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PBW THEOREMS AND FROBENIUS STRUCTURES FOR QUANTUM MATRICES

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Abstract. Let $G \in \{Mat_n(\mathbb{C}), GL_n(\mathbb{C}), SL_n(\mathbb{C})\}$, let $\mathcal{O}_q(G)$ be the quantum function algebra – over $\mathbb{Z}[q, q^{-1}]$ – associated to G, and let $\mathcal{O}_\varepsilon(G)$ be the specialisation of the latter at a root of unity ε, whose order ℓ is odd. There is a quantum Frobenius morphism that embeds $\mathcal{O}(G)$, the function algebra of G, in $\mathcal{O}_\varepsilon(G)$ as a central Hopf subalgebra, so that $\mathcal{O}_\varepsilon(G)$ is a module over $\mathcal{O}(G)$. When $G = SL_n(\mathbb{C})$, it is known by [**3**], [**4**] that (the complexification of) such a module is free, with rank $\ell^{\dim(G)}$. In this note we prove a PBW-like theorem for $\mathcal{O}_q(G)$, and we show that – when G is Mat_n or GL_n – it yields explicit bases of $\mathcal{O}_\varepsilon(G)$ over $\mathcal{O}(G)$. As a direct application, we prove that $\mathcal{O}_\varepsilon(GL_n)$ and $\mathcal{O}_\varepsilon(M_n)$ are free Frobenius extensions over $\mathcal{O}(GL_n)$ and $\mathcal{O}(M_n)$, thus extending some results of [**5**].

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1. The general setup. Let G be a complex semisimple, connected, simply connected affine algebraic group. One can introduce a quantum function algebra $\mathcal{O}_q(G)$, a Hopf algebra over the ground ring $\mathbb{C}[q, q^{-1}]$, where q is an indeterminate, as in [**7**]. If ε is any root of 1, one can specialize $\mathcal{O}_q(G)$ at $q = \varepsilon$, which means taking the Hopf $\mathbb{C}$-algebra $\mathcal{O}_\varepsilon(G) := \mathcal{O}_q(G)/(q-\varepsilon)\mathcal{O}_q(G)$. In particular, for $\varepsilon = 1$ one has $\mathcal{O}_1(G) \cong \mathcal{O}(G)$, the classical (commutative) function algebra over G. Moreover, if the order ℓ of ε is odd, then there exists a Hopf algebra monomorphism $\mathfrak{Fr}\colon \mathcal{O}(G) \cong \mathcal{O}_1(G) \hookrightarrow \mathcal{O}_\varepsilon(G)$, called *quantum Frobenius morphism for* G, which embeds $\mathcal{O}(G)$ inside $\mathcal{O}_\varepsilon(G)$ as a central Hopf subalgebra. Therefore, $\mathcal{O}_\varepsilon(G)$ is naturally a module over $\mathcal{O}(G)$. It is proved in [**4**] and in [**3**] that such a module is free, with rank $\ell^{\dim(G)}$. In the special case of $G = SL_2$, a stronger result was given in [**8**], where an explicit basis was found. We shall give similar results when G is GL_n or $M_n := Mat_n$; namely we provide explicit bases of $\mathcal{O}_\varepsilon(G)$ as a free module over $\mathcal{O}(G)$, where in addition everything is defined replacing $\mathbb{C}$ with $\mathbb{Z}$. The proof is via some (more or less known) PBW theorems for $\mathcal{O}_q(M_n)$ and $\mathcal{O}_q(GL_n)$ – and $\mathcal{O}_q(SL_n)$ as well – as modules over $\mathbb{Z}[q, q^{-1}]$.

Let $M_n := Mat_n(\mathbb{C})$. The algebra $\mathcal{O}(M_n)$ of regular functions on M_n is the unital associative commutative $\mathbb{C}$-algebra with generators $\bar{t}_{i,j}$ $(i, j = 1, \dots, n)$. The semigroup structure on M_n yields on $\mathcal{O}(M_n)$ the natural bialgebra structure given by matrix product – see [**6**], Ch. 7. We can also consider the semigroup-scheme $(M_n)_{\mathbb{Z}}$ associated to M_n, for which a like analysis applies: in particular, its function algebra $\mathcal{O}^{\mathbb{Z}}(M_n)$ is a $\mathbb{Z}$-bialgebra, with the same presentation as $\mathcal{O}(M_n)$ but over the ring $\mathbb{Z}$.

Now we define quantum function algebras. Let R be any commutative ring with unity, and let $q \in R$ be invertible. We define $\mathcal{O}_q^R(M_n)$ as the unital associative R-algebra with generators $t_{i,j}$ $(i, j = 1, \ldots, n)$ and relations

$$\begin{aligned} t_{i,j}t_{i,k} &= q t_{i,k}t_{i,j}, & t_{i,k}t_{h,k} &= q t_{h,k}t_{i,k} & \forall \quad j < k, i < h, \\ t_{i,l}t_{j,k} &= t_{j,k}t_{i,l}, & t_{i,k}t_{j,l} - t_{j,l}t_{i,k} &= \left(q - q^{-1}\right) t_{i,l}t_{j,k} & \forall \quad i < j, k < l. \end{aligned}$$

It is known that $\mathcal{O}_q^R(M_n)$ is a bialgebra, but we do not need this extra structure in the present work (see [**6**] for further details – cf. also [**1**] and [**12**]).

As to specialisations, set $\mathbb{Z}_q := \mathbb{Z}[q, q^{-1}]$, let $\ell \in \mathbb{N}_+$ be odd, let $\phi_\ell(q)$ be the ℓ-th cyclotomic polynomial in q, and let $\varepsilon := \overline{q} \in \mathbb{Z}_\varepsilon := \mathbb{Z}_q/(\phi_\ell(q))$, so that ε is a (formal) primitive ℓ-th root of 1 in $\mathbb{Z}_\varepsilon$. Then

$$\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n) = \mathcal{O}_q^{\mathbb{Z}_q}(M_n) \big/ \big(\phi_\ell(q)\big)\mathcal{O}_q^{\mathbb{Z}_q}(M_n) \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}_q^{\mathbb{Z}_q}(M_n).$$

It is also known that there is a bialgebra isomorphism

$$\mathcal{O}_1^{\mathbb{Z}}(M_n) \cong \mathcal{O}_q^{\mathbb{Z}_q}(M_n)\big/(q-1)\mathcal{O}_q^{\mathbb{Z}_q}(M_n) \lhook\joinrel\twoheadrightarrow \mathcal{O}^{\mathbb{Z}}(M_n), \quad t_{i,j} \bmod (q-1)\mathcal{O}_q^{\mathbb{Z}_q}(M_n) \mapsto \bar{t}_{i,j}$$

and a bialgebra monomorphism, called *quantum Frobenius morphism* (ε and ℓ as above),

$$\mathfrak{Fr}_{\mathbb{Z}} \colon \mathcal{O}^{\mathbb{Z}}(M_n) \cong \mathcal{O}_1^{\mathbb{Z}}(M_n) \lhook\joinrel\longrightarrow \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n), \quad \bar{t}_{i,j} \mapsto t_{i,j}^{\ell}\big|_{q=\varepsilon}$$

whose image is central in $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$. Thus $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n) := \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{Z}}(M_n)$ becomes identified – via $\mathfrak{Fr}_{\mathbb{Z}}$, which clearly extends to $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$ by scalar extension – with a central subbialgebra of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$, so the latter can be seen as an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$-module. By the result in [**4**] and [**3**] mentioned above, we can expect this module to be free, with rank ℓ^{n^2}.

All the previous framework also extends to GL_n and to SL_n instead of M_n. Indeed, consider the *quantum determinant* $D_q := \sum_{\sigma \in \mathcal{S}_n} (-q)^{\ell(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} \in \mathcal{O}_q^R(M_n)$, where $\ell(\sigma)$ denotes the length of any permutation σ in the symmetric group $\mathcal{S}_n$. Then D_q belongs to the centre of $\mathcal{O}_q^R(M_n)$, hence one can extend $\mathcal{O}_q^R(M_n)$ by a formal inverse to D_q, i.e. defining the algebra $\mathcal{O}_q^R(GL_n) := \mathcal{O}_q^R(M_n)[D_q^{-1}]$. Similarly, we can define also $\mathcal{O}_q^R(SL_n) := \mathcal{O}_q^R(M_n)/(D_q - 1)$. Now $\mathcal{O}_q^R(GL_n)$ and $\mathcal{O}_q^R(SL_n)$ are Hopf R-algebras, and the maps $\mathcal{O}_q^R(M_n) \hookrightarrow \mathcal{O}_q^R(GL_n)$, $\mathcal{O}_q^R(GL_n) \twoheadrightarrow \mathcal{O}_q^R(SL_n)$, $\mathcal{O}_q^R(M_n) \twoheadrightarrow \mathcal{O}_q^R(SL_n)$ (the third one being the composition of the first two) given by $t_{i,j} \mapsto t_{i,j}$ are epimorphisms of R-bialgebras, and even of Hopf R-algebras in the second case. The specialisations

$$\begin{aligned} \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n) &= \mathcal{O}_q^{\mathbb{Z}_q}(GL_n)\big/(\phi_\ell(q))\mathcal{O}_q^{\mathbb{Z}_q}(GL_n) \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}_q^{\mathbb{Z}_q}(GL_n) \\ \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(SL_n) &= \mathcal{O}_q^{\mathbb{Z}_q}(SL_n)\big/(\phi_\ell(q))\mathcal{O}_q^{\mathbb{Z}_q}(SL_n) \cong \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}_q^{\mathbb{Z}_q}(SL_n) \end{aligned}$$

enjoy the same properties as above, namely there exist isomorphisms $\mathcal{O}_1^{\mathbb{Z}}(GL_n) \cong \mathcal{O}^{\mathbb{Z}}(GL_n)$ and $\mathcal{O}_1^{\mathbb{Z}}(SL_n) \cong \mathcal{O}^{\mathbb{Z}}(SL_n)$ and there are quantum Frobenius morphisms

$$\begin{aligned} \mathfrak{Fr}_{\mathbb{Z}} \colon \mathcal{O}^{\mathbb{Z}}(GL_n) \cong \mathcal{O}_1^{\mathbb{Z}}(GL_n) &\hookrightarrow \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n), \\ \mathfrak{Fr}_{\mathbb{Z}} \colon \mathcal{O}^{\mathbb{Z}}(SL_n) \cong \mathcal{O}_1^{\mathbb{Z}}(SL_n) &\hookrightarrow \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(SL_n) \end{aligned}$$

described by the same formulæ as for M_n. Moreover, $D_q^{\pm 1} \bmod (q-1) \mapsto D^{\pm 1}$ in the isomorphisms and $D^{\pm 1} \cong D_q^{\pm 1} \bmod (q-1) \mapsto D_q^{\pm \ell} \bmod (q-\varepsilon)$ in the quantum Frobenius morphisms for GL_n (which extend those of M_n). In addition, all these isomorphisms and quantum Frobenius morphisms are compatible (in the obvious sense) with the natural maps which link $\mathcal{O}_q^{\mathbb{Z}_q}(M_n)$, $\mathcal{O}_q^{\mathbb{Z}_q}(GL_n)$ and $\mathcal{O}_q^{\mathbb{Z}_q}(SL_n)$, and their specialisations, to each other.

Like for M_n, the image of the quantum Frobenius morphisms are central in $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$ and in $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(SL_n)$. Thus $\mathcal{O}^{\mathbb{Z}_\varepsilon}(GL_n) := \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{Z}}(GL_n)$ identifies to a central Hopf subalgebra of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$, and $\mathcal{O}^{\mathbb{Z}_\varepsilon}(SL_n) := \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}} \mathcal{O}^{\mathbb{Z}}(SL_n)$ identifies to a central Hopf subalgebra of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(SL_n)$; so $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$ is an $\mathcal{O}^{\mathbb{Z}}(GL_n)$-module and $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(SL_n)$ is an $\mathcal{O}^{\mathbb{Z}}(SL_n)$-module.

In § 2, we shall prove (Theorem 2.1) a PBW-like theorem providing several different bases for $\mathcal{O}_q^R(M_n)$, $\mathcal{O}_q^R(GL_n)$ and $\mathcal{O}_q^R(SL_n)$ as R-modules. As an application, we find (Theorem 2.2) explicit bases of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$ as an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$-module, which then in particular is free of rank $\ell^{\dim(M_n)}$. The same bases are also $\mathcal{O}^{\mathbb{Z}_\varepsilon}(GL_n)$-bases for $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$, which then is free of rank $\ell^{\dim(GL_n)}$. Both results can be seen as extensions of some results in [**4**].

Finally, in § 3 we use the above mentioned bases to prove that $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$ is a free Frobenius extension of its central subalgebra $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$, and to explicitly compute the associated Nakayama automorphism. The same we do for $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$ as well. Everything follows from the ideas and methods in [**5**], now applied to the explicit bases given by Theorem 2.2.

2. PBW–like theorems.

THEOREM 2.1. *(PBW theorem for $\mathcal{O}_q^R(M_n)$, $\mathcal{O}_q^R(GL_n)$ and $\mathcal{O}_q^R(SL_n)$ as R-modules) Assume $(q-1)$ is not invertible in $R_q := \langle q, q^{-1}\rangle$, the subring of R generated by q and q^{-1}.*

(a) Let any total order be fixed in $\{1, \ldots, n\}^{\times 2}$. Then the following sets of ordered monomials are R-bases of $\mathcal{O}_q^R(M_n)$, resp. $\mathcal{O}_q^R(GL_n)$, resp. $\mathcal{O}_q^R(SL_n)$, as modules over R:

$$B_M := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \,\middle|\, N_{i,j} \in \mathbb{N} \forall i, j \right\}$$

$$B_{GL}^{\wedge} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} D_q^{-N} \,\middle|\, N, N_{i,j} \in \mathbb{N} \forall i, j; \min\left(\{N_{i,i}\}_{1\leq i\leq n} \cup \{N\}\right) = 0 \right\}$$

$$B_{GL}^{\vee} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} D_q^{Z} \,\middle|\, Z \in \mathbb{Z}, N_{i,j} \in \mathbb{N} \forall i, j; \min\{N_{i,i}\}_{1\leq i\leq n} = 0 \right\}$$

$$B_{SL} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \,\middle|\, N_{i,j} \in \mathbb{N} \forall i, j; \min\{N_{i,i}\}_{1\leq i\leq n} = 0 \right\}.$$

(b) Let $\preceq$ be any total order fixed in $\{1, \ldots, n\}^{\times 2}$ such that $(i, j) \preceq (h, k) \preceq (l, m)$ whenever $j > n+1-i$, $k = n+1-h$, $m < n+1-l$. Then the following sets of ordered

monomials are R-bases of $\mathcal{O}_q^R(GL_n)$*, resp.* $\mathcal{O}_q^R(SL_n)$*, as modules over R:*

$$B_{GL}^{\wedge,-} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} D_q^{-N} \,\middle|\, N, N_{i,j} \in \mathbb{N} \forall i,j; \min(\{N_{i,n+1-i}\}_{1 \le i \le n} \cup \{N\}) = 0 \right\}$$

$$B_{GL}^{\vee,-} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} D_q^{Z} \,\middle|\, Z \in \mathbb{Z}, N_{i,j} \in \mathbb{N} \forall i,j; \min\{N_{i,n+1-i}\}_{1 \le i \le n} = 0 \right\}$$

$$B_{SL}^{-} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \,\middle|\, N_{i,j} \in \mathbb{N} \forall i,j; \min\{N_{i,n+1-i}\}_{1 \le i \le n} = 0 \right\}.$$

Proof. Roughly speaking, our method is a (partial) application of the diamond lemma (see [**2**]): however, we do not follow it in all details, as we use a specialisation trick as a shortcut.

If we prove our results for the algebras defined over R_q instead of R, then the same results will hold as well by scalar extension. Thus we can assume $R = R_q$, and then we note that, by our assumption, the specialised ring $\overline{R} := R/(q-1)R \neq \{0\}$ *is non-trivial.*

Proof of (a): (see also [**10**]*, Theorem 3.1, and* [**12**]*, Theorem 3.5.1)*

We begin with $\mathcal{O}_q^R(M_n)$. It is clearly spanned over R by the set of all (possibly unordered) monomials in the t_{ij}'s: so we must only prove that any such monomial belongs to the R-span of the ordered monomials. In fact, the latter are linearly independent, since such are their images via specialisation $\mathcal{O}_q^R(M_n) \twoheadrightarrow \mathcal{O}_q^R(M_n)/(q-1)\mathcal{O}_q^R(M_n) \cong \mathcal{O}_1^{\overline{R}}(M_n)$.

Thus, take any (possibly unordered) monomial in the t_{ij}'s, say $\underline{t} := t_{i_1,j_1} t_{i_2,j_2} \cdots t_{i_k,j_k}$, where k is the degree of $\underline{t}$: we associate to it its weight, defined as

$$w(\underline{t}) := (k, d_{1,1}, d_{1,2}, \ldots, d_{1,n}, d_{2,1}, d_{2,2}, \ldots, d_{2,n}, d_{3,1}, \ldots, d_{n-1,n}, d_{n,1}, d_{n,2}, \ldots, d_{n,n})$$

where $d_{i,j} := |\{s \in \{1,\ldots,k\} | (i_s, j_s) = (i,j)\}|$ = number of occurrences of $t_{i,j}$ in $\underline{t}$. Then $w(\underline{t}) \in \mathbb{N}^{n^2+1}$, and we consider $\mathbb{N}^{n^2+1}$ as a totally ordered set with respect to the (total) lexicographic order $\preceq_{lex}$. By a quick look at the defining relations of $\mathcal{O}_q^R(M_n)$, namely

$$t_{i,j} t_{i,k} = q t_{i,k} t_{i,j}, \qquad t_{i,k} t_{h,k} = q t_{h,k} t_{i,k} \qquad \forall \;\; j < k, i < h,$$
$$t_{i,l} t_{j,k} = t_{j,k} t_{i,l}, \quad t_{i,k} t_{j,l} - t_{j,l} t_{i,k} = (q - q^{-1}) t_{i,l} t_{j,k} \qquad \forall \;\; i < j, k < l.$$

one easily sees that the weight defines an algebra filtration on $\mathcal{O}_q^R(M_n)$.

Now, using these same relations, one can re-order the t_{ij}'s in any monomial according to the fixed total order. During this process, only two non-trivial things may occur, namely:

–1) some powers of q show up as coefficients (when a relation in the first line is employed);

–2) a new summand is added (when the bottom-right relation is used);

If only steps of type 1) occur, then the process eventually stops with an ordered monomial in the t_{ij}'s multiplied by a power of q. Whenever instead a step of type 2) occurs, the newly added term is just a coefficient $(q - q^{-1})$ times a (possibly unordered) monomial in the t_{ij}'s, call it $\underline{t}'$: however, by construction $w(\underline{t}') \precneqq_{lex} w(\underline{t})$. Then, by induction on the weight, we can assume that $\underline{t}'$ lies in the R-span of the ordered monomials, so we can ignore the new summand. The process stops in finitely many steps, and we are done with $\mathcal{O}_q^R(M_n)$.

Second, we look at $\mathcal{O}_q^R(GL_n)$. Let us consider $f \in \mathcal{O}_q^R(GL_n)$. By definition, there exists $N \in \mathbb{N}$ such that $f D_q^N \in \mathcal{O}_q^R(M_n)$; therefore, by the result for $\mathcal{O}_q^R(M_n)$ just proved, we can expand $f D_q^N$ as an R-linear combination of ordered monomials, call them $\underline{t} = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}}$. Thus, f itself is an R-linear combination of monomials $\underline{t} D_q^{-N}$, so the latter span $\mathcal{O}_q^R(GL_n)$.

Now consider an ordered monomial $\underline{t} = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}}$ in which $N_{i,i} > 0$ for all i. Then we can re-arrange the $t_{i,i}$'s in $\underline{t}$ so to single out a factor $t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n}$, up to "paying the cost" (perhaps) of producing some new summands *of lower weight*: the outcome reads

$$\underline{t} = q^s \underline{t}_0 t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n} + l.t.\text{'s} \tag{2.1}$$

for some $s \in \mathbb{Z}$, with $\underline{t}_0 := \prod_{i,j=1}^n t_{i,j}^{N_{i,j}-\delta_{i,j}}$ having lower weight than $\underline{t}$, and the expression $l.t.\text{'s}$ standing for an R-linear combination of some monomials $\underline{\check{t}}$ such that $w(\underline{\check{t}}) \lneqq_{lex} w(\underline{t})$. Then we re-write the monomial $t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n}$ using the identity

$$t_{1,1} t_{2,2} \cdots t_{n-1,n-1} t_{n,n} = D_q - \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \neq id}} (-q)^{\ell(\sigma)} t_{1,\sigma(1)} t_{2,\sigma(2)} \cdots t_{n,\sigma(n)} = D_q + l.t.\text{'s} \tag{2.2}$$

and we replace the right-hand side of (2.2) inside (2.1). We get $\underline{t} = q^s \underline{t}_0 D_q + l.t.\text{'s}$ (for D_q is central!), where now $\underline{t}_0$ and all monomials within $l.t.\text{'s}$ have strictly lower weight than $\underline{t}$.

If we look now at $\underline{t} D_q^z$ (for some $z \in \mathbb{Z}$), we can re-write $\underline{t}$ as above, thus getting

$$\underline{t} D_q^z = q^s \underline{t}_0 D_q D_q^z + l.t.\text{'s} = q^s \underline{t}_0 D_q^{z+1} + l.t.\text{'s} \tag{2.3}$$

where $l.t.\text{'s}$ is an R-linear combination of monomials $\underline{\tilde{t}} D_q^{z+1}$ such that $w(\underline{\tilde{t}}) \lneqq_{lex} w(\underline{t})$.

By repeated use of (2.3) as a reduction argument, we can easily show – by induction on the weight – that any monomial of type $\underline{t} D_q^{-N}$ ($N \in \mathbb{N}$) can be expanded as an R-linear combination of elements of $B_{GL}^{\wedge}$ or elements of $B_{GL}^{\vee}$. Thus, both these sets do span $\mathcal{O}_q^R(GL_n)$.

To finish with, both $B_{GL}^{\wedge}$ and $B_{GL}^{\vee}$ are R-linearly independent, as their image through the specialisation epimorphism $\mathcal{O}_q^R(GL_n) \twoheadrightarrow \mathcal{O}_1^{\overline{R}}(GL_n) \cong \mathcal{O}^{\overline{R}}(GL_n)$ are $\overline{R}$-bases of $\mathcal{O}^{\overline{R}}(GL_n)$.

As to $\mathcal{O}_q^R(SL_n)$, we can repeat the argument for $\mathcal{O}_q^R(GL_n)$. First, B_{SL} is linearly independent, for its image through specialisation $\mathcal{O}_q^R(SL_n) \twoheadrightarrow \mathcal{O}_1^{\overline{R}}(SL_n) \cong \mathcal{O}^{\overline{R}}(SL_n)$ is an $\overline{R}$-basis of $\mathcal{O}^{\overline{R}}(SL_n)$. Second, the epimorphism $\mathcal{O}_q^R(M_n) \twoheadrightarrow \mathcal{O}_q^R(SL_n) (t_{i,j} \mapsto t_{i,j})$, and the result for $\mathcal{O}_q^R(M_n)$, imply that the R-span of $S_{SL} := \{\prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \mid N_{i,j} \in \mathbb{N} \forall i, j\}$ is $\mathcal{O}_q^R(SL_n)$. Thus one is only left to prove that each monomial $\underline{t} = \prod_{i,j=1}^n t_{i,j}^{N_{i,j}} \in S_{SL}$ belongs to the R-span of B_{SL}: as before, this can be done by induction on the weight, using the reduction formula $\underline{t} = q^s \underline{t}_0 D_q + l.t.\text{'s}$ (see above), and plugging into the relation $D_q = 1$.

Alternatively, we recall there is an isomorphism $\mathcal{O}_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong \mathcal{O}_q^R(GL_n)$ (of R-algebras) given by $t_{i,j} \otimes x^z \mapsto D_q^{-\delta_{i,1}} t_{i,j} \cdot D_q^z$ (cf. [**11**]). This along with the result about $B_{GL}^{\vee}$ clearly implies that also B_{SL} is an R-basis for $\mathcal{O}_q^R(SL_n)$, as claimed.

<u>*Proof of (b)*</u>*:* First look at $\mathcal{O}_q^R(GL_n)$. If $f \in \mathcal{O}_q^R(GL_n)$, as in the proof of *(a)* we expand $f D_q^N$ as an R-linear combination of ordered (according to $\preceq$) monomials of type $\underline{t} = \underline{t}^- \underline{t}^= \underline{t}^+$, with $\underline{t}^- := \prod_{j>n+1-i} t_{i,j}^{N_{i,j}}$, $\underline{t}^= := \prod_{j=n+1-i} t_{i,j}^{N_{i,j}}$ and $\underline{t}^+ := \prod_{j<n+1-i} t_{i,j}^{N_{i,j}}$. So f is an R-linear combination of monomials $\underline{t}^- \underline{t}^= \underline{t}^+ D_q^{-N}$, hence the latter span $\mathcal{O}_q^R(GL_n)$.

We show that each (ordered) monomial $\underline{t}^- \underline{t}^= \underline{t}^+ D_q^{-N}$ belongs both to the R-span of $B_{GL}^{\wedge,-}$ and of $B_{GL}^{\vee,-}$, by induction on the (total) degree of the monomial $\underline{t}^=$. The basis of induction is $\deg(\underline{t}^=) = 0$, so that $\underline{t}^= = 1$ and $\underline{t}^- \underline{t}^= \underline{t}^+ D_q^{-N} = \underline{t}^- \underline{t}^+ D_q^{-N} \in B_{GL}^{\wedge,-} \cap B_{GL}^{\vee,-}$.

As a matter of notation, let $\mathcal{N}^-$, resp. $\mathcal{H}$, resp. $\mathcal{N}^+$, be the R-subalgebra of $\mathcal{O}_q^R(M_n)$ generated by the $t_{i,j}$'s with $j > n+1-i$, resp. $j = n+1-i$, resp. $j < n+1-i$. Note that $\mathcal{H}$ is Abelian, and $\underline{t}^- \in \mathcal{N}^-$, $\underline{t}^= \in \mathcal{H}$, $\underline{t}^+ \in \mathcal{N}^+$.

Now assume that all the exponents $N_{i,n+1-i}$'s in the factor $\underline{t}^=$ are strictly positive. As $\mathcal{H}$ is Abelian, we can draw out of $\underline{t}^=$ (even out of $\underline{t} = \underline{t}^- \underline{t}^= \underline{t}^+$) a factor $t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n}$. Now recall that D_q can be expanded as $D_q = \sum_{\sigma \in \mathcal{S}_n} (-q)^{\ell(\sigma)} t_{n,\sigma(n)} t_{n-1,\sigma(n-1)} \cdots t_{2,\sigma(2)} t_{1,\sigma(1)}$ (see, e.g., [**12**] or [**10**]). Then we can re-write the monomial $t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n}$ as

$$t_{n,1} t_{n-1,2} \cdots t_{1,n} = (-q)^{-\ell(\sigma_0)} D_q - \sum_{\substack{\sigma \in \mathcal{S}_n \\ \sigma \neq \sigma_0}} (-q)^{\ell(\sigma)-\ell(\sigma_0)} t_{n,\sigma(n)} t_{n-1,\sigma(n-1)} \cdots t_{1,\sigma(1)} \tag{2.4}$$

where $\sigma_0 \in \mathcal{S}_n$ is the permutation $i \mapsto (n+1-i)$. Note also that we can reorder the factors in the summands of (2.4) so that all factors $t_{i,j}$ from $\mathcal{N}^-$ are on the left of those from $\mathcal{N}^+$.

Now we replace the right-hand side of (2.4) in the factor $\underline{t}^=$ within $\underline{t} = \underline{t}^- \underline{t}^= \underline{t}^+$, thus

$$\underline{t}^- \underline{t}^= \underline{t}^+ = (-q)^{-\ell(\sigma_0)} \underline{t}^- \underline{t}_0^= D_q \underline{t}^+ + l.t.\text{'s} = (-q)^{-\ell(\sigma_0)} \underline{t}^- \underline{t}_0^= \underline{t}^+ D_q + l.t.\text{'s}.$$

Here $\underline{t}_0^= := \underline{t}^= (t_{n,1} t_{n-1,2} \cdots t_{2,n-1} t_{1,n})^{-1}$ has lower (total) degree than $\underline{t}^=$, and the expression *l.t.'s* stands for an R-linear combination of some other monomials $\underline{\hat{t}}^- \underline{\hat{t}}^= \underline{\hat{t}}^+$ (like $\underline{t}^- \underline{t}^= \underline{t}^+$ above) in which again the degree of $\underline{\hat{t}}^=$ is lower than the degree of $\underline{t}^=$. In fact, this holds because when any factor $t_{i,\sigma(i)} \in \mathcal{N}^-$ is pulled from the right to the left of any monomial in $\underline{\check{t}}^= \in \mathcal{H}$ the degree of $\underline{\check{t}}^=$ is not increased. By induction on this degree, we can easily conclude that every ordered monomial $\underline{t}^- \underline{t}^= \underline{t}^+ D_q^z$ (with $z \in \mathbb{Z}$) belongs to both the R-span of $B_{GL}^{\wedge,-}$ and the R-span of $B_{GL}^{\vee,-}$. That is, both sets span $\mathcal{O}_q^R(GL_n)$.

Eventually, both $B_{GL}^{\wedge,-}$ and $B_{GL}^{\vee,-}$ are linearly independent, as their image through the specialisation epimorphism $\mathcal{O}_q^R(GL_n) \twoheadrightarrow \mathcal{O}_1^{\overline{R}}(GL_n) \cong \mathcal{O}^{\overline{R}}(GL_n)$ are $\overline{R}$-bases of $\mathcal{O}^{\overline{R}}(GL_n)$.

Second, we look at $\mathcal{O}_q^R(SL_n)$. As for claim *(a)*, we can repeat again – *mutatis mutandis* – the argument for $\mathcal{O}_q^R(GL_n)$, which does work again – one only has to plug in the additional relation $D_q = 1$ too. Otherwise, as an alternative proof, we can note that the isomorphism $\mathcal{O}_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong \mathcal{O}_q^R(GL_n)$ together with the result about $B_{GL}^{\vee,-}$ easily implies that B_{SL}^- too is an R-basis for $\mathcal{O}_q^R(SL_n)$, q.e.d. □

REMARK 2.2. *(1)* Claim *(a)* of Theorem 2.1 for M_n only was independently proved in **[12]** and in **[10]**, but taking a field as ground ring. In **[10]**, claim *(b)* for GL_n only was proved as well. Similarly, the analogue of claim *(b)* for SL_n only was proved in **[9]**, § 7, but taking as ground ring the field $k(q)$ – for any field k of zero characteristic. Our proof then provides an alternative, unifying approach, which yields stronger results over R.

(2) We would better point out a special aspect of the basic assumption of Theorem 2.1 about q and R. Namely, if the subring $\langle 1 \rangle$ of R generated by 1 has prime characteristic (hence it is a finite field) then the condition on $(q-1)$ is equivalent to q being trascendental over R_q or $q = 1$. But if instead the characteristic of $\langle 1 \rangle$ is zero or positive non-prime, then $(q-1)$ might be non-invertible in R_q even though q is algebraic (or even integral) over $\langle 1 \rangle$.

The end of the story is that Theorem 2.1 holds true in the "standard" case of trascendental values of q, but also in more general situations.

(3) The argument used in the proof of Theorem 2.1 to get the result for $\mathcal{O}_q^R(SL_n)$ from those for $\mathcal{O}_q^R(GL_n)$, via the isomorphism $\mathcal{O}_q^R(SL_n) \otimes_R R[x, x^{-1}] \cong \mathcal{O}_q^R(GL_n)$, actually works *both ways*. Therefore, one can also prove the results directly for $\mathcal{O}_q^R(SL_n)$ – as we have sketched above – and from them deduce those for $\mathcal{O}_q^R(GL_n)$. Even more, as we have proved independently the results for $\mathcal{O}_q^R(GL_n)$ – i.e., $B_{GL}^{\vee}$ and $B_{GL}^{\vee,-}$ are R-bases – and for $\mathcal{O}_q^R(SL_n)$ – i.e., B_{SL} and B_{SL}^{-} are R-bases – we can use them to prove that the algebra morphism $\mathcal{O}_q^R(SL_n) \otimes_R R[x, x^{-1}] \longrightarrow \mathcal{O}_q^R(GL_n)$ is in fact bijective.

(4) The orders considered in claim *(b)* of Theorem 2.1 refer to a triangular decomposition of $\mathcal{O}_q^R(GL_n)$ and $\mathcal{O}_q^R(SL_n)$ which is opposite to the standard one. This opposite decomposition was introduced – and its importance was especially pointed out – in **[10]**.

We are now ready to state and prove the main result of this paper:

THEOREM 2.3. (PBW theorem for $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ as an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-module, for $G \in \{M_n, GL_n\}$) *Let any total order be fixed in* $\{1, \dots, n\}^{\times 2}$. *Then the set of ordered monomials*

$$\mathrm{B}_{GL}^{M} := \left\{ \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}} \middle| 0 \leq N_{i,j} \leq \ell - 1, \forall i, j \right\}$$

thought of as a subset of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n) \subset \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$, *is a basis of* $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$ *as a module over* $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$, *and a basis of* $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$ *as a module over* $\mathcal{O}^{\mathbb{Z}_\varepsilon}(GL_n)$.

In particular, both modules are free of rank $\ell^{\dim(G)}$, *with* $G \in \{M_n, GL_n\}$.

Proof. When specialising, Theorem 2.1 *(a)* implies that $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$ is a free $\mathbb{Z}_\varepsilon$-module with $B_M|_{q=\varepsilon} = \{\prod_{i,j=1}^n t_{ij}^{N_{ij}} | N_{ij} \in \mathbb{N} \forall i, j\}$ as basis – where, by abuse of notation, we write again t_{ij} for $t_{ij}|_{q=\varepsilon}$. Now, whenever the exponent N_{ij} is a multiple of ℓ, the power $t_{ij}^{N_{ij}}$ belongs to the isomorphic image $\mathfrak{Fr}_{\mathbb{Z}}(\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n))$ of $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$ inside $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$, hence it is a scalar for the $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$-module structure of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$. Therefore, reducing all exponents modulo ℓ we find that B_{GL}^{M} is a spanning set for the $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$-module $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$. In addition, $\mathcal{O}^{\mathbb{Z}}(M_n)$ clearly admits as $\mathbb{Z}$-basis the set $\overline{B}_M = \{\prod_{i,j=1}^n \bar{t}_{ij}^{N_{ij}} | N_{ij} \in \mathbb{N} \forall i, j\}$. It follows that $\overline{B}_M$ is also a $\mathbb{Z}_\varepsilon$-basis of $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$, so $\mathfrak{Fr}_{\mathbb{Z}}(\overline{B}_M) = \{\prod_{i,j=1}^n t_{ij}^{\ell N_{ij}} | N_{ij} \in \mathbb{N} \forall i, j\}$ is a $\mathbb{Z}_\varepsilon$-basis of $\mathfrak{Fr}_{\mathbb{Z}}(\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n))$. This last fact easily implies that B_{GL}^{M} is also $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$-linearly independent, hence it is a basis of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$ over $\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)$ as claimed.

As to $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$, from definitions and the analysis in § 1 we get (with $D_\varepsilon := D_q|_\varepsilon$)

$$\begin{aligned}\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n) &= \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)\big[D_\varepsilon^{-1}\big] = \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)\big[D_\varepsilon^{-\ell}\big] \\ &= \mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)[D^{-1}] \bigotimes_{\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)} \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n) = \mathcal{O}^{\mathbb{Z}_\varepsilon}(GL_n) \bigotimes_{\mathcal{O}^{\mathbb{Z}_\varepsilon}(M_n)} \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)\end{aligned}$$

thus the result for $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(GL_n)$ follows at once from that for $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(M_n)$. □

3. Frobenius structures.

3.1 Frobenius extensions and Nakayama automorphisms. Following [**5**], we say that a ring R is a *free Frobenius extension* over a subring S, if R is a free S-module of finite rank, and there is an isomorphism $F: R \longrightarrow \mathrm{Hom}_S(R, S)$ of $R - S$-bi-modules. Then F provides a non-degenerate associative S-bilinear form $\mathbb{B}: R \times R \longrightarrow S$, via $\mathbb{B}(r, t) = F(t)(r)$. Conversely, one can characterise Frobenius extensions using such forms. When $S = \mathcal{Z}$ is contained in the centre of R, there is a $\mathcal{Z}$-algebra automorphism $\nu : R \longrightarrow R$, given by $rF(1) = F(1)\nu(r)$ (for all $r \in R$), and such $\mathbb{B}(x, y) = \mathbb{B}(\nu(y), x)$. This is called the *Nakayama automorphism*, and it is uniquely determined by the pair $\mathcal{Z} \subseteq R$, up to $Int(R)$.

Proposition 3.2. *(cf. [5], § 2)*
Let R be a ring, $\mathcal{Z}$ an affine central subalgebra of R. Assume that R is free of finite rank as a $\mathcal{Z}$-module, with a $\mathcal{Z}$-basis $\mathcal{B}$ that satisfies the following condition: there exists a $\mathcal{Z}$-linear functional $\Phi: R \to \mathcal{Z}$ such that for any non-zero $a = \sum_{b\in\mathcal{B}} z_b b \in R$ there exists $x \in R$ for which $\Phi(xa) = uz_b$ for some unit $u \in \mathcal{Z}$ and some non-zero $z_b \in \mathcal{Z}$.
Then R is a free Frobenius extension of $\mathcal{Z}$. Moreover, for any maximal ideal $\mathfrak{m}$ of $\mathcal{Z}$, the finite dimensional quotient $R/\mathfrak{m}R$ is a finite dimensional Frobenius algebra.

This result is used in [**5**] to show that many families of algebras – in particular, some related to $\mathcal{O}_\varepsilon(G)$, where G is a (complex, connected, simply-connected) semisimple affine algebraic group – are indeed free Frobenius extensions. But the authors could not prove the same for $\mathcal{O}_\varepsilon(G)$, as they did not know an explicit $\mathcal{O}(G)$-basis of $\mathcal{O}_\varepsilon(G)$. Now, following their strategy and using Theorem 2.3, I shall now prove that $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ is free Frobenius over $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$ when G is M_n or GL_n.

Theorem 3.3. *Let G be M_n or GL_n. Then $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ is a free Frobenius extension of $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$, with Nakayama automorphism ν given by $\nu(t_{i,j}) = \varepsilon^{2(i+j-n-1)} t_{i,j}$ $(i, j = 1, \ldots, n)$.*

Proof. We prove that there is a suitable $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-linear functional $\Phi: \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G) \longrightarrow \mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$ as required in Proposition 3.2, so that this result applies to $R := \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ and $\mathcal{Z} := \mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$.

Define Φ on the elements of the $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-basis B_{GL}^{M} of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ (see Theorem 2.3) by

$$\Phi\left(\prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}}\right) := \prod_{i,j=1}^{n} \delta_{N_{i,j},\ell-1} = \begin{cases} 1, & \text{if } N_{i,j} = \ell - 1 \forall i, j \\ 0, & \text{if not} \end{cases} \tag{3.1}$$

(for all $0 \leq N_{i,j} \leq \ell - 1$), and extend to all of $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ by $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-linearity. In other words, Φ is the unique $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-valued linear functional on $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ whose value is 1 on the basis element $\underline{t}^{\underline{\ell-1}} := \prod_{i,j=1}^{n} t_{i,j}^{\ell-1}$ and is zero on all other elements of the $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-basis B_{GL}^{M}.

We claim that Φ satisfies the assumptions of Proposition 3.2, so the latter applies and proves our statement. Indeed, let us consider any non-zero $a = \sum_{\underline{t}\in \mathrm{B}_{GL}^{M}} z_{\underline{t}}\underline{t} \in \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$, and let $\underline{t}_0 = \prod_{i,j=1}^{n} t_{i,j}^{N_{i,j}}$ in B_{GL}^{M} be such that $z_{\underline{t}_0} \neq 0$ and $w(\underline{t}_0)$ is maximal (w.r.t. $\leq_{lex}$). Then define $\underline{t}_0^\vee := \prod_{i,j=1}^{n} t_{i,j}^{N'_{i,j}}$ $(\in \mathbb{B}_{GL}^{M})$ with $N'_{i,j} := \ell - 1 - N_{i,j}$ for all $i, j = 1, \dots, n$. Quoting from the proof of Theorem 2.1(a), we know that $\underline{t}_0^\vee \underline{t}_0 = \varepsilon^s \underline{t}^{\underline{\ell-1}} + l.t.'s$, where $s \in \mathbb{Z}$ and the expression $l.t.'s$ now stands for an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-linear combination of monomials $\check{\underline{t}} \in \mathrm{B}_{GL}^{M}$ such that $w(\check{\underline{t}}) \lneqq_{lex} w(\underline{t}^{\underline{\ell-1}})$; in particular, $\Phi(\check{\underline{t}}) = 0$ for all these $\check{\underline{t}}$, hence eventually $\Phi(\underline{t}_0^\vee \underline{t}_0) = \varepsilon^s \Phi(\underline{t}^{\underline{\ell-1}}) = \varepsilon^s$. Similarly, if $\underline{t}' \in \mathbb{B}_{GL}^{M}$ is such that $w(\underline{t}') <_{lex} w(\underline{t})$, then $\underline{t}_0^\vee \underline{t}'$ is an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-linear combination of PBW monomials whose weight is at most $w(\underline{t}_0^\vee \underline{t}')$, hence $\Phi(\underline{t}_0^\vee \underline{t}') = 0$. As we chose $\underline{t}_0$ so that $w(\underline{t}_0)$ is maximal, we eventually find

$$\Phi\big(\underline{t}_0^\vee a\big) = \sum_{\underline{t}\in \mathrm{B}_{GL}^{M}} z_{\underline{t}} \Phi(\underline{t}) = z_{\underline{t}_0}\Phi(\underline{t}_0) = \varepsilon^s z_{\underline{t}_0}$$

where ε^s is a unit in $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$. So Φ satisfies the assumptions of Proposition 3.2, as claimed.

As to the Nakayama automorphism $\nu\colon \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G) \longrightarrow \mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$, it is characterized (see § 3.1) by the property that $\mathbb{B}(x, y) = \mathbb{B}(\nu(y), x)$ for all $x, y \in R$. Here $\mathbb{B}$ is a $\mathcal{Z}$-bilinear form as in § 3.1, which now is related to Φ by the formula $\mathbb{B}(x, y) = \Phi(xy)$ for all $x, y \in R$.

As Φ is an automorphism, and $\mathcal{O}_\varepsilon^{\mathbb{Z}_\varepsilon}(G)$ is generated – over $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$ – by the $t_{i,j}$'s, the claim about ν is proved if we show that

$$\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j}\right) = \Phi\left(\varepsilon^{2(i+j-n-1)} t_{i,j} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}}\right). \tag{3.2}$$

Now, our usual argument shows that the expansions of the product of a generator $t_{i,j}$ and a PBW monomial $\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}}$ (in either order of the factors) as an $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-linear combination of elements of the $\mathcal{O}^{\mathbb{Z}_\varepsilon}(G)$-basis B_{GL}^{M} are of the form

$$\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j} = \varepsilon^{i+j-2n} \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}} + l.t.'s$$

$$t_{i,j} \cdot \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} = \varepsilon^{2-i-j} \prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}} + l.t.'s.$$

This along with (3.1) gives

$$\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j}\right) = \varepsilon^{i+j-2n}\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = \varepsilon^{i+j-2n} \quad \text{if } e_{r,s} = \ell - 1 - \delta_{r,i}\delta_{j,s}$$

$$\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}} \cdot t_{i,j}\right) = \varepsilon^{i+j-2n}\Phi\left(\prod_{r,s=1}^{n} t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right) = 0 \quad \text{if not}$$

and similarly

$$\Phi\left(t_{i,j}\cdot\prod_{r,s=1}^{n}t_{r,s}^{e_{r,s}}\right)=\varepsilon^{2-i-j}\Phi\left(\prod_{r,s=1}^{n}t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right)=\varepsilon^{2-i-j}\quad\text{if } e_{r,s}=\ell-1-\delta_{r,i}\delta_{j,s}$$

$$\Phi\left(t_{i,j}\cdot\prod_{r,s=1}^{n}t_{r,s}^{e_{r,s}}\right)=\varepsilon^{2-i-j}\Phi\left(\prod_{r,s=1}^{n}t_{r,s}^{e_{r,s}+\delta_{r,i}\delta_{j,s}}\right)=0\quad\text{if not.}$$

Direct comparison now shows that (3.2) holds, q.e.d. □

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