APPROXIMATION OF THE HELFRICH'S FUNCTIONAL VIA DIFFUSE INTERFACES

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Abstract. We give a rigorous proof of the approximability of the so-called Helfrich's functional via diffuse interfaces, under a constraint on the ratio between the bending rigidity and the Gauss-rigidity.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be an open connected set with smooth boundary. Define

$$\mathcal{W}_{\rm Hel}(E) := \int_{\Omega\cap\partial E} \left[\frac{\varkappa_b}{2}(H_{\partial E} - H_0)^2 + \varkappa_G K_{\partial E}\right] d\mathcal{H}^2, \tag{1.1}$$

where $E \subset \Omega$ is open, bounded and with boundary ∂E of class C^2 in Ω; $H_{\partial E}$, $K_{\partial E}$ are respectively the mean curvature and the Gaussian-curvature of ∂E (i.e. respectively the sum and the product of the two principal curvatures of ∂E); $\mathcal{H}^2$ is the 2-dimensional Hausdorff-measure; $\varkappa_b$, H_0, $\varkappa_G$ are given constants. For our purposes it is convenient to write $\mathcal{W}_{\rm Hel}$ as

$$\mathcal{W}_{\rm Hel}(E) = \frac{\varkappa_b}{2}\mathcal{H}(E) + \varkappa_G \mathcal{K}(E),$$

where

$$\mathcal{H}(E) := \int_{\Omega\cap\partial E} (H_{\partial E} - H_0)^2 \, d\mathcal{H}^2, \tag{1.2}$$

$$\mathcal{K}(E) := \int_{\Omega\cap\partial E} K_{\partial E} \, d\mathcal{H}^2. \tag{1.3}$$

The functional $\mathcal{W}_{\rm Hel}$ was proposed by Helfrich as a surface energy for closed biological membranes represented by a smooth boundaryless surface (see also [12, 25] and [8, Chapter 7]). Minimizers and critical points of $\mathcal{W}_{\rm Hel}$ in the class of subsets $E \subset \Omega$ satisfying a constraint on the area $\mathcal{H}^2(\Omega \cap \partial E)$ and on the enclosed volume $\mathcal{L}^3(E \cap \Omega)$, are expected to describe approximately the shape of biological membranes such as monolayers or lipid bilayers (see again [8] for an introduction to the subject). Note that the term $\mathcal{K}(E)$ can be neglected when minimizing $\mathcal{W}_{\rm Hel}(E)$ under a topological constraint on E, since by the Gauss-Bonnet theorem it reduces to a constant depending on the fixed topology. On the other hand $\mathcal{K}$ plays an essential role in several recent related models (see e.g. [3, 6, 2]).

The constant $\varkappa_b > 0$ is called the bending rigidity. The constant H_0 is called the spontaneous curvature. It is expected to be non zero when dealing with biological membranes such as bilayers with chemically different interior and exterior layers,

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or when different enviroments inside and outside the membrane are source of asymmetry. Observe that, when $H_0 \neq 0$, the functional $\mathscr{H}$ depends on the orientation of ∂E (and not only on the geometry of ∂E as in the case $H_0 = 0$). The constant $\varkappa_G$ is called the Gauss-rigidity. Although few experimental measurements for $\varkappa_G$ are presently available, it is expected to be negative (see [42], [40], [37, Section 4.5.9], [8, Section 7.2]). Moreover, at least in case of some monolayers (see [42, 40]), $\varkappa_b$ and $\varkappa_G$ satisfy

$$-1 < \frac{\varkappa_G}{\varkappa_b} < 0. \tag{1.4}$$

In this paper we are concerned with the variational approximation of $\mathscr{W}_{\rm Hel}$, under condition (1.4) and with $H_0 = 0$; in Section 9 we briefly discuss how to relax these two constraints. In this respect we note that, for any given $H_0 \in \mathbb{R}$, a condition ensuring compactness and lower semicontinuity of $\mathscr{W}_{\rm Hel}$ in a reasonable topology (see Theorem 3.2 and Remark 3.4) is the existence of two positive numbers c and λ such that

$$\frac{\varkappa_b}{2}\left(H_{\partial E} - H_0\right)^2 + \varkappa_G K_{\partial E} \geq c\left|\mathbf{B}_{\partial E}\right|^2 - \lambda,$$

where $\mathbf{B}_{\partial E}$ denotes the second fundamental form of ∂E. Such a condition is equivalent to the constraint $-2 < \varkappa_G/\varkappa_b < 0$ (see Section 9.1), which is trivially satisfied when (1.4) holds.

Recently several authors have used diffuse interfaces approximations in order to develop efficient numerical simulations for a number of models involving $\mathscr{W}_{\rm Hel}$ (e.g. see [7, 18, 19, 21, 23, 22, 20, 10, 11, 17, 24, 26]). Analytical results have been carried on, mainly by means of formal asymptotics, in [23, 18, 19, 46]. Most of the papers cited above concentrate on the approximation of the term $\mathscr{H}$ which (up to minor modifications) takes the form

$$\mathscr{H}_\varepsilon(u) := \frac{1}{\varepsilon}\int_\Omega \left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon} - \varepsilon|\nabla u| H_0\right)^2 dx, \tag{1.5}$$

where $\varepsilon > 0$ is a small parameter related to the width of the diffuse interface, and $W \in C^2(\mathbb{R})$ is a double-well potential with two equal minima (from now on, throughout the paper, we will make the choice $W(s) := (1-s^2)^2/4$). Actually, in the case $H_0 = 0$, it was firstly conjectured in [14] that functionals similar to (1.5) Γ-converge to $\sigma\mathcal{H}$ as $\varepsilon \to 0^+$, where σ is a suitable positive constant.

At least in the case $H_0 = 0$, the choice of the sequence in (1.5) can be heuristically motivated with the fact that $\mathcal{H}_\varepsilon$ represents a kind of (rescaled) squared " L^2-gradient" of the functional $\mathscr{P}_\varepsilon$ defined as

$$\mathscr{P}_\varepsilon(u) := \int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right) dx, \qquad \text{if } u \in H^1(\Omega),$$

and $\mathscr{P}_\varepsilon(u) := +\infty$ elsewhere in $L^1(\Omega)$. This, together with the well known results that $\mathscr{P}_\varepsilon$ approximate the perimeter functional as $\varepsilon \to 0^+$ (see [32, 9]), and that the "L^2-gradient" of the perimeter is formally given by the mean curvature operator, furnishes a (very) heuristic justification for the choice of $\mathscr{H}_\varepsilon$.

The aim of this paper is twofold: we want to propose a diffuse interface approximation of $\mathscr{K}$ which slightly differs from those proposed until now (see [22, 20] and Remark 2.5). Moreover, we want to prove a rigorous convergence result for our approximating sequence within the framework of Γ-convergence, under the assumptions that $H_0 = 0$, and provided the parameters $\varkappa_b$, $\varkappa_G$ satisfy (1.4).

In order to define the approximating functionals we need some notation. For every $u \in C^2(\Omega)$ we define the vector field $\nu_u \in L^\infty(\Omega)$ by $\nu_u := \nabla u/|\nabla u|$ whenever $\nabla u \neq 0$ and $\nu_u := \mathbf{e}$ on $\{\nabla u = 0\}$, where $\mathbf{e}$ is an arbitrary unit vector (to fix the notation from now on we will choose $\mathbf{e} = \mathbf{e}_3$, $\mathbf{e}_3$ being the third element of the canonical basis of $\mathbb{R}^3$). Then, denoting by $|\cdot|$ the norm of a matrix as defined in (2.1), we propose to approximate $\mathcal{K}$ with the functionals $\mathcal{K}_\varepsilon$ defined as

$$\begin{aligned}\mathcal{K}_\varepsilon(u) :=& \frac{1}{2\varepsilon}\int_\Omega \left[\left(\varepsilon\Delta u - \frac{W'(u)}{\varepsilon}\right)^2 - \left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u\right|^2\right] dx \\ =& \frac{1}{\varepsilon}\int_\Omega \sum_{1\le i<j\le 3} \det\left[\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u\right]_{ij} dx,\end{aligned} \tag{1.6}$$

when $u \in C^2(\Omega)$ and $+\infty$ elsewhere in $L^1(\Omega)$, where, for a 3×3-matrix M, M_{ij} stands for its ij-th principal minor. Eventually, as an approximation of $\mathcal{W}_{\mathrm{Hel}}$, if $\mathcal{H}_\varepsilon$ is as in (1.5) with $H_0 = 0$, we consider

$$\mathcal{W}_\varepsilon(u) := \frac{\varkappa_b}{2}\mathcal{H}_\varepsilon(u) + \varkappa_G\mathcal{K}_\varepsilon(u) \tag{1.7}$$

$$= \int_\Omega \left\{\frac{\varkappa_b+\varkappa_G}{2\varepsilon}\left[\mathrm{tr}\left(\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u\right)\right]^2 - \frac{\varkappa_G}{2\varepsilon}\left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u\right|^2\right\} dx.$$

We can roughly summarize our main results as follows. Suppose that (1.4) holds, that $H_0 = 0$, and let $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ satisfy

$$\sup_{0<\varepsilon<1} \mathcal{P}_\varepsilon(u_\varepsilon) < +\infty, \qquad \sup_{0<\varepsilon<1} \mathcal{W}_\varepsilon(u_\varepsilon) < +\infty. \tag{1.8}$$

Then

(*Compactness*, see Theorems 4.1 and 4.4). Up to a (not relabelled) subsequence, there exists a function $u = 2\chi_E - 1 \in BV(\Omega, \{-1,1\})$ such that $\lim_{\varepsilon\to 0^+} u_\varepsilon = u$ in $L^1(\Omega)$. Furthermore, the measures μ_{u_ε} associated with the density of the functionals $\mathcal{P}_\varepsilon(u_\varepsilon)$ (see (2.14)) concentrate, as $\varepsilon \to 0^+$, on a generalized surface $\mathcal{M} \supseteq \Omega\cap\partial E$, for which a weak notion of second fundamental form is defined. Actually, for almost every $s \in (-1,1)$ the oriented varifolds associated with the level sets $\{u_\varepsilon = s\}$ converge to the same limit.

(*Lower bound*, see Theorem 4.1). The $\liminf_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u_\varepsilon)$ is bounded from below by a suitable positive constant c_0 times the value of (a suitable extension of) $\mathcal{W}_{\mathrm{Hel}}$ evaluated on $\mathcal{M}$. In particular if E has C^2-boundary in Ω we have

$$\liminf_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u_\varepsilon) \ge c_0\mathcal{W}_{\mathrm{Hel}}(E). \tag{1.9}$$

(*Upper bound*, see Theorem 4.2). For every bounded open set $E \subset \Omega$ with C^2-boundary there exists a sequence $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that $\lim_{\varepsilon\to 0^+} u_\varepsilon \to 2\chi_E - 1$ in $L^1(\Omega)$, and $\lim_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u_\varepsilon) = c_0\mathcal{W}_{\mathrm{Hel}}(E)$.

($\Gamma(L^1)$-*Limit on smooth points*, see Corollary 4.3). By the $L^1(\Omega)$-lower semicontinuity of $\mathcal{W}_{\mathrm{Hel}}$ (see Theorem 3.2) we can conclude that if the bounded set E has C^2-boundary in Ω, then

$$\Gamma(L^1) - \lim_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u) = c_0\mathcal{W}_{\mathrm{Hel}}(E).$$

As we already said, in [22, 20] slightly different approximations of the Gaussian curvature have been proposed and used in numerical experiments to retrieve topological informations for the diffuse interface. The functional $\mathcal{K}_\varepsilon$ in (1.6) might have some advantages, at least from the analytical point of view. Firstly $\mathcal{W}_\varepsilon$ can be expressed in terms of the trace and the norm of $\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u \otimes \nu_u$, and for every $x_0 \in \Omega$ such that $\nabla u(x_0) \neq 0$, the matrix $\varepsilon\nabla^2 u(x_0) - \frac{W'(u(x_0))}{\varepsilon}\nu_u(x_0) \otimes \nu_u(x_0)$ has an explicit relation with the second fundamental form of the level line $\{u = u(x_0)\}$ times $|\nabla u(x_0)|$ (see (5.8)). Secondly, if (1.4) is satisfied, from (1.8) we can derive the bound

$$\sup_{0<\varepsilon<1} \frac{1}{\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\nu_{u_\varepsilon} \otimes \nu_{u_\varepsilon}\right|^2 dx < +\infty.$$

From this latter relation we can deduce two rather interesting further properties. The first is that, as already stated above, the energy measures μ_{u_ε} concentrate on a generalized surface with second fundamental form in L^2 (namely a *Hutchinson's curvature varifold*, see Lemmata 5.1 and 5.3). As a consequence we get better regularity for the limit of the μ_{u_ε} with respect to the case when only a uniform bound on $\mathcal{H}_\varepsilon(u_\varepsilon)$ is available; indeed, under this latter uniform bound, the measures μ_{u_ε} concentrate on a *rectifiable integral Allard's varifold* with squared integrable generalized mean curvature (see [38, 45], and Appendix B for the definitions of varifold and curvature varifold). The second property is an improved convergence to zero of the discrepancies $\xi^\varepsilon_{u_\varepsilon}$ defined in (2.16). In fact, we obtain that $\lim_{\varepsilon\to 0^+} \|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon}\|_{L^p(\Omega)} = 0$, for every $p \in [1, 3/2)$ (see Proposition 4.6). Let us stress that the improved convergence of the discrepancies may indicate a good behaviour of $\mathcal{W}_\varepsilon$ in numerical experiments. Indeed, given $\{u_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ such that $\lim_{\varepsilon\to 0^+} u_\varepsilon = 2\chi_E - 1$ in $L^1(\Omega)$, the condition $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon} = O(\varepsilon)$ is one of the characteristics for a sequence to be a "good" recovery sequence (like, for example, the one constructed in Theorem 4.2). In other words, one of the properties that suggests a "good" convergence to the sharp interface functional is that $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon}$ vanishes rapidly enough as $\varepsilon \to 0^+$. In numerical applications, a penalizing term of the form $\|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{W(u_\varepsilon)}{\varepsilon}\|^p_{L^p(\Omega)}$ is often added to the diffuse interface functional to force such a "fast" decay of $|\xi^\varepsilon_{u_\varepsilon}|$.

Let us conclude by remarking the fact that, although an approximation via diffuse interfaces seems to be reasonable for numerical purposes, our result does not establish any physical derivation of the Helfrich's energy as a mesoscale limit, as for example it has been recently done in [36].

The paper is organized as follows. In Section 2 we fix some notation, recall some basic definitions from differential geometry and briefly comment on the definition of $\mathcal{W}_\varepsilon$, as well as on the relation of $\mathcal{K}_\varepsilon$ with [22, 20]. In Section 3 we summarize the main results proved in [38], that represent one of the pillars on which our paper rests. In Section 4 we state our main results. The proofs are postponed to Sections 5-8. In Section 9 we collect some additional results, and we show how the assumptions on the parameters $\varkappa_b$, $\varkappa_G$, can be weakened; we briefly discuss the possibility of proving a full Γ-convergence result and the problems arising in the case $H_0 \neq 0$. Eventually in Appendices A-B we collect some definitions and results on measure-function pairs and geometric measure theory, needed in the proofs of the main results.

2. Notation

2.1. **Linear algebra.** We endow the space of the (3×3) matrices $M=(m_{ij})\in\mathbb{R}^{3\times 3}$ (resp. 3^3 tensors $T=(t_{ijk})\in\mathbb{R}^{3^3}$) with the norm

$$|M|^2 := \operatorname{tr}(M^TM)=\sum_{i,j=1}^{3}(m_{ij})^2 \qquad \left(\text{resp. } |T|^2 := \sum_{i,j,k=1}^{3}(t_{ijk})^2\right), \tag{2.1}$$

where M^T is the transposed of M.

If $M\in\mathbb{R}^{3\times 3}$ is symmetric, $O=(o_{il})\in O(3)$ and $D=\operatorname{diag}(d_{11},d_{22},d_{33})$ are such that $M=O^TDO$, then $|M|^2=\operatorname{tr}(O^TD^2O)=\sum_{l=1}^3(d_{ll})^2\sum_{i=1}^3(o_{li})^2=\sum_{l=1}^3(d_{ll})^2$. Moreover, still for a symmetric matrix $M\in\mathbb{R}^{3\times 3}$, we have $\frac{1}{2}\left[(\operatorname{tr}(M))^2-\operatorname{tr}(M^TM)\right]=\sum_{1\le i<j\le 3}\det(M_{ij})$, where M_{ij} is the ij-principal minor of M.

Remark 2.1. If $P\in\mathbb{R}^{3\times 3}$ is a (symmetric) orthogonal projection matrix onto some subspace of $\mathbb{R}^3$ and M is symmetric, then

$$|P^TMP|^2\le |M|^2. \tag{2.2}$$

Indeed

$$|P^TMP|^2=\sum_{j=1}^3\sum_{i=1}^3\left(\sum_{l=1}^3 p_{il}\left(\sum_{k=1}^3 m_{lk}p_{kj}\right)\right)^2=\sum_{j=1}^3\left|P\,(MP)^{(j)}\right|^2, \tag{2.3}$$

where the column vector $(MP)^{(j)}\in\mathbb{R}^3$ has components $(\sum_{k=1}^3 m_{1k}p_{kj},\sum_{k=1}^3 m_{2k}p_{kj},\sum_{k=1}^3 m_{3k}p_{kj})$, and $|\cdot|$ on the right hand side of (2.3) is the euclidean norm of a vector. Since P is an orthogonal projection we have

$$|P^TMP|^2\le\sum_{j=1}^3\left|(MP)^{(j)}\right|^2=\sum_{i,j=1}^3\left(\sum_{k=1}^3 m_{ik}p_{kj}\right)^2=\sum_{i=1}^3\left|(M)_{(i)}\,P\right|^2,$$

where $(M)_{(i)}=(m_{i1},m_{i2},m_{i2})\in\mathbb{R}^3$. Using again the fact that P is a projection we have

$$|P^TMP|^2\le\sum_{i=1}^3\left((M)_{(i)}\right)^2=\sum_{i=1}^3\sum_{j=1}^3(m_{ij})^2=|M|^2.$$

By $G_{2,3}$ (resp. $G^0_{2,3}$) we denote the Grassmannian of the unoriented 2-planes in $\mathbb{R}^3$ (resp. the Grassmannian of the oriented 2-planes in $\mathbb{R}^3$).

We denote by $\mathbf{q}$ the standard 2-fold covering map $\mathbf{q}:G^0_{2,3}\to G_{2,3}$. We often identify $G^0_{2,3}$ with the set of simple unit 2-vectors $\tau\in\Lambda_2(\mathbb{R}^3)$. Moreover

$$\star:\Lambda^1(\mathbb{R}^3)\to\Lambda_2(\mathbb{R}^3)$$

denotes the Hodge operator. Often vectors and covectors will be identified. For every $\tau\in G^0_{2,3}$ we define $\nu^\tau\in\mathbb{R}^3\simeq\Lambda^1(\mathbb{R}^3)$ as the unique unit vector such that $\star\nu^\tau=\tau$.

We endow $G_{2,3}$ with the distance induced by the norm $|S|$, where S is the matrix associated with the orthogonal projection of $\mathbb{R}^3$ onto $S\in G_{2,3}$. Moreover, for every open set $\Omega\subseteq\mathbb{R}^3$ we let $G_2(\Omega):=\Omega\times G_{2,3}$, endowed with the product distance.

In the same way, we endow $G^0_{2,3}$ with the distance induced by $|\tau|$, where τ is the simple unit 2-vector associated with $\tau\in G^0_{2,3}$. Moreover, for every open set $\Omega\subseteq\mathbb{R}^3$ we let $G^0_2(\Omega):=\Omega\times G^0_{2,3}$, endowed with the product distance. Finally, we let $\mathbb{S}^2 := \{\xi \in \mathbb{R}^3 : |\xi| = 1\}$, and we denote by $\triangle$ the symmetric difference between sets.

2.2. **Differential Geometry.** Let Σ be a smooth, compact oriented surface without boundary embedded in $\mathbb{R}^3$. If $x \in \Sigma$, we denote by $P_\Sigma(x)$ the orthogonal projection onto the tangent plane $T_x\Sigma$ to Σ at x. Often we identify the linear operator $P_\Sigma(x)$ with the symmetric (3×3)-matrix $\mathrm{Id} - \nu_x \otimes \nu_x$ where $x \to \nu_x \in (T_x\Sigma)^\perp$ is a smooth unit covector field orthogonal to $T_x\Sigma$.

Let us recall that, when Σ is given as a level surface $\{v = t\}$ of a smooth function v such that $\nabla v \neq 0$ on $\{v = t\}$, we can take at $x \in \{v = t\}$

$$\nu_x = \frac{\nabla v(x)}{|\nabla v(x)|}, \qquad P_\Sigma(x) = \mathrm{Id} - \frac{\nabla v(x) \otimes \nabla v(x)}{|\nabla v(x)|^2}.$$

The second fundamental form $\mathbf{B}_\Sigma$ of Σ has the expression

$$\mathbf{B}_\Sigma = \Big(P_\Sigma^T \frac{\nabla^2 v}{|\nabla v|} P_\Sigma\Big) \otimes \frac{\nabla v}{|\nabla v|},$$

where $P_\Sigma^T = (P_\Sigma)^T$. The definition of $\mathbf{B}_\Sigma$ depends only on Σ and not on the particular choice of the function v. Moreover $\mathbf{B}_\Sigma(x)$, if restricted to $T_x\Sigma$ and considered as a bilinear map from $T_x\Sigma \times T_x\Sigma$ with values in $(T_x\Sigma)^\perp$, coincides with the usual notion of second fundamental form. By

$$\mathbf{H}_\Sigma(x) = (H_1(x), H_2(x), H_3(x)) = \mathrm{tr}\Big(P_\Sigma^T \frac{\nabla^2 v}{|\nabla v|} P_\Sigma\Big)\nu_x,$$

we denote the mean curvature vector of Σ at $x \in \Sigma$. We define the (scalar) mean curvature of Σ at x with respect to ν_x as

$$H_\Sigma(x) := \mathbf{H}_\Sigma(x) \cdot \nu_x.$$

Notice that $\mathbf{H}_\Sigma$ does not depend on the choice of ν, while the sign of H_Σ does. Observe also that H_Σ is the sum of the two principal curvatures of Σ: sometimes H_Σ is also referred to as the total curvature. When $\Sigma = \partial E$, where $E \subset \mathbb{R}^3$ is open and bounded, we define $\nu_{\partial E}$ to be the interior normal to $\partial E = \Sigma$ and $H_{\partial E} := \mathbf{H}_{\partial E} \cdot \nu_{\partial E}$, which turns out to positive on convex surfaces.

Let us also define $A^\Sigma(x) := (A^\Sigma_{ijk}(x))_{1\le i,j,k\le 3} \in \mathbb{R}^{3^3}$ as

$$A^\Sigma_{ijk} = \delta^\Sigma_i P_{\Sigma jk} \qquad \text{on } \Sigma, \tag{2.4}$$

where $\delta^\Sigma_i := P_{\Sigma ij}\frac{\partial}{\partial x_j}$.

To better understand definition (2.4), it is useful to recall the links netween $\mathbf{B}_\Sigma$ and A^Σ (see [31, Proposition 2.3]).

Proposition 2.2. *Set $A = A^\Sigma$, $\mathbf{B} = \mathbf{B}_\Sigma$ and $\mathbf{H} = \mathbf{H}_\Sigma$. For $i, j, k \in \{1, 2, 3\}$ the following relations hold:*

$$B^k_{ij} = P_{jl}A_{ikl}, \tag{2.5}$$

$$A_{ijk} = B^k_{ij} + B^j_{ik}, \tag{2.6}$$

$$\mathbf{H}_i = A_{jij} = B^j_{ji} + B^i_{jj}. \tag{2.7}$$

The next proposition shows some of the relations between the curvatures of Σ and the derivatives of the signed distance function from Σ itself.

Proposition 2.3. *Let E be a bounded open subset of $\mathbb{R}^3$ with C^2-boundary. Then there exists an open neighborhood U of ∂E such that, denoting by $d : U \to \mathbb{R}$ the signed distance from ∂E positive inside E, we have $d \in C^2(U)$ and, for $y \in U$ and $\pi(y) := y - d(y)\nabla d(y) \in \partial E$ the unique orthogonal projection point of y on ∂E,*

$$\Delta d(y) = H_{\partial E}(\pi(y)) + o(d(y)) \tag{2.8}$$

$$\sum_{1\le i<j\le 3} \det\left([\nabla^2 d(y)]_{ij}\right) = K_{\partial E}(\pi(y)) + o(d(y)), \tag{2.9}$$

where $o(t) \to 0$ as $t \to 0$.

Proof. It is well known (see for example [27]) that d is of class C^2 in a suitable tubular neighborhood U of ∂E where π is single valued, and moreover that, for every $y \in U$, the eigenvalues of $\nabla^2 d(y)$ are

$$\lambda_1(y) = \frac{k_1(\pi(y))}{1 - d(y)k_1(\pi(y))}, \qquad \lambda_2(y) = \frac{k_2(\pi(y))}{1 - d(y)k_2(\pi(y))}, \qquad \lambda_3(y) = 0,$$

where $k_1(x), k_2(x)$ are the principal curvatures of ∂E at x. Then (2.8) follows, and

$$\begin{aligned}\sum_{1\le i<j\le 3} \det\left([\nabla^2 d(y)]_{ij}\right) =& \frac{\Big(\mathrm{tr}(\nabla^2 d(y))\Big)^2 - |\nabla^2 d(y)|^2}{2} \\ =& \lambda_1(y)\lambda_2(y) = K_{\partial E}(\pi(y)) + o(d(y)).\end{aligned}$$

□

2.3. **The Helfrich's Functional $\mathscr{W}_{\mathrm{Hel}}$.** Throughout the paper $\Omega \subseteq \mathbb{R}^3$ is an open connected set with smooth boundary ($\Omega = \mathbb{R}^3$ is allowed). If $E \subseteq \mathbb{R}^3$, χ_E is the characteristic function of E equal to 1 on E and 0 elsewhere. Let $E \subseteq \Omega$ be an open set. We say that E has C^k-boundary in Ω ($k \in \mathbb{N} \cup \{\infty\}$) if for every $x \in \Omega \cap \partial E$ the set $\Omega \cap \partial E$ can be written, locally around x, as the graph of a C^k function, and $\Omega \cap E$ is locally the subgraph of the same function.

By assumption (1.4) it follows that $\frac{\varkappa_b - \varkappa_G}{\varkappa_b + \varkappa_G}$ is a positive real number. We set

$$l^2 := (H_0)^2 \varkappa_b \, \frac{\varkappa_b - \varkappa_G}{2(\varkappa_b + \varkappa_G)}. \tag{2.10}$$

We claim that, whenever E is bounded with smooth boundary in Ω, then

$$\mathscr{W}_{\mathrm{Hel}}(E) \ge -l^2 \mathcal{H}^2(\Omega \cap \partial E).$$

To prove the claim, write

$$\begin{aligned}&\mathscr{W}_{\mathrm{Hel}}(E) \\ &= \int_{\Omega\cap\partial E} \left[-\frac{\varkappa_G}{2}|\mathbf{B}_{\partial E}|^2 + \left(\frac{\varkappa_b + \varkappa_G}{2}\right)(H_{\partial E})^2 + \varkappa_b H_0 H_{\partial E} + \frac{\varkappa_b}{2}(H_0)^2\right] d\mathcal{H}^2.\end{aligned}$$

If $\alpha := \frac{\varkappa_b + \varkappa_G}{2} > 0$, $\beta := \varkappa_b H_0$, and $\gamma := \varkappa_b \frac{(H_0)^2}{2}$, since $\alpha t^2 + \beta t + \gamma \ge \frac{\alpha}{2} t^2 - l^2$ for any $t \in \mathbb{R}$ and $l^2 = \frac{\beta^2}{2\alpha} - \gamma$, we have the inequality

$$\mathscr{W}_{\mathrm{Hel}}(E) \ge \int_{\Omega\cap\partial E} \left[-\frac{\varkappa_G}{2}|\mathbf{B}_{\partial E}|^2 + \frac{(\varkappa_b + \varkappa_G)}{4}(H_{\partial E})^2 - l^2\right] d\mathcal{H}^2. \tag{2.11}$$

Thanks to (1.4), the first two addenda inside the integral on the right hand side of (2.11) are nonnegative, hence the claim follows.

2.4. **Definitions of μ_u^ε, $\widetilde{\mu}_u^\varepsilon$, ξ_u^ε, R_u^ε, $\mathbf{B}_u$, A^u, V_u^ε, $V_u^{0,\varepsilon}$, f_u^ε, B_u^ε and H_u^ε.** We set

$$W(r) := \frac{1}{4}(1-r^2)^2, \qquad r \in \mathbb{R},$$

and

$$c_0 := \int_{-1}^{1} \sqrt{2W(s)}\, ds. \tag{2.12}$$

If $\gamma(s) := \tanh(s)$ we have $\ddot{\gamma} = \frac{d}{ds}(W(\gamma))$,

$$\int_{\mathbb{R}} |\dot{\gamma}|^2\, ds = \int_{\mathbb{R}} 2W(\gamma)\, ds = c_0,$$

and

$$c_0 = \min \left\{ \int_{\mathbb{R}} \left(\frac{|\dot{v}|^2}{2} + W(v) \right) ds : \, v \in H^1_{\rm loc}(\mathbb{R}), \, \lim_{s \to \pm\infty} v(s) = \pm 1 \right\}. \tag{2.13}$$

For $u \in \mathcal{C}^2(\Omega)$ and $\mathcal{L}^3$ the Lebesgue measure in $\mathbb{R}^3$, we define the following Radon measures:

$$\mu_u^\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{W(u)}{\varepsilon} \right) \mathcal{L}^3 \llcorner \Omega, \tag{2.14}$$

$$\widetilde{\mu}_u^\varepsilon := \varepsilon |\nabla u|^2 \mathcal{L}^3 \llcorner \Omega, \tag{2.15}$$

$$\xi_u^\varepsilon := \left(\frac{\varepsilon}{2} |\nabla u|^2 - \frac{W(u)}{\varepsilon} \right) \mathcal{L}^3 \llcorner \Omega, \tag{2.16}$$

where $\llcorner$ is the restriction. ξ_u^ε is usually called discrepancy measure, while μ_u^ε is the density of the Allen-Cahn functional $\mathcal{P}_\varepsilon$. With a small abuse of notation, when necessary we still denote by ξ_u^ε the density of the discrepancy measure, i.e., $\xi_u^\varepsilon = \frac{\varepsilon}{2}|\nabla u|^2 - \frac{W(u)}{\varepsilon}$. Note that

$$\nabla \xi_u^\varepsilon = \varepsilon \nabla^2 u \nabla u - \frac{W'(u)}{\varepsilon} \nabla u. \tag{2.17}$$

For $u \in C^2(\Omega)$ define $R_u^\varepsilon : G_2(\Omega) \to \mathbb{R}^3$ as

$$R_u^\varepsilon(x, S) = R_u^\varepsilon(x) := \frac{1}{\varepsilon |\nabla u(x)|^2} \nabla \xi_u^\varepsilon(x), \tag{2.18}$$

with the convention that $R_u^\varepsilon := 0$ on the set $\{\nabla u = 0\}$.

Let $u \in C^2(\Omega)$. We will often look at geometric properties of the *ensemble of the level sets* of u. We define

$$\nu_u := \frac{\nabla u}{|\nabla u|}, \qquad P^u := \mathrm{Id} - \nu_u \otimes \nu_u, \qquad P^u_{ij} = \delta_{ij} - (\nu_u)_i (\nu_u)_j, \tag{2.19}$$

on $\{\nabla u \neq 0\}$ and $\nu_u := \mathbf{e}_3$, $P^u := \mathrm{Id} - \mathbf{e}_3 \otimes \mathbf{e}_3$ on $\{\nabla u = 0\}$. Moreover we define the second fundamental form of the ensemble of the level sets of u by

$$\mathbf{B}_u = \frac{(P^u)^T \nabla^2 u P^u}{|\nabla u|} \otimes \nu_u, \tag{2.20}$$

on $\{\nabla u \neq 0\}$ and $\mathbf{B}_u := \otimes^3 \mathbf{e}_3$ on $\{\nabla u = 0\}$. Similarly we define

$$A^u_{ijk} := -P^u_{il} \big[\partial_l ((\nu_u)_j (\nu_u)_k) \big], \tag{2.21}$$

on $\{\nabla u \neq 0\}$ and $A^u := \otimes^3 \mathbf{e}_3$ on $\{\nabla u = 0\}$.

It will be convenient to consider $\mathbf{B}_u$ and A^u as defined on $G_2(\Omega)$ (resp. on $G_2^0(\Omega)$) by $\mathbf{B}_u(x,S) := \mathbf{B}_u(x)$, $A^u(x,S) := A^u(x)$ (resp. $\mathbf{B}_u(x,\tau) := \mathbf{B}_u(x)$, $A^u(x,\tau) := A^u(x)$).

By V_u (resp. V_u^0) we denote the varifold (resp. oriented varifold)

$$V_u^\varepsilon(\phi) = c_0^{-1} \int \phi(x, P^u)\, d\widetilde{\mu}_u^\varepsilon \qquad \forall \phi \in C_c^0(G_2(\Omega)), \tag{2.22}$$

$$V_u^{0,\varepsilon}(\phi) = c_0^{-1} \int \phi(x, \star\nu_u)\, d\widetilde{\mu}_u^\varepsilon \qquad \forall \phi \in C_c^0(G_2^0(\Omega)), \tag{2.23}$$

see Appendix B.

We also set

$$f_u^\varepsilon := \varepsilon \Delta u - \frac{W'(u)}{\varepsilon}. \tag{2.24}$$

Definition 2.4. *Let $u \in \mathcal{C}^2(\Omega)$ and $x \in \Omega$. We define*

$$B_u^\varepsilon(x) := \begin{cases} \dfrac{1}{\varepsilon|\nabla u(x)|}\left(\varepsilon\nabla^2 u(x) - \dfrac{W'(u(x))}{\varepsilon}\nu_u(x)\otimes\nu_u(x)\right) & \text{if } \nabla u(x) \neq 0,\\ 0 & \text{otherwise,}\end{cases} \tag{2.25}$$

$$H_u^\varepsilon(x) := \operatorname{tr}\Big(B_u^\varepsilon(x)\Big) = \begin{cases} \dfrac{f_u^\varepsilon(x)}{\varepsilon|\nabla u(x)|} & \text{if } \nabla u(x) \neq 0,\\ 0 & \text{otherwise.}\end{cases} \tag{2.26}$$

We can informally think of $B_u^\varepsilon \otimes \nu_u$ and $H_u^\varepsilon \nu_u$ as the *approximate* second fundamental form and the approximate mean curvature vector of the level sets of u, respectively.

Note that

$$R_u^\varepsilon = B_u^\varepsilon \frac{\nabla u}{|\nabla u|} \qquad \text{on } \{\nabla u \neq 0\}.$$

2.5. **The functionals $\mathscr{W}_\varepsilon$.** We recall that our approximating sequences of functionals is defined in (1.7), where $\mathscr{H}_\varepsilon, \mathscr{K}_\varepsilon$ are as in (1.5), (1.6).

Observe that

$$\int (H_u^\varepsilon)^2\, d\widetilde{\mu}_u^\varepsilon \leq \mathscr{H}_\varepsilon(u),$$

with equality if $\mathcal{L}^3(\{f_u^\varepsilon \neq 0\} \cap \{\nabla u = 0\}) = 0$, and

$$\int_\Omega |B_u^\varepsilon|^2\, d\widetilde{\mu}_u^\varepsilon \leq \frac{1}{\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u\right|^2 dx,$$

with equality if

$$\mathcal{L}^3\left(\left\{\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u\otimes\nu_u \neq 0\right\} \cap \{\nabla u = 0\}\right) = 0.$$

Moreover

$$\int_\Omega \frac{(H_u^\varepsilon)^2 - |B_u^\varepsilon|^2}{2}\, d\widetilde{\mu}_u^\varepsilon = \int_{\{\nabla u \neq 0\}} \sum_{1\leq i<j\leq 3}^{3} \det\Big([B_u^\varepsilon]_{ij}\Big)\, d\widetilde{\mu}_u^\varepsilon,$$

where $[B_u^\varepsilon]_{ij}$ is the ij-th principal minor of B_u^ε, and

$$\det\Big([B_u^\varepsilon]_{ij}\Big) = \frac{1}{\varepsilon^2|\nabla u|^2}\Bigg[\Big(\varepsilon\partial^2_{ii}u - \frac{W'(u)}{\varepsilon}\frac{(\partial_i u)^2}{|\nabla u|^2}\Big)\Big(\varepsilon\partial^2_{jj}u - \frac{W'(u)}{\varepsilon}\frac{(\partial_j u)^2}{|\nabla u|^2}\Big) - \Big(\varepsilon\partial^2_{ij}u - \frac{W'(u)}{\varepsilon}\frac{\partial_i u\partial_j u}{|\nabla u|^2}\Big)^2\Bigg]. \tag{2.27}$$

Remark 2.5. Let us notice that

$$\begin{aligned}
\frac{(H_u^\varepsilon)^2 - |B_u^\varepsilon|^2}{2} &= \frac{(f_u^\varepsilon)^2 - \mathrm{tr}[(\varepsilon\nabla^2 u - \frac{1}{\varepsilon}W'(u)\nu_u\otimes\nu_u)^2]}{2\varepsilon^2|\nabla u|^2}\\
&= \frac{1}{2\varepsilon^2|\nabla u|^2}\Big((f_u^\varepsilon)^2 - \mathrm{tr}\Big[\varepsilon^2(\nabla^2 u)^2 - 2W'(u)\nabla^2 u\ \nu_u\otimes\nu_u + \frac{(W'(u))^2}{\varepsilon^2}\nu_u\otimes\nu_u\Big]\Big)\\
&= \frac{\varepsilon^2\left\{(\Delta u)^2 - \mathrm{tr}[(\nabla^2 u)^2]\right\} - 2W'(u)(\Delta u - \partial^2_{\nu_u\nu_u}u)}{2\varepsilon^2|\nabla u|^2}\\
&= \frac{1}{2\varepsilon^2|\nabla u|^2}\Big\{\varepsilon^2\mathrm{div}(\Delta u\nabla u - \nabla^2 u\nabla u) - 2W'(u)\mathrm{tr}\left[(\mathrm{Id} - \nu_u\otimes\nu_u)\nabla^2 u\right]\Big\},
\end{aligned}$$

where we used

$$\mathrm{div}(\nabla^2 u\nabla u) = \mathrm{tr}[(\nabla^2 u)^2] + \nabla u\cdot\nabla(\Delta u).$$

Suppose that $\Omega \subset\subset \mathbb{R}^3$ is open, and $u \in C^2(\Omega)$ verifies $\nabla u \equiv 0$ on $\Omega\setminus\Omega'$, for some $\Omega' \subset\subset \Omega$. By Sard's Lemma we can find a sequence of $t_k \in \mathbb{R}^+$ such that $t_k \to 0$ as $k\to\infty$, and, setting $N_k := \{|\nabla u| > t_k\}$ we have

$$\partial N_k \subseteq \{|\nabla u| = t_k\} \text{ is a smooth, embedded surface,}$$

$$\lim_{k\to\infty}\mathcal{L}^3\Big(\{\nabla u\neq 0\}\setminus N_k\Big) = 0.$$

Thus we have

$$\begin{aligned}
\left|\int_{\{\nabla u\neq 0\}}\mathrm{div}(\Delta u\nabla u - \nabla^2 u\nabla u)\,dx\right| &= \lim_{k\to\infty}\left|\int_{N_k^\varepsilon}\mathrm{div}(\Delta u\nabla u - \nabla^2 u\nabla u)\,dx\right|\\
&= \lim_{k\to\infty}\left|\int_{\partial N_k}(\Delta u\nabla u - \nabla^2 u\nabla u)\cdot\nu_{\partial N_k^\varepsilon}\,d\mathcal{H}^2\right|\\
&\le \lim_{k\to\infty}\|u\|_{C^2}\mathcal{H}^2(\partial N_k)t_k = 0.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int\frac{(H_u^\varepsilon)^2 - |B_u^\varepsilon|^2}{2}\,d\widetilde{\mu}_u^\varepsilon\\
&= \frac{1}{2\varepsilon}\int_{\{\nabla u\neq 0\}}\Big(\varepsilon^2\mathrm{div}(\Delta u\nabla u - \nabla^2 u\nabla u) - 2W'(u)\ \mathrm{tr}\left[(\mathrm{Id} - \nu_u\otimes\nu_u)\nabla^2 u\right]\Big)\,dx\\
&= -\int_{\{\nabla u\neq 0\}}\frac{W'(u)}{\varepsilon}\mathrm{tr}\left[P^u\nabla^2 u\right]\,dx.
\end{aligned}$$

When $u_\varepsilon(x) = \gamma_\varepsilon(d(x)) + g_\varepsilon(x)$, where γ_ε is as in Section 6 and $g_\varepsilon \in C^2(\Omega)$ is such that $\|g_\varepsilon\|_{C^2(\Omega)} = O(\varepsilon)$, this formula coincides (up to an error of order $O(\varepsilon)$) with the one proposed in [20] in order to approximate $\mathcal{K}$.

3. Preliminary known results

In this section we recall some recent results about a modified conjecture of De Giorgi concerning the variational approximation of the Willmore functional (see [14]). More precisely, the so-called $\Gamma - \limsup$ inequality has been proved in [5] in any dimension on smooth boundaries; in [4] the $\Gamma - \liminf$ inequality has been proved in any dimension, under a rather strong ansatz on the u_ε (namely $u_\varepsilon = v_\varepsilon(d)$, where d is the signed distance from the boundary of the limit set). An ansatz-free proof of the $\Gamma - \liminf$ inequality has been given in dimension 2 and 3 in [38], and independently, but only in two-dimensions, in [45] (by means of a different proof which makes use of generalized varifolds introduced in [34]).

The following theorem has been proved in [38] and is one of the key ingredients in the proofs of our results.

Theorem 3.1. *Let $\{u_\varepsilon\} \subset C^2(\Omega)$ be a sequence such that*

$$\sup_{0<\varepsilon<1} \left\{ \mu^\varepsilon_{u_\varepsilon}(\Omega) + \frac{1}{\varepsilon} \int_\Omega \left(\varepsilon \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \right)^2 dx \right\} < +\infty.$$

Then there exists a subsequence (still denoted by $\{u_\varepsilon\}$) converging to $u = 2\chi_E - 1$ in $L^1(\Omega)$, where E is a finite perimeter set. Moreover

(A) *$\mu^\varepsilon_{u_\varepsilon} \rightharpoonup \mu$ as $\varepsilon \to 0^+$ weakly* in Ω as Radon measures and μ verifies*

$$\mu \geq c_0 \mathcal{H}^2 \llcorner \partial E.$$

In addition

$$\lim_{\varepsilon \to 0^+} |\xi^\varepsilon_{u_\varepsilon}| = 0 \qquad \text{as Radon measures,} \tag{3.1}$$

where $|\xi^\varepsilon_{u_\varepsilon}|$ denotes the total variation of the measure $\xi^\varepsilon_{u_\varepsilon}$, and hence

$$\mu = \lim_{\varepsilon \to 0^+} \mu^\varepsilon_{u_\varepsilon} = \lim_{\varepsilon \to 0^+} \widetilde{\mu}^\varepsilon_{u_\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{2W(u_\varepsilon)}{\varepsilon} \mathcal{L}^3 \llcorner \Omega \qquad \text{as Radon measures.} \tag{3.2}$$

(B) *The sequence $\{V^\varepsilon_{u_\varepsilon}\}$ converges in the varifolds sense to an integral-rectifiable varifold $V \in \mathbf{IV}_2(\Omega)$ with generalized mean curvature $\mathbf{H}_V \in L^2(\mu)$ and such that $\mu_V = c_0^{-1}\mu$.*

(C) *For any $Y \in C^1_c(\Omega; \mathbb{R}^n)$ we have*

$$c_0 \lim_{\varepsilon \to 0^+} \delta V^\varepsilon_{u_\varepsilon}(Y) = \lim_{\varepsilon \to 0^+} -\int_\Omega f^\varepsilon_{u_\varepsilon} \nabla u_\varepsilon \cdot Y \, dx = -\int_\Omega \mathbf{H}_V \cdot Y \, d\mu, \tag{3.3}$$

and

$$c_0 \int_\Omega |\mathbf{H}_V|^2 \, d\mu_V \leq \liminf_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega \left(\varepsilon \Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \right)^2 dx. \tag{3.4}$$

An important point in order to establish the $\Gamma(L^1(\Omega))$-convergence of $\mathcal{W}_\varepsilon$ to $\mathcal{W}_{\mathrm{Hel}}$ is the lower-semicontinuity of $\mathcal{W}_{\mathrm{Hel}}$ on smooth sets. This is the aim of the following theorem, which is a consequence of [16, Theorem 5.1].

Theorem 3.2. *Let $H_0 \in \mathbb{R}$ and suppose that* (1.4) *holds. Let $E \subset \Omega$ be a bounded open set with smooth boundary in Ω. Let $\{E_h\}$ be a sequence of bounded open* *subsets of* Ω *with smooth boundary in* Ω*, such that*

$$\sup_{h\in\mathbb{N}} \mathcal{H}^2(\Omega \cap \partial E_h) < +\infty, \tag{3.5}$$

$$\lim_{h\to\infty} \mathcal{L}^3(\Omega \cap (E_h \triangle E)) = 0. \tag{3.6}$$

Then

$$\mathcal{W}_{\rm Hel}(E) \le \liminf_{h\to\infty} \mathcal{W}_{\rm Hel}(E_h). \tag{3.7}$$

Remark 3.3. Theorem 3.2 holds under the weaker assumption $-2 < \varkappa_G/\varkappa_b < 0$.

Remark 3.4. The bound (3.5) is necessary in order to gain sufficient compactness on the sequence $\{\partial E_h\}$, since the bound $\sup_h \mathcal{W}_{\rm Hel}(E_h) < +\infty$ alone does not imply any uniform control on the area of ∂E_h. This is seen with the following example: $\Omega = \mathbb{R}^3$, $H_0 = 2$, E_h the union, over $n \in \{1, \dots, h\}$, of the balls of radius 1 and centered at $(2n, 0, 0)$, so that $\mathcal{W}_{\rm Hel}(E_h) = 4\pi^2 \varkappa_G h < 0$.

4. Statements of the main results

We can now state our Γ-convergence results.

Theorem 4.1 (Equicoercivity and Γ-liminf inequality)**.** *Let* $H_0 = 0$ *and suppose that* (1.4) *holds. Let* $\{u_\varepsilon\} \subset C^2(\Omega)$ *be a sequence satisfying* (1.8)*. Then there exists a (not relabelled) subsequence satisfying the theses of Theorem 3.1. Moreover, the varifold* V *in Theorem 3.1 is a curvature varifold with generalized second fundamental form* $\mathbf{B}_V$ *in* L^2 *(see Definition B.3), and*

$$\lim_{\varepsilon\to0^+} (V^\varepsilon_{u_\varepsilon}, A^{u_\varepsilon}) = (V, A_V) \tag{4.1}$$

as measure-function pairs on $G_2(\Omega)$ *with values in* $\mathbb{R}^{3^3}$*. Eventually*

$$\liminf_{\varepsilon\to0^+} \mathcal{W}_\varepsilon(u_\varepsilon) \ge c_0 \int \left[\frac{\varkappa_b}{2}|\mathbf{H}_V|^2 + \frac{\varkappa_G}{2}(|\mathbf{H}_V|^2 - |\mathbf{B}_V|^2)\right] dV. \tag{4.2}$$

Theorem 4.2 (Γ-limsup inequality)**.** *Let* $H_0 = 0$ *and* $E \subset \Omega$ *be a bounded open set with boundary of class* C^2*. Then there exists a sequence* $\{u_\varepsilon\} \subset C^2(\Omega)$ *such that*

$$\lim_{\varepsilon\to0^+} u_\varepsilon = 2\chi_E - 1 \text{ in } L^1(\Omega), \tag{4.3}$$

$$\lim_{\varepsilon\to0^+} \mu^\varepsilon_{u_\varepsilon} = c_0 \mathcal{H}^2 \llcorner \partial E \text{ as Radon measures}, \tag{4.4}$$

$$\lim_{\varepsilon\to0^+} \mathcal{W}_\varepsilon(u_\varepsilon) = c_0 \mathcal{W}_{\rm Hel}(E). \tag{4.5}$$

As a consequence of Theorems 4.2, 4.1 and 3.2 we obtain the following

Corollary 4.3 (Γ-limit on smooth sets)**.** *Let* $H_0 = 0$ *and suppose that* (1.4) *holds. Let* $E \subset \Omega$ *be a bounded open set with boundary of class* C^2*. Then*

$$\left[\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to0^+} \mathcal{W}_\varepsilon\right](2\chi_E - 1) = c_0 \mathcal{W}_{\rm Hel}(E). \tag{4.6}$$

Next theorem shows that actually from the hypotheses of Theorem 4.1 we can prove a stronger compactness result, since the oriented varifold (see Appendix B) associated with almost every level line converge to the same limit.

Theorem 4.4 (Enhanced compactness)**.** *Let* $H_0 = 0$ *and suppose that* (1.4) *holds. Let* $\{u_\varepsilon\} \subset C^2(\Omega)$ *be a sequence satisfying* (1.8)*. Then there exists a (not relabelled) subsequence such that*

(A) *the sequence* $\{V^{0,\varepsilon}_{u_\varepsilon}\}$ *converges in the sense of oriented varifolds to an oriented varifold* $V^0 \in \mathbf{IV}^0_2(\Omega)$ *such that* $\mathbf{q}_\sharp V^0 = V$, *where* $V \in \mathbf{IV}_2(\Omega)$ *is as in Theorem 4.1.*

(B) *For every* $\psi \in C^1_c(\Omega \times \mathbb{S}^2)$ *the sequence* $\{g^\psi_\varepsilon\} \subset W^{1,1}((-1,1))$, *defined by*

$$g^\psi_\varepsilon(s) := \int_{\{u_\varepsilon = s\}} \psi(y, \nu_{u_\varepsilon}(y))\; \varepsilon|\nabla u_\varepsilon(y)|\, d\mathcal{H}^2(y),$$

converges strongly in $W^{1,1}((-1,1))$ *to the function* $g^\psi(s) := \sqrt{2W(s)}V^0(\psi)$. *Moreover, for* $\mathcal{L}^1$*-almost every* $s \in [-1,1]$ *we have*

$$\begin{aligned}\lim_{\varepsilon\to 0^+} \mathbf{v}\left(\{u_\varepsilon = s\}, \star\nu_{u_\varepsilon}, \varepsilon|\nabla u_\varepsilon|\right) &= \lim_{\varepsilon\to 0^+} \mathbf{v}\left(\{u_\varepsilon = s\}, \star\nu_{u_\varepsilon}, \sqrt{2W(s)}\right)\\ &= \sqrt{2W(s)}V^0\end{aligned} \tag{4.7}$$

as oriented varifolds in Ω.

Remark 4.5. We can adapt the proof of Theorem 4.4 to show that, under the weaker assumption that the hypothesis of Theorem 3.1 hold, the sequence g^ψ_ε converges strongly to g^ψ in $W^{1,1}_{\mathrm{loc}}((-1,1))$ as $\varepsilon \to 0^+$ for every $\psi \in C^1_c(\Omega)$.

The next proposition shows that a stronger convergence to zero of the discrepancies $\xi^\varepsilon_{u_\varepsilon}$ defined in (2.16) holds, assuming the bounds in (1.8). Similar estimates have been obtained in [35], when u_ε is a local minimizer for $\mathcal{P}_\varepsilon$.

Proposition 4.6 (Improved convergence of the discrepancies)**.** *Suppose that* $\{u_\varepsilon\} \subset \mathcal{C}^2(\Omega)$ *is such that* (1.8) *holds. Then there exists a (not relabelled) subsequence such that*

$$\nabla \xi^\varepsilon_{u_\varepsilon}\, \mathcal{L}^3 \rightharpoonup 0 \qquad \text{as Radon measures on } \Omega, \tag{4.8}$$

$$\lim_{\varepsilon\to 0^+} \|\xi^\varepsilon_{u_\varepsilon}\|_{L^p(\Omega)} = 0 \qquad \text{for every } p \in [1, 3/2). \tag{4.9}$$

5. Proof of Theorem 4.1

The present section is organized as follows. We start by proving two technical lemmata, namely Lemma 5.1 and Lemma 5.3. Then in Section 5.1 we prove that $V := \lim_{\varepsilon\to 0} V^\varepsilon_{u_\varepsilon}$ is a curvature varifold with generalized second fundamental form in L^2, we show (4.1) and inequality (4.2).

Lemma 5.1. *Suppose that* $\{u_\varepsilon\} \subset \mathcal{C}^2(\Omega)$ *is such that*

$$\sup_{0<\varepsilon<1} \left\{ \mu^\varepsilon_{u_\varepsilon}(\Omega) + \frac{1}{\varepsilon}\int_\Omega \left| \varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\right|^2 dx \right\} < +\infty. \tag{5.1}$$

Then there exists a (not relabelled) subsequence such that

$$\lim_{\varepsilon\to 0^+} (V^\varepsilon_{u_\varepsilon}, R^\varepsilon_{u_\varepsilon}) = (V, 0) \tag{5.2}$$

as measures function pairs on $G_2(\Omega)$ *with values in* $\mathbb{R}^3$, *where the varifold* V *is defined in Theorem 3.1 (B).*

Proof. Since $f^\varepsilon_{u_\varepsilon} = \mathrm{tr}\big(\varepsilon\nabla^2 u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\big)$, we have

$$\frac{1}{\varepsilon}\int_\Omega (f_{u_\varepsilon})^2\, dx \le \frac{3}{\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\right|^2 dx.$$

Hence, by (5.1), we can apply Theorem 3.1, and select a (not relabelled) subsequence such that $V^\varepsilon_{u_\varepsilon} \to V$ as $\varepsilon \to 0^+$ in the sense of varifolds, with $V = \mathbf{v}(\mathcal{M}, \theta) \in \mathbf{IV}_2(\Omega)$. Since on $\{\nabla u_\varepsilon \neq 0\}$ we have

$$R^\varepsilon_{u_\varepsilon} = \frac{\nabla \xi^\varepsilon_{u_\varepsilon}}{\varepsilon |\nabla u_\varepsilon|^2} = B^\varepsilon_{u_\varepsilon} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}, \tag{5.3}$$

we conclude that

$$c_0 \int |R^\varepsilon_{u_\varepsilon}|^2 \, dV^\varepsilon_{u_\varepsilon} = \int \left| \frac{\nabla \xi^\varepsilon_{u_\varepsilon}}{\varepsilon |\nabla u_\varepsilon|^2} \right|^2 d\widetilde{\mu}^\varepsilon_{u_\varepsilon} \leq 3 \int |B^\varepsilon_{u_\varepsilon}|^2 \, d\widetilde{\mu}^\varepsilon_{u_\varepsilon},$$

which is uniformly bounded with respect to ε in view of (5.1). By Theorem A.4 (i), we can select a further (not relabelled) subsequence such that $(V^\varepsilon_{u_\varepsilon}, R^\varepsilon_{u_\varepsilon})$ converge weakly as measure-function pairs on $G_2(\Omega)$ with values in $\mathbb{R}^3$ to (V, R), for a certain $R \in L^2(V, \mathbb{R}^3)$. In order to prove (5.2) we closely follow [43, page 10]. Let $\phi \in C^1_c(\Omega)$ and R_i (resp. $R^\varepsilon_{u_\varepsilon, i}$) be the i-th component of R (resp. of $R^\varepsilon_{u_\varepsilon}$). By (3.1) we have

$$c_0 \int R_i(x,S) \phi(x) \, dV(x,S) = \lim_{\varepsilon \to 0^+} \int R^\varepsilon_{u_\varepsilon, i} \phi \, d\widetilde{\mu}^\varepsilon_{u_\varepsilon} = - \lim_{\varepsilon \to 0^+} \int \partial_i \phi \, d\xi^\varepsilon_{u_\varepsilon} = 0, \tag{5.4}$$

where in the two last equalities we used (5.3), (2.17) and (3.1) respectively.

From (5.4), using that $V^\varepsilon_{u_\varepsilon} \to V = \mathbf{v}(\mathcal{M}, \theta) \in \mathbf{IV}_2(\Omega)$ as varifolds, it follows

$$\int R_i(x,S)\phi(x) \, dV(x,S) = 0 = \int_M R_i(x, T_x M) \phi(x) \, \theta(x) d\mathcal{H}^2(x).$$

This implies that $R(x, T_x M) = 0$ for $\mu_V = \theta \mathcal{H}^2 \llcorner M$-a.e. x, and (5.2) follows. □

Remark 5.2. We need to consider R^ε_u as a function on $G_2(\Omega)$ and not just on Ω because R^ε_u appears in the "ε-formulation" of (B.1) (see (5.12)), which characterizes Hutchinson's curvature varifolds via an "integration by parts" formula involving test functions in $C^1_c(G_2(\Omega))$.

The following lemma shows that if (5.1) holds then the varifold V limit of the $V^\varepsilon_{u_\varepsilon}$ is a curvature varifold with generalized second fundamental form in L^2.

Lemma 5.3. *Suppose that* (5.1) *holds. Then*

$$\sup_{0<\varepsilon<1} \int |\mathbf{B}_{u_\varepsilon}|^2 \, dV_{u_\varepsilon} < +\infty. \tag{5.5}$$

Moreover the varifold V in Lemma 5.1 is a curvature varifold with generalized second fundamental form $\mathbf{B}_V$ in L^2 and, up to a subsequence,

$$\lim_{\varepsilon \to 0^+} (V_{u_\varepsilon}, A^{u_\varepsilon}) = (V, A_V), \tag{5.6}$$

$$\lim_{\varepsilon \to 0^+} (V_{u_\varepsilon}, \mathbf{B}_{u_\varepsilon}) = (V, \mathbf{B}_V), \tag{5.7}$$

as measure-function pairs on $G_2(\Omega)$ with values in $\mathbb{R}^{3^3}$.

Proof. From the definitions of $\mathbf{B}_{u_\varepsilon}$ and $B^\varepsilon_{u_\varepsilon}$ given in (2.20) and (2.25) respectively, we have

$$\begin{aligned}|\mathbf{B}_{u_\varepsilon}|^2 &= \sum_{i,j,k=1}^{3}\left[\left(\frac{(P^{u_\varepsilon})^T\nabla^2 u_\varepsilon P^{u_\varepsilon}}{|\nabla u_\varepsilon|}\right)_{ij}\right]^2\left(\frac{\partial_k u_\varepsilon}{|\nabla u_\varepsilon|}\right)^2\\ &= \sum_{i,j=1}^{3}\left[\left(\frac{(P^{u_\varepsilon})^T\nabla^2 u_\varepsilon P^{u_\varepsilon}}{|\nabla u_\varepsilon|}\right)_{ij}\right]^2 = \left|\frac{(P^{u_\varepsilon})^T\nabla^2 u_\varepsilon P^{u_\varepsilon}}{|\nabla u_\varepsilon|}\right|^2\\ &= \left|\frac{(P^{u_\varepsilon})^T\left[\varepsilon\nabla^2 u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\nabla u_\varepsilon\otimes\nabla u_\varepsilon/|\nabla u_\varepsilon|^2\right]P^{u_\varepsilon}}{\varepsilon|\nabla u_\varepsilon|}\right|^2 \leq |B^\varepsilon_{u_\varepsilon}|^2,\end{aligned} \tag{5.8}$$

where in the last inequality we use (2.2). Integrating (5.8) with respect to dV_{u_ε} (see (2.22) and (2.16)) and using (5.1), we conclude that (5.5) holds. Notice that by (5.1) the conclusions of Theorem 3.1 hold.

By (2.21) and (5.1) we obtain also

$$\sup_{0<\varepsilon<1}\int |A^{u_\varepsilon}|^2\, dV_{u_\varepsilon} < +\infty.$$

This latter estimate together with $\sup_{0<\varepsilon<1}\mu^\varepsilon_{u_\varepsilon}(\Omega) < +\infty$, enables us to apply Theorem A.4 and conclude that, passing to a subsequence, there is $\widehat{A}\in L^2(V, R^{3^3})$ such that

$$\lim_{\varepsilon\to 0^+}(V_{u_\varepsilon}, A^{u_\varepsilon}) = (V, \widehat{A}) \tag{5.9}$$

as measure-function pairs on $G_2(\Omega)$ with values on R^{3^3}.

Now we want to prove that actually $\widehat{A}(x,S)$ verifies equation (B.1) and hence that V is a curvature varifold with generalized second fundamental form in L^2, and $\widehat{A} = A^V$. In doing this we closely follow [43, Proposition 2].

Fix $1\leq i\leq 3$ and $\phi\in C^1_c(\Omega)$. Multiply equation (2.24) by $\phi\partial_i u_\varepsilon$. Integrating by parts we firstly obtain

$$\int_\Omega\left[\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2\partial_i\phi - \varepsilon\partial_i u_\varepsilon\partial_j u_\varepsilon\partial_j\phi + \frac{W(u_\varepsilon)}{\varepsilon}\partial_i\phi\right]dx = \int_\Omega f^\varepsilon_{u_\varepsilon}\;\phi\;\partial_i u_\varepsilon\, dx. \tag{5.10}$$

Hence

$$\int_\Omega\left[(\partial_i\phi - (\nu_{u_\varepsilon})_i(\nu_{u_\varepsilon})_j\partial_j\phi)\;\varepsilon|\nabla u_\varepsilon|^2 + \phi\partial_i\xi^\varepsilon_{u_\varepsilon}\right]\;dx = \int_\Omega f^\varepsilon_{u_\varepsilon}\;\phi\;\partial_i u_\varepsilon\, dx. \tag{5.11}$$

Let now $\varphi\in C^1_c(\Omega\times\mathbb{R}^{3\times 3})$, $\sigma>0$, and define $\phi^\sigma\in C^1_c(\Omega)$ by

$$\phi^\sigma(x) := \varphi\left(x, \mathrm{Id} - \frac{\nabla u_\varepsilon(x)\otimes\nabla u_\varepsilon(x)}{\sigma^2 + |\nabla u_\varepsilon(x)|^2}\right), \qquad x\in\Omega.$$

Using ϕ^σ in place of ϕ in (5.11) and letting $\sigma\to 0^+$ we obtain

$$\begin{aligned}&\int_\Omega\left[P^{u_\varepsilon}_{ij}\left(\partial_j\varphi - \partial_j\left[(\nu_{u_\varepsilon})_l(\nu_{u_\varepsilon})_k\right]D_{m_{lk}}\varphi\right) - \frac{f^\varepsilon_{u_\varepsilon}}{\varepsilon|\nabla u_\varepsilon|}\frac{\partial_i u_\varepsilon}{|\nabla u_\varepsilon|}\varphi\right]d\widetilde{\mu}_{u_\varepsilon}\\ &= -\int_\Omega\varphi\partial_i\xi^\varepsilon_{u_\varepsilon}\, dx.\end{aligned} \tag{5.12}$$

In (5.12) the integration is only on the subset of Ω where $\nabla u_\varepsilon\neq 0$, the function φ is evaluated at $(x, \mathrm{Id} - \nu_{u_\varepsilon}(x)\otimes\nu_{u_\varepsilon}(x))$, and $D_{m_{lk}}\varphi$ is the derivative of $\varphi(x,\cdot)$ with respect to its lk-entry variable. Next we notice that, by the definition of $f^\varepsilon_{u_\varepsilon}$ and A^{u_ε} in (2.21) we have

$$\begin{aligned}\frac{f^\varepsilon_{u_\varepsilon}}{\varepsilon|\nabla u_\varepsilon|}\frac{\partial_i u_\varepsilon}{|\nabla u_\varepsilon|} =&\mathrm{div}\left(\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\right)\frac{\partial_i u_\varepsilon}{|\nabla u_\varepsilon|}\\ &+\frac{1}{\varepsilon|\nabla u_\varepsilon|^2}\left[\frac{\varepsilon\nabla^2 u_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon-\varepsilon^{-1}W'(u_\varepsilon)|\nabla u_\varepsilon|^2}{|\nabla u_\varepsilon|^2}\right]\partial_i u_\varepsilon \\ =&A^{u_\varepsilon}_{jij}(x,P^{u_\varepsilon})+\frac{1}{\varepsilon|\nabla u_\varepsilon|^2}\left(\nabla\xi^\varepsilon_{u_\varepsilon}\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\right)_i.\end{aligned} \tag{5.13}$$

Inserting (5.13) into (5.12), and recalling the definition of $V^\varepsilon_{u_\varepsilon}$, A^{u_ε} and $R^\varepsilon_{u_\varepsilon}$ given in (2.22), (2.21) and (2.18) respectively, we have that equality (5.12) becomes

$$\begin{aligned}&\int(S_{ij}\partial_j\varphi+A^{u_\varepsilon}_{ijk}D_{m_{jk}}\varphi-A^{u_\varepsilon}_{jij}\varphi)\,dV^\varepsilon_{u_\varepsilon}(x,S)\\ =&-\int(R^\varepsilon_{u_\varepsilon}(x,S)\ S)_i\varphi(x,S)\,dV^\varepsilon_{u_\varepsilon}(x,S),\end{aligned}$$

where φ on the left hand side is evaluated at (x,S). Passing to the limit as $\varepsilon\to 0^+$, by the convergence of $\{V^\varepsilon_{u_\varepsilon}\}$ to V, (5.9) and Lemma 5.1, we get

$$\int\left(S_{ij}\partial_j\varphi+\widehat{A}_{ijk}D_{m_{jk}}\varphi-\widehat{A}_{jij}\varphi\right)dV(x,S)=0,$$

that is V is a curvature varifold with generalized second fundamental form in L^2, and $A^V=\widehat{A}$.

In order to get (5.7) we proceed as follows. Let $V=\mathbf{v}(\mathcal{M},\theta)$. We define

$$\begin{aligned}&\overline{P^{u_\varepsilon}}:G_2(\Omega)\to\mathbb{R}^{3\times 3}, &&(x,S)\to P^{u_\varepsilon}(x),\\ &\overline{P^V}:G_2(\Omega)\to\mathbb{R}^{3\times 3}, &&(x,S)\to P^{\mathcal{M}}(x),\end{aligned}$$

where $P^{\mathcal{M}}(x)$ is the orthogonal projection matrix of $\mathbb{R}^3$ onto the tangent plane $T_x\mathcal{M}\in G_{2,3}$ to $\mathcal{M}$ at x (recall that $T_x\mathcal{M}$ is well defined $\mathcal{H}^2\llcorner\mathcal{M}$-almost everywhere by the 2-rectifiability of $\mathcal{M}$, see [1]). By Remark B.2 we have that the convergence of $V^\varepsilon_{u_\varepsilon}$ to V as varifolds implies that $(V^\varepsilon_{u_\varepsilon},\overline{P^{u_\varepsilon}})\to(V,\overline{P^V})$ as $\varepsilon\to 0^+$ in the L^2-strong convergence as measure-function pairs on $G_2(\Omega)$ with values in $\mathbb{R}^{3\times 3}$. Hence, by (2.5) and Lemma A.6 we obtain (5.7). □

Note that the left hand side of (5.10) can also be written as $\int_\Omega T^{ij}_\varepsilon\partial_j\phi\ dx$, where T^{ij}_ε is the so-called energy-momentum tensor, defined as $T^{ij}_\varepsilon:=\left(\frac{\varepsilon}{2}|\nabla u|^2+\frac{1}{\varepsilon}W(u)\right)\delta_{ij}-\varepsilon\partial_i u\,\partial_j u$.

5.1. **Proof of** (4.2)**.** From the definition of $\mathcal{W}_\varepsilon$ in (1.7) we have

$$\mathcal{W}_\varepsilon(u_\varepsilon)=-\frac{\varkappa_G}{2\varepsilon}\int_\Omega\left|\varepsilon\nabla^2 u_\varepsilon-\frac{W'(u_\varepsilon)}{\varepsilon}\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\right|^2 dx+\frac{\varkappa_b+\varkappa_G}{2\varepsilon}\int_\Omega(f^\varepsilon_{u_\varepsilon})^2\ dx. \tag{5.14}$$

From (1.4), (1.8) and (5.14) it follows that (5.1) holds. Hence by Lemma 5.3 we can conclude that V is a curvature varifold with generalized second fundamental form $\mathbf{B}_V$ in L^2, and $A_V \in L^2(\mu_V)$ and also that (4.1) is verified. In order to prove the $\Gamma - \liminf$ inequality (4.2) we observe that, by (5.8), we have

$$
\begin{aligned}
\mathscr{W}_\varepsilon(u_\varepsilon) \geq & \int \left[\frac{-\varkappa_G}{2}|\mathbf{B}_{u_\varepsilon}|^2 + \frac{\varkappa_b + \varkappa_G}{2}(H^\varepsilon_{u_\varepsilon})^2\right] d\widetilde{\mu}^\varepsilon_{u_\varepsilon} \\
= & c_0 \int \frac{-\varkappa_G}{2}|\mathbf{B}_{u_\varepsilon}(x, P^{u_\varepsilon})|^2 \, dV^\varepsilon_{u_\varepsilon} + \int \frac{\varkappa_b + \varkappa_G}{2}(H^\varepsilon_{u_\varepsilon})^2 \, d\widetilde{\mu}^\varepsilon_{u_\varepsilon}.
\end{aligned}
\tag{5.15}
$$

By (5.15), (5.7), and Theorem A.4, we have

$$
\begin{aligned}
\liminf_{\varepsilon\to 0^+} \mathscr{W}_\varepsilon(u_\varepsilon) \geq & c_0 \liminf_{\varepsilon\to 0^+} \int \frac{-\varkappa_G}{2}|\mathbf{B}_{u_\varepsilon}|^2 \, dV^\varepsilon_{u_\varepsilon} + \liminf_{\varepsilon\to 0^+} \int \frac{\varkappa_b + \varkappa_G}{2}(H^\varepsilon_{u_\varepsilon})^2 \, d\widetilde{\mu}^\varepsilon_{u_\varepsilon} \\
\geq & c_0 \int \left[\frac{\varkappa_b}{2}|\mathbf{H}_V|^2 + \frac{\varkappa_G}{2}\left(|\mathbf{H}_V|^2 - |\mathbf{B}_V|^2\right)\right] dV,
\end{aligned}
$$

which proves (4.2).

6. Proofs of Theorem 4.2 and of Corollary 4.3

We prove Theorem 4.2 in the case $\Omega = \mathbb{R}^3$. The case of a bounded Ω can be proved almost in the same way.

We will construct a sequence $\{u_\varepsilon\} \subset H^2(\mathbb{R}^3)$ satisfying the thesis. To conclude the proof it is enough to mollify each u_ε and use a standard diagonal argument to obtain a new sequence $\{\widehat{u}_\varepsilon\} \subset C^2(\mathbb{R}^3)$ still satisfying (4.3), (4.4), (4.5).

We consider $u_\varepsilon \in H^2(\mathbb{R}^3)$ as in [5]. Let $d(\cdot)$ be the signed distance function from ∂E, as defined in Proposition 2.3, and let $\gamma(s) := \tanh(s)$. For any $0 < \varepsilon < 1$ and $s \in \mathbb{R}$, let $\gamma_\varepsilon(s) := \gamma(s/\varepsilon)$ and $\widetilde{\gamma}_\varepsilon$ be defined as follows: $\widetilde{\gamma}_\varepsilon := \gamma_\varepsilon$ in $(0, \varepsilon|\log\varepsilon|)$, $\widetilde{\gamma}_\varepsilon := p_\varepsilon$ in $(\varepsilon|\log\varepsilon|, s^0_\varepsilon)$, $\widetilde{\gamma}_\varepsilon := +1$ in $(s^0_\varepsilon, +\infty)$, and $\widetilde{\gamma}_\varepsilon(s) := -\widetilde{\gamma}_\varepsilon(-s)$ if $s < 0$. Here, p_ε is an arc of parabola on $(\varepsilon|\log\varepsilon|, s^0_\varepsilon)$ connecting the points $(\varepsilon|\log\varepsilon|, \gamma_\varepsilon(\varepsilon|\log\varepsilon|))$ and $(s^0_\varepsilon, 1)$, that is $p_\varepsilon(s) := -a_\varepsilon(s - s^0_\varepsilon)^2 + 1$, $a_\varepsilon > 0$. To find a_ε and s^0_ε, we impose the condition $\widetilde{\gamma}_\varepsilon \in H^2(\mathbb{R})$, that gives $s^0_\varepsilon = \varepsilon + \varepsilon^3 + \varepsilon|\log\varepsilon|$ and $a_\varepsilon = \frac{2}{(1+\varepsilon^2)^3}$.

We define

$$u_\varepsilon(x) := \widetilde{\gamma}_\varepsilon(d(x)). \tag{6.1}$$

Then (4.3) and (4.4) follow directly from [5], and it remains to prove only (4.5).

To this aim we notice that, since $\nabla^2 u_\varepsilon = \widetilde{\gamma}'_\varepsilon(d)\nabla^2 d + \widetilde{\gamma}''_\varepsilon(d)\nabla d \otimes \nabla d$, we have

- in $U_\varepsilon := \{-\varepsilon|\log\varepsilon| < d(x) < \varepsilon|\log\varepsilon|\}$

$$B^\varepsilon_{u_\varepsilon} = \frac{\gamma'(d/\varepsilon)\nabla^2 d + \varepsilon^{-1}\Big(\gamma''(d/\varepsilon) - W'(\gamma(d/\varepsilon))\Big)\nabla d \otimes \nabla d}{|\gamma'(d/\varepsilon)|} = \nabla^2 d, \tag{6.2}$$

$$H^\varepsilon_{u_\varepsilon} = \Delta d; \tag{6.3}$$

- in $\mathcal{V}_\varepsilon := \{\varepsilon|\log\varepsilon| < |d(x)| < s^0_\varepsilon\}$

$$B^\varepsilon_{u_\varepsilon} = \nabla^2 d + \frac{1}{\varepsilon p'_\varepsilon(d)}\left(\varepsilon p''_\varepsilon(d) - \frac{W'(p_\varepsilon(d))}{\varepsilon}\right)\nabla d \otimes \nabla d, \tag{6.4}$$

$$H^\varepsilon_{u_\varepsilon} = \Delta d + \frac{1}{\varepsilon p'_\varepsilon(d)}\left(\varepsilon p''_\varepsilon(d) - \frac{W'(p_\varepsilon(d))}{\varepsilon}\right). \tag{6.5}$$

Let us now derive some estimates in $\mathcal{V}_\varepsilon$. Let $x \in \mathcal{V}_\varepsilon$; then $1 \geq u_\varepsilon(x) \geq p_\varepsilon(\varepsilon|\log\varepsilon|) = 1 - \frac{2\varepsilon^2}{1+\varepsilon^2}$. Hence $|W'(u_\varepsilon(x))| = |4u_\varepsilon(x)(1-u_\varepsilon(x))(1+u_\varepsilon(x))| \leq \frac{16\varepsilon^2}{1+\varepsilon^2}$, so that $\varepsilon^{-1}W'(u_\varepsilon) = O(\varepsilon)$. Moreover $\varepsilon p''_\varepsilon(d) = O(\varepsilon)$, so that

$$-\varepsilon p''_\varepsilon(d) + \frac{W'(u_\varepsilon)}{\varepsilon} = O(\varepsilon). \tag{6.6}$$

Moreover since $\varepsilon|p'_\varepsilon(s)|^2 = \frac{8\varepsilon(s-\varepsilon-\varepsilon^3-\varepsilon|\log\varepsilon|)^2}{(1+\varepsilon^2)^6}$, making the change of variable $\sigma = s - \varepsilon|\log\varepsilon|$, it follows

$$\int_{\varepsilon|\log\varepsilon|}^{\varepsilon+\varepsilon^3+\varepsilon|\log\varepsilon|} \varepsilon|p'_\varepsilon(s)|^2\, ds = \frac{32\varepsilon}{(1+\varepsilon^2)^6}\int_0^{\varepsilon+\varepsilon^3} (\tau-\varepsilon-\varepsilon^3)^2\, d\tau = O(\varepsilon^4), \tag{6.7}$$

as $\varepsilon \to 0^+$

By [5] it follows that

$$\lim_{\varepsilon\to 0^+} \mathscr{H}_\varepsilon(u_\varepsilon) = c_0 \int_{\partial E} (H_{\partial E})^2\, d\mathcal{H}^2. \tag{6.8}$$

Eventually we have

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \mathscr{K}_\varepsilon(u_\varepsilon) = {} & \lim_{\varepsilon\to 0^+} \left\{ \int_{U_\varepsilon} \sum_{1\leq i<j\leq j} \det([B^\varepsilon_{u_\varepsilon}]_{ij})\,\varepsilon|\nabla u_\varepsilon|^2 dx + \int_{\mathcal{V}_\varepsilon} \frac{(H^\varepsilon_{u_\varepsilon})^2 - |B^\varepsilon_{u_\varepsilon}|^2}{2}\,\varepsilon|\nabla u_\varepsilon|^2 dx \right\} \\
= {} & \lim_{\varepsilon\to 0^+} \left\{ \int_{U_\varepsilon} \sum_{1\leq i<j\leq 3} \det([\nabla^2 d]_{ij}) \frac{1}{\varepsilon}|\gamma'(d/\varepsilon)|^2\, dx \right. \\
& + \frac{1}{2\varepsilon}\int_{\mathcal{V}_\varepsilon} \left[\varepsilon p'_\varepsilon \Delta d + \left(\varepsilon p''_\varepsilon(d) - \frac{W'(p_\varepsilon(d))}{\varepsilon}\right)\right]^2 dx \\
& \left. - \frac{1}{2\varepsilon}\int_{\mathcal{V}_\varepsilon} \left|\varepsilon p'_\varepsilon \nabla^2 d + \left(\varepsilon p''_\varepsilon(d) - \frac{W'(p_\varepsilon(d))}{\varepsilon}\right)\nabla d\otimes\nabla d\right|^2 dx \right\} \\
= {} & \lim_{\varepsilon\to 0^+} \left(\int_{U_\varepsilon} \sum_{1\leq i<j\leq 3} \det([\nabla^2 d]_{ij}) \frac{1}{\varepsilon}|\gamma'(d/\varepsilon)|^2\, dx + O(\varepsilon) \right) \\
= {} & c_0 \int_{\partial E} K_{\partial E}\, d\mathcal{H}^2,
\end{aligned} \tag{6.9}$$

where in the last equality we use Proposition 2.3. Hence, by (6.8) and (6.9) we deduce that (4.5) holds.

6.1. **Proof of Corollary 4.3.** If E has smooth boundary in Ω, as in the proof of Theorem 3.2, we can use the locality of the generalized second fundamental form for Hutchinson's curvature varifolds (see [31]) together with

$$c_0\mathcal{H}^2 \llcorner \partial E \leq \mu = c_0\mu_V \qquad \text{as Radon measures,}$$

to conclude that

$$c_0 \int \left[\frac{\varkappa_b}{2}|\mathbf{H}_V|^2 + \frac{\varkappa_G}{2}\left(|\mathbf{H}_V|^2 - |\mathbf{B}_V|^2\right)\right] dV \geq c_0 \mathscr{W}_{\mathrm{Hel}}(E).$$

The thesis is then a direct consequence of Theorems 4.1, 4.2 and 3.2.

7. Proof of Theorem 4.4

Firstly we notice that we can assume (up to selecting a subsequence) that $V^{\varepsilon}_{u_\varepsilon}$ converge as varifolds to the curvature varifold $V \in \mathbf{IV}_2(\Omega)$ and that (4.1) holds. Moreover, since $V^{0,\varepsilon}_{u_\varepsilon}(G^0_2(\Omega)) = \mu^{\varepsilon}_{u_\varepsilon}(\Omega)$, by (1.8), we can extract a further subsequence such that $V^{0,\varepsilon}_{u_\varepsilon}$ converge as Radon measures to a Radon measure V^0 on $G^0_2(\Omega)$, and also that $\mathbf{q}_\sharp V^0 = V$ (notice that for the moment V^0 is rectifiable but not necessarily integral). Eventually, without loss of generality, we can also assume that

$$\liminf_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0^+} \mathcal{W}_\varepsilon(u_\varepsilon) < +\infty.$$

The present section is organized as follows. We firstly prove Lemma 7.1, from which Theorem 4.4-(B) follows. Then, in Proposition 7.3, we conclude the proof of Theorem 4.4-(A) showing that $V^0 \in \mathbf{IV}^0_2(\Omega)$.

Lemma 7.1. *Let $u_\varepsilon \in C^2(\Omega)$ be such that* (5.1) *holds and* $\liminf_{\varepsilon\to 0^+} \mu^{\varepsilon}_{u_\varepsilon}(\Omega) > 0$. *Suppose V^0 is such that $\lim_{\varepsilon\to 0^+} V^{0,\varepsilon}_{u_\varepsilon} = V^0$ as oriented varifolds. Then there exists a (not relabelled) subsequence of $\{u_\varepsilon\}$ such that for $\mathcal{L}^1$-almost every $s \in [-1,1]$ we have*

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \mathbf{v}\left(\{u_\varepsilon = s\}, \star\nu_{u_\varepsilon}, \varepsilon|\nabla u_\varepsilon|\right) = \lim_{\varepsilon\to 0^+} \mathbf{v}\left(\{u_\varepsilon = s\}, \star\nu_{u_\varepsilon}, \sqrt{2W(s)}\right)\\ &=\sqrt{2W(s)}V^0\end{aligned} \tag{7.1}$$

as oriented varifolds on Ω.

Remark 7.2. When $\liminf_{\varepsilon\to 0^+} \mu^{\varepsilon}_{u_\varepsilon}(\Omega) > 0$, by [13, Lemma 4.4] (see also [38, Proposition 3.4]) we can conclude that, up to a subsequence, $\{u_\varepsilon = s\} \neq \emptyset$ for every $s \in (-1,1)$.

Proof. Let us firstly remark that on one hand for $\psi \in C^0_c(\Omega\times\mathbb{S}^2)$ we can define $\psi^* \in C^0_c(G^0_2(\Omega))$ as $\psi^\star(x,\tau) := \psi(x,\nu^\tau)$. On the other hand for $\phi \in C^0_c(G^0_2(\Omega))$ we can define $\phi_\star \in C^0_c(\Omega\times\mathbb{S}^2)$ as $\phi_\star(x,\xi) := \phi(x,\star\xi)$. This means that the convergence as oriented varifolds of $\mathbf{v}(\{u_\varepsilon = s\}, \star\nu_{u_\varepsilon}, 1)$ is equivalent to the convergence of $\mathcal{H}^2 \llcorner \{u_\varepsilon = s\} \otimes \delta_{\nu_{u_\varepsilon}}$ as measures on $\Omega\times\mathbb{S}^2$. Moreover for a given $\psi \in C^1_c(\Omega\times\mathbb{S}^2)$ we can find $\overline{\psi} \in C^1_c(\Omega\times\mathbb{R}^3)$ such that $\psi(x,\xi) = \overline{\psi}(x,\xi)$ for every $\xi\in\mathbb{S}^2$, and $\|\overline{\psi}\|_{L^\infty(\Omega\times\mathbb{R}^3)} \le \|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}$.

Let $\psi \in C^1_c(\Omega\times\mathbb{S}^2)$ and define $g^\psi_\varepsilon : \mathbb{R} \to [0,+\infty)$ as in the statement, i.e.,

$$g^\psi_\varepsilon(s) := \int_{\{u_\varepsilon = s\}} \psi(y, \nu_{u_\varepsilon}(y))\, \varepsilon|\nabla u_\varepsilon(y)|\, d\mathcal{H}^2(y).$$

We extend ψ to a function of class $C^1_c(\Omega\times B)$, where $B := \{\xi\in\mathbb{R}^3 : \frac{1}{2} < |\xi| < 2\}$, and we still denote by $\psi = \psi(x,\xi)$ such an extension. Fixed $\delta \in (0,1/2]$ we set $I_\delta := [-1+\delta, 1-\delta]$. Let $\eta \in C^\infty_c(I_\delta)$. For fixed $\varepsilon > 0$ and $\sigma \neq 0$, we define $\psi^\sigma \in C^1_c(\Omega\times\mathbb{R}^3)$ as

$$\psi^\sigma(x) := \psi\left(x, \frac{\nabla u_\varepsilon(x)}{\sigma^2 + |\nabla u_\varepsilon(x)|}\right),$$

so that, since $\psi \in C^1_c(\Omega \times B)$, we obtain $\psi^\sigma \equiv 0$ on $\{\nabla u_\varepsilon = 0\}$. We then have, using the coarea formula,

$$\int_{\mathbb{R}} \eta' g_\varepsilon^{\psi^\sigma} \, ds = \int_\Omega \varepsilon \eta'(u_\varepsilon) \psi^\sigma |\nabla u_\varepsilon|^2 \, dx = \int_\Omega \varepsilon \psi^\sigma \nabla(\eta(u_\varepsilon)) \cdot \nabla u_\varepsilon \, dx$$
$$= - \int_\Omega \varepsilon \psi^\sigma \eta(u_\varepsilon) \Delta u_\varepsilon \, dx - \int_\Omega \varepsilon \eta(u_\varepsilon) \nabla \psi^\sigma \cdot \nabla u_\varepsilon \, dx.$$

Letting $\sigma \to 0$ we obtain

$$\begin{aligned} \int_{\mathbb{R}} \eta' g_\varepsilon^{\psi} \, ds = & - \int_{\Omega_\varepsilon} \varepsilon \eta(u_\varepsilon) \ \psi \Delta u_\varepsilon \, dx \\ - \int_{\Omega_\varepsilon} \varepsilon \eta(u_\varepsilon) \nabla \psi \cdot \nabla u_\varepsilon \, dx & - \int_{\Omega_\varepsilon} \varepsilon \eta(u_\varepsilon) D_{\xi_j} \psi(x, \nu_{u_\varepsilon}) \partial_k (\nu^{u_\varepsilon})_j \partial_k u_\varepsilon \, dx, \end{aligned} \tag{7.2}$$

where $\Omega_\varepsilon := \Omega \cap \{\nabla u_\varepsilon \neq 0\}$.

Adding and subtracting the term $\int_{\Omega_\varepsilon} \eta(u_\varepsilon) \psi \frac{W'(u_\varepsilon)}{\varepsilon} \, dx$, observing that the last addendum on the right hand side of (7.2) can be written as

$$- \int_{\Omega_\varepsilon} \eta(u_\varepsilon) D_\xi \psi(x, \nu_{u_\varepsilon}) P^{u_\varepsilon} \varepsilon \nabla^2 u_\varepsilon \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \, dx,$$

and since $P^{u_\varepsilon} \nu_{u_\varepsilon} \otimes \nu_{u_\varepsilon} = 0$, from (7.2) we obtain

$$\begin{aligned} \int_{\mathbb{R}} \eta' g_\varepsilon^{\psi} \, ds = & \int_{\Omega_\varepsilon} \eta(u_\varepsilon) \psi \left(-\varepsilon \Delta u_\varepsilon + \frac{W'(u_\varepsilon)}{\varepsilon} \right) dx - \int_{\Omega_\varepsilon} \varepsilon \eta(u_\varepsilon) \nabla \psi \cdot \nabla u_\varepsilon \, dx \\ - \int_{\Omega_\varepsilon} & \eta(u_\varepsilon) D_\xi \psi(x, \nu_{u_\varepsilon}) \Big(P^{u_\varepsilon} \Big(\varepsilon \nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \nu_{u_\varepsilon} \otimes \nu_{u_\varepsilon} \Big) \Big) \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \, dx \\ - \int_{\Omega_\varepsilon} & \eta(u_\varepsilon) \psi \frac{W'(u_\varepsilon)}{\varepsilon} \, dx. \end{aligned} \tag{7.3}$$

Since for every $t \in I_\delta$ we have $|W'(t)| = |t(1-t^2)| \leq \frac{4(1-\delta)}{\delta} W(t)$, we can conclude that

$$\begin{aligned} \left| \int_{\mathbb{R}} \eta' g_\varepsilon^{\psi} \, ds \right| \leq & \|\eta\|_{L^\infty(I_\delta)} \|\psi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \|f^\varepsilon_{u_\varepsilon}\|_{L^1(\Omega)} \\ & + \varepsilon^{1/2} \|\eta\|_{L^\infty(I_\delta)} \|\nabla \psi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \left(\int_\Omega \varepsilon |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \\ & + \|\eta\|_{L^\infty(I_\delta)} \|D_\xi \psi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \int_\Omega \left| \varepsilon \nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon} \nu_{u_\varepsilon} \otimes \nu_{u_\varepsilon} \right| dx \\ & + \|\eta\|_{L^\infty(I_\delta)} \|\psi\|_{L^\infty(\Omega \times \mathbb{S}^2)} \frac{4(1-\delta)}{\delta} \int_\Omega \frac{W(u_\varepsilon)}{\varepsilon} \, dx. \end{aligned}$$

From this inequality we can deduce that there exists $g^\psi \in BV_{\rm loc}([-1,1])$ such that $g_\varepsilon^\psi \to g^\psi$ in $L^1_{\rm loc}([-1,1])$ and $\mathcal{L}^1$-almost everywhere in $[-1,1]$.

Next, for any fixed $\psi \in C^1_c(\Omega)$, we consider the functions $\widehat{g}_\varepsilon^\psi : \mathbb{R} \to [0, +\infty)$ defined as

$$\widehat{g}_\varepsilon^\psi(s) := \sqrt{2W(s)} \int_{\{u_\varepsilon = s\}} \psi(y, \nu_{u_\varepsilon}(y)) \, d\mathcal{H}^2(y),$$

and we claim that as $\varepsilon \to 0^+$ the sequence $\{\widehat{g}_\varepsilon^\psi\}$ converges in $L^1_{\rm loc}([-1,1])$ and $\mathcal{L}^1$-almost everywhere to g^ψ. In order to prove the claim, let $\delta > 0$. By (3.2) we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0^+}\int_{I_\delta}\left|\widehat{g}^{\psi}_{\varepsilon}-g^{\psi}\right| ds \le \lim_{\varepsilon\to 0^+}\left(\int_{I_\delta}\left|\widehat{g}^{\psi}_{\varepsilon}-g^{\psi}_{\varepsilon}\right| ds+\int_{I_\delta}|g^{\psi}_{\varepsilon}-g^{\psi}|\, ds\right)\\
&=\lim_{\varepsilon\to 0^+}\left(\int_{I_\delta}\Big|\int_{\{u_\varepsilon=s\}}\psi(\sqrt{2W(s)}-\varepsilon|\nabla u_\varepsilon|)\, d\mathcal{H}^2\Big| ds+O(\varepsilon)\right)\\
&\le \lim_{\varepsilon\to 0^+}\int_{I_\delta}\int_{\{u_\varepsilon=s\}}\left|\psi(\sqrt{2W(s)}-\varepsilon|\nabla u_\varepsilon|)\right| d\mathcal{H}^2\, ds\\
&\le 2\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\lim_{\varepsilon\to 0^+}\int_{\Omega\cap\{u_\varepsilon\in I_\delta\}}\left|\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}-\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|\right|\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|\, dx\\
&\le 2\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\lim_{\varepsilon\to 0^+}\int_{\Omega}\left|\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}-\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|\right|\left(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|+\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right) dx\\
&= 2\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\lim_{\varepsilon\to 0^+}\int_{\Omega}|\xi^{\varepsilon}_{u_\varepsilon}|\, dx=0,
\end{aligned}
$$

which shows the claim. Since on I_δ we have $\sqrt{(2\delta-\delta^2)/2}\le\sqrt{2W(s)}\le\sqrt{2}$, we can also conclude that the sequence of functions

$$
h^{\psi}_{\varepsilon}:\mathbb{R}\to[0,+\infty),\qquad h^{\psi}_{\varepsilon}(s):=\frac{\widehat{g}^{\psi}_{\varepsilon}(s)}{\sqrt{2W(s)}}=\int_{\{u_\varepsilon=s\}}\psi(y,\nu_{u_\varepsilon}(y))d\mathcal{H}^2(y),
$$

is equibounded in $L^1_{\rm loc}([-1,1])$ and converges in $L^1_{\rm loc}([-1,1])$ to

$$
h^{\psi}=\frac{g^{\psi}}{\sqrt{2W}}. \tag{7.4}
$$

Next we refine formula (7.2), by proving that, for every $\delta>0$, every $\psi\in C^1_c(\Omega)$ and $\eta\in C^\infty_c(I_\delta)$, we have

$$
\lim_{\varepsilon\to 0^+}\int_{I_\delta}\eta' g^{\psi}_{\varepsilon}\, ds=\int_{I_\delta}\eta\Big(\frac{d}{ds}\sqrt{2W}\Big)h^{\psi}\, ds. \tag{7.5}
$$

To this aim we start noticing that

$$
\begin{aligned}
&\left|\int_{\Omega_\varepsilon}\eta(u_\varepsilon)\psi\frac{W'(u_\varepsilon)}{\varepsilon}\, dx-\int_{\Omega}\eta(u_\varepsilon)\psi\frac{W'(u_\varepsilon)}{\sqrt{2W(u_\varepsilon)}}|\nabla u_\varepsilon|\, dx\right|\\
&\le \|\eta\|_{L^\infty(I_\delta)}\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\int_{\Omega\cap\{u_\varepsilon\in I_\delta\}}\frac{|W'(u_\varepsilon)|}{\varepsilon^{1/2}\sqrt{W(u_\varepsilon)}}\left|\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}-\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|\right| dx\\
&\le \|\eta\|_{L^\infty(I_\delta)}\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\frac{4(1-\delta)}{\delta}\left(\int_\Omega\frac{W(u_\varepsilon)}{\varepsilon}\, dx\right)^{1/2}\left\|\sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}-\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|\right\|_{L^2(\Omega)},
\end{aligned}
$$

which, by (3.1), vanishes as $\varepsilon \to 0^+$. Then, by the $L^1(I_\delta)$ convergence of h_ε^ψ, the coarea formula and the Lebesgue's Dominated Convergence theorem, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon}\eta(u_\varepsilon)\psi\frac{W'(u_\varepsilon)}{\varepsilon}\,dx = \lim_{\varepsilon\to 0^+}\int_{\Omega\cap\{u_\varepsilon\in I_\delta\}}\eta(u_\varepsilon)\psi\frac{W'(u_\varepsilon)}{\sqrt{2W(u_\varepsilon)}}|\nabla u_\varepsilon|\,dx\\ &= \lim_{\varepsilon\to 0^+}\int_{I_\delta}\eta\left(\frac{d}{ds}\sqrt{2W}\right)h_\varepsilon^\psi\,ds = \int_{I_\delta}\lim_{\varepsilon\to 0^+}\left(\eta\left(\frac{d}{ds}\sqrt{2W}\right)h_\varepsilon^\psi\right)\,ds\\ &= \int_{I_\delta}\eta\left(\frac{d}{ds}\sqrt{2W}\right)h^\psi\,ds.\end{aligned}$$

In order to obtain (7.5) it is then enough to plug the following estimates in (7.3):

$$\left|\int_\Omega \eta(u_\varepsilon)\psi f_{u_\varepsilon}^\varepsilon\,dx\right| \le \varepsilon^{1/2}\|\eta\|_{L^\infty(I_\delta)}\|\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\sqrt{\mathcal{L}^n(\Omega)}\|\varepsilon^{-1}f_{u_\varepsilon}^\varepsilon\|_{L^2(\Omega)},$$

$$\left|\int_\Omega \varepsilon\eta(u_\varepsilon)\nabla\psi\cdot\nabla u_\varepsilon\,dx\right| \le \varepsilon^{1/2}\|\eta\|_{L^\infty(I_\delta)}\|\nabla\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\sqrt{\mathcal{L}^n(\Omega)}\left(\int_\Omega\varepsilon|\nabla u_\varepsilon|^2\,dx\right)^{1/2},$$

and

$$\begin{aligned}&\left|\int_\Omega \eta(u_\varepsilon)D_\xi\psi(x,\nu_{u_\varepsilon})\left(P^{u_\varepsilon}\left(\varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\nu_{u_\varepsilon}\otimes\nu_{u_\varepsilon}\right)\right)\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}\,dx\right|\\ \le&\|\eta\|_{L^\infty(I_\delta)}\|D_\xi\psi\|_{L^\infty(\Omega\times\mathbb{S}^2)}\varepsilon^{1/2}\|B_{u_\varepsilon}^\varepsilon\|_{L^2(\widetilde{\mu}_{u_\varepsilon}^\varepsilon)}.\end{aligned}$$

We are now in a position to prove that the distributional derivative of the function h^ψ in (7.4) is zero in I_δ. In fact by (7.5), the definition of h_ε^ψ and Lebesgue's Dominated Convergence Theorem we have

$$\int_{I_\delta}\eta'\sqrt{2W}h^\psi\,ds = \lim_{\varepsilon\to 0^+}\int_{I_\delta}g_\varepsilon^\psi\,ds = -\int_{I_\delta}\eta\left(\frac{d}{ds}\sqrt{2W}\right)h^\psi\,ds,$$

that is, for every $\eta \in C_c^\infty(I_\delta)$ we have

$$\int_{I_\delta}\frac{d}{ds}\left(\eta\sqrt{2W}\right)h^\psi\,ds = 0. \tag{7.6}$$

Since $\sqrt{2W} \ge \sqrt{\frac{2\delta-\delta^2}{2}}$ on I_δ, from (7.6) we can conclude that the distributional derivative of h^ψ is zero in I_δ. This means that there exists a real number $\beta(\psi)$ such that

$$h^\psi(s) = \beta(\psi), \qquad \text{for } \mathcal{L}^1 - \text{a.e. } s\in I_\delta. \tag{7.7}$$

Let $\Omega' \subset\subset \Omega$, and select $\{\psi_i\} \subset C_c^1(\Omega\times\mathbb{S}^2)$ such that $\{\psi_i\}$ is dense in $C^0(\overline{\Omega'}\times\mathbb{S}^2)$. Fix ψ_i, and choose $\eta_\delta \in C_c^\infty([-1,1])$ such that $0\le\eta_\delta\le 1$ on $[-1,1]$, $\eta_\delta\equiv 1$ on $I_{\delta/2}$. Before proceeding any further, let us recall that, by [38, Proposition 3.4] (see also [13, Lemma 4.4]) there exists $\delta_0 > 0$ independent of ε, such that if $\delta\le\delta_0$

$$\mu_{u_\varepsilon}^\varepsilon(\Omega\cap\{|u_\varepsilon| > 1-\delta\}) \le C\delta,$$

where C depends on Ω', but not on ε.

We then have

$$
\begin{aligned}
&\int_{-1}^{1} \eta_\delta \sqrt{2W}\beta(\psi_i)\,ds = \int_{-1}^{1} \eta_\delta \sqrt{2W} \lim_{\varepsilon\to 0^+} h_\varepsilon^{\psi_i}\,ds = \lim_{\varepsilon\to 0^+} \int_{-1}^{1} \eta_\delta g_\varepsilon^{\psi_i}\,ds \\
=&\lim_{\varepsilon\to 0^+} \int_{\Omega\cap\{|u_\varepsilon|<1-\frac{\delta}{2}\}} \psi_i \sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,dx \\
&+ \int_{\Omega\cap\{1-\frac{\delta}{2}<|u_\varepsilon|<1-\delta\}} \eta_\delta(u_\varepsilon)\psi_i\sqrt{W(u_\varepsilon)}|\nabla u_\varepsilon|\,dx \\
&+ \int_{\Omega\cap\{|u_\varepsilon|>1-\frac{\delta}{2}\}} \psi_i\varepsilon\sqrt{2W(u_\varepsilon)}|\nabla u_\varepsilon|\,dx - \int_{\Omega\cap\{|u_\varepsilon|>1-\frac{\delta}{2}\}} \psi_i\sqrt{W(u_\varepsilon)}|\nabla u_\varepsilon|\,dx \\
=&c_0 \int \psi_i(y,\xi)\,dV^0(y,\star\xi) + O(\delta) \\
=&\int_{-1}^{1} \sqrt{2W}\,ds V^0(\psi_i) + O(\delta) = \int_{-1}^{1} \eta_\delta\sqrt{2W} V^0(\psi_i)\,ds \\
&+ \left(\int_{-1}^{-1+\frac{\delta}{2}} (1-\eta_\delta)\sqrt{2W}\,ds + \int_{1-\frac{\delta}{2}}^{1} (1-\eta_\delta)\sqrt{2W}\,ds \right) V^0(\psi_i) + O(\delta) \\
=&\int_{-1}^{1} \eta_\delta\sqrt{2W} V^0(\psi_i)\,ds + O(\delta).
\end{aligned}
$$

Sending $\delta \to 0^+$ we obtain

$$
\int_{-1}^{1} \sqrt{2W}\,ds\ \beta(\psi_i) = \int_{-1}^{1} \sqrt{2W}\,ds\ V^0(\psi_i). \tag{7.8}
$$

Repeating the same argument for every ψ_i, by the density of $\{\psi_i\}$ in $C^0(\overline{\Omega'}\times\mathbb{S}^2)$ and (7.8) we deduce that $\beta = V^0$ as measures on $G_2^0(\Omega')$. □

Let $\psi \in C_c^1(\Omega\times\mathbb{S}^2)$. From the estimates on $(d/ds)g_\varepsilon^\psi$ obtained in the proof of Lemma 7.1 we can conclude that $g_\varepsilon^\psi \to g^\psi$ strongly in $W^{1,1}_{\mathrm{loc}}((-1,1))$ as $\varepsilon \to 0^+$. The proof of Theorem 4.4-(B) is complete.

We are now in a position to conclude the proof of Theorem 4.4-(A).

Proposition 7.3. *There exists a (not relabelled) subsequence* $\{V^{0,\varepsilon}_{u_\varepsilon}\}$ *converging, as oriented varifolds, to* $V^0 = \mathbf{v}(\mathcal{M},\tau,\theta_1,\theta_2) \in \mathbf{IV}_2^0(\Omega)$*, with* $\mathbf{q}_\sharp V^0 = V$*.*

Proof. As we already noticed at the beginning of the present section, by (1.8), we can extract a subsequence such that $V^{0,\varepsilon}_{u_\varepsilon}$ converge as Radon measures to a Radon measure V^0 on $G_2^0(\Omega)$, and also that $\mathbf{q}_\sharp V^0 = V$. Hence, in order to conclude it remains to show that $V^0 \in \mathbf{IV}_2^0(\Omega)$. To this aim we will make use of Lemma 7.1.

Fix $\Omega' \subset\subset \Omega$ with smooth boundary. By Sard's Lemma and Lemma 7.1 we can find a subsequence $\{V^{0,\varepsilon_k}_{u_{\varepsilon_k}}\}_k$ and a subset $J \subset [-1,1]$, with $\mathcal{L}^1(J) = 0$, such that for every $s \in [-1,1]\setminus J$,

$$
\begin{gathered}
\{u_{\varepsilon_k} = s\} \text{ is a smooth embedded surface and } \{u_{\varepsilon_k} = s\} \cap \{\nabla u_{\varepsilon_k} = 0\} = \emptyset, \\
\partial [\![\mathbf{v}(\{u_{\varepsilon_k} = s\}, \star\nu_{u_{\varepsilon_k}}, 1)]\!] (\Omega') = 0, \\
\lim_{k\to\infty} \mathbf{v}(\{u_{\varepsilon_k} = s\}, \star\nu_{u_{\varepsilon_k}}, 1) = V^0 \ \text{ as oriented varifolds on } \Omega'.
\end{gathered}
$$

Next we fix $\delta > 0$ and set $I_\delta := [-1+\delta, 1-\delta]$. Since we have

$$\int_{I_\delta \setminus J} \left|\delta \mathbf{v}(\{u_{\varepsilon_k} = s\}, \star\nu_{u_{\varepsilon_k}}, 1)\right|(\Omega')\, ds = \int_{I_\delta \setminus J} \int_{\{u_{\varepsilon_k}=s\}\cap\Omega'} \left|\mathrm{div}\left(\nu_{u_{\varepsilon_k}}\right)\right| \, d\mathcal{H}^2 ds$$

$$\leq \frac{1}{(2\delta-\delta^2)} \int_{\Omega'} \left|\mathrm{div}\left(\nu_{u_{\varepsilon_k}}\right)\right| \sqrt{2W(u_{\varepsilon_k})}|\nabla u_{\varepsilon_k}|\, dx \leq \frac{2}{(2\delta-\delta^2)} \int_{\Omega'} |\mathbf{B}_{u_{\varepsilon_k}}| \sqrt{2W(u_{\varepsilon_k})}|\nabla u_{\varepsilon_k}|\, dx$$

$$\leq \frac{2}{(2\delta-\delta^2)} \left(\int_\Omega |\mathbf{B}_{u_{\varepsilon_k}}|^2 \, d\widetilde{\mu}^{\varepsilon_k}_{u_{\varepsilon_k}}\right)^{1/2} \left\{\left[\widetilde{\mu}^{\varepsilon_k}_{u_{\varepsilon_k}}(\Omega)\right]^{1/2} + 2\left[|\xi^{\varepsilon_k}_{u_{\varepsilon_k}}|(\Omega)\right]^{1/2}\right\},$$

by the choice of the ε_k, the set J and (5.1), we can conclude that there exists $s = s_{\varepsilon_k} \in I_\delta \setminus J$ such that

$$\limsup_{k\to\infty} \left|\delta \mathbf{v}(\{u_{\varepsilon_k} = s_{\varepsilon_k}\}, \star\nu_{u_{\varepsilon_k}}, 1)\right|(\Omega') < +\infty.$$

The thesis is then a direct consequence of the properties of $\{u_{\varepsilon_k} = s_{\varepsilon_k}\}$ for $s \in I_\delta \setminus J$ and Theorem B.1. □

8. Proof of Proposition 4.6

As in Section 7, by (1.8) we deduce that (5.1) holds. Hence we can apply Theorem 3.1 and conclude that, up to selecting a further subsequence, (3.1) holds. In addition, the densities of the discrepancy measures are uniformly bounded in $L^1(\Omega)$, and we have

$$\int_\Omega \left|\nabla \xi^\varepsilon_{u_\varepsilon}\right| \, dx = \int_\Omega \left|\varepsilon\nabla^2 u_\varepsilon \nabla u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\nabla u_\varepsilon\right| \, dx$$

$$= \int_{\{\nabla u_\varepsilon \neq 0\}} \left|\left[\varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right]\nabla u_\varepsilon\right| \, dx$$

$$\leq 3^{1/4}\left(\frac{1}{\varepsilon}\int_{\{\nabla u_\varepsilon \neq 0\}} \left|\varepsilon\nabla^2 u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\frac{\nabla u_\varepsilon \otimes \nabla u_\varepsilon}{|\nabla u_\varepsilon|^2}\right|^2 \, dx\right)^{1/2} \left(\widetilde{\mu}^\varepsilon_{u_\varepsilon}(\Omega)\right)^{1/2}$$

$$= 3^{1/4}\left(\int_\Omega |B^\varepsilon_{u_\varepsilon}|^2 \, d\widetilde{\mu}^\varepsilon_{u_\varepsilon}\right)^{1/2} \left[\widetilde{\mu}^\varepsilon_{u_\varepsilon}(\Omega)\right]^{1/2} \leq C,$$

where C is a positive constant independent of ε.

By the compactness theorem in BV (see [1]) and Theorem 3.1 we can select a further subsequence such that $\xi^\varepsilon_{u_\varepsilon} \rightharpoonup 0$ weakly in $BV(\Omega)$ as $\varepsilon \to 0^+$. Moreover (4.9) holds by Rellich-Kondrachov compactness theorem (see [1]).

9. Final Comments

9.1. **Relaxing the constraints on $\varkappa_b$, $\varkappa_G$.** As already stated in Remark 3.3, Theorem 3.2 still holds when replacing (1.4) with the more general constraint $-2 < \varkappa_G/\varkappa_b < 0$. Although we cannot prove Theorem 4.1 (and hence Corollary 4.3) when $-2 < \varkappa_G/\varkappa_b < 0$, we can relax condition (1.4) to

$$\varkappa_G < 0 < \frac{3}{2}\varkappa_b + \varkappa_G. \tag{9.1}$$

In fact, in this case we can still derive (5.1) using the inequality

$$(f^\varepsilon_u)^2 = (\mathrm{tr}(B^\varepsilon_u))^2 \leq 3|B^\varepsilon_u|^2.$$

Hence, in particular, Theorem 4.1 holds for $\varkappa_b = -\varkappa_G = 1$, which gives the usual isotropic bending energy

$$\mathscr{W}_{\mathrm{Hel}}(E) = \frac{1}{2}\int_{\Omega\cap\partial E} |\mathbf{B}_{\partial E}|^2 \, d\mathcal{H}^2,$$

$$\mathscr{W}_\varepsilon(u) = \frac{1}{2\varepsilon}\int_\Omega \left|\varepsilon\nabla^2 u - \frac{W'(u)}{\varepsilon}\nu_u \otimes \nu_u\right|^2 dx.$$

9.2. **Full Γ-convergence and convergence of constrained minimizers.** Corollary 4.3 shows that the Γ-limit with respect to the L^1-topology of $\mathscr{W}_\varepsilon$ is given by $\mathscr{W}_{\mathrm{Hel}}$ on smooth points. However, since Γ-limits are always lower semicontinuous, the natural candidate for a full Γ-convergence result is the L^1-lower semicontinuous envelope $\overline{\mathscr{W}_{\mathrm{Hel}}}$ of $\mathscr{W}_{\mathrm{Hel}}$ defined by

$$\overline{\mathscr{W}_{\mathrm{Hel}}}(E) := \inf\Big\{ \liminf_{h\to\infty} \mathscr{W}_{\mathrm{Hel}}(E_h) : \, E_h \subset \Omega \text{ bounded with } \partial E_h \in C^2, \\ \lim_{h\to\infty} \chi_{E_h} = \chi_E \text{ in } L^1(\Omega)\Big\}.$$

Let us recall some facts about $\overline{\mathscr{W}_{\mathrm{Hel}}}$ (see for example [16]). Define

$$\mathcal{D} := \Big\{ W \in \mathbf{IV}_2(\Omega) : \, W = \lim_{h\to\infty} \mathbf{v}(\partial E_h, 1),\, E_h \subset \Omega \text{ bounded with } \partial E_h \in C^2, \\ \sup_{h\in\mathbb{N}} \int_{\Omega\cap\partial E_h} \left[1 + |\mathbf{B}_{\partial E_h}|^2\right] d\mathcal{H}^2 < +\infty \Big\},$$

and

$$\mathcal{A}(E) := \Big\{ W \in \mathcal{D} : \, W = \lim_{h\to\infty} \mathbf{v}(\partial E_h, 1),\, E_h \subset \Omega \text{ bounded with } \partial E_h \in C^2, \\ \lim_{h\to\infty} \chi_{E_h} = \chi_E \text{ in } L^1(\Omega)\Big\}.$$

Eventually, we recall that if $W \in \mathcal{D}$ then $W \in \mathcal{A}(E_W)$ where E_W is an open, bounded subset with finite perimeter in Ω, such that the essential boundary of E coincides with the set of points of odd 2-density with respect to μ_W.

From [16, Corollary 5.4], we obtain

$$\overline{\mathscr{W}_{\mathrm{Hel}}}(E) = \min\{\mathscr{W}_{\mathrm{Hel}}(V) : V \in \mathcal{A}(E)\}.$$

Hence, if we would be able to prove that $V = \lim_{\varepsilon\to 0^+} V^\varepsilon_{u_\varepsilon} \in \mathcal{A}(E)$, by (4.2) we would have

$$\liminf_{\varepsilon\to 0^+} \mathscr{W}_\varepsilon(u_\varepsilon) \geq c_0 \mathscr{W}_{\mathrm{Hel}}(V) \geq c_0 \overline{\mathscr{W}_{\mathrm{Hel}}}(E),$$

which, together with $\overline{\mathscr{W}_{\mathrm{Hel}}}(E) = \mathscr{W}_{\mathrm{Hel}}(E)$ for $E \subset \Omega$ bounded with boundary of class C^2, would imply that $\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0^+} \mathscr{W}_\varepsilon = \overline{\mathscr{W}_{\mathrm{Hel}}}$. Although Theorem 4.4-(B) seems to represent a signicative step in this direction, in order to prove that $V \in \mathcal{A}(E)$ we miss an estimate similar to the one proved in [43, Lemma 2], [44, Theorem 1]. Actually, we are able to prove that $V \in \mathcal{A}(E)$ under the stronger assumption

$$\sup_{0<\varepsilon<1} \widetilde{\mathscr{W}_\varepsilon}(u_\varepsilon) < +\infty, \tag{9.2}$$

$$\widetilde{\mathscr{W}_\varepsilon}(u_\varepsilon) := \mathscr{W}_\varepsilon(u_\varepsilon) + \int_\Omega |B^\varepsilon_{u_\varepsilon}|^2 \frac{W(u_\varepsilon)}{\varepsilon}\, dx.$$

Indeed, assuming that (9.2) holds, we have

$$\sup_{0<\varepsilon<1} \int_{-1}^{1} \int_{\{u_\varepsilon=s\}} |\mathbf{B}_{\{u_\varepsilon=s\}}|^2 \, d\mathcal{H}^2 ds \le c_0^{-1} \sup_{0<\varepsilon<1} \widetilde{\mathscr{W}_\varepsilon}(u_\varepsilon) < +\infty,$$

which, by Lemma 7.1, gives $V \in \mathcal{A}(E)$. Moreover, this means that we can conclude that chosen $\widetilde{u}_\varepsilon$ so that

$$\widetilde{\mathscr{W}_\varepsilon}(\widetilde{u}_\varepsilon) = \min\left\{ \widetilde{\mathscr{W}_\varepsilon}(u) : \mathscr{P}_\varepsilon(u) = \Lambda_1, \int_\Omega \frac{1+u}{2} \, dx = \Lambda_2 \right\}$$

we have, up to a subsequence,

$$V^\varepsilon_{\widetilde{u}_\varepsilon} \to \widetilde{V} \in \mathcal{D}, \qquad u_\varepsilon \to u = 2\chi_{\widetilde{E}} - 1 \qquad \text{as } \varepsilon \to 0^+,$$

where

- $\widetilde{V}$ solves

$$\min\left\{ \mathscr{W}_{\mathrm{Hel}}(V) : V \in \mathcal{D}, \ \mu_V(\Omega) = \Lambda_1, \ \mathcal{L}^3(\Omega \cap E_V) = \Lambda_2 \right\}$$

- $\widetilde{E} \subset \Omega$ solves

$$\min\left\{ \overline{\mathscr{W}_{\mathrm{Hel}}}(E) : \forall W \in \mathcal{A}(E) \text{ we have } \mu_W(\Omega) = \Lambda_1, \, \mathcal{L}^3(\Omega \cap E) = \Lambda_2 \right\}.$$

- $\mathscr{W}_{\mathrm{Hel}}(\widetilde{V}) = \overline{\mathscr{W}_{\mathrm{Hel}}}(\widetilde{E})$.

9.3. **The case of non-zero spontaneous curvature.** As we already remarked in the introduction, when $H_0 \neq 0$ the functional

$$\int_{\partial E \cap \Omega} (H_{\partial E} - H_0)^2 \ d\mathcal{H}^2 \tag{9.3}$$

not only depends on the surface ∂E but also on the orientation of ∂E. Moreover such a functional is not lower semicontinuous with respect to the varifolds convergence. In fact, as an example due to Karsten Große-Brauckmann shows (see [28], [29] and [39]), there exists a sequence $\{E_h\}_h$ of smooth sets in $\Omega := B(0,1)$, such that for every $h \in \mathbb{N}$ the surface ∂E_h has constant (scalar) mean curvature equal to 1, and at the same time the sequence of varifolds $\mathbf{v}(\partial E_h, 1)$ converges to the varifold $\mathbf{v}(\langle \mathbf{e}_3 \rangle^\perp, 2)$ in Ω. Hence, assuming $H_0 = 1$, we have

$$0 = \lim_{h\to\infty} \int_{\Omega\cap\partial E_h} (H_{\partial E_h} - H_0)^2 \, d\mathcal{H}^2 < 2\pi = 2 \int_{\langle \mathbf{e}_3\rangle^\perp \cap B(0,1)} (H_0)^2 \ d\mathcal{H}^2.$$

However if we consider the complete Helfrich's energy

$$\mathscr{W}_{\mathrm{Hel}}(E) = \int_{\Omega\cap\partial E} \left[\frac{\varkappa_b}{2}\Big(H_{\partial E} - H_0\Big)^2 + \varkappa_G K_{\partial E} \right] \ d\mathcal{H}^2, \tag{9.4}$$

and assume (as in the case of zero spontaneous curvature) that $-2 < \varkappa_b/\varkappa_G < 0$, the results of [16] still apply and Theorem 3.2 holds also in this case. Moreover the functional is lower semicontinuous with repect to the convergence of the oriented varifolds and, whenever $\sup_{h\in\mathbb{N}} \mathscr{W}_{\mathrm{Hel}}(E_h) < +\infty$, the oriented varifolds $\mathbf{v}(\partial E_h, \star\nu_{\partial E_h}, 1)$ converge (up to a subsequence) to an *oriented curvature varifold* $V^0 \in \mathbf{IV}^0_2(\Omega)$ in the sense of [15].

Possible diffuse-interface approximating functionals for (9.3) are

$$\frac{1}{\varepsilon} \int_\Omega \Big(f_u^\varepsilon - H_0 \varepsilon |\nabla u| \Big)^2 dx, \quad \frac{1}{\varepsilon} \int_\Omega \Big(f_u^\varepsilon - H_0 \sqrt{2W(u)} \Big)^2 dx, \tag{9.5}$$

the latter being the one proposed in [18]. Consequently a natural candidate for the diffuse-interface approximation of (9.4) is

$$\widehat{\mathscr{W}}_\varepsilon(u) := \frac{\varkappa_b}{2}\widehat{\mathscr{H}}_\varepsilon(u) + \varkappa_G \mathscr{K}_\varepsilon(u),$$

where $\widehat{\mathscr{H}}_\varepsilon(u)$ is given by one of the two expressions in (9.5). If (1.4) is satisfied, by a direct calculation we can show that (5.1) holds as soon as

$$\sup_{0<\varepsilon<1}\Big(\mu^\varepsilon_{u_\varepsilon}(\Omega) + \widehat{\mathscr{W}}_\varepsilon(u_\varepsilon)\Big) < +\infty.$$

Hence we can conclude that also Lemma 5.1 and Lemma 5.3 apply and, with minor modifications to the arguments of Sections 7-8, we can prove that Theorem 4.4 and Proposition 4.6 hold also for $\widehat{\mathscr{W}}_\varepsilon$. Moreover we can use the same sequence $\{\widehat{u}_\varepsilon\}_\varepsilon \subset C^2(\Omega)$ constructed in Section 6 to show that also an analog of Theorem 4.2 holds for $\widehat{\mathscr{W}}_\varepsilon$. However, in order to prove that the lower bound estimate corresponding to (4.2) holds, we should prove that

$$\lim_{\varepsilon\to 0^+}\int_\Omega \nabla\xi^\varepsilon_{u_\varepsilon}\cdot\nu_{u_\varepsilon}\,dx = 0. \tag{9.6}$$

Unfortunately we are not able to prove (9.6) unless additional hypothesis are made on u_ε (for example if $\mu^\varepsilon_{u_\varepsilon} \to 2c_0|\nabla\chi_E|$, then (9.6) follows from (5.2), Theorem 4.4 and Lemma A.6). However, a possible strategy to obtain (9.6) might be trying to use Proposition 4.6 on each of the "well-separated transition layers" that can be obtained via an appropriate blow-up procedure (see [38, Proposition 5.3]), and then conclude via a covering argument.

Appendix A. Measure-function pairs

Let $D \subset \mathbb{R}^l$; we say that (μ, f) is a *measure-function pair over D with values in $\mathbb{R}^m$*, if μ is a positive Radon measure on D, $f : D \to \mathbb{R}^m$ is defined μ-almost everywhere and $f \in L^1_{\mathrm{loc}}(\mu)$.

Let us recall the definition of measure-function pairs convergence (see [30])

Definition A.1. *Let (μ_k, f_k), (μ, f) be measure-function pairs on D with values in $\mathbb{R}^m$ for every $k \in \mathbb{N}$. We say that (μ_k, f_k) converge weakly to (μ, f) as measure-function pairs as $k \to \infty$ if*

$$\lim_{k\to\infty}\int f_k\cdot Y\,d\mu_k = \int f\cdot Y\,d\mu \qquad \forall Y \in C^0_c(D,\mathbb{R}^m).$$

Definition A.2. *We say that a function $F : \mathbb{R}^m \to [0, +\infty)$ is a standard integrand provided F is strictly convex on $\mathbb{R}^m$, and*

$$g(|q|) \le F(q) \qquad \forall q \in \mathbb{R}^m,$$

where $g \in C^0([0, +\infty))$ is non-negative, increasing and $g(t) \to +\infty$ as $t \to +\infty$.

Definition A.3. *Let (μ_k, f_k) and (μ, f) be measure-function pairs over D with values on $\mathbb{R}^m$. Suppose $\mu_k \rightharpoonup \mu$ as $k \to \infty$ as Radon measures. We say that (μ_k, f_k) converge to (μ, f) in the F-strong sense in D if*

(i) $\int F(f_k)\,d\mu_k < +\infty$ *for every $k \in \mathbb{N}$;*

(ii) *setting* $D_{kj} := \{y \in D : |f_k(y)| \geq j\}$ *we have*

$$\lim_{k\to\infty} \int_{D_{kj}} F(f_k)\, d\mu_k = 0,$$

uniformly in $k \in \mathbb{N}$*;*

(iii) *for every* $\psi \in C^0_c(D \times \mathbb{R}^m)$ *we have*

$$\lim_{k\to\infty} \int \psi(y, f_k)\, d\mu_k = \int \psi(y, f)\, d\mu.$$

We say that a sequence of measure-function pairs converges L^p-strongly ($p \in [1,\infty)$) if it converges strongly in the F_p-sense, with $F_p(q) := |q|^p$.

The following result has been proved in [30, Theorem 4.4.2].

Theorem A.4. *Let* $(\mu_k, f_k)_{k\in\mathbb{N}}$ *be measure-function pairs over* D *with values in* $\mathbb{R}^m$*. Suppose that* μ *is a Radon measure on* D *and* $\mu_k \rightharpoonup \mu$ *in* D *as* $k \to \infty$*. Let* $F : \mathbb{R}^m \to [0, +\infty)$ *be a standard integrand. The following assertions hold.*

(i) *If*

$$\sup_{k\in\mathbb{N}} \int F(f_k)\, d\mu_k < +\infty, \tag{A.1}$$

then there exists $f \in L^1_{\rm loc}(\mu)$ *and a (not relabelled) subsequence* $\{(\mu_k, f_k)\}$ *such that*

$$\lim_{k\to\infty} (\mu_k, f_k) = (\mu, f), \tag{A.2}$$

weakly as measure-function pairs on D *with values on* $\mathbb{R}^m$*.*

(ii) *If* $\{(\mu_k, f_k)\}$ *and* (μ, f) *satisfy* (A.1), (A.2)*, then*

$$\int F(f)\, d\mu \leq \liminf_{k\to\infty} \int F(f_k)\, d\mu_k. \tag{A.3}$$

Remark A.5. We can adapt the notions and results proved until this point in the present Appendix to the case where D is an open subset of a smooth manifold embedded in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. In particular, in our applications we will often consider $D = G_2(\Omega)$ or $D = G^0_2(\Omega)$.

The following lemma is a particular case of [33, Proposition 3.2].

Lemma A.6. *Let* (μ_k, g_k) *and* (μ, g) *be measure-function pairs on* D *with values in* $\mathbb{R}^m$ *such that*

$$\sup_{k\in\mathbb{N}} \|g_k\|_{L^2(\mu_k)} < +\infty,$$

and (μ_k, g_k) *weakly converge to* (μ, g) *as measure-function pairs.*

Moreover let (μ_k, f_k)*,* (μ, f) *be measure-function pairs on* D *with values in* R^m *such that* (μ_k, f_k) *converges* L^2*-strongly to* (μ, f)*. Then*

$$\lim_{k\to\infty} (\mu_k, f_k \cdot g_k) = (\mu, f \cdot g),$$

weakly as measure-function pairs on D *with values in* $\mathbb{R}$*.*

Appendix B. Geometric Measure Theory: varifolds

Let us recall some basic fact in the theory of varifolds, the main bibliographic sources being [41] and [30].

We call *varifold* (resp. *oriented varifold*) any positive Radon measure on $G_2(\Omega)$ (resp. on $G_2^0(\Omega)$). In this paper we are confined to surfaces, hence we use the terms varifold and oriented varifold to mean a 2-varifold in Ω.

If V^0 is an oriented varifold then the push-forward $\mathbf{q}_\sharp V^0$ is the corresponding unoriented varifold associated with V^0 by projection onto $G_2(\Omega)$.

For any varifold (or oriented varifold) V we define μ_V to be the Radon measure on Ω obtained by projecting V onto Ω.

Let $\mathcal{M}$ be a 2-rectifiable subset of $\mathbb{R}^3$ with finite $\mathcal{H}^2$-measure and let θ, θ_1, θ_2 : $\mathcal{M} \to \mathbb{R}^+$ be $\mathcal{H}^2 \llcorner \mathcal{M}$-measurable functions. Suppose $\tau : \mathcal{M} \to G_{2,3}^0$ is $\mathcal{H}^2 \llcorner \mathcal{M}$-measurable and $\mathbf{q}(\tau(x)) = T_x\mathcal{M}$ for $\mathcal{H}^2 \llcorner \mathcal{M}$-almost everywhere x (τ is called an orientation function on $\mathcal{M}$). Then we define the *rectifiable* (unoriented and oriented respectively) *varifolds*

$$V = \mathbf{v}(\mathcal{M}, \theta), \qquad V^0 = \mathbf{v}(\mathcal{M}, \tau, \theta_1) + \mathbf{v}(\mathcal{M}, -\tau, \theta_2) =: \mathbf{v}(\mathcal{M}, \tau, \theta_1, \theta_2),$$

by

$$V(\phi) := \int_{\mathcal{M}} \phi(x, T_x\mathcal{M})\, \theta(x) d\mathcal{H}^2 \qquad \forall \phi \in C_c^0(G_2(\Omega)),$$

$$V^0(\varphi) := \int_{\mathcal{M}} \Big[\varphi(x, \tau(x))\theta_1(x) + \varphi(x, -\tau(x))\theta_2(x)\Big]\, d\mathcal{H}^2 \qquad \forall \varphi \in C_c^0(G_2^0(\Omega)).$$

With the notation $\mathbf{v}(\mathcal{M}, \tau, \theta)$ we mean $\mathbf{v}(\mathcal{M}, \tau, \theta, 0)$.

When θ (resp. θ_1, θ_2) take values in $\mathbb{N}$ we say that $V = \mathbf{v}(\mathcal{M}, \theta)$ (resp. $V^0 = \mathbf{v}(\mathcal{M}, \tau, \theta_1, \theta_2)$) is a *rectifiable integer* unoriented (resp. oriented) *varifold* and we write $V \in \mathbf{IV}_2(\Omega)$ (resp. $V^0 \in \mathbf{IV}_2^0(\Omega)$). If $V^0 = \mathbf{v}(\mathcal{M}, \tau, \theta_1, \theta_2) \in \mathbf{IV}_2^0(\Omega)$ the integral rectifiable 2-current $[\![V^0]\!]$ is defined as

$$[\![V^0]\!](\omega) := \int_M \langle \omega(x), \tau(x) \rangle\ (\theta_1(x) - \theta_2(x))\, d\mathcal{H}^2(x) \qquad \forall \omega \in C^0(\Omega, \Lambda_2(\mathbb{R}^3)).$$

As usual $\partial[\![V^0]\!]$ denotes the boundary of the current $[\![V^0]\!]$, and $|\partial[\![V^0]\!]|$ is the mass of $[\![\partial V^0]\!]$ (see [41]).

Let V be an unoriented varifold on Ω; we define *the first variation of* V as the linear operator

$$\delta V : C_c^1(\Omega, \mathbb{R}^3) \to \mathbb{R}, \qquad Y \to \int \operatorname{tr}(S\nabla Y(x))\, dV(x, S).$$

We say that V has *bounded first variation* (resp. *generalized mean curvature in* L^p, $p > 1$) if δV can be extended to a linear continuous operator on $C_c^0(\Omega, \mathbb{R}^3)$ (resp. on $L^p(\mu_V, \mathbb{R}^3)$). In this case $|\delta V|$ denotes the total variation of δV. Whenever the varifold V has bounded first variation we call *the generalized mean curvature vector of* V the vector field

$$\mathbf{H}_V = \frac{d\delta V}{d\mu_V},$$

where the right-hand side denotes the Radon-Nikodym derivative.

By varifold convergence (resp. oriented varifold convergence) we mean the convergence as Radon measures on $G_2(\Omega)$ (resp. on $G_2^0(\Omega)$). The following compactness theorem for oriented varifolds is proved in [30, Theorem 3.1].

Theorem B.1. *Let $C > 0$ and let $\{\Omega_i\}$ be a sequence of open subsets with smooth boundary invading Ω. The set*

$$\left\{V^0 \in \mathbf{IV}_2^0(\Omega) : \forall i \in \mathbb{N},\, \mu_{\mathbf{q}_\sharp(V^0)}(\Omega_i) + |\delta(\mathbf{q}_\sharp V^0)|(\Omega_i) + |\partial [\![V^0]\!]|(\Omega_i) \leq C\right\}$$

is sequentially compact with respect to the oriented varifolds convergence.

Remark B.2. Let $\{V_h\}$ be a sequence of varifolds converging to a varifold V, and suppose that there exist μ_{V_h}-measurable maps $S^h_\cdot$ and a μ_V-measurable map $S_\cdot$ such that

$$V_h(\Psi) = \int \Psi(x, S^h_x)\, d\mu_{V_h}(x) \qquad \forall \Psi \in C^0_c(G_2(\Omega)),\ \forall h \in \mathbb{N}$$

$$V(\Psi) = \int \Psi(x, S_x)\, d\mu_V(x) \qquad \forall \Psi \in C^0_c(G_2(\Omega)).$$

Then it can be checked that the measure function pair $(\mu_{V_h}, S^h_\cdot)$ converge L^p-strongly to $(\mu_V, S_\cdot)$ as measure function pairs on Ω with values in $G_2(\Omega)$, for every $p \in (1, +\infty)$.

Following [30] we define the notion of Hutchinson's curvature varifold with generalized second fundamental form.

Definition B.3. *Let $V \in \mathbf{IV}_2(\Omega)$. We say that V is a curvature varifold with generalized second fundamental form in L^2, if there exists $A^V = A^V_{ijk} \in L^2(V, \mathbb{R}^{3^3})$ such that for every function $\phi \in C^1_c(G_2(\Omega))$ and $i = 1, 2, 3$,*

$$\int_{G_2(\Omega)} (S_{ij}\partial_j \phi + A^V_{ijk} D_{m_{jk}} \phi + A^V_{jij}\phi)\, dV(x, S) = 0, \tag{B.1}$$

where $D_{m_{jk}}\phi$ denotes the derivative of $\phi(x, \cdot)$ with respect to its jk-entry variable.

Moreover we define the generalized second fundamental form $\mathbf{B}_V = (B^k_{ij})_{1\leq i,j,k\leq 3}$ of V as

$$B^k_{ij}(x, S) := S_{jl} A^V_{ikl}(x, S). \tag{B.2}$$

Remark B.4. Every curvature varifold V with generalized second fundamental form in L^2 has bounded first variation. Moreover

$$\mathbf{H}_V(x) = (A_{j1j}(x, T_x\mu_V), A_{j2j}(x, T_x\mu_V), A_{j3j}(x, T_x\mu_V)) \in L^2(\mu_V, \mathbb{R}^3), \tag{B.3}$$

for μ_V almost every $x \in \Omega$.

Remark B.5. If $V = \mathbf{v}(\Sigma, 1)$, where Σ is a smooth, compact surface without boundary, the generalized second fundamental form as well as the mean curvature and the tensor A_V coincide with the classical quantities defined in Section 2.2, and the same is true for the oriented varifold associated with Σ. Moreover the generalized second fundamental form and the functions A^V_{ijk} verify Proposition 2.2.

Next we give a definition of convergence for Hutchinson's curvature varifolds.

Definition B.6. *Let $\{V_h\}$ be a sequence of curvature varifolds with generalized second fundamental form in L^2, and let V be a curvature varifold with generalized second fundamental form in L^2. We say that V_h converge as curvature varifolds to V if*

$$\lim_{h\to\infty} V_h = V \qquad \textit{as varifolds},$$

$$\lim_{h\to\infty} (V_h, A_{V_h}) = (V, A_V) \qquad \textit{as measure} - \textit{function pairs}.$$

Remark B.7. By Remark B.2, Lemma A.6 and the definition of generalized second fundamental form $\mathbf{B}_{V_h}$, we have that if $V_h \to V$ as curvature varifolds then

$$(V_h, \mathbf{B}_{V_h}) \to (V, \mathbf{B}_V)$$

as measure-function pairs on $G_2(\Omega)$ with values in $\mathbb{R}^{3^3}$.

As a consequence of Definition B.6 and Theorem A.4 we have the following

Proposition B.8. *Let $\{V_h\} \subset \mathbf{IV}_2(\Omega)$ be a sequence of curvature varifolds with generalized second fundamental form in L^2 satisfying*

$$\sup_{h\in\mathbb{N}} \left\{ \mu_{V_h}(\Omega) + \int \sum_{i,j,k=1}^{3} (A_{ijk}^{V_h})^2 \, dV_h < +\infty \right\}.$$

Then $\{V_h\}$ has a subsequence converging to $V \in \mathbf{IV}_2(\Omega)$ as curvature varifolds.

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