A MODEL FOR SHAPE MEMORY ALLOYS WITH THE
POSSIBILITY OF VOIDS

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Abstract. The paper is devoted to the study of a mathematical model for
the thermomechanical evolution of metallic shape memory alloys. The main
novelty of our approach consists in the fact that we include the possibility for
these materials to exhibit voids during the phase change process. Indeed, in the
engineering paper [60] has been recently proved that voids may appear when
the mixture is produced by the aggregations of powder. Hence, the composition
of the mixture varies (under either thermal or mechanical actions) in this way:
the martensites and the austenite transform into one another whereas the voids
volume fraction evolves. The first goal of this contribution is hence to state a
PDE system capturing all these modelling aspects in order then to establish
the well-posedness of the associated initial-boundary value problem.

1. **Introduction.** Shape memory alloys are mixtures of many martensites variants
and of austenite. They exhibit an unusual behavior: even if they are permanently
deformed, they can totally recover their initial shape just by thermal or mechanical
means. There may be voids in the mixture, which may appear when the mixture
is produced by the aggregations of powders, as it has been recently proved in the
engineering paper [60]. Of course, the voids are filled either with gas or air when
appearing or when aggregating powders. We do not take into account the gas
phase mechanical properties (which are mainly described in terms of pressure and
temperature) because we focus on the mechanical behaviour of the solid mixture,
i.e., we assume the volume fraction of voids is small. The composition of the mixture
varies: the martensites and the austenite transform into one another whereas the
voids volume fraction evolves. These phase changes can be produced either by
thermal actions or by mechanical actions. The striking properties of shape memory
alloys result from interactions between mechanical and thermal actions (cf., e.g.,
[9, 42]).

We assume that the phases can coexist at each point and we suppose that,
besides austenite, only two martensitic variants are present. However, this choice
provides a sufficiently good description of the phenomenon, as we want describe
a macroscopic predictive theory which can be used for engineering purposes. The

phase volume fractions, which are state quantities, are subjected to constraints. In particular, their sum must be lower than 1, but not necessarily equal to 1, due to the presence of voids (cf. also [40] and [41] where this property is introduced in order to treat solid-liquid phase transitions with the possibility of voids and [1] for the corresponding numerical results). It is shown that most of the properties of shape memory alloys result from careful treatment of those internal constraints (cf., e.g., [36]–[39]). All these quoted references are related to a three dimensional model taking the temperature, the macroscopic deformation and the volumetric proportion of austenite and martensite as state variables. Moreover, let us note that in solid mechanics there are not too many mathematical models describing phase transitions in which the interactions between different types of substances and the possibility of having voids is taken into account: we can quote only the two contributions [40, 41]. In fluid mechanics the contributions are numerous due to the cavitation phenomenon which is a liquid-gas phase change. Let us quote the paper by J.J. Moreau [53], which introduces basic mechanical ideas.

It is beyond our purposes to give a complete description of the existing literature on models for SMA. However, restricting ourselves to the macroscopic description of these phenomena, we can refer to the main contributions [36]–[39], [4, 5, 34, 35, 58] and [3, 11, 12, 18, 19, 49, 57, 65] (and references therein) describing full thermo-mechanical models and studying the resulting PDEs from mathematical viewpoint respectively. We shall instead focus here on a generalization of the Frémond model for SMA introduced in [36]–[39].

Let us then explain in detail which is the main aim of this contribution, compare it with the results already present in the literature, and show the main mathematical difficulties encountered. As already mentioned, in this paper we deal with a generalization of the model introduced in [36]–[39] and later on studied in many contributions starting from the pioneering paper [25], where an existence and uniqueness result has been proved for the solution of a simplified problem, where all the nonlinearities in the balance of energy are neglected and the momentum balance equation is considered in the quasi-stationary form and fourth order terms (related to the second gradient theory) are taken into account. In the case when the fourth-order term is omitted an existence result dealing with the linearized energy balance equation has been proved in [21], while [22] one can find the proof of the existence of solutions to the linearized problem by including an inertial term in the momentum balance. We can report also of some results when some or all the nonlinearities are kept in the energy balance. The full one-dimensional model is shown to admit a unique solution both in the quasi-stationary case in [30] and in the case of a hyperbolic momentum equation in [31, 62]. Existence results have been proved also for the three-dimensional model (cf. [23, 28, 43]). Finally, let us mention the uniqueness result for the full quasi-static three-dimensional model proved in [17] and an updated and detailed presentation of the Frémond model and related system of equations and conditions, applying to the multidimensional case as well, which is provided in [11, 12], [37, Chapter 13], and [39]. Let us also point out [11, 12] for recent existence and uniqueness results in the three-dimensional situation, where the various nonlinear terms arising in the derivation of the model are accounted. The large time behavior of solutions is investigated in [26] in connection with the convergence to steady-state solutions and in [27, 24] where the authors characterize the large time behavior according to the theory of dissipative dynamical systems.

However, all these contributions were dealing with the case in which no voids can occur between phases. To model this possibility and to solve rigorously the results PDE system is just our aim here. First, in the next Section 2, we derive a model taking the possibility of having voids into account, introducing rigorously the pressure which has a paramount importance on the mechanical behaviour. In order to do that, we follow the ideas of [40] in which this was done in case of a two-phase transition phenomenon. Then, in Section 3, we give a rigorous formulation of initial boundary value problem associated with the resulting PDEs and we state our main results: existence, uniqueness, and continuous dependence of solutions from the data. The continuous dependence of the solution from the data is proved under appropriate regularity assumption on the nonlinearities. The proofs are carried over in Section 4 and Section 5. The main mathematical difficulties are due to the nonlinear and singular coupling between the equations. In particular, in order to describe the evolution of the absolute temperature variable, we shall use the entropy balance equation (cf. [13]–[15] for a complete derivation and motivation of this equation). This equation turns out to be singular in ϑ but the main advantage of using it is that once one has proved that a solution component ϑ does exist then it turns automatically out to be positive and the proof of positivity of the absolute temperature is historically one of the main difficulties of these types of problems. The idea here is to approximate the nonlinearities with regular functions, to solve the regularized system by means of a Banach fixed point argument and then to use compactness and lower semicontinuity arguments in order to pass to the limit and obtain a solution of the original problem.

2. **The derivation of the model.** In this section we explicitly derive a macroscopic model describing the evolution of SMA with the possibility of voids. The model is obtained by properly choosing the state quantities, the balance laws and the constitutive relations in agreement with the principle of thermodynamics and with experimental evidence.

2.1. **The state quantities.** We deal only with macroscopic phenomena and macroscopic quantities. To describe the deformations of the alloy, the macroscopic small deformation $\varepsilon(\mathbf{u})$, ($\mathbf{u}$ being the small displacement) and the temperature ϑ are chosen as state quantities.

The properties of shape memory alloys result from martensite–austenite phase changes produced either by thermal actions (as usual) or by mechanical actions. On the macroscopic level, some quantities are needed to take those phase changes into account. For this purpose, the volume fractions β_i of the martensites and austenite are chosen as state quantities. For simplicity, we assume that only two martensites exist together with austenite. The volume fractions of the martensites are β_1 and β_2. The volume fraction of austenite is β_3. These volume fractions are not independent: they satisfy the following *internal* constraints

$$0 \le \beta_i \le 1, \quad i = 1, 2, 3 \tag{1}$$

due to the definition of volume fractions. Since we assume that voids can appear in the martensite–austenite mixture, then the β's must satisfy an other internal constraint

$$\beta_1 + \beta_2 + \beta_3 \le 1 \tag{2}$$

the quantity $v = 1 - (\beta_1 + \beta_2 + \beta_3)$ being the voids volume fraction. This is the case when the alloy is produced by aggregating powders as shown in [60]. In case no

voids are considered, this sum should be equal to 1 and this considerably simplifies the analysis (cf., e.g., [25]).

We denote by $\boldsymbol{\beta}$ the vector of components β_i ($i = 1, 2, 3$) and the set of the state quantities is

$$E = \{\varepsilon(\mathbf{u}), \boldsymbol{\beta}, \nabla\boldsymbol{\beta}, \vartheta\}$$

while the quantities which describe the evolution and the thermal heterogeneity are

$$\delta E = \{\varepsilon(\mathbf{u})_t, \boldsymbol{\beta}_t, \nabla\boldsymbol{\beta}_t, \nabla\vartheta\}\,.$$

The gradient of $\nabla\boldsymbol{\beta}$ accounts for local interactions of the volume fractions at their neighborhood points.

2.2. **The mass balance.** Assuming the same constant density ρ (the reader can refer to [41] for a model in which different densities of the substances are taken into account in a general two-phase change phenomenon) and the same velocity $\mathbf{U} = \mathbf{u}_t$ for each phase, the mass balance reads

$$\rho(\beta_1 + \beta_2 + \beta_3)_t + \rho(\beta_1 + \beta_2 + \beta_3)\mathrm{div}\,\mathbf{U} = 0\,.$$

Within the small perturbation assumption, this equation gives

$$\rho(\beta_1 + \beta_2 + \beta_3)_t + \rho(\beta_1^0 + \beta_2^0 + \beta_3^0)\mathrm{div}\,\mathbf{U} = 0$$

where the β_i^0's are the initial values of the β_i. In agreement with the assumption that the voids volume fraction is small, we assume

$$\beta_1^0 + \beta_2^0 + \beta_3^0 = 1 \tag{3}$$

and have $\partial_t(\beta_1 + \beta_2 + \beta_3) + \mathrm{div}\,\mathbf{U} = 0$, hence

$$\partial_t(\beta_1 + \beta_2 + \beta_3) + \mathrm{div}\,\mathbf{u}_t = 0\,. \tag{4}$$

Mass balance is a relationship between the quantities of δE, indeed, its effects will be included in the cinematic relations (cf. (9) in the following subsections).

2.3. **The equations of motion.** They result from the principle of virtual power involving the power of the internal forces, (cf, e.g., [37])

$$-\int_\Omega \{\sigma : D(\mathbf{V}) + \mathbf{B} \cdot \delta + H : \nabla\delta\}\, d\Omega$$

where $\mathbf{V}$ and δ are virtual velocities, the actual velocities being $\mathbf{U}$ and $\boldsymbol{\beta}_t$. The internal forces are the stress σ, the phase change work vector $\mathbf{B}$, and the phase change work flux tensor H. The equations of motion are

$$\rho\mathbf{U}_t = \mathrm{div}\,\sigma + \mathbf{f}, \quad 0 = \mathrm{div}\,H - \mathbf{B} + \mathbf{A} \quad \text{in } \Omega \tag{5}$$

$$\sigma\mathbf{n} = \mathbf{g},\ H\mathbf{n} = \mathbf{a} \quad \text{on } \partial\Omega \tag{6}$$

where ρ is the density, $\mathbf{U}_t$ the acceleration of the alloy which occupies the domain Ω, with boundary $\partial\Omega$ and outward normal vector $\mathbf{n}$. The alloy is loaded by body forces $\mathbf{f}$ and by surface tractions $\mathbf{g}$, and submitted to body sources of phase change work $\mathbf{A}$ and surfaces sources of phase change work $\mathbf{a}$ (for instance, electric, magnetic or radiative actions producing the evolution of the alloys without macroscopic motion). In the following we will suppose, for simplicity, $\mathbf{A} = \mathbf{a} = \mathbf{0}$.

2.4. **The free energy.** As explained above, a shape memory alloy is considered as a mixture of the martensite and austenite phases with volume fractions β_i. The volume free energy of the mixture we choose is

$$\Psi = \Psi(E) = \sum_{i=1}^{3} \beta_i \Psi_i(E) + h(E) \tag{7}$$

where the Ψ_i's are the volume free energies of the i phases and h is a free energy describing interactions between the different phases. We have assumed that internal constraints are physical properties, hence, we decide to choose properly the two functions describing the material, i.e., the free energy Ψ and the pseudopotential of dissipation Φ, in order to take these constraints into account. Since, the pseudopotential describes the kinematic properties (i.e., properties which depend on the velocities) and the free energy describes the state properties, obviously the internal constraints (1) and (2) are to be taken into account with the choice of the free energy Ψ.

For this purpose, we assume the Ψ_i's are defined over the whole linear space spanned by β_i and the free energy is defined by

$$\Psi(E) = \beta_1 \Psi_1(E) + \beta_2 \Psi_2(E) + \beta_3 \Psi_3(E) + h(E) \, .$$

We choose the very simple interaction free energy

$$h(E) = I_C(\boldsymbol{\beta}) + \frac{k}{2} |\nabla \boldsymbol{\beta}|^2$$

where I_C is the indicator function of the convex set

$$C = \{(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3; 0 \le \gamma_i \le 1; \gamma_1 + \gamma_2 + \gamma_3 \le 1\} \, . \tag{8}$$

Moreover, and by $(k/2)|\nabla \boldsymbol{\beta}|^2$ we mean the product of two tensors $\nabla \boldsymbol{\beta}$ multiplied by the *interfacial energy coefficient* $(k/2) > 0$. The terms $I_C(\boldsymbol{\beta}) + (k/2)|\nabla \boldsymbol{\beta}|^2$ may be seen as a *mixture or interaction free-energy*.

The only effect of $I_C(\boldsymbol{\beta})$ is to guarantee that the proportions β_1, β_2 and β_3 take admissible physical values, i.e. they satisfy constraints (1) and (2) (cf. also (8)). The interaction free energy term $I_C(\boldsymbol{\beta})$ is equal to zero when the mixture is physically possible ($\boldsymbol{\beta} \in C$) and to $+\infty$ when the mixture is physically impossible ($\boldsymbol{\beta} \notin C$).

Let us note even if the free energy of the voids phase is 0, the voids phase has physical properties due to the interaction free energy term $(\nu/2)|\nabla \boldsymbol{\beta}|^2$ which depends on the gradient of $\boldsymbol{\beta}$. It is known that this gradient is related to the interfaces properties: $\nabla \beta_1$, $\nabla \beta_2$ describes properties of the voids-martensites interfaces and $\nabla \beta_3$ describes properties of the voids-austenite interface. In this setting, the voids have a role in the phase change and make it different from a phase change without voids. The model is simple and schematic but it may be upgraded by introducing sophisticated interaction free energy depending on $\boldsymbol{\beta}$ and on $\nabla \boldsymbol{\beta}$.

For the volume free energies, we choose

$$\Psi_1(E) = \frac{1}{2}\varepsilon(\mathbf{u}) : K_1 : \varepsilon(\mathbf{u}) + \sigma_1(\vartheta) : \varepsilon(\mathbf{u}) - C_1 \vartheta \log \vartheta,$$

$$\Psi_2(E) = \frac{1}{2}\varepsilon(\mathbf{u}) : K_2 : \varepsilon(\mathbf{u}) + \sigma_2(\vartheta) : \varepsilon(\mathbf{u}) - C_2 \vartheta \log \vartheta,$$

$$\Psi_3(E) = \frac{1}{2}\varepsilon(\mathbf{u}) : K_3 : \varepsilon(\mathbf{u}) - \frac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) - C_3 \vartheta \log \vartheta,$$

where K_i are the volume elastic tensors and C_i the volume heat capacities of the phases. Stresses $\sigma_i(\vartheta)$ depend on temperature ϑ and the quantity l_a is the latent

heat martensite-austenite volume phase change at temperature ϑ_0 (see Remark 2.1 below).

Remark 2.1. To make the model more realistic, we should introduce two temperatures to characterize the transformation: ϑ_0, the temperature at the beginning of the transformation and ϑ_f the temperature at the end. The interaction free energy is completed by $h(\boldsymbol{\beta}) = (l_a/\vartheta_0)(\vartheta_0 - \vartheta_f)(\beta_3)^2$ (cf. [6], [37], [58], [59]). However, we prefer not to investigate this case in the present contribution which is only a first mathematical approach to this problem. This realistic term could be treated in further analysis on the topic.

Because we want to describe the main basic properties of the shape memory alloys with voids, we assume that the elastic matrices K_i and the heat capacities C_i are the same for all of the phases:

$$C_i = \bar{C}, \; K_i = K \quad i = 1, 2, 3 \,.$$

Always for the sake of simplicity, we assume that

$$\sigma_1(\vartheta) = -\sigma_2(\vartheta) = -\tau(\vartheta)\mathbb{I}$$

where $\mathbb{I}$ stands for the identity matrix. Concerning the stress $\tau(\vartheta)$, it is known that at high temperature the alloy has a classical elastic behaviour. Thus $\tau(\vartheta) = 0$ at high temperature, and we choose the schematic simple expression

$$\tau(\vartheta) = (\vartheta - \vartheta_c)\overline{\tau}, \text{ for } \vartheta \leq \vartheta_c, \tau(\vartheta) = 0, \text{ for } \vartheta \geq \vartheta_c$$

with $\overline{\tau} \leq 0$ and assume the temperature ϑ_c is greater than ϑ_0. With those assumptions, it results

$$\Psi(E) = \frac{(\beta_1 + \beta_2 + \beta_3)}{2}\{\varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u})\}$$

$$-(\beta_1 - \beta_2)\tau(\vartheta)\mathbb{I} : \varepsilon(\mathbf{u}) - \beta_3\frac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) - C\vartheta\log\vartheta + \frac{k}{2}|\nabla\boldsymbol{\beta}|^2 + I_C(\boldsymbol{\beta}) \,.$$

2.5. **The pseudo-potential of dissipation.** The dissipative forces are defined via a pseudo-potential of dissipation Φ introduced by J.J. Moreau (it is a convex, positive function with value zero at the origin, [33], [51], [52]). As already remarked, the mass balance (4) is a relationship between velocities of δE. Thus we take it into account in order to define the pseudo-potential and introduce the indicator function I_0 of the origin of $\mathbb{R}$ as follows

$$I_0(\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div} \mathbf{u}_t) \,.$$

From experiments, it is known that the behaviour of shape memory alloys depends on time, i.e., the behaviour is dissipative. We define a pseudopotential of dissipation

$$\Phi(\vartheta, \boldsymbol{\beta}_t, \nabla\boldsymbol{\beta}_t, \nabla\vartheta) = \frac{c}{2}|\boldsymbol{\beta}_t|^2 + \frac{v}{2}|\nabla\boldsymbol{\beta}_t|^2 + \frac{\lambda}{2\vartheta}|\nabla\vartheta|^2 + I_0(\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div}\mathbf{u}_t) \quad (9)$$

where $\lambda \geq 0$ represents the thermal conductivity and $c \geq 0$, $v \geq 0$ stand for phase change viscosities.

2.6. **The constitutive laws.** The internal forces are split between non-dissipative forces σ^{nd}, $\mathbf{B}^{nd}$ and H^{nd} depending on (E, x, t) and dissipative forces by

$$\left\{ \sigma^d, \mathbf{B}^d, H^d, -\mathbf{Q}^d \right\},$$

depending on $\delta E = \{\varepsilon(\mathbf{u})_t, \boldsymbol{\beta}_t, \nabla\boldsymbol{\beta}_t, \nabla\vartheta\}$ and (E, x, t)

$$\sigma = \sigma^{nd} + \sigma^d, \quad \mathbf{B} = \mathbf{B}^{nd} + \mathbf{B}^d, \ H = H^{nd} + H^d$$

with the entropy flux vector $\mathbf{Q}$ being

$$\mathbf{Q} = \mathbf{Q}^d.$$

The nondissipative forces are defined with the free energy

$$\sigma^{nd}(E) = \frac{\partial\Psi}{\partial\varepsilon(\mathbf{u})}(E) = (\beta_1 + \beta_2 + \beta_3)K : \varepsilon(\mathbf{u}) - (\beta_1 - \beta_2)\tau(\vartheta)\mathbb{I} \qquad (10)$$

$$\mathbf{B}^{nd}(E, x, t) = \frac{\partial\Psi}{\partial\boldsymbol{\beta}}(E) = \frac{1}{2} \begin{pmatrix} \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) - 2\tau(\vartheta) : \varepsilon(\mathbf{u}) \\ \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) + 2\tau(\vartheta) : \varepsilon(\mathbf{u}) \\ \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) - 2\frac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) \end{pmatrix} + \mathbf{B}^{ndr}(E, x, t)$$

$$(11)$$

$$\mathbf{B}^{ndr}(E, x, t) \in \partial I_C(\boldsymbol{\beta}) \qquad (12)$$

$$H^{nd} = k\nabla\boldsymbol{\beta} \qquad (13)$$

and the dissipative forces are defined with the pseudo-potential of dissipation

$$\left\{ \sigma^d, \mathbf{B}^d, H^d, -\mathbf{Q}^d \right\} = \partial\Phi(E, \delta E) \qquad (14)$$

where the subdifferential of Φ is with respect to δE. Relationship (14) gives

$$\sigma^d = -p\mathbb{I} \qquad (15)$$

$$\mathbf{B}^d = -p \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c\boldsymbol{\beta}_t \qquad (16)$$

$$H^d = \upsilon\nabla\boldsymbol{\beta}_t \qquad (17)$$

$$-\mathbf{Q}^d = \frac{\lambda}{\vartheta}\nabla\vartheta \qquad (18)$$

where p is the pressure in the mixture and it results

$$-p \in \partial I_0(\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div}\mathbf{u}_t). \qquad (19)$$

The state laws (10)–(13), besides implying that the internal constraints are satisfied, give also the value of the reactions, during the evolution, to these internal constraints.

Relationships (10)–(13) and (15)–(19) give the constitutive laws

$$\sigma = (\beta_1 + \beta_2 + \beta_3)K : \varepsilon(\mathbf{u}) - ((\beta_1 - \beta_2)\tau(\vartheta) + p)\,\mathbb{I}, \tag{20}$$

$$\mathbf{B}(E, \delta E, x, t) = \frac{1}{2}\left\{ \begin{array}{c} \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) - 2\tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) + 2\tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ \varepsilon(\mathbf{u}) : K : \varepsilon(\mathbf{u}) - 2\dfrac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) - p \end{array} \right\} + \mathbf{B}^{ndr}(E, x, t) + c\boldsymbol{\beta}_t, \tag{21}$$

$$\mathbf{B}^{ndr}(E, x, t) \in \partial I_C(\boldsymbol{\beta}) \tag{22}$$

$$-p \in \partial I_0(\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div} \mathbf{u}_t) \tag{23}$$

$$H = k\nabla\boldsymbol{\beta} + \upsilon\nabla\boldsymbol{\beta}_t \tag{24}$$

$$-\mathbf{Q}(E, \delta E) = -\mathbf{Q}^d(E, \delta E) = -\frac{\lambda}{\vartheta}\nabla\vartheta\,. \tag{25}$$

It can be proved that our choice is such that the internal constraints and the second law of thermodynamics are satisfied (cf., e.g., [39, 37] and the next Subsection 2.7).

2.7. **The entropy balance.** By denoting

$$s = -\frac{\partial\Psi}{\partial\vartheta} = \bar{C}(1 + \log\vartheta) + \beta_3\frac{l_a}{\vartheta_0} \tag{26}$$

the entropy balance is

$$\frac{ds}{dt} + \operatorname{div}\mathbf{Q} = R + \frac{1}{\vartheta}\left\{ \sigma^d : \varepsilon(\mathbf{u})_t + \mathbf{B}^d\frac{\partial\boldsymbol{\beta}}{\partial t} + H^d : \nabla\boldsymbol{\beta}_t - \mathbf{Q}\cdot\nabla\vartheta \right\}$$

$$= R + \frac{1}{\vartheta}\left\{ c|\boldsymbol{\beta}_t|^2 + \upsilon|\nabla\boldsymbol{\beta}_t|^2 + \frac{\lambda}{\vartheta}|\nabla\vartheta|^2 \right\},\ \text{in}\ \Omega \tag{27}$$

$$-\mathbf{Q}\cdot\mathbf{n} = \Pi,\ \text{in}\ \Omega \tag{28}$$

because

$$p\left((\beta_1 + \beta_2 + \beta_3)_t + \operatorname{div}\mathbf{U}\right) = 0$$

due to (19), $\vartheta\mathbf{Q}$ is the heat flux vector, $R\vartheta$ is the exterior volume rate of heat that is supplied to the alloy, $\vartheta\pi$ is the rate of heat that is supplied by contact action, $\varepsilon(\mathbf{u}_t)$ is the strain rate. The constitutive laws, within the small perturbation assumption and (3), become

$$\sigma = K : \varepsilon(\mathbf{u}) - ((\beta_1 - \beta_2)\tau(\vartheta) + p)\,\mathbb{I} \tag{29}$$

$$-p \in \partial I_0((\beta_1 + \beta_2 + \beta_3)_t + \operatorname{div}\mathbf{U}) \tag{30}$$

$$\mathbf{B} = \left\{ \begin{array}{c} -\tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ \tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ -\dfrac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) - p \end{array} \right\} + \mathbf{B}^{ndr} + c\boldsymbol{\beta}_t \tag{31}$$

$$\mathbf{B}^{ndr} \in \partial I_C(\boldsymbol{\beta}) \tag{32}$$

$$H = k\nabla\boldsymbol{\beta} + \upsilon\nabla\boldsymbol{\beta}_t \tag{33}$$

$$\mathbf{Q}(E, \delta E) = \mathbf{Q}^d(E, \delta E) = -\frac{\lambda}{\vartheta}\nabla\vartheta\,. \tag{34}$$

2.8. **The set of partial differential equations.** We assume also quasi-static evolution and, using again the small perturbation assumption (i.e. neglecting the higher order contributions in the velocities in (27) which are smaller than the other quantities in applications), we get the following set of partial differential equations coupling the equations of motion (5), the entropy balance (27) and constitutive laws (29)–(34)

$$\operatorname{div}\left((K : \varepsilon(\mathbf{u}) - ((\beta_1 - \beta_2)\tau(\vartheta) + p)\,\mathbb{I})\right) + \mathbf{f} = 0 \tag{35}$$

$$-p \in \partial I_0(\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div}\mathbf{u}_t) \tag{36}$$

$$c\boldsymbol{\beta}_t - \upsilon\Delta\boldsymbol{\beta}_t - k\Delta\boldsymbol{\beta} + \begin{pmatrix} -\tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ \tau(\vartheta) : \varepsilon(\mathbf{u}) - p \\ -\frac{l_a}{\vartheta_0}(\vartheta - \vartheta_0) - p \end{pmatrix} + \mathbf{B}^{ndr} = 0 \tag{37}$$

$$\mathbf{B}^{ndr} \in \partial I_C(\boldsymbol{\beta}) \tag{38}$$

$$\bar{C}\frac{\partial \log\vartheta}{\partial t} + \frac{l_a}{\vartheta_0}\partial_t\beta_3 - \lambda\Delta\log\vartheta = R. \tag{39}$$

This set is completed by suitable initial conditions and the following boundary conditions:

$$\sigma\mathbf{n} = \mathbf{g} \quad \text{on } \Sigma_1 := \Gamma_1 \times [0, T] \tag{40}$$

$$\mathbf{u} = \mathbf{u}_t = 0 \quad \text{on } \Sigma_0 := \Gamma_0 \times [0, T] \tag{41}$$

$$\partial_{\mathbf{n}}\dot{\boldsymbol{\beta}} + \partial_{\mathbf{n}}\boldsymbol{\beta} = 0 \quad \text{on } \Sigma := \partial\Omega \times [0, T] \tag{42}$$

$$\partial_{\mathbf{n}}(\ln\vartheta) = \Pi \quad \text{on } \Sigma \tag{43}$$

where $\partial_{\mathbf{n}}$ is the normal outward derivative to the surface $\partial\Omega$, $\mathbf{g}$ is the exterior contact force applied to Γ_1, where (Γ_0, Γ_1) is a partition of $\partial\Omega$ and Γ_0 has positive measures.

2.9. **Remarks on the model.** The evolution of a structure made of shape memory alloys, i.e., the computation of

$$E(x,t) = (\varepsilon(\mathbf{u})(x,t), \beta_1(x,t), \beta_2(x,t), \beta_3(x,t), \vartheta(x,t)),$$

depending on the point x of the domain Ω occupied by the structure and on time t, can be performed by solving numerically the set of partial differential equations resulting from the equations of motion (5), (6) the energy balance (27), (28) and the constitutive laws (20)–(25), completed by convenient initial and boundary conditions (cf., e.g., [30], [17], [64], [56]). The model we have described here is able to account for the different features of the shape memory alloys: in particular, their macroscopic, mechanical and thermal properties. We have used schematic free energies and schematic pseudopotentials of dissipation.

There are still many possibilities to upgrade the basic choices we have made to take into account the practical properties of shape memory alloys. Let us, for instance, mention that the pseudopotential of dissipation can be modified in order to describe more precisely the hysteretical properties of the materials. There is no difficulty in having more than two martensites, for instance, to take care of 24 possible martensites! In the same way, it is possible to take into account of the different forms of a single martensite variant, as explained in [58].

Note that the physical quantities for characterizing an educated shape memory alloys are K, $\bar{C}$, l_a, ϑ_0, ϑ_c, τ, the two martensite volume fractions (for the free energy) and c, k, λ (for the pseudopotential of dissipation). They are indeed not

so many in order to have a complete multidimensional model which can be used for engineering purposes.

Other models and results may be found in [4, 5, 10, 46, 55, 45]. Let us note the very important role of internal constraints and of the reaction $\mathbf{B}^{ndr}$ to those internal constraints which are responsible for many properties. Let us also note that the pressure is the reaction to the kinematic constraint resulting from the mass balance. From this point of view, pressure p is involved in the equations in a logic and clear way.

3. **Main results.** In order to give a precise formulation of our problem, let us denote by Ω a bounded, convex set in $\mathbb{R}^n$ ($n = 1, 2, 3$) with Lipschitz boundary Γ, by T a positive final time, and by Q the space-time cylinder $\Omega \times (0, T)$. Let (Γ_0, Γ_1) be a partition of $\partial\Omega$ into two measurable sets such that both Γ_0 has positive surface measure. Finally, denote by $\Sigma := \partial\Omega \times [0, T]$, $\Sigma_j := \Gamma_j \times [0, T]$ ($j = 0, 1$) and introduce the Hilbert triplet (V, H, V') where

$$H := L^2(\Omega) \quad \text{and} \quad V := W^{1,2}(\Omega) \tag{44}$$

and identify, as usual, H (which stands either for the space $L^2(\Omega)$ or for $(L^2(\Omega))^2$ or for $(L^2(\Omega))^3$) with its dual space H', so that $V \hookrightarrow H \hookrightarrow V'$ with dense and continuous embeddings. Moreover, we denote by $\| \cdot \|_X$ the norm in some space X and by $\langle \cdot, \cdot \rangle$ the duality pairing between V and V' and by $(\cdot, \cdot)$ the scalar product in H.

Moreover, let r, s be two nonnegative real numbers, then, we introduce the following notation (cf. also [48, Def. (2.1), p. 6])

$$H^{r,s}(Q) := L^2(0, T; W^{r,2}(\Omega)) \cap W^{s,2}(0, T; H),$$
$$H^{r,s}(\Sigma) := L^2(0, T; W^{r,2}(\Gamma)) \cap W^{s,2}(0, T; L^2(\Gamma)).$$

Set, for simplicity of notation and without any loss of generality

$$c = k = l_a/\vartheta_0 = \bar{C} = \lambda = \upsilon = 1 \,.$$

Then, in order to write the variational formulation of our problem (35–39), we need to generalize the relationship $\mathbf{B}^{ndr} \in \partial I_C(\boldsymbol{\beta})$ stated in (38) (cf. also [8] for similar generalizations). Hence, we need to introduce the following ingredients

$$j : \mathbb{R}^3 \to [0, +\infty] \text{ a proper, convex, lower semicontinuous function}$$
$$\text{such that } j(0) = 0 \quad \text{and its subdifferential} \tag{45}$$

$$\alpha = \partial j : \mathbb{R}^3 \to 2^{\mathbb{R}^3} \tag{46}$$

$$\tau \in W^{1,\infty}(\mathbb{R}) \,. \tag{47}$$

Moreover, we consider the associate functionals

$$J_H(\mathbf{v}) = \int_\Omega j(\mathbf{v}(x))dx \quad \text{if} \quad \mathbf{v} \in H \quad \text{and} \quad j(\mathbf{v}) \in L^1(\Omega) \tag{48}$$

$$J_H(\mathbf{v}) = +\infty \quad \text{if} \quad \mathbf{v} \in H \quad \text{and} \quad j(\mathbf{v}) \notin L^1(\Omega) \tag{49}$$

$$J_V(\mathbf{v}) = J_H(\mathbf{v}) \quad \text{if} \quad \mathbf{v} \in V^3 \tag{50}$$

with their subdifferentials (cf. [7, Chap. II, p. 52])

$$\partial_{V,V'} J_V : V^3 \to 2^{(V')^3} \tag{51}$$

and (cf. [16, Ex. 2.1.4, p. 21])

$$\partial_H J_H : H \to 2^H. \tag{52}$$

Denote by $D(\partial_{V,V'} J_V) := \{\mathbf{v} \in V^3 : \partial_{V,V'} J_V(\mathbf{v}) \neq \emptyset\}$ the domain of $\partial_{V,V'} J_V$. Then, for $\boldsymbol{\chi}, \boldsymbol{\xi} \in H$, we have (see, e.g., [16, Ex. 2.1.3, p. 52]) that

$$\boldsymbol{\xi} \in \partial_H J_H(\boldsymbol{\chi}) \quad \text{if and only if } \boldsymbol{\xi} \in \alpha(\boldsymbol{\chi}) \quad \text{a.e. in } \Omega$$

and, thanks to (50) and to the definitions of $\partial_{V,V'} J_V$ and $\partial_H J_H$, we have

$$\partial_H J_H(\boldsymbol{\chi}) \subseteq H \cap \partial_{V,V'} J_V(\boldsymbol{\chi}) \quad \forall \boldsymbol{\chi} \in V^3. \tag{53}$$

Now we denote by W the following space

$$W := \{\mathbf{v} \in V^3 : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0\} \tag{54}$$

endowed with the usual norm. In addition, we introduce on $W \times W$ a bilinear symmetric continuous form $a(\cdot, \cdot)$ defined by

$$a(\mathbf{u}, \mathbf{v}) := \sum_{i,j=1}^{3} \int_{\Omega} \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}).$$

Note here that (since Γ_0 has positive measure), thanks to Korn's inequality (cf., e.g., [20], [32, p. 110]), there exists a positive constant c such that

$$a(\mathbf{v}, \mathbf{v}) \geq c \|\mathbf{v}\|_W^2 \quad \forall \mathbf{v} \in W. \tag{55}$$

Next, in order to rewrite the problem (35–39) in an abstract framework, let us introduce the operators

$$\mathcal{B} : V \to V', \quad \langle \mathcal{B}u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \quad u, v \in V \tag{56}$$

$$\mathcal{A} : W \to W', \quad {}_{W'}\langle \mathcal{A}\mathbf{u}, \mathbf{v} \rangle_W = a(\mathbf{u}, \mathbf{v}) \quad \mathbf{u}, \mathbf{v} \in W \tag{57}$$

$$\mathcal{H} : H \to W', \quad {}_{W'}\langle \mathcal{H}u, \mathbf{v} \rangle_W = \int_{\Omega} u \operatorname{div} \mathbf{v} \quad u \in H, \mathbf{v} \in W. \tag{58}$$

Moreover, we make the following assumptions on the data

$$\mathbf{u}_0 \in W, \quad \vartheta_0 \in L^1(\Omega), \tag{59}$$

$$\vartheta_0 > 0 \quad \text{a.e. in } \Omega, \quad w_0 := \log \vartheta_0 \in W^{2,2}(\Omega), \tag{60}$$

$$\boldsymbol{\beta}_0 = (\beta_1^0, \beta_2^0, \beta_3^0) \in D(\partial_{V,V'} J_V), \tag{61}$$

$$\mathbf{f} \in W^{1,1}(0, T; H), \quad \mathbf{g} \in W^{1,1}(0, T; (L^2(\Gamma_1))^3), \tag{62}$$

$$R \in L^2(0, T; V) \cap L^1(0, T; L^\infty(\Omega)), \quad \Pi \in L^\infty(\Sigma) \cap H^{3/2, 3/4}(\Sigma). \tag{63}$$

Then, we introduce the functions $\mathcal{R} \in L^2(0, T; V')$ and $\mathcal{F} \in W^{1,1}(0, T, W')$ such that

$$\langle \mathcal{R}(t), v \rangle = \int_{\Omega} R(t)v + \int_{\partial\Omega} \Pi(t)v_{|\partial\Omega} \quad v \in V, \quad \text{for a.e. } t \in [0, T], \tag{64}$$

$${}_{W'}\langle \mathcal{F}(t), \mathbf{v} \rangle_W = \int_{\Omega} \mathbf{f}(t) \cdot \mathbf{v} + \int_{\Gamma_1} \mathbf{g}(t) \cdot \mathbf{v}_{|\partial\Omega} \quad \mathbf{v} \in W, \quad \text{for a.e. } t \in [0, T]. \tag{65}$$

Now take the function

$$\gamma(r) := \exp(r) \quad \text{for } r \in \mathbb{R} \tag{66}$$

and take J_V as in (50), then we are ready to introduce the variational formulation of our problem as follows.

PROBLEM (P). Find $(\mathbf{u}, w, \beta_1, \beta_2, \beta_3)$ and (ξ_1, ξ_2, ξ_3, p) with the regularities

$$\mathbf{u} \in L^\infty(0, T, W), \quad \operatorname{div} \mathbf{u} \in W^{1,2}(0, T; V) \tag{67}$$

$$w \in W^{1,2}(0, T; H) \cap L^2(0, T; V) \cap L^\infty(Q) \tag{68}$$

$$\beta_1, \beta_2, \beta_3 \in W^{1,2}(0, T; V) \cap L^\infty(0, T; V), \quad \boldsymbol{\beta} \in D(\partial_{V,V'} J_V) \quad \text{a.e. in } (0, T) \tag{69}$$

$$\xi_1, \xi_2, \xi_3 \in L^2(0, T; V'), \quad p \in L^2(Q) \tag{70}$$

satisfying

$$\mathcal{A}\mathbf{u} - \mathcal{H}(p + (\beta_1 - \beta_2)\tau(\gamma(w))) = \mathcal{F} \quad \text{in } W' \text{ a.e. in } [0, T] \tag{71}$$

$$\partial_t(\beta_1 + \beta_2 + \beta_3) + \operatorname{div} \mathbf{u}_t = 0 \quad \text{a.e. in } Q \tag{72}$$

$$\partial_t w + \partial_t(\beta_3) + \mathcal{B}w = \mathcal{R} \quad \text{in } V' \text{ a.e. in } [0, T] \tag{73}$$

$$\partial_t \boldsymbol{\beta} + \begin{pmatrix} \mathcal{B}\partial_t\beta_1 \\ \mathcal{B}\partial_t\beta_2 \\ \mathcal{B}\partial_t\beta_3 \end{pmatrix} + \begin{pmatrix} \mathcal{B}\beta_1 \\ \mathcal{B}\beta_2 \\ \mathcal{B}\beta_3 \end{pmatrix} + \boldsymbol{\xi} = \begin{pmatrix} -\tau(\gamma(w)) : \varepsilon(\mathbf{u}) - p \\ \tau(\gamma(w)) : \varepsilon(\mathbf{u}) - p \\ -(\gamma(w) - \vartheta_0) - p \end{pmatrix}$$

$$\text{in } (V')^3 \text{ a.e. in } [0, T] \tag{74}$$

$$\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \partial_{V,V'} J_V(\boldsymbol{\beta}) \quad \text{a.e. in } [0, T] \tag{75}$$

and such that

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega \tag{76}$$

$$w(0) = w_0 \quad \text{a.e. in } \Omega \tag{77}$$

$$\boldsymbol{\beta}(0) = \boldsymbol{\beta}_0 \quad \text{a.e. in } \Omega. \tag{78}$$

Remark 3.1. Obviously we can take $j = I_C$ in (45) with C as in (8) (cf. [7, Ex. 3, p. 54]) and recover the problem already stated in (35–39) as a particular case of our more general formulation. Note moreover that we can write down equation (74) only in $(V')^3$ (and by consequence we need to introduce the notion (51) of subdifferential in $(V')^3$) because of the V' regularity of $(\mathcal{B}\dot\beta_1, \mathcal{B}\dot\beta_2, \mathcal{B}\dot\beta_3)$ in (74). Indeed in this case the subdifferential has sense only in $(V')^3$ as it is physically meaningful because equation (74) comes from the principle of virtual power, which is written down as duality with virtual velocities (cf. also the modelling Section 2). Notice moreover that, in case $j = I_C$, we can still recover the physically meaningful constraint $\boldsymbol{\beta} \in C$ a.e. due to the last relation of (69). The difficult point in the proof of our result will be in fact the passage to the limit in the two non-smooth nonlinearities in equation (74). By the contrary, we aim to remark that this dissipative term in (74) gives more spatial regularity to $\partial_t\boldsymbol{\beta}$ which furnish more regularity to w in (73) (cf. the following Theorem 3.2) and consequently to $\vartheta = \gamma(w)$ in (74). This regularity is needed in order to prove well-posedness for our problem. However, we can also notice that, from the mechanical viewpoint, since we have introduced in the model the elastic (non-dissipative) local interaction term $-\Delta\boldsymbol{\beta}$ (in (36)), it seems also reasonable to include in the model the dissipative local interaction term $-\Delta\boldsymbol{\beta}_t$, as it is done also in other solid-solid phase transition models (cf., e.g., [11, 12]).

We are now ready to state our main result which is the following global existence and uniqueness theorem.

Theorem 3.2. *Let the assumptions on the data (45–65) hold and let T be a positive final time. Then* PROBLEM (P) *has a unique solution on the whole time interval* $[0, T]$.

Remark 3.3. Let us note that assumption (63) on the data are needed (in particular) in order to get the L^∞-bounds on the $w(= \log \vartheta)$-component of our solution (cf. Lemma 4.4 in Section 4.2). Moreover, let us notice that proving the L^∞-bound on $\log \vartheta$ ensures that ϑ is also bounded and bounded away from 0.

Moreover, in the last Section 5, we will get a proof for the following continuous dependence result for PROBLEM (P).

Theorem 3.4. *Let T be a positive final time, $(\mathbf{u}_0^i, w_0^i, \boldsymbol{\beta}_0^i)$ $(i = 1, 2)$ be two sets of initial data satisfying conditions (59–61), $(\mathcal{R}^i, \mathcal{F}^i)$ $(i = 1, 2)$ be two data of PROBLEM (P) satisfying assumptions (64–65) with $(\mathbf{f}^i, \mathbf{g}^i, R^i, \Pi^i)$ $(i = 1, 2)$ as in (62–63). Let $(\mathbf{u}^i, w^i, \beta_1^i, \beta_2^i, \beta_3^i)$ $(i = 1, 2)$ be two solutions of PROBLEM (P) corresponding to these data. Moreover, besides conditions (45–46), suppose that the following hypothesis*

$$\alpha \in C^{0,1}(\mathbb{R}^3) \tag{79}$$

holds. Then, there exists a positive constant M, depending on the parameters of the problem, such that the following continuous dependence estimate

$$\|w^1 - w^2\|_{L^\infty(0,t;H) \cap L^2(0,t;V)}^2 + \|\mathbf{u}^1 - \mathbf{u}^2\|_{W^{1,2}(0,t;W)}^2 + \sum_{j=1}^{3} \|\beta_j^1 - \beta_j^2\|_{W^{1,2}(0,t;V)}^2$$

$$\leq M \Big(\|w_0^1 - w_0^2\|_H^2 + \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_W^2 + \sum_{j=1}^{3} \|\beta_j^{01} - \beta_j^{02}\|_V^2$$

$$+ \|\mathcal{F}^1 - \mathcal{F}^2\|_{L^2(0,t;W')}^2 + \|\mathcal{R}^1 - \mathcal{R}^2\|_{L^2(0,t;V')}^2 \Big) \tag{80}$$

holds for any $t \in (0, T)$.

Remark 3.5. Let us note that in this paper we can treat the difficult coupling between the phase-equations (74) in which appears the temperature ϑ $(= \gamma(w))$ and the entropy balance equation (73) in which only the function $\log \vartheta$ $(= w)$ plays some role, using the L^∞-bound on the w-component of solution to PROBLEM (P) (cf. (68)). Indeed it was just due to the lack of regularity of solutions that in [13] (where there was the same type of coupling without the $\Delta \partial_t \beta$-term in (74)) the authors did not obtain uniqueness of solutions (cf. also [13, Remark 5.2]).

However, let us observe that the main advantage of taking the entropy balance equation instead of the internal energy balance equation is that once one has solved the problem in some sense and has found the temperature $\vartheta := \gamma(w)$, it is automatically positive because it stands in the image of the function γ (cf. (66)). Indeed in many cases it is difficult to deduce this fact only from the internal energy balance equation (cf., e.g., [29] in order to see one example of these difficulties). Let us note that within the small perturbations assumption the entropy balance and the classical heat equation are equivalent in mechanical terms (cf. [13, 14, 15]).

Finally, note that the uniqueness and continuous dependence result is obtained only in case the subdifferential α is Lipschitz continuous in $\mathbb{R}^3$, and so it cannot be applied to the case $\alpha = \partial I_C$, but only to some regularizations of it. Moreover the result is given in terms of the phase, displacement and temperature variables, but not in the selections p and ξ (as usual for these kind of problems) (cf. (80)).

4. **Proof of Theorem 3.2.** The following section is devoted to the proof of Theorem 3.2. First we approximate our PROBLEM (P) by a more regular PROBLEM

(P^ε), then (fixed $\varepsilon > 0$) we find well-posedness for the approximating problem using a iterated Banach contraction fixed-point argument and then we perform some a-priori estimates (independent of ε) on its solution, which allow us to pass to the limit in Problem (P^ε) as $\varepsilon \searrow 0$, recovering a solution to Problem (P).

4.1. **The approximating problem.** We take a small positive parameter $\varepsilon > 0$ and approximate $\partial_{V,V'} J_V$ in (74–75) , let us take the Lipschitz continuous Yosida-Moreau approximation $\alpha^\varepsilon = \partial j^\varepsilon = (j^\varepsilon)'$ (cf. [16, Prop. 2.11, p. 39]) of α and the associated functional $J_{H,\varepsilon}(\mathbf{v}) = \int_\Omega j^\varepsilon(\mathbf{v}(x))\,dx$, whose differential $(\partial_H J_{H,\varepsilon})$ is the Yosida-Moreau approximation of $\partial_H J_H$ (cf. [16, Prop. 2.16, p. 47]). Now we aim to recall some properties of this approximation which will be useful in order to pass to the limit as $\varepsilon \searrow 0$. Note that the proof of the following lemma is a consequence of [2, Thm. 3.20, p. 289], [2, Thm. 3.62, p. 365], and of the Lebesgue theorem of passage to the limit under the sign of integral.

Lemma 4.1. *If $\partial_{V,V'} J_{V,\varepsilon}$ is the Yosida-Moreau approximation of $\partial_{V,V'} J_V$ (cf. [7, Thm. 2.2, p. 57]), then the following inclusion*

$$\partial_H J_{H,\varepsilon} \subseteq \partial_{V,V'} J_{V,\varepsilon} \tag{81}$$

holds true. Moreover, the following properties hold true (for $\varepsilon \searrow 0$)

$$J_{V,\varepsilon} \to \sup_{\varepsilon > 0} J_{V,\varepsilon} \text{ in the sense of Mosco (cf. [2, Def. 3.17, p. 295]),} \tag{82}$$

$$\sup_{\varepsilon > 0} J_{V,\varepsilon} = \lim_{\varepsilon \searrow 0} \int_\Omega j^\varepsilon = \int_\Omega j, \quad hence \tag{83}$$

$$J_{V,\varepsilon} \to J_V \text{ in the sense of Mosco and, in particular,} \tag{84}$$

$$\forall \mathbf{v} \in V^3, \ \forall \mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } V^3 \ J_V(\mathbf{v}) \le \liminf J_{V,\varepsilon}(\mathbf{v}_\varepsilon). \tag{85}$$

Finally, for $\varepsilon \searrow 0$, it holds

$$(\mathbf{u}, \partial_{V,V'} J_{V,\varepsilon}(\mathbf{u})) \to \partial_{V,V'} J_V \text{ in the graph sense (cf. [2, Def. 3.58, p. 360]).} \tag{86}$$

Then, let us call γ^ε the following Lipschitz continuous approximation of the function $\gamma(w) = \exp(w)$, i.e. the function

$$\gamma^\varepsilon(r) := \begin{cases} \exp r & \text{if } r \le 1/\varepsilon \\ (r - 1/\varepsilon)\exp(1/\varepsilon) + \exp(1/\varepsilon) & \text{if } r \ge 1/\varepsilon. \end{cases} \tag{87}$$

Moreover let δ^ε be the inverse function of γ^ε, i.e.

$$\delta^\varepsilon(\gamma^\varepsilon(r)) = r \quad \forall r \in \mathbb{R} \tag{88}$$

and let $\widehat{\gamma}^\varepsilon$ be a primitive of the function γ^ε, i.e.

$$\widehat{\gamma}^\varepsilon(r) = 1 + \int_0^r \gamma^\varepsilon(s)\,ds \quad \forall r \in \mathbb{R}. \tag{89}$$

Then the following properties of γ^ε hold true (cf. also [13, Lemma 5.1]).

Lemma 4.2. *There holds*

$$\widehat{\gamma}^\varepsilon(r) \ge \gamma^\varepsilon(r), \quad r(\delta^\varepsilon)'(r) \ge 1 \quad \forall r \in \mathbb{R}.$$

We are ready now to introduce the approximating Problem (P^ε) as follows.

PROBLEM (P^ε). Find $(\mathbf{u}^\varepsilon, w^\varepsilon, \beta_1^\varepsilon, \beta_2^\varepsilon, \beta_3^\varepsilon, p^\varepsilon)$ with the regularities

$$\mathbf{u}^\varepsilon \in L^\infty(0,T,W), \quad \operatorname{div}\mathbf{u}^\varepsilon \in W^{1,2}(0,T;V), \quad p^\varepsilon \in L^2(0,T;H) \tag{90}$$

$$w^\varepsilon \in W^{1,2}(0,T;H) \cap L^2(0,T;V) \cap L^\infty(Q) \tag{91}$$

$$\beta_1^\varepsilon, \beta_2^\varepsilon, \beta_3^\varepsilon \in W^{1,2}(0,T;V) \tag{92}$$

satisfying

$$\mathcal{A}\mathbf{u}^\varepsilon - \mathcal{H}(p^\varepsilon + (\beta_1^\varepsilon - \beta_2^\varepsilon)\tau(\gamma^\varepsilon(w^\varepsilon))) = \mathcal{F} \quad \text{in } W' \text{ a.e. in } [0,T] \tag{93}$$

$$\partial_t(\beta_1^\varepsilon + \beta_2^\varepsilon + \beta_3^\varepsilon) + \operatorname{div}(\mathbf{u}^\varepsilon)_t = \varepsilon p^\varepsilon \quad \text{a.e. in } Q \tag{94}$$

$$\partial_t w^\varepsilon + \partial_t\beta_3^\varepsilon + \mathcal{B}w^\varepsilon = \mathcal{R} \quad \text{in } V' \text{ a.e. in } [0,T] \tag{95}$$

$$\partial_t\boldsymbol{\beta}^\varepsilon + \begin{pmatrix} \mathcal{B}\partial_t\beta_1^\varepsilon \\ \mathcal{B}\partial_t\beta_2^\varepsilon \\ \mathcal{B}\partial_t\beta_3^\varepsilon \end{pmatrix} + \begin{pmatrix} \mathcal{B}\beta_1^\varepsilon \\ \mathcal{B}\beta_2^\varepsilon \\ \mathcal{B}\beta_3^\varepsilon \end{pmatrix} + \partial_H J_{H,\varepsilon}(\boldsymbol{\beta}^\varepsilon) = \begin{pmatrix} -\tau(\gamma^\varepsilon(w^\varepsilon)) : \varepsilon(\mathbf{u}^\varepsilon) - p^\varepsilon \\ \tau(\gamma^\varepsilon(w^\varepsilon)) : \varepsilon(\mathbf{u}^\varepsilon) - p^\varepsilon \\ -(\gamma^\varepsilon(w^\varepsilon) - \vartheta_0) - p^\varepsilon \end{pmatrix}$$

$$\text{in } (V')^3 \text{ a.e. in } [0,T] \tag{96}$$

and such that

$$\mathbf{u}^\varepsilon(0) = \mathbf{u}_0 \quad \text{a.e. in } \Omega \tag{97}$$

$$w^\varepsilon(0) = w_0 \quad \text{a.e. in } \Omega \tag{98}$$

$$\boldsymbol{\beta}^\varepsilon(0) = \boldsymbol{\beta}_0 \quad \text{a.e. in } \Omega. \tag{99}$$

Concerning this approximating problem, we prove hereafter the following well-posedness result.

Theorem 4.3. *Let the assumptions (45–65) hold true. Let T be a positive final time and $\varepsilon > 0$. Then the* PROBLEM (P^ε) *has a unique solution in $[0,T]$.*

Proof. Here we are going first to prove local existence (and uniqueness) in a finite time interval $[0,\bar{t}]$ for some $\bar{t} \in [0,T]$, then we will extend the solution to the whole interval $[0,T]$ proving global existence (and uniqueness) of solution to PROBLEM (P^ε). Hence, let us take $\bar{t} \in [0,T]$ (we will choose it later) and denote by $\mathcal{X} := (W^{1,2}(0,\bar{t};V))^3$. Fix for the moment $(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon) \in \mathcal{X}$ in the equations (93–95), then, by well-known results (cf. also (62–63)), we find a unique $w^\varepsilon = \mathcal{T}_2(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon) \in W^{1,2}(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;W^{2,2}(\Omega))$ solution of (95). By [32, Thm. 6.2, p. 168], it is possible to find a unique $\mathbf{u}^\varepsilon = \mathcal{T}_1(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon, w^\varepsilon) \in L^\infty(0,T;W)$ solution of (93–94) such that $\operatorname{div}(\mathbf{u}^\varepsilon)_t \in L^2(0,T;H)$.

Moreover, if we take these values of $\mathbf{u}^\varepsilon$ and w^ε in (96), we can find a solution $(\beta_1^\varepsilon, \beta_2^\varepsilon, \beta_3^\varepsilon) \in \mathcal{X}$, depending on

$$\mathbf{u}^\varepsilon =: \mathcal{T}_1(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon, w^\varepsilon)$$

and

$$w^\varepsilon =: \mathcal{T}_2(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon),$$

of the equation (96) again by standard results.

In this way, we have defined an operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ such that $(\beta_1^\varepsilon, \beta_2^\varepsilon, \beta_3^\varepsilon) =: \mathcal{T}(\bar{\beta}_1^\varepsilon, \bar{\beta}_2^\varepsilon, \bar{\beta}_3^\varepsilon)$. What we have to do now is to prove that $\mathcal{T}$ is a contraction mapping on $\mathcal{X}$ for a sufficiently small $\bar{t} \in [0,T]$ and moreover, repeating the procedure step by step in time (this is possible thanks to the regularities properties of the solution listed above), we can prove well-posedness for the PROBLEM (P^ε) on the whole time interval $[0,T]$ and conclude the proof of Theorem 4.3. In order to prove that $\mathcal{T}$ is contractive, let us proceed by steps and forget of the apices ε.

First step. Let $(\bar{\beta}_1^{\ i}, \bar{\beta}_2^{\ i}, \bar{\beta}_3^{\ i}) \in \mathcal{X}$, $\mathbf{u}^i = \mathcal{T}_1(\bar{\beta}_1^{\ i}, \bar{\beta}_2^{\ i}, \bar{\beta}_3^{\ i})$, $w^i = \mathcal{T}_2(\bar{\beta}_1^{\ i}, \bar{\beta}_2^{\ i}, \bar{\beta}_3^{\ i})$, and $(\bar{\beta}_1^{\ i}, \bar{\beta}_2^{\ i}, \bar{\beta}_3^{\ i}) = \mathcal{T}(\bar{\beta}_1^{\ i}, \bar{\beta}_2^{\ i}, \bar{\beta}_3^{\ i})$ $(i = 1, 2, 3)$. Then, writing two times (95) with $\bar{\beta}_3^{\ i}$ $(i = 1, 2)$ instead of $\partial_t(\beta_3)$, making the difference, testing the resulting equation with $(w^1 - w^2)_t$, and integrating on $(0, t)$ with $t \in [0, T]$, we get the following inequality

$$\|(w^1 - w^2)_t\|_{L^2(0,t;H)}^2 + \|(w^1 - w^2)(t)\|_V^2 \leq C_0 \|(\bar{\beta}_3^1)_t - (\bar{\beta}_3^2)_t\|_{L^2(0,t;H)}^2 \qquad (100)$$
$$+ \frac{1}{2}\|w_1 - w_2\|_{L^2(0,t;H)}^2,$$

for some positive constant C_0 independent of t. Hence, we get, for all $t \in [0, T]$,

$$\|(w^1 - w^2)_t\|_{L^2(0,t;H)}^2 + \frac{1}{2}\|w^1 - w^2\|_{C^0([0,t];V)}^2 \leq C_1 \|(\bar{\beta}_3^1)_t - (\bar{\beta}_3^2)_t\|_{L^2(0,t;H)}^2 \quad (101)$$

being $C_1 := C_0 \mathrm{e}^{T/2}$.

Second step. Let us take $\vartheta^i = \gamma^\varepsilon(w^i)$ and $\varepsilon p^i := \partial_t(\bar{\beta}_1^i + \bar{\beta}_2^i + \bar{\beta}_3^i) + \mathrm{div}\,(\mathbf{u}^i)_t$ $(i = 1, 2)$ and write (93) with $\bar{\beta}_1^i$ and $\bar{\beta}_2^i$, make the difference, test the resulting equation with $\frac{\varepsilon}{2}(\mathbf{u}^1 - \mathbf{u}^2)_t$, integrate on $(0, t)$ with $t \in [0, T]$, and use equation (94), getting the following inequality

$$\frac{\varepsilon}{2}\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_W^2 + \frac{\varepsilon^2}{2}\|p^1 - p^2\|_{L^2(0,t;H)}^2$$

$$\leq \frac{\varepsilon}{2}\int_Q \sum_{j=1}^3 (\bar{\beta}_j^1 - \bar{\beta}_j^2)_t (p^1 - p^2)$$

$$- \frac{\varepsilon}{2}\int_Q \left(\tau(\vartheta^1)\left((\bar{\beta}_1^1 - \bar{\beta}_2^1) - (\bar{\beta}_2^1 - \bar{\beta}_2^2)\right)\right)\left(\varepsilon(p^1 - p^2) - \sum_{j=1}^3 (\bar{\beta}_j^1 - \bar{\beta}_j^2)_t\right)$$

$$+ \frac{\varepsilon}{2}\int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2))(\bar{\beta}_2^1 - \bar{\beta}_2^2)\right)\left(\varepsilon(p^1 - p^2) - \sum_{j=1}^3 (\bar{\beta}_j^1 - \bar{\beta}_j^2)_t\right)$$

$$\leq \frac{\varepsilon^2}{4}\|p^1 - p^2\|_{L^2(0,t;H)}^2 + \frac{1}{2}\sum_{j=1}^3 \|(\bar{\beta}_j^1)_t - (\bar{\beta}_j^2)_t\|_{L^2(0,t;H)}^2$$

$$+ C_2\left(t^2 + t\right)\varepsilon \sum_{j=1}^3 \|(\bar{\beta}_j^1)_t - (\bar{\beta}_j^2)_t\|_{L^2(0,t;H)}^2$$

$$+ C_3 t \exp(2/\varepsilon)\|w^1 - w^2\|_{C^0([0,t];V)}^2. \qquad (102)$$

Third step. Write equation (96) for ϑ^i and $\mathbf{u}^i$, make the difference between the two equations written for $i = 1$ and $i = 2$, and test the resulting vectorial equation by the vector $\frac{\varepsilon}{4}\left((\beta_1^1)_t - (\beta_1^2)_t, (\beta_2^1)_t - (\beta_2^2)_t, (\beta_3^1)_t - (\beta_3^2)_t\right)$. Summing up the two lines and integrating on $(0, t)$ with $t \in [0, T]$, we have (exploiting the Lipschitz continuity

of $\partial_H J_{H,\varepsilon}$ (cf. [16, Prop. 2.6, p. 28]))

$$\frac{\varepsilon}{4}\sum_{j=1}^{3}\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 + \frac{\varepsilon}{4}\sum_{j=1}^{3}\|\nabla(\beta_j^1 - \beta_j^2)(t)\|_H^2$$

$$\leq -\frac{\varepsilon}{4}\int_Q \sum_{j=1}^{3}(\beta_j^1 - \beta_j^2)_t(p^1 - p^2)$$

$$-\frac{\varepsilon}{4}\int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2))\mathrm{div}\,\mathbf{u}^1 + (\mathrm{div}\,\mathbf{u}^1 - \mathrm{div}\,\mathbf{u}^2)\tau(\vartheta^2)\right)(\beta_1^1 - \beta_1^2)_t$$

$$+\frac{\varepsilon}{4}\int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2))\mathrm{div}\,\mathbf{u}^1 + (\mathrm{div}\,\mathbf{u}^1 - \mathrm{div}\,\mathbf{u}^2)\tau(\vartheta^2)\right)(\beta_2^1 - \beta_2^2)_t$$

$$-\frac{\varepsilon}{4}\int_Q(\vartheta^1 - \vartheta^2)(\beta_3^1 - \beta_3^2)_t + \frac{\varepsilon}{16}\sum_{j=1}^{3}\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;H)}^2$$

$$+\frac{1}{4\varepsilon}\sum_{j=1}^{3}\|\beta_j^1 - \beta_j^2\|_{L^2(0,t;H)}^2 .$$

Moreover, using the definition (4.2) of γ^ε and the assumption (47) on τ, we get the following inequality

$$\frac{3\varepsilon}{16}\sum_{j=1}^{3}\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 + \frac{\varepsilon}{4}\sum_{j=1}^{3}\|\nabla(\beta_j^1 - \beta_j^2)(t)\|_H^2$$

$$\leq \frac{\varepsilon^2}{8}\|p^1 - p^2\|_{L^2(0,t;H)}^2 + C_4\varepsilon^2\|(\beta_j^1 - \beta_j^2)_t\|_{L^2(0,t;H)}^2$$

$$+ C_5 t\exp(2/\varepsilon)\|w^1 - w^2\|_{C^0([0,t];V)}^2$$

$$+ C_6\|\mathbf{u}^1 - \mathbf{u}^2\|_{L^2(0,T;W)}^2$$

$$+ t\exp(2/\varepsilon)\|w^1 - w^2\|_{C^0([0,t];H)}^2$$

$$+ \frac{t^2}{4\varepsilon}\sum_{j=1}^{3}\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 . \tag{103}$$

Fourth step. Summing up the two inequalities (102) and (103), using (100), and choosing t sufficiently small, we get

$$\varepsilon\sum_{j=1}^{3}\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 + \varepsilon\sum_{j=1}^{3}\|\nabla(\beta_j^1 - \beta_j^2)(t)\|_H^2 + \varepsilon\|\mathbf{u}^1 - \mathbf{u}^2\|_{C^0([0,t];W)}^2$$

$$\leq C_\varepsilon\left(t + t^2 + \frac{1}{2}\right)\sum_{j=1}^{2}\|(\bar\beta_j^1)_t - (\bar\beta_j^2)_t\|_{L^2(0,t;V)}^2 \tag{104}$$

where C_ε does not depend on t. Hence, choosing t sufficiently small (this is our $\bar t$), we recover the contractive property of $\mathcal{T}$. Moreover, applying the Banach fixed point theorem to $\mathcal{T}$, we get a unique solution for the PROBLEM (P^ε) on the time interval $[0, \bar t]$. Due to this estimate it is easy to prove that there exists $m \in \mathbb{N}$ such that $\mathcal{T}^m$ is a contraction on X. Hence we have a unique solution on the whole time interval $[0, T]$. This concludes the proof of Theorem 4.3. $\qquad\square$

4.2. **A priori estimates.** In this subsection we perform a-priori estimates on PROBLEM (P^ε) uniformly in ε which will lead us pass to the limit as $\varepsilon \searrow 0$ and recover a solution of PROBLEM (P). We denote by c all the positive constants (which may also differ from line to line) independent of ε and depending on the data of the problem. For simplicity, we omit the subscript ε when it is not necessary.

Let us prove here the following regularity result for parabolic equations which will be useful in the sequel.

Lemma 4.4. *Consider the operator $\mathcal{B}$ already introduced in* (56), *take $\mathcal{R}$ as in* (64), *and w_0 as in* (60). *Then, if $R \in L^2(0,T;V)$ and $\Pi \in H^{3/2,3/4}(\Sigma)$, the solution w of the following initial-boundary value problem*

$$w_t + \mathcal{B}w = \mathcal{R} \quad in \ V' \quad and \ a.e. \ in \ (0,T) \tag{105}$$

$$w(0) = w_0 \quad a.e. \ in \ \Omega \tag{106}$$

is bounded in $L^\infty(Q)$.

Proof. First of all let us note that the problem (105–106) is the weak formulation of the parabolic equation

$$w_t - \Delta w = R \quad a.e. \ in \ Q \tag{107}$$

coupled with the Neumann non-homogeneous boundary condition

$$\partial_{\mathbf{n}} w = \Pi \quad \text{a.e. on } \Sigma \tag{108}$$

with the initial condition

$$w(0) = w_0 \quad \text{a.e. in } \Omega. \tag{109}$$

In order to prove this lemma, we need to reduce (107–109) to the case of Neumann homogeneous boundary condition, i.e. to the problem

$$v_t + \mathcal{B}v = \mathcal{G} \quad in \ V' \quad \text{and a.e. in } (0,T), \tag{110}$$

$$v(0) = v_0 \quad \text{a.e. in } \Omega \tag{111}$$

with

$$\langle \mathcal{G}, v \rangle = \int_\Omega Gv \quad \text{and } G \in L^2(0,T;V). \tag{112}$$

This is possible with the choice

$$G := R - \bar{w}_t + \Delta\bar{w}, \quad v_0 = w_0 - \bar{w}(0) \text{ and } \bar{w} \in H^{3,3/2}(Q) : \partial_{\mathbf{n}}\bar{w} = \Pi \text{ a.e. on } \Sigma. \tag{113}$$

Note that $\bar{w}$ is well-defined in $H^{3,3/2}(Q)$ because $\Pi \in H^{3/2,3/4}(\Sigma)$ and Theorem 5.3, p. 32 in [48] holds true. Indeed, if we take $w = \bar{w} + v$ and put it in (107), we find that v solves

$$v_t - \Delta v = G \quad \text{a.e. in } Q, \tag{114}$$

$$\partial_{\mathbf{n}} v = 0 \quad \text{a.e. on } \Sigma, \tag{115}$$

$$v(0) = w_0 - \bar{w}(0) \quad \text{a.e. in } \Omega. \tag{116}$$

Then, (110–111) is the weak formulation of (114–116) if one takes $\mathcal{G}$ as in (112). Note that (cf. (113)) $G \in L^2(0,T;V)$ because $R \in L^2(0,T;V)$ and, since $\bar{w} \in H^{3,3/2}(Q)$, $\Delta\bar{w} \in L^2(0,T;V)$ by definition and $\bar{w}_t \in L^2(0,T;V)$ because $H^{3,3/2}(Q)$ is embedded in the interpolation space $(L^2(0,T;W^{3,2}(\Omega)), H^{3/2}(0,T;L^2(\Omega)))_{\zeta,2} = W^{(3/2)\zeta,2}(0,T;W^{3(1-\zeta),2}(\Omega))$ for $\zeta \in (0,1)$ (cf. [48, Prop. 2.1, p.7] and [47, Prop. 2.3, p. 19]). Choosing $\zeta = 2/3$, we get exactly $\bar{w} \in W^{1,2}(0,T;V)$. Hence, we may apply the regularity result [44, Thm. 7.1, p.181] with $n = 3$, $r = 2$, and $q = 6$ to (110–111),

in order to find that v is bounded in $L^\infty(Q)$. Note that we have to do just one modification in order to apply [44, Thm. 7.1, p.181], i.e. their argument works with Dirichlet boundary conditions and uses inequality [44, (3.4), p. 75] to derive the estimate. On the other hand, this inequality still holds even though the functions involved do not vanish on the boundary provided that Ω is bounded and Lipschitz and we allow the constants to depend also on Ω. Moreover, by using the fact that $\bar{w}$ is bounded in $H^{3,3/2}(Q)$ and by the previous interpolation results, we have that $H^{3,3/2}(Q) \hookrightarrow L^\infty(Q)$. Hence $w(=\bar{w}+v)$ is bounded in $L^\infty(Q)$ and this concludes the proof of our lemma. $\qquad\square$

First a-priori estimate. Test (93) by $\mathbf{u}_t$, (94) by p, (95) by $\gamma^\varepsilon(w)+w$, (96) by $\boldsymbol{\beta}_t$, sum up the resulting equations and integrate over $(0,t)$ $(t \in [0,T])$. The result is

$$\int_\Omega \left(\widehat{\gamma}^\varepsilon(w(t)) + \frac{1}{2}w^2(t)\right) + \int_Q \nabla\delta^\varepsilon(\vartheta^\varepsilon)\nabla\vartheta^\varepsilon + \int_Q |\nabla w|^2 + \int_0^t \|\boldsymbol{\beta}_t\|_V^2$$

$$+\frac{1}{2}\int_\Omega |\nabla\boldsymbol{\beta}(t)|^2 + J_{H,\varepsilon}(\boldsymbol{\beta}(t)) + \frac{1}{2}\|\mathbf{u}(t)\|_W^2 + \varepsilon\|p\|_{L^2(0,t;H)}^2 = \int_\Omega \widehat{\gamma}^\varepsilon(w_0)$$

$$+\frac{1}{2}\int_\Omega w_0^2 + \frac{1}{2}\|\nabla\boldsymbol{\beta}_0\|_H^2 + \frac{1}{2}\|\mathbf{u}_0\|_W^2 + \int_0^t {}_{W'}\langle\mathcal{F},\mathbf{u}_t\rangle_W + \int_0^t \langle\mathcal{R},(\gamma^\varepsilon(w)+w)\rangle$$

$$-\int_Q \partial_t\beta_3(w - \vartheta_0) + \int_Q \tau(\gamma^\varepsilon(w))\left(\operatorname{div}\mathbf{u}\left(\partial_t\beta_1 - \partial_t\beta_2\right) + (\beta_1 - \beta_2)\operatorname{div}\mathbf{u}_t\right).$$

Now, following the line of [13, (5.5)–(5.7), p. 1583], we can deal with the source term $\mathcal{R}$ recalling (64) and using a well-known compactness inequality (cf. [47, Theorem 16.4]) in this way

$$\int_Q R\gamma^\varepsilon(w) \le \int_0^t \|R(s)\|_{L^\infty(\Omega)}\|\gamma^\varepsilon(w(s))\|_{L^1(\Omega)}\,ds \tag{117}$$

$$\int_0^t \int_{\partial\Omega} \Pi\gamma^\varepsilon(w) \le c\|\Pi\|_{L^\infty(\Sigma)}\|(\gamma^\varepsilon(w))^{1/2}\|_{L^2(\Sigma)}^2$$

$$\le \frac{1}{2}\|\nabla(\gamma^\varepsilon(w))^{1/2}\|_{(L^2(0,t;H))^3}^2 + c\|(\gamma^\varepsilon(w))^{1/2}\|_{L^2(0,t;H)}^2. \tag{118}$$

Moreover, using assumptions (62) and (65), we get (integrating by parts in time)

$$\int_0^t {}_{W'}\langle\mathcal{F},\mathbf{u}_t\rangle_W = -\int_0^t {}_{W'}\langle\mathcal{F}_t,\mathbf{u}\rangle_W + {}_{W'}\langle\mathcal{F}(t),\mathbf{u}(t)\rangle_W - {}_{W'}\langle\mathcal{F}(0),\mathbf{u}(0)\rangle_W$$

$$\le c + \frac{1}{4}\|\mathbf{u}(t)\|_W^2 + \int_0^t \|\mathcal{F}_t\|_{W'}\|\mathbf{u}\|_W. \tag{119}$$

Now, collecting estimates (117–119), using Lemma 4.2 and (94) in order to estimate the term containing $\operatorname{div}\mathbf{u}_t$ and employing assumptions (47) on τ and (59–65) on the data, we get the following inequality

$$\int_\Omega \left(\gamma^\varepsilon(w(t)) + \frac{1}{2}w^2(t)\right) + \int_Q \frac{|\nabla\gamma^\varepsilon(w)|^2}{\gamma^\varepsilon(w)} + \int_Q |\nabla w|^2 + \frac{1}{4}\int_0^t \|\boldsymbol{\beta}_t\|_V^2$$

$$+\frac{1}{2}\|\boldsymbol{\beta}(t)\|_V^2 + J_{H,\varepsilon}(\boldsymbol{\beta}(t)) + \frac{1}{4}\|\mathbf{u}(t)\|_W^2 + \varepsilon\|p\|_{L^2(0,t;H)}^2 \le c + \int_Q w^2 + \int_0^t \|\mathbf{u}\|_W^2$$

$$+c\int_0^t \|\boldsymbol{\beta}\|_H^2 + \frac{\varepsilon^2}{4}\int_0^t \|p\|_H^2 + c\|(\gamma^\varepsilon(w))^{1/2}\|_{L^2(0,t;H)}^2 + \int_0^t \|\mathcal{F}_t\|_{W'}\|\mathbf{u}\|_W$$

which - via Gronwall lemma - and choosing ε small, leads to the following first estimate

$$\|\mathbf{u}(t)\|_W^2 + \|(\gamma^\varepsilon(w))^{1/2}(t)\|_H^2 + \int_0^t \|\nabla(\gamma^\varepsilon(w))^{1/2}\|_H^2 + \|w(t)\|_H^2 + \|w\|_{L^2(0,t;V)}^2$$

$$+\varepsilon\|p\|_{L^2(0,t;H)}^2 + \|\partial_t\boldsymbol{\beta}\|_{L^2(0,t;V)}^2 + \|\nabla\boldsymbol{\beta}(t)\|_H^2 + J_{H,\varepsilon}(\boldsymbol{\beta}^\varepsilon(t)) \leq c. \qquad (120)$$

Then, applying the standard regularity results for linear parabolic equations to (95), we get

$$\|w\|_{W^{1,2}(0,T;H)\cap L^\infty(0,T;V)\cap L^2(0,T;W^{2,2}(\Omega))} \leq c. \qquad (121)$$

Now, thanks to assumptions (60) and (63), we may apply [40, Lemma 4.3] to equation (94), getting the further bound

$$\|w\|_{L^\infty(Q)} \leq c. \qquad (122)$$

Thanks to (87) we can immediately recover

$$\|\gamma^\varepsilon(w)\|_{L^\infty(Q)} \leq c \qquad (123)$$

and, due to Lemma 4.2 and (120–123), we have

$$\int_Q |\nabla\gamma^\varepsilon(w)|^2 \leq c\int_Q \frac{|\nabla\gamma^\varepsilon(w)|^2}{\gamma^\varepsilon(w)} \leq c\int_Q \nabla\delta^\varepsilon(\gamma^\varepsilon(w))\nabla\gamma^\varepsilon(w)$$

$$\leq c\int_Q |\nabla w|^2 \leq c$$

and

$$\int_Q |(\gamma^\varepsilon(w))_t|^2 \leq c\int_Q \left|\frac{(\gamma^\varepsilon(w))_t}{\gamma^\varepsilon(w)}\right|^2 \leq c\int_Q |(\delta^\varepsilon)'(\gamma^\varepsilon(w))(\gamma^\varepsilon(w))_t|^2$$

$$\leq c\int_Q |w_t|^2 \leq c.$$

Hence, from these two inequalities it follows that

$$\|\gamma^\varepsilon(w)\|_{W^{1,2}(0,T;H)\cap L^2(0,T;V)\cap L^\infty(Q)} \leq c. \qquad (124)$$

Second a-priori estimate. In order to pass to the limit (as $\varepsilon \searrow 0$) in (96), we need to pass to the limit in p^ε. We will use the following Lions' lemma which is stated in this form, e.g., in [63, Rem. 1.1, p. 17] (its proof is due to [50, Note (27), p. 320] in case of a $\mathcal{C}^1$ class domain Ω and to [54] when Ω is only Lipschitz). For further comments on this topic the reader can refer to [40, Remark 4.1].

Lemma 4.5. *Let Ω be a bounded and Lipschitz set in $\mathbb{R}^3$ and let m be a continuous seminorm on H and a norm on the constants. Then there exists a positive constant $c(\Omega)$ (depending only on Ω) such that the following inequality*

$$\|u\|_H \leq c(\Omega)\{m(u) + \|\nabla u\|_{V'}\} \qquad (125)$$

holds for all $u \in H$ with $\nabla u \in V'$.

We want to apply this result in order to find the uniform (in ε) bound on p^ε. First of all let us note that from comparison in (93), using also the bound (120) on $\mathbf{u}$ with the assumption (62) on $\mathcal{F}$, we immediately deduce that

$$\|\nabla p^\varepsilon\|_{L^2(0,T;V')} \leq c. \qquad (126)$$

Moreover, always by comparison in (93), we have that

$$\left| \int_Q p^\varepsilon \operatorname{div} \mathbf{v} \right| \le c \quad \forall \mathbf{v} \in W. \tag{127}$$

Following the idea of [40], we can choose $\mathbf{v}_\star \in W$ such that

$$\int_\Omega \operatorname{div}(\mathbf{v}_\star) \, dx = \int_{\partial\Omega} \mathbf{v}_\star \cdot \mathbf{n} \, ds \ne 0. \tag{128}$$

Note that, since Ω is regular (it suffices for Ω to be a Lipschitz domain), we can always find a $\mathbf{v}_\star \in W$ such that (128) is satisfied, because, if we take $B_\varepsilon(x)$ the ball in $\mathbb{R}^3$ centered in $x \in \Gamma_1$ with radius ε such that $B_\varepsilon(x) \cap \Gamma_0 = \emptyset$ and consider the parametrization of $\Gamma_1 \cap B_\varepsilon(x)$ through the Lipschitz function $(x_1, x_2) \mapsto (x_1, x_2, \varphi(x_1, x_2))$, then the normal unit vector associated is

$$\mathbf{n} = \frac{(\partial_{x_1}\varphi, -\partial_{x_2}\varphi, 1)}{\sqrt{1 + |\nabla\varphi|^2}}.$$

Then, if we take $\mathbf{v}_\star = (0, 0, \zeta)$ with

$$\zeta(y) = \begin{cases} \exp\left(-\dfrac{1}{1 - \frac{|x-y|^2}{\varepsilon^2}} \right) & \text{if} \quad |x - y| \le \varepsilon \\ 0 & \text{otherwise}, \end{cases}$$

then $\mathbf{v}_\star \in W$ and moreover we can show that (128) holds because

$$\frac{1}{\sqrt{1 + |\nabla\varphi|^2}} \ge \frac{1}{\sqrt{1 + L^2}},$$

where L is the Lipschitz constant of φ, and hence

$$\int_\Omega \operatorname{div} \mathbf{v}_\star = \int_{\Gamma_1} \mathbf{v}_\star \cdot \mathbf{n} \, ds = \int_{\Gamma_1 \cap B_\varepsilon(x)} \frac{\zeta}{\sqrt{1 + |\nabla\varphi|^2}} \, ds \ne 0 \,.$$

Take now m in Lemma 4.5 as

$$m(v) = \left| \int_\Omega v \operatorname{div} \mathbf{v}_\star \right|.$$

Then, $m(v)$ is a seminorm on H and a norm on the constants because of (128). Hence, we can apply Lemma 4.5 to p^ε with the choices done above and, thanks to (126–127), we get the bound

$$\|p^\varepsilon\|_{L^2(Q)} \le c. \tag{129}$$

Finally, by comparison in (96) and using the estimates (120) and (124–129), we deduce that also $\partial_H J_{H,\varepsilon}(\boldsymbol{\beta})$ is bounded in $L^2(0, T; V')$. Then, testing (96) with $\mathcal{B}\boldsymbol{\beta}^\varepsilon$ and then by using again (120) and (124–129) and the monotonicity properties of α^ε, we get also

$$\|\beta_j^\varepsilon\|_{L^\infty(0,T;W^{2,2}(\Omega))} \le c \quad (j = 1, 2, 3). \tag{130}$$

Now it remains only to pass to the limit in (93–96) as $\varepsilon \searrow 0$. This will be the aim of the next subsection.

4.3. **Passage to the limit and uniqueness.** As we have just mentioned, we want to conclude the proof of Theorem 3.2 passing to the limit in the well-posed (cf. Subsection 4.1) PROBLEM (P^ε) as $\varepsilon \searrow 0$ using the previous uniform (in ε) estimates on its solution (cf. Subsection 4.2) and exploiting some compactness-monotonicity argument. Let us list before the weak or weak-star convergence coming directly from the previous estimates and well-known weak-compactness results. Note that the following convergences hold only up to a subsequence of $\varepsilon \searrow 0$ (let us say $\varepsilon_k \searrow 0$). We denote it again with ε only for simplicity of notation. From the estimates (120–130) and the property (81) of $\partial_H J_{H,\varepsilon}$, we deduce that

$$\mathbf{u}^\varepsilon \to \mathbf{u} \quad \text{weakly star in } L^\infty(0,T;W) \tag{131}$$

$$\operatorname{div} \mathbf{u}_t^\varepsilon \to \operatorname{div} \mathbf{u}_t \quad \text{weakly in } L^2(0,T;H) \tag{132}$$

$$w^\varepsilon \to w \quad \text{weakly in } W^{1,2}(0,T;H) \cap L^2(0,T;W^{2,2}(\Omega))$$
$$\text{and weakly star in } L^\infty(Q) \cap L^\infty(0,T;V) \tag{133}$$

$$\gamma^\varepsilon(w^\varepsilon) \to \vartheta \quad \text{weakly star in } W^{1,2}(0,T;H) \cap L^\infty(0,T;V) \cap L^\infty(Q) \tag{134}$$

$$\beta_j^\varepsilon \to \beta_j \quad \text{weakly star in } W^{1,2}(0,T;V) \cap L^\infty(0,T;W^{2,2}(\Omega))$$
$$(j = 1,2,3) \tag{135}$$

$$p^\varepsilon \to p \quad \text{weakly in } L^2(Q) \tag{136}$$

$$\xi_j^\varepsilon \to \xi_j \quad \text{weakly in } L^2(0,T;V') \quad (j = 1,2,3) \,. \tag{137}$$

Moreover, employing [61, Cor. 5, p. 86], we get also

$$w^\varepsilon \to w \quad \text{strongly in } L^2(0,T;V) \cap C^0([0,T];H)$$
$$\text{and hence a.e. in } Q \tag{138}$$

$$\vartheta^\varepsilon \to \vartheta \quad \text{strongly in } C^0([0,T];H) \tag{139}$$

$$\beta_j^\varepsilon \to \beta_j \quad \text{strongly in } C^0([0,T];V) \quad (j = 1,2,3) \,. \tag{140}$$

Note that (138–139) imply immediately the convergence

$$\gamma^\varepsilon(w^\varepsilon) \to \vartheta = \gamma(w) \quad \text{and} \quad \tau(\gamma^\varepsilon(w^\varepsilon)) \to \tau(\vartheta) \quad \text{a.e. in } Q \,.$$

Moreover, the two convergences (137) and (140) along with the property (86) and [2, Thm. 3.66, p. 373] give immediately the identification of the maximal monotone graph $\partial_{V,V'} J_V$, i.e.

$$\boldsymbol{\xi} \in \partial_{V,V'} J_V(\boldsymbol{\beta}) \quad \text{in } (V')^3 \text{ and a.e. in } [0,T]$$

with $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ and ξ_j $(j = 1,2,3)$ are the weak limits defined in (137). All these convergences with the identifications made above make us able to pass to the limit (as $\varepsilon \searrow 0$ or at least for a subsequence of it) in PROBLEM (P^ε) finding a solution to PROBLEM (P) and concluding in this way the proof of Theorem 3.2. Note that the convergences hold for all subsequences ε_k of ε tending to 0 because of uniqueness of solutions. Indeed we may prove it in this way.

Consider two solutions of PROBLEM (P) $(\mathbf{u}^i, w^i, \beta_1^i, \beta_2^i, \beta_3^i, p^i)$ $(i = 1,2,3)$ corresponding to the same data. Moreover let us take the *mass balance* equation (72) in the following integrated form

$$\beta_1 + \beta_2 + \beta_3 + \operatorname{div} \mathbf{u} = \beta_1(0) + \beta_2(0) + \beta_3(0) + \operatorname{div} \mathbf{u}(0) \quad \text{a.e. in } Q \,. \tag{141}$$

Then, integrate equation (73) over $(0,t)$ (let us call it $1 * (73)$ with a little abuse of notation) and write down two times equations (71–72), $1 * (73)$, (74) with

$(\mathbf{u}^i, w^i, \beta_1^i, \beta_2^i, \beta_3^i, p^i)$, make the difference between the two equations $1 * (73)$, and test the result with $(w^1 - w^2)$. Make the difference between the two equations (71), test the result with $(\mathbf{u}^1 - \mathbf{u}^2)$. Make the difference between the two equations (74), written for $i = 1$ and $i = 2$, and test the resulting vectorial equation by the vector $\left((\beta_1^1) - (\beta_1^2), (\beta_2^1) - (\beta_2^2), (\beta_3^1) - (\beta_3^2)\right)$. Finally, summing up the three resulting equations, integrating over $(0,t)$, with $t \in [0,T]$, exploiting the monotonicity of $\partial_{V,V'} J_V$, using equation (141) in order to get rid of the p-terms, and using the fact that γ, defined in (66), is a locally Lipschitz continuous function, $\vartheta^i = \gamma(w^i)$ $(i = 1, 2)$, and w^i are bounded in $L^\infty(Q)$ (cf. (68)), we get the following inequality

$$\frac{1}{2}\|(\mathbf{u}^1 - \mathbf{u}^2)\|_{L^2(0,t;W)}^2 + \frac{1}{2}\|w^1 - w^2\|_{L^2(0,t;H)}^2 + \|1 * (w^1 - w^2)(t)\|_V^2$$

$$+ \sum_{j=1}^{3} \left(\|(\beta_j^1 - \beta_j^2)(t)\|_V^2 + \|(\beta_j^1 - \beta_j^2)\|_{L^2(0,t;V)}^2\right)$$

$$\leq c \int_0^t \sum_{j=1}^{3} \left(1 + \|\operatorname{div} \mathbf{u}^1\|_V^2 + \|\operatorname{div} \mathbf{u}^2\|_V^2\right) \|\beta_j^1 - \beta_j^2\|_V^2$$

for some positive constant c depending on the data of the problem. Let us notice that we have estimated the terms containing the nonlinearity τ in (71) and (74) on the right hand side as follows

$$\int_Q \left((\beta_1^1 - \beta_2^1)\tau(\vartheta^1) - (\beta_1^2 - \beta_2^2)\tau(\vartheta^2)\right)\left(\operatorname{div}(\mathbf{u}^1 - \mathbf{u}^2)\right)$$

$$+ \int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2))\operatorname{div} \mathbf{u}^1 + \tau(\vartheta^2)(\operatorname{div} \mathbf{u}^1 - \operatorname{div} \mathbf{u}^2)\right)\left(\beta_2^1 - \beta_2^2 - \beta_1^1 + \beta_1^2\right)$$

$$\leq \frac{1}{4}\|w^1 - w^2\|_{L^2(0,t;H)}^2 + \frac{1}{2}\|(\mathbf{u}^1 - \mathbf{u}^2)\|_{L^2(0,t;W)}^2$$

$$+ c \int_0^t \left(1 + \|\operatorname{div} \mathbf{u}^1\|_V^2 + \|\operatorname{div} \mathbf{u}^2\|_V^2\right)\left(\|\beta_1^1 - \beta_2^1\|_V^2 + \|\beta_1^2 - \beta_2^2\|_V^2\right).$$

The application of the standard Gronwall lemma together with the regularity (67) leads to uniqueness of solutions to PROBLEM (P) and concludes to proof of Theorem 3.2. $\qquad\square$

5. **Proof of Theorem 3.4.** In this section we give the proof of Theorem 3.4. We will use here the same symbol c for some positive constants (depending only on the data of the problem), which may also be different from line to line.

Then, let us take two sets of data $(\mathbf{u}_0^i, w_0^i, \boldsymbol{\beta}_0^i)$, $(\mathcal{R}^i, \mathcal{F}^i)$ $(i = 1, 2)$ of *Problem (P)* and let $(\mathbf{u}^i, w^i, \beta_1^i, \beta_2^i, \beta_3^i, p^i)$ $(i = 1, 2)$ be two solutions of *Problem (P)* corresponding to these data.

Then, write two times equations (71–74) with $(\mathbf{u}^i, w^i, \beta_1^i, \beta_2^i, \beta_3^i, p^i)$, make the difference between the two equations (73), and test the result with $2(w^1 - w^2)$. Make the difference between the two equations (71), test the result with $2(\mathbf{u}^1 - \mathbf{u}^2)_t$. Make the difference between the two equations (74), written for $i = 1$ and $i = 2$, and test the resulting vectorial equation by the vector $\left((\beta_1^1)_t - (\beta_1^2)_t, (\beta_2^1)_t - (\beta_2^2)_t, (\beta_3^1)_t - (\beta_3^2)_t\right)$.

Finally, summing up the three resulting equations, integrating over $(0,t)$, with $t \in [0,T]$, and exploiting the Lipschitz continuity of α (cf. assumption (79)), we get

the following inequality

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_W^2 + \|w^1(t) - w^2(t)\|_H^2 + \|w^1 - w^2\|_{L^2(0,t;V)}^2$$

$$+ \sum_{j=1}^3 \left(\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 + 1/2\|\nabla(\beta_j^1 - \beta_j^2)(t)\|_H^2 \right)$$

$$\leq c \int_0^t \left(1 + \|\operatorname{div} \mathbf{u}_1\|_V^2 + \|\operatorname{div} \mathbf{u}_2\|_V^2\right) \|\vartheta^1 - \vartheta^2\|_H^2$$

$$+ \sum_{j=1}^3 \left(c\|\beta_j^1 - \beta_j^2\|_{L^2(0,t;H)}^2 + 1/2\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 \right)$$

$$+ \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_W^2 + \|w_0^1 - w_0^2\|_H^2 + 1/2 \sum_{j=1}^3 \|\nabla(\beta_j^{01} - \beta_j^{02})\|_H^2$$

$$+ c\|\mathcal{F}^1 - \mathcal{F}^2\|_{L^2(0,T;W')}^2 + c\|\mathcal{R}^1 - \mathcal{R}^2\|_{L^2(0,t;V')}^2. \tag{142}$$

Let us notice that we have estimated the terms containing the nonlinearity τ in (71) and (74) on the right hand side using (72) and the fact that γ, defined in (66), is a locally Lipschitz continuous function, $\vartheta^i = \gamma(w^i)$ $(i = 1, 2)$, and w^i are bounded in $L^\infty(Q)$ (cf. (68)), as follows

$$\int_Q \left((\beta_1^1 - \beta_2^1)\tau(\vartheta^1) - (\beta_1^2 - \beta_2^2)\tau(\vartheta^2) \right) \left(\operatorname{div}(\mathbf{u}^1 - \mathbf{u}^2)_t \right)$$

$$+ \int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2)) \operatorname{div} \mathbf{u}^1 + \tau(\vartheta^2)(\operatorname{div} \mathbf{u}^1 - \operatorname{div} \mathbf{u}^2) \right) \left((\beta_2^1 - \beta_2^2)_t - (\beta_1^1 - \beta_1^2)_t \right)$$

$$= -\int_Q \left((\beta_1^1 - \beta_2^1)\left(\tau(\vartheta^1) - \tau(\vartheta^2)\right) + \left(\beta_1^1 - \beta_1^2 - \beta_2^1 + \beta_2^2\right)\tau(\vartheta^2) \right) \sum_{j=1}^3 \left(\beta_j^1 - \beta_j^2\right)_t$$

$$+ \int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2)) \operatorname{div} \mathbf{u}^1 + \tau(\vartheta^2)(\operatorname{div} \mathbf{u}^1 - \operatorname{div} \mathbf{u}^2) \right) \left(\beta_2^1 - \beta_2^2\right)_t$$

$$- \int_Q \left((\tau(\vartheta^1) - \tau(\vartheta^2)) \operatorname{div} \mathbf{u}^1 + \tau(\vartheta^2)(\operatorname{div} \mathbf{u}^1 - \operatorname{div} \mathbf{u}^2) \right) \left(\beta_1^1 - \beta_1^2\right)_t$$

$$\leq \frac{1}{4} \sum_{j=1}^3 \|(\beta_j^1 - \beta_j^2)_t\|_V^2 + c \sum_{j=1}^3 \|\beta_j^1 - \beta_j^2\|_{L^2(0,t;H)}^2$$

$$+ c \int_0^t \left(1 + \|\operatorname{div} \mathbf{u}^1\|_V^2 + \|\operatorname{div} \mathbf{u}^2\|_V^2\right) \left(\|w^1 - w^2\|_H^2 + \|(\mathbf{u}^1 - \mathbf{u}^2)\|_W^2\right).$$

Moreover, by adding to both sides in the inequality (142)

$$\frac{1}{2} \sum_{j=1}^3 \|(\beta_j^1)(t) - (\beta_j^2)(t)\|_H^2 = \frac{1}{2} \sum_{j=1}^3 \|\beta_j^{01} - \beta_j^{02}\|_H^2 + \int_0^t \sum_{j=1}^3 ((\beta_j^1)_t - (\beta_j^2)_t, \beta_j^1 - \beta_j^2)$$

we get the following inequality

$$\|\mathbf{u}^1(t) - \mathbf{u}^2(t)\|_W^2 + \|w^1(t) - w^2(t)\|_H^2 + \|w^1 - w^2\|_{L^2(0,t;V)}^2$$

$$+ \sum_{j=1}^{3} \left(\|(\beta_j^1)_t - (\beta_j^2)_t\|_{L^2(0,t;V)}^2 + \|(\beta_j^1 - \beta_j^2)(t)\|_V^2 \right)$$

$$\leq c \int_0^t \left(1 + \|\operatorname{div}\mathbf{u}_1\|_V^2 + \|\operatorname{div}\mathbf{u}_2\|_V^2\right) \left(\|w^1 - w^2\|_H^2 + \|(\mathbf{u}^1 - \mathbf{u}^2)\|_W^2\right)$$

$$+ c\Big(\sum_{j=1}^{3} \|\beta_j^1 - \beta_j^2\|_{L^2(0,t;H)}^2 + \|\mathbf{u}_0^1 - \mathbf{u}_0^2\|_W^2 + \|w_0^1 - w_0^2\|_H^2 + \sum_{j=1}^{3} \|\beta_j^{01} - \beta_j^{02}\|_V^2$$

$$+ \|\mathcal{F}^1 - \mathcal{F}^2\|_{L^2(0,T;W')}^2 + \|\mathcal{R}^1 - \mathcal{R}^2\|_{L^2(0,t;V')}^2 \Big).$$

Applying now a standard version of Gronwall's lemma (cf. [16, Lemme A.4, p. 156]), we get the desired continuous dependence estimate (80). This concludes the proof of Theorem 3.4. □

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