gamma\xi)$, $\gamma \in \Gamma$, $z, \xi \in \mathbb{H}$).

2) The trace $\tau_A(k)$ is computed by

$$\frac{1}{\nu_0(F)} \int_F k(\overline{z}, z)\, d\nu_0(z).$$

3) More general, let P_t be the projection from $L^2(\mathbb{H}, d\nu_t)$ onto H_t. Let f be a bounded measurable function on $\mathbb{H}$, that is Γ-equivariant and let M_f be the multiplication operator on $L^2(\mathbb{H}, d\nu_t)$ by f. Let $T_f^t = P_t M_f P_t$ be the Toeplitz operator on H_t with symbol M_f.

Then T_f^t belongs to $\mathcal{A}_t$ and

$$\tau(T_f^t A) = \frac{1}{\nu_0(F)} \int_F k_A(\overline{z}, z) f(z)\, d\nu_0(z).$$

4) $L^2(\mathcal{A}_t)$ is identified with the space of all bivariable functions k on $\mathbb{H} \times \mathbb{H}$, that are analytic in the second variable, antianalytic in the first variable and diagonally Γ-invariant. The norm of such an element k is given by the formula

$$\|k\|_{2,t} = \frac{1}{\mathrm{area}\,(F)} c_t \iint\limits_{F \times \mathbb{H}} |k(\overline{z}, \eta)|^2 d(z, \eta)^{2t} d\nu_0(z)\, d\nu_0(\eta).$$

We also note that the algebras $\widehat{B(H_t)}$, and the map $\Psi_{s,t}$, $s \geq t$, have obvious counterparts for $\mathcal{A}_t$. Obviously $\Psi_{s,t}$ maps $\mathcal{A}_t$ into $\mathcal{A}_s$ for $s \geq t$.

Definition 2.7 ([27]). *Let $\mathcal{A}_t = \widehat{B(H_t)} \cap \mathcal{A}_t$. Then $\widehat{\mathcal{A}_t}$ is a weakly dense involutive, unital subalgebra of $\mathcal{A}_t$.*

Moreover $\widehat{\mathcal{A}_t}$ is closed under any of the operations $*_s$, for $s \geq t$. This means that $\Psi_{s,t}(k)\Psi_{s,t}(l) \in \Psi_{s,t}(\widehat{\mathcal{A}_t})$ for all k, l in $\widehat{\mathcal{A}_t}$, $s \geq t$.

More generally, $\mathcal{A}_s$ is contained in $\widehat{\mathcal{A}_t}$ if $s < t - 2$, and $\widehat{\mathcal{A}_r}$ is weakly dense in $\mathcal{A}_t$ if $r \leq t$. (and hence $\widehat{\mathcal{A}_r}$ is weakly dense in $\mathcal{A}_t$ if $r \leq t$) ([27], Proposition 4.6).

We also note that, as a consequence of the previous lemma, we can define for $1 < t_0 < t$

$$\mathcal{C}_t(k, l) = \frac{\mathrm{d}}{\mathrm{d}s}(k *_s l)\bigg|_{s=t} \quad \text{for } k, l \text{ in } \widehat{\mathcal{A}_{t_0}}$$

and we have the expression (0.1) of the kernel.

Another way to define $\mathcal{C}_t(k,l)$ is to fix vectors ξ,η in H_t and to consider the derivative

$$\frac{\mathrm{d}}{\mathrm{d}s}\langle (k *_s l)\xi,\eta\rangle_{H_t} = \langle \mathcal{C}_t(k,l)\xi,\eta\rangle|_{H_t},\ \xi,\eta\in H_t.$$

For k,l in $\widehat{\mathcal{A}_{t_0}}$, $t_0<t$ this makes sense because $k *_s l$ is already the kernel of an operator in $\widehat{\mathcal{A}_{t_0}}$.

3. Outline of the paper

The paper is organized as follows:

In Section 4, we show, based on the facts proved in [27], that the symbol maps $\Psi_{s,t}:\mathcal{A}_t\to\mathcal{A}_s$, for $s\geq t$, are completely positive, unital and trace preserving. Consequently the derivative of the multiplication operation (keeping the symbols fixed) is a positive, 2-Hochschild cocycle, (see [13]). In particular the trace of this Hochschild cocycle is a (noncommutative) Dirichlet form (see [31]).

In Section 5 we analyze positivity properties for families of symbols induced by intertwining operators. As in [18], let S_{Δ^ε} is the multiplication operator by Δ^ε, viewed as an operator from H_t into $H_{t+12\varepsilon}$. Then S_{Δ^ε} is an intertwiner between $\pi_t|_\Gamma$ and $\pi_{t+12\varepsilon}|_\Gamma$, with $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$. Here we use the following branch for $\ln(cz+d)=\ln(j(\gamma,z))$, which appears in the definition of $\pi_t(\gamma)$, $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $\mathrm{PSL}(2,\mathbb{Z})$, $\gamma\in\Gamma$. We define $\ln(j(\gamma,z))=\ln(\Delta(\gamma^{-1}z))-\ln\Delta(z)$, which is possible since there is a canonical choice for $\ln\Delta(z)$.

We use the fact that $S_{\Delta^\varepsilon}S^*_{\Delta^\varepsilon}$ is a decreasing family of operators, converging to the identity as $\varepsilon\to 0$. Let

$$\varphi(\overline{z},\xi)=\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}.$$

Then

$$\ln\varphi(\overline{z},\xi)=\overline{\ln\Delta(z)}+\ln\Delta(\xi)+12\ln[(\overline{z}-\xi)/(-2\mathrm{i})],$$

has the property that

$$\left[\left(-\frac{1}{12}\ln\varphi(\overline{z}_i,z_j)+\frac{c_t'}{c_t}\right)[(\overline{z}_i-z_j)/(-2\mathrm{i})]^{-t}\right]_{i,j}$$

is a positive matrix for all $z_1, z_2, \dots, z_n$ in $\mathbb{H}$ and for all $t > 1$.

In Section 6 we use the positivity proven in Section 5 to check that the operator of symbol multiplication by $(\ln\varphi)\,(\varphi^{\varepsilon}+C_t)$ (for a suitable constant C_t, depending only on t) is well defined on a weakly dense subalgebra of $\mathcal{A}_t$. This operator gives a completely positive map on this subalgebra.

By a principal value procedure, valid in a in type II_1 factor, we deduce that multiplication by $(-\ln\varphi+c_{t,\widetilde{\mathcal{A}}})$ is a completely positive map Λ, on a weakly dense unital subalgebra $\widetilde{\mathcal{A}}$ of $\mathcal{A}_t$ ($c_{t,\widetilde{\mathcal{A}}}$ is a constant that only depends on t and $\mathcal{A}$). The multiplication by $(\ln\varphi)$ maps $\widetilde{\mathcal{A}}$ into the operator affiliated with $\mathcal{A}_t$.In particular $\Lambda(1)$ is affiliated with $\mathcal{A}_t$. We obtain this results by checking that the kernels $-\dfrac{\varphi^{\varepsilon}-\mathrm{Id}}{\varepsilon}$, are decreasing as $\varepsilon\downarrow\varepsilon_0$, $\varepsilon_0>0$ (up to a small linear perturbation) to $\varphi^{\varepsilon_0}\ln\varphi$, plus a suitable constant.

This is not surprising as $\Lambda(1)=\ln\varphi(\overline{z},\xi)$ fails shortly the summability criteria for $L^1(\mathcal{A}_t)$.

In Section 7, we analyze the derivatives X_t, at t, of the intertwining maps $\theta_{s,t}:\mathcal{A}_t\to\mathcal{A}_s$, $s\geq t$, with $\theta_{s,t}(k)=S_{\Delta^{(s-t)/12}}kS^*_{\Delta^{(s-t)/12}}$. The derivatives (X_t) are, up to a multiplicative constant, the operators defined in Section 6. The operator X_t is defined on a weakly dense unital subalgebra of $\mathcal{A}_t$.

We take the derivative of the identity satisfied by $\theta_{s,t}$, which is

$$\theta_{s,t}(k *_t T^t_{\varphi^{(s-t)/12}} *_t l)=\theta_{s,t}(k)*_s\theta_{s,t}(l).$$

This gives the identity

$$X_t(k*_t l)+k*_t T^t_{\ln\varphi}*_t\varphi=\mathcal{C}_t(k,l)+X_tk*_t l+k*_t X_tl$$

which holds on a weakly dense (nonunital) subalgebra.

Based on an estimate, on the growth of the function $|\ln\Delta(z)\Delta^{\varepsilon}(z)|$, $z\in\mathbb{H}$, for fixed $\varepsilon>0$, we prove in Section 8 that the positive, affiliated operators $-\Lambda(1)$ and $-T^t_{\ln\varphi}$ are equal operators. We prove this by showing that there is an increasing family A_ε in $\mathcal{A}_t$ and dense domains $\mathcal{D}_0, \mathcal{D}_1$ (where $\mathcal{D}_0$ is affiliated to $\mathcal{A}_t$) such that $\langle A_\varepsilon \xi, \xi\rangle \to \langle -\Lambda(1)\xi, \xi\rangle$ for ξ in $\mathcal{D}_0$ and $\langle A_\varepsilon \xi, \xi\rangle \to \langle -T_{\ln\varphi}\xi, \xi\rangle$ for ξ in $\mathcal{D}_1$.

In Section 9 we analyze the cyclic cocycle associated with the deformation, which is obtained from the positive Hochschild cocycle, by discarding a trivial part.

The precise formula is

$$\Psi_t(k,l,m) = \tau_{\mathcal{A}_t}([\mathcal{C}_t(k,l) - Y_t(kl) + (Y_t k)l + k(Y_t l)]m),$$

for k, l, m in a dense subalgebra, and

$$\langle Y_t k, l\rangle = -1/2\tau_{\mathcal{A}_t}(\mathcal{C}_t(k, l^*)).$$

We reprove a result in [27], that the cyclic cohomology cocycle $\Psi(k,l,m) -$ cst $\tau(klm)$ is implemented by $\chi_t(k, l^*) = \langle X_t k, l\rangle - \langle k, X_t l\rangle$ for k, l in a dense subalgebra. Since the constant in the above formula is nonzero, this corresponds to nontriviality of Ψ_t on this dense subalgebra.

In Section 10 we analyze a dual form of the coboundary for $\mathcal{C}_t(k,l)$, in which multiplication by φ is rather replaced by the Toeplitz operator of (compressed to $L^2(\mathcal{A}_t)$) multiplication by $\overline{\varphi}$. It turns out that the roles of $\Lambda(1)$ and $T^t_{\ln\varphi}$ are reversed in the functional equation verified by the coboundary.

In the appendix, giving up to the complete positivity requirement, and on the algebra requirement on the domain of the corresponding maps, we find some more general coboundaries for $\mathcal{C}_t$, that were hinted in [27].

4. Complete positivity for the 2-Hochschild cocycle associated with the deformation

In this section we prove the positivity condition on the 2-Hochschild cocycle associated with the Berezin's deformation.

Denote for z, η in $\mathbb{H}$ the expression

$$d(\overline{z}, \eta) = \frac{(\mathrm{Im} z)^{1/2}(\mathrm{Im}\eta)^{1/2}}{[(\overline{z} - \eta)/(-2\mathrm{i})]}$$

and recall that $|d(\overline{z},\eta)|^2$ is the hyperbolic cosine of the hyperbolic distance between $z,\eta \in \mathbb{H}$.

In [27] we introduced the following seminorm, defined for $A \in B(H_s)$, given by the kernel $k = k_A(\overline{z},\xi)$, $z,\xi \in \mathbb{H}$.
$\|A\|\widehat{}_s = \|k\|\widehat{}_s =$

$$= max\left(\sup_{z\in\mathbb{H}}\int |k(\overline{z},\eta)||d(z,\eta)|^s \mathrm{d}\nu_0(\eta), \sup_{z\in\mathbb{H}}\int |k(z,\eta)||d(z,\eta)|^s \mathrm{d}\nu_0(z)\right).$$

The subspace of all elements A in $B(H_s)$ (respectively $\mathcal{A}_s$) such that $\|A\|\widehat{}_s$ is finite is a closed, involutive Banach subalgebra of $B(H_s)$, (respectively $\mathcal{A}_s$) that we denote by $\widehat{B(H_s)}$ (respectively $\widehat{\mathcal{A}}_s$).

In [27] we proved that in fact $\widehat{\mathcal{A}}_s$ (or $\widehat{B(H_s)}$) is also closed under any of the products $*_t$, for $t \geq s$, and that there is a universal constant $c_{s,t}$, depending on s,t, such that

$$\|k *_t l\|\widehat{}_s \leq c_{s,t}\|k\|\widehat{}_s\|l\|\widehat{}_s,\ k,l \in \widehat{\mathcal{A}}_s.$$

Also $\widehat{\mathcal{A}}_s$ (or $\widehat{B(H_s)}$) is weakly dense in $\mathcal{A}_s$ (respectively $B(H_s)$).

Let $\Psi_{s,t}$, $s \geq t > 1$, be the map that associates to any A in $B(H_t)$ (respectively $\mathcal{A}_t$) the corresponding element in $B(H_s)$ (respectively $\mathcal{A}_s$) having the same symbol (that is $\Psi_{s,t}(A) \in \mathcal{A}_s$ has the same symbol as A in $\mathcal{A}_t$). Then $\Psi_{s,t}$ maps continuously $\mathcal{A}_t$ in $\mathcal{A}_s$.

In the next proposition we prove that $\Psi_{s,t}$ is a completely positive map. This is based on the following positivity criteria proved in [27].

Lemma 4.1 (Positivity criterion). *A kernel $k(\overline{z},\xi)$ defines a positive bounded operator in $B(H_t)$, of norm less then 1, if and only if for all N in $\mathbb{N}$ and for all $z_1,z_2,\ldots,z_N$ in $\mathbb{H}$ we have that the following matrix inequality holds*

$$0 \leq \left[\frac{k(\overline{z}_i,z_j)}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}\right]_{i,j=1}^N \leq \left[\frac{1}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}\right]_{i,j=1}^N.$$

This criterion obviously holds at the level of matrices of elements in $M_p(\mathbb{C}) \otimes \mathcal{A}_t$.

Lemma 4.2 (Matrix positivity criteria). *If $[k_{p,q}]_{p,q=1}^p$ is a positive matrix of elements in $\mathcal{A}_t$ then for all N in $\mathbb{N}$, all $z_1,z_2,\ldots,z_N$ in $\mathbb{H}$ the following* *matrix is positive definite:*

$$\left[\frac{k_{p,q}(\overline{z}_i, z_j)}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}\right]_{(i,p),(j,q)\in\{1,2,\dots,P\}\times\{1,2,\dots,N\}}.$$

Conversely, if the entries $k_{p,q}$ represent an element in $\mathcal{A}_t$ and if the above matrix is positive, then $[k_{p,q}]_{p,q}$ is a positive matrix in $\mathcal{A}_t$.

Proof. Let $[k_{p,q}]_{p,q=1}^N$ be a matrix in $\mathcal{A}_t$. Then $k = [k_{p,q}]$ is positive if and only if for all vectors $\xi_1, \xi_2, \dots \xi_N$ in H_t we have that

$$\sum_{p,q} \langle k_{p,q}\xi_p, \xi_q\rangle_{H_t} \geq 0.$$

Since $k_{p,q} = P_t k_{p,q} P_t$, where P_t is the projection from $L^2(\mathbb{H}, \nu_t)$ onto H_t it turns out that this is equivalent with the same statement which new must be valid for all $\xi_1, \xi_2, \dots \xi_N$ in $L^2(\mathbb{H}, \nu_t)$.

Thus we have that

$$\sum_{p,q} \iint\limits_{\mathbb{H}^2} \frac{k_{p,q}(\overline{z}, w)}{[(\overline{z} - w)/(-2\mathrm{i})]^t} \xi_p(z)\overline{\xi_q(w)}\mathrm{d}\nu_t(z)\mathrm{d}\nu_t(w) \geq 0$$

for all $\xi_1, \xi_2, \dots \xi_N$ in $L^2(\mathbb{D}, \nu_t)$. We let the vectors ξ_p converge to the Dirac distributions, for all $p = 1, 2, ...N$, $\sum\limits_i \lambda_{ip}\delta_{z_i}(\mathrm{Im}\, z_i)^{-(t-2)}$, for all $p = 1, 2, ...N$. By the above inequality we get

$$\sum_{i,j,p,q} \frac{k_{p,q}(\overline{z}_i, z_j)}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t} \overline{\lambda}_{ip}\lambda_{jq} \geq 0$$

for all choices of $\{\lambda_{ij}\}$ in $\mathbb{C}$. This corresponds exactly to the fact that the matrix

$$\left[\frac{k_{p,q}(\overline{z}_i, z_j)}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}\right]_{(p,i),(q,j)}$$

is positive.

Proposition 4.3. *The map $\Psi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ which sends an element A in $\mathcal{A}_t$ into the corresponding element in $\mathcal{A}_s$, having the same symbol, is unital and completely positive.*

Proof. This is a consequence of the fact ([32], [30]) that the matrix

$$\left[\frac{1}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^{\varepsilon}}\right]_{i,j}$$

is a positive matrix for all ε, all N, all $z_1, z_2, \ldots, z_n$ in $\mathbb{H}$. Indeed

$$\frac{1}{[(\overline{z} - \xi)/(-2\mathrm{i})]^{\varepsilon}}$$

(or $1/(1-\overline{z}\xi)^{\varepsilon}$) is a reproducing kernel for a space of analytic functions, even if $\varepsilon < 1/2$.

We will now follow Lindblad's ([22]) argument to deduce that $\mathcal{C}_t(k,l)$ is a completely positive Hochschild 2-cocycle. We recall first the definition of the cocycle $\mathcal{C}_t$ associated with the deformation.

Definition 4.4. *Fix $t > s_0 > 1$. Then the following formula defines a Hochschild 2-cocycle on $\widehat{\mathcal{A}}_{s_0}$.*

$$\mathcal{C}_t(k,l)(\overline{z},\xi) = \frac{d}{ds}\bigg|_{\substack{s=t\\ s<t}} (k *_s l)(\overline{z},\xi) =$$

$$= \frac{c_t'}{c_t}\,(k *_t l)\,(\overline{z},\xi) + c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi) [\overline{z},\eta,\overline{\eta},\xi]^t \ln[\overline{z},\eta,\overline{\eta},\xi]\, d\nu_0(\eta).$$

Indeed, it was proven in [27] that the above integral is absolutely convergent for k, l in $\widehat{\mathcal{A}}_{s_0}$, for any $s_0 < t$.

The above definition may be taught of also in the following way. Fix vectors ξ, η in H_t and fix k, l in $\mathcal{A}_{s_0}$. Then $k *_t l$, and $k *_s l$ make sense for all $s_0 < s < t$ and they represent bounded operators in $\mathcal{A}_t$. Thus the following derivative makes sense:

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{\substack{s=t\\ s<t}} \langle k *_s l\xi, \eta\rangle_{H_t}$$

and it turns out to be equal to

$$\langle \mathcal{C}_t(k,l)\xi, \eta\rangle_{H_t}.$$

In the following lemma we use the positivity of $\Psi_{s,t}$ to deduce the complete positivity of $\mathcal{C}_t$. We recall the following formal formula for $\mathcal{C}_t$ that was proved in [28], [29].

LEMMA 4.5. [27] *Let* $1 < t_0 < t$ *and let* k, l, m *belong to* $\widehat{\mathcal{A}}_{t_0}$. *Then the following holds:*

$$\tau_{\mathcal{A}_t}(\mathcal{C}_t(k,l) *_t m) = \frac{d}{ds}\tau_{\mathcal{A}_s}((k *_s l *_s m) - (k *_t l) *_s m)\Big|_{\substack{s=t\\ s>t}}.$$

In a more precise notation, the second term is

$$\frac{d}{ds}\tau_{\mathcal{A}_s}([\Psi_{s,t}(k) *_s \Psi_{s,t}(l) - \Psi_{s,t}(k *_t l)] *_s \Psi_{s,t}(m))\Big|_{\substack{s=t\\ s>t}}.$$

The proof of the lemma is trivial, as long as one uses the absolute convergence of the integrals, which follows from the fact that the kernels belong to an algebra $\widehat{\mathcal{A}}_{t_0}$, for some $t_0 < t$.

The positivity property that we are proving for $\mathcal{C}_t(k,l)$, is typical for coboundaries of the form $D(ab) - D(a)b - aD(b)$, where D is the generator of dynamical semigroup. It is used by Sauvageot to construct the cotangent bimodule associated with a dynamical semigroup, and much of the properties in [31] can be transferred to $\mathcal{C}_t$ with the same proof. Such positive (or negative) cocycles appear in the work of Cunz and Connes (see also [9]).

PROPOSITION 4.6. *Fix* $1 < t_0 < t$ *and for* k, l *in* $\widehat{\mathcal{A}}_{t_0}$ *define*

$$\mathcal{C}_t(k,l) := \frac{d}{ds}(k *_s l)\Big|_{\substack{s>t\\ s=t}}.$$

Then for all $k_1, k_2, \ldots, k_N$ *in* $\widehat{\mathcal{A}}_{t_0}$, $l_1, l_2, \ldots, l_N$ *in* $\mathcal{A}_t$, *we have that*

$$\sum_{i,j} \tau_{\mathcal{A}_t}(l_i^* \mathcal{C}_t(k_i^*, k_j) l_j) \geq 0.$$

This is the same as requiring for the matrix $(\mathcal{C}_t(k_i^*, k_j))_{i,j}$ *to be negative in* $M_N(\mathcal{A}_t)$.

Proof. For $s \geq t$, let $f(s)$ be defined by the formula

$$f(s) = \tau\left(\sum_{i,j}(k_i^* *_s k_j - k_i^* *_t k_j) *_t (l_i^* *_t l_j)\right).$$

By using the $\Psi_{s,t}$ notation, this is

$$f(s) = \sum_{i,j} \tau\left((\Psi_{s,t}(k_i^*)\Psi_{s,t}(k_j) - \Psi_{s,t}(k_i^* k_j))\, \Psi_{s,t}(l_i^* l_j)\right).$$

By the previous lemma, $f'(t)$ is equal to $\tau(\mathcal{C}_t(k_i^*, k_j)l_j l_i^*)$. In this terms, to prove the statement we must to prove that $f'(t) \geq 0$. Clearly $f(t) = 0$.

By the generalized Cauchy–Schwarz-Stinespring inequality for completely positive maps, and since $\Psi_{s,t}$ is unital, we get that the matrix

$$D_{ij} = [\Psi_{s,t}(k_i^*)\Psi_{s,t}(k_j) - \Psi_{s,t}(k_i^* k_j)]$$

is non-positive. Since $Z_{ij} = \Psi_{s,t}(l_j l_i^*)$ is another positive matrix in $M_n(\mathcal{A}_s)$, we obtain that

$$f(s) = \tau_{\mathcal{A}_s \otimes M_N(\mathbb{C})}(DZ)$$

is negative.

So $f(s) \leq 0$ for all $s \geq t$, $f(0) = 0$. Hence $\left.\dfrac{\mathrm{d}}{\mathrm{d}s} f(s)\right|_{s=t;s>t}$ is negative.

Appendix (to Section 4)

We want to emphasize the properties of the trace $\mathcal{E}_t(k,l) = -\tau(\mathcal{C}_t(k,l))$, $k, l \in \mathcal{A}_t$. Clearly $\mathcal{E}_t$ is a positive form on $\mathcal{A}_t$, and in fact it is obviously positive definite. Following [31], one can prove that $\mathcal{E}_t$ is a Dirichlet form. The following expression holds for $\mathcal{E}_t$.

Lemma 4.7. *For $1 < t_0 < t$, $k, l \in \widehat{\mathcal{A}}_{t_0}$ we have that*

$$\mathcal{E}_t(k,l) = \iint\limits_{F \times \mathbb{H}} k(\overline{z}, \eta)\overline{l(\overline{z}, \eta)} |d(\overline{z}, \eta)|^{2t} \ln |d(z, \eta)| d\nu_0(z, \eta),$$

where F is a fundamental domain for Γ in $\mathbb{H}$, and

$$|d(\overline{z}, \eta)| = \left| \frac{Im\, z^{1/2} Im\, \eta^{1/2}}{[(\overline{z} - \eta)/(-2i)]} \right|$$

is the hyperbolic cosine of the hyperbolic distance between z and η in $\mathbb{H}$.

Recall that $L^2(\mathcal{A}_t)$ is identified ([27]) with the Bargmann type Hilbert space of functions $k(\overline{z},\eta)$ on $\mathbb{H}\times\mathbb{H}$ that are antianalytic in the first variable, analytic in the second, diagonally Γ-invariant, (that is $k(\overline{\gamma}z,\gamma\eta)=k(\overline{z},\eta)$, $\gamma\in\Gamma$, z,η in $\mathbb{H}$), and square summable:

$$\|k\|^2_{L^2(\mathcal{A}_t)} = c_t \iint\limits_{F\times\mathbb{H}} |k(\overline{z},\eta)|^2 |d(\overline{z},\eta)|^{2t} \mathrm{d}\nu_0(z,\eta).$$

Let $\mathcal{P}_t$ be the projection from the Hilbert space of square summable functions f on $\mathbb{H}\times\mathbb{H}$, that are Γ-invariant and square summable:

$$c_t \iint\limits_{F\times\mathbb{H}} |f(\overline{z},\eta)|^2 |d(\overline{z},\eta)|^{2t} \mathrm{d}\nu_0(\eta)\mathrm{d}\nu_0(z) < \infty.$$

The following proposition is easy to prove, but we won't make any use of it in this paper.

Proposition 4.8 *Let φ be a bounded measurable Γ-invariant function on $\mathbb{H}\times\mathbb{H}$. Let $\mathcal{T}_\varphi$ be the Toeplitz operator of multiplication by φ on the Hilbert space $L^2(\mathcal{A}_t)$, that is $\mathcal{T}_\varphi k = \mathcal{P}(\varphi k)$, $k\in L^2(\mathcal{A}_t)$. Then $\mathcal{T}_\varphi k = \mathcal{P}_t k \mathcal{P}_t$, where the last composition is in $\mathcal{A}_t$, by regarding k as an element affiliated to $\mathcal{A}_t$.*

Remark. *In this setting the positive form $\mathcal{E}_t$ may be identified with the quadratic form on $L^2(\mathcal{A}_t)$ induced by the unbounded operator $\mathcal{T}_{\ln d}$ where $d = |d(\overline{z},\eta)|$ is defined as above.*

5. Derivatives of some one parameter families of positive operators

In this section we consider some parametrized families of completely positive maps that are induced by automorphic forms (and fractional powers of thereof). The automorphic forms are used as intertwining operators between the different representation spaces of $\mathrm{PSL}(2,\mathbb{Z})$, consisting of analytic functions.

It was proved in [18] that automorphic forms f for $\mathrm{PSL}(2,\mathbb{Z})$ of weight-k, provide bounded multiplication operators $S_f : H_t \to H_{t+k}$. The boundedness property comes exactly from the fact that one of the conditions for an automorphic forms f of order k is

$$\sup_{z\in\mathbb{H}} |f(z)|^2 \mathrm{Im}\, z^k \leq M,$$

which is exactly the condition that the operator of multiplication by f from H_t into H_{t+k} be norm bounded by M.

Secondly, the automorphic forms, have the (cocycle) Γ-invariance property as functions on $\mathbb{H}$, that is

$$f(\gamma^{-1}z) = (cz+d)^{-k} f(z),\ z \in \mathbb{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2,\mathbb{Z}).$$

Since $\pi_t(\gamma)$, $\pi_{t+k}(\gamma)$ act on the corresponding Hilbert space of analytic functions on $\mathbb{H}$, by multiplication with the automorphic factor $(cz+d)^{-t}$, respectively $(cz+d)^{-t-k}$, this implies exactly that

$$\pi_{t+k}(\gamma) S_f = S_f \pi_t(\gamma).$$

Let f, g be automorphic forms of order k. Let F be a fundamental domain for the group $PSL(2,\mathbb{Z})$ in $\mathbb{H}$. It was proved in [18] that the trace (in $\mathcal{A}_t$) of $S_f^* S_g$ is equal to the Petterson scalar product

$$\frac{1}{\mathrm{area} F} \langle f, g \rangle = \frac{1}{\mathrm{area} F} \int_F f(z) \overline{g(z)} (\mathrm{Im}\, z)^k \mathrm{d}\nu_0(z). \qquad (*)$$

In the next lemma we will prove that the symbol of $S_f S_g^*$, as an operator on H_t, belonging to $\mathcal{A}_t$, (the commutant of $\mathrm{PSL}(2,\mathbb{Z})$) is (up to a normalization constant) $f(z)g(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^k$.

In particular this shows that the above formula $(*)$ is explained by the trace formula $\tau_{\mathcal{A}_t}(k) = \dfrac{1}{\mathrm{area} F} \displaystyle\int_F k(\overline{z}, z) \mathrm{d}\nu_0(z)$, applied to the operator $k = S_f S_g^*$.

The role of the factor $[(\overline{z}-\xi)/(-2\mathrm{i})]^k$ is to make the function $\overline{f(z)} g(\xi) [(\overline{z}-\xi)/(-2\mathrm{i})]^k$ diagonally $\mathrm{PSL}(2,\mathbb{Z})$-invariant. It is easy to observe that $S_f^* S_g$ is the Toeplitz operator on H_t with symbol $\overline{f(z)} g(z) (\mathrm{Im}\, z)^k$. Note that, to form

$S_f S_g^*$, we have the restriction $k < t-1$, because S_g^* has to map H_t into a space H_{t-k} that makes sense.

We observe that the symbol of $S_f k S_g^*$ for an operator k on H_{t+k} of symbol $k = k(\overline{z}, \xi)$ is

$$\frac{c_{t-p}}{c_t}\overline{f(z)}g(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^p k(\overline{z}, \xi),$$

if f, g are automorphic forms of order p.

This also explains the occurrence of operators of multiplication with symbols $\Phi(\overline{z}, \xi)$ on the space $L^2(\mathcal{A}_t)$, in this setting. In the terminology of the Appendix in the previous chapter those are the Toeplitz operators with analytic symbol $\Phi(\overline{z}, \xi)$, a diagonally a $\mathrm{PSL}(2,\mathbb{Z})$-invariant function. In the present setting, to get a bounded operator, we map $L^2(\mathcal{A}_{t+k})$ into $L^2(\mathcal{A}_t)$, by multiplying by $\overline{f(z)}g(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^k$. In this chapter we will analyze the derivatives of a families of such operators.

Let $\Delta(z)$ be the unique automorphic form for $\mathrm{PSL}(2,\mathbb{Z})$ in dimension 12 (this is the first order for which there is a non zero space of automorphic form).

We rescale this form by considering the normalized function $\Delta_1 = \dfrac{\Delta}{c}$, where the constant c is chosen so that

$$\sup_{z \in \mathbb{H}} |\Delta_1(z)|^2 (\operatorname{Im} z)^{12} \leq 1.$$

In the sequel we will omit the subscript 1 from Δ. This gives that the norm $\|S_\Delta\|$, as on operator from H_t into H_{t+12} is bounded by 1.

As Δ is a non zero analytic function on the upper halfplane, one can choose an analytic branch for $\ln \Delta$. Consider the Γ-invariant function

$$\varphi(\overline{z}, \xi) = \ln \overline{\Delta(z)} + \ln \Delta(\xi) + 12 \ln[(\overline{z}-\xi)/(-2\mathrm{i})]$$

which we also write as

$$\varphi(\overline{z}, \xi) = \ln(\overline{\Delta(z)}\Delta(\xi) \cdot [(\overline{z}-\xi)/(-2\mathrm{i})]^{12}).$$

Defining $\pi_t(\gamma)$, for γ in $\mathrm{PSL}(2,\mathbb{Z})$ involves a choice of a branch for $\ln(cz+d)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We define for $\pi_t(\gamma)$, $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$, by using the factor $(cz+d)^{-t}$ corresponding to the following choice of the logarithm for $\ln(cz+d)$:

$$\ln \Delta(\gamma^{-1}z) - \ln \Delta(z) = \ln(cz+d),$$

$z \in \mathbb{H},\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{PSL}(2,\mathbb{Z})$.

By making this choice for π_t, restricted to Γ, we do not change the algebra $\mathcal{A}_t$, which is the commutant of $\{\pi_t(\Gamma)\}$, but we have the following.

With the above choice for $\ln(cz+d)$ and thus for $\pi_t(\gamma)$, $\gamma \in \mathrm{PSL}(2,\mathbb{Z})$, and for any $\varepsilon > 0$, we have that S_{Δ^ε} is a bounded operator between H_t and $H_{t+12\varepsilon}$, that intertwines π_t and $\pi_{t+12\varepsilon}$, for all $t > 1$, $\varepsilon > 0$.

In the following lemma we make the symbol computation for operators of the form $S_f S_g^*$. Recall that H_t, $t > 1$ is the Hilbert space analytic functions on $\mathbb{H}$, that are square-summable under $\mathrm{d}\nu_t = (\operatorname{Im} z)^{t-2}\mathrm{d}\overline{z}\mathrm{d}z$.

Lemma 5.1. *Let f, g be an analytic functions on $\mathbb{H}$, k a strictly positive integer and $t > 1$. Assume that $M_f = \sup_z |f(z)|^2 Im\, z^k$, $M_g = \sup_z |g(z)|^2 Im\, z^k$ are finite quantities.*

Let S_f, S_g be the multiplication operators from H_t into H_{t+k} by the functions f, g. Then S_f, S_g are bounded operators of norm at most M_f, M_g respectively.

Moreover the symbol of $S_f S_g^* \in B(H_t)$ is given by the formula

$$\frac{c_{t-k}}{c_t}\overline{f(z)}f(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^k.$$

Proof. Before starting the proof we'll make the following remark that should explain the role of the constant c_t $(= (t-1)/4\pi)$ in this computation.

Remark. *The quantity c_t is a constant that appears due to the normalization in the definition of H_t, where we have chosen*

$$\|f\|_{H_t}^2 = \int_{\mathbb{H}} |f(z)|^2 (Im\, z)^{t-2} d\overline{z}\, dz.$$

Consequently the reproducing vectors e_z^t, (defined by $\langle f, e_z^t \rangle = f(z)$, $f \in H_t$, $z \in \mathbb{H}$), are given by the following formula: ([5], [25])

$$e_z^t(\xi) = \langle e_z^t, e_\xi^t \rangle = \frac{c_t}{[(\overline{z}-\xi)/(-2i)]^t},\ \ z, \xi \in \mathbb{H}.$$

Consequently the normalized symbol of an operator A in $B(H_t)$ is given by the formula $k_A(\overline{z},\xi) = \dfrac{\langle Ae^t_z, e^t_\xi\rangle}{\langle e^t_z, e^t_\xi\rangle}$, z,ξ in H.

In the product formula we have that the symbol $k_{AB}(\overline{z},\xi)$ of the product of two operators A, B on H_t with symbols k_A, k_B is given by

$$\langle e^t_z, e^t_\xi\rangle k_{AB}(\overline{z},\eta) = \langle ABe^t_z e^t_\xi\rangle = \int_{\mathbb{H}} \langle Ae^t_z, e^t_\eta\rangle\langle Be^t_\eta, e^t_\xi\rangle \mathrm{d}\nu_t(\eta).$$

Thus

$$\begin{aligned} k_{AB}(\overline{z},\xi) &= \langle e^t_z, e^t_\xi\rangle \int_{\mathbb{H}} k_A(\overline{z},\eta)\langle e^t_z, e^t_\eta\rangle k_B(\overline{\eta},\xi)\langle e^t_\eta, e^t_\xi\rangle \mathrm{d}\nu_t(\eta) \\ &= \frac{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}{c_t} \int_{\mathbb{H}} k_A(\overline{z},\eta)\frac{c_t}{[(\overline{z}-\eta)/(-2\mathrm{i})]^t} k_B(\overline{\eta},\xi) \\ &\quad \frac{c_t}{[(\overline{\eta}-\xi)/(-2\mathrm{i})]^t}\mathrm{d}\nu_t(\eta) \\ &= c_t \int_{\mathbb{H}} k_A(\overline{z},\eta)k_B(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]\mathrm{d}\nu_0(\eta). \end{aligned}$$

This accounts for the constant c_t that occurs in front of the product formula (otherwise if we proceed as in [5] and include the constant c_t in the measure $\mathrm{d}\nu_t$, the constant will still show up in the product formula).

In the proof of the lemma we use the following observation.

Observation 5.2. *Let f, H_t, S_f be as in the statement of Lemma 5.1. Let e^t_z, e^{t+k}_z be the evaluation vectors at z, in the spaces H_t and H_{t+k}. Then*

$$S^*_f e^{t+k}_z = f(z)e^t_z,\ z \in \mathbb{H}.$$

Proof. Indeed, since we will prove the boundedness of S_f, we can check this by evaluating on a vector g in H_t. We have

$$\langle S^*_f e^{t+k}_z, g\rangle_{H_t} = \langle e^{t+k}_z, (S_f)g\rangle_{H_{t+k}} = \langle e^{t+k}_z, fg\rangle = \overline{\langle fg, e^{t+k}_z\rangle} = \overline{fg(z)}.$$

On the other hand:

$$\langle \overline{f(z)}, e^t_z, g\rangle_{H_t} = \overline{f(z)}\langle e^t_z, g\rangle_{H_t} = \overline{f(z)}\overline{\langle g, e^t_z\rangle}_{H_t} = \overline{f(z)g(z)}.$$

This shows the equality of the two vectors.

We can now go on with the proof of Lemma 5.1.

It is obvious that S_f, S_g unbounded operators of norms M_f, M_g. Indeed for S_f we have that

$$\begin{aligned}
|S_f g\|^2_{H_{t+k}} &= \int_{\mathbb{H}} |(S_f g)(z)| \mathrm{d}\nu_{t+k}(z) \\
&= \int_{\mathbb{H}} |(fg)(z)|^2 \mathrm{d}\nu_{t+k}(z) \\
&= \int_{\mathbb{H}} |f(z)|^2 |g(z)|^2 (\operatorname{Im} z)^k (\operatorname{Im} z)^{t-2} \mathrm{d}\overline{z}\mathrm{d}z \\
&= \int_{\mathbb{H}} |g(z)|^2 (|f(z)|^2 (\operatorname{Im} z)^k) \mathrm{d}\nu_t(z) \\
&\leq M_f \int_{\mathbb{H}} |g(z)|^2 \mathrm{d}\nu_t(z).
\end{aligned}$$

Hence $\|S_f\| \leq M_f$.

To prove the second assertion, observe that the symbol $k(\overline{z}, \xi)$ of $S_f S_g^*$, as an operator on H_t is given by the following formula:

$$\begin{aligned}
k(\overline{z}, \xi) &= \frac{\langle S_f S_g^* e_z^t, e_\xi^t \rangle_{H_t}}{\langle e_z^t, e_\xi^t \rangle_{H_t}} \\
&= \frac{\langle S_g^* e_z^t, S_f^* e_\xi^t \rangle}{\langle e_z^t, e_\xi^t \rangle_{H_t}} \\
&= \frac{\overline{g(z)} \langle e_z^{t-k}, \overline{f(\xi)} e_\xi^{t-k} \rangle_{H_{t-k}}}{\langle e_z^t, e_\xi^t \rangle_{H_t}} \\
&= \overline{g(z)} f(\xi) \frac{\langle e_z^{t-k}, e_\xi^{t-k} \rangle_{H_{t-k}}}{\langle e_z^t, e_\xi^t \rangle} \\
&= \overline{g(z)} f(\xi) \frac{\dfrac{c_{t-k}}{\overline{[(\overline{z} - \xi)/(-2\mathrm{i})]^{t-k}}}}{\dfrac{c_t}{\overline{[(\overline{z} - \xi)/(-2\mathrm{i})]^t}}} \\
&= \frac{c_{t-k}}{c_t} \overline{g(z)} f(\xi) [(\overline{z} - \xi)/(-2\mathrm{i})]^k, \quad z, \xi \text{ in } \mathbb{H}.
\end{aligned}$$

This also works also for k not an integer (as $\ln[(\overline{z} - \xi)/(-2\mathrm{i})]$ is chosen once for all).

Let us finally note that the same arguments might be used to prove the following more general statement.

Remark. *Let f, g be analytic function as in the statement of the lemma, and let k be an operator in $\mathcal{A}_t$. Then $S_f k S_g^*$ which belongs to $\mathcal{A}_{t+k}$ (if we think of S_f, S_g as bounded operators mapping H_t into H_{t+k}) has the following symbol*

$$\frac{c_t}{c_{t+k}} f(\xi)\overline{g(z)}[(\overline{z}-\xi)/(-2\mathrm{i})]^k k(\overline{z},\xi).$$

Proof. We have to evaluate

$$\begin{aligned}
\frac{\langle S_f k S_g^* e_z^{t+k}, e_\xi^{t+k}\rangle_{H_{t+k}}}{\langle e_z^{t+k}, e_\xi^{t+k}\rangle_{H_t}} &= \frac{f(\xi)\overline{g(z)}\langle k e_z^t, e_\xi^t\rangle_{H_t}}{\langle e_z^{t+k}, e_\xi^{t+k}\rangle_{H_{t+k}}} \\
&= f(\xi)\overline{g(z)}\frac{\langle k e_z^t, e_\xi^t\rangle_{H_t}}{\langle e_z^t, e_\xi^t\rangle_{H_t}} \cdot \frac{\langle e_z^t, e_\xi^t\rangle_{H_t}}{\langle e_z^{t+k}, e_\xi^{t+k}\rangle_{H_{t+k}}} \\
&= \frac{c_t}{c_{t+k}} f(\xi)\overline{g(z)}[(\overline{z}-\xi)/(-2\mathrm{i})]^k k(\overline{z},\xi).
\end{aligned}$$

In the next lemma we will deduce a positivity condition for kernels of operators that occur as generators of parametrized families $S_{f^\varepsilon} S_{g^\varepsilon}^*$, where f, g are supposed to have a logarithm on $\mathbb{H}$.

Lemma 5.3. *Assume that f is a function as in Lemma 5.1. f is analytic on $\mathbb{H}$ and we assume $M_f = \sup\limits_{z\in\mathbb{H}} |f(z)|^2 (Im\, z)^k$ is less than 1. Assume that f is nonzero on $\mathbb{H}$, and choose a branch for $\ln f$ and hence for f^ε, ε being strictly positive. Let $\varphi(\overline{z},\eta)$ be the function $\overline{\ln f(z)} + \ln f(\xi) + k\ln[(\overline{z}-\xi)/(-2i)]$ and use this as a choice for $\ln[\overline{f(z)} f(\xi)[(\overline{z}-\xi)/(-2i)]^k] = \varphi(\overline{z},\xi)$. Then for all $\varepsilon > 0$ the kernel:*

$$k_\varphi(\overline{z},\eta) = k_{\varphi,t,\varepsilon}(\overline{z},\xi) = \varphi^\varepsilon(\overline{z},\xi)\left[\frac{c_{t-k\varepsilon}}{c_t}\ln\varphi - k\frac{c_t'}{c_t}\right]$$

is nonpositive in the sense of $\mathcal{A}_t$, that is $\dfrac{k_\varphi(\overline{z}_i, z_j)}{[(\overline{z}_i - z_j)/(-2i)]^t}$ is a nonpositive matrix for all choices of $N \in \mathbb{N}$, $z_1, z_2, \ldots, z_N \in \mathbb{N}$.

Proof. By the choice we just made it is clear that the norm of the operator S_{f^ε} is always less that 1. We will also denote by $S^t_{f^\varepsilon}$ the corresponding operators, which act as a contraction from H_t into $H_{t+k\varepsilon}$.

Consider the following operator valued functions, with values in H_t

$$f(\varepsilon) = S^{t-k\varepsilon}_{f^\varepsilon}\left(S^{t-k\varepsilon}_{f^\varepsilon}\right)^*$$

Obviously the symbol of $f(\varepsilon)$ is $(c_{t-k\varepsilon}/c_t)\varphi^\varepsilon(\overline{z},\xi)$ and moreover $f(0) = 1$, $f(\varepsilon)$ is a decreasing map because for $0 \leq \varepsilon \leq \varepsilon'$, we have that

$$f(\varepsilon') = S^{t-k\varepsilon'}_{f^{\varepsilon'}}\left(S^{t-k\varepsilon'}_{f^{\varepsilon'}}\right)^* = S^{t-k\varepsilon}_{f^\varepsilon}\left[S^{t-k\varepsilon'}_{f^{\varepsilon'-\varepsilon}}\left(S^{t-k\varepsilon'}_{f^{\varepsilon'-\varepsilon}}\right)^*\right]\left(S^{t-k\varepsilon}_{f^\varepsilon}\right)^*$$

But the operator in the middle has norm less than 1, and hence we get that

$$f(\varepsilon') \leq S^{t-k\varepsilon}_{f^\varepsilon}\left(S^{t-k\varepsilon}_{f^\varepsilon}\right)^* = f(\varepsilon).$$

Fix N, and $z_1, z_2, \ldots, z_N$ in $\mathbb{H}$. Then (since the corresponding operators form a decreasing familly)

$$g(\varepsilon) = \left[\frac{f(\varepsilon)(\overline{z_i}, z_j)}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}\right]_{i,j}$$

is a decreasing family of matrices, and $g(0) = \mathrm{Id}$. Hence $g'(\varepsilon)$ must be a negative (nonpositive) matrix. Note that $f(\varepsilon)(\overline{z_i}, z_j) = (c_{t-k\varepsilon}/c_t)\varphi^\varepsilon(\overline{z_i}, z_j)$.

But $g'(\varepsilon)$ has exactly the formula stated above, that is

$$g'(\varepsilon) = \frac{\varphi^\varepsilon(\overline{z_i}, z_j)\left[\ln\varphi(\overline{z_i}, z_j)\dfrac{c_t - k\varepsilon}{c_t} - k\dfrac{c'_t}{c_t}\right]}{[(\overline{z}_i - z_j)/(-2\mathrm{i})]^t}$$

This completes the proof.

By collecting the terms together we, and since $\dfrac{c'_t}{c_t} = \dfrac{1}{t-1}$, we obtain, for all $\varepsilon \geq 0$ the following

LEMMA 5.4. *With the notations from the previous lemma, for all $\varepsilon \geq 0$, the kernel*

$$k_\varphi = k_{\varphi,\varepsilon,t} = \varphi^\varepsilon\left[\ln\varphi - \frac{1}{t-1-k\varepsilon}\right]$$

is nonpositive in $\mathcal{A}_t$. Precisely this means that is all choices of N in $\mathbb{N}$ and $z_1, z_2, \ldots, z_N$ in $\mathbb{H}$ we have that

$$\frac{k_\varphi(\overline{z_i}, z_j)}{[(\overline{z}_i - z_j)/(-2i)]^t}$$

is a nonpositive matrix.

The following remark will be used later in the proofs.

Remark 5.5 *For any $s > 1$, the identity*

$$\tau_{\mathcal{A}_s}(S_g^* S_g) = \frac{c_s}{c_{s+k}} \tau_{\mathcal{A}_t}(S_g S_g^*),$$

holds true for any automorphic form g of order k.

Proof. We have that $(S_g^* S_g)$ is the Toeplitz operator (on H_s) with symbol $|g(z)|^2(\mathrm{Im}\ z)^k$. Hence the trace $\tau_{\mathcal{A}_s}(S_g^* S_g)$ is $\frac{1}{\mathrm{area}(F)} \int\limits_F |g|^2 (\mathrm{Im}\ z)^k \mathrm{d}\nu_0(z)$.

On the other hand the symbol of $(S_g S_{g^*})$ (which is viewed here as an operator on $H_{s+k},$) is equal to

$$(z, \xi) \to \frac{\langle S_{g^*} e_z^{s+k}, S_{g^*} e_\xi^{s+k} \rangle}{\langle_z^{S+k}, e_\xi^{e+k} \rangle} = \frac{c_s}{c_{s+k}} \overline{g(z)} g(\xi) [(\overline{z} - \xi)/(-2\mathrm{i})]^k$$

and hence the trace of later symbol is

$$\frac{c_s}{c_{s+k}} \frac{1}{\mathrm{area}(F)} \int\limits_F |g(z)|^2 (\mathrm{Im}\ z)^k \mathrm{d}\nu_0(z).$$

6. Properties of the (unbounded) multiplication maps by $\ln[\overline{\Delta(z)}\Delta(\xi)[(\overline{z} - \xi)/(-2\mathrm{i})]^{12}]$ on different spaces of kernels

Let $\varphi(\overline{z}, \xi) = \ln(\overline{\Delta(z)}\Delta(\xi)[(\overline{z} - \xi)/(-2\mathrm{i})]^{12})$. In this chapter we want to exploit the negativity properties of the kernels

$$\varphi^{\varepsilon/12} \left(\frac{1}{12} \ln \varphi - \frac{1}{t - 1 - \varepsilon} \right).$$

By $\odot$ we denote the operation of pointwise multiplication of symbols. It is the analogue of Schurr multiplication on matrices or on a group algebra. When no confusion is possible we will omit the symbol $\odot$ and just replace it by $\cdot$.

We want to draw conclusions on the properties of the multiplication maps, defined on a suitable dense subspace of $L^2(\mathcal{A}_t)$, by the formula

$$\Lambda_\varepsilon(k)=k \odot \left(\varphi^{\varepsilon/12}\left[\frac{1}{12}\ln\varphi - \frac{1}{t-1-\varepsilon}\right]\right).$$

For functions $k(\overline{z},\eta)$ on $\mathbb{H}\times\mathbb{H}$, that are positive, but do not necessary represent a positive operator we will introduce the following definition.

Definition 6.1. *A function $k(\overline{z},\eta)$ on $\mathbb{H}\times\mathbb{H}$ that is analytic for η and antianalytic for z will be called* positive, for *$\mathcal{A}_t$, if the following matrix*

$$\left[\frac{k(\overline{z}_i,z_j)}{[(\overline{z}_i - z_j)/(-2i)]^t}\right]_{i,j=1}^n$$

is positive, for all choices of $N\in\mathbb{N}$ and $z_1,z_2,\dots,z_N$ in $\mathbb{H}$. The space of such kernels will be denoted by $\mathcal{S}_t$.

The following remark is a trivial consequence of the fact that the Schurr product of two positive matrices is positive, and a consequence of the description for positivity of kernels of operators in $\mathcal{A}_t$ given in Section 4.

Proposition 6.2 *For all numbers $r,s > 1$, the vector space $(\mathcal{A}_r)_+\odot\mathcal{S}_s$ is contained in $\mathcal{S}_{r+s}$ and $(\mathcal{A}_r)_+\subseteq\mathcal{S}_r$.*

Proof. Just observe that in fact $\mathcal{S}_s\odot\mathcal{S}_r$ is contained in $\mathcal{S}_{s+r}$.

The problem that we address in this chapter comes from the fact that the operator

$$-\Lambda_{\varepsilon,r,s}(k) = k\odot\varphi^{\varepsilon/12}\left[-\frac{1}{12}\ln\varphi + \frac{1}{r-1-\varepsilon}\right],$$

maps $k\in(\mathcal{A}_s)_+$ into $\mathcal{S}_{r+s}$. Also $\Lambda_{\varepsilon,r,s}(k)(\overline{z},z)$ is integrable on F, so it is tempting to infer that $\Lambda_{\varepsilon,r,s}(k)$ belongs to $L^1(\mathcal{A}_{r+s})$. In fact we conjecture that a kernel $k(\overline{z},\eta)$ in $\mathcal{S}_t$, that is also diagonally integrable on F, corresponds to an element in $L^1(\mathcal{A}_t)_t$. Since we are unable to prove directly the conjecture, we will use monotonicity properties for the derivatives of φ^ε.

If no constants were involved, we would simply say that $\varphi^{\varepsilon}(-\ln\varphi)$ is the increasing limit of the derivatives, since the second derivative would be negative. This doesn't hold exactly, but the constants involved are small enough and have a neglectable effect on the previous line of reasoning. This is done in the following lemma.

LEMMA 6.3. *Let k be a positive kernel in $\mathcal{A}_s$. Fix $v > 1$ and let $\varepsilon > 0$ be small enough. Consider the following elements in $\mathcal{A}_{s+\varepsilon}$ defined by the kernels*

$$\lambda_{\varepsilon,v,s}(k)(\overline{z},\xi) = \left[\frac{v-1-\varepsilon}{v-1}\varphi^{\varepsilon/12}(\overline{z},\xi)\right] k(\overline{z},\xi).$$

Note that up to a multiplicative constant $\lambda_{\varepsilon,v,s}(k)$ is the kernel of $S_{\Delta^{\varepsilon/12}} k S^*_{\Delta^{\varepsilon/12}}$ in $\mathcal{A}_{s+\varepsilon}$. Let $\widetilde{\lambda}_{\varepsilon,v,s}(k)$ be the image (through $\Psi_{v+2s,v+\varepsilon}$) of this kernel in $\mathcal{A}_{v+2s}$.

Then $\widetilde{\lambda}_{\varepsilon,v,s}(k)$ is a decreasing family of positive kernels representing elements in $\mathcal{A}_{v+2s}$ and there exists a negative element $M(k) = M_{\varepsilon,v,s}(k)$ in $-L^1(\mathcal{A}_{v+2s})_+$, such that $M(k)$ is the derivative with respect to ε:

$$M(k) = \frac{d}{d\varepsilon}\widetilde{\lambda}_{\varepsilon,v,s}(k) \qquad (*)$$

The derivative is computed in the strong convergence topology, on a dense domain $\mathcal{D} \subseteq H_{v+2s}$, affiliated with $\mathcal{A}_{v+2s}$.

The symbol of $M_{\varepsilon,v,s}(k)$ as an operator in H_{v+2s} is equal to

$$\Lambda_{\varepsilon,v,s}(k)(\overline{z},\xi) = \frac{v-1-\varepsilon}{v-1} k(\overline{z},\xi)\varphi^{\varepsilon/12}\left[\frac{1}{12}\ln\varphi - \frac{1}{v-1-\varepsilon}\right].$$

Proof. For simplicity of the proof well use the notation $\varphi_1 = \varphi^{1/12}$. We prove first that the family $\lambda_{\varepsilon,v,s}(k)$ is a decreasing family in $\mathcal{A}_{v+s+\varepsilon}$ and hence in $\mathcal{A}_{v+2s}$.

Indeed $\dfrac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon}(\overline{z},\xi)$ is a decreasing family of operators in $\mathcal{A}_v$, and hence by Proposition 6.2 it follows that

$$\frac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon}(\overline{z},\xi)k(\overline{z},\xi)$$

is a decreasing family in $\mathcal{S}_{v+s}$, and hence in $\mathcal{S}_{v+s+\varepsilon}$. Since we know that these operators are already bounded in $\mathcal{A}_{v+\varepsilon}$, the first part of the statement follows immediately.

Denote by $G(\varepsilon) = G(\varepsilon)(\overline{z}, \xi)$ the kernel represented by

$$\frac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon}(\overline{z}, \xi)k(\overline{z}, \xi),$$

which represent therefore a (decreasing) family in $\mathcal{A}_{v+s+\varepsilon}$ and hence in $\mathcal{A}_{v+2s}$. Fix $\varepsilon_0 > 0$ and let

$$g_{\varepsilon}(\overline{z}, \xi) = \frac{G(\varepsilon)(\overline{z}, \xi) - G(\varepsilon_0)(\overline{z}, \xi)}{\varepsilon - \varepsilon_0}.$$

Then g_{ε} is a negative (nonpositive) element in $\mathcal{A}_{v+2s}$. We want to find a formula for $g_{\varepsilon'} - g_{\varepsilon}$. Obviously when ε converges to ε_0, the kernel g_{ε} converges (at least pointwise) to the kernel $\Lambda_{\varepsilon_0, v, s}(k)(\overline{z}, \xi)$.

It is elementary calculus to find the following pointwise expression for $H_{\varepsilon', \varepsilon}(\overline{z}, \xi) = g_{\varepsilon'}(\overline{z}, \xi) - g_{\varepsilon}(\overline{z}, \xi)$.

$$H_{\varepsilon, \varepsilon'} = (\varepsilon - \varepsilon') \int_0^1 \left(\int_0^1 t G''(\varepsilon(t, s)) \mathrm{d}s \right) \mathrm{d}t \tag{6.2}$$

where $\varepsilon(t, s) = s[(1-t)\varepsilon_0 + t\varepsilon] + (1-s)[(1-t)\varepsilon_0 + t\varepsilon']$ belongs to the interval determined by $\varepsilon, \varepsilon', \varepsilon_0$.

This formula holds, at the level of kernels (that is by evaluating both sides on any given points $z, \xi \in \mathbb{H}$).

On the other hand we may compute immediately because

$$G(\varepsilon) = \frac{1}{v-1}[(v-1-\varepsilon)\varphi_1^{\varepsilon}]k$$

that

$$G'(\varepsilon) = \frac{1}{v-1}[(v-1-\varepsilon)\varphi_1^{\varepsilon}\ln\varphi_1 - \varphi_1^{\varepsilon}]k$$

$$G''(\varepsilon) = \frac{1}{v-1}[(v-1-\varepsilon)\ln^2\varphi_1 - 2\ln\varphi_1]\varphi_1^{\varepsilon} \cdot k.$$

Furthermore we have the following expression for $G''(\varepsilon)$

$$\begin{aligned} G''(\varepsilon) &= \frac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon} \cdot k \left[\ln^2\varphi_1 - \frac{2}{v-1-\varepsilon}\ln\varphi_1\right] \\ &= \frac{v-1-\varepsilon}{v-1}\left\{ k \left[\varphi_1^{\varepsilon/2}\left(-\ln\varphi_1 + \frac{1}{v-1-\varepsilon} \right) \right]^2 - \frac{k\varphi_1^{\varepsilon}}{(v-1-\varepsilon)^2} \right\}. \end{aligned}$$

Thus we obtain further that

$$G''(\varepsilon) = \frac{v-1-\varepsilon}{v-1} k \left[\varphi_1^{\varepsilon/2} \left(-\ln \varphi_1 + \frac{1}{v-1-\varepsilon} \right) \right]^2 - \frac{k\varphi_1^{\varepsilon}}{(v-1-\varepsilon)(v-1)}.$$

But because of the previous Propositions 6.2 and Lemma 6.4 we have that

$$\left[\varphi_1^{\varepsilon/2} \left(-\ln \varphi_1 + \frac{1}{v-1-\varepsilon} \right) \right]^2 = \left[\varphi_1^{\varepsilon/2} \left(-\ln \varphi_1 + \frac{1}{v-\varepsilon/2-1-\varepsilon/2} \right) \right]^2$$

represents the square of an element:

$$\varphi_1^{\varepsilon/2} \left(-\ln \varphi_1 + \frac{1}{v-1-\varepsilon} \right)$$

in $\mathcal{S}_{v-\varepsilon/2}$. The square of the above element consequently belongs to $\mathcal{S}_{2v-\varepsilon}$.

Hence

$$\mathcal{R}(\varepsilon) = k \left[\varphi_1^{\varepsilon/2} \left(-\ln \varphi_1 + \frac{1}{v-1-\varepsilon} \right) \right]^2,$$

as a kernel, belongs to $\mathcal{S}_{s+2v-\varepsilon} \subseteq \mathcal{S}_{s+2v}$.

In conclusion we have just verified that

$$G''(\varepsilon) = \mathcal{R}(\varepsilon) - \frac{k\varphi_1^{\varepsilon}}{(v-1)(v-1-\varepsilon)},$$

where $\mathcal{R}(\varepsilon)$ belongs to $\mathcal{S}_{2v+s}$.

Moreover, it is obvious that

$$Q(\varepsilon) = \frac{k\varphi_1^{\varepsilon}}{(v-1)(v-1-\varepsilon)}$$

is a bounded element in $\mathcal{A}_{v+2s}$, and that $Q(\varepsilon)$ is consequently bounded by a constant C, independent of all the variables v, s, ε.

$$Q(\varepsilon) \leq C \cdot \mathrm{Id} \qquad \text{in } \mathcal{A}_{2v+s}$$

and hence

$$Q(\varepsilon) \leq C \cdot \mathrm{Id} \qquad \text{in } \mathcal{S}_{2v+s}.$$

We put this into the integral formula for

$$H_{\varepsilon,\varepsilon'} = g(\varepsilon) - g(\varepsilon') = \frac{G(\varepsilon) - G(\varepsilon_0)}{\varepsilon - \varepsilon_0} - \frac{G(\varepsilon') - G(\varepsilon_0)}{\varepsilon' - \varepsilon_0}$$

to obtain that $H_{\varepsilon,\varepsilon'}$ is of the form $(\varepsilon - \varepsilon')\,[\mathcal{R} - \mathcal{Q}]$ where $\mathcal{R}$ belongs to $\mathcal{S}_{2v+s}$ and $\mathcal{Q}$ belongs to $(\mathcal{A}_{v+2s})_+$ and $0 \leq \mathcal{Q} \leq C \cdot \mathrm{Id}$.

But then $\mathcal{R}$ belongs to $\mathcal{A}_{v+2s} \cap \mathcal{S}_{v+2s}$ and hence $\mathcal{R} \in (\mathcal{A}_{v+2s})_+$.

Thus, we obtain that in $\mathcal{A}_{v+2s}$, we have that (assuming $\varepsilon - \varepsilon' \geq 0$)

$$H_{\varepsilon,\varepsilon'} \geq -(\varepsilon - \varepsilon')\mathcal{Q} \geq -(\varepsilon - \varepsilon')\mathcal{C}$$

so

$$H_{\varepsilon,\varepsilon'} \geq -(\varepsilon - \varepsilon')C$$

and therefore

$$g_\varepsilon - g_{\varepsilon'} \geq C(\varepsilon' - \varepsilon)$$

Hence for $\varepsilon \geq \varepsilon' \geq \varepsilon_0$ we have that $g_\varepsilon + C_\varepsilon \geq g_{\varepsilon'} + C_{\varepsilon'}$ in $\mathcal{A}_{v+2s}$, for a fixed, positive constant C.

Now recall that

$$g(\varepsilon) = \frac{G(\varepsilon) - G(\varepsilon_0)}{\varepsilon - \varepsilon_0}$$

and that $G(\varepsilon)$ itself, was a decreasing family in $\mathcal{A}_{v+2s}$, so that $g(\varepsilon)$ are negative elements in $\mathcal{A}_{v+2s}$.

Denote for simplicity $h(\varepsilon) = -g(\varepsilon)$. Then what we just obtained is the following:

The operators $h(\varepsilon)$ are positive elements in $(\mathcal{A}_{v+2s})_+$. Moreover $h(\varepsilon) - C\varepsilon \leq h(\varepsilon') - C\varepsilon'$ if $\varepsilon \geq \varepsilon'$, i.e. $h(\varepsilon) - C\varepsilon$ is a decreasing family. By adding a big constant k to $h(\varepsilon)$ we have that $K + h(\varepsilon) - C\varepsilon$ is a decreasing family of positive elements in $(\mathcal{A}_{s+2v})_+$.

Thus as ε decreases to ε_0 we have that $K + h(\varepsilon) - C\varepsilon$ is an increasing family of positive operators in $\mathcal{A}_{v+s}$.

Moreover, as the trace of $h(\varepsilon)$ is equal to $-\tau(g(\varepsilon))$, which is

$$-\int_F \frac{\left[\dfrac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon}(\overline{z}, z) - \varphi_1^{\varepsilon_0}(\overline{z}, z)\dfrac{v-1-\varepsilon_0}{v-1}\right] k(\overline{z}, z)}{\varepsilon - \varepsilon_0} \mathrm{d}\nu.$$

This integral converges, (in $L^1(F\mathrm{d}\nu_0)$), to

$$\int_F \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{v-1-\varepsilon}{v-1}\varphi_1^{\varepsilon}(\overline{z},z)\Big|_{\varepsilon=\varepsilon_0} k(\overline{z},z)\mathrm{d}\nu_0(z)$$

$$= -\frac{v-1-\varepsilon_0}{v-1}\int_F \varphi_1^{\varepsilon_0}(\overline{z},z)\left[\ln\varphi_1 - \frac{1}{v-1-\varepsilon_0}\right] k(\overline{z},z)\mathrm{d}\nu_0(z)$$

which is finite (the convergence is dominated here for example by $C\varphi_1^{\varepsilon'}$, for some $\varepsilon' \leq \varepsilon_0$).

Thus $K+h(\varepsilon)-C(\varepsilon)$ are an increasing family in $\mathcal{A}_t$ (as ε decreases to ε_0) and the supremum of the traces (in $L^1(\mathcal{A}_t)$) is finite. By Lesbegue's Dominated Convergence Theorem in $L^1(\mathcal{A}_t)$, the limit of $K+h(\varepsilon)-C(\varepsilon)$ exists in $L^1(\mathcal{A}_t)$ and convergence is in the strong operator topology on a dense domain, affiliated with $\mathcal{A}_t$.

Corollary 6.4. *Let $\Lambda_{\varepsilon,v,s}(k)$, be the map, defined in the previous lemma, that associates to any positive k in $\mathcal{A}_s$, a positive element in $\mathcal{A}_{v+2s}$, whose kernel is given by the formula:*

$$\Lambda_{\varepsilon,v,s}(k)(\overline{z},\xi) = \frac{v-1-\varepsilon}{v-1}k(\overline{z},\xi)\varphi^{\varepsilon/12}\left[-\frac{1}{12}\ln\varphi + \frac{1}{v-1-\varepsilon}\right].$$

Then $-\Lambda_{\varepsilon,v,s}$ is a completely positive map from $\mathcal{A}_s$ into $L^1(\mathcal{A}_{v+2s})$.

Proof. From the previous lemma we know that $\Lambda_{\varepsilon,v,s}(k)$ is well defined and belongs to $L^1(\mathcal{A}_{v+2s})$. On the other hand $\Lambda_{\varepsilon,v,s}(k)$ is obtained by multiplication with a positive kernel in $\mathcal{S}_v$, and hence (as in the proof of the complete positivity for $\Psi_{s,t}$) we obtain that $\left[\Lambda_{\varepsilon,v,s}\left(k_{pq}\right)\right]_{p,q}$ is a positive in $M_N(\mathbb{C})\otimes L^1(\mathcal{A}_{v+2s})$, if $[k_{p,q}]$ is a positive matrix in $M_N(\mathbb{C})\otimes\mathcal{A}_s$.

Corollary 6.5. *Let $\varepsilon_0>0$ and $t>3+\varepsilon_0$. Let Λ_ε be defined, on the space of all symbols k representing operators in $\bigcup\limits_{1<s<t-2-\varepsilon_0}\mathcal{A}_s$, by the formula*

$$\Lambda_{\varepsilon_0}(k) = \frac{d}{d\varepsilon}(k\odot\varphi^{\varepsilon})\Big|_{\varepsilon=\varepsilon_0}.$$

Note that pointwise derivative of kernels is $(\varphi^{\varepsilon_0}\ln\varphi)\odot k$.

Then $\Lambda_{\varepsilon_0}(k)$ belongs to $L^1(\mathcal{A}_t)$ and moreover the derivative is valid in the sense of strong operator topology, on a dense domain, affiliated to $\mathcal{A}_t$.

Fixing $1 < s < t-2-\varepsilon_0$, there exists a sufficiently large constant C (depending on s, t, ε_0), such that $-[\Lambda_{\varepsilon_0} + Ck \odot \varphi^{\varepsilon}]$ (and hence $-[\Lambda_{\varepsilon_0} + C \cdot \mathrm{Id}]$) becomes a completely positive operator from $\mathcal{A}_s$ into $\mathcal{A}_t$.

Proof. Because of the condition $s < t-2-\varepsilon_0$, we can always find a constant C, by the previous lemma such that the previous lemma applies to $\Lambda_{\varepsilon_0} + Ck \odot \varphi^{\varepsilon}$.

Corollary 6.6. *Fix $t > 3$. For every $1 < s < t-2$ and for every k in $\mathcal{A}_s$ there exists an (eventually unbounded) operator $\Lambda(k)$ (of symbol multiplication by $\ln \varphi$) that is affiliated with $\mathcal{A}_t$, and there exists a dense domain $\mathcal{D}$ in H_t, that is affiliated with $\mathcal{A}_t$, such that the derivative*

$$\frac{d}{d\varepsilon}\langle k \odot \varphi^{\varepsilon}\xi, \eta\rangle_{H_t}\bigg|_{\varepsilon=0}$$

exists for all ξ, η in $\mathcal{D}$ and it is equal to

$$\langle \Lambda(k)\xi, \eta\rangle.$$

Moreover there exists a constant, C, depending only on s, t, such that for any positive matrix $[k_{p,q}]_{p,q=1}^{p}$ in $M_N(\mathcal{A}_s)_+$, the operator matrix

$$-\left[(\Lambda + C \cdot \mathrm{Id})(k_{p,q})\right]_{p,q=1}^{N}$$

represents a positive operator, affiliated with $\mathcal{A}_t$.

Remark. *The operator $k \odot \varphi^{\varepsilon}$ appearing in the previous statement is bounded. Indeed, modulo a multiplicative constant $k \odot \varphi^{\varepsilon}$ is the symbol of $S_{\Delta^{\varepsilon}} k S_{\Delta^{\varepsilon}}^*$. If $k \in \mathcal{A}_s$, then $S_{\Delta^{\varepsilon}} k S_{\Delta^{\varepsilon}}^*$ belongs to $\mathcal{A}_{s+12\varepsilon}$, and since $s < t$, by choosing ε small enough, we can assume that $S_{\Delta^{\varepsilon}} k S_{\Delta^{\varepsilon}}^*$ represents a bounded operator on H_t, and hence that the expression $\langle k \odot \varphi^{\varepsilon}\xi, \eta\rangle_{H_t}$ makes sense for all ξ, η in H_t.*

Before going to the proof of the statement of Corollary 6.6, we prove the following lemma, (that will be used in the proof of Corollary 6.6), concerning the operator $k \odot \varphi^{\varepsilon}$ and the range of the operator $\Lambda_{\varepsilon,v,s}(k)$ defined in Lemma 6.3.

Lemma 6.7. *With the notations from Lemma 6.3, let k be an operator in $\mathcal{A}_s$, $v > 1$, $\varepsilon > 0$. Let $M_{\varepsilon,v,s}(k)$ be the derivative (at ε), which belongs to $L^1(\mathcal{A}_{v+2s})$, of the decreasing family*

$$\widetilde{\lambda}_{\varepsilon,v,s}(k) = \frac{v-1-\varepsilon}{v-1}\varphi^{\varepsilon}(\overline{z},\xi)k(\overline{z},\xi).$$

Then the range and init space of the unbounded operator $M_{\varepsilon,v,s}(k)$ are contained (and dense) in the closure of the range of $S_{\Delta^\varepsilon} \subseteq H_{2v+s}$, (more precisely in closure of the range of $S_{\Delta^\varepsilon}^{2v+s-\varepsilon}$).

Proof. Indeed, by what we have just proved, $M_{\varepsilon,v,s}(k)$ is the strong operator topology limit (on a dense domain affiliated with the von Neumann algebra), as ε' decreases to ε, of the operators

$$\frac{G_{\varepsilon'}(\overline{z},\xi) - G_{\varepsilon}(\overline{z},\xi)}{\varepsilon'-\varepsilon}.$$

Recall that $G_{\varepsilon'}(\overline{z},\xi)$ was the symbol (modulo a multiplicative constant) of the operator $S_{\Delta^{\varepsilon'}} k S^*_{\Delta^{\varepsilon'}}$.

Then by applying $(G_{\varepsilon'} - G_{\varepsilon'})/(\varepsilon'-\varepsilon)$ to any vector ξ in H_t, the outcome is already a vector in the closure of the range of S_{Δ^ε}. This property is preserved in the limit. By selfadjointness the same is valid for the init space.

We proceed now to the proof of Corollary 6.6.

Proof of Corollary 6.6. We start by constructing first the domain $\mathcal{D}$. For $\varepsilon_0 > 0$ let $\mathcal{D}_{\varepsilon_0} \subseteq H_t$ be the range of $(S^t_{\Delta^{\varepsilon_0}})^*$, considered as an operator from $H_{t+12\varepsilon_0}$ into H_t. $\mathcal{D}$ will be the increasing union (after ε_0) of $\mathcal{D}_{\varepsilon_0}$.

Let B_{ε_0} be a right inverse, as an unbounded operator for the operator $S_{\Delta^{\varepsilon_0}}$. Thus B_{ε_0} acts from a domain dense in the closure of range $S^t_{\Delta^{\varepsilon_0}}$ into H_t. It is clear that B_{ε_0} is an intertwiner affiliated with the von Neumann algebras $\mathcal{A}_t$ and $\mathcal{A}_{t+12\varepsilon_0}$ (by von Neumann's theory of unbounded operators, affiliated to a II_1 factor ([24])).

Thus, by denoting P_{ε_0} to the projection onto the closure of the range of $S^t_{\Delta^{\varepsilon_0}}$ in $H_{t+12\varepsilon_0}$, the following properties hold true:

$$(S^t_{\Delta^{\varepsilon_0}})B_{\varepsilon_0} = P_{\varepsilon_0}.$$

By taking the adjoint, we obtain

$$B^*_{\Delta_{\varepsilon_0}}\left(S^t_{\Delta^{\varepsilon_0}}\right)^* = P_{\varepsilon^\circ}.$$

All compositions make sense in the algebra of unbounded operators affiliated with $\mathcal{A}_t$, and $\mathcal{A}_{t+12\varepsilon_0}$. On $H_{t+12\varepsilon_0}$, we let $M_{\varepsilon_0}(k)$ be the L^1 operator, given by Corollary 3.4, whose symbol is

$$k \odot \varphi^{\varepsilon_0} \ln \varphi,$$

for k in $\mathcal{A}_s$.

We define $\Lambda_{\varepsilon_0}(k)$ by the following composition:

$$\Lambda_{\varepsilon_0}(k) = \frac{c_{t+12\varepsilon_0}}{c_t} B_{\varepsilon_0} M_{\varepsilon_0}(k) B^*_{\varepsilon_0}.$$

We want to prove that Λ_{ε_0} does not depend on ε_0. Obviously (by [24]), the operator $\Lambda_{\varepsilon_0}(k)$ is affiliated with $\mathcal{A}_t$.

Moreover for ξ, η in $\mathcal{D}_{\varepsilon_0}$, which are thus of the form

$$\xi = S^*_{\Delta^{\varepsilon_0}} \xi_1, \qquad \eta = S^*_{\Delta^{\varepsilon_0}} \eta_1,$$

for some ξ_1, η_1 in $H_{t+12\varepsilon_0}$, we have that

$$\langle \Lambda_{\varepsilon_0}(k)\xi, \eta\rangle = \langle \Lambda_{\varepsilon_0}(k) S^*_{\Delta^{\varepsilon_0}} \xi_1, S^*_{\Delta^{\varepsilon_0}} \eta_1\rangle_{H_t}.$$

This is equal to

$$\begin{aligned}&\frac{c_{t+12\varepsilon_0}}{c_t} \langle B_{\varepsilon_0} M_{\varepsilon_0}(k) B^*_{\varepsilon_0} S^*_{\Delta^{\varepsilon_0}} \xi_1, S^*_{\Delta^{\varepsilon_0}} \eta_1\rangle_{H_t}\\ &= \frac{c_{t+12\varepsilon_0}}{c_t} \langle P_{\varepsilon_0} M_{\varepsilon_0}(k) P_{\varepsilon_0} \xi_1, \eta_1\rangle_{H_{t+12\varepsilon_0}}.\end{aligned}$$

Because of Lemma 6.7, we know that this is further equal to

$$\frac{c_{t+12\varepsilon_0}}{c_t} \langle M_{\varepsilon_0}(k)\xi_1, \eta_1\rangle.$$

We use the above chain of equalities to deduce that the definition of $\Lambda_{\varepsilon_0}(k)$ is independent on the choice of ε_0.

Indeed assume we use another ε_0', which we assume to be bigger than ε_0. Assume $\xi = S^*_{\Delta^{\varepsilon_0'}} \xi_2$. This is further equal to $S^*_{\Delta^{\varepsilon_0}} S^*_{\Delta^{(\varepsilon_0' - \varepsilon_0)}} \xi_2$.

Then, by redoing the previous computations we arrive to the term

$$\frac{c_{t+12\varepsilon_0'}}{c_t} \langle M_{\varepsilon_0'}(k)\xi_2, \eta_2\rangle.$$

But on the other hand in this situation

$$\begin{aligned}\langle M_{\varepsilon_0}(k)\xi_1,\eta_1\rangle &= \frac{c_{t+12\varepsilon_0}}{c_t}\langle M_{\varepsilon_0}(k)S^*_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}\xi_2, S^*_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}\eta_2\rangle\\ &= \frac{c_{t+12\varepsilon_0}}{c_t}\langle S_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}M_{\varepsilon_0}(k)S^*_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}\xi_2,\eta_2\rangle.\end{aligned}$$

To show independence on the choice of ε_0, we need consequently to prove that

$$\frac{c_{t+12\varepsilon_0}}{c_t}\langle S_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}M_{\varepsilon_0}(k)S^*_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}\xi_2,\eta_2\rangle$$

is equal to $\frac{c_{t+12\varepsilon_0}}{c_t}\langle M_{\varepsilon_0'}(k)\xi_2,\eta_2\rangle$.

Now all the operators are in L^1. Moreover the symbol of

$$S_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}M_{\varepsilon_0}(k)S^*_{\Delta^{(\varepsilon_0'-\varepsilon_0)}}$$

is

$$\frac{c_{t+12\varepsilon_0}}{c_{t+12\varepsilon_0'}}\varphi^{\varepsilon_0'-\varepsilon_0}$$

times the symbol of $M_{\varepsilon_0}(k)$.

But the symbol of $M_{\varepsilon_0}(k)$ is $\varphi^{\varepsilon_0}\ln\varphi$ divided by $\frac{c_{t+12\varepsilon_0}}{c_t}$.

This shows independence of the choice on ε_0 (some care has to be taken when choosing ξ_1,η_1, ξ_2,η_2 given ξ,η). We always choose them in the init space of $S^*_{\Delta^{\varepsilon_0}}$, respectively $S^*_{\Delta^{\varepsilon_0'}}$. By the von Neumann theorem we will be able to choose a common intersection domain for these operators.

Consequently to check that the derivative of $\langle h\odot\varphi^{\varepsilon}\xi,\eta\rangle_{H_t}$ at $\varepsilon=0$ is equal to the operator $\Lambda(k)$ introduced in the statement of Corollary 6.6, we only have to check this for vectors ξ,η, that we assume to be of the form

$$\xi = S^*_{\Delta^{\varepsilon_0}}\xi_1, \eta = S^*_{\Delta^{\varepsilon_0}}\eta_1.$$

Then, modulo a multiplicative constant $\langle k\odot\varphi^{\varepsilon}\xi,\eta\rangle_{H_t}$ becomes

$$\langle k\odot\varphi^{\varepsilon+\varepsilon_0}\xi_1,\eta_1\rangle_{H_{t+\varepsilon}}.$$

By a change of variables the derivative at 0 of $\langle k\odot\varphi^{\varepsilon}\xi,\eta\rangle_{H_t}$ becomes the derivative at ε_0 of the later expression: $\langle k\odot\varphi^{\varepsilon}\xi_1,\eta_1\rangle_{H_{t+\varepsilon}}$. Up to a multiplicative constant, this derivative exists and it is equal to $\langle M(k)\xi_1,\eta_1\rangle$, which is by definition $\langle\Lambda(k)\xi_1,\eta_1\rangle$.

Finally observe that for any constant C, $\langle(\Lambda(k)+C)\xi_1, \eta_1\rangle$ is equal to

$$B_{\varepsilon_0}(M_{\varepsilon_0}(k) + C' S_{\Delta^{\varepsilon_0}} k S^*_{\Delta^{\varepsilon_0}}) B^*_{\varepsilon_0}$$

for a constant C' obtained from C by multiplication by a normalization factor depending on t and ε_0.

Consequently, if $[k_{p,q}]_{p,q}$ is a positive matrix in $\mathcal{A}_s$, then by using the complete positivity result of Lemma 6.3, we infer that the matrix

$$-\left[M_{\varepsilon_0}\left(k_{p,q}\right) + C'\varphi^{\varepsilon_0} \odot k_{p,q}\right]_{p,q}$$

represents a positive operator in $M_p(\mathbb{C}) \otimes (\mathcal{A}_{t+\varepsilon_0})_+$.

Since $\varphi^{\varepsilon_0} \odot k_{p,q}$ is $S_{\Delta^{\varepsilon_0}} k_{p,q} S^*_{\Delta^{\varepsilon_0}}$, we get that $-\left[\Lambda(k_{p,q})\right]^p_{p,q=1}$ is a positive matrix of operators affiliated to $M_p(\mathbb{C}) \otimes \mathcal{A}_t$.

Remark 6.8. *If want to deal with less general operators, (paying the price of not including the identity operator in the domain of Λ), then we can take operators of the form $S_{\Delta^{\varepsilon_0}} k S_{(\Delta^{\varepsilon_0})^*}$ that belong to $\mathcal{A}_s$, $s < t-2$, $s-\varepsilon_0 > 1$, and then $\Lambda(k)$ will be in $L^1(\mathcal{A}_t)$, for such a kernel k, directly from the Lemma 3.3.*

7. Construction of an (unbounded) coboundary for the Hochschild cocycle in the Berezin's deformation

In this section we analyze the 2-Hochschild cocycle

$$\mathcal{C}_t(k,l) = \frac{\mathrm{d}}{\mathrm{d}s}(k *_s l)\bigg|_{\substack{s=t\\ s>t}}$$

that arrises in the Berezin deformation. We prove that the operator introduced in the previous section (6) may be used to construct an operator $\mathcal{L}$ (defined on a dense subalgebra of $\mathcal{A}_t$), taking values in the algebra of unbounded operators affiliated with $\mathcal{A}_t$. $\mathcal{L}$ will be defined on a dense subalgebra of $\mathcal{A}_t$.

The equation satisfied by $\mathcal{L}$ is

$$\mathcal{C}_t(A,B) = \mathcal{L}_t(A *_t B) - A *_t \mathcal{L}_t(B) - \mathcal{L}_t(A) *_t B$$

and this will be fulfilled in the form sense (that is by taking the scalar product with some vectors ξ, η in both sides).

The fact that $\mathcal{L}$ takes its values in the unbounded operators affiliated with $\mathcal{A}_t$ presents some inconvenience, but we recall that in the setting of type II_1 factors, by von Neumann theory [24], the algebra of unbounded (affiliated) operators is a well behaved algebra (with respect composition, sum and the adjoint operations).

In fact we will prove that $\mathcal{L}$ comes with two summands

$$\mathcal{L}(k) = \Lambda(k) - 1/2\{T, k\}$$

where $-T$ is positive affiliated with $\mathcal{A}_t$ and $-\Lambda$ a completely positive (unbounded) map. In the next chapter we prove that T is $\Lambda(1)$.

For technical reasons (to have an algebra domain for $\mathcal{L}$), we require that $k \in \widehat{\mathcal{A}_s}$, $s < t-2$, since we know (by [27]) that the space of operators in $\mathcal{A}_t$, represented by such kernels, is closed under taking the $*_t$ multiplication (the multiplication in $\mathcal{A}_t$).

The operator Λ will be (up to an additive multiple of the identity), multiplication of the symbol by $\ln\varphi$. This operation is made more precise in 3.6.

If k is already of the form $S_{\Delta^{\varepsilon_0}} k S^*_{\Delta^{\varepsilon_0}}$, for some k in $\mathcal{A}_{s-\varepsilon_0}$, $s - \varepsilon_0 > 1$, $s < t-2$, then $\Lambda(k)$ is an operator in $L^1(\mathcal{A}_t)$. In order to have the identity I in the domain) we allow Λ to take its values in the operators affiliated with $\mathcal{A}_t$.

Consequently $\Lambda(1)$ is just positive operator, affiliated with $\mathcal{A}_t$, which corresponds to the symbol $\ln\varphi = \ln(\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12})$ plus a suitable multiple of the identity.

To deduce the expression for $\mathcal{C}_t(k,l)$ one could argue formally as follows:

$$\begin{aligned}\mathcal{C}_t(k,l)(\overline{z},\xi) = {} & \frac{c_t'}{c_t}(k *_t l)(\overline{z},\xi) \\ & + c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{z},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t \ln[\overline{z},\eta,\overline{\eta},\xi] \mathrm{d}\nu_0(\eta).\end{aligned} \tag{7.1}$$

At this point to get a Γ-invariant expression, we should decompose $\ln[\overline{z},\eta,\overline{\eta},\xi] = \ln\left[((\overline{z}-\xi)(\overline{\eta}-\eta))/((\overline{z}-\eta)(\overline{\eta}-\xi))\right]$ as a sum of Γ-invariant functions. The easier way to do that would be to write

$$\ln\varphi[\overline{z},\eta,\overline{\eta},\xi]=\frac{1}{12}[\ln\varphi(\overline{z},\xi)+\ln\varphi(\overline{\eta},\eta)-\ln\varphi(\overline{\eta},\xi)-\ln\varphi(\overline{z},\eta)].$$

If we use this expression back in (7.1) we would get four terms which are described as follows.

The term corresponding to $\ln\varphi(\overline{z},\xi)$ will come in front of the integral and give

$$\frac{1}{12}\ln\varphi(\overline{z},\xi)(k*_t l)(\overline{z},\xi).$$

The term corresponding to $\ln\varphi(\overline{z},\eta)$ would multiply $k(\overline{z},\eta)$ and would correspond formally to $\frac{1}{12}[(\ln\varphi)k]*_t l$.

The term corresponding to $\ln\varphi(\overline{\eta},\eta)$ would give the following integral:

$$c_t\int_{\mathbb{H}}k(\overline{z},\eta)(\ln\varphi(\overline{\eta},\eta))l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t\mathrm{d}\nu_0(\eta).$$

This is formally $\dfrac{1}{12}k*_t T^t_{\ln\varphi}*_t l$. If $\ln\varphi$ were a bounded function and $T^t_{\ln\varphi}$ the Toeplitz operator with this symbol, this expression would make perfect sense.

Putting this together we would get

$$\begin{aligned}k*'_t l=\mathcal{C}_t(k,l)=\frac{c'_t}{c_t}k*_t l+\frac{1}{12}\ln\varphi(k*_t l)-\left[\left(\frac{1}{12}\ln\varphi\right)k\right]*_t l\\ -k*_t\left[\left(\frac{1}{12}\ln\varphi\right)l\right]+k*_t T^t_{1/12\ln\varphi}*_t l.\end{aligned}$$

This would give that $\mathcal{C}_t(k,l)$ is implemented by the operator

$$\mathcal{L}(k)=\left(\frac{1}{12}\ln\varphi-\frac{c'_t}{c_t}\right)k-\frac{1}{2}\left\{T_{(1/12)\ln\varphi},k\right\}$$

where, by $\{a,b\}$, we denote the Jordan product $ab+ba$.

This means that $\mathcal{C}_t(k,l)$ is implemented by the operator $\mathcal{L}(k)$, which resembles to the canonical form of a generator of a dynamical semigroup: a positive map ($-\ln\varphi$ is positive kernel, when adding a constant) minus a Jordan product.

To justify such a formula and the convergence of the integrals involved seems to be a difficult task, so we will follow a somehow different but more rigorous approach, which consists into defining the operator $(-\ln\varphi)k$, as in the previous section, as a strong operator topology derivative.

To that end we introduce a family of completely positive maps that canonically connect the fibers of the deformation. These maps arise from automorphic forms, viewed (as in [18]) as intertwining operators.

In the next lemma we give a precise meaning for the operator $T^t_{\ln\varphi}$, which is the (unbounded) Toeplitz operator acting on H_t with symbol $\ln\varphi$.

LEMMA 4.1. *We define $T = T^t_{\ln\varphi}$, as a quadratic form, by*

$$\langle T^t_{\ln\varphi}\xi, \xi\rangle_{H_t} = \int_{\mathbb{H}} (\ln\varphi)|\xi|^2\, d\nu_t,$$

on the domain

$$\mathcal{D} = \left\{\xi \in H_t \mid \int (\ln\varphi)|\xi|^2\, d\nu_t < \infty\right\}.$$

Clearly $\mathcal{D}$ is dense in H_t as it contains $\mathcal{D}_0 = \bigcup\limits_{\varepsilon>0} \mathrm{Range}\, S_{\Delta^\varepsilon}$, where S_{Δ^ε} is viewed as the operator of multiplication by Δ^ε from $H_{t-\varepsilon}$ into H_t.

Moreover $T^t_{\ln\varphi}$ is the restriction to H_t of the multiplication operator by $\ln\varphi$ on $L^2(\mathbb{H}, \nu_t)$. For ξ, η in $\mathcal{D}_0$ we have that

$$\langle T^t_{\ln\varphi}\xi, \eta\rangle_{H_t} = \frac{d}{d\varepsilon}\, \langle T^t_{\varphi^\varepsilon}\xi, \eta\rangle_{H_t}\Big|_{\varepsilon=0}.$$

Proof. All what stated above is obvious: the last statement is justified because, if $S_{\Delta^{\varepsilon_0}} : H_{t-12\varepsilon_0} \to H_t$, then $S^*_{\Delta^{\varepsilon_0}} T^t_{\ln\varphi} S_{\Delta^{\varepsilon_0}}$ is obviously equal to $T^{t-\varepsilon_0}_{\ln\varphi\varphi^{\varepsilon_0}}$.

In the next lemma we explain the role of automorphic form as comparison operators between different algebras $\mathcal{A}_t$ (it is a sort of tool for making a differentiable field out of the algebras $\mathcal{A}_t$).

DEFINITION 7.2. *For $s \geq t$, let $\theta_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ be the completely positive map associating to k in $\mathcal{A}_t$ the bounded operator in $\mathcal{A}_s$ defined as*

$$\theta_{s,t}(k) = (S_{\Delta^{(s-t)/12}})\, k\, (S_{\Delta^{(s-t)/12}})^*$$

Clearly the symbol of $\theta_{s,t}(k)$ *is*

$$\frac{c_t}{c_s} k(\overline{z}, \xi)(\varphi(\overline{z}, \xi))^{(s-t)/12}.$$

Also we have $\theta_{s,t}(\theta_{t,v}(k)) = \theta_{s,v}(k)$ *for all* $s \geq t \geq v$.

The following property is a trivial consequence of the definition of $\theta_{s,t}$. It expresses the fact that $\theta_{s,t}$ has an almost multiplicative structure, as follows.

LEMMA 7.3. *For* $s \geq t$ *the following holds for all* k, l *in* $\mathcal{A}_t$*:*

$$\theta_{s,t}\left(k *_t T^t_{\varphi^{(s-t)/12}} *_t l\right) = \theta_{s,t}(k) *_s \theta_{s,t}(l).$$

Proof. This is obvious since $\theta_{s,t}(k)\theta_{s,t}(l)$ (with product in $\mathcal{A}_s$) is equal to

$$S_{\Delta^{(s-t)/12}} k S^*_{\Delta^{(s-t)/12}} S_{\Delta^{(s-t)/12}} l S^*_{\Delta^{(s-t)/12}}.$$

But an obvious formula (see, e.g., [27]) shows that $S^*_{\Delta^{(s-t)/12}} S_{\Delta^{(s-t)/12}}$ is equal to $T^t_{\varphi^{(s-t)/12}}$.

We intend next to differentiate the above formula, in s, by keeping t-fixed. In order to do this we will need to differentiate $\theta_{s,t}(k)$. One problem that arrises, is the fact that *a priori* $\theta_{s,t}(k)$ belongs rather to $\mathcal{A}_s$ than $\mathcal{A}_t$. But if k belongs to some $\mathcal{A}_{t_0}$, with $t_0 < t$, and s is sufficiently closed to s, then $\theta_{s,t}(k)$ will be (up to a multiplicative constant) represented by the symbol of $\theta_{s+t-t_0,t_0}(k)$. Since s was small, this defines (via $\Psi_{t,t-t_0+s}$) a bounded operator in $\mathcal{A}_t$. Thus for such k it makes sense to define $\langle \theta_{s,t}(k)\xi, \eta\rangle_{H_t}$ for all vectors ξ, η in H_t.

We derivate this expression after s. The existence of the derivative, in the strong operator topology, was already done in the previous chapter. We reformulate Corollary 6.6, in the new setting.

LEMMA 7.4. *Let* $t > 3$ *and let* k *belong to* $\mathcal{A}_{s_0}$, *where* $1 < s_0 < t - 2$. *Then there exists a dense domain* $\mathcal{D}_0$ *(eventually depending on* k*), that is affiliated with* $\mathcal{A}_t$ *such that the following expression:*

$$\langle X_t(k)\xi, \eta\rangle_{H_t} = \left.\frac{d}{ds}\right|_{s=t} \langle \theta_{s,t}(k)\xi, \eta\rangle_{H_t}$$

defines a linear operator X_t *on* $\mathcal{D}$*, that is affiliated (and hence closable) with* $\mathcal{A}_t$*.*

Moreover, for a sufficiently large constant C*, (depending on* s, t*)* $-X_t + C \cdot \mathrm{Id}$ *becomes a completely positive map with values in the operators affiliated to* $\mathcal{A}_t$*.*

Consider the (non-unital) subalgebra $\widetilde{\mathcal{A}}_{s_0} \subseteq \mathcal{A}_{s_0}$*, which is also weakly dense, consisting of all operators* k *in* $\mathcal{A}_{s_0}$ *that are of the form* $S_{\Delta^{\varepsilon_0}} k S^*_{\Delta^{\varepsilon_0}}$ *(where* $S_{\Delta^{\varepsilon_0}}$ *maps* $H_{s_0-12\varepsilon_0}$ *into* H_s*),* k *belongs to* $\mathcal{A}_{s_0-12\varepsilon_0}$ *and* $s_0 - 12\varepsilon_0$ *is assumed bigger than 1.*

Then X_t *also maps* $\widetilde{\mathcal{A}}_s$ *into* $L^1(\mathcal{A}_t)$*. For such a* k *the limit in the definition of* X_t *is the strong operator topology on a dense, affiliated domain.*

Before going into the proof we make the following remark. (which is not required for the proof).

Remark. *Since the kernel of the operator* $\theta_{s,t}(k)$ *(in* $\mathcal{A}_s$*) is equal to*

$$\frac{c_t}{c_s} \cdot k(\overline{z}, \xi)[\varphi(\overline{z}, \xi)]^{(s-t)/12}$$

it follows that $X_t(k)$ *is associated (in a sense that doesn't have to be made precise for the proof) to*

$$\left(-\frac{c_t}{c_t'} + \frac{1}{12}\ln\varphi\right) k$$

which appeared in the formula in the introduction.

Proof of Lemma 4.4. Because of the form of the symbol we may use the Corollary 6.6.

The main result of our paper shows that, by accepting an unbounded coboundary, the 2-Hochschild cocycle appearing in the Berezin's deformation, is trivial, and the coboundary (which is automatically dissipative) has a form very similar to the canonical expression of a generator of a quantum dynamical semigroup.

First we deduce a direct consequence out of the formula in Lemma 7.3.

Proposition 7.5. *Fix a number* $t > 3$*. Consider the algebra* $\widetilde{\mathcal{A}}_t \subseteq \mathcal{A}_t$ *consisting of all* $k \in \mathcal{A}_s$ *for some* $s < t - 2$ *that are of the form* $S_{\Delta^{\varepsilon_0}} k_1 S^*_{\Delta^{\varepsilon_0}}$*,* *for some* ε_0*, (such that* $s-\varepsilon_0>1$*) and* $k_1\in\mathcal{A}_{s_0-\varepsilon}$*. Let* X_t *be the operator defined in the previous lemma. Then*

$$\left.\frac{d}{ds}\right|_{s=t}\theta_{s,t}\left(k*_t T^t_{\varphi(s-t)/12}*_t l\right)=X_t(k*_t l)+k*_t T^t_{\frac{1}{12}\ln\varphi}*_t l, \tag{7.2}$$

$$\left.\frac{d}{ds}\right|_{s=t}[\theta_{s,t}(k)*_s\theta_{s,t}(l)]=X_t(k)*_t l+\mathcal{C}_t(k,l)+k*_t X_t(l), \tag{7.3}$$

for all $k,l\in\widetilde{\mathcal{A}}_t$*. Consequently the two terms on the right hand side of* (7.2) *and* (7.3) *are equal, that is*

$$X_t(k*_t l)+k*_t T^t_{(1/12)\ln\varphi}*_t l=X_t(k)*_t l+\mathcal{C}_t(k,l)+k*_t X_t(l).$$

Before proceeding to the the proof of Proposition 7.5, we note that $\widetilde{\mathcal{A}}_t$ is indeed an algebra (see also the end of this chapter). Assume k,l are given, but that they correspond to two different choices of s, say s,s', with $s'<s$. Because $\Psi_{s,s'}$ maps $\mathcal{A}_{s'}$ into $\mathcal{A}_s$, we can assume $s=s'$. Then when $k=S_{\Delta^{\varepsilon_0}}k_1S^*_{\Delta^{\varepsilon_0}}$, $l=S'_{\Delta^{\varepsilon'_0}}l_1S^*_{\Delta^{\varepsilon'_0}}$, and say $\varepsilon'_0>\varepsilon_0$. Then we replace the expression of l as $S_{\Delta^{\varepsilon_0}}\left[S_{\Delta^{\varepsilon'_0-\varepsilon_0}}l_1\left(S_{\Delta^{\varepsilon'_0-\varepsilon_0}}\right)^*\right]S^*_{\Delta^{\varepsilon_0}}$, and choose the new l_1 to be $S_{\Delta^{\varepsilon'_0-\varepsilon_0}}l_1\left(S_{\Delta^{\varepsilon'_0-\varepsilon_0}}\right)^*$.

Proof of Proposition 7.5. We will give separate proofs for each of the equalities (7.2), (7.3). Of course these are the product formula for derivatives, but the complicated nature of the operator functions, obliges us to work on the nonunital algebra $\widetilde{\mathcal{A}}_t$. This might be just a technical condition, that perhaps could be dropped.

Proof of equality (7.2).

$$\frac{\mathrm{d}}{\mathrm{d}s}\theta_{s,t}\left(k*_t T^t_{\varphi^{(s-t)/12}}*_t l\right)=X_t(k*_t l)+k*_t T^t_{(1/12)\ln\varphi^*}*_t l$$

We start with left hand side: Denote $P_s=k*_t T^t_{\varphi(s-t)/12}*_t l$, for fixed $h,l\in\widetilde{\mathcal{A}}_t$.

We have to evaluate (against $\langle\cdot,\xi\rangle\eta$, where ξ,η belong to a suitable dense domain $\mathcal{D}$ affiliated with $\mathcal{A}_t$), the expression:

$$\frac{\theta_{s,t}(P_s)-P_t}{s-t}=\theta_{s,t}\left(\frac{P_s-P_t}{s-t}\right)+\frac{\theta_{s,t}(P_t)-P_t}{s-t}.$$

The second term converges when $s \searrow t$, since $P_t = k *_t l$, to $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \theta_{s,t}(P_t)$ which is by definition $X_t(k *_t l)$. Here we rely on the fact that $\widetilde{\mathcal{A}}_t$ is an algebra, so that $k *_t l$ belongs to the domain of X_t.

For the limit of the expression $\theta_{s,t}\left(\frac{P_s - P}{s-t}\right)$, we note that

$$\frac{P_s - P_t}{s-t} = k *_t T^l_{h_{s,t}} *_t l,$$

with

$$h_{s,t} = \frac{\varphi^{(s-t)/12} - \mathrm{Id}}{s-t}.$$

Assume now that $k = S_{\Delta^{\varepsilon_0}} k_1 S^*_{\Delta^{\varepsilon_0}}$, $l = S_{\Delta^{\varepsilon_0}} l_1 S^*_{\Delta^{\varepsilon_0}}$.

Then $\theta_{s,t}(k *_t T^t_{h_{s,t}} *_t l)$ is equal to

$$S_{\Delta^{(s-t)/12}} \Big((S_{\Delta^{\varepsilon_0}} k_1 S^*_{\Delta^{\varepsilon_0}}) *_t T^t_{h_{s,t}} *_t (S_{\Delta^{\varepsilon_0}} l_1 S^*_{\Delta^{\varepsilon_0}}) \Big) S^*_{\Delta^{(s-t)/12}}.$$

This is easily seen to be equal to

$$S_{\Delta^{((s-t)/12)+\varepsilon_0}} \left[k_1 *_{t-\varepsilon_0} T^t_{\varphi^{\varepsilon_0} h_{s,t}} *_{t-\varepsilon} l_1 \right] S^*_{\Delta^{((s-t)/12)+\varepsilon_0}} =$$

$$= \theta_{s,t-\varepsilon_0}\Big(k_1 *_{t-\varepsilon_0} T^t_{\varphi^{\varepsilon_0} h_{s,t}} *_{t-\varepsilon_0} l_1 \Big).$$

Denote $\widetilde{P}_s = k_1 *_{t-\varepsilon_0} T^t_{\varphi^{\varepsilon_0} h_{s,t}} *_{t-\varepsilon_0} l_1$.

As s decreases to t, we have (as $\varphi^{\varepsilon_0} \ln \varphi$ is bounded) that $\widetilde{P}_s$ converges in the uniform operator topology, to $k_1 *_{t-\varepsilon_0} T^t_{1/12\varphi^{\varepsilon_0} \ln \varphi} *_{t-\varepsilon_0} l_1$. This is because $\varphi^{\varepsilon_0}(\varphi^{(s-t)/12} - \mathrm{Id})/(s-t)$ converges uniformly to $\frac{1}{12}\varphi^{\varepsilon_0} \ln \varphi$; since φ is a bounded function.

Also if s is sufficiently closed to t, $\theta_{s,t-\varepsilon}(k_1)$ defines for every $k_1 \in \mathcal{A}_{t-\varepsilon_0}$ a bounded operator on $\mathcal{A}_t$. Indeed $\theta_{s,t-\varepsilon_0}(k_1)$ has symbol (up to a multiplicative constant) equal to $\varphi^{(s-t+\varepsilon_0/12} k_1$. This is well defined as the kernel of an operator in $\mathcal{A}_t$, since $k_1 \in \mathcal{A}_{t_0-\varepsilon_0}$,

Thus $\theta_{s,t_0-\varepsilon}$ can be taught of as a completely positive map from $\mathcal{A}_{t_0-\varepsilon}$ into $\mathcal{A}_t$. Moreover, $\theta_{s,t-\varepsilon_0}(1)$, which is $S_{\Delta^{(s-t_0+\varepsilon)/12}} S^*_{\Delta(s-t_0+\varepsilon)/12}$, is less than a constant C (not depending on s) times the identity.

Hence the linear maps $\theta_{s,t_0-\varepsilon}$, acting from $\mathcal{A}_{t_0-\varepsilon}$ into $\mathcal{A}_t$, are uniformly bounded. Consequently, when evaluating

$$\left|\left\langle \left(\theta_{s,t_0-\varepsilon}(\widetilde{P}_s) - \theta_{t,t_0-\varepsilon}(\widetilde{P}_t)\right) \xi, \eta \right\rangle\right|$$

we can majorize by

$$\left|\langle \theta_{s,t_0-\varepsilon}(\widetilde{P}_s - \widetilde{P}_t)\xi, \eta\rangle\right| + \left|\langle (\theta_{t,t_0-\varepsilon} - \theta_{s,t_0-\varepsilon})\,(\widetilde{P}_t)\xi, \eta\rangle\right|.$$

The first term goes to zero by uniform continuity of the $\theta_{s,t_0+\varepsilon}$ after s, (and since $\|\widetilde{P}_s - \widetilde{P}_t\| \to 0$). The second goes to zero because of pointwise strong operator topology continuity of the map $s \to \theta_{s,t_0-\varepsilon}$.

Thus $\theta_{s,t}\left((P_s - P_t)/(s-t)\right)$ converges to $\theta_{t,t_0-\varepsilon}(\widetilde{P}_t)$ which was

$$S_{\Delta^{\varepsilon_0}}\left(k_1 *_{t-\varepsilon_0} T^t_{(1/12)\varphi^{\varepsilon_0}\ln\varphi} * l_1\right) S^*_{\Delta^{\varepsilon_0}}$$

which is equal to $k *_t T^t_{\ln\varphi} *_t l$.

Proof of the equality (7.3).

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \theta_{s,t}(k) *_s \theta_{s,t}(l) = X_t k *_t l + \mathcal{C}_t(k,l) + k *_t X_t.$$

We verify this equality by evaluating it on $\langle\cdot,\xi\rangle\eta$, $\xi,\eta \in \mathcal{D}$, where $\mathcal{D}$ is a dense domain, affiliated to $\mathcal{A}_t$.

We write the expression

$$\frac{\theta_{s,t}(k) *_s \theta_{s,t}(l) - k *_t l}{s-t}$$

as

$$\frac{\theta_{s,t}(k) *_s \theta_{s,t}(l) - \theta_{s,t}(k) *_t \theta_{s,t}(l)}{s-t} + \frac{\theta_{s,t}(k) *_t \theta_{s,t}(l) - k *_t l}{s-t}.$$

We will analyze first the first summand and prove that

$$\frac{\theta_{s,t}(k) *_s \theta_{s,t}(l) - \theta_{s,t}(k) *_t \theta_{s,t}(l)}{s-t} \tag{7.4}$$

converges to $\mathcal{C}_t(k *_t l)$.

We use the symbols of k, l, and then the symbols of $\theta_{s,t}(k)$, $\theta_{s,t}(l)$ are $\varphi^{(s-t)/12}k$, $\varphi^{(s-t)/12}l$, up to multiplicative constants, that we ignore here (because the argument has a qualitative nature).

Then the symbol of the expression in (7.4) is

$$\int_{\mathbb{H}} \left(k\varphi^{(s-t)/12}\right)(\overline{z},\eta)\left(l\varphi^{(s-t)/12}\right)(\overline{\eta},\xi)\frac{[\overline{z},\eta,\overline{\eta},\xi]^s-[\overline{z},\eta,\overline{\eta},\xi]^t}{s-t}\mathrm{d}\nu_0(\eta). \quad (7.5)$$

By the mean value theorem, with $\alpha_s(v) = sv + (1-v)t$, the expression becomes

$$\int_0^1\int_{\mathbb{H}} \left(k\varphi^{(s-t)/12}\right)(\overline{z},\eta)\left(l\varphi^{(s-t)/12}\right)(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^{\alpha_s(v)}\ln[\overline{z},\eta,\overline{\eta},\xi]\mathrm{d}\nu_0(\eta)\mathrm{d}v.$$

Similarly (by ignoring the numerical factors due to the constants c_s) we have that

$$\mathcal{C}_t(\theta_{s,t}(k),\theta_{s,t}(l))$$

contains the integral

$$\int_{\mathbb{H}} \left(k\varphi^{(s-t)/12}\right)(\overline{z},\eta)\left(l\varphi^{(s-t)/12}\right)(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t\ln[\overline{z},\eta,\overline{\eta},\xi]\mathrm{d}\nu_0(\eta). \quad (7.6)$$

Taking the difference, we obtain the following integral:

$$\int_{\mathbb{H}} \left(k\varphi^{(s-t)/12}\right)(\overline{z},\eta)\left(l\varphi^{(s-t)/12}\right)(\overline{\eta},\xi)\cdot$$
$$\cdot\left(\frac{[\overline{z},\eta,\overline{\eta},\xi]^s-[\overline{z},\eta,\overline{\eta},\xi]^t}{s-t}-[\overline{z},\eta,\overline{\eta},\xi]^t\ln[\overline{z},\eta,\overline{\eta},\xi]\right)\mathrm{d}\nu_0(\eta). \quad (7.7)$$

By Taylor expansion, this is $(s-t)$ times a term involving the integral:

$$\int_{\mathbb{H}} \left(k\varphi^{(s-t)/12}\right)(\overline{z},\eta)\left(l\varphi^{(s-t)/12}\right)(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^{s'}\left(\ln[\overline{z},\eta,\overline{\eta},\xi]\right)^2\mathrm{d}\nu_0(\eta) \quad (7.8)$$

where s' is the interval determined by s and t.

We have to prove that the integral of the absolute values of the integrands in the above integral, are bonded by a constant independent on the choices of s' (and s).

We write $|[\overline{z},\eta,\overline{\eta},\xi]| = d(\overline{z},\eta)d(\overline{\eta},\xi)/d(\overline{z},\xi)$. Then

$$|\ln|[\overline{z},\eta,\overline{\eta},\xi]|| \leq |\ln|d(\overline{z},\eta)|| + |\ln|d(\overline{\eta},\xi)|| + |\ln|d(\overline{z},\xi)||.$$

Also we note that the logarithm in $\ln[\overline{z},\eta,\overline{\eta},\xi]$ has bounded imaginary part, as the branches in

$$\ln[(\overline{z}-\eta)/(-2\mathrm{i})],\ \ln[(\overline{\eta}-\xi)/(-2\mathrm{i})],\ \ln[(\overline{z}-\xi)/(-2\mathrm{i})]$$

have imaginary part in the fixed segment $[0,2\pi]$.

Thus the term that we have to evaluate will involve terms of the form

$$\int\limits_H |k'(\overline{z},\eta)||l'(\overline{\eta},\xi)||\mathrm{d}(\overline{z},\eta)|^{s'}|\mathrm{d}(\overline{\eta},\xi)|^{s}\mathrm{d}\nu_0(\eta)$$

where $k'(\overline{z},\eta)$ could be $k\varphi^{\frac{s-t}{12}}(\overline{z},\eta)$, eventually multiplied by a power (1 on 2) of $\ln d(\overline{z},\eta)$ A similar assumption holds for l.

By the Cauchy Schwarz inequality, this expression is bounded by products of:

$$\left(\int |k'(\overline{z},\eta)|^2|d(\overline{z},\eta)|^{2s}|\mathrm{d}(s)\right)^{1/2}\left(\int |l'(\overline{z},\xi)|^2|d(\overline{\eta},\xi)|^{2s}\mathrm{d}\nu_0\right)^{1/2}. \qquad (7.9)$$

But such expressions are finite, because we know that k,l are in $\mathcal{A}_{t_0}$ for some fixed $t_0<t$, and hence in $L^2(\mathcal{A}_{t_0})$ and consequently the integral

$$\int\limits_H |k(\overline{z},\eta)|^2|d(\overline{z},\eta)|^{2t_0}\mathrm{d}\nu_0(\eta) \qquad (7.10)$$

is finite.

Moreover $\varphi(\overline{z},\xi)=\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}$. This term is bounded in absolute value by $|d(\overline{z},\xi)|^{-12}$, so that $|\varphi(\overline{z},\xi)|^{(s-t)/12}$ is bounded by $|d(\overline{z},\xi)|^{s-t}$ Also $|\ln|d(\overline{z},\eta)|||d(\overline{z},\eta)|^{\varepsilon}$ is bounded for a any choice of ε.

Thus, by choosing ε small enough, the finiteness of the integral in 7.10 implies the finiteness of the integral in 7.9

Thus the integral in 7.7 tends to zero. This completes the proof that

$$\frac{\theta_{s,t}(k)x_s\theta_{s,t}(l)-\theta_{s,t}(k)x_t\theta_{s,t}(l)}{s-t}$$

converges to $\mathcal{C}_t(k,l)$

The remaining term, to be analyzed, is

$$\frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=t} \frac{\theta_{s,t}(k) *_t \theta_{s,t}(l) - k *_t l}{s-t}.$$

We have to show that the limit is $(X_t(k)) *_t l + k *_t (X_t(l))$ (evaluated on vectors ξ, η in a dense domain affiliated to $\mathcal{A}_t$).

Fix the vector ξ, η. Then we have to analyze the following sum

$$\left\langle \frac{\theta_{s,t}(l) - l}{s-t} \xi, \theta_{s,t}(k^*)\eta \right\rangle + \left\langle \frac{\theta_{s,t}(k) - k}{s-t} l\xi, \eta \right\rangle$$

The second term obviously converges to $< [X_t(k) *_t] l\xi, \eta >$ and the first term, is also convergent to $< X_t(l)\xi k^*, \eta >$, because $((\theta_{s,t}(l) - l)/(s-t))\xi$, for ξ in a dense domain $\mathcal{D}$ converges in norm to $X_t(l)\xi$. Indeed in Corollary 6.6 we proved that $(\theta_{s,t}(l) - l)/(s-t)$ converges strongly to X_t, on a dense domain topology, because the convergence (for $l = S_{\Delta^{\varepsilon_0}} l_1 (S^*_{\Delta^{\varepsilon_0}})$, $l_1 \in \mathcal{A}_{t_0 - \varepsilon_0}$) comes by proving that the partial fractions $-(\theta_{s,t}(l) - l)/(s-t)$ increase (modulo$(s-t)$ times a constant) to $-X_t l$.

This completes the proof.

We are now able to formulate of our main result. We recall first the context of this result. The algebras $\mathcal{A}_t$ are the von Neumann algebras (type II_1 factors) associated with the Berezin's deformation of $\mathbb{H}/\mathrm{PSL}(2,\mathbb{Z})$. These algebras can be realized as subalgebras of $B(H_t)$ where H_t is the Hilbert space $H^2(\mathbb{H},\ (\mathrm{Im} z^{t-2}\ \mathrm{d}\overline{z}\mathrm{d}z)$.

As such, every operator A in $\mathcal{A}_t$ (or $B(H_t)$) is given by a reproducing kernel: k_A, which is a bivariable function on $\mathbb{H}$, analytic in the second variable, antianalytic in the first and $\mathrm{PSL}(2,\mathbb{Z})$-invariant. The symbols are normalized so that the symbol of the identity is the constant function 1.

By using these symbols (that represent the deformation) we can define the $*_t$ product of two symbols k, l by letting $k *_t l$ be the product symbol, in the algebra $\mathcal{A}_t$.

The 2- Hochschild cocycle associated with the deformation is defined by

$$\mathcal{C}_t(k,l) = \frac{\mathrm{d}}{\mathrm{d}s}(k *_s l)\bigg|_{s=t}.$$

The Hochschild cocycle condition is obtained by differentiation of the associativity identity.

The cocycle $\mathcal{C}_t$ is well defined on a weakly dense, unital subalgebra $\widehat{\mathcal{A}_t}$ of $\mathcal{A}_t$. A sufficient condition that an element in $\mathcal{A}_t$, represented by a symbol k, belongs to $\widehat{\mathcal{A}}_t$, is that the quantity $\|k\|_{\widehat{t}}$, defined as the maximum of

$$\sup_{z\in H}\int |k(\overline{z},\eta)||d(\overline{z},\eta)|^t \mathrm{d}\nu_0(\eta)$$

and

$$\sup_{\eta\in H}\int |k(\overline{z},\eta)||d(\overline{z},\eta)|^t \mathrm{d}\nu_0(z),$$

be finite.

The algebra $\widehat{\mathcal{A}_t}$ is the analogue of Jolissaint algebra [19] for discrete groups.

We proved in Section 5 that the applications $\Psi_{s,t}$ which map the operator A in $\mathcal{A}_t$ into the corresponding operator in $\mathcal{A}_s$, having the same symbol, are completely positive.

This property proves that $\mathcal{C}_t$ is completely negative, that is for all $l_1, l_2 \dots l_N$ in $\mathcal{A}_t$, for all $k_1, k_2 \dots k_N$ in $\widehat{\mathcal{A}}_t$, we have that

$$\sum l_i^* c(k_i^*, k_j) l_j \leq 0$$

This property could be used to construct, as in [31], the cotangent bundle. In fact, here $\mathcal{C}_t$, or rather $-\mathcal{C}_t$, plays the role of ∇L, where L should be a generator of a quantum dynamical semigroup Φ_t, (thus $L = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s\Big|_{s=0}$) and we have $\nabla L(a,b) = L(a,b) - aL(b) - L(a)b$.

It is well know that ∇L is completely negative ([22]). In our case, the role of the quantum dynamical semigroup is played by the completely positive maps $\Psi_{s,t}$ that have the property $\Psi_{s,t}\Psi_{t,v} = \Psi_{s,v}$, $s \geq t \geq v$. The generator L doesn't make sense here, since $\Psi_{s,t}$ takes its values in different algebras, depending on s.

Instead we use the derivative of the multiplication operation, which formally is

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi_{s,t}^{-1}(\Psi_{s,t}(k) *_s \Psi_{s,t}(l))\Big|_{s=t}.$$

as a substitute for ∇L.

All the above is valid for the general Berezin's deformation of H/Γ, where Γ is any discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$, of finite covolume.

When specializing to $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ we construct also the diffusive operator $\mathcal{L}$, which plays the role of the generator of a dynamical semigroup.

In the next theorem we formulate our main result. We construct explicitly an operator $\mathcal{L}$ such that

$$\mathcal{L}(ab) - \mathcal{L}(a)b - a\mathcal{L}(b) = \mathcal{C}_t(a,b).$$

We will show that $\mathcal{L}$ is well defined on a weakly dense (non-unital) subalgebra $\mathcal{D}_t^0$ and the above relation holds for $a,b \in \mathcal{D}_t^0$. (which is obtained by considering suitable subalgebras of $\widehat{\mathcal{A}_s}, s < t-2$). Moreover $\mathcal{L}$ has an expression that is very similar to the Lindblad ([22],[8],[16], [21]) form of the generator L of a uniformly continuos semigroup. Recall that this expression is in the uniform continuous case

$$L(x) = \Phi(x) - \frac{1}{2}\{\Phi(1), x\} + i[H,x],$$

where Φ is completely positive and H is selfadjoint.

In our case (which is certainly not ([14]) corresponding to the uniformly continuous case) the generator $\mathcal{L}(x)$ is defined rather as a an unbounded operator (which is the approach taken in ([14], [7], [20], [17], [23]).

We prove that there exists a weakly dense, *unital* algebra $\mathcal{D}_t$ containing $\mathcal{D}_t^0$, and a linear map Λ from $\mathcal{D}_t$ into the operators affiliated with $\mathcal{A}_t$, and a positive operator that is also a affiliated to $\mathcal{A}_t$, such that

$$\mathcal{L}(x) = \Lambda(x) - \frac{1}{2}\{T, x\}.$$

Also Λ maps $\mathcal{D}_t^0$ into $L^1(\mathcal{A}_t)$

Moreover Λ has properties that are very similar to a completely positive map. We prove that there exists an increasing filtration $(\mathcal{B}_{rt})_{1<r<t-2}$ of $\mathcal{D}_t$, consisting of weakly dense subalgebras, such that, for a constant C_{rt}^0 depending on r, $-[\Lambda + C_{rt}^0{\cdot}\mathrm{Id}]$ is a completely positive map on $\mathcal{B}_{rt}$

This means that when restricted to $\mathcal{B}_{rt}$, $\mathcal{L}$ has the form $\mathcal{L}(x) = \Lambda'(x) - (1/2)\{T', x\}$, where $-\Lambda' = -[\Lambda + C_r^0\ \mathrm{Id}]$ is a completely positive map and $T = T + C_{rt}^0 \cdot \mathrm{Id}$.

Theorem 7.6. *Let $\mathcal{A}_t$, $t > 1$, with product operation $*_t$ be the von Neumann algebra (a type II_1 factor) associated with the Berezin's deformation of $\mathbb{H}/PSL(2,\mathbb{Z})$.*

Let $\mathcal{C}_t$ be the 2- Hochschild cocycle associated with the deformation

$$\mathcal{C}_t(k,l) = \frac{d}{ds} k *_s l \bigg|_{s=t},$$

which is defined on the weakly dense subalgebra $\widehat{\mathcal{A}}_t$.

Then there exists a weakly dense (non-unital) subalgebra $\mathcal{D}_t^0$ in $\widehat{\mathcal{A}}_t \subseteq \mathcal{A}_t$ and $\mathcal{L}_t$, a linear operator on $\mathcal{D}_t^0$, with values in the algebra of operators affiliated with $\mathcal{A}_t$, such that

$$\mathcal{C}_t(k,l) = \mathcal{L}_t(kl) - k\mathcal{L}_t(l) - \mathcal{L}_t(k)l, \qquad k,l \in \mathcal{D}_t^0.$$

Note that $-\mathcal{L}_t$ is automatically completely dissipative.

Moreover $\mathcal{L}_t$ has the following expression. There exists a weakly dense, unital *subalgebra $\mathcal{D}_t$, such that $\mathcal{D}_t^0 \subseteq \mathcal{D}_t \subseteq \widehat{\mathcal{A}_t}$, there exists Λ_t defined on $\mathcal{D}_t$ with values in the operators affiliated to $\mathcal{A}_t$, and there there exists T, a positive unbounded operator, affiliated with $\mathcal{A}_t$ such that*

$$\mathcal{L}_t(k) = \Lambda_t(k) - \frac{1}{2}\{T, k\}, \quad k \in \mathcal{D}_t^0.$$

Moreover Λ_t has the following completely positivity properties

1) Λ_t maps $\mathcal{D}_t^0$ into $L^1(\mathcal{A}_t)$

2) There exists an increasing filtration of weakly dense, unital subalgebras $(\mathcal{B}_{s,t})_{1<s<t-2}$ of $\mathcal{D}_t$, with $\cup_s \mathcal{B}_{s,t} = \mathcal{D}_t$ and there exist constants $C_{s,t}$, such that $-[\lambda_t + C_{s,t} \cdot Id]$ is completely positive on $\mathcal{B}_{s,t}$.

Remark. *At the level of symbols the operator Λ_t has a very easy expression, namely $\Lambda_t(k)$ is the pointwise multiplication (the analogue of Schurr multiplication) of k with the Γ- eqivariant symbol*

$$\ln(\overline{\Delta(z)}\Delta(\xi)[(\overline{z} - \xi)/(-2i)]^{12}).$$

We identify as in Section 4, $L^2(\mathcal{A}_t)$ with a Hilbert space of Γ bivariable functions, analytic in the first variable, and antianalytic in the second. Then Λ corresponds to the (unbounded) analytic Toeplitz operator with symbol $\ln(\overline{\Delta(z)}\Delta(\xi)[(\overline{z} - \xi)/(-2\mathrm{i})]^{12})$.

Proof of Theorem 7.6. This was almost proved in Lemma 6.3 and Proposition 7.5, but we have to identify the ingredients. Here the algebra $\mathcal{D}_t^0$ is the union (with respect to s, ε_0)

$$\bigcup_{1<s-\varepsilon_0<s<t-2} S_{\Delta^{\varepsilon_0}} \mathcal{A}_{s-\varepsilon_0} S_{\Delta^{\varepsilon_0}}^*.$$

It is obvious that $\mathcal{D}_t^0$ is an algebra (under the product on $\mathcal{A}_t$. The algebra $\mathcal{D}_t$ is the union $\underset{1<s<t-2}{\cup} \mathcal{A}_t$, viewed as an algebra of $\mathcal{A}_t$). The algebra $\mathcal{B}_{s,t}$ is the union (after ε) of $\underset{1<s-\varepsilon_0}{\cup} S_{\Delta^{\varepsilon_0}} \mathcal{A}_{s-\varepsilon_0} S^*_{\Delta^{\varepsilon_0}}$.

The operator T is the Toeplitz operator with symbol $(1/12)\ln\varphi$, while Λ_t is X_t, where X_t was defined in Lemas 7.6 and 7.4. In proposition 7.5 we also proved that

$$\mathcal{C}_t(a,b) = X_t(a *_t b) - X_t(a) *_t b - a *_t X_t b + a *_t T^t_{(\ln\varphi)/12} *_t b, \text{for all } a, b \in \mathcal{D}_t^0.$$

Clearly the term $a *_t T^t_{(\ln\varphi)/12} *_t b$ is a cohomolgicaly trivial term, and hence $\mathcal{C}_t(a,b)$ is implemented by $\mathcal{L}_t(a) = X_t(a) - 1/2\{a, T^t_{\ln(\ln\varphi)/12}\}$. Hence $\mathcal{C}_t$ is implemented by $\mathcal{L}_t = \Lambda_t(a) - 1/2\{a, T^t_{(\ln\varphi)/12}\}$. All the other properties for Λ_t where proven in Section 6.

One also needs to show that the vector spaces $\mathcal{D}_t = \underset{s<t-2}{\cup} \widehat{\mathcal{A}_s}$ and

$$\mathcal{D}_t^0 = \underset{1<s-\varepsilon_0<s<t-2}{\cup} S_{\Delta^{\varepsilon_0}} \widehat{\mathcal{A}}_{s-12\varepsilon_0} S^*_{\Delta^{\varepsilon_0}}$$

are indeed algebras (in $\mathcal{A}_t$). $\mathcal{D}_t$ is obviously an algebra, since we proved ([27]) that $\widehat{\mathcal{A}_s}$ is closed under $*_v$ for all $v \geq s$. Of course, if we take the product of different $\widehat{\mathcal{A}}_{s_1}$ and $\widehat{\mathcal{A}}_{s_2}$ we may embedded them in $\widehat{\mathcal{A}}_{\max(s_1,s_2)}$.

To prove that $\mathcal{D}_t^0$ is an algebra (in $\mathcal{A}_t$) we will need to show first that we are reduced to proving that $S_{\Delta^{\varepsilon_0}} \widehat{\mathcal{A}}_{S-12\varepsilon_0} S^*_{\Delta^{\varepsilon_0}}$, for fixed s and ε_0 is closed under the product $*_t$ in $\mathcal{A}_t$.

Indeed if we do product for different s, we may simply take the maximum of $s,' s$. If we do a product corresponding to different $\varepsilon_0' s$, say ε_0 and ε_1, then we choose ε_1, to be the largest.

Then observe that for $k \in \mathcal{A}_{s-12\varepsilon_1}$

$$S_{\Delta^{\varepsilon_1}} k S^*_{\Delta^{\varepsilon_1}} = S_{\Delta^{\varepsilon_0}} (S_{\Delta^{\varepsilon_1-\varepsilon_0}} k S^*_{\Delta^{\varepsilon_1-\varepsilon_0}}) S^*_{\Delta^{\varepsilon_0}}.$$

Now $S_{\Delta^{\varepsilon_1-\varepsilon_0}} k S^*_{\Delta^{\varepsilon_1-\varepsilon_0}}$ has symbol equal to, (modulo a multiplicative constant) $\varphi^{\varepsilon_1-\varepsilon_0} k$. Since $|\varphi| \leq d^{-12}$, if follows $|\varphi|^{\varepsilon} \leq d^{-12\varepsilon}$ and hence that $\varphi^{\varepsilon_1-\varepsilon_0} k$ belongs to $\widehat{\mathcal{A}}_{s-12\varepsilon_1+12(\varepsilon_1-\varepsilon_0)}$ which is $\widehat{\mathcal{A}}_{s-12\varepsilon_0}$.

Thus $S_{\Delta^{\varepsilon_1}} \widehat{\mathcal{A}}_{s-12\varepsilon_1} S^*_{\Delta^{\varepsilon_1}}$ is contained in $S_{\Delta^{\varepsilon_0}} \widehat{\mathcal{A}}_{s-12\varepsilon_0} S^*_{\Delta^{\varepsilon_0}}$.

Now we are reduced to show that the product of two elements: $S_{\Delta^{\varepsilon_0}} k_1 S^*_{\Delta^{\varepsilon_0}}$ and $S_{\Delta^{\varepsilon_0}} l_1 S^*_{\Delta^{\varepsilon_0}}$, $k, l_1 \in \widehat{\mathcal{A}}_{s-12\varepsilon_0}$ is again an element in $S_{\Delta^{\varepsilon_0}} \widehat{\mathcal{A}}_{s-12\varepsilon_0} S^*_{\Delta^{\varepsilon_0}}$.

But

$$(S_{\Delta^{\varepsilon_0}})k_1 S^*_{\Delta^{\varepsilon_0}}) *_t (S_{\Delta^{\varepsilon_0}} l_1 S^*_{\Delta^{\varepsilon_0}})$$

coincides with

$$S_{\Delta^{\varepsilon_0}}[k_1 *_{t-12\varepsilon_0} S^*_{\Delta^{\varepsilon_0}} S_{\Delta^{\varepsilon_0}} *_{t-12\varepsilon_0} l_1] S_{\Delta^{\varepsilon_0}}$$

Because $\widehat{\mathcal{A}}_{s-12\varepsilon_0}$ is closed under the product $*_{t-12\varepsilon_0}$ it is sufficient to show that

$$T^{t-\varepsilon_0}_{\varphi^{\varepsilon_0}} = S^*_{\Delta^{\varepsilon_0}} S_{\Delta^{\varepsilon_0}}$$

belongs to $\widehat{\mathcal{A}}_{s-12\varepsilon_0}$. But this is a general fact contained in the following lemma.

Lemma 7.7. *Assume f is a bounded, measurable, Γ- equivariant function on $\mathbb{H}$. Let T^t_f be the Toeplitz operator on H_t, with symbol f. Then T^t_f belongs to $\widehat{\mathcal{A}_t}$. Moreover $\widehat{\|T^t_f\|}_t \leq \|f\|_\infty$, where C is a constant depending on t.*

Proof. Note the symbol of T^t_f is given by the formula [27]

$$s_f(\overline{z}, \xi) = \int\limits_{\mathbb{H}} f(a)[z, a, \overline{a}, \xi]^t \mathrm{d}\nu_0(a), \qquad z, \xi \in \mathbb{H}$$

We have to check that the quantity:

$$\sup_{z \in \mathbb{H}} \int\limits_{\mathbb{H}} |S_f(z, \xi)| |d(z, \xi)|^t \mathrm{d}\nu_0(\xi) \leq \|f\|_\infty$$

(and a similar one) is finite.

But the above integral is bounded by

$$\iint\limits_{\mathbb{H}^2} |f(a)| \, |[\overline{z}, a, \overline{a}, \xi]|^t \; |d(z, \xi)|^t \, \mathrm{d}\nu_0(a, \xi)$$

$$= \iint\limits_{\mathbb{H}^2} |f(a)| (d(\overline{z}, a))^t (d(\overline{a}, \xi))^t \mathrm{d}\nu_0(a, \xi)$$

$$= \int_{\mathbb{H}^2} f(a)(d(\overline{z}, a))^t \left(\int_{\mathbb{H}^2} d(\overline{a}, \xi)^t \mathrm{d}\nu_0(\xi) \right) \mathrm{d}\nu_0(a).$$

But the inner integral is a constant K_t, depending just on t and not on z. Thus we get

$$K_t \int_{\mathbb{H}^2} f(a)(d(\overline{z}, a))^t \mathrm{d}\nu_0(a) \leq K_t^2 \|f\|_\infty.$$

8. Comparison of $T^t_{\ln \varphi}$ and $\Lambda(1)(\overline{z}, \xi) = $ " $\ln \varphi(\overline{z}, \xi) - (c_t'/c_t)$"

In this chapter we compare $\Lambda(1)$, which was constructed in Section 6, with $T^t_{\ln \varphi}$

We recall that $\Lambda(1)$ is (up to an additive constant depending on the deformation parameter t)

$$(S_{\Delta^{\varepsilon_0}})^{-1} \left(\frac{\mathrm{d}}{\mathrm{d}\varepsilon} S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon} \Big|_{\varepsilon = \varepsilon_0, \varepsilon > \varepsilon_0} \right) ((S_{\Delta^{\varepsilon_0}})^*)^{-1}$$

where S_{Δ^ε} is acting on $H_{t+12\varepsilon_0}$, while $S_{\Delta^{\varepsilon_0}}$ acts from H_t into $H_{t+12\varepsilon_0}$. The inverse $(S_{\Delta^{\varepsilon_0}})^{-1}$ is an unbounded operator with domain dense is closure of range of $S_{\Delta^{\varepsilon_0}}$. We have explained in Section 7 that $\Lambda(1)$ corresponds, in a non-specified way, to the kernel: $\ln \varphi(\overline{z}, \xi) - (c_t'/c_t)$.

Both $\Lambda(1)$ and $T^t_{\ln \varphi}$ are positive and affiliated with $\mathcal{A}_t$. Also recall from Section 6, that the above definition for $\Lambda(1)$ translates into the fact that for $\mathcal{W} = \bigcup\limits_{\varepsilon_0} \mathrm{Range}\, (S^t_{\Delta^{\varepsilon_0}})^*$ we have that (up to constant)

$$\Lambda(1) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \langle S^t_\Delta \varepsilon \left(S^t_\Delta \varepsilon \right)^* w, w \rangle_{H_t} = \lim_{\varepsilon \searrow 0} \langle \frac{S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon} - \mathrm{Id}}{\varepsilon} w, w \rangle$$

Our main result proves that there exists (a possibility different domain) where $T^t_{\ln \varphi}$ is given by the same formula.

The main results is as follows:

Proposition 8.1. *There exists a densely defined $\mathcal{S}_0 \subseteq H_t$, which is a core for $T^t_{\ln \varphi}$ (though not affiliated with $\mathcal{A}_t$) such that the following holds true:*

Let G_ε be the bounded operator in $\mathcal{A}_t$ given by $(1/\varepsilon)\,(S_{\Delta^\varepsilon}S^*_{\Delta^\varepsilon} - \mathrm{Id})$. Clearly G_ε has kernel $\widehat{G_\varepsilon}(\overline{z},\xi) = (1/\varepsilon)\,((c_t/(c_t+\varepsilon))\varphi(\overline{z},\xi) - \mathrm{Id})$, and the kernels converge pointwise (as ε tends to 0) to $\ln\varphi(\overline{z},\xi) - (c_t'/c_t)$.

Then, for all v_1, v_2 in $\mathcal{S}_0$, we have that

$$\langle T_{\ln\varphi} v_1, v_2\rangle = \lim_{\varepsilon\searrow 0}\langle G_\varepsilon v_1, v_2\rangle$$

Remark. *By comparison, the same holds true for $\Lambda(1)$, the only difference is that this happens on a different domain $\mathcal{W}$ (in place of $\mathcal{S}_0$) which is affiliated to $\mathcal{A}_t$.*

This will be proved in several steps, divided in the following lemmas.

Lemma 8.2. *Let*

$$\mathcal{S} = \left\{ \sum_{i=1}^{N} \frac{\lambda_i}{(z-\overline{a}_i)}\alpha_i e^{i\varepsilon_i z} \;\middle|\; Re\,\alpha_i > 3,\ \varepsilon_i > 0,\ \lambda_i \in \mathbb{C}, N \in \mathbb{N} \right\}.$$

Then $\mathcal{S}$ is contained in all H_t, and dense in all H_t, $t > 1$.

Proof. Actually $\mathrm{Re}\,\alpha_i > 1$ would be sufficient for the convergence, but for latter considerations we take 3 instead of 1. It is sufficient to consider a single term (so $N = 1$). We omit all the indices for α, a, ε and let $\lambda = 1$. We prove first that $f(z) = \dfrac{1}{(z-\overline{a})^\alpha}\mathrm{e}^{i\varepsilon z}$ belongs to any H_t. Indeed we have

$$\int_{\mathbb{H}} \left| \frac{1}{(z-\overline{a})^\alpha}\mathrm{e}^{i\varepsilon z}\right|^2 \mathrm{d}\nu_t(z) = \int_{\mathbb{H}} \frac{1}{|z-\overline{a}|^{\mathrm{Re}\,\alpha}}\mathrm{e}^{-(\mathrm{Im}\,z)\varepsilon}(\mathrm{Im}\,z)^{t-2}\mathrm{d}\overline{z}\mathrm{d}z$$

which is obviously convergent as $\mathrm{Re}\,\alpha \geq 2$.

In the next lemma, we enlarge that space $\mathcal{S}$ to exhaust the range of all S_{Δ^ε}.

Lemma 8.3. *Let $\mathcal{S}_{0,t} = \bigcup_{\varepsilon>0} \Delta^\varepsilon \mathcal{S}$, $t - \varepsilon > 1$. Then $\mathcal{S}_{0,t}$ is dense in all H_t, $t > 1$.*

Proof. We need only look at $\mathcal{S} \subseteq H_{t-\varepsilon}$ and apply the operator $S_{\Delta^{\varepsilon}}$.

Next we need a bound on $\mathrm{Im}(\ln \Delta(z))$. Recall that we are using a choice for $\ln \Delta(z)$ which comes from that fact that $\Delta(z)$ is non zero in $\mathbb{H}$.

LEMMA 8.4. *Let* $\ln \Delta(z)$ *be the principal branch of the logarithm of the function* Δ*. Then* $|\mathrm{Im} \ln(\Delta(z))|$ *is bounded by a constant times* $C\left(\mathrm{Re}\, z + (1/(\mathrm{Im} z)^2)\right)$*, as* $\mathrm{Im} z \downarrow 0$.

Proof. We let $q = \mathrm{e}^{2\pi i z}$ and use the following expansion for $\ln \Delta(z)$

$$\ln \Delta(z) = \frac{\pi i z}{12} + \sum_{n \geq 1} \ln(1 - q^n).$$

When $r = |q| = |\mathrm{e}^{2\pi i z}| = \mathrm{e}^{-\pi y}$ tends to 1 we have, with $q = r\mathrm{e}^{i\theta}$, $z = x + \mathrm{i}y$ that

$$\begin{aligned} \mathrm{Im} \ln \Delta(z) &= \frac{\pi x}{12} + \sum_{n \geq 1} \arg((1 - r^n \cos n\theta) + \mathrm{i} r^n \sin n\theta) = \\ &= \frac{\pi x}{12} + \sum_{n \geq 1} \tan^{-1} \left[\frac{r^n \sin(n\theta)}{1 - r^n \cos n\theta} \right]. \end{aligned}$$

As $r \to 1$ this is dominated by

$$\frac{\pi x}{12} + \sum_{n \geq 1} \frac{r^n \sin n\theta}{1 - r^n \cos n\theta}$$

which in turn is dominated by

$$\frac{\pi x}{12} + \sum_{n \geq 1} \frac{r^n}{1 - r^n}$$

This turns out to be

$$\begin{aligned} \frac{\pi x}{12} &+ (r + r^2 + r^3 + \dots) \\ &+ (r^2 + r^4 + r^6 + r^8) \\ &+ (r^3 + r^6 + r^9 + \dots) \\ &+ (r^4 + r^8 + \dots) \\ &+ (r^5 + r^{10} + \dots) \\ &+ \dots \end{aligned}$$

and this is dominated by
$$\frac{\pi x}{12} + \frac{c}{(1-r)^2},$$
for some constant c.

Letting $r = \mathrm{e}^{-2\pi y}$, and using that $\lim\limits_{y\to 0} \dfrac{1-\mathrm{e}^{-2\pi y}}{y}$ is finite, it follows that
$$|\mathrm{Im}\ln\Delta(x)| \leq c\left(x + \frac{1}{y^2}\right) = c\left(\mathrm{Re}\, z + \frac{1}{(\mathrm{Im} z)^2}\right)$$

Corollary 8.5.. *For any $\varepsilon > 0$, then exists a constant c_ε such that*
$$|\Delta^\varepsilon(z)\ln\Delta(z)| \leq c_\varepsilon\left(1 + \frac{1}{\mathrm{Im} z}\right)\left(1 + \mathrm{Re}\ z + \frac{1}{(\mathrm{Im} z)^2}\right)$$

Proof. We write
$$|\Delta^\varepsilon(z)\ln(\Delta(z))| \leq |\Delta|^\varepsilon \ln|\Delta(z)| + |\Delta(z)|^\varepsilon |\mathrm{Im}(\ln\Delta(z))$$
We note that $|\Delta(z)|^2 \mathrm{Im} z^{12}$ is a bounded function and hence
$$|\Delta(z)| \leq \frac{c_1}{(\mathrm{Im} z)^6}.$$
Also, since $|x^\varepsilon \ln x| \leq \mathrm{const}\,([x^{\varepsilon_1}, x^{\varepsilon_2}])$ for $x > 0$, where $\varepsilon_1 > \varepsilon > \varepsilon_2$, we have that
$$\begin{aligned}|\Delta(z)|^\varepsilon \ln|\Delta(z)| &\leq \mathrm{const}\,(|\Delta(z)|^{\varepsilon_1}, |\Delta(z)|^{\varepsilon_2}) \\ &\leq c\max\left(\frac{1}{(\mathrm{Im} z)^{6\varepsilon_1}}, \frac{1}{(\mathrm{Im} z)^{6\varepsilon_2}}\right) \leq c\left(1 + \frac{1}{(\mathrm{Im} z)}\right)\end{aligned}$$
Similarly
$$|\Delta(z)|^\varepsilon |\mathrm{Im}(\ln(\Delta(z)))| \leq \frac{c}{(\mathrm{Im} z)^{6\varepsilon}}\left[x + \frac{1}{(\mathrm{Im} z)^2}\right]$$
Putting the two inequalities together we get
$$|\Delta^\varepsilon(z)\ln(\Delta(z)))| \leq c\left(1 + \frac{1}{(\mathrm{Im} z)} + \frac{1}{(\mathrm{Im} z)^{6\varepsilon}}\left[x + \frac{1}{(\mathrm{Im} z)^2}\right]\right)$$

which is thus smaller than

$$c\left(\left(1+\frac{1}{(\mathrm{Im}z)}\right)+x\left(1+\frac{1}{(\mathrm{Im}z)}\right)+\left(1+\frac{1}{(\mathrm{Im}z)}\right)\frac{1}{(\mathrm{Im}z)^2}\right)$$
$$=c\left(1+\frac{1}{(\mathrm{Im}z)}\right)\left(1+\mathrm{Re}\ z+\frac{1}{(\mathrm{Im}z)^2}\right)$$

Corollary 8.6. *Because* $|\Delta(z)|$ *has the order of growth of* $|e^{2\pi \mathrm{i} z}| = e^{-2\pi y}$, $y = \mathrm{Im}z$, *it follows, by first splitting* $\Delta^{\varepsilon}(z) = \Delta^{\varepsilon_1}(z)\Delta^{\varepsilon_2}(z)$, *that the growth of* $|\Delta^{\varepsilon}(z)\ln\Delta(z)|$ *will come from* $1/\mathrm{Im}z$ *as* $\mathrm{Im}z \to 0$. *Thus the above estimate can be improved to*

$$|\Delta^{\varepsilon}(z)\ln\Delta(z)| \leq c\frac{\mathrm{Re}z}{(\mathrm{Im}z)^3}\left(e^{-\varepsilon_1\mathrm{Im}z}\right).$$

In the next lemma we establish the integral formula for $\langle T^t_{\ln\varphi}v, v\rangle$.

Lemma 8.7. *Fix* $t \geq 10$. *For* v *in* $\mathcal{S}_{0,t}$, *the integral*

$$\iint\limits_{\mathbb{H}^2}\frac{\ln\varphi(\overline{z},\xi)}{(\overline{z}-\xi)^t}v(z)\overline{v(\xi)}\,d\nu_t(z,\xi),$$

is absolutely convergent and equal to

$$\int\limits_{\mathbb{H}}\ln\varphi(\overline{z},z)|v(z)|^2\,d\nu_t(z) = \langle T^t_{\ln\varphi}v, v\rangle.$$

Proof. We will make use of the fact that $v \in \mathcal{S}_{0,t}$, so that

$$v(z) = \Delta^{\varepsilon}(z)v_1(z),$$

for some $\varepsilon > 0$ and for some $v_1 \in \mathcal{S}$, (which is contained in $H_{t-\varepsilon}$).

We start by establishing the absolute convergence of the integral. The integral of the absolute value of the integrands is

$$\iint\limits_{\mathbb{H}^2}\frac{|\ln\varphi(\overline{z},\xi)||\Delta^{\varepsilon}(z)|}{|\overline{z}-\xi|^t}|v_1(z)|\Delta^{\varepsilon}(\xi)||v_1(\xi)|\mathrm{d}\nu_t(z,\xi).$$

We expand this into three terms, by using the expression

$$\ln \varphi(\overline{z}, \xi) = \ln \overline{\Delta(z)} + \ln \Delta(\xi) + 12 \ln(\overline{z} - \xi), \text{ for } z, \xi \in \mathbb{H}.$$

We will analyse each term separetely. Since the situations are similar will do only the computation for the term involving $|\ln \Delta(z)|$. The corresponding integral is

$$\iint\limits_{\mathbb{H}^2} \frac{|\ln \Delta(z)||\Delta^{\varepsilon}(z)|}{|\overline{z} - \xi|^t} |v_1(z)||\Delta^{\varepsilon}(\xi)||v_1(\xi)| \mathrm{d}\nu_t(z, \xi) \qquad (8.1).$$

Because $(\mathrm{Im} z)^{t/2} (\mathrm{Im} \xi)^{t/2} / |\overline{z} - \xi|^t$ is bounded above 1, the previous integral is in turn bounded by the integral

$$\iint\limits_{\mathbb{H}^2} |\ln \Delta(z) \Delta^{\varepsilon}(z)||v_1(z)||\Delta^{\varepsilon}(\xi)||v_2(\xi)|(\mathrm{Im} z)^{t/2-1} (\mathrm{Im} \xi)^{t/2-1} \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}\xi \mathrm{d}\overline{\xi}$$

We use the estimate from Corollary 8.5 to obtain that this integral is further bounded (up to a multiplicative constant c) by

$$c \iint\limits_{\mathbb{H}^2} \frac{\mathrm{Re}\ z}{(\mathrm{Im} z)^3} |v_1(z)||v_2(\xi)| \mathrm{e}^{-\varepsilon_1 \mathrm{Im} z} \mathrm{e}^{-\varepsilon (\mathrm{Im} \xi)} (\mathrm{Im} z)^{t/2-2} (\mathrm{Im} \xi)^{t/2} \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}\overline{\xi} \mathrm{d}\xi.$$

This comes to

$$c \iint\limits_{\mathbb{H}^2} (\mathrm{Re} z) |v_1(z)||v_2(\xi)| \mathrm{e}^{-\varepsilon_1 (\mathrm{Im} z)} \mathrm{e}^{-\varepsilon (\mathrm{Im} \xi)} (\mathrm{Im} z)^{t/2-5} (\mathrm{Im} \xi)^{t/2} \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}\overline{\xi} \mathrm{d}\xi.$$

As long as $t/2 - 5 \geq 0$, the term $\mathrm{e}^{-\varepsilon_1 (\mathrm{Im} z)} (\mathrm{Im} z)^{t/2-5}$ will be bounded by some $\mathrm{e}^{-\varepsilon_1' (\mathrm{Im} z)}$.

Thus if $t \geq 0$, and with the price of replacing $\varepsilon, \varepsilon_1$ with some smaller ones, in order to kill growth of $(\mathrm{Im} z)^{t/2-5}$ and $(\mathrm{Im} \xi)^{t/2-2}$, we get a multiple of

$$\iint\limits_{\mathbb{H}^2} (\mathrm{Re}\, z) |v_1(z)||v_2(\xi)| \mathrm{e}^{-\varepsilon_1 \mathrm{Im} z} \mathrm{e}^{-\varepsilon_2 \mathrm{Im} \xi} \mathrm{d}z \mathrm{d}\overline{z} \mathrm{d}\xi \mathrm{d}\overline{\xi}.$$

But for $z = x + \mathrm{i} y$, $|v_1(z)|$ involves powers of $1/x^3$ which makes the integral absolutely convergent. Hence the integral in (8.1) is absolutely convergent.

In the next lemma we will prove that for $v \in \mathcal{S}_0$, the integral

$$c_t \iint\limits_{\mathbb{H}^2} \frac{\ln[(\overline{z}-\xi)/(-2\mathrm{i})]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t} v_1(z)\overline{v_2(\xi)}\mathrm{d}\nu_t(\overline{z},\xi)$$

is absolutely convergent and equal to

$$\int v_1(z)\overline{v_2(\xi)}\ln(\overline{z}-z)\mathrm{d}\nu_t(z) + \frac{c_t'}{c_t}.$$

We now complete the proof of Lemma 8.7:

$$\int\limits_{\mathbb{H}} \ln\varphi(\overline{z},z)|v(z)|^2\mathrm{d}\nu_t(z) = \int [\overline{\ln\Delta(z)} + \ln\Delta(z) + 12\ln(\overline{z}-z)]|v(z)|^2\mathrm{d}\nu_t(z).$$

We analyze each term separately:

We have

$$\int \ln\Delta(z)v(z)\overline{v(z)}\mathrm{d}\nu_t(z) = \int \ln\Delta(z)\overline{v(z)}c_t\left(\int \frac{v(\xi)}{(z-\overline{\xi})^t}\mathrm{d}\nu_t(\xi)\right)\mathrm{d}\nu_t(z) =$$

$$= c_t\iint\limits_{\mathbb{H}^2} \frac{\ln(\Delta(z))\overline{v(z)}v(\xi)}{(z-\overline{\xi})^t}\mathrm{d}\nu_t(z,\xi) = c_t\iint\limits_{\mathbb{H}^2} \frac{\ln\Delta(\xi)\overline{v(\xi)}v(z)}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}\mathrm{d}\nu_t(z,\xi).$$

Similarly

$$\int \ln\overline{\Delta(z)}v(z)\overline{v(z)}\mathrm{d}\nu_t(z) = c_t\iint\limits_{\mathbb{H}^2} \frac{\overline{\ln\Delta(z)v(\xi)}v(z)}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}\mathrm{d}\nu_t(z,\xi).$$

We know that the integrals are absolutely convergent and that we may integrate in any order. Finally using the next lemma, we will have that

$$\int\limits_{\mathbb{H}} \ln(\mathrm{Im}z)|v(z)|^2\mathrm{d}\nu_t(z) = c_t\iint\limits_{\mathbb{H}^2} \frac{\ln[(\overline{z}-\xi)/(-2\mathrm{i})]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v(z)\overline{v(\xi)}\mathrm{d}\nu_t(z,\xi) - \frac{c_t'}{c_t}\langle v,v\rangle_{H_t}.$$

Putting this together we get that

$\langle T_{(1/12)\ln\varphi}v, v\rangle$

$$= c_t\iint\limits_{\mathbb{H}^2} \frac{[1/12\ln(\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v(z)\overline{v(\xi)}\mathrm{d}\nu_t(z,\xi) - \frac{c_t'}{c_t}\langle v,v\rangle_{H_t}.$$

This completes the proof of Lemma 8.7.

The following lemma was used above.

Lemma 8.8. *For v in $\mathcal{S}_{0,t}$, we have that*

$$\int_{\mathbb{H}} \ln(\operatorname{Im} z)|v(z)|^2 \mathrm{d}\nu_t = c_t \iint_{\mathbb{H}^2} \frac{\ln[(\overline{z}-\xi)/(-2\mathrm{i})]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t} v(z)\overline{v(\xi)} \mathrm{d}\nu_t(z,\xi) - \frac{c_t'}{c_t}\langle v, v\rangle_{H_t}.$$

Proof. Start with the identity

$$v(\xi) = c_s \int_{\mathbb{H}} \frac{v(z)}{[(\overline{z}-\xi)/(-2\mathrm{i})]^s} \mathrm{d}\nu_t(z) = \langle v, \mathrm{e}_\xi^t\rangle_{H_t}.$$

We differentiate this after s, at $s=t$ (which is allowed because of the fast decay of the functions in $\mathcal{S}_{0,t}$).

This gives us

$$0 = \frac{c_t'}{c_t} v(\xi) + c_t \int_{\mathbb{H}} \frac{v(z)[\ln(\operatorname{Im} z) - \ln[(\overline{z}-\xi)/(-2\mathrm{i})]]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^s} \mathrm{d}\nu_t(s).$$

Now we integrate on $\mathbb{H}$, with respect to the measure $\overline{v(\xi)} \cdot \mathrm{d}\nu_t(\xi)$.

We get

$$\begin{aligned} 0 =& \frac{c_t'}{c_t}\|v\|_{H_t}^2 + c_t \iint_{\mathbb{H}^2} \frac{v(z)\overline{v(\xi)} \ln \operatorname{Im} z}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t} \mathrm{d}\nu_t(z,\xi) \\ & - \iint_{\mathbb{H}^2} \frac{\ln[(\overline{z}-\xi)/(-2\mathrm{i})] v(z)\overline{v(\xi)}}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t} \mathrm{d}\nu_t(z,\xi). \end{aligned}$$

The second integral is

$$\int_{\mathbb{H}} |v(z)|^2 \ln(\operatorname{Im} z) \mathrm{d}\nu_t(z).$$

So we get the required identity.

This completes the Lemma 8.7 and also the proof of Lemma 8.8.

We now prove that the reproducing kernel

$$\frac{1}{12}\ln\left(\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}\right)-\frac{c_t'}{c_t}$$

is the derivative of $(c_t/(c_{t+12(s-t)}))S_{\Delta^{(s-t)/12}}S^*_{\Delta^{(s-t)/12}}$ on the space $\mathcal{S}_{0,t}$.

Lemma 8.9. *For $v_1, v_2 \in \mathcal{S}_{0,t}$, we have that*

$$\iint\limits_{\mathbb{H}^2} \frac{\ln\varphi(\overline{z},\xi)v_1(z)\overline{v_2(\xi)}}{[(\overline{z}-\xi)/(-2i)]^t}d\nu_t(z,\xi)$$

is the limit, when $\varepsilon \searrow 0$, of

$$\iint\limits_{\mathbb{H}^2} \frac{1/\varepsilon(\varphi(\overline{z},\xi)^\varepsilon-\mathrm{Id})}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v_1(z)\overline{v_2(\xi)}\mathrm{d}\nu_t(z,\xi).$$

Proof. The convergence of the integrals involved in the limits was proved in the Lemma 8.7. To check the value of the limit we will evaluate the difference. This is

$$\iint\limits_{\mathbb{H}^2} \frac{[1/\varepsilon(\varphi(\overline{z},\xi)^\varepsilon-\mathrm{Id})-\ln\varphi(\overline{z},\xi)]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v_1(z)\overline{v_2(\xi)}\mathrm{d}\nu_t(z,\xi).$$

We use the Taylor formula to express

$$\iint\limits_{\mathbb{H}^2}[1/\varepsilon(\varphi(\overline{z},\xi)^\varepsilon-1)-\ln\varphi(\overline{z},\xi)]v_1(z)\overline{v_2(\xi)}\mathrm{d}\nu_t(z,\xi)=$$

$$=\iint_{\mathbb{H}^2}\varepsilon\int\limits_0^1\frac{\varphi^{\varepsilon r}(\overline{z},\xi)\ln^2\varphi(\overline{z},\xi)]}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v_1(z)\overline{v_2(\xi)}\mathrm{d}r\mathrm{d}\nu_t(\overline{z},\xi).$$

The same type of arguments as in Lemma 8.7, because of the rapid decay of the vectors v_1, v_2 in $\mathcal{S}_{0,t}$, proves that the integral

$$\int\limits_0^1\iint\limits_{\mathbb{H}^2}\frac{\varphi^{\varepsilon r}(\overline{z},\xi)\ln^2\varphi(\overline{z},\xi)}{[(\overline{z}-\xi)/(-2\mathrm{i})]^t}v_1(z)\overline{v_2(\xi)}\mathrm{d}\nu_t(z,\xi)\mathrm{d}r$$

is absolutely convergent with a bound independent of ε. This completes the proof of Lemma 8.9.

To complete the proof of Proposition 8.1, it remains to check the fact the operators $G_\varepsilon = (S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon} - \mathrm{Id})/\varepsilon$ are decreasing (after making a correction of the form $-G_\varepsilon + \varepsilon K$, for a constant K. This is done in the following lemma

LEMMA 8.10 *Consider the bounded operators $G_\varepsilon = (S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon} - Id)/\varepsilon$, which are represented by the kernels*

$$\frac{1}{\varepsilon}\left[\frac{c_{t-12\varepsilon}}{c_t}\varphi(\overline{z},\xi)^\varepsilon - 1\right].$$

Then, there exists a constant K such that $-G_\varepsilon + K\varepsilon$ is, (as ε decreses to 0), an increasing family of positive operators in $\mathcal{A}_{2t+1}$.

Proof. Note that the kernel of $S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon}$ is

$$\frac{c_{t-12\varepsilon}}{c_t}\left[\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}\right]^\varepsilon$$

Hence the derivative is

$$-12\frac{c_t'}{c_t} + \ln\left[\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{12}\right].$$

Clearly $S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon}$ is a decreasing family. We will proceed as in Lemma 6.3. Let $s_\varepsilon = S(\varepsilon)$ be the kernel of $S_{\Delta^\varepsilon} S^*_{\Delta^\varepsilon}$ (as an operator on H_t). Then

$$s_\varepsilon(\overline{z},\xi) = \frac{c_{t-12\varepsilon}}{c_t}(\varphi(z,\xi))^\varepsilon.$$

Let $G_\varepsilon = (s_\varepsilon - \mathrm{Id})/\varepsilon$.

The first derivative of s_ε (after ε) is

$$-12\frac{c_t'}{c_t}\varphi^\varepsilon + \frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon \ln\varphi.$$

The second derivative is

$$24\frac{c_t'}{c_t}\varphi^\varepsilon \ln\varphi - \frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon(\ln\varphi)^2.$$

This is equal to (as $c_t' = 1$)

$$\frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon\left[(\ln\varphi)^2 - \frac{24\ln\varphi}{c_{t-12\varepsilon}}\right] = \frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon\left[(\ln\varphi) + \frac{12\ln\varphi}{c_{t-12\varepsilon}}\right]^2 - \frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon\frac{144}{(c_{t-12\varepsilon})^2}.$$

This is further equal to

$$\frac{c_{t-12\varepsilon}}{c_t}\varphi^\varepsilon\left[\ln\varphi + \frac{12\ln\varphi}{c_{t-12\varepsilon}}\right]^2 - \frac{144}{c_t(c_{t-12\varepsilon})}\varphi^\varepsilon.$$

For every $r > 1$, $S^r_{\Delta^\varepsilon}\,(S^r_{\Delta^\varepsilon})^* = f(\varepsilon)$ is a decreasing family in $\mathcal{A}_r$. By evaluating the kernel, which is

$$f(\varepsilon)(\overline{z},\xi) = \frac{c_{r-12\varepsilon}}{c_r}[\varphi(z,\xi)]^\varepsilon$$

we get that $\dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}f(\varepsilon)(\overline{z},\xi)$ is a positive kernel for $\mathcal{A}_r$. Since

$$c_{r-12\varepsilon} = \frac{r-12\varepsilon-1}{\pi},$$

we obtain that

$$\frac{c_{r-12\varepsilon}}{c_r}\varphi^\varepsilon\ln\varphi - 12\frac{c_r'}{c_r}\varphi^\varepsilon$$

represents a negative kernel for $\mathcal{A}_r$.

We recall, from Section 6, that a kernel $k = k(\overline{z},\xi)$ is positive for $\mathcal{A}_r$ (even if k does not necessary represent an operator in $\mathcal{A}_r$), if $\left[\dfrac{k(\overline{z_i},z_j)}{(\overline{z_i}-z_j)^r}\right]_{i,j=1}^N$ is a positive matrix for all choices of $z_1, z_2 \dots z_N$ in H, and for all N in $\mathbb{N}$

We get that

$$\varphi^\varepsilon\left(\ln\varphi - \frac{1}{c_{r+12\varepsilon}}\right)$$

represents a negative (nonpositive) kernel for $\mathcal{A}_r$.

Thus $\varphi^{\varepsilon/2}\,[\ln\varphi + (12/(r-12(\varepsilon/2)-(\varepsilon/2)-1))]$ is negative for $\mathcal{A}_{r+12\frac{\varepsilon}{2}}$, and hence the square

$$\varphi^\varepsilon\left[\ln\varphi + \frac{12}{r-12\frac{\varepsilon}{2}-\frac{\varepsilon}{2}-1}\right]^2 = \varphi^\varepsilon\left[\ln\varphi\frac{12}{r-12\frac{\varepsilon}{2}-\frac{\varepsilon}{2}-1}\right]^2$$

is positive for $\mathcal{A}_{2r-12\varepsilon}$.

Consequently the kernels

$$\frac{c_t - 12\varepsilon}{c_t}\varphi^{\varepsilon}\left[\ln\varphi - \frac{12}{t-12\varepsilon-1}\right]^2$$

are positive for $\mathcal{A}_{2t+13\varepsilon}$

Now we note the trivial calculus formulae

$$G_{\varepsilon} = \frac{S(\varepsilon)-\mathrm{Id}}{\varepsilon} = \int_0^1 S'(\varepsilon v)\mathrm{d}v$$

$$G_{\varepsilon'} = \frac{S(\varepsilon')-\mathrm{Id}}{\varepsilon'} = \int_0^1 S'(\varepsilon' v)\mathrm{d}v$$

The above equalities hold pointwise, that is when evaluating the corresponding kernels on points in $\mathbb{H}^2$. Hence

$$\begin{aligned} G_{\varepsilon} - G_{\varepsilon'} &= \int_0^1 (S'(\varepsilon v) - S'(\varepsilon' v))\mathrm{d}v \\ &= \int_0^1 (\varepsilon v - \varepsilon' v)\int_0^1 S''(p(\varepsilon v) + (1-p)\varepsilon' v)\mathrm{d}p\mathrm{d}v \\ &= (\varepsilon - \varepsilon')\int_0^1\int_0^1 vS''(\alpha(v,p))\mathrm{d}p\mathrm{d}v \end{aligned}$$

where $\alpha(v,p) = p(\varepsilon v) + (1-p)\varepsilon' v \leq \max(\varepsilon, \varepsilon')$.

We haved proved that $S''(\alpha(v,p))$ is represented by a positive kernel R, from which one has to subtract a quantity Q (which is precisely $\dfrac{\mathrm{const}}{c_{v-12\alpha(v,p)}}\varphi^{\alpha(v,p)}$).

As such by integration we obtain

$$G_{\varepsilon} - G_{\varepsilon'} = (\varepsilon - \varepsilon')[\mathcal{R} - \mathcal{Q}],$$

where $\mathcal{R}$ represents a positive kernel for $\mathcal{A}_{2t-12\min(\varepsilon,\varepsilon')}$. Moreover Q is positive element in $\mathcal{A}_{2t} + 12\min(\varepsilon,\varepsilon'))$ and $\mathcal{Q}$ is bounded by $c{\cdot}\mathrm{Id}$, where c is a universal constant.

Assume that $\varepsilon \geq \varepsilon'$, then in the sense of inequalities in $\mathcal{A}_{2r+1}$ we have that

$$G_{\varepsilon} - G_{\varepsilon'} \geq (\varepsilon - \varepsilon')(-\mathcal{Q})$$

Since $0 \leq \mathcal{Q} \leq c \cdot \mathrm{Id} \cdot \mathrm{Id}$, we have that $0 \geq -\mathcal{Q} \geq c \cdot \mathrm{Id} \cdot -\mathrm{Id}$ (in $\mathcal{A}_{2t+1}$), Consequently, in $\mathcal{A}_{2t+1}$, we have that

$$G_\varepsilon - G_{\varepsilon'} \geq (\varepsilon - \varepsilon')(-c)$$

Therefore, the following inequality holds in $\mathcal{A}_{2t+1}$.

$$G_\varepsilon + \varepsilon c \geq G_{\varepsilon'} + \varepsilon' c.$$

If we take in account that G_ε was negative and replace G_ε by $H_\varepsilon = -G_\varepsilon$ then we get that in $\mathcal{A}_{2t+1}$ we have that

$$(-G_\varepsilon) - \varepsilon c \leq (-G_{\varepsilon'}) - \varepsilon' c$$

i.e. that if $\varepsilon \geq \varepsilon'$

$$H_\varepsilon - \varepsilon c \leq H_{\varepsilon'} - \varepsilon c$$

We have consequently proved that, in $\mathcal{A}_{2t+1}$, the kernels

$$H_\varepsilon(\overline{z}, \xi) = -G_\varepsilon(\overline{z}, \xi) = -\frac{\dfrac{c_t - 12\varepsilon}{c_t}\varphi^\varepsilon + \mathrm{Id}}{\varepsilon}$$

are positive and they increase (when ε decreases to zero, modulo an infintesimal term) to $-(c_t'/c_t) + \ln \varphi(\overline{z}, \xi)$.

Lemma 8.11. Let $M \subseteq B(H)$ be a type II_1 factor and assume that $(H_n)_{n\in\mathbb{N}}$ is an increasing family of positive operators in M. Let

$$\mathcal{D}(X) = \{\xi \in H \mid \sup \langle H_n\xi, \xi\rangle < \infty\}$$

and assume that $\mathcal{D}(X)$ is weakly dense in H. Then $\mathcal{D}(X)$ is affiliated with M, and $\langle X\xi, \xi\rangle = \sup_n \langle H_n\xi, \xi\rangle, \xi \in \mathcal{D}(X)$, defines an operator affiliated with M.

Proof. Clearly $\mathcal{D}(X) = \left\{\xi \in H \mid \sup_n ||H_n^{1/2}\xi|| < \infty\right\}$ and such that $\mathcal{D}(X)$ is a subspace, because if $||H_n^{1/2}\xi|| \leq A$, $||H_n^{1/2}\eta|| \leq B$ for all n then $||H_n^{1/2}(\xi+\eta)|| \leq A+B$. Moreover $\mathcal{D}(X)$ is clearlyinvariant under $u' \in \mathcal{U}(M')$, and hence $\mathcal{D}(X)$ is affiliated with M.

The quadratic linear form $q_X(\xi) = \sup_n \langle H_n\xi, \xi\rangle$ is weakly lower semicontinous, thus q_X defines a positive unbounded operator X, affiliated with M, with domain $\mathcal{D}(X)$.

Corollary 8.12. *The following holds:*

$$T^t_{\ln\varphi} + \frac{c'_t}{c_t} \cdot \mathrm{Id} = \Lambda(1).$$

Proof. Let $H_\varepsilon = -G_\varepsilon + K \cdot \mathrm{Id} + C_\varepsilon$, where G_ε are as in Lemma 8.10. Then by definition $X = -\Lambda(1) + K \cdot \mathrm{Id}$ coincides with the supremum of H_ε on $S_0 = \cup S^*_{\Delta^\varepsilon}$.

On S_0, which is a core for $T^t_{\ln\varphi} + K \cdot \mathrm{Id}$, the same holds for $T^t_{\ln\varphi} + K \cdot \mathrm{Id}$.

Thus $T^t_{\ln\varphi}|_{S_0} \subseteq X$, hence $\overline{T^t_{\ln\varphi}} \subseteq X$ and so $T^t_{\ln\varphi} = X = -\Lambda(1)$, by [24].

9. The cyclic cocycle associated to the deformation

In [27] we introduced a cyclic cocycle Ψ_t, which lives on the algebra $\underset{s<t}{\cup} \widehat{\mathcal{A}}_s$, and we proved a certain form of non-triviality for this cocycle.

We recall first the definition of the cocycle Ψ_t and then we will show the non-triviality of Ψ_t by using a quadratic form deduced from the operator introduced in Lemma 6.6 and Lemma 7.4. The main result of this paragraph will be the following:

Theorem 9.1. *Let $t > 1$, let $\mathcal{B}_t = \underset{s<t}{\cup} \widehat{\mathcal{A}}_s$, which is a weakly dense subalgebra of $\mathcal{A}_t$ and let R_t be defined on $\mathcal{B}_t$ (with values in $\mathcal{B}_t$) by the formula*

$$\langle R_t k, l\rangle_{L^2(\mathcal{A}_t)} = -\frac{1}{2}\tau_{\mathcal{A}_t}(\mathcal{C}_t(k, l^*)), \quad k, l \in \mathcal{B}_t$$

(that is R_t implements the Dirichlet form $\tau_t(\mathcal{C}_t(k, l^*))$.

Let $(\nabla R_t)(k, l) = R_t(k, l) - kR_t l - R_t(k)l$, which belongs to $\mathcal{B}_t$, if $k, l \in \mathcal{B}_t$, and let Ψ_t be the cyclic cocycle associated with the deformation ([27])

$$\Psi_t(k, l, m) = \tau_{\mathcal{A}_t}([\mathcal{C}_t(k, l) - (\nabla R_t)(k, l)]m), \; k, l, m \in \mathcal{B}_t.$$

Let Λ_0 be the operator, on the weakly dense (non-unital subalgebra) $\mathcal{D}_t^0 \subseteq \mathcal{A}_t$, introduced in Theorem 7.6, by requiring that $\Lambda_0(k)$ is the derivative at 0, *of the operator represented in $\mathcal{A}_t$ by the kernel $\varphi^\varepsilon(\overline{z},\xi)k(\overline{z},\xi)$. Thus $\Lambda_0(k)(\overline{z},\xi)$ is formally $k(\overline{z},\xi)\ln\varphi(\overline{z},\xi)$.*

Let $\chi_t(k,l) = \langle \Lambda_0 k, l^*\rangle_{L^2(\mathcal{A}_t)} - \langle k, \Lambda_0(l^*)\rangle_{L^2(\mathcal{A}_t)}$ be the antisymmetric form associated with Λ_0. Then

$$\Psi_t(k,l,m) = \frac{c_t'}{c_t}\tau_{\mathcal{A}_t}(klm) + \chi_t(k *_t l, m) + \chi_t(l *_t m, k) + \chi_t(m *_t k, l)$$

for $k,l,m \in \mathcal{D}_t^0$.

We will split the proof of this result in several steps: First we prove some properties about Λ_0 and its formal adjoint Λ^+. We start with the definition of Λ^+. The first lemma collects the definition and basic properties of Λ^+.

Lemma 9.2. *Let f be a bounded measurable function, that is* $\mathrm{PSL}(2,\mathbb{Z})$-*equivariant .*

We define $\Lambda^+(T_f^t) = T_{f\ln\varphi}^t$. Then Λ^+ has the following properties:

1) *Assume in addition that $f\ln\varphi(\overline{z},z)$ is a bounded function. Then $(\Lambda_0|\mathcal{D}_t)^* \subseteq \Lambda^+$ and*

$$\tau_{\mathcal{A}_t}(\Lambda_0(k)(T_f^t)^*) = \tau_{\mathcal{A}_t}(k\Lambda^+(T_f^t)) = \frac{1}{\text{area } F}\int_F k(\overline{z},z)f(z)\ln\varphi(\overline{z},z)d\nu_0(z).$$

2) *For k,l be in $\mathcal{D}_t^0$, we have that*

$$\tau(k\Lambda^+(T_f^t)l) = \tau(\Lambda_0(l*k)T_f^t).$$

Proof. The proof of this propositions is obvious, since integrals are absolutely summable. For part 2 we remark that $k\Lambda^+(T_f^t)l$ has symbol

$$c_t\int_{\mathbb{H}} k(\overline{z},\eta)[f(\overline{\eta},\eta)\ln\varphi(\overline{\eta},\eta)]\varphi(\overline{\eta},\xi)\mathrm{d}\nu_t(\eta).$$

Hence by summability, the trace is

$$\frac{c_t}{\text{area } F}\iint_{F\times\mathbb{H}} k(\overline{z},\eta)l(\overline{\eta},z)f(\overline{\eta},\eta)\ln\varphi(\overline{\eta},\eta)\mathrm{d}\nu_t(z,\eta)$$

which is exactly $\tau\left(\Lambda(l*k)T^t_{f\ln\varphi}\right)$. This completes the proof.

Recall that in Section 5 we introduced the, densely defined, operator $\mathcal{T}_{\ln d}$, on $L^2(\mathcal{A}_t)$, given by the formula

$$\langle \mathcal{T}_{\ln d}k, l\rangle = \frac{c_t}{\text{area } F}\iint\limits_{F\times\mathbb{H}} k(\overline{z},\eta)\overline{l(\overline{z},\eta)}\ln d(\overline{z},\eta)|d(\overline{z},\eta)|^{2t}\mathrm{d}\nu_0(t,\eta),$$

which is well defined for k,l in algebra $\hat{\mathcal{B}}_t$. We note that $\mathcal{T}_{\ln d}$ acts like a Toeplitz operator on $L^2(\mathcal{A}_t)$, with symbol $\ln d$. In the next lemma we establish the relation between the operator $\mathcal{T}_{\ln d}$ and the operator R_t.

Lemma 9.3. *The operator R_t, defined by the property*

$$\langle R_t k, l\rangle = -\frac{1}{2}\tau(\mathcal{C}_t(k,l))$$

has the following simple expression in terms of $\mathcal{T}_{\ln d}$:

$$R_t = -\frac{1}{2}\frac{c_t'}{c_t} - \langle \mathcal{T}_{\ln d}k, l\rangle, \quad k,l\in\widehat{\mathcal{B}}_t.$$

Proof. Indeed we have that for $k,l\in\mathcal{B}_t\subseteq\widehat{\mathcal{A}}_t$

$$\mathcal{C}_t(k,l) = \frac{c_t'}{c_t}(k*_t l) + c_t\int_{\mathbb{H}} k(\overline{z},\eta)l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t\ln[\overline{z},\eta,\overline{\eta},\xi]\mathrm{d}\nu_t(\eta).$$

If we make $\xi = z$ in the above expression and then integrate over F to get the trace of $\mathcal{C}_t(k,l)$, we get

$$\begin{aligned}\tau(\mathcal{C}_t(k,l)) &= \frac{c_t'}{c_t}\tau(k*_t l) + \frac{c_t}{\text{area}\,F}\iint\limits_{F\times\mathbb{H}} k(\overline{z},\eta)\overline{l(\overline{z},\eta)}|d(\overline{z},\eta)|^{2t}\ln d^2\mathrm{d}\nu_t(\eta)\\ &= \frac{c_t'}{c_t}\tau(k*_t l) + 2\langle \mathcal{T}_{\ln\varphi}k, l\rangle.\end{aligned}$$

This completes the proof.

In the next lemma we prove a relation between $\Lambda_0+\Lambda^+$ and the other terms (remark that Λ^+ is not necessary the adjoint of Λ_0, rather we define $\Lambda^+(T^t_f) = T^t_{f\ln\varphi}$ whenever possible).

PROPOSITION 9.4. *For all k,l in $\mathcal{D}_t^0$ we have:*

$$\langle \Lambda_0 k, l^* \rangle + \langle k, \Lambda_0 l^* \rangle = \tau(k T^t_{f \ln \varphi} l) + \tau(l T^t_{f \ln\varphi} k) - 2\langle T_{\ln\varphi} k, l\rangle. \tag{9.1}$$

Consequently if we define "$Re\Lambda_0$" (formwise) by the relation

$$\langle (Re\ \Lambda_0)k, l\rangle = \frac{1}{2}(\langle \Lambda_0 k, l\rangle + \langle k, \Lambda_0(l)\rangle),$$

then

$$\langle (\mathrm{Re}\,\Lambda_0)k, l^*\rangle = \frac{1}{2}\tau(kTl + lTk) + \langle R_t k, l^*\rangle + 1/2\frac{c_t'}{c_t}\langle k, l^*\rangle_{L^2(\mathcal{A}_t)} \tag{9.2}$$

Proof. We prove first the relation (9.1). For $k, l \in \mathcal{D}_t^0$, we have that

$$\langle \Lambda_0 k, l^*\rangle_{L^2(\mathcal{A}_t)} + \langle k, \Lambda_0 l^*\rangle_{L^2(\mathcal{A}_t)}$$

is equal to

$$\frac{c_t}{\mathrm{area}\, F}\iint\limits_{F\times\mathbb{H}} [\ln\varphi(\overline{z},\eta) + \ln\varphi(\overline{\eta}, z)] k(\overline{z},\eta) l(\overline{\eta}, z) d(\overline{z},\eta)^{2t} \mathrm{d}\nu_0(z,\eta).$$

Since

$$\ln\varphi(\overline{z},\eta) + \ln\varphi(\overline{\eta}, z) = \ln\varphi(\overline{z}, z) + \ln\varphi(\overline{\eta},\eta) - \ln[d(\overline{z},\eta)]^2$$

we get the relation (9.1).

Dividing by 2 we get

$$\frac{1}{2}[\langle \Lambda_0 k, l^*\rangle + \langle k, \Lambda_0(l)^*\rangle] = \frac{1}{2}[\tau(k T^t_{\ln\varphi} l) + \tau(l T^t_{\ln\varphi} k)] - \langle T_{\ln d} k, l^*\rangle_{L^2(\mathcal{A}_t)}.$$

The definition of R_t and previous lemma completes the proof.

Recall that in Section 7 we proved that for all k,l in $\mathcal{D}_t^0$ we have that

$$\mathcal{C}_t(k,l) = \frac{c_t'}{c_t}\mathrm{Id}\ + k T^t_{\ln\varphi} l + \Lambda_0(kl) - \Lambda_0(k)l - k\Lambda_0(l). \tag{9.3}$$

We want to use (9.3) to find an expression for

$$\mathcal{C}_t(k,l) - (\Delta R_t)(k,l)$$

by taking the trace of the product of $m \in \mathcal{D}_t^0$ with the previous expression.

Notation. We denote $T = T^t_{\ln\varphi}$ and let $\langle \mathrm{Sym}_\varphi k, l\rangle = 1/2[\tau(kTl^*) + \tau(l^*Tk)]$, for $k, l \in \mathcal{D}_t^0$. Hence

$$\tau_{\mathcal{A}_t}(\mathrm{Sym}_\varphi(k)l) = \frac{1}{2}[\tau(kTl) + \tau(lTk)].$$

In this terminology the relation in Proposition 9.4 becomes

$$\langle(\mathrm{Re}\,\Lambda_0)k, l^*\rangle = \langle \mathrm{Sym}_\varphi k, l\rangle + \langle R_t k, l^*\rangle + \frac{1}{2}\frac{c_t'}{c_t}\langle k, l^*\rangle.$$

Note that in the relation above, the scalar product refers to the scalar product on $L^2(\mathcal{A}_t)$. Moreover the following relations hold true.

$$\tau(\mathrm{Sym}_\varphi(kl)m) = \frac{1}{2}[\tau(klTm) + \tau(mTkl)] \tag{9.4}$$

$$\tau((\mathrm{Sym}_\varphi k)lm) = \tau(\mathrm{Sym}_\varphi(k)(lm)) = \frac{1}{2}[\tau(klTm) + \tau(lmTk)] \tag{9.5}$$

$$\tau(k(\mathrm{Sym}_\varphi(l))m) = \tau(\mathrm{Sym}_\varphi(mk)l) = \frac{1}{2}[\tau(lTmk) + \tau(mkTl)]. \tag{9.6}$$

LEMMA 69.5. *For all k, l, m in $\mathcal{D}_t^0$ we have that*

$$E = Sym_\varphi(kl) - (Sym_\varphi k)l - k(Sym_\varphi l) + kT^t l = 0.$$

To check this, one has to verify that $\tau(Em) = 0$ for all m in $\mathcal{D}_t^0$.

Proof. We have to check that the expression

$$\tau(klTm) + \tau(mTkl) - \tau(kTlm) - \tau(lmTk) - \tau(lTmk) - \tau(mkTl) + 2\tau(kTlm)$$

vanishes. But

$$\tau(mTkl) = \tau(lmTk),$$
$$\tau(klTm) = \tau(lTmk).$$

After cancelling the above terms we are left to check that

$$-\tau(kTlm) - \tau(mkTl) + 2\tau(kTlm)$$

is equal to zero, which is obvious since $k, l, m \in \mathcal{D}_t^0$. This completes the proof.

We now decompose $\tau(\Lambda_0(k)l)$ in the following way

$$\tau(\Lambda_0(k), l^*) = \langle (\operatorname{Re}\Lambda_0)(k), l\rangle + \mathrm{i}\langle (\operatorname{Im}\Lambda_0)(k), l\rangle$$

where

$$\langle (\operatorname{Im}\Lambda_0)(k), l^*\rangle = (1/2\mathrm{i})[\langle \Lambda_0(k), l^*\rangle - \langle k, \Lambda_0(l^*)\rangle].$$

We now can proceed to the proof of Theorem 9.1.

Proof of Theorem 9.1. We have

$$\mathcal{C}_t(k,l) = \frac{c_t'}{c_t} kl + kTl + \Lambda_0(kl) - \Lambda_0(k)l - k\Lambda_0(l).$$

Hence by taking scalar product with an m in $\mathcal{A}_t$, i.e. computing $\tau(\mathcal{C}_t(k,l)m)$ we obtain

$$\begin{aligned}
\tau(\mathcal{C}_t(k,l)m) &= \frac{c_t'}{c_t}\tau(klm) + \tau(kTlm) + \tau(\Lambda_0(kl)m) - \tau(\Lambda_0(k)lm) - \tau(\Lambda_0(k)mk) \\
&= \frac{c_t'}{c_t}\tau(klm) + \tau([kTl]m) + \langle \operatorname{Re}\Lambda_0(kl), m^*\rangle \\
&- \langle \operatorname{Re}\Lambda_0(k), (lm)^*\rangle - \langle \operatorname{Re}\Lambda_0(l), (mk)^*\rangle \\
&+ \mathrm{i}\langle \operatorname{Im}\Lambda_0(kl), m^*\rangle - \mathrm{i}\langle \operatorname{Im}\Lambda_0(k), (lm)^*\rangle - \mathrm{i}\langle \operatorname{Im}\Lambda_0(l), (mk)^*\rangle.
\end{aligned}$$

By using the relation

$$\langle \operatorname{Re}\Lambda_0(k), l^*\rangle = \langle R_t k, l^*\rangle + \frac{1}{2}\frac{c_t'}{c_t}\langle k, l^*\rangle + \langle \operatorname{Sym}_\varphi k, l^*\rangle.$$

we obtain that $\tau(\mathcal{C}_t(k,l)m)$ is equal to

$$\tau([(\Delta R_t)(k,l)]m)$$

plus the following terms

$$\langle \operatorname{Sym}_\varphi(kl), m^*\rangle - \tau(\operatorname{Sym}_\varphi(k)lm) - \tau(\operatorname{Sym}_\varphi(l)mk), \qquad (9.8)$$

plus the terms

$$\frac{c_t'}{c_t}\tau(klm) + \left(\frac{1}{2}\frac{c_t'}{c_t} - \frac{1}{2}\frac{c_t'}{c_t} - \frac{1}{2}\frac{c_t'}{c_t}\right)\tau(klm), \tag{9.9}$$

plus the terms

$$\mathrm{i}\langle \operatorname{Im}\Lambda_0 kl, m^*\rangle - \mathrm{i}\langle \operatorname{Im}\Lambda_0(k), (lm)^*\rangle - \mathrm{i}\langle \operatorname{Im}\Lambda_0(l), (mk)^*\rangle. \tag{9.10}$$

The terms in (9.8) add up to zero, as it was proved in Lemma 9.5. The terms in (9.9) add up to $\frac{1}{2}\frac{c_t'}{c_t}\tau(klm)$.

Since $\chi_t(k,l) = \frac{1}{2}[\langle \Lambda_0 k, l^*\rangle - \langle k, \Lambda_0 l^*\rangle] = \mathrm{i}\langle \operatorname{Im}\Lambda_0(k), l^*\rangle$ we obtain by adding the terms from (9.8), (9.9) (9.10) that

$$\tau(\mathcal{C}_t(k,l)m) = \tau([(\Delta R_t)(k,l)]m) + \frac{1}{2}\frac{c_t'}{c_t}\tau(klm) + \chi_t(kl,m) - \chi_t(k,lm) - \chi_t(l,mk).$$

Thus

$$\Psi_t(k,l,m) = \frac{1}{2}\frac{c_t'}{c_t} + \chi_t(kl,m) - \chi_t(k,lm) - \chi_t(l,mk).$$

Lemma 9.6. *Let $t > 1$. Assume that k,l are such that $k = k_1 *_t k_2$, $l = l_1 *_t l_2$, $k_i, l_i \in \mathcal{D}_t^0$. Then*

$$\tau(k_2\mathcal{L}_t(l)k_1) + \tau(l_2\mathcal{L}_t(l)l_1) + \tau(\mathcal{C}_t(k,l)) = -\frac{c_t'}{c_t}\tau(k *_t l).$$

Proof. Recall that

$$\mathcal{L}_t = \left(\Lambda_0 - \frac{c_t'}{c_t}\cdot \mathrm{Id}\right) - \frac{1}{2}\{T, \cdot\}. \tag{9.11}$$

Also $\tau(\mathcal{C}_t(k,l)) = (c_t'/c_t)\tau(k *_t l) + 2\langle \mathcal{T}_{\ln d}k, l\rangle$. Also

$$\tau(\Lambda_0(k)\cdot l) + \tau(k\Lambda_0(l)) = \tau(kTl) + \tau(lTk) - 2\langle \mathcal{T}_{\ln d}k, l\rangle. \tag{9.12}$$

Moreover,

$$\tau(kTl) = \tau(\Lambda_0(l *_t k)) = \int_F \ln\varphi\cdot(l *_t k)(\bar{z}, z)\mathrm{d}\nu_0(z). \tag{9.13}$$

Hence by (9.11),

$$\tau(k_2\mathcal{L}_t(l)k_1) = \tau(k_2\Lambda_0(l)k_1) - \frac{c_t'}{c_t}\tau(kl) - \frac{1}{2}\tau(k_2Tlk_1) - \frac{1}{2}\tau(k_2lTk_1)$$

and

$$\tau(l_2\mathcal{L}_t(k)l_1) = \tau(l_2\Lambda_0(k)l_1) - \frac{1}{2}(-\tau(l_2Tkl_1) - \tau(l_2kTl_1)) - \frac{c_t'}{c_t}\tau(kl).$$

So by (9.11), (9.12),

$$\begin{aligned}
&\tau(k_2\mathcal{L}_t(l)k_1) + \tau(l_2\mathcal{L}_t(k)l_1) + \tau(\mathcal{C}_t(k,l))\\
&= \tau(\Lambda_0(k)l) + \tau(\Lambda_0(l)k) - 2\frac{c_t'}{c_t}\tau(kl) - \tau(kTl) - \tau(lTk) + \tau(\mathcal{C}_t(k,l))\\
&= \tau(kTl) + \tau(lTk) - 2\frac{c_t'}{c_t}\tau(kl) - \tau(kTl) - \tau(lTk)\\
&-2\langle \mathcal{T}_{\ln d}k, l\rangle + 2\langle \mathcal{T}_{\ln d}k, l\rangle + \frac{c_t'}{c_t}\tau(kl)\\
&= -\frac{c_t'}{c_t}\tau(kl).
\end{aligned}$$

10. A dual solution; closability of Λ

In this chapter we analyze the Hilbert space dual of the operator $\Lambda(k), k \in D_0^t$, introduced in the Section 7. This is achieved by analyzing the derivative of the one parameter family of completely positive maps $\chi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$, $1 < s \leq t$, defined as follows:

$$\chi_{s,t}(k) = S_{\Delta^{\frac{s-t}{12}}} k \left(S_{\Delta^{\frac{s-t}{12}}}\right)^*, \qquad k \in \mathcal{A}_t$$

Recall that Hilbert space $L^2(\mathcal{A}_t)$ is naturally identified with the Hilbert space of all kernels $k = k(\overline{z}, \xi)$ on $\mathbb{H} \times \mathbb{H}$, that are diagonally Γ- equivariant, $\Gamma = PSL(2,\mathbb{Z})$. The kernels are also required to be square summable with respect to the the measure $[|d(z,\xi)|]^{2t}\mathrm{d}\nu_0(z)\mathrm{d}\nu_0(\xi)$ on $F \times \mathbb{H}$. (Recall that F is a fundamental domain for Γ in $\mathbb{H}$).

Consider the Hilbert space L_t, of all measurable functions on $\mathbb{H} \times F$, that are, square summable with respect to the measure $d^{2t}\mathrm{d}\nu_0 \times \mathrm{d}\nu_0$. This space is obviously identified with a space of Γ- invariant (diagonally) functions on $\mathbb{H} \times \mathbb{H}$, square summable over $F \times \mathbb{H}$.

We let $\mathcal{P}$ be the orthogonal projection from L^t into $L^2(A_t)$. Let Φ be a measurable, (diagonally), Γ- equivariant function on $\mathbb{H} \times \mathbb{H}$. With the above identification let M_Φ be the (eventually unbounded operator) on L_t, defined by multiplication with Φ on L_t. Correspondingly, there is a Toeplitz operator $\mathcal{T}_\Phi = \mathcal{P}\mathcal{M}_\Phi\mathcal{P}$, densely defined on $L^2(A_t)$.

For example, the map Λ, constructed in Section 7, is $\mathcal{T}_\Phi$ with $\Phi = \ln \varphi$. In Section 9, Lemma 9.3, we have proved that the operator R_t defined by

$$< R_t k, l >_{L^2(\mathcal{A}_t)} = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\tau_{\mathcal{A}_s}(k *_s l^*),$$

defined for k, l in an algebra, is exactly $-\mathcal{T}_{\ln d} - \frac{1}{2}(c_t'/c_t) \cdot \mathrm{Id}$.

Let also P_t be the orthogonal projection from $L^2(\mathbb{H}, \mathrm{d}\nu_t)$ onto H_t. Recall that the formula for π_t has a trivial extension to a projective unitary representation, $\widetilde{\pi}_t$ (given by the same formula as π_t), on functions on $L^2(\mathbb{H}, d\nu_t)$. Moreover $P_t = P_t\widetilde{\pi}_t = \widetilde{\pi}_t P_t = P_t\widetilde{\pi}_t P_t$. Let $\widetilde{\mathcal{A}_t} \subseteq B(L^2(H, \nu_t))$ be the commutant of $\widetilde{\pi}_t(\Gamma)$. By [A], this is a type II_∞ factor, such that $L^2(\widetilde{\mathcal{A}_t})$ is a canonically identified with L_t. Consequently, at least, for k in $L^2(\widetilde{\mathcal{A}_t}) \cap \widetilde{\mathcal{A}}_t$, it makes sense to consider $\mathcal{P}(k) = P_t k P_t$.

Lemma 10.1. *Let $\mathcal{P}_t$ be the orthogonal projection from L_t (identified with $L^2(\widetilde{\mathcal{A}_t})$) into $L^2(\mathcal{A}_t)$.*

Then $\mathcal{P}_t(k)$ is given by the formula $P_t k P_t$, which is well defined for $k \in L^2(\widetilde{\mathcal{A}_t}) \cap \widetilde{\mathcal{A}}_t$ and then extended by continuity. For such a k, the kernel of $(\mathcal{P}_t k)(\overline{z}, \xi)$, z, ξ in $\mathbb{H}$, is given by the formula

$$(\mathcal{P}_t)(\overline{z}, \xi) = c_t^2[(\overline{z} - \xi)/(-2\mathrm{i})]^t \iint\limits_{\mathbb{H}^2} \frac{k(\eta_1, \eta_2)}{(\overline{z} - \eta_1)^t(\overline{\eta_1} - \eta_2)^t(\overline{\eta_2} - \xi)^t}\mathrm{d}\nu_t(\eta_1)\mathrm{d}\nu_t(\eta_2)$$

Proof of the lemma. One can check imediately that the map $\mathcal{P}(k) = P_t k P_t$, for k in $L^2(\widetilde{\mathcal{A}_t})$, defines an orthogonal projection on $L^2(\mathcal{A}_t)$.

The formula for $\mathcal{P}(k)$ follows by writing down the corresponding kernels, and it holds as long as k is in $L^2(\widetilde{\mathcal{A}}_t)$.

In the next lemma we will prove that the Toeplitz operators of symbols $\ln \varphi$ and $\ln \overline{\varphi}$, have dense domain (in $L^2(\mathcal{A}_t)$) and that they are adjoint to each other.

LEMMA 10.2. *Let* $\varphi(\overline{z},\xi) = \dfrac{1}{12}\ln[\overline{\Delta(z)}\Delta(\xi)[(\overline{z}-\xi)/(-2i)]^{12}]$ *(as in Section 7). Let* $\widetilde{\mathcal{D}}_t^0$ *be the union of* $S_{\Delta^\varepsilon}\hat{\mathcal{A}}_{t-3-\varepsilon}S^*_{\Delta^\varepsilon}$, *after* $\varepsilon > 0$, $t-3-\varepsilon > 1$.

Then $Dom(\mathcal{M}_\varphi) \cap H_t$ *contains the weakly dense subalgebra* $\widetilde{\mathcal{D}}_t^0$. *Consequently* $\widetilde{\mathcal{D}}_t^0$ *is contained in the domain of* $\mathcal{P}\mathcal{M}_\varphi\mathcal{P}$, *which is the Toeplitz operator* $\mathcal{T}_\varphi$.

Before beginning the proof of the lemma, we note the following consequence:

COROLLARY 10.3 *The operator* Λ *introduced in Section 7, (restricted to* $\widetilde{\mathcal{D}}_t^0$*) coincides* $\mathcal{M}_{(1/12)\ln\varphi}k$, *acting on the same domain. Morever, the operators* $\mathcal{T}_{\ln\overline{\varphi}}$ *and* $\mathcal{T}_{\ln\varphi}$ *are densely defined and* $\mathcal{T}_{\ln\overline{\varphi}} \subseteq (\mathcal{T}_{\ln\varphi})^*$, $\mathcal{T}_{\ln\varphi} \subseteq (\mathcal{T}_{\ln\overline{\varphi}})^*$. *Consequently, this operators are closable, in* $L^2(\mathcal{A}_t)$.

Proof. The fact that $\Lambda(k)$ is equal to $\mathcal{M}_{(1/12)\ln\varphi}k$ for k in $\mathcal{D}_t^0$, is a consequence of the fact that $\mathcal{M}_\varphi(k)$ is the L^2-valued derivative at 0, of the differentiable, L_t-valued function $\varepsilon \to \mathcal{M}_{\varphi^\varepsilon}(k)$. This is based on the arguments in the proof (below) of Lemma 10.2. Hence, $\mathcal{T}_{\ln\varphi}(k)$ is the derivative at 0, of the differentiable, $L^2(\mathcal{A}_t)$-valued function function $\varepsilon \to \mathcal{T}_{\varphi^\varepsilon}$.

Proof of Lemma 10.2. We have to check that for k in $\widetilde{\mathcal{D}}_t^0$ having the expression $k = S_{\Delta^\varepsilon}k_1S^*_{\Delta^\varepsilon}$, with $k_1 \in \widehat{A}_{t-2-\varepsilon}$ (so that up to a constant $k(\overline{z},\xi) = \varphi^\varepsilon k_1(\overline{z},\xi), z,\xi \in \mathbb{H}$), the following integral:

$$\iint\limits_{F\times\mathbb{H}} |\ln\varphi(z,\xi)|^2|\overline{\Delta(z)}\Delta(\xi)(\overline{z}-\xi)^{12}|^\varepsilon|k_1(z,\xi)|d^{2t}d\nu_0(z) \tag{10.1}$$

is (absolutely) convergent.

Since k_1 belongs to $\widehat{A}_{t-2-\varepsilon}$ we may free up a small power α of d, so that the integral

$$\iint\limits_{F\times\mathbb{H}} |k_1(\overline{z},\xi)|d^{2t-4-\alpha}\mathrm{d}\nu_o(z,\xi)$$

is still convergent. We proved in Section 8 that for any $\varepsilon < \varepsilon'$, there exists a positive constant $C_{\varepsilon,\varepsilon'}$, such that

$$|\ln\Delta(z)\Delta^\varepsilon(z)| \leq C_{\varepsilon,\varepsilon'}\frac{\mathrm{Re}z}{(\mathrm{Im}\ z)^2}e^{-\varepsilon'\ \mathrm{Im}\ \mathrm{z}}.$$

When evaluating the integral in (10.1), we will have to find an estimate for each of the terms that arrise by writting

$$\ln \varphi(\overline{z}, \xi) = \ln \overline{\Delta(z)} + \ln \Delta(\xi) + 12 \ln(\overline{z} - \xi)$$

After taking the square, we see that it remains to prove that the integrals, containing the following quadratic terms, are finite:

$$|\ln \Delta(z)|^2 |\Delta(z)|^{\varepsilon} d^{\alpha},$$

$$|\ln \Delta(\xi)|^2 |\Delta(\xi)|^{\varepsilon} d^{\alpha},$$

$$|\ln[(\overline{z} - \xi)/(-2\mathrm{i})]|^2 |\Delta^{\varepsilon}(\xi)|.$$

We analyze for example the term involving $|\ln \Delta(z)|^2$. By using Corollary 8.6, we note that the integral is consequently bounded by

$$\iint\limits_{F \times \mathbb{H}} \frac{(\mathrm{Re}\ z)^2}{(\mathrm{Im}\ z)^6} e^{-\varepsilon' \mathrm{Im}\ z} e^{-\varepsilon \mathrm{Im}\ \xi} |\overline{z} - \xi|^{12\varepsilon} \cdot (d(z, \xi))^{2t} |k_1(\overline{z}, \xi)|^2 d\nu_0(z)$$

We write $(d(z,\xi))^{2t} = (d(z,\xi))^{2(t-3)} \cdot d(z,\xi)^6$ to get that the above integral is bounded by

$$\iint\limits_{F \times \mathbb{H}} e^{-\varepsilon' \mathrm{Im}\ z} e^{-\varepsilon \mathrm{Im}\ \xi} \frac{(\mathrm{Im}\ \xi)^6}{|\overline{z} - \xi|^{12 - 12\varepsilon}} \cdot (d(z, \xi))^{2(t-3)} |k_1(z, \xi)|^2 d\nu_0(z, \xi)$$

Because of the term $e^{-\varepsilon \mathrm{Im}\ \xi}$, by eventually multiplying with a a constant, we can neglect the term $(\mathrm{Im}\ \xi)^6$.

Thus we are led to analyze the following integral

$$\iint\limits_{F \times \mathbb{H}} (\mathrm{Re}\ z)^2 e^{-\varepsilon' \mathrm{Im}\ z} e^{-\varepsilon \mathrm{Im}\ \xi} \frac{1}{|\overline{z} - \xi|^{12 - 12\varepsilon}} (d(z, \xi))^{2(t-3)} |k_1(z, \xi)|^2 d\nu_0(z)$$

Because $(z, \xi) \in F \times \mathbb{H}$, it follows that there is a constant C such that $|\overline{z} - \xi| \geq C$, for $z, \xi \in F \times \mathbb{H}$. Also $(\mathrm{Re}\ z)^2 / |\overline{z} - \xi|^2$ is bounded from above on this region.

Thus the above integral is bounded by a constant times

$$\iint\limits_{F\times\mathbb{H}} (d(z,\xi))^{2(t-3)}|k_1(\overline{z},\xi)|^2 d\nu_0(z,\xi)$$

which is finite if $k_1 \in \mathcal{A}_{t-3-\varepsilon}$.

The terms involving $|\ln[(\overline{z}-\xi)/(-2\mathrm{i})]|$ are solved by absorbing $|\ln[(\overline{z}-\xi)/(-2\mathrm{i})]|$ into some power of $|\overline{z}-\xi|$.

Clearly $\mathcal{PM}_{\ln\overline{\varphi}}$ has the same domain $\mathcal{PM}_{\ln\varphi}$. This is precisely the vector space of all $k \in L^2(\mathcal{A}_t)$ such that $|k(\ln\varphi)|^2 = |k\overline{(\ln\varphi)}|^2$ is summable on $F\times\mathbb{H}$, with respect to the measure $d^t d\nu_0 \times d\nu_0$. This completes the proof.

We introduce the following definition which will be used in the dual solution for the cohomology problem, corresponding to $\mathcal{C}_t$.

Definition 10.4. *Let $\chi_{s,t} : \mathcal{A}_t \to \mathcal{A}_s$ be defined by the formula*

$$\chi_{s,t}k = S^*_{\Delta^{(t-s)/12}} k S_{\Delta^{(t-s)/12}},$$

for k in $\mathcal{A}_t$. Here $s \le t$.

In the next proposition we analyze the relation between the derivative of $\chi_{s,t}$ at $s = t$, $s \nearrow t$ with the derivative of $\theta_{s',t}$, at $s' = t, (s' \searrow t)$ introduced in Section 7.

Definition 10.5 *For $t > 1$, we let $\mathcal{D}^+_t$ be the algebra consisting of all k in $\mathcal{A}_s$ that for some $s < t$, are of the form $S^*_{\Delta^\varepsilon} k_1 S_{\Delta^\varepsilon}$, for some $\varepsilon > 0$, such that $s + \varepsilon < t$, and $k_1 \in \mathcal{A}_{s+\varepsilon}$.*

Clearly $\mathcal{D}^+_t$ is a weakly dense, unital subalgebra of $\mathcal{A}_t$.

Lemma 10.6. *Fix $t > 1$. Assume that k in $\mathcal{D}^+_t$ has the expression $k = S^*_{\Delta^\varepsilon} k_1 S_{\Delta^\varepsilon}$, $k_1 \in \mathcal{A}_s, \varepsilon > 0, \varepsilon + s < t$. Then*

$$k = \mathcal{T}_{\overline{\varphi}^\varepsilon}(k_1)\mathcal{T}_{\overline{\Delta^\varepsilon(z)\Delta^\varepsilon(\xi)[(\overline{z}-\xi)/(-2i)]^{12\varepsilon}}}(k_1).$$

Remark *Note that by putting the variables $\overline{z}, \xi$, we indicated that k_1 is multiplied by a function, that contrary to k_1, is antianalytic in the second variable and analytic in the first. Thus $\mathcal{T}_{\overline{\varphi}^\varepsilon}$ corresponds to a Toeplitz operator with an "antianalytic" symbol.*

Proof of Lemma 10.6. Let $T(k) = S^*_{\Delta^\varepsilon} k S_{\Delta^\varepsilon}$. The statement follows immediately from the fact that the adjoint of T, as a map on $L^2(\mathcal{A}_t)$, is $l \to S_{\Delta^\varepsilon} l S^*_{\Delta^\varepsilon}$.

In the next proposition we clarify the relation between the operator $Y_t k$ defined as $\frac{\mathrm{d}}{\mathrm{d}s}\chi_{s,t}(k)\Big|_{\substack{s=t\\ s\nearrow t}}$ and the operator X_t introduced in Section 7.

First we recall that the "real part" associated with the deformation is given by the Dirichlet form $\mathcal{E}_s(k,l) = \frac{\mathrm{d}}{\mathrm{d}s}\tau_{\mathcal{A}_s}(k *_s l) = \tau_{\mathcal{A}_s} c_s(k,l)$.

Definition 10.7 *Recall (from Section 9), that the real part of the cocycle $\mathcal{C}_t$ is the operator R_t given by by the formula:*

$$< R_t k, l^* >= -\frac{1}{2}\frac{d}{ds} \cdot \tau_{\mathcal{A}_s}(k *_s l)\Big|_{\substack{s=t\\ s\searrow t}} = -\frac{1}{2}\mathcal{E}_s(k,l).$$

This holds for all k,l in $\underset{r<t}{\cup} L^2(\mathcal{A}_r)$, (where $L^2(\mathcal{A}_r)$) is identified with a vector subspace of $L^2(\mathcal{A}_t)$ via the symbol map $\Psi_{t,r}$

Moreover, in Section 9 we proved that R_t has the following expression:

$$R_t = \mathcal{T}_{\ln d} - \frac{1}{2}\frac{c_t'}{c_t} \cdot \mathrm{Id}.$$

In the next proposition we construct the dual object for the generator used in Section 7.

Proposition 10.8 *For any k in $\mathcal{D}_t^+ \subseteq \mathcal{A}_t$ the limit*

$$Y_t(k) = \frac{d}{ds}\Psi_{t,s}(\chi_{s,t}k))\Big|_{\substack{s=t\\ s\nearrow t}},$$

exists in $L^2(\mathcal{A}_t)$. Moreover, we have that

$$Y_t = -\left(\mathcal{T}_{(1/12)\ln\varphi}\right)^* - \frac{c_t'}{c_t}\mathrm{Id} \ + 2R_t.$$

The adjoint $(\mathcal{T}_{(1/12)\ln\varphi})^*$ is obtained by first restricting $\mathcal{T}_{(1/12)\ln\varphi}$ to $\widetilde{\mathcal{D}}_t^0$ and then taking the adjoint.

Proof. Indeed $\chi_{s,t}(k)$ may be identified with the *Toeplitz operator (on $L^2(\mathcal{A}_s)$)* with symbol

$$\overline{\varphi^{(t-s)/12}} = \overline{[\Delta^{(t-s)/12}(z)\Delta^{(t-s)/12}(\xi)[(\overline{z}-\xi)/(-2\mathrm{i})]^{(t-s)}]}$$

Thus $\chi_{s,t}(k)$ is (modulo a multiplicative constant)

$$\mathcal{P}_s[\mathcal{M}_{\overline{\varphi(t-s)/12}}k]$$

and hence $\Psi_{t,s}\chi_{s,t}(k)$ is

$$\Psi_{t,s}\mathcal{P}_s[\mathcal{M}_{\overline{\varphi^{(t-s)/12}}}(k)]$$

The derivative at $s=t$ involves consequently two components:

One component is the derivative $\left.\frac{\mathrm{d}}{\mathrm{d}s}\Psi_{t,s}(\mathcal{P}_s k)\right|_{s\nearrow t}$ which gives the summand corresponding to R_t, i.e. $-(c_t'/c_t)+2R_t$.

The other component is $\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{P}_t(\mathcal{M}_{\overline{\varphi}(t-s)/12}(k))$ which gives the multiplication by φ part. Indeed, recall that k belongs to $\mathcal{D}_t^+\subseteq\mathcal{A}_t$, and hence k is of the form $S^*_{\Delta^\varepsilon}k_1S_{\Delta_0^\varepsilon}$, for some $\varepsilon_0>0$, such that $s+\varepsilon_0<t$, and $k_1\in\mathcal{A}_{s+\varepsilon_0}$. But then

$$\mathcal{P}_t(\mathcal{M}_{\overline{\varphi^{(t-s)/12}}}(k)) = \mathcal{P}_t(\mathcal{M}_{\overline{\varphi^{(t-s)/12}}}(\mathcal{P}_s(\mathcal{M}_{\overline{\varphi^{\varepsilon_0}}}(k_1)))).$$

Since, $\overline{\varphi}$ plays the role of an antianalytic symbol, it follows that this is further equal to

$$\mathcal{P}_t\left(\mathcal{M}_{\overline{\varphi^{[(t-s)/12+\varepsilon_0]}}}(k_1)\right).$$

The derivative (in the s variable) of $s\to\overline{\varphi}^{[(t-s)/12+\varepsilon_0]}$ at $s=t$ exists, by the method in Lemma 10.2 in L_t and it is equal to

$$-\frac{1}{12}\ln\overline{\varphi}\cdot\overline{\varphi}^{\varepsilon_0}\cdot k_1.$$

Thus, in the Hilbert space $L^2(\mathcal{A}_t)$, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{P}_t(\mathcal{M}_{\varphi^{(t-s)/12}}(k)) &= -\mathcal{P}_t(\mathcal{M}_{(1+/12)\ln\overline{\varphi}\overline{\varphi}^{\varepsilon_0}}(k_1))\\ &= -\mathcal{P}_t(\mathcal{M}_{(1+/12)\ln\overline{\varphi}}(\mathcal{P}_{t+12\varepsilon_0}\overline{\varphi}^{\varepsilon_0}(k_1)))\\ &= -\mathcal{P}_t(\mathcal{M}_{(1+/12)\ln\overline{\varphi}}(k)).\end{aligned}$$

This completes the proof.

We use above arguments to prove that also the operator $Y_t = \frac{\mathrm{d}}{\mathrm{d}s}\chi_{s,t}$ implements a coboundary for $\mathcal{C}_t$.

Lemma 10.9. *For k, l in $\mathcal{D}_t^+$, we have that* $\left.\dfrac{d}{ds}\chi_{s,t}(k *_t \varphi^{t-s}(\overline{z}, \xi) *_t l)\right|_{s\nearrow t}$ *is equal to* $Y_t(k *_t l) - k *_t \Lambda(1) *_t l$.

Proof. Since k, l are in $\mathcal{D}_t^+$, there exists $\varepsilon_0 > 0$ and there are $k_1, l_1 \in \mathcal{A}_{t+12\varepsilon_0}$ such that $k = S^*_{\Delta_0^\varepsilon} k_1 S_{\Delta^{\varepsilon_0}}$, $l = S^*_{\Delta_0^\varepsilon} l_1 S_{\Delta^{\varepsilon_0}}$. This gives that

$$k *_t \Lambda(1) *_t l = S^*_{\Delta^\varepsilon}[k_1 *_{t+\varepsilon} \varphi^\varepsilon \ln \varphi *_{t+\varepsilon} l_1] S_{\Delta^\varepsilon}$$

where by $\varphi^\varepsilon \ln \varphi$ we understand the unbounded operator defined in Section 6, corresponding to

$$\varphi(\overline{z}, \xi)^\varepsilon \ln \varphi(\overline{z}, \xi)$$

As in the proof of Proposition 7.6, when computing this derivative, we have a trivial summand plus a more complicated summand, corresponding to the symbol

$$\lim_{s\nearrow t} \Psi_{t,s}\left[\mathcal{P}_s \overline{\varphi^{t-s}}\left[k *_t \frac{\varphi^{t-s} - \mathrm{Id}}{t-s} *_t l\right]\right]$$

Because of the assumptions, the inside term

$$k *_t \frac{\varphi^{t-s} - \mathrm{Id}}{t-s} *_t \varphi$$

is equal to

$$\begin{aligned} & S_{\Delta^{\varepsilon_0}}\left[k_1 *_{t+\varepsilon_0} \varphi^{\varepsilon_0} \frac{\varphi^{t-s} - \mathrm{Id}}{t-s} *_{t+\varepsilon_0} l_1\right] S^*_{\Delta^{\varepsilon_0}} \\ &= \mathcal{P}_t\left(\overline{\varphi^{\varepsilon_0}}\left(k_1 *_{t+\varepsilon_0} \varphi^{\varepsilon_0} \frac{\varphi^{t-s} - \mathrm{Id}}{t-s} *_{t+\varepsilon_0} l_1\right)\right) \end{aligned} \tag{10.2}$$

But the methods in the proof of the density of the domain of $\mathcal{M}_{\ln \varphi}$, may also be used to prove that

$$\varphi^{\varepsilon_0}\left(\frac{\varphi^{t-s} - \mathrm{Id}}{t-s}\right)$$

converges, as $s \nearrow t$ in $L^2(\mathcal{A}_{t+12\varepsilon_0})$ to $-\varphi^{\varepsilon_0} \ln \varphi$.

Since $\Psi_{t,s}\mathcal{P}_s$ converges strongly to the identity, and the norm of $\Psi_{t,s}\mathcal{P}_s$ as an operator from $L^2(\widehat{\mathcal{A}_s})$ into $L^2(\mathcal{A}_t)$ is bounded by 1, it follows that the expression in 10.2 converges to

$$\mathcal{P}_t(\overline{\varphi}_{\varepsilon_0}[k_1 *_{t+\varepsilon_0} \varphi^{\varepsilon_0}(-\ln \varphi) *_{t+\varepsilon_0}])$$

which is $k *_t (-\Lambda(1)) *_t l$.

Similarly we have the following lemma

Lemma 10.10. *For k, l in $\mathcal{D}_t^+$ we have that*

$$\frac{d}{ds}[\chi_{t,s}(k) *_s \chi_{t,s}(l)]\bigg|_{s \nearrow t} = Y_t(k) *_t l + \mathcal{C}_t(k,l) + k *_t Y_t(l).$$

Proof. Again this derivative has three summands: the first summand is

$$\lim_{s \nearrow t} \frac{\chi_{t,s}(k) *_s \chi_{t,s}(l) - \chi_{t,s}(k) *_t \chi_{t,s}(l)}{t-s}$$

The same type of argument as in Proposition 7.5 gives that this is $\mathcal{C}_t(k,l)$.

From the remaining two summands, the only one that is complicated is

$$\chi_{s,t}(k) *_t \frac{\chi_{s,t}(l) - l}{t-s}.$$

Because for l in $\mathcal{D}_+^t$, we have that $\dfrac{\chi_{s,t}(l) - l}{t-s}$ converges in $L^2(\mathcal{A}_t)$ to $Y_t l$, and since $\chi_{s,t}(k)$ is bounded in $L^2(\mathcal{A}_t)$ as $s \nearrow t$, it follows that this term converges too, to $k *_t Y_t l$. The remaining term trivially converges to $Y_t k *_t l$.

This completes the proof of the lemma.

As a corollary we obtain the following result:

Proposition 10.11. *Let $\mathcal{D}_t^+$ be as in Definition 10.5. Assume $t > 3$, then for all k, l in $\mathcal{D}_t^+$, we have that*

$$Y_t(k *_t l) - Y_t k *_t l - k *_t Y_t l - k *_t \frac{1}{12} \ln \varphi *_t l = \mathcal{C}_t(k,l).$$

Here by $\dfrac{1}{12}\varphi(\overline{z}, \xi)$ we understand $\Lambda(1) = \mathcal{M}_{\ln \varphi}(1)$, the operator constructed in Corollary 6.6 and in Lemma 7.4.

Proof. Indeed the following identity:

$$\chi_{s,t}(k *_t \varphi^{t-s} *_t l) = \chi_{s,t}(k) *_s \chi_{s,t}(l)$$

is obvious, valid for all k, l in $\mathcal{D}_t^+$, $s \leq t$.

By differentiation, and using the two previous lemmas, we get our result.

Remark. *Recall that in Proposition 9.3 we proved that if $\Lambda_0(k) = \mathcal{M}_{\ln \varphi} k$, (for k in $\mathcal{D}_0^t$), then, denoting $S = Sym_\varphi$, we have*

$$< \Lambda_0(k), l > + \langle k, \Lambda_0(l) > = 2 < Sk, l > +2 < R_t k, l > + \frac{c_t'}{c_t} < k, l >$$

If k would belong to $\widetilde{\mathcal{D}}_t^0$ (which is the domain of Λ_0) and also to the domain of Y_t, which is $\mathcal{D}_t^+$, then the above relation could be rewritten as

$$\Lambda_0 + \Lambda_0^* = 2S + 2R_t + \frac{c_t'}{c_t} \qquad (10.3).$$

Recal that $S = \mathrm{Sym}_\varphi$ is the operator defined by

$$< Sk, l >= \tau_{\mathcal{A}_t}(kTl^*) + \tau_{\mathcal{A}_t}(l^*Tk)$$

But on the intersection of the domains we have (from Proposition 10.8) that

$$Y_t = -\Lambda_0^* + 2R_t - \frac{c_t'}{c_t}\mathrm{Id}. \qquad (10.4)$$

Consequently $\Lambda_0^* = -Y_t + 2R_t - \frac{c_t'}{c_t}\mathrm{Id}$. Thus, by (10.3), for k in $\widetilde{\mathcal{D}}_t^0 \cap \mathcal{D}_t^+$ we get that $\Lambda_0 = 2S + Y_t + 2\frac{c_t'}{c_t}\mathrm{Id}$. and hence that

$$X_t = \Lambda_0 - \frac{c_t'}{c_t}\mathrm{Id} = 2S + Y_t + \frac{c_t'}{c_t}\mathrm{Id} \qquad (10.5)$$

where equality holds on $\widetilde{\mathcal{D}}_t^0 \cap \mathcal{D}_t^+$

Now we compare the way X_t Y_t implement a coboundary for $\mathcal{C}_t(k, l)$ Recall the notation $(\nabla\Phi)(k, l) = \Phi(k, l) - k\Phi(l) - \Phi(k)l$

Thus we have proved that

$$\nabla X_t(k,l) = \mathcal{C}_t(k,l) - kT^t_{\ln\varphi}\varphi, \qquad k,l \text{ in} \widetilde{\mathcal{D}}^0_t \tag{10.6}$$

$$\nabla Y_t(k,l) = \mathcal{C}_t(k,l) - k\Lambda(1)l, \qquad k,l \text{ in } \mathcal{D}^+_t \tag{10.7}$$

Now if k,l would be in $\widetilde{\mathcal{D}}^0_t \cap \mathcal{D}^+_t$, it would follows, by substituting (10.5) in (10.6), that

$$2\nabla S_t(k,l) + \nabla Y_t(k,l) - \frac{c'_t}{c_t} k *_t l = \mathcal{C}_t(k,l) - kT^t_{\ln\varphi}l. \tag{10.8}$$

By using (10.7) in (10.8) we get

$$2(\nabla S_t)(k,l) - k\Lambda(1)l - \frac{c'_t}{c_t}(k *_t l) = -kT^t_{\ln\varphi}l,$$

and thus that for k,l in $\widetilde{\mathcal{D}}^0_t \cap \mathcal{D}^+_t$ we would get that

$$k\left[T^t_{\ln\varphi} - \left(\Lambda(1) + \frac{c'_t}{c_t}\right)\right]l = 2\nabla S_t(k,l) \tag{7.9}$$

for all k,l in $\widetilde{\mathcal{D}}^0_t \cap \mathcal{D}^+_t$. But recall that $< S_t(k), l^* >= \tau(kTl^* + l^*Tk)$ This corresponds, at least formally, to the fact that $S_t k = kT + Tk$ and hence $(\nabla S_t)(k,l)$ is $2kT^t_{\ln\varphi}l$.

Thus (10.9) would imply directly that $T^t_{\ln\varphi} = \Lambda(1) + \dfrac{c'_t}{c_t}$ if $\widetilde{\mathcal{D}}^0_t \cap \mathcal{D}^+_t$ is nonzero.

11. Appendix
A more general coboundary for $\mathcal{C}_t$

In this appendix, we want to construct a more general solution for a coboundary (which is necessary unbounded, see ([27]) for $\mathcal{C}_t$. This will be constructed out a measurable function g, that has the same Γ- invariance properties as $\ln\Delta(z)$. By this construction we will lose the completely positivity properties of the solution.

Recall that L_t consists of all kernels k or $\mathbb{H}\times\mathbb{H}$, that are diagonally Γ-invariant and square summable on $F\times\mathbb{H}$, against the measure $d^t \mathrm{d}\nu_0 \times \mathrm{d}\nu_0$.

Also recall that the elements in L_t are canonically identified with operators in the II_∞ factor, of all operators that commute with $\widetilde{\pi}_t(\Gamma)$, acting on $L^2(\mathbb{H}, \mathrm{d}\nu_t)$.

Proposition 11.1. *Let g be a measurable function $\mathbb{H}$ such that the bivariable function θ in $\mathbb{H} \times \mathbb{H}$ defined by $\theta(z,\xi) = \overline{g(z)} + g(\xi) + \ln[(\overline{z} - \xi)/(-2i)]$ be Γ-invariant. (It is this point which makes the problem sovable, by this method, only for* $\mathrm{PSL}(2,\mathbb{Z})$.*)*

Let $\mathcal{D}_g, \widetilde{\mathcal{D}_g}$ consist of all k in $L^2(\mathcal{A}_t)$ (respectively L_t) such that $k \cdot \theta$ still belongs to L_t. Let $\mathcal{M}_\theta$ the (unbounded operator) with domain $\widetilde{\mathcal{D}_g}$, of multiplication by θ. Let $\mathcal{T}_\theta = \mathcal{P}_t \mathcal{M}_\theta|_{L^2(\mathcal{A}_t)}$ and let T_θ^t be the Toeplitz operator with symbol $\theta(z,z) = Reg(z) + \ln(\overline{z} - z)$.

Let k, l be in $\mathcal{D}_g$ such that k, l also belong to the domain of $\mathcal{C}_t(k,l)$. Then

$$\mathcal{M}_\theta(k *_t l) - \mathcal{M}_\theta k *_t l - k *_t \mathcal{M}_\theta l + M_\theta(k,l) = \mathcal{C}_t(k,l),$$

where $M_\theta(k,l)$ is a bimodule map, equal to $k *_t T_\theta^t *_t l$, if T_θ^t exists.

Consequently, by taking P_t on the left and right hand side, the same will hold true for $\mathcal{T}_\theta = \mathcal{P}_t \mathcal{M}_\theta|_{L^2(\mathcal{A}_t)}$.

Proof. Indeed $\mathcal{C}_t(k,l)$ is given by the kernel

$$\mathcal{C}_t(k,l)(\overline{z},\xi) = \frac{c_t'}{c_t}(k *_t l)(\overline{z},\xi) + c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{z},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t \ln[\overline{z},\eta,\overline{\eta},\xi] \mathrm{d}\nu_0(\eta).$$

On the other hand $\mathcal{M}_\theta(k *_t l) - \mathcal{M}_\theta k *_t l - k *_t \mathcal{M}_\theta l$ has the kernel

$$c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{z},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t (\theta(z,\xi) - \theta(z,\eta) - \theta(\eta,\xi)) \mathrm{d}\nu_0(\eta).$$

Since $\theta(z,\xi) - \theta(z,\eta) - \theta(\eta,\xi)$ is equal to $\theta(\eta,\eta)$, it follows that

$$\mathcal{C}_t(k,l) - [\mathcal{M}_\theta(k *_t l) - \mathcal{M}_\theta k *_t l - k *_t \mathcal{M}_\theta l]$$

is given by the kernel

$$[(\overline{z} - \xi)/(-2\mathrm{i})]^t \int_{\mathbb{H}} \frac{k(\overline{z},\eta) l(\overline{\eta},\xi)}{[(\overline{z} - \eta)/(-2\mathrm{i})]^t [(\overline{\eta} - \xi)/(-2\mathrm{i})]^t} \left(\theta(\eta,\eta) + \frac{c_t'}{c_t}\right) \mathrm{d}\nu_t(\eta)$$

which indeed corresponds to $T_\theta^t + (c_t'/c_t)\cdot \mathrm{Id}$, as long as we can make sense of the unbounded Toeplitz operator T_θ^t.

A dual version could be obtained if we consider

$$\langle 2R_t k, l^* \rangle_{L^2(\mathcal{A}_t)} = -\frac{\mathrm{d}}{\mathrm{d}s}\tau_{\mathcal{A}_s}(k *_s l^*)\Big|_{s=t},$$

which is in other terms $2R_t = -\dfrac{c_t'}{c_t} - 2\mathcal{T}_{\ln \mathrm{d}}$

One can check immediately that

$$\begin{aligned}
&[\mathcal{C}_t(k,l) - (\nabla 2R_t)(k,l)](\overline{z},\xi)\\
&\quad = \frac{c_t'}{c_t}\tau(k *_t l) + c_t \int_{\mathbb{H}} (k(\overline{z},\eta) l(\overline{\eta},\xi) \ln[\overline{z},\eta,\overline{\eta},\xi][\overline{z},\eta,\overline{\eta},\xi]^t \mathrm{d}\nu_0(\eta)\\
&\quad + [\int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi)[-\ln(d(z,\xi))^2 + \ln(d(z,\eta))^2 + \ln(d\eta,\xi))^2]\\
&\quad - \left(-\frac{c_t'}{c_t} + \frac{c_t'}{c_t} + \frac{c_t'}{c_t}\right)\tau(k *_t l)\\
&\quad = c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]^t\\
&\quad \cdot \{\ln([\overline{z},\eta,\overline{\eta},\xi] - 2\ln d(z,\xi) + 2\ln d(z,\eta) + 2\ln d(\eta,\xi)\}\mathrm{d}\nu_0(\eta)\\
&\quad = -c_t \int_{\mathbb{H}} k(\overline{z},\eta) l(\overline{\eta},\xi)[\overline{z},\eta,\overline{\eta},\xi]. \ln[\overline{\overline{z},\eta,\overline{\eta},\xi}]\mathrm{d}\nu_0\eta
\end{aligned}$$

Then consider g such that

$$\theta(z,\xi) = g(z) + \overline{g(\xi)} + \ln(z - \overline{\xi})$$

is Γ-invariant.

The same argument as above gives that for k, l in $\mathcal{D}(\mathcal{M}_\theta)$, s.t. $k *_t l \in \mathcal{D}(\mathcal{M}_\theta)$ and k, l in $\mathrm{Dom}(R_t)$, $k *_t l$ in $\mathrm{Dom}(R_t)$ we have that

$$\mathcal{M}_\theta(k *_t l) - \mathcal{M}_\theta k *_t l - k *_t \mathcal{M}_\theta l + k *_t T_\theta^t *_t l$$

is equal to $\mathcal{C}_t(k,l) - 2\nabla R_t(k,l)$.

Finally remark that we have proved that for k, l in $\mathcal{D}_t^+$, which is the vector space of all k that are of the form $\mathcal{P}_t(k_1\overline{\varphi^\varepsilon})$ we have that the expression:

$$\mathcal{M}_{\overline{\varphi}}(k *_t l) - \mathcal{M}_{\overline{\varphi}}k *_t l - k *_t \mathcal{M}_{\overline{\varphi}}l + k *_t \Lambda(1) *_t l - \mathcal{C}_t(k, l)$$

is orthogonal to $\mathcal{P}_t(L^t)$ (otherwise if we apply P_t to the left and right we get 0).

If we could extend the above relation to all k, l in $\mathrm{Dom}(\mathcal{M}_{\overline{\varphi}}) \cap \mathrm{Dom} R_t$, such that $k *_t l$ belongs to the same domain, then the above relation, by the considerations at the end of Section 10, would imply that $\Lambda(1) - (c_t'/c_t) \cdot \mathrm{Id}$ coincides (on an affiliated domain) with $T^t_{\ln\varphi}$.

Note that this corresponds formally to the fact that

$$T^t_{\ln\varphi} = T^t_{\ln(\overline{\eta}-\eta)} + T^t_{\overline{\ln\Delta}} + T^t_{\ln\Delta}.$$

On the other hand $T^t_{\ln(\overline{\eta}-\eta)} = \mathcal{P}_{\ln(\overline{z}-\xi)} - (c_t'/c_t) \cdot \mathrm{Id}$, while $T^t_{\overline{\ln\Delta}}$, is clearly, on its domain, $S^t_{\overline{\ln\Delta}}$ (and similarly for $T^t_{\ln\Delta}$).

If the domains would have nonzero intersection one could directly conclude that

$$S^t_{\overline{\ln\Delta}} + S^t_{\ln\Delta} + \mathcal{P}_{\ln(\overline{z}-\xi)} = \Lambda(1).$$

References

[1]Accardi, L., Hudson, R., *Quantum Stochastic flows and Non Abelian Cohomology*, in Quantum Probability and applications V, L. Accardi, W. von Waldenfelds (Eds.), Lect. Notes in Math., 1442, Springer Verlag, 1990.

[2] Atiyah, M. F. *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "Analyze et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), pp. 43–72. Asterisque, No. 32-33, Soc. Math. France, Paris, 1976.

[3] Atyiah, M.F., Schmidt, W., *A geometric construction of the discrete series for semisimple Lie groups*, Invent. Math. **42**, (1977), pp. 1-62.

[4] Berezin, F.A., *General Concept of Quantisation*, Comm. Math. Phys, **40**(1975), 153–174.

[5] Bargmann, V., *Irreducible unitary representations of the Lorenz group, Annals of Mathematics*, **48** (1947), 568-640.

[6] Barge, J.; Ghys, E. *Surfaces et cohomologie borne*, Invent. Math. 92 (1988), no. 3, 509–526.

[7] Chebotarev, A. M. ; Fagnola, F. *Sufficient conditions for conservativity of minimal quantum dynamical semigroups*, J. Funct. Anal. **153** (1998), 382–404.

[8] Christensen, E; Evans, D. E. *Cohomology of operator algebras and quantum dynamical semigroups*, J. London Math. Soc, **20** (1979), 358–368.

[9] Cohen, P. Beazley,; Hudson, R. L., *Generators of quantum stochastic flows and cyclic cohomology*, Math. Proc. Cambridge Philos. Soc. **123** (1998), 345–363.

[10] Connes, A., *Sur la Theorie Non Commutative de l'Integration.* Springer Verlag, 725.

[11] Connes, A., *Noncommutative Geometry*, Academic Press, Harcourt Brace & Company Publishers, 1995.

[12] Connes, A., *On the spatial theory of von Neumann algebras*, J. Funct. Anal. 35 (1980), 153-164.

[13] Connes, A.; Cuntz, J., *Quasi homomorphismes, cohomologie cyclique et positivit. [Quasihomomorphisms, cyclic homology and positivity* Comm. Math. Phys. **114** (1988), 515–526.

[14] Davies, E. B. *Uniqueness of the standard form of the generator of a quantum dynamical semigroup*, Rep. Math. Phys. **17** (1980), 249–264.

[15] Dykema, K., *Interpolated free group factors*, Pacific J. Math. **163** (1994), no. 1, 123–135.

[16] Gorini, V.; Kossakowski, A.; Sudarshan, E. C. G., *Completely positive dynamical semigroups of N-level systems*, J. Mathematical Phys. **17** (1976), 821–825.

[17] Goswami, D.; Sinha, K. B., *Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra.* Comm. Math. Phys. **205** (1999), 377–403.

[18] Goodman, F., de la Harpe, P., V.F.R. Jones, Coxeter Graphs and Towers of Algebras, Springer Verlag, New York, Berlin, Heidelberg, 1989.

[19] Jolissaint, P.; Valette, A., it Normes de Sobolev et convoluteurs bornés sur $L^2(G)$, Ann. Inst. Fourier (Grenoble) **41** (1991), 797–822.

[20] Holevo, A. S. *Covariant quantum dynamical semigroups: unbounded generators*, in Irreversibility and causality (Goslar, 1996), 67–81, Lecture Notes in Phys., **504**, Springer, Berlin, 1998.

[21] Hudson, R. L.; Parthasarathy, K. R., *Quantum Ito's formula and*

stochastic evolutions, Comm. Math. Phys. **93**, (1984), 301–323.

[22] Lindblad, G. *On the generators of quantum dynamical semigroups*, Comm. Math. Phys. **48** (1976), 119–130.

[23] Mohari, A.; Sinha, K. B., *Stochastic dilation of minimal quantum dynamical semigroup*, Proc. Indian Acad. Sci. Math. Sci. **102** (1992), 159–173.

[24] Murray, F. J.; von Neumann, J., *On ring of Operators,IV*, Annals of Mathematics, **44** (1943), 716-808.

[25] Miyake, T., *Modular forms*, Springer-Verlag, Berlin-New York, 1989.

[26] Pukanszky, L., *The Plancherel formula for the universal covering group of* SL(R,2), Math. Ann. 156 (1964), 96-143.

[27] Rădulescu, F., *The* Γ*-equivariant form of the Berezin quantization of the upper half plane*, Mem. Amer. Math. Soc. **133**, (1998), no. 630,

[28] Rădulescu, F., *On the von Neumann Algebra of Toeplitz Operators with Automorphic Symbol*, in Subfactors, Proceedings of the Taniguchi Symposium on Operator Algebras, edts H. Araki, Y. Kawahigashi, H. Kosaki, World Scientific, Singapore-New Jersey,1994.

[29] Rădulescu, F., *Quantum dynamics and Berezin's deformation quantization*, in Operator algebras and quantum field theory (Rome, 1996), 383–389, Internat. Press, Cambridge, MA, 1997.

[30] Sally, P., *Analytic Continuation of the Irreducible Unitary Representations of the Universal Covering Group*, Memoirs A. M. S. , 1968.

[31] Sauvageot, J.-L., *Quantum Dirichlet forms, differential calculus and semigroups*, in Quantum probability and applications, V (Heidelberg, 1988), 334–346, Lecture Notes in Math., **1442**, Springer, Berlin, 1990.

[32] Shapiro, H. S.; Shields, A. L. *On the zeros of functions with finite Dirichlet integral and some related function spaces* Math. Z., **80**, (1962), 217–229.

[33] Voiculescu, D., *Limit laws for random matrices and free products*, Invent. Math. **104** (1991), 201–220.

Author:

Florin Rădulescu
Dipartimento Matematica, Universita Tor Vergata,
00133 Roma, Italy
email: radulesc@mat.uniroma2.it