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Behaviour of symmetric solutions of a nonlinear elliptic field equation in the semi-classical limit: Concentration around a circle *

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Abstract

In this paper we study the existence of concentrated solutions of the nonlinear field equation

$$-h^2\Delta v + V(x)v - h^p\Delta_p v + W'(v) = 0\,,$$

where $v:\mathbb{R}^N\to\mathbb{R}^{N+1}$, $N\geq 3$, $p>N$, the potential V is positive and radial, and W is an appropriate singular function satisfying a suitable symmetric property. Provided that h is sufficiently small, we are able to find solutions with a certain spherical symmetry which exhibit a concentration behaviour near a circle centered at zero as $h\to 0^+$. Such solutions are obtained as critical points for the associated energy functional; the proofs of the results are variational and the arguments rely on topological tools. Furthermore a penalization-type method is developed for the identification of the desired solutions.

1 Introduction

This paper carries on the study started in [2, 3, 13], which considers the nonlinear elliptic equation

$$-h^2\Delta v + V(x)v - h^p\Delta_p v + W'(v) = 0\,, \tag{1.1}$$

where $h>0$, $v:\mathbb{R}^N\to\mathbb{R}^{N+1}$, $N\geq 3$, $p>N$, $V:\mathbb{R}^N\to\mathbb{R}$, $W:\Omega\to\mathbb{R}$ with $\Omega\subset\mathbb{R}^{N+1}$ an open set, denoting W' the gradient of W.

Here $\Delta v = (\Delta v_1,\ldots,\Delta v_{N+1})$, being Δ the classical Laplacian operator, while $\Delta_p v$ denotes the $(N+1)$-vector whose j-th component is given by

$$\mathrm{div}(|\nabla v|^{p-2}\nabla v_j).$$

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By making the change of variables $x \to hx$, (1.1) can be rewritten as

$$-\Delta u + V_h(x)u - \Delta_p u + W'(u) = 0 \tag{1.2}$$

where $V_h(x) = V(hx)$ and $u(x) = v(hx)$.

The interest in studying this equation lies in its relationship with the particle physics and the relativistic quantum field theory. More precisely, equations like (1.1) or (1.2) have been introduced in a set of recent papers (see [4]-[11]); in such works the authors look for soliton-like solutions, i.e. solutions whose energy is finite and which preserve their shape after interactions; in this respect the solitons resemble as closely as possible classical particles and their dynamics is studied in order to provide some examples of classical models which exhibit a quantistic behaviour. We refer to [5, 7, 10] for a more precise description of such developments.

The presence of a small diffusion parameter h in equation (1.1) leads to the problem of finding bound states (i.e. solutions with finite energy) at least for small h. Once one has obtained existence results, other natural questions arise: do these bound states exhibit some notable behaviour in the semi-classical limit, i.e. as $h \to 0^+$? If so, is it possible to locate their concentration points? In [2, 3, 13] the authors have given a partial positive answer. Let us recall the results obtained in the quoted papers. In [2], under the assumption $\liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x) > 0$, it was proved that if h is small enough (1.1) possesses at least a solution obtained as a minimum for the associated energy functional; furthermore this solution concentrates around an absolute minimum of V as $h \to 0^+$, in the sense that its shape is a sharp peak near that point, while it vanishes everywhere else. In [3] the authors removed any global assumption on V except for $\inf_{x\in\mathbb{R}^N} V(x) > 0$ and constructed solutions with multiple peaks which concentrate at any prescribed finite set of local minimum points of V in the semi-classical limit. Again such solutions are captured as minima for the energy functional and the technique is based on the analysis of the behaviour of sequences with bounded energy, in the spirit of the concentration-compactness principle ([20]). Finally [13], under the assumption $\lim_{|x|\to+\infty} V(x) = +\infty$, deals with the existence of critical points, instead of minima, for small h; variational methods based on variants of the Mountain Pass Theorem are used to obtain a critical value for the energy functional characterized by a mini-max argument; moreover the associated solutions to equation (1.1) vanish uniformly outside a bounded set in $\mathbb{R}^N$. This paper intends to go further in the analysis begun in [2, 3, 13]: in particular we deal with the existence of solutions which satisfy some symmetry properties and which are not necessarily with least energy; we are also interested in finding a sufficient condition for the concentration of such "radial" bound states. What will derive is the appearance of a new phenomenon, i.e. the concentration near a circle; this provides one of the first example of solutions concentrating around a curve and no more around a finite set of points as for most of the literature concerning with concentration phenomena. In what follows we recall the results contained in some of a large number of works which has been devoted in studying single and multiple spike solutions.

The existence of concentrated solutions and related problems has attracted considerable attention in recent years, particularly in relation with the study of standing waves for the nonlinear scalar Schrödinger equation:

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2m}\Delta\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi, \tag{1.3}$$

where $\gamma > 0$, $p > 1$ and $\psi : \mathbb{R}^N \to \mathbb{C}$. Looking for standing waves of (1.3), i.e. solutions of the form $\psi(x,t) = \exp(-iEt/h)v(x)$, the equation for v becomes

$$-h^2\Delta v + V(x)v - |v|^{p-1}v = 0 \tag{1.4}$$

where we have assumed $\gamma = 2m = 1$ and the parameter E has been absorbed by V. The first result in this line, at our knowledge, is due to Floer and Weinstein ([17]). These authors considered the one-dimensional case and constructed for small $h > 0$ such a concentrating family via a Lyapunov-Schmidt reduction around any non-degenerate critical point of the potential V, under the condition that V is bounded and $p = 3$. In [21] and [22] Oh generalized this result to higher dimensions when $1 < p < \frac{N+2}{N-2}$ $(N \geq 3)$ and V exhibits "mild oscillations" at infinity. Variational methods based on variants of Mountain-Pass Lemma are used in [23] to get existence results for (1.4) where V lies in some class of highly oscillatory V's which are not allowed in [21, 22]. Under the condition $\liminf_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x)$ in [25] Wang established that these mountain-pass solutions concentrate at global minimum points of V as $h \to 0^+$; moreover a point at which a sequence of solutions concentrates must be critical for V. This line of research has been extensively pursued in a set of papers by Del Pino and Felmer ([14]-[16]). We also recall the nonlinear finite dimensional reduction used in [1] and a recent paper by Grossi ([18]). The most complete and general results for this kind of problems seem due to Del Pino and Felmer ([15]) and Li ([19]). As for radial positive bound states, in [25] Wang has worked on equation (1.4) in case when V is radial: he proved the existence of a family $\{v_h\}$ of positive radial solutions with least energy among all nontrivial radial solutions; such a family must concentrate at the origin as $h \to 0^+$.

In most of the above examples (but with some exceptions such as [14], [15], [16]), the method employed, local in nature, seems to use in an essential way the splitting of the functional space into a direct sum of good invariant subspaces of the linearized operator; in such a linearization process the non-degeneracy of the concentration points plays a basic role, even though this assumption can be somewhat relaxed. However the finite dimensional reduction does not seem possible in the study of equation (1.1) because of the presence of the p-Laplacian operator. Instead the direct use of variational methods, relying on topological tools, permits to obtain good results under relatively minimal assumptions and this is exactly the direction we will follow. Indeed W is chosen to be a suitable singular function so that the presence of the term $W'(u)$ in (1.2) implies that the solutions have to be searched among the maps which take value in a certain open set $\Omega \subset \mathbb{R}^{N+1}$ (see hypotheses a)-g) below). So the nontrivial topological properties of Ω allow us to give a topological classification of such maps. This classification is carried out by means of a topological invariant, the topological charge, which is an integer number depending only on the behaviour of the function on a bounded set (see definition 3.1). Let us see more precisely the class of the nonlinearity W and of the potential V we deal with. First we introduce the following notation: for every $\xi \in \mathbb{R}^{N+1}$ we write

$$\xi = (\xi_0, \xi_1), \quad \xi_0 \in \mathbb{R}, \quad \xi_1 \in \mathbb{R}^N.$$

Throughout this paper we always make the following assumptions:

a) $V \in C^1(\mathbb{R}^N, \mathbb{R})$, $V_0 \equiv \inf_{x \in \mathbb{R}^N} V(x) > 0$ and V is a radially symmetric function;

b) $W \in C^1(\Omega, \mathbb{R})$ where $\Omega = \mathbb{R}^{N+1} \setminus \{\overline{\xi}\}$ with $\overline{\xi} = (1, 0)$;

c) $W(\xi) \geq W(0) = 0$ for all $\xi \in \Omega$; W is two times differentiable in 0;

d) there exist c, $r > 0$ such that

$$|\xi| < r \Rightarrow W(\overline{\xi} + \xi) > c|\xi|^{-q}$$

where

$$\frac{1}{q} = \frac{1}{N} - \frac{1}{p}, \quad N \geq 3, \quad p > N;$$

e) for every $\xi = (\xi_0, \xi_1) \in \Omega$ and for every $g \in O(N)$ there results

$$W(\xi_0, g\xi_1) = W(\xi_0, \xi_1).$$

With obvious notation by $O(N)$ we have denoted the group of the rotation matrices which acts on $\mathbb{R}^N$. Hence in assumption a) by radially symmetric function we mean that for all $g \in O(N)$ and $x \in \mathbb{R}^N$ it holds: $V(gx) = V(x)$.

In most of this paper equations (1.1) and (1.2) will be considered for a more restricted class of W's; more precisely we will study (1.2) when W satisfies the further assumptions:

f) $|W''(0)| < \frac{V_0}{(2(N+1))^{1/2}+2}$;

g) there exists $\overline{\varepsilon} \in (0, 1)$ such that for every $\xi \in \Omega$ with $|\xi| \leq \overline{\varepsilon}$:

$$W(\lambda\xi) \leq W(\xi) \quad \forall \lambda \in [0, 1].$$

We notice that no restriction on the global behaviour of V and W is required other than a). In particular they are not required to be bounded or to satisfy some assumption at infinity. A simple function W satisfying hypotheses b)-g) is the following

$$W(\xi) = \frac{|\xi|^4}{|\xi - \overline{\xi}|^q}.$$

Under the regularity assumptions on V and W it is standard to check that the weak solutions of (1.2) correspond to the critical points for the associated energy functional:

$$E_h(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_h(x)|u|^2\right)\, dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} W(u)\, dx. \quad (1.5)$$

In order to sum up our main arguments we observe that, since we are interested in maps u which lie in some open subset of $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \cap W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$, it makes sense to adopt the following convention

$$u = (u_0, u_1), \quad \text{with} \quad u_0 : \mathbb{R}^N \to \mathbb{R}, \quad u_1 : \mathbb{R}^N \to \mathbb{R}^N. \quad (1.6)$$

Following the idea used in [6], we look for solutions u of (1.2) which satisfy the following symmetry property:

$$u_0(gx) = u_0(x), \;\; g^{-1}u_1(gx) = u_1(x) \quad \forall g \in O(N), \;\; \forall x \in \mathbb{R}^N. \quad (1.7)$$

In [6] the authors showed that under assumptions b)-e) equation like (1.1), but without the potential term $V(x)v$, admits infinitely many solutions. Obviously equation (1.1) deprived of the potential term exhibits an invariance under the group of the rotations and translations in $\mathbb{R}^N$, while the presence of V leads to a loss of such invariance and then the arguments in [6] partially fall. However observe that if $u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \cap W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$ verifies (1.7), then $|u|$ is a radially symmetric real valued function; hence, by using the compactness properties of the radial functions, we are able to overcome the difficulty due to the addition of V and under the only hypotheses a)-e) we prove (in section 5) the existence of infinitely many solutions for arbitrary $h > 0$; furthermore such solutions concentrate near the origin as $h \to 0^+$ (see theorem 5.1).

Most of this paper is devoted to a problem which raises naturally from the symmetry properties of equation (1.1), i.e. that of seeking solutions satisfying (1.7) and concentrating near a circle centered in 0 in the semi-classical limit. We summerize the result achieved in section 8 as follows: assuming that there exists $r_0 > 0$ such that $V(x)$ is sufficiently big for $|x| = r_0$, then at least for small h we can find a solution v_h to equation (1.1) with the following property: for every sequence $h_n \to 0^+$ there exists a subsequence, still denoted by h_n, such that $|v_{h_n}|$ has a circle of local maximum points $\{x \in \mathbb{R}^N : |x| = r_{h_n}\}$ with $r_{h_n} \to \overline{r} > r_0$ as $n \to +\infty$, while v_{h_n} vanishes to zero away from the circle $\{x \in \mathbb{R}^N : |x| = \overline{r}\}$. We also provide an implicit formula which permits us to locate exactly the concentration set of the solutions and simultaneously gives us an estimate of how big must be $V(x)$ for $|x| = r_0$ to make the above result available, even though we cannot express explicitly the radius $\overline{r}$ in terms of the potential V. Our solutions are obtained by using variational methods and the method employed requires some restrictions of a technical nature on the nonlinear term which are specified in assumptions f)-g). For the exact statement of the above result see theorem 8.1 which provides the main result of this paper about existence and concentration of solutions.

We now briefly outline the organization of the contents of this paper. In section 2 we introduce the abstract setting, i.e. the functional set in which it is convenient to study the functional E_h and we point out some of its compactness properties. Section 3 is devoted to the definition of a topological device, the "topological charge", which we will need in order to give a classification of the maps we deal with. In section 4 we study the energy functional E_h and provide an important invariance property under a certain class of transformations. The first existence and concentration result for equation (1.1) is given in section 5, where we find a family $\{v_h\}$ of solutions concentrating near zero as $h \to 0^+$. Section 6 and 7 pave the way for the achievement, in last section, of the second result of the paper, i.e. the existence of solutions exhibiting a concentration behaviour near a circle.

Notation. For the rest of this paper we use the following notation.

- $x\,y$ is the standard scalar product between $x,\, y \in \mathbb{R}^N$. $|x|$ is the Euclidean norm of $x \in \mathbb{R}^N$. Analogously $|M|$ is the Euclidean norm of a $m \times n$ real matrix M.
- $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ and $W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$ are the standard Sobolev spaces.
- For $u : \mathbb{R}^N \to \mathbb{R}^{N+1}$, ∇u is the $(N+1) \times N$ real matrix whose rows are given by the gradient of each component function u_j.
- For any $U \subset \mathbb{R}^N$, $int(U)$ is its internal part, $\overline{U}$ its closure and ∂U its boundary. Furthermore χ_U denotes the characteristic function of U, while by $meas\,(U)$ we intend the Lebesgue measure of U. Finally $\mathrm{dist}(x, U)$ is the Euclidean distance between a point $x \in \mathbb{R}^N$ and U, i.e. $\mathrm{dist}(x, U) = \inf_{y \in U} |x - y|$.
- ω_N is the surface measure of the unit sphere S^{N-1} of $\mathbb{R}^N$.
- If $x \in \mathbb{R}^N$ and $r > 0$, then $B_r(x)$ or $B(x, r)$ is the open ball with center in x and radius r.
- Given $R_2 > R_1 > 0$, by $C(R_1, R_2)$ we denote the ring in $\mathbb{R}^N$ centered in 0 and with internal radius R_1 and external radius R_2, i.e. $C(R_1, R_2) = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$.

2 Functional Setting

To obtain critical points for the functional E_h we choose a suitable Banach space: for every $h > 0$ let H_h denote the subspace of $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ consisting of functions u such that

$$\|u\|_{H_h} \equiv \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V_h(x)|u|^2 \right) dx \right)^{1/2} + \left(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \right)^{1/p} < +\infty. \quad (2.1)$$

The space H_h can also be defined as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{R}^{N+1})$ with respect to the norm (2.1). The main properties of H_h are summarized in the following lemma.

Theorem 2.1 *For every $h > 0$ the following statements hold:*

i) H_h is continuously embedded in $W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1})$ and $W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$.

ii) There exist two constants C_0, $C_1 > 0$ such that, for every $u \in H_h$, $\|u\|_{L^\infty} \leq C_0 \|u\|_{H_h}$ and

$$|u(x) - u(y)| \leq C_1\, |x-y|^{(p-N)/p} \|\nabla u\|_{L^p} \quad \forall\, x, y \in \mathbb{R}^N. \tag{2.2}$$

iii) For every $u \in H_h$

$$\lim_{|x| \to +\infty} u(x) = 0\,. \tag{2.3}$$

iv) If $\{u_n\}$ converges weakly in H_h to some function u, then it converges uniformly on every compact set in $\mathbb{R}^N$.

The proof is a direct consequence of the Sobolev embedding theorems (see[13] for the proof of the continuous immersion $H_h \subset W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$).

Note that from (2.2) we derive the following property we are going to use several times in the proofs of our results: given $\{u_\alpha\} \subset H_h$ a family of functions verifying $\|\nabla u_\alpha\|_{L^p} \leq M$ for some $M \geq 0$, then there results: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| \leq \delta \Rightarrow |u_\alpha(x) - u_\alpha(y)| \leq \varepsilon \quad \forall\, \alpha.$$

We refer to the above property as to the "equi-uniform continuity" of the family $\{u_\alpha\}$.

Now let $H_{h,r}$ denote the subset of H_h given by

$$H_{h,r} = \{u \in H_h : |u| \ \text{ is a radially symmetric function}\}.$$

By *iv*) of lemma 2.1 it follows that $H_{h,r}$ is weakly closed in H_h. We next prove an easy useful radial lemma concerning a compactness property of the functions in $H_{h,r}$.

Theorem 2.2 *For every $h > 0$ $H_{h,r}$ is relatively compact in $L^s(\mathbb{R}^N, \mathbb{R}^{N+1})$ for every $s \in (2, +\infty]$.*

Proof. Let $\{u_n\} \subset H_{h,r}$ be a sequence such that $\|u_n\|_{H_h}$ is bounded. Then, up to a subsequence, it is

$$u_n \rightharpoonup u \quad \text{weakly in} \quad H_h$$

for some $u \in H_{h,r}$. Fix $\gamma \in (0, 1)$ arbitrarily and consider $R > 0$ such that, according to (2.3),

$$\forall x \in \mathbb{R}^N \setminus \overline{B}_R(0) : \ \ |u(x)| < \frac{\gamma}{2}.$$

The objective is to prove that, for n sufficiently large,

$$\forall x \in \mathbb{R}^N \setminus \overline{B}_R(0): \quad |u_n(x)| < \frac{\gamma}{2}. \tag{2.4}$$

Assume by the contrary that, up to a subsequence,

$$\forall n \in \mathbb{N} \;\; \exists x_n \in \mathbb{R}^N \setminus \overline{B}_R(0) \;\text{ s.t. }\; |u_n(x_n)| \geq \frac{\gamma}{2}.$$

First we'll prove that the sequence $\{x_n\}$ is bounded in $\mathbb{R}^N$. Indeed, arguing by contradiction, we suppose that, up to a subsequence,

$$|x_n| \to +\infty \quad \text{as} \quad n \to +\infty\,. \tag{2.5}$$

By (2.2), since the sequence $\{\|\nabla u_n\|_p\}$ is bounded, there exists $\delta > 0$ such that

$$\forall\, n \in \mathbb{N}, \quad \forall\, x,\, y \in \mathbb{R}^N: \quad |x - y| < \delta \quad \Rightarrow \quad |u_n(x) - u_n(y)| < \frac{\gamma}{4}. \tag{2.6}$$

¿From (2.6) we immediately get

$$B_\delta(x_n) \subset \left\{x \in \mathbb{R}^N : |u_n(x)| > \frac{\gamma}{4}\right\} \quad \forall n \in \mathbb{N}.$$

Because of (2.5) we may assume $|x_n| - \delta > 0$ for every $n \in \mathbb{N}$. Now, taking into account that $\{|u_n|\}$ is a sequence of radial functions, putting $C_n \equiv C(|x_n| - \delta, |x_n| + \delta)$ (see the notations at the end of the introduction), it holds

$$|u_n(x)| > \frac{\gamma}{4} \quad \forall x \in C_n \quad \forall n \in \mathbb{N}.$$

Then we can write

$$\int_{\mathbb{R}^N} V_h(x)|u_n|^2\,dx \geq V_0 \int_{C_n} |u_n|^2\,dx \geq V_0 \frac{\gamma^2}{16}\,\mathrm{meas}(C_n) \to +\infty \quad \text{as} \quad n \to +\infty,$$

but this contradicts the fact that $\{u_n\}$ is bounded in H_h. Therefore, the sequence $\{x_n\}$ turns out to be bounded in $\mathbb{R}^N$; up to a subsequence we obtain

$$x_n \to x \in \mathbb{R}^N \quad \text{as} \quad n \to +\infty.$$

Now we have

$$|u_n(x_n) - u(x)| \leq |u_n(x_n) - u_n(x)| + |u_n(x) - u(x)|.$$

For n large enough both terms on the right side are arbitrary small, the first because of the equi-uniform continuity of $\{u_n\}$ and the second because of *iv)* of lemma 2.1. This implies $u_n(x_n) \to u(x)$ as $n \to +\infty$ and then $|u(x)| \geq \frac{\gamma}{2}$. The choice of γ yields $|x| < R$, which is a contradiction since $|x_n| \geq \overline{R}$ for every $n \in \mathbb{N}$. So (2.4) holds.

Now combine (2.4) with iv) of lemma 2.1: what we deduce is the existence of a subsequence $\{u_n^1\}$ such that

$$|u_n^1(x) - u(x)| < \gamma \quad \forall x \in \mathbb{R}^N, \quad \forall n \in \mathbb{N}.$$

Now we apply a diagonal method and construct a subsequence $\{\widehat{u}_n\}$ of $\{u_n\}$ such that

$$|\widehat{u}_n(x) - u(x)| < \gamma^n \quad \forall x \in \mathbb{R}^N, \quad \forall n \in \mathbb{N}$$

which implies that $\{\widehat{u}_n\}$ converges uniformly to u in $\mathbb{R}^N$; then we have proved that $H_{h,r}$ is relatively compact in $L^\infty(\mathbb{R}^N, \mathbb{R}^{N+1})$.

Our thesis will follow by showing that $H_{h,r}$ is relatively compact in $L^s(\mathbb{R}^N, \mathbb{R}^{N+1})$ for every $s \in (2, 2^*)$, where $2^* = \frac{2N}{N-2}$; the proof of this fact can be found in [6]. $\diamondsuit$ Since the functions in H_h are continuous, we can consider the set

$$\Lambda_h = \left\{u \in H_h : \forall\, x \in \mathbb{R}^N : u(x) \neq \overline{\xi} \equiv (1,0)\right\}.$$

By ii) and iii) of Lemma 2.1, it is easy to obtain that Λ_h is open in H_h. The boundary of Λ_h is given by $\partial\Lambda_h = \left\{u \in H_h : \exists\, \overline{x} \in \mathbb{R}^N : u(\overline{x}) = (1,0)\right\}$.

Following the notation introduced in (1.6), in the space H_h we can consider the following $O(N)$-action: for every $u \in H_h$ and $g \in O(N)$:

$$T_g u(x) = (u_0(gx), g^{-1}u_1(gx)). \tag{2.7}$$

Now for every $h > 0$ let F_h denotes the subspace of fixed points:

$$F_h = \{u \in H_h : T_g(u) = u \ \ \forall g \in O(N)\}.$$

An easy computation shows that $F_h \subset H_{h,r}$. Furthermore F_h is weakly closed in H_h, i.e. for every $\{u_n\} \subset F_h$ and $u \in H_h$ it is

$$u_n \rightharpoonup u \ \text{ weakly in } \ H_h \Rightarrow u \in F_h. \tag{2.8}$$

Then define the set $\Lambda_F(h) = F_h \cap \Lambda_h$. Part i) of lemma 2.1 assures that for every $h > 0$ the space H_h is continuously embedded in $H \equiv W^{1,2}(\mathbb{R}^N, \mathbb{R}^{N+1}) \cap W^{1,p}(\mathbb{R}^N, \mathbb{R}^{N+1})$; setting $\Lambda = \{u \in H : u(x) \neq \overline{\xi} \equiv (1,0) \ \ \forall x \in \mathbb{R}^N\}$, in the course of the paper we will often make use of the following two sets

$$F = \{u \in H : T_g(u) = u \ \ \forall g \in O(N)\}, \quad \Lambda_F = F \cap \Lambda.$$

Now we want to give a topological classification of the maps $u \in \Lambda$. More precisely we introduce a topological invariant with suitable "localization" properties in the sense that, roughly speaking, it depends on the compact region where u is concentrated. This invariant consists of an integer number called "topological charge" and it will be defined by means of the topological degree.

3 Topological Charge

In this section we take from [6]some crucial results. With the help of the notation introduced in (1.6), we give the following definition.

Definition Given $u = (u_0, u_1) \in \Lambda$ and $U \subset \mathbb{R}^N$ an open set such that $u_1(x) \neq 0$ if $x \in \partial U$, then we define the (topological) charge of u in U as the following integer number

$$ch(u, U) = \deg\left(u_1,\, K(u) \cap U,\, 0\right)$$

where the open set $K(u)$ is defined as follows:

$$K(u) = \left\{x \in \mathbb{R}^N : u_0(x) > 1\right\}.$$

We recall the convention $\deg\left(u_1,\, \emptyset,\, 0\right) = 0$. Furthermore given $u \in \Lambda$ we define the (topological) charge of u as the integer number

$$ch(u) = \deg\left(u_1,\, K(u),\, 0\right).$$

We note that the topological charge is well defined thanks to (2.3) and the definition of Λ. ¿From well known properties of the topological degree we get other useful properties of the topological charge. For example notice that if $U \subset \mathbb{R}^N$ is open and such that $0 \notin u_1(\partial U)$ and if U consists of m connected components $U_1, \dots, U_m$, i.e. u has the energy concentrated in different regions of the space, then by the additivity property of the degree we get

$$ch(u, U) = \sum_{j=1}^{m} ch\left(u,\, U_j\right). \tag{3.1}$$

The next lemma shows how the topological charge exhibits a sort of invariance under a class of transformations in Λ.

Theorem 3.1 *Let $w \in \Lambda$ compactly supported with $w \equiv 0$ in a neighbourhood of the origin and for $r > 0$ arbitrarily fixed define $\widetilde{w} : \mathbb{R}^N \to \mathbb{R}^{N+1}$ by setting*

$$\widetilde{w}(x) = \begin{cases} 0 & \textit{if } |x| \leq r, \\ w\left(x - r\frac{x}{|x|}\right) & \textit{if } |x| > r. \end{cases}.$$

Then there results: $ch(w) = ch(\widetilde{w})$.

Proof. First observe that by definition $\widetilde{w}$ belongs to Λ too. It is immediate to show that if $K(w) = \emptyset$ it follows $K(\widetilde{w}) = \emptyset$ and then $ch(w) = ch(\widetilde{w}) = 0$. Now assume $K(w) \neq \emptyset$; it makes sense to choose $m \geq 1$ and $0 < R_1 < R_2 < \cdots < R_m$ such that

$$w_0(x) > 1 \quad \forall x \in \cup_{i=1}^{m-1} C(R_i, R_{i+1}),$$
$$w_0(x) \leq 1 \quad \forall x \in \mathbb{R}^N \setminus \cup_{i=1}^{m-1} C(R_i, R_{i+1}).$$

Remember that for $a < b$ $C(a, b)$ is the ring centered at 0 and with internal radius a and external radius b. In other words the sets $C(R_i, R_{i+1})$ represent the connected components of $\{x \in \mathbb{R}^N : w_0(x) > 1\}$. By construction we infer

$$\widetilde{w}_0(x) > 1 \quad \forall x \in \cup_{i=1}^{m} C\left(R_i + r, R_{i+1} + r\right).$$

To simplify notation, set

$$C_i = C(R_i, R_{i+1}), \quad \widetilde{C}_i = C\left(R_i + r, R_{i+1} + r\right) \quad \text{for } i = 1, \dots, m-1.$$

According to (3.1),

$$ch(w) = \sum_{i=1}^{m-1} \deg(w_1, C_i, 0), \quad ch(\widetilde{w}) = \sum_{i=1}^{m-1} \deg(\widetilde{w}_1, \widetilde{C}_i, 0).$$

Hence, in order to conclude, it will be sufficient to show that

$$\deg(w_1, C_i, 0) = \deg(\widetilde{w}_1, \widetilde{C}_i, 0) \quad \forall i = 1, \dots, m-1.$$

Take for example $i = 1$ and consider the functions G, $\widetilde{G} : C\left(R_1, r + R_2\right) \to \mathbb{R}^{N+1}$ defined by setting

$$G(x) = \begin{cases} w_1(x) & \text{if } R_1 \le |x| \le R_2, \\ w_1\left(\frac{x}{|x|} R_2\right) & \text{if } R_2 \le |x| \le r + R_2. \end{cases},$$

$$\widetilde{G}(x) = \begin{cases} \widetilde{w}_1(x) & \text{if } r + R_1 \le |x| \le r + R_2, \\ \widetilde{w}_1\left(\frac{x}{|x|}(r + R_1)\right) & \text{if } R_1 \le |x| \le r + R_1. \end{cases}$$

It is obvious that G and $\widetilde{G}$ are continuous and moreover $G(x) = \widetilde{G}(x)$ for $x \in \partial C\left(R_1, r + R_2\right)$. Furthermore, since by construction $w_0(x) = 1$ for $|x| = R_2$ and $\widetilde{w}_0(x) = 1$ for $|x| = r + R_1$, the definition of Λ implies $w_1(x) \neq 0$ for $|x| = R_2$ and $\widetilde{w}_1(x) \neq 0$ for $|x| = r + R_1$ and then

$$G(x) \neq 0 \quad \forall R_2 \le |x| \le r + R_2, \quad \widetilde{G}(x) \neq 0 \quad \forall R_1 \le |x| \le r + R_1.$$

Then, because of the excision property and the homotopy invariance of the topological degree, we conclude

$$\begin{aligned} \deg(w_1, C_1, 0) &= \deg\left(G, C\left(R_1, r + R_2\right), 0\right) \\ &= \deg\left(\widetilde{G}, C\left(R_1, r + R_2\right), 0\right) \\ &= \deg(\widetilde{w}_1, \widetilde{C}_1, 0), \end{aligned}$$

and hence the proof is complete. $\diamondsuit$

Another consequence is that the topological charge is stable under uniform convergence as the following lemma states.

Theorem 3.2 *Let $\{u_n\} \subset \Lambda$, $u \in \Lambda$ and $U \subset \mathbb{R}^N$ an open set such that $u_n \to u$ uniformly in U and*

$$(u_n)_1(x) \neq 0, \quad u_1(x) \neq 0 \quad \forall x \in \partial U, \ \forall n \in \mathbb{N}.$$

Then, for n large enough, $ch(u, U) = ch(u_n, U)$.

Corollary 3.3 *For every $u \in \Lambda_h$ there exists $\varrho = \varrho(u) > 0$ such that, for every $v \in \Lambda_h$,*

$$\|u - v\|_{L^\infty} \le \varrho \Rightarrow ch(u) = ch(v).$$

By definition we easily deduce that for every $u \in \Lambda$ and $U \subset \mathbb{R}^N$ an open set with $u_1(x) \neq 0$ for $x \in \partial U$,

$$ch(u, U) \neq 0 \Rightarrow \|u\|_{L^\infty(U)} > 1. \tag{3.2}$$

Finally we define the set

$$\Lambda_F^*(h) = \{u \in \Lambda_F(h) : ch(u) \neq 0\}.$$

In [6] the authors proved the existence a function in Λ_F with $ch(u) \neq 0$. Now multiply such u for a cut-off radially symmetric function $\tau : \mathbb{R}^N \to \mathbb{R}$ verifying

$$\tau \in C^\infty(\mathbb{R}^N, \mathbb{R}), \quad \tau = 1 \text{ in } B_R(0), \quad \tau = 0 \text{ in } \mathbb{R}^N \setminus B_{R+1}(0), \quad 0 \le \tau \le 1$$

with R sufficiently large such that $K(u) \subset B_R(0)$; we immediately obtain that $\tau u \in \Lambda_F^*(h)$ for all $h > 0$. In particular we get $\Lambda_F^*(h) \neq \emptyset$ for every $h > 0$. Our aim is to find critical points of the functional E_h defined by (1.5) in the class $\Lambda_F^*(h)$ to obtain the existence of solutions for (1.2) in the set of the fields u with nontrivial charge. This is what we'll do in the next sections.

4 The Energy Functional

Now we are going to study the properties of the functional E_h; first recall that under the hypotheses a), b) and c) in section 4 of [2] the authors proved that E_h is well defined in the space Λ_h, i.e. for every $u \in \Lambda_h$ we have $E_h(u) < +\infty$.

Obviously E_h is bounded from below and is coercive in the H_h-norm:

$$\lim_{\|u\|_{H_h} \to +\infty} E_h(u) = +\infty\,. \tag{4.1}$$

Moreover in [2] it is proved that the energy functional E_h belongs to the class $C^1(\Lambda_h, \mathbb{R})$ under the assumptions a)-d). An immediate corollary is that the critical points $u \in \Lambda_h$ (particularly the minima) for the functional E_h are weak solutions of equation (1.2).

The next two propositions deal with some other properties of the functional E_h; we omit the proofs because they are the same as in [11], provided that we substitute E_h for E and Λ_h for Λ. The first deals with the behaviour of E_h when u approaches the boundary of Λ_h.

Proposition 4.1 ([11], Lemma 3.7, p. 326) *Let $\{u_n\} \subset \Lambda_h$ be bounded in the H_h-norm and weakly converging to $u \in \partial\Lambda_h$, then*

$$\int_{\mathbb{R}^N} W(u_n)\, dx \to +\infty \quad as \quad n \to +\infty.$$

As a consequence, if $\{u_n\} \subset \Lambda_h$ is weakly converging to u and such that $E_h(u_n)$ is bounded, then $u \in \Lambda_h$.

The second proposition states the weakly lower semi-continuity of the energy functional E_h.

Proposition 4.2 ([11], Prop. 3.10, p. 328) *For every $u \in \Lambda_h$ and for every sequence $\{u_n\} \subset \Lambda_h$, if $\{u_n\}$ weakly converges to u, then*

$$\liminf_{n\to+\infty} E_h(u_n) \geq E_h(u).$$

Taking into account of the radial assumptions a) and e), an easy calculation shows that the open set Λ_h and the functional E_h are invariant under the action (2.7), i.e. for every $u \in \Lambda_h$ and $g \in O(N)$,

$$T_g(u) \in \Lambda_h, \quad E_h(T_g(u)) = E_h(u).$$

This fact suggests the idea that the set $\Lambda_F(h)$ is a natural constrain to obtain critical points of E_h: more precisely the following important result holds:

Theorem 4.3 *Let $f : \Lambda_h \to \mathbb{R}$ a C^1 functional verifying $f(T_g(u)) = f(u)$ for every $u \in \Lambda_h$. Then for every $u \in \Lambda_h$ and $v \in H_h$ it is*

$$\langle f'(u), v\rangle = \langle f'(u), P_h v\rangle,$$

where P_h is the projection of H_h onto F_h. As a corollary, if $u \in \Lambda_F(h)$ is such that, for any $v \in F_h$, $\langle f'(u), v\rangle = 0$, then $f'(u) = 0$, i.e. u is a critical point for f.

The proof can be found in [6].

By applying lemma 4.3 to our functional E_h we deduce that every local minimum point of E_h in $\Lambda_F(h)$ is also a critical point of E_h. This fact suggests a useful technique to obtain solutions for equation (1.2), i.e. that of minimizing the functional E_h in some open subset of $\Lambda_F(h)$; this is exactly what we'll do in the next sections to achieve our existence and concentration results.

5 First Existence Result: Concentration Near the Origin

In this section we will show the existence of a family of solutions $\{v_h\}$ to equation (1.1) which concentrate near the origin as $h \to 0^+$. First observe that $\Lambda^*_F(h)$ is open in $\Lambda_F(h)$: indeed $\Lambda^*_F(h) = \Lambda_F(h) \cap \{u \in \Lambda_h : ch(u) \neq 0\}$, and the set $\{u \in \Lambda_h : ch(u) \neq 0\}$ is open in Λ_h because of *ii*) of lemma 2.1 and corollary 3.1. Then, according to last section, our solutions v_h will be obtained by minimizing the functional E_h in the set $\Lambda^*_F(h)$. To this aim put

$$E^*_h = \inf_{u\in\Lambda^*_F(h)} E_h(u).$$

In what follows we will consider the functional E_0 defined on Λ by setting

$$E_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(0)|u|^2\right)\, dx + \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p\, dx + \int_{\mathbb{R}^N} W(u)\, dx.$$

E_0 is the functional associated with equation (1.2) replacing V_h by $V(0)$. Now we are able to provide the first existence and concentration result of this paper.

Theorem 5.1 *Assume that a)–e) hold. Then for every $h>0$ the minimum E_h^* is attained in the set $\Lambda_F^*(h)$. Furthermore if we put*

$$v_h(x) = u_h\left(\frac{x}{h}\right) \tag{5.1}$$

where $u_h \in \Lambda_F^*(h)$ is the minimizing function for E_h^*, then v_h is a solution of (1.1) and there exist at least a points $x_h \in \mathbb{R}^N$ such that $|v_h(x_h)| > 1$. Finally the family $\{v_h\}$ concentrates at the origin in the following sense: for every $\delta > 0$, it holds:

$$v_h \to 0 \quad as \quad h \to 0^+ \quad uniformly\ \ in\ \ the\ \ set \quad \{|x| \geq \delta\}.$$

In particular this implies $x_h \to 0$ as $h \to 0^+$.

Proof. Fix $h>0$ and consider $\{u_n^h\}$ a minimizing sequence in $\Lambda_F^*(h)$. Remember that $\Lambda_F^*(h) \neq \emptyset$ for every $h>0$. Applying lemma 2.2 we have, up to a subsequence:

$$u_n^h \rightharpoonup u_h \quad \text{weakly in} \quad H_h \quad \text{as} \quad n \to +\infty$$

$$u_n^h \to u_h \quad \text{uniformly in} \quad \mathbb{R}^N \quad \text{as} \quad n \to +\infty,$$

for some $u_h \in F_h$. Proposition 4.2 implies $u_h \in \Lambda_F(h)$. The continuity of the topological degree with respect to the uniform convergence implies that the sequence $\{ch(u_n^h)\}$ is constant for large n's, and then

$$ch(u_h) = \lim_{n\to+\infty} ch(u_n^h) \neq 0,$$

hence $u_h \in \Lambda_F^*(h)$. By the weakly lower semi-continuity of the energy functional E_h,

$$E_h^* \leq E_h(u_h) \leq \liminf_{n\to+\infty} E_h(u_n^h) = E_h^*,$$

i.e. $E_h(u_h) = E_h^*$. Hence we have proved that each u_h is a minimum point of E_h in $\Lambda_F^*(h)$ and, by applying lemma 4.3, it is a critical point of E_h and hence a solution to equation (1.2); by rescaling, we immediately obtain that the family $\{v_h\}$ related to $\{u_h\}$ by (5.1) provides solutions to equation (1.1). We recall that for every $x \in K(u_h)$ with $(u_h)_1(x) = 0$ it is $|u_h(x)| > 1$; then the definition of $\Lambda_F^*(h)$ implies the existence of at least a point x_h verifying $|v_h(x_h)| > 1$. It remains to prove that $\{v_h\}$ satisfies the concentration property announced in the theorem. To this aim we first establish some important facts in the following two steps.

Step 1. $\limsup_{h\to 0^+} E_h^* < +\infty$.

Take $u \in \Lambda_F$ compactly supported such that $ch(u) \neq 0$. In the last section we have pointed out that such a function exists. Furthermore we obviously have $u \in \Lambda_F^*(h)$ for all $h > 0$. Now observe:

$$E_h^* \leq E_h(u) = E_0(u) + \frac{1}{2}\int_{\mathbb{R}^N} (V_h(x) - V(0))\,|u|^2\,dx. \tag{5.2}$$

Since V_h converges to $V(0)$ uniformly on compact sets as $h \to 0^+$, then, if we take h sufficiently small, the integral in (5.2) is close to zero. This implies

$$\limsup_{h\to 0^+} E_h^* \leq E_0(u)$$

and the conclusion follows.

Step 2. Let $h_n \to 0^+$ an arbitrary sequence. Then for every $\gamma > 0$ there exists $R_\gamma > 0$ such that for n sufficiently large:

$$|u_{h_n}(x)| < \gamma \quad \forall x \in \mathbb{R}^N \setminus B_{R_\gamma}(0).$$

Fix $\gamma > 0$. By Step 1 E_h^* is bounded for $h > 0$ small. In particular, according to (2.2), the sequence $\{u_{h_n}\}$ is equi-uniformly continuous; then there exists $\delta > 0$ such that

$$\forall\, n \in \mathbb{N}, \quad \forall\, x,\, y \in \mathbb{R}^N: \quad |x-y| < \delta \quad \Rightarrow \quad |u_{h_n}(x) - u_{h_n}(y)| < \frac{\gamma}{2}.$$

Assume by absurd that, passing to a subsequence, we can find a sequence $\{z_{h_n}\} \subset \mathbb{R}^N$ such that

$$|u_{h_n}(z_{h_n})| \geq \gamma, \quad |z_{h_n}| \to +\infty \quad \text{as} \quad n \to +\infty;$$

we immediately get

$$B_\delta(z_{h_n}) \subset \left\{x \in \mathbb{R}^N : |u_{h_n}(x)| > \frac{\gamma}{2}\right\} \quad \forall n \in \mathbb{N}.$$

Without loss of generality we may assume $|z_{h_n}| - \delta > 0$ for every $n \in \mathbb{N}$. Now, taking into account that $\{|u_{h_n}|\}$ is a sequence of radial functions, putting $C_{h_n} \equiv C(|z_{h_n}| - \delta, |z_{h_n}| + \delta)$ (see the notations at the end of the introduction), it holds

$$|u_{h_n}(x)| > \frac{\gamma}{2} \quad \forall x \in C_{h_n} \quad \forall n \in \mathbb{N}.$$

Then we can write

$$\int_{\mathbb{R}^N} V_{h_n}(x)|u_{h_n}|^2\,dx \geq V_0 \int_{C_{h_n}} |u_{h_n}|^2\,dx \geq \frac{\gamma^2}{4}\,\text{meas}(C_{h_n}) \to +\infty$$

as $n \to +\infty$, but this contradicts the fact that $\{E_h^*\}$ is bounded for small $h > 0$. The proof of Step 2 is complete.

¿From Step 2 it directly follows that, taken an arbitrary sequence $h_n \to 0^+$, $\{u_{h_n}\}$ has uniform behaviour at infinity, i.e. $\lim_{|x|\to+\infty} |u_{h_n}(x)| = 0$ uniformly with respect to n. By re-scaling it is easy to prove that $\{v_{h_n}\}$ decays uniformly to zero for x outside every fixed neighbourhood of the origin as $n \to +\infty$. Finally the fact that h_n is arbitrary allows us to conclude. $\diamondsuit$

6 Second Existence Result: Preliminaries

This section is devoted to establish some preliminary results concerning the study of the behaviour of sequences $\{u_{h_n}\}$, with $u_{h_n} \in \Lambda_F(h_n)$, whose energy grows like $1/h_n^{N-1}$. First we put

$$\mathcal{V}(|x|) = V(x) \quad \forall x \in \mathbb{R}^N. \tag{6.1}$$

We notice that by hypothesis V is a radially symmetric function, so the above definition is well posed. Toward our aims the following lemma constitutes a crucial step; it uses in an essential way the radial properties of the functions in $\Lambda_F(h)$.

Theorem 6.1 *Let $h_n \to 0^+$ an arbitrary sequence and $u_{h_n} \in \Lambda_F(h_n)$ such that*

$$\limsup_{n\to+\infty} h_n^{N-1} \int_{\mathbb{R}^N} \left(|\nabla u_{h_n}|^2 + |u_{h_n}|^2\right) dx < +\infty.$$

For every $n \in \mathbb{N}$ and $x \in \mathbb{R}^N$ set $\varphi_n(|x|) \equiv |u_{h_n}(x)|$. Then for all $t > 0$ it holds

$$\limsup_{n\to+\infty} \int_{t/(2h_n)}^{+\infty} \left(|\varphi_n'|^2 + |\varphi_n|^2\right) dr < +\infty\,. \tag{6.2}$$

Furthermore the sequence $\{\varphi_n\}$ is equi-uniformly continuous in $\left[\frac{t}{2h_n}, +\infty\right)$ in the following sense: for every $\eta > 0$ there exists $\delta > 0$ such that :

$$\forall n \in \mathbb{N}, \quad \forall s, s' \in \left[\frac{t}{2h_n}, +\infty\right) : \quad |s - s'| \le \delta \Rightarrow |\varphi_n(s) - \varphi_n(s')| \le \eta\,.$$

Proof. An immediate computation leads to

$$\begin{aligned}
&\limsup_{n\to+\infty} h_n^{N-1} \int_{\mathbb{R}^N} \left(|\nabla u_{h_n}|^2 + |u_{h_n}|^2\right) dx \\
&\ge \limsup_{n\to+\infty} h_n^{N-1} \int_{\mathbb{R}^N} \left(|\nabla |u_{h_n}||^2 + |u_{h_n}|^2\right) dx \\
&= \limsup_{n\to+\infty} \omega_N \int_0^{+\infty} (h_n r)^{N-1} \left(|\varphi_n'|^2 + |\varphi_n|^2\right) dr \qquad (6.3)\\
&\ge \omega_N \limsup_{n\to+\infty} \int_{\frac{t}{2h_n}}^{+\infty} (h_n r)^{N-1} \left(|\varphi_n'|^2 + |\varphi_n|^2\right) dx \\
&\ge \omega_N \frac{t^{N-1}}{2^{N-1}} \limsup_{n\to+\infty} \int_{\frac{t}{2h_n}}^{+\infty} \left(|\varphi_n'|^2 + |\varphi_n|^2\right) dx.
\end{aligned}$$

It is obvious that each φ_n belongs to the class $W^{1,2}\left(\left(\frac{t}{2h_n}, +\infty\right), \mathbb{R}\right)$. Then the theory of the one-dimensional Sobolev spaces leads to

$$|\varphi_n(s) - \varphi_n(s')| \le \left(\int_{t/(2h_n)}^{+\infty} |\varphi_n'(r)|^2\, dr\right)^{1/2} |s - s'|^{1/2} \quad \forall s, s' \in \left(\frac{t}{2h_n}, +\infty\right). \tag{6.4}$$

Using (6.3) it immediately follows that the sequence $\int_{t/(2h_n)}^{+\infty} |\varphi_n'|^2\, dr$ is bounded, hence from (6.4) we easily deduce the equi-uniform continuity of $\{\varphi_n\}$. $\diamondsuit$

A second preliminary result we will need is given in the following lemma inspired by the "splitting lemma" of [11] in the spirit of the concentration compactness principle of [20].

Theorem 6.2 *Let $h_n \to 0^+$ an arbitrary sequence and $u_{h_n} \in \Lambda_F(h_n)$ such that*

$$\limsup_{n\to+\infty} h_n^{N-1} \int_{\mathbb{R}^N} \left(|\nabla u_{h_n}|^2 + |u_{h_n}|^2\right)\, dx < +\infty, \tag{6.5}$$

$$\sup_{|x|\geq \frac{t}{h_n}} |u_{h_n}(x)| > \frac{\varepsilon}{2} \quad \forall n \in \mathbb{N} \tag{6.6}$$

and

$$|u_{h_n}(x)| \leq \frac{\varepsilon}{2} \quad \forall x \in C\left(\frac{t}{2h_n}, \frac{t}{h_n}\right) \quad \forall n \in \mathbb{N}, \tag{6.7}$$

for some $t, \varepsilon > 0$. Then, setting $\varphi_{h_n}(|x|) \equiv |u_{h_n}(x)|$, there exist $\ell \in \mathbb{N}$, $R_1, \ldots, R_\ell > 0$ and ℓ sequences of positive numbers $\{r_{h_n}^1\}, \ldots, \{r_{h_n}^\ell\}$, with $r_{h_n}^i > \frac{t}{h_n}$, such that, up to subsequence

$$\{h_n r_{h_n}^i\} \quad \textit{is bounded for } i = 1, \ldots, \ell; \tag{6.8}$$

$$\varphi_{h_n}(r_{h_n}^i) > \frac{\varepsilon}{2} \quad \forall\, n \in \mathbb{N}, \quad \forall i = 1, \ldots, \ell; \tag{6.9}$$

$r_{h_n}^i$ is maximal point for φ_{h_n} in

$$\Big[\frac{t}{2h_n}, +\infty\Big) \setminus \cup_{j<i}(r_{h_n}^j - R_j, r_{h_n}^j + R_j); \tag{6.10}$$

$$\forall\, r \notin \left(\left[0, \frac{t}{2h_n}\right] \cup \cup_{i=1}^{\ell}(r_{h_n}^i - R_i, r_{h_n}^i + R_i)\right): \quad \varphi_{h_n}(r) \leq \frac{\varepsilon}{2}; \tag{6.11}$$

$$|r_{h_n}^i - r_{h_n}^j| \to +\infty \quad \textit{as} \quad n \to +\infty \quad \forall i, j \in \{1, \ldots, \ell\} \quad \textit{with} \quad i \neq j. \tag{6.12}$$

The proof is just slight variation of the that in [11] (see lemma 4.1). Last lemma represents a crucial step for the proof of the second existence result in section 8.

7 Construction of the Auxiliary Functional

The objective is now to define a suitable auxiliary functional which will prove very useful for the achievements of our results. Since this auxiliary functional will be defined through integral terms, in order to make its form easier we will use the change of variables in the space $\mathbb{R}^N$ from Cartesian into polar coordinates: more precisely we will consider the transformation $x \to (\rho, \theta_1, \ldots, \theta_{N-1})$ defined by:

$$x_i = \rho f_i(\theta_1, \ldots, \theta_{N-1}), \quad i = 1, \ldots, N,$$

where each f_i is a real C^1 function. Here we have set $\rho = |x|$, while by $\theta_1, \dots, \theta_{N-1}$ we denote the angles which locate through the functions f_i the position of $\frac{x}{|x|}$ (for $x \neq 0$) on the unit sphere S^{N-1} of $\mathbb{R}^N$. It is well known that the transformation above can be inverted in $\mathbb{R}^N \setminus \{0\}$ and hence we can write

$$\theta_i \equiv \psi_i(x) \quad \forall x \neq 0, \quad \psi_i \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}) \quad \forall i = 1, \dots, N-1.$$

It is obvious that each ψ_i is constant on the rays $\{\varrho x : \varrho > 0\}$ for a fixed $x \neq 0$. For sake of simplicity we will use the following abbreviations:

$$\theta \equiv (\theta_1, \dots, \theta_{N-1}),\ \psi(x) \equiv (\psi_1(x), \dots, \psi_{N-1}(x)),\ f(\theta) \equiv (f_1(\theta), \dots, f_N(\theta)).$$

For every $u \in \Lambda_F$ put

$$\Phi_u(\rho, \theta) \equiv u(\rho f(\theta)) \tag{7.1}$$

and define

$$\frac{\partial u}{\partial \rho}(x) \equiv \frac{\partial \Phi_u}{\partial \rho}(|x|, \psi(x)) \quad \forall x \neq 0.$$

Next consider the set

$$\Upsilon_F \equiv \{u \in \Lambda_F : u \equiv 0 \text{ in a neighbourhood of } 0,\ \ u \text{ compactly supported}\};$$

then for $r > 0$ and $u \in \Upsilon_F$ the auxiliary functional $\mathcal{E}_r$ is defined as

$$\mathcal{E}_r(u) = \int_{\mathbb{R}^N} \frac{1}{|x|^{N-1}} \Big(\frac{1}{2} \Big(\mathcal{V}(r)|u|^2 + \Big|\frac{\partial u}{\partial \rho}\Big|^2 \Big) + \frac{1}{p} \Big|\frac{\partial u}{\partial \rho}\Big|^p \, dx + W(u) \Big) \, dx$$

where $\mathcal{V}$ has been defined in (6.1). We notice that $\mathcal{E}_r$ is well defined in Υ_F. Using the Cauchy-Schwartz inequality and the relation $\sum_{i=1}^N |f_i(\psi(x))|^2 \equiv 1$ for all $x \neq 0$, we obtain

$$\Big|\frac{\partial u}{\partial \rho}(x)\Big| = \Big|\frac{\partial \Phi_u}{\partial \rho}(|x|, \psi(x))\Big| = \Big|\sum_{i=1}^N \frac{\partial u}{\partial x_i}(x) f_i(\psi(x))\Big| \leq |\nabla u(x)| \quad \text{a.e. in } \mathbb{R}^N.$$

Hence, taking $\gamma = 2, p$, we obtain that for $u \in \Lambda_F$

$$\int_{\mathbb{R}^N} \frac{1}{|x|^{N-1}} \Big|\frac{\partial u}{\partial \rho}\Big|^\gamma \equiv \int_{\mathbb{R}^N} \frac{1}{|x|^{N-1}} \Big|\frac{\partial \Phi_u}{\partial \rho}(|x|, \psi(x))\Big|^\gamma \leq \int_{\mathbb{R}^N} \frac{1}{|x|^{N-1}} |\nabla u|^\gamma \, dx. \tag{7.2}$$

To study the properties of the functional $\mathcal{E}_r$ we will need the elementary results given by the following two lemmas.

Lemma 7.1 *Let $r, R, \tau > 0$ with $r - R > 0$ and $f \in L^1(C(r-R, r+R), \mathbb{R})$. Then there results*

$$\begin{aligned} &\int_{C(\tau, \tau+2R)} f\Big(x + (r - R - \tau)\frac{x}{|x|}\Big) \, dx \\ &= \int_{C(r-R, r+R)} \Big(\frac{|x| - r + R + \tau}{|x|}\Big)^{N-1} f(x) \, dx \end{aligned}$$

and

$$\begin{aligned}&\int_{C(\tau,\tau+2R)}\Big(\frac{|x|+r-R-\tau}{|x|}\Big)^{N-1} f\big(x+(r-R-\tau)\frac{x}{|x|}\big)\,dx\\ &= \int_{C(r-R,r+R)} f(x)\,dx\,.\end{aligned}$$

Theorem 7.2 *Let $A\subset\mathbb{R}^N$ a measurable set, $\{f_n\}\subset C(A,\mathbb{R})$ and $g_n : A\to\mathbb{R}$ two sequences of functions such that g_n measurable, $g_n(x)\ge 0$ a.e. in $\mathbb{R}^N$, and $f_n\to f$ uniformly in A as $n\to+\infty$ for some $f\in C(A,\mathbb{R})$ with $\inf_A|f(x)| = \nu>0$. Then*

$$\limsup_{n\to+\infty}\int_A f_n g_n\,dx = \limsup_{n\to+\infty}\int_A f\,g_n\,dx,$$
$$\liminf_{n\to+\infty}\int_A f_n g_n\,dx = \liminf_{n\to+\infty}\int_A f\,g_n\,dx.$$

The usefulness of having introduced the auxiliary functional $\mathcal{E}_r$ will be clear in the next lemma which establish a relation between the behaviour of the two functional E_h and $\mathcal{E}_r$ with respect to suitable sequences of functions in the set Υ_F.

Theorem 7.3 *The following two statements hold:*

1. *Let $h_n\to 0$ be an arbitrary sequence and consider $\{u_{h_n}\}\subset\Upsilon_F$ and $\{r_{h_n}\}\subset(0,+\infty)$ such that*

$$u_{h_n}\equiv 0 \quad in \quad \mathbb{R}^N\setminus C(r_{h_n}-R, r_{h_n}+R) \tag{7.3}$$

and

$$h_n r_{h_n}\to r \quad as \quad n\to+\infty \tag{7.4}$$

for some $R,r>0$. For $n\in\mathbb{N}$ define $w_{h_n}(x)\equiv u_{h_n}\Big(x+(r_{h_n}-2R)\frac{x}{|x|}\Big)$, in $C(R,3R)$ and $w_{h_n}\equiv 0$ everywhere else. Then there results

$$\liminf_{n\to+\infty} h_n^{N-1}E_{h_n}(u_{h_n})\ge r^{N-1}\liminf_{n\to+\infty}\mathcal{E}_r(w_{h_n}).$$

2. *Let $w\in\Upsilon_F$ and $r>0$ and for every $h>0$ set $v_h(x)\equiv w\Big(x-\frac{r}{h}\frac{x}{|x|}\Big)$ for $|x|>\frac{r}{h}$ and $v_h\equiv 0$ everywhere else. Then it is*

$$\limsup_{h\to 0^+} h^{N-1}E_h(v_h)\le r^{N-1}\mathcal{E}_r(w).$$

Proof. **Part 1.** To avoid triviality, assume $\liminf_{n\to+\infty} h_n^{N-1}E_{h_n}(u_{h_n}) < +\infty$. According to (7.4) it is $r_{h_n}\to+\infty$ as $n\to+\infty$ and then it makes sense to assume $r_{h_n}-R>0$ for every $n\in\mathbb{N}$. Then by lemma 7.1,

$$\begin{aligned}
&\liminf_{n\to+\infty}\int_{C(R,3R)}|w_{h_n}|^2\,dx\\
&= \liminf_{n\to+\infty}\int_{C(r_{h_n}-R,r_{h_n}+R)}\Big(\frac{|x|-r_{h_n}+2R}{|x|}\Big)^{N-1}|u_{h_n}|^2\,dx\\
&\le \liminf_{n\to+\infty} h_n^{N-1}\Big(\frac{3R}{h_n(r_{h_n}-R)}\Big)^{N-1}\int_{\mathbb{R}^N}|u_{h_n}|^2\,dx\\
&= \frac{2}{V_0}\big(\frac{3R}{r}\big)^{N-1}\liminf_{n\to+\infty} h_n^{N-1}E_{h_n}(u_{h_n})<+\infty\,.
\end{aligned}$$

Observe that each w_{h_n} belongs to Υ_F, so that $\mathcal{E}_r(w_{h_n})$ is well defined. Using lemma 7.1 again, there follows

$$\begin{aligned}
&\int_{C(r_{h_n}-R,r_{h_n}+R)}\Big(\frac{1}{2}\mathcal{V}_{h_n}(|x|)|u_{h_n}|^2+W(u_{h_n})\Big)\,dx\\
&= \int_{C(R,3R)}\Big(\frac{|x|+r_{h_n}-2R}{|x|}\Big)^{N-1}\\
&\qquad\times\Big(\frac{1}{2}\mathcal{V}_{h_n}(|x|+r_{h_n}-2R)|w_{h_n}|^2+W(w_{h_n})\Big)\,dx.
\end{aligned}\tag{7.5}$$

Using the notation in (7.1), we infer that $\Phi_{w_{h_n}}(|x|,\psi(x))=\Phi_{u_{h_n}}(|x|+r_{h_n}-2R,\psi(x))$ for all $x\neq0$. Then, by applying again lemma 7.1 and using (7.2), for $\gamma=2,p$, we can write

$$\begin{aligned}
&\int_{C(r_{h_n}-R,r_{h_n}+R)}\frac{1}{\gamma}|\nabla u_{h_n}|^\gamma\,dx\\
&\ge \int_{C(r_{h_n}-R,r_{h_n}+R)}\frac{1}{\gamma}\left|\frac{\partial u_{h_n}}{\partial\rho}\right|^\gamma\,dx\\
&= \int_{C(R,3R)}\frac{1}{\gamma}\left(\frac{|x|+r_{h_n}-2R}{|x|}\right)^{N-1}\left|\frac{\partial w_{h_n}}{\partial\rho}\right|^\gamma\,dx.
\end{aligned}\tag{7.6}$$

Note that we have used the equality $\psi\left(x+(r_{h_n}-2R)\frac{x}{|x|}\right)=\psi(x)$. Now combine (7.5) with (7.6) and obtain

$$\begin{aligned}
&\liminf_{n\to+\infty} h_n^{N-1}E_{h_n}(u_{h_n})\\
&\ge \liminf_{n\to+\infty}\Big[\int_{C(R,3R)}\big(\frac{h_n(|x|+r_{h_n}-2R)}{|x|}\big)^{N-1}\frac{1}{2}\big(\mathcal{V}(r)|w_{h_n}|^2+|\frac{\partial w_{h_n}}{\partial\rho}|^2\big)\,dx\\
&\quad+\int_{C(R,3R)}\big(\frac{h_n(|x|+r_{h_n}-2R)}{|x|}\big)^{N-1}\big(\frac{1}{p}|\frac{\partial w_{h_n}}{\partial\rho}|^p+W(w_{h_n})\big)\,dx\Big]\\
&\quad+\frac{1}{2}\liminf_{n\to+\infty}\int_{C(R,3R)}\big(\frac{h_n(|x|+r_{h_n}-2R)}{|x|}\big)^{N-1}
\end{aligned}$$

$$\times\big(\mathcal{V}_{h_n}(|x|+r_{h_n}-2R)-\mathcal{V}(r)\big)|w_{h_n}|^2\,dx\,.$$

It's immediate to prove that $\left(\frac{h_n(|x|+r_{h_n}-2R)}{|x|}\right)^{N-1}\to\frac{1}{|x|^{N-1}}r^{N-1}$ uniformly in $C(R,3R)$; hence lemma 7.2 applies in both integral and gives

$$\begin{aligned}&\liminf_{n\to+\infty} h_n^{N-1}E_{h_n}(u_{h_n})\\&\quad\geq\ r^{N-1}\liminf_{n\to+\infty}\mathcal{E}_r(w_{h_n})\\&\qquad+\frac{r^{N-1}}{2}\liminf_{n\to+\infty}\int_{C(R,3R)}\frac{1}{|x|^{N-1}}\left(\mathcal{V}_{h_n}(|x|+r_{h_n}-2R)-\mathcal{V}(r)\right)|w_{h_n}|^2\,dx\,.\end{aligned}$$

On the other hand $\mathcal{V}_{h_n}(|x|+r_{h_n}-2R)\to\mathcal{V}(r)$ uniformly in $C(R,3R)$, while in the first part of the proof we have obtained $\liminf_{n\to+\infty}\int_{C(R,3R)}|w_{h_n}|^2\,dx<+\infty$; then we easily deduce

$$\liminf_{n\to+\infty}\int_{C(R,3R)}\frac{1}{|x|^{N-1}}\left(\mathcal{V}_{h_n}(|x|+r_{h_n}-2R)-\mathcal{V}(r)\right)|w_{h_n}|^2\,dx=0$$

and the desired conclusion follows.

Part 2. Since w is compactly supported, for every $h>0$ it is $v_h\in\Lambda_F(h)$. Then we can write

$$E_h(v_h)=\frac{1}{2}\int_{\mathbb{R}^N}\left(\mathcal{V}_h(|x|)|v_h|^2+|\nabla v_h|^2\right)\,dx+\frac{1}{p}\int_{\mathbb{R}^N}|\nabla v_h|^p\,dx+\int_{\mathbb{R}^N}W(v_h)\,dx. \tag{7.7}$$

We begin by analyzing the behaviour of the terms which do not involve the gradient. The definition of Υ_F assure the existence of $\tau>0$ and $R>0$ such that $w\equiv 0$ in $\mathbb{R}^N\setminus C(\tau,R)$. Then by applying lemma 7.1 we deduce

$$\begin{aligned}&\int_{C\left(\frac{r}{h}+\tau,\frac{r}{h}+R\right)}\left(\frac{1}{2}\mathcal{V}_h(|x|)|v_h|^2+W(v_h)\right)\,dx\\&=\ \frac{1}{2}\int_{C(\tau,R)}\left(\frac{|x|+\frac{r}{h}}{|x|}\right)^{N-1}\mathcal{V}_h\left(|x|+\frac{r}{h}\right)|w|^2\,dx\\&\quad+\int_{C(\tau,R)}\left(\frac{|x|+\frac{r}{h}}{|x|}\right)^{N-1}W(w)\,dx.\end{aligned}$$

Since $\mathcal{V}\left(h|x|+r\right)$ and $h^{N-1}\left(\frac{|x|+\frac{r}{h}}{|x|}\right)^{N-1}$ converge respectively to $\mathcal{V}(r)$ and to $r^{N-1}/|x|^{N-1}$ uniformly on $C(\tau,R)$ as $h\to 0^+$,

$$\begin{aligned}&\lim_{h\to 0}h^{N-1}\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)}\left(\frac{1}{2}\mathcal{V}_h(|x|)|v_h|^2+W(v_h)\right)\,dx\\&=\ r^{N-1}\int_{C(\tau,R)}\frac{1}{|x|^{N-1}}\left(\frac{1}{2}\mathcal{V}(r)|w|^2+W(w)\right)\,dx\,.\end{aligned} \tag{7.8}$$

Now we estimate of the integrals in (7.7) containing gradient terms. According to the notation in (7.1), it is immediate to show that $\Phi_{v_h}(|x|,\psi(x)) \equiv \Phi_w\left(|x|-\frac{r}{h},\psi(x)\right)$. Then compute

$$\begin{aligned}
&\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} |\nabla v_h|^2\,dx \\
&= \int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} \sum_{\alpha=1}^{N} \Big|\frac{\partial v_h}{\partial x_\alpha}\Big|^2\,dx \qquad (7.9)\\
&= \int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} \sum_{\alpha=1}^{N} \Big|\frac{\partial \Phi_{v_h}}{\partial \rho}(|x|,\psi(x))\frac{x_\alpha}{|x|} + \sum_{i=1}^{N-1}\frac{\partial \Phi_{v_h}}{\partial \theta_i}(|x|,\psi(x))\frac{\partial \psi_i(x)}{\partial x_\alpha}\Big|^2\,dx \\
&= \int_{C(\tau,R)} \Big(\frac{|x|+\frac{r}{h}}{|x|}\Big)^{N-1} \sum_{\alpha=1}^{N} \Big|\frac{\partial \Phi_w}{\partial \rho}(|x|,\psi(x))\frac{x_\alpha}{|x|} \\
&\quad + \sum_{i=1}^{N-1}\frac{\partial \Phi_w}{\partial \theta_i}(|x|,\psi(x))\frac{\partial \psi_i}{\partial x_\alpha}\big(x+\frac{r}{h}\frac{x}{|x|}\big)\Big|^2 dx\,.
\end{aligned}$$

In the above equalities we have used the fact that $\psi(x)\equiv\psi(\lambda x)$ for every $x\neq 0$ and $\lambda>0$. Observe that another consequence of the invariance of the functions ψ_i over the rays is the following

$$\left(1+\frac{r}{h|x|}\right)\frac{\partial \psi_i}{\partial x_\alpha}\left(x+\frac{r}{h}\frac{x}{|x|}\right) \equiv \frac{\partial \psi_i}{\partial x_\alpha}(x) \quad \forall x\neq 0.$$

By inserting this equality in (7.9) one has

$$\begin{aligned}
&\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} |\nabla v_h|^2\,dx \\
&= \int_{C(\tau,R)} \left(\frac{|x|+\frac{r}{h}}{|x|}\right)^{N-1} \\
&\quad \times \sum_{\alpha=1}^{N} \Big|\frac{\partial \Phi_w}{\partial \rho}(|x|,\psi(x))\frac{x_\alpha}{|x|} + \frac{h|x|}{h|x|+r}\sum_{i=1}^{N-1}\frac{\partial \Phi_w}{\partial \theta_i}(|x|,\psi(x))\frac{\partial \psi_i}{\partial x_\alpha}(x)\Big|^2 dx.
\end{aligned}$$

Then, since $\left(\frac{h|x|+r}{|x|}\right)^{N-1} \to \frac{1}{|x|^{N-1}}r^{N-1}$ uniformly in $C(\tau,R)$, by taking the limit as $h\to 0^+$ and using lemma 7.2, we get

$$\begin{aligned}
&\limsup_{h\to 0^+} h^{N-1}\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} |\nabla v_h|^2\,dx \\
&= \limsup_{h\to 0^+}\int_{C(\tau,R)} \frac{r^{N-1}}{|x|^{N-1}} \qquad (7.10)\\
&\quad \times \sum_{\alpha=1}^{N} \left|\frac{\partial \Phi_w}{\partial \rho}(|x|,\psi(x))\frac{x_\alpha}{|x|} + \frac{h|x|}{h|x|+r}\sum_{i=1}^{N-1}\frac{\partial \Phi_w}{\partial \theta_i}(|x|,\psi(x))\frac{\partial \psi_i}{\partial x_\alpha}(x)\right|^2 dx\,.
\end{aligned}$$

¿From (7.2) we deduce $\frac{\partial\Phi_w}{\partial\rho}(|x|,\psi(x)) \in L^2(C(\tau,R),\mathbb{R}^{N+1})$ and, as a consequence, $\frac{\partial\Phi_w}{\partial\rho}(|x|,\psi(x))\frac{x_\alpha}{|x|} \in L^2(C(\tau,R),\mathbb{R}^{N+1})$. On the other hand an elementary calculus shows that

$$\sum_{i=1}^{N-1}\frac{\partial\Phi_w}{\partial\theta_i}(|x|,\psi(x))\frac{\partial\psi_i}{\partial x_\alpha}(x) = \frac{\partial w}{\partial x_\alpha}(x) - \frac{\partial\Phi_w}{\partial\rho}(|x|,\psi(x))\frac{x_\alpha}{|x|}$$

by which $\sum_{i=1}^{N-1}\frac{\partial\Phi_w}{\partial\theta_i}(|x|,\psi(x))\frac{\partial\psi_i}{\partial x_\alpha}(x) \in L^2(C(\tau,R),\mathbb{R}^{N+1})$. Thus we derive

$$\frac{h|x|}{h|x|+r}\sum_{i=1}^{N-1}\frac{\partial\Phi_w}{\partial\theta_i}(|x|,\psi_j(x))\frac{\partial\psi_i}{\partial x_\alpha}(x) \to 0 \quad \text{in} \quad L^2(C(\tau,R),\mathbb{R}^{N+1}). \qquad (7.11)$$

Combining (7.10) together with (7.11) we obtain

$$\begin{aligned}
&\limsup_{h\to 0^+} h^{N-1}\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} |\nabla v_h|^2\,dx \\
&= r^{N-1}\int_{C(\tau,R)}\frac{1}{|x|^{N-1}}\sum_{\alpha=1}^{N}\left|\frac{\partial\Phi_w}{\partial\rho}(|x|,\psi(x))\frac{x_\alpha}{|x|}\right|^2 dx \\
&\le r^{N-1}\int_{C(\tau,R)}\frac{1}{|x|^{N-1}}\left|\frac{\partial w}{\partial\rho}\right|^2\sum_{\alpha=1}^{N}\frac{x_\alpha^2}{|x|^2}\,dx \\
&= r^{N-1}\int_{C(\tau,R)}\frac{1}{|x|^{N-1}}\left|\frac{\partial w}{\partial\rho}\right|^2 dx\,.
\end{aligned}$$

In the same way one obtains

$$\limsup_{h\to 0^+} h^{N-1}\int_{C(\frac{r}{h}+\tau,\frac{r}{h}+R)} |\nabla v_h|^\gamma\,dx \le r^{N-1}\int_{C(\tau,R)}\frac{1}{|x|^{N-1}}\left|\frac{\partial w}{\partial\rho}\right|^\gamma dx.$$

The thesis follows from (7.8) and the above inequalities. ◇

Before going on with the statements of our results for every $r>0$ set

$$\widetilde{\mathcal{E}}_r^* = \inf_{u\in\Upsilon_F^*}\mathcal{E}_r(u),$$

where, with obvious notation, we have put $\Upsilon_F^* = \{u\in\Upsilon_F : ch(u)\neq 0\}$. We point out that in [6] the authors exhibited some examples of functions $u\in\Lambda_F$ with $ch(u)\neq 0$ and such that $K(u)\subset C(R_1,R_2)$ for some $0<R_1<R_2$. Then by multiplying such u for a suitable cut-off radially function τ, we obtain that $\tau u\in\Upsilon_F^*$, i.e. $\Upsilon_F^*\neq\emptyset$. Then we conclude this section with the following lemma.

Theorem 7.4 *The following statements hold:*

i) The function $r>0\longmapsto\widetilde{\mathcal{E}}_r^*$ *is continuous.*

ii) For every $M>0$ there exists $\sigma=\sigma(M)>0$ such that for every $r>0$,

$$\widetilde{\mathcal{E}}_r^* < M \Rightarrow \widetilde{\mathcal{E}}_r^* \geq \mathcal{V}(r)\sigma\,.$$

As a corollary,

$$\inf_{r>0} \widetilde{\mathcal{E}}_r^* > 0\,. \tag{7.12}$$

iii) If $\{r_n\}\subset(0,+\infty)$ is a sequence verifying $\mathcal{V}(r_n)\to+\infty$ as $n\to+\infty$, then there results $\limsup_{n\to+\infty}\widetilde{\mathcal{E}}_{r_n}^*=+\infty$.

Proof. i) Choose $b>a>0$. From the continuity of $\mathcal{V}$ we deduce that the function $r\in[a,b]\longmapsto\widetilde{\mathcal{E}}_r^*$ is bounded; then choose $M>0$ such that $\widetilde{\mathcal{E}}_r^*<M$ for all $r\in[a,b]$. Hence the definition of $\mathcal{E}_r$ implies that each $\widetilde{\mathcal{E}}_r^*$ can be also written in the form

$$\widetilde{\mathcal{E}}_r^* \equiv \inf\left\{\mathcal{E}_r(u) : u\in\Upsilon_F^*,\ \int_{\mathbb{R}^N}\frac{1}{|x|^{N-1}}|u|^2\,ds \leq \frac{2M}{V_0}\right\}.$$

Thus for every $u\in\Upsilon_F^*$ with $\int_{\mathbb{R}^N}|u|^2\,ds\leq\frac{2M}{V_0}$ and for every $s,s'\in[a,b]$ it is

$$\widetilde{\mathcal{E}}_s^* \leq \mathcal{E}_s(u) \leq \mathcal{E}_{s'}(u)+|\mathcal{V}(s)-\mathcal{V}(s')|\frac{M}{V_0}.$$

By taking the infimum on the right side we get $\widetilde{\mathcal{E}}_s^*\leq\widetilde{\mathcal{E}}_{s'}^*+|\mathcal{V}(s)-\mathcal{V}(s')|\frac{M}{V_0}$. By changing the role of s and s' we obtain the symmetric inequality, by which, taking into account of the continuity of the potential V, the part i) of the thesis.
ii) Fix $M>0$ arbitrarily. By the definition of $\mathcal{E}_r$ there follows that for every $r>0$ with $\widetilde{\mathcal{E}}_r^*<M$ we can write

$$\widetilde{\mathcal{E}}_r^* \equiv \inf\left\{\mathcal{E}_r(u) : u\in\Gamma^*\right\} \tag{7.13}$$

where the set Γ^* is given by $\Gamma^*\equiv\{u\in\Upsilon_F^* : \int_{\mathbb{R}^N}\frac{1}{|x|^{N-1}}|\nabla u|^2\,ds\leq 2M\}$. Now take $u\in\Gamma^*$ and set $\varphi_{[u]}(|x|)=|u(x)|$. Then we easily compute:

$$2M \geq \int_{\mathbb{R}^N}\frac{1}{|x|^{N-1}}|\nabla u|^2\,dx \geq \int_{\mathbb{R}^N}\frac{1}{|x|^{N-1}}|\nabla|u|\,|^2\,dx = \int_0^{+\infty}|\varphi_{[u]}'|^2\,dr.$$

Since each $\varphi_{[u]}$ belongs to $W^{1,2}((0,+\infty),\mathbb{R})$, the theory of the one-dimensional Sobolev spaces yields

$$|\varphi_{[u]}(s)-\varphi_{[u]}(s')| \leq \left(\int_{s'}^{s}|\varphi_{[u]}'|^2\right)^{1/2}|s-s'|^{1/2} < \sqrt{2M}|s-s'|^{1/2},$$

for all $u\in\Gamma^*$, and all $s>s'>0$, by which it follows that the set of functions $\{\varphi_{[u]} : u\in\Gamma^*\}$ is equi-uniformly continuous in $(0,+\infty)$. This fact provides the existence of $\sigma>0$ such that

$$\forall u\in\Gamma^*,\quad \forall s,s'\in(0,+\infty),\ \ |s-s'|<\sigma \Rightarrow |\varphi_{[u]}(s)-\varphi_{[u]}(s')| \leq \frac{1}{2}. \tag{7.14}$$

On the other hand, taking into account that if $u \in \Upsilon_F^*$ it is $ch(u) \neq 0$, by (3.2) we deduce that for every $u \in \Gamma^*$ there is $s_u \in (0,+\infty)$ verifying $\varphi_{[u]}(s_u) > 1$. From (7.14) there follows

$$|\varphi_{[u]}(s)| > \frac{1}{2} \quad \forall s \in (s_u - \sigma, s_u + \sigma).$$

By using (7.13), an elementary calculus shows that for every $r > 0$ with $\widetilde{\mathcal{E}}_r^* < M$ it is:

$$\widetilde{\mathcal{E}}_r^* \geq \mathcal{V}(r)\frac{\sigma}{4}$$

and the proof of the first part of ii) ends. As regards (7.12), assume by absurd that $\inf_{r>0} \widetilde{\mathcal{E}}_r^* = 0$ and take $\{r_n\} \subset (0,+\infty)$ such that $\widetilde{\mathcal{E}}_{r_n}^* \to 0$. Then, since the sequence $\{\widetilde{\mathcal{E}}_{r_n}^*\}$ is bounded, we can apply the first part of ii) and obtain for every $n \in \mathbb{N}$ and for some $\sigma > 0$: $\widetilde{\mathcal{E}}_{r_n}^* \geq \mathcal{V}(r_n)\sigma \geq V_0\sigma$, which is a contradiction.

iii) Assume by absurd that $\limsup_{n\to+\infty} \widetilde{\mathcal{E}}_{r_n}^* < +\infty$ and fix $M > 0$ such that $\widetilde{\mathcal{E}}_{r_n}^* < M$ for every $n \in \mathbb{N}$. By part ii) we get the existence of $\sigma = \sigma(M) > 0$ such that

$$\widetilde{\mathcal{E}}_{r_n}^* \geq \mathcal{V}(r_n)\sigma \to +\infty \ \text{ as } \ n \to +\infty$$

and we achieve the desired contradiction. $\diamondsuit$

Now we have in our hands all the instruments to prove in the next section the second existence and concentration result of this work.

8 Concentration Around a Circle

In this section we solve the problem of finding a family of solutions $\{v_h\}$ to equation (1.1) exhibiting a concentration behaviour around a circle $\{x \in \mathbb{R}^N : |x| = \overline{r}\}$. Toward this end suppose assumptions a)-g) hold and assume $r_0 > 0$ such that

$$r_0^{N-1}\widetilde{\mathcal{E}}_{r_0}^* > \inf_{r \geq r_0} r^{N-1}\widetilde{\mathcal{E}}_r^*. \tag{8.1}$$

To achieve our results we need a suitable modification of the nonlinear term W: because of assumption f) choose $\overline{c} > 1$ such that $\overline{c}|W''(0)| < V_0/(2(N+1))^{1/2}+2$. Then take $\alpha \in (0,\overline{\varepsilon})$, with $\overline{\varepsilon}$ given by hypothesis g), such that

$$|W'(\xi)| \leq \frac{V_0}{(2(N+1))^{1/2}+2}|\xi| \quad \forall \xi \in \Omega \quad \text{with} \quad |\xi| \leq \alpha \tag{8.2}$$

and

$$\overline{c}|W(\xi)| \leq \frac{1}{2}\frac{V_0}{(2(N+1))^{1/2}+2}|\xi|^2 \quad \forall \xi \in \Omega \quad \text{with} \quad |\xi| \leq \alpha. \tag{8.3}$$

We notice that every $\alpha' \leq \alpha$ satisfies (8.2) and (8.3). Then (8.2) and (8.3) yield

$$\begin{gathered} \sup_{|\xi|\leq\alpha} |W'(\xi)| \leq \tfrac{V_0}{(2(N+1))^{1/2}+2}\alpha, \\ \overline{c}\sup_{|\xi|\leq\alpha} |W(\xi)| \leq \tfrac{1}{2}\tfrac{V_0}{(2(N+1))^{1/2}+2}\alpha^2\,. \end{gathered} \tag{8.4}$$

Now consider a function $K_\alpha : \mathbb{R}_+ \to \mathbb{R}$ with the following properties

$$K_\alpha \in C^1(\mathbb{R}_+, \mathbb{R}), \quad K_\alpha(s) = 1 \text{ if } s \le \frac{\alpha^2}{2},$$
$$K_\alpha(s) = 0 \text{ if } s \ge \alpha^2, \quad 0 \le K_\alpha \le 1, \quad |K'_\alpha| \le \frac{2\overline{c}}{\alpha^2}.$$

Let

$$G(\cdot, \xi) = \chi_{B(0,r_0/2)}\, K_\alpha\left(|\xi|^2\right) W(\xi) + \left(1 - \chi_{B\left(0,\frac{r_0}{2}\right)}\right) W(\xi), \quad \xi \in \mathbb{R}^{N+1}.$$

Then the modified functional $J_h : \Lambda_h \to \mathbb{R}$ is defined as

$$J_h(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}\left(|\nabla u|^2 + V_h(x)|u|^2\right) + \frac{1}{p}|\nabla u|^p + G(hx, u) \right) dx.$$

Obviously J_h is well defined in the open set Λ_h since $G(\cdot, u)$ differs from $W(u)$ only on the compact $\{x \in \mathbb{R}^N : |u(x)| \ge \alpha/\sqrt{2}\}$. Note that for every $x \in \mathbb{R}^N$ the function $G(x, \cdot)$ belongs to $C^1(\mathbb{R}^{N+1}, \mathbb{R})$ and is two times differentiable in 0; this implies that J_h belongs to $C^1(\Lambda_h, \mathbb{R})$ and the proof is identical to that of lemma 4.1 in [2].

We point out that J_h is weakly lower semi-continuous and is coercive in the H_h-norm, i.e.

$$\lim_{\|u\|_{H_h} \to +\infty} J_h(u) = +\infty\,. \tag{8.5}$$

The critical points of J_h correspond to solutions of the equation

$$-\Delta u + V_h(x)u - \Delta_p(u) + G'(hx, u) = 0, \tag{8.6}$$

where $G'(x, \cdot)$ denotes the gradient of the function $G(x, \cdot)$. Observe how by construction and by assumption g) we get

$$G(x, (\xi_0, g\xi_1)) = G(x, (\xi_0, \xi_1)) \quad \forall \xi = (\xi_0, \xi_1) \in \mathbb{R}^{N+1}, \quad \forall g \in O(N).$$

Then an immediate calculus shows that J_h, just like E_h, is invariant under the action of (2.7), i.e. $J_h(T_g(u)) = J_h(u)$ for every $u \in \Lambda_h$, hence lemma 4.3 holds for J_h too, and we deduce that every minimum of the functional J_h in the set $\Lambda_F(h)$ is a critical point of J_h and hence provides a solution to equation (8.6). Furthermore by construction a solution $u \in \Lambda_h$ of (8.6) satisfying $|u(x)| \le \alpha/\sqrt{2}$ for $|x| \le \frac{r_0}{2h}$ will also be a solution for (1.2). Then the strategy we will follow is clear: we will find a local minimum of J_h in the set $\Lambda_F(h)$ and this will yield a solution to (8.6). Next we will show that, provided h is sufficiently small, such solution will satisfy the property $|u(x)| \le \frac{\alpha}{\sqrt{2}}$ for $|x| \le \frac{r_0}{2h}$ and thus will solve the original equation (1.2). Before going on let

$$\begin{aligned}&\widetilde{\Lambda}^*_F(h)\\ &= \left\{ u \in \Lambda_F(h) : |u(x)| < \frac{\overline{\varepsilon}}{2}\ \forall x \in \overline{C}\big(\frac{r_0}{2h}, \frac{r_0}{h}\big),\ ch\big(u, \mathbb{R}^N \setminus \overline{B}(0, \frac{r_0}{h})\big) \ne 0 \right\}\end{aligned}$$

and $\widetilde{c}_h^* = \inf_{u\in\widetilde{\Lambda}_F^*(h)} J_h(u)$. The continuous immersion $H_h \subset L^\infty$ and lemma 3.2 assures that $\widetilde{\Lambda}_F^*(h)$ is open in $\Lambda_F(h)$. We notice that $\widetilde{\Lambda}_F^*(h) \neq \emptyset$ for all $h > 0$: indeed it is sufficient to take $u \in \Upsilon_F^*$ and set $\widetilde{u}(x) = u(x - \frac{r_0}{h}\frac{x}{|x|})$ if $|x| \geq \frac{r_0}{h}$ and $\widetilde{u}(x) = 0$ everywhere else. By lemma 3.1 we get $ch(\widetilde{u}) = ch(u) \neq 0$, by which $\widetilde{u} \in \widetilde{\Lambda}_F^*(h)$ for every $h > 0$. The first object is to minimize the energy functional J_h in the set $\widetilde{\Lambda}_F^*(h)$. To this aim we need the estimate provided by the following lemma.

Lemma 8.1 $\limsup_{h\to 0^+} h^{N-1}\widetilde{c}_h^* \leq \inf_{r\geq r_0} r^{N-1}\widetilde{\mathcal{E}}_r^*$.

Proof. Consider $r \geq r_0$ arbitrarily and $w \in \Upsilon_F^*$. For every $h > 0$, define $\widetilde{w}_h : \mathbb{R}^N \to \mathbb{R}^{N+1}$ by setting

$$\widetilde{w}_h(x) = \begin{cases} 0 & \text{if } |x| \leq r/h, \\ w\big(x - \frac{r}{h\,|x|}x\big) & \text{if } |x| > r/h\ . \end{cases}$$

It is obvious that $\widetilde{w}_h \in \Lambda_F(h)$ for every $h > 0$ and furthermore $\widetilde{w}_h(x) = 0$ if $|x| \leq r_0/h$. By lemma 3.1 there follows $ch\left(\widetilde{w}_h, \mathbb{R}^N \setminus \overline{B}\,(0, r_0/h)\right) = ch(\widetilde{w}_h) = ch(w) \neq 0$ so that $\widetilde{w}_h \in \widetilde{\Lambda}_F^*(h)$. Note that by construction it is $J_h(\widetilde{w}_h) = E_h(\widetilde{w}_h)$ for every $h > 0$. From part 2 of lemma 7.3 we have:

$$\limsup_{h\to 0^+} h^{N-1}\widetilde{c}_h^* \leq \limsup_{h\to 0^+} h^{N-1}E_h(\widetilde{w}_h) \leq r^{N-1}\mathcal{E}_r(w).$$

By taking the infimum on the right side of last inequality first for $w \in \Upsilon_F^*$ and then for $r \geq r_0$ we achieve the desired conclusion. $\diamondsuit$

Now we are able to give the main result of this section.

Theorem 8.2 *Assume that assumptions a)-g) hold and, in addition, (8.1). Then there exists $h_0 > 0$ such that for every $h \in (0, h_0)$ there is a solution $u_h \in \widetilde{\Lambda}_F^*(h)$ to equation (1.2). Furthermore if for every $h \in (0, h_0)$ we set*

$$v_h(x) = u_h\left(\frac{x}{h}\right), \tag{8.7}$$

then v_h is a solution of (1.1) and the family $\{v_h\}$ exhibits the following concentration behaviour as $h \to 0^+$: for each sequence $h_n \to 0^+$ there exists a subsequence, still denoted by h_n, such that $|v_{h_n}|$ has a circle of local maximum points $\{x \in \mathbb{R}^N : |x| = r_{h_n}\}$ with $|v_{h_n}(x)| > 1$ for $|x| = r_{h_n}$ and

$$r_{h_n} \to \overline{r}, \quad \overline{r}^{N-1}\widetilde{\mathcal{E}}_{\overline{r}}^* = \inf_{r\geq r_0} r^{N-1}\widetilde{\mathcal{E}}_r^*;$$

also, for every $\delta > 0$, it holds $v_{h_n} \to 0$ as $n \to +\infty$ uniformly in the set $\{x : \big||x| - \overline{r}\big| \geq \delta\}$.

Proof. Fix $h>0$ arbitrarily and consider u_k^h a minimizing sequence in $\widetilde{\Lambda}_F^*(h)$ for J_h; it has obviously bounded energy because of (8.5). Then up to a subsequence we have $u_k^h \rightharpoonup u_h$ weakly in H_h as $k\to+\infty$, for some $u\in F_h$ and, from lemma 2.2,

$$u_k^h \to u_h \quad \text{uniformly in} \quad \mathbb{R}^N \quad \text{as} \quad k\to+\infty\,. \tag{8.8}$$

Proposition 4.2 implies $u_h\in\Lambda_F(h)$. By (8.8) we deduce

$$|u_h(x)|\le\frac{\overline{\varepsilon}}{2}\quad\forall x\in\overline{C}\left(\frac{r_0}{2h},\frac{r_0}{h}\right).$$

Using lemma 3.2 we infer that the sequence $\left\{ch\left(u_k^h,\mathbb{R}^N\setminus\overline{B}\left(0,\frac{r_0}{h}\right)\right)\right\}$ is definitively constant and moreover, for k large enough,

$$ch\left(u_h,\mathbb{R}^N\setminus\overline{B}\left(0,\frac{r_0}{h}\right)\right)=ch\left(u_k^h,\mathbb{R}^N\setminus\overline{B}\left(0,\frac{r_0}{h}\right)\right)\ne 0\,. \tag{8.9}$$

¿From now on we focus our attention on a generic sequence $h_n\to0^+$.

For every $n\in\mathbb{N}$ let $\varphi_{h_n}:[0,+\infty)\to\mathbb{R}$ such that

$$\varphi_{h_n}(|x|)=|u_{h_n}(x)|\quad\forall x\in\mathbb{R}^N.$$

The weakly lower semi-continuity of J_h leads to the inequality

$$J_{h_n}(u_{h_n})\le\liminf_{k\to+\infty}J_{h_n}(u_k^{h_n})=\widetilde{c}^*_{h_n}. \tag{8.10}$$

Taking into account of lemma 8.1, the sequence $\{h_n^{N-1}\widetilde{c}^*_{h_n}\}$, and consequently $\{h_n^{N-1}J_{h_n}(u_{h_n})\}$, is bounded. Taken $\overline{\varepsilon}\in(0,1)$ given by hypothesis g), by (3.2) and (8.9) we immediately get $\sup_{|x|\ge\frac{r_0}{h_n}}|u_{h_n}(x)|>1>\overline{\varepsilon}$; then the sequence $\{u_{h_n}\}$ satisfies all the hypotheses of lemma 6.2 with $t\equiv r_0$ and $\overline{\varepsilon}\equiv\varepsilon$. Then we obtain the existence of $\ell\in\mathbb{N}$, $R_1,\dots,R_\ell>0$ and ℓ sequences of positive numbers $\{r^1_{h_n}\},\dots,\{r^\ell_{h_n}\}$, with $r^i_{h_n}>\frac{r_0}{h_n}$, such that, up to a subsequence

$$\{h_nr^i_{h_n}\}\quad\text{is bounded}\quad\forall i=1,\dots,\ell; \tag{8.11}$$

$$\varphi_{h_n}(r^i_{h_n})>\tfrac{\overline{\varepsilon}}{2}\quad\forall n\in\mathbb{N},\quad\forall i=1,\dots,\ell;$$

$$r^i_{h_n}\text{is the maximum point for }\varphi_{h_n}\text{in}$$

$$\left[\tfrac{r_0}{2h_n},+\infty\right)\setminus\cup_{j<i}(r^j_{h_n}-R_j,r^j_{h_n}+R_j); \tag{8.12}$$

$$\forall\, r\in\left[\tfrac{r_0}{2h_n},+\infty\right)\setminus\cup_{i=1}^{\ell}(r^i_{h_n}-R_i,r^i_{h_n}+R_i):\quad\varphi_{h_n}(r)\le\tfrac{\overline{\varepsilon}}{2}; \tag{8.13}$$

$$|r^i_{h_n}-r^j_{h_n}|\to+\infty\quad\text{as}\quad n\to+\infty,\quad\forall i,j\in\{1,\dots,\ell\}\quad\text{with } i\ne j\,. \tag{8.14}$$

Because of the inequality $r^i_{h_n}>\frac{r_0}{h_n}$, we have $r^i_{h_n}-\frac{r_0}{2h_n}\to+\infty$. Hence, taking into account of (8.14), it makes sense to assume $C\left(r^i_{h_n}-R_i,r^i_{h_n}+R_i\right)\cap C\left(r^j_{h_n}-R_j,r^j_{h_n}+R_j\right)=\emptyset$ for $i\ne j$ and $C\left(r^i_{h_n}-R_i,r^i_{h_n}+R_i\right)\subset\mathbb{R}^N\setminus\overline{B}\left(0,\frac{r_0}{2h_n}\right)$ for every $n\in\mathbb{N}$.

Then from definition 3.1 and (3.1) there follows

$$ch\left(u_{h_n}, \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_n}\right)\right) = \sum_{i=1}^{\ell} ch(u_{h_n}, C(r^i_{h_n} - R_i, r^i_{h_n} + R_i)).$$

This equality implies the existence of $\widehat{i} \in \{1, \dots, \ell\}$ such that

$$ch(u_{h_n}, C(r^{\widehat{i}}_{h_n} - R_{\widehat{i}}, r^{\widehat{i}}_{h_n} + R_{\widehat{i}})) \neq 0. \tag{8.15}$$

In particular, according to (3.2) and (8.12), this implies that $\varphi_{h_n}(r^{\widehat{i}}_{h_n}) > 1$ and, because of (8.13), $|u_{h_n}(x)| \leq \frac{\overline{\varepsilon}}{2} < 1$ for every $x \in \partial C(r^{\widehat{i}}_{h_n} - R_{\widehat{i}}, r^{\widehat{i}}_{h_n} + R_{\widehat{i}})$. Hence we have

$$|u_{h_n}(x)| > 1 \quad \forall x \in \mathbb{R}^N \quad \text{with} \quad |x| = r^{\widehat{i}}_{h_n} \quad \forall\, n \in \mathbb{N} \tag{8.16}$$

and

$$\left\{x \in \mathbb{R}^N : |x| = r^{\widehat{i}}_{h_n}\right\} \tag{8.17}$$

is a circle of local maximum points for $|u_{h_n}|$ for all n in $\mathbb{N}$. By the inequality $r^{\widehat{i}}_{h_n} > \frac{r_0}{h_n}$ and by (8.11) we get the existence of $\overline{r} \geq 0$ verifying, up to subsequence,

$$h_n r^{\widehat{i}}_{h_n} \to \overline{r} \quad \text{as} \quad n \to +\infty, \quad \overline{r} \geq r_0. \tag{8.18}$$

Now for every $R > 2R_{\widehat{i}}$ and $n \in \mathbb{N}$ consider $\eta^R_{h_n} \in C^\infty_0(\mathbb{R}^N, \mathbb{R})$ a radial function so that

$$\eta^R_{h_n} \equiv 0 \ \text{ on } \ \mathbb{R}^N \setminus C(r^{\widehat{i}}_{h_n} - R, r^{\widehat{i}}_{h_n} + R),$$
$$\eta^R_{h_n} \equiv 1 \ \text{ on } \ C\left(r^{\widehat{i}}_{h_n} - \frac{R}{2}, r^{\widehat{i}}_{h_n} + \frac{R}{2}\right), \quad 0 \leq \eta^R_{h_n} \leq 1, \quad |\nabla \eta^R_{h_n}| \leq \frac{c}{R}$$

where c is a constant independent of R and n.

Hereafter, for the sake of simplicity, for every $u \in \Lambda_F(h)$ and $A \subset \mathbb{R}^N$ measurable we set

$$(J_h)\big|_A(u) = \frac{1}{2}\int_A \left(|\nabla u|^2 + V_h(x)|u|^2\right)\, dx + \frac{1}{p}\int_A |\nabla u|^p\, dx + \int_A G(hx, u)\, dx.$$

The key steps in the proof of the theorem are the following 6 claims.

Claim 1. For every couple of positive numbers $\delta, M > 0$ there exists $R = R(\delta, M) > 2R_{\widehat{i}}$ and $\overline{n} \in \mathbb{N}$ such that for every $n \geq \overline{n}$ and for every $w_n \in \Lambda_F(h_n)$ satisfying

$$|w_n(x)| \leq \overline{\varepsilon} \ \ for \ \ x \in C^R_n, \quad h_n^{N-1} J_{h_n}(w_n) \leq M, \tag{8.19}$$

there results

$$h_n^{N-1} J_{h_n}(\eta^R_{h_n} w_n) \leq h_n^{N-1} (J_{h_n})\big|_{C^R_n}(w_n) + \delta,$$

where

$$C^R_n \equiv C\big(r^{\widehat{i}}_{h_n} - R, r^{\widehat{i}}_{h_n} - \frac{R}{2}\big) \cup C\big(r^{\widehat{i}}_{h_n} + \frac{R}{2}, r^{\widehat{i}}_{h_n} + R\big).$$

Proof of Claim 1. Fix $\delta, M>0$ arbitrarily. The basic tool is the following assertion: for every $R>2R_{\widehat{i}}$ there exists $n_R>0$ such that for every $n\geq n_R$ and for every $w_n\in\Lambda_F(h_n)$ satisfying (8.19), it holds:

$$h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^2\,dx\leq h_n^{N-1}\int_{C_n^R}|\nabla w_n|^2\,dx+C_1\big(\frac{1}{R}+\frac{1}{R^2}\big) \tag{8.20}$$

and

$$h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^p\,dx\leq h^{N-1}\int_{C_n^R}|\nabla w_n|^p\,dx+C_2\big(\frac{1}{R^{(p-1)/p}}+\frac{1}{R^{p-1}}\big) \tag{8.21}$$

where C_1 and C_2 are constants depending only on $\overline{r}$, M, $\overline{\varepsilon}$, p and N. Before proving the above assertion, observe how, assuming (8.20) and (8.21), the thesis of claim 1 easily follows. Indeed hypothesis g) and the first of (8.19) give $W(\eta_{h_n}^R w_n)\leq W(w_n)$ in C_n^R and this implies

$$\begin{aligned}&\frac12\int_{\mathbb{R}^N}V_{h_n}(x)|\eta_{h_n}^R w_n|^2\,dx+\int_{\mathbb{R}^N}W(\eta_{h_n}^R w_n)\,dx \\ &\leq \frac12\int_{\mathbb{R}^N}V_{h_n}(x)|w_n|^2\,dx+\int_{\mathbb{R}^N}W(w_n)\,dx.\end{aligned} \tag{8.22}$$

Hence, taking $R=R(\delta,M)>2R_{\widehat{i}}$ sufficiently large and combining (8.20), (8.21) and (8.22) we obtain the thesis.

The rest of this proof will be devoted (8.20) and (8.21). Let us begin with (8.20). Observe

$$\begin{aligned}&h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^2\,dx \\ &\leq h_n^{N-1}\Big(\int_{C_n^R}|\nabla w_n|^2\,dx+\frac{c^2}{R^2}\int_{C_n^R}|w_n|^2\,dx+2\frac{c}{R}\int_{C_n^R}|\nabla w_n||w_n|\,dx\Big).\end{aligned}$$

The Hölder inequality yields

$$\begin{aligned}h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^2\,dx &\leq h_n^{N-1}\big(\int_{C_n^R}|\nabla w_n|^2\,dx+\frac{c^2}{R^2}\int_{C_n^R}|w_n|^2\,dx\big)\\ &\quad +h_n^{N-1}2\frac{c}{R}\big(\int_{C_n^R}|\nabla w_n|^2\,dx\big)^{1/2}\big(\int_{C_n^R}|w_n|^2\,dx\big)^{1/2}\end{aligned}$$

by which, since $h_n^{N-1}\int_{C_n^R}|\nabla w_n|^2\,dx,\leq 2M$ and $h_n^{N-1}\int_{C_n^R}|w_n|^2\,dx\leq\frac{2M}{V_0}$, (8.20) immediately follows. As regards (8.21) the proof is more delicate; we will use the following numerical inequality

$$(a+b)^p\leq a^p+(p+1)a^{p-1}b+C_pb^p$$

which is valid if $p\geq 3$, $C_p\geq\frac1p(p+1)^{p-1}(p-2)^{p-2}$ and $a,b\geq 0$. Then we get

$$\begin{aligned}h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^p\,dx &\leq h_n^{N-1}\int_{C_n^R}|\nabla w_n|^p\,dx\\ &\quad +h_n^{N-1}\Big((p+1)\frac{c}{R}\int_{C_n^R}|\nabla w_n|^{p-1}|w_n|\,dx+C_p\frac{c^p}{R^p}\int_{C_n^R}|w_n|^p\,dx\Big).\end{aligned}$$

Again from Hölder inequality, we obtain

$$h_n^{N-1}\int_{C_n^R}|\nabla\eta_{h_n}^R w_n|^p\,dx \ \le\ h_n^{N-1}\int_{C_n^R}|\nabla w_n|^p\,dx + h_n^{N-1}\Big((p+1)\frac{c}{R} \\ \times\Big(\int_{C_n^R}|\nabla w_n|^p\,dx\Big)^{(p-1)/p}\overline{\varepsilon}\,(\operatorname{meas}C_n^R)^{1/p} + C_p\frac{c^p}{R^p}\overline{\varepsilon}^p\operatorname{meas}(C_n^R)\Big). \tag{8.23}$$

Taking into account of (8.18), an elementary calculus shows

$$\lim_{n\to+\infty} h_n^{N-1}\operatorname{meas}(C_n^R)=\omega_N\overline{r}^{N-1}R.$$

Then, by inserting this in (8.23) and proceeding as for the other inequality we conclude. Claim 1 is completely proved.

Now for every $n\in\mathbb{N}$ put $\widetilde{u}_{h_n}^R\equiv\eta_{h_n}^R u_{h_n}$.

Claim 2. For every $R>2R_{\widehat{i}}$ there exists $\widehat{n}\in\mathbb{N}$ such that for $n\ge\widehat{n}$,

$$|u_{h_n}(x)|\le\frac{\overline{\overline{\varepsilon}}}{2}\quad\forall x\in C_n^R,\quad \widetilde{u}_{h_n}^R\in\Lambda_F^*(h_n).$$

Proof of Claim 2. According to (8.14) for n sufficiently large,

$$C(r_{h_n}^{\widehat{i}}-R,r_{h_n}^{\widehat{i}}+R)\cap C(r_{h_n}^{i}-R_i,r_{h_n}^{i}+R_i)=\emptyset\quad\forall i\in\{1,\dots,\ell\},\ \ i\ne\widehat{i}$$

and, since $r_{h_n}^{\widehat{i}}>\frac{r_0}{h_n}$, $(r_{h_n}^{\widehat{i}}-R,r_{h_n}^{\widehat{i}}+R)\subset\big[\frac{r_0}{2h_n},+\infty\big)$. Then, by (8.13),

$$|u_{h_n}(x)|\le\frac{\overline{\overline{\varepsilon}}}{2}\quad\forall x\in C(r_{h_n}^{\widehat{i}}-R,r_{h_n}^{\widehat{i}}+R)\setminus C(r_{h_n}^{\widehat{i}}-R_{\widehat{i}},r_{h_n}^{\widehat{i}}+R_{\widehat{i}}). \tag{8.24}$$

It is easy to show that $C_n^R\subset C(r_{h_n}^{\widehat{i}}-R,r_{h_n}^{\widehat{i}}+R)\setminus C(r_{h_n}^{\widehat{i}}-R_{\widehat{i}},r_{h_n}^{\widehat{i}}+R_{\widehat{i}})$, then the first part of the thesis follows. By construction there results $\widetilde{u}_{h_n}^R\in F_{h_n}$. On the other hand $\widetilde{u}_{h_n}^R\equiv 0$ in $\mathbb{R}^N\setminus C(r_{h_n}^{\widehat{i}}-R,r_{h_n}^{\widehat{i}}+R)$ and, since $R>2R_{\widehat{i}}$, it is $\widetilde{u}_{h_n}^R\equiv u_{h_n}$ in $C(r_{h_n}^{\widehat{i}}-R_{\widehat{i}},r_{h_n}^{\widehat{i}}+R_{\widehat{i}})$, hence we easily deduce $\widetilde{u}_{h_n}^R\in\Lambda_F(h_n)$; furthermore from (8.15) and (8.24) we get

$$ch(\widetilde{u}_{h_n}^R)=ch(\widetilde{u}_{h_n}^R,C(r_{h_n}^{\widehat{i}}-R_{\widehat{i}},r_{h_n}^{\widehat{i}}+R_{\widehat{i}}))=ch(u_{h_n},C(r_{h_n}^{\widehat{i}}-R_{\widehat{i}},r_{h_n}^{\widehat{i}}+R_{\widehat{i}}))\ne 0,$$

by which, provided n is sufficiently large, $\widetilde{u}_{h_n}^R\in\Lambda_F^*(h_n)$.

Claim 3. $\overline{r}^{N-1}\widetilde{\mathcal{E}}_{\overline{r}}^*=\inf_{r>r_0}r^{N-1}\widetilde{\mathcal{E}}_r^*$

Proof of claim 3. Assume on the contrary that $\overline{r}^{N-1}\widetilde{\mathcal{E}}_{\overline{r}}^*\ne\inf_{r>r_0}r^{N-1}\widetilde{\mathcal{E}}_r^*$. In particular, because of (8.18), it would be $\overline{r}^{N-1}\widetilde{\mathcal{E}}_{\overline{r}}^*>\inf_{r>r_0}r^{N-1}\widetilde{\mathcal{E}}_r^*$; then take $\delta>0$ such that

$$\overline{r}^{N-1}\widetilde{\mathcal{E}}_{\overline{r}}^*>\inf_{r>r_0}r^{N-1}\widetilde{\mathcal{E}}_r^*+\delta. \tag{8.25}$$

Using (8.10) and lemma 8.1, choose $M > 0$ such that $h_n^{N-1} J_{h_n}(u_{h_n}) < M$, for $n \in \mathbb{N}$. Then let $R = R\left(\frac{\delta}{4}, M\right)$ given by claim 1. According to claims 1 and 2 there exists $\overline{n} \in \mathbb{N}$ such that for every $n \geq \overline{n}$:

$$\begin{aligned} &\widetilde{u}_{h_n}^R \in \Lambda_F^*(h_n), \quad |u_{h_n}(x)| \leq \tfrac{\overline{\varepsilon}}{2} \ \ \forall x \in C_n^R, \\ &h_n^{N-1} J_{h_n}(\widetilde{u}_{h_n}^R) \leq h_n^{N-1} J_{h_n}(u_{h_n}) + \tfrac{\delta}{4}\,. \end{aligned} \tag{8.26}$$

For every $n \in \mathbb{N}$ let $w_{h_n} : \mathbb{R}^N \to \mathbb{R}^{N+1}$ defined as

$$w_{h_n}(x) = \begin{cases} 0 & \text{if } |x| \leq R, \\ \widetilde{u}_{h_n}^R\left(x + (r_{h_n}^{\widehat{i}} - 2R)\frac{x}{|x|}\right) & \text{if } |x| > R. \end{cases}$$

It is obvious that $w_{h_n} \in \Upsilon_F$ for every $n \geq \overline{n}$.

Since $\widetilde{u}_{h_n}^R(x) = w_{h_n}\left(x - (r_{h_n}^{\widehat{i}} - 2R)\frac{x}{|x|}\right)$ if $|x| \geq r_{h_n}^{\widehat{i}} - R$ and $\widetilde{u}_{h_n}^R \equiv 0$ in $B(0, r_{h_n}^{\widehat{i}} - R)$, from lemma 3.1 we obtain $ch(w_{h_n}) = ch(\widetilde{u}_{h_n}^R) \neq 0$, which implies $w_{h_n} \in \Upsilon_F^*$.

Note that, since $r_{h_n}^{\widehat{i}} > \frac{r_0}{h_n}$, provided that $\overline{n}$ is large enough we have $C(r_{h_n}^{\widehat{i}} - R, r_{h_n}^{\widehat{i}} + R) \subset \mathbb{R}^N \setminus B\left(0, \frac{r_0}{2h_n}\right)$ for every $n \geq \overline{n}$ so that, by construction, $E_{h_n}(\widetilde{u}_{h_n}^R) = J_{h_n}(\widetilde{u}_{h_n}^R)$ for all $n \geq \overline{n}$. From part 1 of lemma 7.3 we obtain

$$\liminf_{n \to +\infty} h_n^{N-1} J_{h_n}(\widetilde{u}_{h_n}^R) \geq \overline{r}^{N-1} \liminf_{n \to +\infty} \widetilde{\mathcal{E}}_{\overline{r}}(w_{h_n}).$$

Then it makes sense to assume $\overline{n}$ sufficiently large so that for every $n \geq \overline{n}$

$$h_n^{N-1} J_{h_n}(\widetilde{u}_{h_n}^R) \geq \overline{r}^{N-1} \liminf_{n \to +\infty} \mathcal{E}_{\overline{r}}(w_{h_n}) - \frac{\delta}{2}, \tag{8.27}$$

and, by lemma 8.1,

$$h_n^{N-1} \widetilde{c}_{h_n}^* \leq \inf_{r \geq r_0} r^{N-1} \widetilde{\mathcal{E}}_r^* + \frac{\delta}{4}. \tag{8.28}$$

Now (8.10), (8.26), (8.27) and (8.28) yield, for $n \geq \overline{n}$,

$$\begin{aligned} \inf_{r \geq r_0} r^{N-1} \widetilde{\mathcal{E}}_r^* &\geq h_n^{N-1} \widetilde{c}_{h_n}^* - \frac{\delta}{4} \geq h_n^{N-1} J_{h_n}(u_{h_n}) - \frac{\delta}{4} \\ &\geq h_n^{N-1} J_{h_n}(\widetilde{u}_{h_n}^R) - \frac{\delta}{2} \geq \overline{r}^{N-1} \liminf_{n \to +\infty} \widetilde{\mathcal{E}}_{\overline{r}}(w_{h_n}) - \delta \\ &\geq \overline{r}^{N-1} \widetilde{\mathcal{E}}_{\overline{r}}^* - \delta\,, \end{aligned}$$

where we have used the fact that each w_{h_n} belongs to Υ_F^*, the above inequalities and the definition of $\widetilde{\mathcal{E}}_r^*$. The contradiction follows and the proof of claim 3 is now complete.

Then Claim 3, (8.1), and (8.18) imply $\overline{r} > r_0$; therefore, by (8.18) we get

$$r_{h_n}^{\widehat{i}} - \frac{r_0}{h_n} \to +\infty \quad \text{as} \quad n \to +\infty. \tag{8.29}$$

Claim 4. For every $\delta > 0$ there exists $R = R(\delta) > 2R_{\widehat{i}}$ such that, for large n,

$$s \in \big[\frac{r_0}{2h_n}, +\infty\big) \setminus [r^{\widehat{i}}_{h_n} - R, r^{\widehat{i}}_{h_n} + R] \Rightarrow \varphi_{h_n}(s) \le \delta.$$

Proof of claim 4. Fix $\delta > 0$ arbitrarily. According to lemma 8.1 choose $M > 0$ such that

$$\sup_{n\in\mathbb{N}} h_n^{N-1} \widetilde{c}^*_{h_n} < M.$$

By (8.10) we get $\limsup_{n\to+\infty} h_n^{N-1} J_{h_n}(u_{h_n}) < M$, hence by lemma 6.1 we can find $\sigma > 0$ such that

$$\forall n \in \mathbb{N}, \quad \forall s, s' \in \big[\frac{r_0}{2h_n}, +\infty\big): \quad |s-s'| \le \sigma \Rightarrow |\varphi_{h_n}(s) - \varphi_{h_n}(s')| \le \frac{\delta}{2}. \tag{8.30}$$

As Claim 1 assures the existence of $R > 2R_{\widehat{i}}$ and $\overline{n} \in \mathbb{N}$ such that for $n \ge \overline{n}$ and for all $w_n \in \Lambda_F(h_n)$ satisfying

$$h_n^{N-1} J_{h_n}(w_n) \le M, \quad |w_n(x)| \le \overline{\varepsilon} \ \text{ in } \ C_n^R,$$

there results

$$h_n^{N-1} J_{h_n}(\eta^R_{h_n} w_n) \le h_n^{N-1} (J_{h_n})_{\big|C_n^R}(w_n) + V_0 \frac{\delta^2}{32} \omega_N \left(\frac{r_0}{2}\right)^{N-1} \sigma. \tag{8.31}$$

Furthermore, by Claim 2, we may assume

$$\widetilde{u}^R_{h_n} \in \Lambda^*_F(h_n), \quad |u_{h_n}(x)| \le \frac{\overline{\varepsilon}}{2} \quad \forall x \in C_n^R \quad \forall n \ge \overline{n}. \tag{8.32}$$

Observe that $\lim_{n\to+\infty} \frac{1}{h_n} \big((\frac{r_0}{2} + h_n\sigma)^N - (\frac{r_0}{2})^N \big) = N\sigma (\frac{r_0}{2})^{N-1}$. Then, taking also into account of (8.29), without loss of generality we may assume $\overline{n}$ chosen sufficiently large so that for every $n \ge \overline{n}$

$$\frac{(\frac{r_0}{2} + h_n\sigma)^N - (\frac{r_0}{2})^N}{h_n} > \frac{N}{2} \sigma \left(\frac{r_0}{2}\right)^{N-1}, \quad \text{and} \quad r^{\widehat{i}}_{h_n} - \frac{r_0}{h_n} > R + \sigma. \tag{8.33}$$

Arguing by contradiction, assume that there exists $\nu \ge \overline{n}$ and $s_\nu \in [\frac{r_0}{2h_\nu}, +\infty) \setminus [r^{\widehat{i}}_{h_\nu} - R, r^{\widehat{i}}_{h_\nu} + R]$ such that $\varphi_{h_\nu}(s_\nu) > \delta$. (8.30) allows us to infer

$$\forall s \in [s_\nu - \sigma, s_\nu + \sigma]: \quad \varphi_{h_\nu}(s_\nu) > \frac{\delta}{2}.$$

We recall that u_{h_ν} has been obtained as weak limit of a minimizing sequence $\{u_k^{h_\nu}\}_{k\in\mathbb{N}}$ in $\widetilde{\Lambda}^*_F(h_\nu)$. Lemma 2.2 yields $u_k^{h_\nu} \to u_{h_\nu}$ uniformly in $\mathbb{R}^N$, then for k large enough there results

$$|u_k^{h_\nu}(x)| > \frac{\delta}{2} \ \ \forall x \in C(s_\nu - \sigma, s_\nu + \sigma), \quad |u_k^{h_\nu}(x)| \le \overline{\varepsilon} \ \ \forall x \in C_\nu^R$$

and, since $J_{h_\nu}(u_k^{h_\nu}) \to \widetilde{c}^*_{h_\nu}$, it follows that $h_\nu^{N-1} J_{h_\nu}(u_k^{h_\nu}) < M$. Then, taking into account of (8.31),

$$h_\nu^{N-1} J_{h_\nu}(\eta^R_{h_\nu} u_k^{h_\nu}) \le h_\nu^{N-1} (J_{h_\nu})_{\big| C^R_\nu}(u_k^{h_\nu}) + V_0 \frac{\delta^2}{32}\omega_N \left(\frac{r_0}{2}\right)^{N-1} \sigma. \qquad (8.34)$$

Since $s_\nu \in [\frac{r_0}{2h_\nu}, +\infty) \setminus [r^{\widehat{i}}_{h_\nu} - R, r^{\widehat{i}}_{h_\nu} + R]$ one of the following two eventualities occur: either $s_\nu > r^{\widehat{i}}_{h_\nu} + R$ and then $C(s_\nu, s_\nu + \sigma) \subset \mathbb{R}^N \setminus B(0, r^{\widehat{i}}_{h_\nu} + R)$, or $s_\nu \in [\frac{r_0}{2h_\nu}, r^{\widehat{i}}_{h_\nu} - R)$; in the latter case, by the second relation of (8.33), we get that the length of the interval $(s_\nu - \sigma, s_\nu + \sigma) \cap \big[\frac{r_0}{2h_\nu}, r^{\widehat{i}}_{h_\nu} - R\big)$ is bigger then σ. Hence in both cases it happens that for large k $|u_k^{h_\nu}|$ is bigger than $\frac{\delta}{2}$ in an ring contained in $\mathbb{R}^N \setminus \Big(B(0, \frac{r_0}{2h_\nu}) \cup C^R_\nu\Big)$ whose Lebesgue measure is bigger then $\frac{\omega_N}{N}((\frac{r_0}{2h_\nu} + \sigma)^N - (\frac{r_0}{2h_\nu})^N)$; from the definition of J_{h_ν} we obtain

$$J_{h_\nu}(u_k^{h_\nu}) \ge (J_{h_\nu})_{\big| C^R_\nu}(u_k^{h_\nu}) + V_0 \frac{\delta^2}{8}\frac{\omega_N}{N}\Big(\big(\frac{r_0}{2h_\nu} + \sigma\big)^N - \big(\frac{r_0}{2h_\nu}\big)^N\Big).$$

Inserting the last inequality in (8.34), one obtains

$$\begin{aligned} & h_\nu^{N-1} J_{h_\nu}(\eta^R_{h_\nu} u_k^{h_\nu}) \\ \le \quad & h_\nu^{N-1} J_{h_\nu}(u_k^{h_\nu}) + V_0 \frac{\delta^2}{32}\omega_N \left(\frac{r_0}{2}\right)^{N-1} \sigma - V_0 \frac{\delta^2}{8}\frac{\omega_N}{N} \frac{(\frac{r_0}{2} + h_\nu \sigma)^N - (\frac{r_0}{2})^N}{h_\nu}. \end{aligned}$$

By the first of (8.33) we conclude with the following inequality which holds for k sufficiently large

$$J_{h_\nu}(\eta^R_{h_\nu} u_k^{h_\nu}) < J_{h_\nu}(u_{h_\nu}) - \frac{V_0}{h_\nu^{N-1}} \frac{\delta^2}{16}\omega_N \left(\frac{r_0}{2}\right)^{N-1} \sigma. \qquad (8.35)$$

What we are going to prove now is that at least for large k,

$$\eta^R_{h_\nu} u_k^{h_\nu} \in \widetilde{\Lambda}^*_F(h_\nu). \qquad (8.36)$$

But first observe how, assuming (8.36), the thesis of Step 4 easily follows. Indeed by (8.35) we immediately obtain that $\{\eta^R_{h_\nu} u_k^{h_\nu}\}$, just like $\{u_k^{h_\nu}\}$, is a minimizing sequence in $\widetilde{\Lambda}^*_F(h_\nu)$; now, putting $\gamma = \frac{V_0}{h_\nu^{N-1}} \frac{\delta^2}{16}\omega_N (\frac{r_0}{2})^{N-1}\sigma > 0$, it is obvious that γ is independent on k. Then, by passing to the limit as $k \to +\infty$ in (8.35), we achieve the contradiction $\widetilde{c}^*_{h_\nu} \le \widetilde{c}^*_{h_\nu} - \gamma$. Thus it remains to prove (8.36). By construction we obtain $\eta^R_{h_\nu} u_k^{h_\nu} \in F_{h_\nu}$. Note that, since $u_k^{h_\nu} \to u_{h_\nu}$ uniformly in $\mathbb{R}^N$, we also have

$$\eta^R_{h_\nu} u_k^{h_\nu} \to \eta^R_{h_\nu} u_{h_\nu} \equiv \widetilde{u}^R_{h_\nu} \quad \text{uniformly as} \quad k \to +\infty. \qquad (8.37)$$

Since $\inf_{x \in \mathbb{R}^N} |\widetilde{u}^R_{h_\nu}(x) - \overline{\xi}| > 0$, by (8.37) for k large enough $\eta^R_{h_\nu} u_k^{h_\nu} \ne \overline{\xi}$, by which $\eta^R_{h_\nu} u_k^{h_\nu} \in \Lambda_F(h_\nu)$. Combining (8.37) with lemma 3.2 we deduce

$$ch\left(\eta^R_{h_\nu} u_k^{h_\nu}, \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_\nu}\right)\right) = ch\left(\widetilde{u}^R_{h_\nu}, \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_\nu}\right)\right)$$

at least for k sufficiently large. The second inequality of (8.33) leads to the inclusion $C(r^{\widehat{i}}_{h_\nu}-R, r^{\widehat{i}}_{h_\nu}+R) \subset \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_\nu}\right)$, while $\widetilde{u}^R_{h_\nu} \equiv 0$ in $\mathbb{R}^N \setminus C(r^{\widehat{i}}_{h_\nu}-R, r^{\widehat{i}}_{h_\nu}+R)$; hence, taking into account of (8.32), we obtain $ch\left(\widetilde{u}^R_{h_\nu}, \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_\nu}\right)\right) = ch(\widetilde{u}^R_{h_\nu}) \neq 0$. On the other hand it is obvious that

$$|\eta^R_{h_\nu} u^{h_\nu}_k| \leq |u^{h_\nu}_k| < \frac{\overline{\varepsilon}}{2} \quad \forall x \in \overline{C}\left(\frac{r_0}{2h_\nu}, \frac{r_0}{h_\nu}\right), \quad \forall k \in \mathbb{N}$$

and this concludes the proof.

Claim 5. For large n there results $u_{h_n} \in \widetilde{\Lambda}^*_F(h_n)$, and $J_{h_n}(u_{h_n}) = \widetilde{c}^*_{h_n}$. Furthermore

$$x \in B\left(0, \frac{r_0}{2h_n}\right) \Rightarrow |u_{h_n}(x)| \leq \frac{\alpha}{\sqrt{2}},$$

where $\alpha \in (0, \overline{\varepsilon})$ has been defined at the beginning of section 6. As a corollary, for n sufficiently large u_{h_n} is a solution to equation (1.2).

Proof of Claim 5. By applying last step we have the existence of $R = R(\alpha) > 0$ such that, for large n

$$x \in \mathbb{R}^N \setminus \left(B\left(0, \frac{r_0}{2h_n}\right) \cup C\left(r^{\widehat{i}}_{h_n} - R, r^{\widehat{i}}_{h_n} + R\right)\right) \Rightarrow |u_{h_n}(x)| \leq \frac{\alpha}{(2(N+1))^{1/2}}.$$

In particular by (8.29) we may assume $C\left(\frac{r_0}{2h_n}, \frac{r_0}{h_n}\right) \cap C\left(r^{\widehat{i}}_{h_n} - R, r^{\widehat{i}}_{h_n} + R\right) = \emptyset$ so that for n sufficiently large there results

$$|u_{h_n}(x)| \leq \frac{\alpha}{(2(N+1))^{1/2}} < \frac{\overline{\varepsilon}}{2} \quad \forall x \in C\left(\frac{r_0}{2h_n}, \frac{r_0}{h_n}\right). \tag{8.38}$$

On the other hand in the first part of the proof we have already showed that for large n $u_{h_n} \in \Lambda_F(h_n)$ and $ch\left(u_{h_n}, \mathbb{R}^N \setminus \overline{B}\left(0, \frac{r_0}{h_n}\right)\right) \neq 0$. The definition of $\widetilde{\Lambda}^*_F(h_n)$ leads to $u_{h_n} \in \widetilde{\Lambda}^*_F(h_n)$ for n sufficiently large. But u_{h_n} is the weak limit of a minimizing sequence $u^{h_n}_k$ in $\widetilde{\Lambda}^*_F(h_n)$ for the functional J_{h_n}. Then the weakly lower semi-continuity of J_{h_n} implies

$$\widetilde{c}^*_{h_n} \leq J_{h_n}(u_{h_n}) \leq \liminf_{k \to +\infty} J_{h_n}(u^{h_n}_k) = \widetilde{c}^*_{h_n}$$

i.e. u_{h_n} is the desired minimizing function. Then u_{h_n} is a local minimum for the functional J_{h_n} in $\Lambda_F(h_n)$; as we have pointed out at the beginning of the section, u_{h_n} turns out to be a critical point of J_{h_n}. Then the function u_{h_n} solves the equation

$$-\Delta u_{h_n} + V_{h_n}(x) u_{h_n} - \Delta_p u_{h_n} + G'(hx, u_{h_n}) = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{8.39}$$

To simplify notation, for every $a > 0$ we consider the function $T_a : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ defined by

$$T_a(z) = \begin{pmatrix} \operatorname{sign} z_1 \, \max\{|z_1| - \frac{a}{(2(N+1))^{1/2}}, 0\} \\ \vdots \\ \operatorname{sign} z_{N+1} \max\{|z_{N+1}| - \frac{a}{(2(N+1))^{1/2}}, 0\} \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ \vdots \\ z_{N+1} \end{pmatrix} \in \mathbb{R}^{N+1}.$$

Since each u_{h_n} solves (8.39), we can choose as a test function

$$\psi_{h_n}(x) = \begin{cases} T_\alpha(u_{h_n}(x)) & \text{if } x \in B\left(0, \frac{r_0}{2h_n}\right), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B\left(0, \frac{r_0}{2h_n}\right). \end{cases}$$

By (8.38) we easily infer that ψ_{h_n} is continuous on the circle $\{x \in \mathbb{R}^N : |x| = r_0/2h_n\}$, hence it is continuous everywhere. It is standard to check that $\psi_{h_n} \in H_{h_n}$. By construction there results $\nabla u_{h_n} \equiv \nabla \psi_{h_n}$ a.e. in $\mathbb{R}^N$. Hence, after integration by parts, one gets

$$\begin{aligned} &\int_{B\left(0,\frac{r_0}{2h_n}\right)} \left(|\nabla\psi_{h_n}|^2 + |\nabla\psi_{h_n}|^p\right) \, dx + \int_{B\left(0,\frac{r_0}{2h_n}\right)} \Big(V_{h_n}(x) u_{h_n} \\ &+ W'(u_{h_n}) K_\alpha\left(|u_{h_n}|^2\right) + 2W(u_{h_n}) K'_\alpha\left(|u_{h_n}|^2\right) u_{h_n}\Big) \psi_{h_n} \, dx \quad = \quad 0\,. \end{aligned} \tag{8.40}$$

We want to write the function in the second integral in a more suitable way. First observe that, denoting by $\psi^i_{h_n}$ the component functions of ψ_{h_n}, it holds $u_{h_n}\psi_{h_n} = |\psi_{h_n}|^2 + \frac{\alpha}{(2(N+1))^{1/2}} \sum_{i=1}^{N+1} |\psi^i_{h_n}|$; hence for every $x \in \mathbb{R}^N$, we have

$$\begin{aligned} &\Big(V_{h_n}(x) u_{h_n}(x) + W'(u_{h_n}(x)) K_\alpha\left(|u_{h_n}(x)|^2\right) \\ &+2W(u_{h_n}) K'_\alpha\left(|u_{h_n}(x)|^2\right) u_{h_n}(x)\Big) \psi_{h_n}(x) \\ &\geq \quad V_{h_n}(x)|\psi_{h_n}(x)|^2 + \frac{\alpha}{(2(N+1))^{1/2}} V_{h_n}(x) \\ &\qquad \times \sum_{i=1}^{N+1} |\psi^i_{h_n}(x)| - |W'(u_{h_n}(x))| K_\alpha(|u_{h_n}(x)|^2) \, |\psi_{h_n}(x)| \\ &\qquad +2W(u_{h_n}(x)) K'_\alpha\left(|u_{h_n}(x)|^2\right) |\psi_{h_n}(x)|^2 \\ &\qquad +2W(u_{h_n}(x)) K'_\alpha\left(|u_{h_n}(x)|^2\right) \frac{\alpha}{(2(N+1))^{1/2}} \sum_{i=1}^{N+1} |\psi^i_{h_n}(x)| \\ &\geq \quad \big(V_0 - 2 \sup_{|\xi|\leq\alpha} |W(\xi)| \frac{2\overline{c}}{\alpha^2}\big) |\psi_{h_n}(x)|^2 + \Big(V_0 \frac{\alpha}{(2(N+1))^{1/2}} - \sup_{|\xi|\leq\alpha} |W'(\xi)| \\ &\qquad -2 \sup_{|\xi|\leq\alpha} |W(\xi)| \frac{2c_\alpha}{\alpha^2} \frac{\alpha}{(2(N+1))^{1/2}}\Big) \sum_{i=1}^{N+1} |\psi^i_{h_n}(x)| \\ &= \quad \left(V_0 - \sup_{|\xi|\leq\alpha} |W(\xi)| \frac{4\overline{c}}{\alpha^2}\right) |\psi_{h_n}(x)|^2 \end{aligned}$$

$$+\left(V_0\frac{\alpha}{(2(N+1))^{1/2}+2}-\sup_{|\xi|\le\alpha}|W'(\xi)|\right)\sum_{i=1}^{N}|\psi^i_{h_n}(x)|$$
$$+\frac{2\alpha}{(2(N+1))^{1/2}}\left(\frac{1}{(2(N+1))^{1/2}+2}V_0-\sup_{|\xi|\le\overline{\varepsilon}}|W(\xi)|\frac{2\overline{c}}{\alpha^2}\right)\sum_{i=1}^{N}|\psi^i_{h_n}(x)|.$$

To obtain the last equality we have used the obvious identity

$$\frac{V_0}{(2(N+1))^{1/2}}=\frac{V_0}{(2(N+1))^{1/2}+2}+\frac{2V_0}{(2(N+1))^{1/2}((2(N+1))^{1/2}+2)}.$$

Then (8.4) implies that the terms in the last inequality, which multiply $|\psi_{h_n}(x)|^2$ and $\sum_{i=1}^{N+1}|\psi^i_{h_n}(x)|$ are nonnegative. Then both integrals in (8.40) are nonnegative, hence they must be all zero. We conclude in particular $\psi_{h_n}\equiv 0$ by which $|u_{h_n}(x)|\le\frac{\alpha}{\sqrt{2}}$ for every $x\in B\left(0,\frac{r_0}{2h_n}\right)$. Hence u_{h_n} solves equation (1.2).

Claim 6. End of the proof. To conclude we have just only to combine the results obtained in the last steps. In particular by (8.16), (8.17), (8.18), claims 3, 4 and 5 we have that for small $h>0$ there is a solution u_h of equation (1.2); furthermore, setting $v_h(x)=u_h\left(\frac{x}{h}\right)$ and using a re-scaling argument we infer that for h small enough,

$$\begin{gathered}
Equation (1.1) \text{ has a solution } v_h,\\
|v_h| \text{ has a circle of local maximum points } \{x\in\mathbb{R}^N : |x|=r_h\}, \quad (8.41)\\
|v_h(x)|>1 \ \ \forall x \ \text{ with } \ |x|=r_h; \quad (8.42)\\
r_h>r_0,\ \ r_h^{N-1}\widetilde{\mathcal{E}}^*_{r_h}\to\inf_{r\ge r_0}r^{N-1}\widetilde{\mathcal{E}}^*_r,\\
\limsup_{h\to0^+}\sup_{x\notin C(r_h-\delta,r_h+\delta)}|v_h(x)|\le\alpha\quad\forall\delta>0\,. \quad (8.43)
\end{gathered}$$

Note that in order to deduce (8.42) we have used the continuity of the function $r>r_0\mapsto r^{N-1}\widetilde{\mathcal{E}}^*_r$. The construction of the family of solutions $\{v_h\}$ depends on the particular $\alpha\in(0,\overline{\varepsilon})$ chosen at the beginning of Section 8. To emphasize this fact we denote this family as $\{v_h^\alpha\}$. Let α_j be any sequence of positive numbers such that $\alpha_j\to 0$. We have already observed that each $\alpha'\le\alpha$ still verifies (8.2) and (8.3). Then we can repeat the same arguments we have used for α and obtain that there is a decreasing sequence of positive numbers $h_j\to 0$ such that for all $0<h<h_j$ there is a solution $v_h^{\alpha_j}$ to equation (1.1); furthermore $|v_h^{\alpha_j}|$ has a circle of local maximum $\left\{x\in\mathbb{R}^N : |x|=r_h^j\right\}$ with $|v_h^{\alpha_j}(x)|>1$ for $|x|=r_h^j$; furthermore $r_h^j>r_0$ and

$$\inf_{r\ge r_0}r^{N-1}\widetilde{\mathcal{E}}^*_r-\frac{1}{j}<(r_h^j)^{N-1}\widetilde{\mathcal{E}}^*_{r_h^j}<\inf_{r\ge r_0}r^{N-1}\widetilde{\mathcal{E}}^*_r+\frac{1}{j}\quad\forall h\in(0,h_j)$$

and

$$\sup_{x\notin C(r_h^j-\alpha_j,r_h^j+\alpha_j)}|v_h^j(x)|\le\alpha_j\quad\forall h\in(0,h_j).$$

Then for $h_j \geq h > h_{j-1}$, we just define $v_h = v_h^{\alpha_j}$ and $r_h = r_h^j$. Then by the definition above we clearly have

$$r_h^{N-1}\widetilde{\mathcal{E}}^*_{r_h} \to \inf_{r\geq r_0} r^{N-1}\widetilde{\mathcal{E}}^*_r$$

and

$$\limsup_{h\to 0^+} \sup_{x\notin C(r_h-\delta, r_h+\delta)} |v_h(x)| = 0 \quad \forall \delta > 0. \tag{8.44}$$

Observe that, by (7.12), the function $r > r_0 \mapsto r^{N-1}\widetilde{\mathcal{E}}^*_r$ is coercive; hence the family $\{r_h\}$ is bounded in $\mathbb{R}^N$ for small $h > 0$; hence, considered a generic sequence $h_n \to 0^+$, up to a subsequence, it converges to a point $\overline{r}$. The continuity of the function $r > r_0 \mapsto r^{N-1}\widetilde{\mathcal{E}}^*_r$ implies

$$\overline{r}^{N-1}\widetilde{\mathcal{E}}^*_{\overline{r}} = \inf_{r\geq r_0} r^{N-1}\widetilde{\mathcal{E}}^*_r.$$

Finally taking into account of (8.44), it is easy to show that $\{v_{h_n}\}$ decays uniformly outside every fixed neighbourhood of the circle $\{|x| = \overline{r}\}$. $\diamondsuit$

Remark By i) of lemma 7.4 we immediately obtain that the function $r \geq r_0 \mapsto r^{N-1}\widetilde{\mathcal{E}}^*_r$ is continuous. Moreover, by (7.12), we get $r^{N-1}\widetilde{\mathcal{E}}^*_r \geq r^{N-1}\nu \to +\infty$ as $r \to +\infty$ for some $\nu > 0$. This fact, combined with (8.1), implies that the infimum $\inf_{r\geq r_0} r^{N-1}\mathcal{E}^*_r$ is attained by some $\overline{r} > r_0$. If such minimum point $\overline{r}$ is unique, then all the bound states $\{v_h\}$ we have found in last theorem concentrate at the circle $\{x \in \mathbb{R}^N : |x| = \overline{r}\}$ as $h \to 0^+$ in the sense specified in theorem 8.2. In general $r_h^{N-1}\widetilde{\mathcal{E}}^*_{r_h} \to \inf_{r\geq r_0} r^{N-1}\widetilde{\mathcal{E}}^*_r$ as $h \to 0^+$, where the circle $\{x \in \mathbb{R}^N : |x| = r_h\}$ is a set of local maxima of $|v_h|$ for small h.

Remark We conclude by observing that if we require the function $\mathcal{V}$ to be sufficiently big in r_0, then (8.1) holds true. Indeed, fixed $\overline{r} > r_0$ and $\mathcal{V}(\overline{r})$ arbitrarily, then iii) of lemma 7.4 assures that provided that $\mathcal{V}(r_0)$ is large enough the inequality $\widetilde{\mathcal{E}}^*_{r_0} > (\frac{\overline{r}}{r_0})^{N-1}\widetilde{\mathcal{E}}^*_{\overline{r}}$ is satisfied.

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