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# Covering a line segment with variable radius discs

Alessandro Agnetis\*    Enrico Grande†    Pitu B. Mirchandani†    Andrea Pacifici‡

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## Abstract

The paper addresses the problem of locating sensors with a circular field of view so that a given line segment is under full surveillance, which is termed as the *Disc Covering Problem on a Line*. The cost of each sensor includes a fixed component  $f$ , and a variable component that is a convex function of the diameter of the field-of-view area. When only one type of sensor or, in general, one type of disc, is available, then a simple polynomial algorithm solves the problem. When there are different types of sensors, the problem becomes hard. A branch-and-bound algorithm as well as an efficient heuristic are developed for the special case in which the variable cost component of each sensor is proportional to the square of the measure of the field-of-view area. The heuristic very often obtains the optimal solution as shown in extensive computational testing.

## Scope and Purpose

Problems of locating facilities to cover sets of points on networks and planes have been widely studied. This paper focuses on a new covering problem that is motivated by an application where a line segment is to be kept under surveillance using different types of radars. Using reasonable assumptions, some nonlinear covering problems are formulated. Efficient exact algorithms and heuristics are developed and analyzed for “easy” and “hard” cases, respectively.

**Keywords:** Sensor location, network covering problems, mixed integer nonlinear programming.

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# 1 Introduction

In this paper we introduce and study a new locational decision problem: given a set of discs with variable radii with costs depending on their radii and fixed costs, find a subset covering a unit length segment at minimum cost.

This problem was motivated by the following application, part of which was an industry-funded radar surveillance project at The University of Arizona. We have a river over which we need to track possible activities of non-collaborative or antagonistic objects or people (e.g., unauthorized boats, dangerous floating objects, swimmers, etc). For this purpose, we need to locate a set of radars so that every point on the river is under surveillance by at least one radar. It is assumed that the river can be modeled as a tree network consisting of line segments and that each radar has a field of view defined by a radius and an angle of view (a pie-shaped coverage), with this angle large enough so that the coverage area may be approximated as a disc. Although the problem is relatively easily stated, the actual locational decision is complicated due to several additional factors. Coverage depends not only on the river topology, radar type and power, but also on several parameters such as width of river and obstacles over it, potentially forbidden areas where radars may not be located, elevation of the potential location sites, and other characteristics associated with the physical environment, dealing with, for example, the atmospheric and water conditions. Further details on this scenario and the scope of the project are reported in [20].

This radar sensors location model relates to several broad classes of geometric locational problems. Many important land-use planning decisions deal with locating facilities at sites, choosing from a given set of potential sites, so that another given set of points are “covered” (i.e., they are within a specified distance from the closest located facility) while optimizing a specified objective. Models for locating at points within continuous spaces, as well as locating among set of discrete points or on a network, are widely used by geographers, regional scientists, network planners, and others facing locational decisions problems which can be modeled as such covering problems (for a comprehensive review of the literature see, for example, [7, 13, 15, 18]). From the methodological viewpoint, the radars location problem relates closely to the class of geometric covering problems where potential facilities and demand points are embedded on a Euclidean plane, for which there is considerable literature. We briefly review below results that are most relevant for our application.

Problems related to *Covering with discs* consists of identifying the minimum number of discs with fixed radius to cover a given set of points in the plane. A number of articles have appeared in the last three decades addressing this NP-hard problem. In 1975, Chvátal introduced the *Art Gallery Problem* in [3], where one has to find the minimum number of watchmen (or cameras) needed to observe every wall of an art gallery room. The art gallery is assumed to be a  $n$ -sided polygon, possibly with polygonal holes. It has been shown that an art gallery with  $h$  holes and  $n$  walls (including holes’ sides) requires at most  $\lfloor (n+h)/3 \rfloor$  watchmen (the bound is tight, see [11, 19]). Another important paper, by Hochbaum and Maas [10], presents polynomial approximation algorithms for different versions of geometric covering problems, including covering by discs. Subsequently, several papers have appeared with improved approximation factors and running times (see for example, [2, 4]).

The problem of partial covering, also referred to as the robust  $k$ -center problem, is analyzed in [22], where computational complexity is discussed and approximation algorithms together with computational evidence of their performance are provided.

The geometric disc covering problem relates also to the deployment of wireless transmission networks. Surveys on coverage problems dealing with this particular application can be found in [12] and [21]. We limit our literature review to a few papers dealing with applications similar to the radar sensors location problem. Alt *et al.* [1] consider a problem where a set of points demand connectivity. Their goal is to locate a set of sensors, modeled as discs with variable radii, covering the demand points at minimum total cost. Each sensor's transmission cost has the form  $f(r) = r^\alpha$  where  $r$  is the covering radius of the sensor. Several results are presented in [1], including complexity characterization and approximation algorithms. Although different scenarios are addressed, depending on possible restrictions on discs' locations and demand points, their analysis is limited to discrete sets of points.

Article [6] addresses the problem of locating base stations for wireless communication where the demands and potential facilities are represented by a discrete set of points and each station can broadcast up to a maximum distance. A polynomial approximation scheme is given, together with complexity results. The following disc-covering geometric problem applied to wireless communication is addressed by Franceschetti *et al.* [5]: given an infinite square grid  $G$ , determine how many discs, centered at the vertices of  $G$ , with a given radius  $r$ , are required, in the worst case, to completely cover a disc with the same radius arbitrarily placed on the plane. The authors show that this number is an integer in  $\{3, 4, 5, 6\}$  depending on  $r$  and on the grid spacing. In addition, they discuss the applicability of this model to the design of approximation algorithms for facility locations on regular grids and to base station placement for wireless communication. The expected quality of service (level of surveillance) of a given sensor network is analyzed in [17] and [14], where the authors exploit computational geometry and graph theoretic techniques, such as Voronoi diagrams, Delaunay triangulation and graph search, to design exact polynomial algorithms for some special cases.

Location of railway stops is another application of the disc covering problem. In [9], the effect of introducing additional stops in the existing railway network is addressed. The problem is comprised of covering a set of points in the plane by discs with the restriction that their centers have to lie on a set of line segments that represents the railway tracks. A similar problem is addressed in [16], where the discs must be centered on two intersecting lines.

The location problem we address in this paper is a special disc covering problem in the following ways:

1. There are different types of facilities, which in our case are radar sensors, where the area covered by each radar is a disc with a diameter  $x$  that depends on the power of the radar unit.
2. The cost  $c_i$  of locating disc  $i$  includes a nonnegative fixed cost  $f_i$  and a variable cost, which may be approximated by an homogeneous polynomial function  $g_i(x_i)$ . In particular,  $g_i(x_i)$  is modeled as a second-order polynomial, that is  $g_i(x_i) = b_i x_i^2$ , where  $x_i$  is the the field of view of radar  $i$  and  $b_i$  is a positive real number.

3. A line segment with negligible width has to be covered by the discs. As it will be clear in Section 3, we may assume that the segment has unit length with no loss of generality.

We refer to our problem as the *Disc Covering Problem on a Line*.

The paper is organized as follows. In Section 2 some preliminary results are presented for the case of identical disc (radar) types and convex cost functions; in the sequel of the paper (Sections 3–5) we focus on the problem with assumptions 1–3. In Section 3 a quadratic programming formulation is developed. A Lagrangean relaxation of the problem and a technique to solve such a relaxation is also proposed. Section 4 presents a branch-and-bound algorithm for the problem: upper and lower bounding techniques are illustrated and a branching strategy is discussed. Some computational results are given in Section 5. Finally, some concluding remarks are made in Section 6.

## 2 Notation and preliminary results

We denote by  $Q$  the set of the  $q$  available discs (radars). For all  $i \in Q$ , *at most one copy* of disc  $i$  may be used for covering the line segment and any power level is allowed so that we can have any disc coverage distance  $0 \leq x_i \leq 1$ . These assumptions may appear restrictive for real applications but note that (i) usage of multiple copies of the same disc type may be modeled by including in  $Q$  a suitable number of items with the same characteristics and (ii) if a limit  $D$  exists on the coverage distance, then the problem may be decomposed by splitting the segment into pieces whose lengths are not greater than  $D$  and solving the problems for each segment separately (this may be an effective heuristic approach).

For any selected disc  $i \in Q$ , the coverage distance is the diameter of the disc  $x_i \in \mathbb{R}_+$ , and its contribution in the total cost function is

$$c_i(x_i) = \begin{cases} 0 & \text{if } x_i = 0 \\ f_i + g_i(x_i) & \text{if } x_i > 0 \end{cases},$$

where  $g_i(\cdot)$  is convex with  $g_i(0) = 0$  and the *setup cost*  $f_i$  is nonnegative. Although, because of the fixed cost component, the cost function  $c_i(x_i)$  is nonconvex in  $0 \leq x_i \leq 1$ , when the set of selected discs  $S$ , i.e., those for which  $x_i > 0$ , is fixed, then total coverage is in fact convex and the problem of determining the covering diameters is easily solved using KKT conditions (see Section 3).

Note that once  $x_i$  is given for all  $i \in Q$  (we will have  $x_i = 0$  for those radars that are not selected), it is trivial to find the set of optimal locations: just align the discs so that they do not intersect and they cover the entire line. For this purpose we choose the diameters in such a way that their sum is equal to the length of the line, which in our case is equal to 1. An illustration of a feasible solution to our problem is given in Figure 1.

We now present some simple results concerning the case when all available discs are of the same type, that is, for all  $i \in Q$  and  $x > 0$ ,  $c_i(x) = c(x)$  is a general nonnegative convex function. To the best of

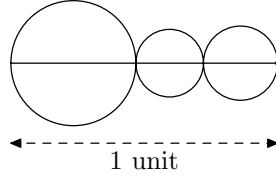


Figure 1: Example of a feasible solution

our knowledge, these results, though straightforward, are not present in the literature. However, it is worthwhile to point out that, differently from [1], where the objective function is of the form  $r^\alpha$  and both the potential facility locations and the demand points are discrete sets, we exploit the fact that we deal with a continuous line segment to obtain efficient solutions for even more general cost functions.

When the discs are all of the same type, our problem reduces to finding the optimal number  $k \leq q$  of copies and the optimal coverage area for each copy.

**Proposition 1** *When all the discs are of the same type having the cost functions  $c_i(\cdot) = c(\cdot)$ , for all  $i \in Q$ , if an optimal solution consists of locating  $k$  discs, then there is one solution where each of the  $k$  discs has the same diameter.*

**Proof.** Based on the convexity of  $c(x)$ , for any  $k$ -uple of nonnegative numbers  $x_1, \dots, x_k$ , with  $\sum_{i=1}^k x_i = 1$ , we have:

$$k \cdot c\left(\frac{1}{k}\right) \leq \sum_{i=1}^k c(x_i). \quad (1)$$

Hence, the cost of locating  $k$  discs (of the same type) with equal diameters—that is, each disc covers an equal portion of the line segment—does not exceed the cost of any other feasible solution that uses  $k$  discs. ■

Proposition 1 clearly indicates the optimal locations of the discs since they need to be uniformly spaced over the line segment.

The next natural question we need to ask is “What is an optimal number of such discs, that is the best value for  $k$ ?”.

First we write  $c(x) = f + g(x)$  with  $f \geq 0$  and  $g(0) = 0$ . Note that we may install at most  $q$  discs of the same type on the line. It is easy to observe that with zero setup costs ( $f = 0$ ) the cheapest solution consists of installing the largest possible number ( $q$ ) of facilities.

A solution that uses  $k + 1$  discs costs no more than a solution with  $k$  discs if and only if the following is true:

$$f + (k + 1) \cdot g\left(\frac{1}{k + 1}\right) \leq k \cdot g\left(\frac{1}{k}\right). \quad (2)$$

If  $f = 0$ , the last inequality is always valid, because of the convexity of  $g(\cdot)$ . Therefore, it is cost-effective to locate another disc if the additional setup cost does not exceed the gain in the variable costs.

The effective cost of locating  $k$  discs is

$$F(k) = kc\left(\frac{1}{k}\right) = kg\left(\frac{1}{k}\right) + kf. \quad (3)$$

Since  $g(\cdot)$  is convex,

$$\frac{\partial^2 F}{\partial k^2} = \left(\frac{1}{k^3}\right) \frac{\partial^2 g}{\partial k^2} \Big|_{\frac{1}{k}} \geq 0 \quad (4)$$

Hence, there must be  $1 \leq k^* \leq q$  such that

$$F(1) \geq F(2) \geq \dots \geq F(k^*) \text{ and } F(k^*) \leq F(k^* + 1) \leq \dots \leq F(q). \quad (5)$$

Therefore, since a binary search can be used to efficiently find the  $k^*$ , the following proposition holds.

**Proposition 2** *When the  $q$  discs are all of the same type with the cost functions  $c_i(\cdot) = c(\cdot)$ , for all  $i \in Q$ , the problem is solvable in  $O(C \log(q))$  time, where  $C$  is the maximum computational effort for calculating  $c(\frac{1}{k})$ . ■*

### 3 Problem formulation

In this section we develop a quadratic programming formulation of the general problem where the  $q$  discs (radars) may have different properties. Using the notation introduced in Section 2, the cost contribution of any disc  $i \in Q$  that covers an area having diameter  $x_i$  is a quadratic polynomial

$$c_i(x_i) = f_i + b_i x_i^2$$

with  $f_i \geq 0$ ,  $b_i > 0$ . Then the location problem can be formulated as the following *Mixed Integer Quadratic Program*.

$$\begin{aligned} z^* = \min \quad & \sum_{i \in Q} f_i y_i + b_i x_i^2 \\ \text{s.t.} \quad & x_i \leq y_i, \text{ for all } i \in Q \quad (c1) \\ (P) \quad & \sum_{i \in Q} x_i = 1 \quad (c2) \\ & x \in \mathbb{R}_+^q \quad (c3) \\ & y \in \{0, 1\}^q \quad (c4) \end{aligned}$$

In the solution of  $P$ ,  $y$  is the vector to indicate selected discs (radars) in  $Q$  where  $y_i = 1$  if disc  $i$  is used,  $y_i = 0$  otherwise. Constraints (c1) force the disc coverage diameter  $x_i$  to be zero when the corresponding disc  $i$  is not selected (and therefore the corresponding cost contribution is zero). Constraint (c2) is the *coverage constraint* that assures that the whole line segment is covered.

As stated before, the unit length assumption does not introduce any loss of generality. It is clear that  $P$  can be equivalently used for a problem  $\tilde{P}$  where the line has length  $\ell \neq 1$ . Let  $\tilde{I}$  be an instance of

$\tilde{P}$  where the cost coefficients for the disc  $i$  are  $\tilde{f}_i$  and  $\tilde{b}_i$ . Then we can solve  $\tilde{I}$  by solving an equivalent instance  $I$  of the unit length problem  $P$  having cost coefficients  $f_i = \tilde{f}_i$  and  $b_i = \ell^2 \tilde{b}_i$ . If  $(x, y)$  is an optimal solution of  $I$ , then  $(\ell x, y)$  is optimal for  $\tilde{I}$ .

In the remainder of the paper we propose methods to solve problem  $P$  and discuss the results of some computational experiments to evaluate the performance of these methods.

Our first observation concerns the existence of efficient methods to find the optimal coverage when the set  $S \subseteq Q$  of selected discs (i.e. active radars) is given or known *a priori*. Under this assumption, the variables  $y_i = 1$  for all  $i \in S$  in problem  $P$  and the resulting problem is easily solved by applying Karush-Kuhn-Tucker optimality (KKT) conditions. In fact, this restriction of the problem can be written as

$$(RP) \quad z(S) = \min \left\{ \sum_{i \in S} f_i + b_i x_i^2 : \sum_{i \in S} x_i = 1; x_i \in \mathbb{R}_+, i \in S \right\}.$$

Problem  $RP$  is a convex optimization problem. Define the following Lagrangean function (without loss of generality, the constant term  $\sum_{i \in S} f_i$  has been omitted in the objective function below):

$$L(x, \mu, \lambda) = \sum_{i \in S} b_i x_i^2 + \lambda \left( 1 - \sum_{i \in S} x_i \right) - \sum_{i \in S} \mu_i x_i$$

The KKT conditions, for the triple  $(x^*, \mu^*, \lambda^*)$ , are

$$\begin{aligned} \nabla_x L(x^*, \lambda^*, \mu^*) &= 0_q \\ \sum_{i \in S} x_i &= 1 \\ \mu^{*T} x^* &= 0 \\ x^* &\geq 0_q \\ \mu^* &\geq 0_q \end{aligned}$$

It follows from  $x_i^* > 0$  that  $\mu_i^* = 0$  for all  $i \in S$ . Then,

$$\lambda^* = \frac{1}{\sum_{j \in S} \frac{1}{2b_j}}$$

and

$$x_i^* = \frac{\lambda^*}{2b_i} = \frac{\frac{1}{2b_i}}{\sum_{j \in S} \frac{1}{2b_j}}, \quad i \in S$$

satisfy the KKT conditions and, therefore, are a global optimum for problem  $RP$ .

Although coverage diameters may be computed in a closed-form, choosing the subset  $S \subseteq Q$  of active radars is a tedious computational task. The branch-and-bound algorithm described in Section 4 relies on a dual bound estimation which is developed in the next subsection.



### 3.1 Lagrangean relaxation of $P$

We will use Lagrangean relaxation to obtain a lower bound on  $z^*$ , the optimal solution value of problem  $P$ . Relaxing constraints (c1) of  $P$  using nonnegative Lagrangean multipliers  $\kappa_i, i = 1, \dots, |Q|$ , we obtain the following problem:

$$(LRP) \quad \begin{aligned} z_{LRP}(\kappa) = \min \quad & \sum_{i \in Q} (f_i - \kappa_i) y_i + b_i x_i^2 + \kappa_i x_i \\ \text{s.t.} \quad & \sum_{i \in Q} x_i = 1 & (c5) \\ & x \in \mathbb{R}_+^q & (c6) \\ & y \in \{0, 1\}^q & (c7) \end{aligned}$$

Problem  $LRP$ , a relaxation of  $P$  for any  $\kappa \geq 0_q$ , is decomposable since optimal values for the  $y_i$  variables are independent of the values of the  $x_i$  variables. In particular, we may choose the following optimal values for  $y$ :

$$y_i^* = \begin{cases} 1 & \text{if } f_i < \kappa_i \\ 0 & \text{if } f_i \geq \kappa_i \end{cases} \quad \text{for all } i \in Q.$$

The remaining convex program, which depends on the  $x$  variables only, is:

$$(LRP') \quad \begin{aligned} z_{LRP'}(\kappa) = \min \quad & b_i x_i^2 + \kappa_i x_i \\ \text{s.t.} \quad & \sum_{i \in Q} x_i = 1 & (c8) \\ & x \in \mathbb{R}_+^q & (c9) \end{aligned}$$

and therefore  $z_{LRP}(\kappa) = z_{LRP'}(\kappa) + \sum_{i \in Q} (f_i - \kappa_i) y_i^*$ . In order to solve problem  $LRP'$ , we define the following Lagrangean function, where we use multiplier  $\lambda \in \mathbb{R}$  for constraint (c8) and multipliers  $\mu \in \mathbb{R}_+^q$  for nonnegativity constraints (c9):

$$L_\kappa(\lambda, \mu) = \min \sum_{i \in Q} (b_i x_i^2 + \kappa_i x_i - \mu_i x_i) + \lambda \left( 1 - \sum_{i \in Q} x_i \right).$$

Then the KKT conditions are

$$\nabla_{x_i} L_\kappa(x^*, \lambda^*, \mu^*) = 2b_i x_i^* + \kappa_i - \mu_i^* - \lambda^* = 0 \quad \text{for all } i \in Q \quad (6)$$

$$\sum_{i \in Q} x_i^* = 1 \quad (7)$$

$$\mu_i^{*T} x_i^* = 0 \quad \text{for all } i \in Q \quad (8)$$

$$x_i^* \geq 0, \mu_i^* \geq 0 \quad \text{for all } i \in Q. \quad (9)$$

From (6) we have

$$x_i^* = \frac{1}{2b_i} (\lambda^* + \mu_i^* - \kappa_i^*) \quad \text{for all } i \in Q. \quad (10)$$

In order to find values for  $x_i^*, \lambda^*, \mu^*$  that satisfy KKT conditions (6)–(9), let  $S$  (so far unknown) include the set of indices that correspond to positive covering diameters in the optimal solution, that is  $S = \{i \in Q : x_i^* > 0\}$ . Given  $S$ , we have from (8) that  $\mu_i^* = 0$  for all  $i \in S$  and we obtain the following relations:

$$x_i^* = \frac{1}{2b_i}(\lambda^* - \kappa_i), \quad \mu_i^* = 0 \quad \text{for all } i \in S \quad (11)$$

$$x_i^* = 0, \quad \mu_i^* = \kappa_i - \lambda^* \quad \text{for all } i \in Q \setminus S. \quad (12)$$

Furthermore, from (11) and (12), we have that

$$x_i^* > 0 \Rightarrow \lambda^* > \kappa_i \quad \text{for all } i \in S \quad (13)$$

$$\mu_i^* \geq 0 \Rightarrow \lambda^* \leq \kappa_i \quad \text{for all } i \in Q \setminus S. \quad (14)$$

Suppose now, without loss of generality, that the  $\kappa_i$  values are in nondecreasing order. From relations (13) and (14), we have:

$$\underbrace{\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_h}_{S} < \lambda^* \leq \underbrace{\kappa_{h+1} \leq \dots \leq \kappa_q}_{Q \setminus S}. \quad (15)$$

Hence,  $S$  has the form  $S = \{1, \dots, h^*\}$ ,  $h^* \leq q$ , and we obtain the following expression for  $\lambda^*$  (using equations (7), (11), (12)):

$$\lambda^* = \frac{1 + \sum_{i=1}^{h^*} \frac{\kappa_i}{2b_i}}{\sum_{i=1}^{h^*} \frac{1}{2b_i}} > 0. \quad (16)$$

Since the feasible region of  $LRP'$  is a closed convex set and its objective function is convex, this problem admits a (finite) optimal solution. In particular any local optimum that satisfies the KKT conditions is an optimal solution for  $LRP'$  and vice versa. Therefore, there must exist an optimal solution  $x^*$  of  $LRP'$ , together with corresponding optimal multipliers  $\lambda^* \in \mathbb{R}$ ,  $\mu \in \mathbb{R}_+^q$ , that satisfy the KKT conditions (6)–(9).

Therefore, once the  $\kappa_i$  are arranged in nondecreasing order, a set of indices  $S = \{1, \dots, h\}$  ( $1 \leq h \leq q$ ) necessarily exists such that (15) is satisfied, and expressions (11), (12), and (16), return an optimal solution  $(x^*, \lambda^*, \mu^*)$  to  $LRP'$ .

We may find  $h$ , that is, the set  $S = \{1, \dots, h\}$  of indices corresponding to selected discs, with a  $O(\log q)$  binary search. Because it is the sum of at most  $q$  elements, it is possible to compute  $\lambda^*$  by (16) in  $O(q)$  time. The same time is required to compute the values  $x_i^*$ , which are at most  $q$  and each is computable in constant time (using expression (11)). Additional  $O(q)$  steps are necessary to determine the value for the  $y$  variables. Hence the following proposition holds:

**Proposition 3** *Given a set of  $q$  nonnegative Lagrangean multipliers  $\kappa \in \mathbb{R}_+^q$ , the solution of the Lagrangean problem  $LRP'$  can be found in time  $O(q \log q)$ , which is the computational cost of ordering the multipliers. ■*

Recalling that  $z_{LRP}(\kappa) = z_{LRP'}(\kappa) + \sum_{i \in Q} (f_i - \kappa_i)y_i^*$ , we observe that a solution of problem  $LRP$ —and therefore a lower bound for the optimal solution value of  $P$ —can be found in time  $O(q \log q)$ .

## 4 An exact algorithm for $P$

In this section, we develop a branch-and-bound algorithm that finds an optimal solution of  $P$ . This implicit enumeration scheme exploits the lower bounds (LB) obtained by a subgradient optimization algorithm described in Section 4.1 and upper bounds (UB) by an efficient local search-based heuristic (described in Section 4.2). Branching and subproblem solution strategies are discussed in Section 4.3.

### 4.1 Lower bound via a subgradient algorithm

For any  $\kappa \in \mathbb{R}_+^q$ , the optimal solution value  $z_{LRP}(\kappa)$  of  $LRP$  provides a lower bound on the value of the optimal solution value of  $P$ . We are now interested in obtaining the best (largest) lower bound by solving the following *Lagrangian Dual Problem*:

$$(DP) \quad z_{LRP}^* = z_{LRP}(\kappa^*) = \max \{z_{LRP}(\kappa) : \kappa \in \mathbb{R}_+^q\}$$

In our approach, the solution of  $DP$  is obtained by a standard subgradient optimization algorithm that is summarized in Figure 2. (The actual values of parameters  $\alpha$  and  $t_i$  used in the implementation are reported later.) The proposed Lagrangean relaxation method not only provides the lower bounds that

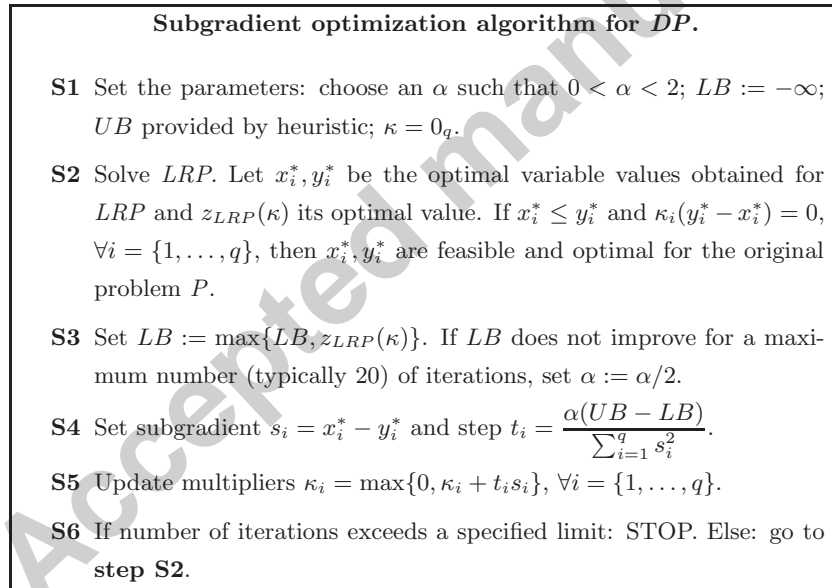


Figure 2: Subgradient algorithm for  $DP$ .

we use in our enumeration scheme but it is also exploited in an efficient heuristic procedure which is presented in the next section.

## 4.2 Upper bound via a heuristic algorithm

The basic idea for this heuristic algorithm based on the Lagrangean Relaxation is to obtain a feasible solution of  $LRP$  by establishing all the discs (radars) corresponding to  $x_i > 0$  (i.e.,  $i \in S$ ) and, possibly, removing unused discs (i.e., switching off all the unnecessary radars) ( $i \notin S$ ).

In general, given a subset  $S \subseteq Q$ , we may easily compute feasible values for the coverage diameters  $x_i$ , for all  $i \in S$ , using the KKT conditions—as described in Section 3. Note that the values  $x_i$  provided by the solution of  $LRP'$  are feasible but, in general, they may not be optimal since the corresponding set of discs  $S$  may not be optimal. We may further refine the set  $S$  using a simple local search which exploits the KKT conditions to find the cheapest location and coverage for a given set of selected discs. Figure 3 summarizes the heuristic.

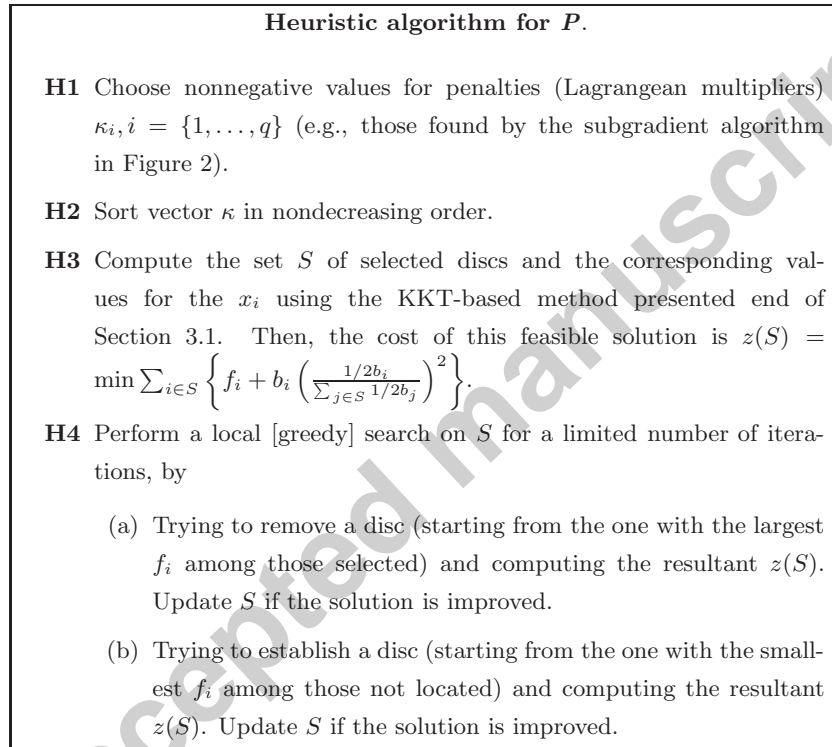


Figure 3: Heuristic procedure for  $P$ .

## 4.3 Exact branch-and-bound algorithm

In this section, we present a branch-and-bound algorithm for  $P$  that uses the lower and upper bounds developed above in Sections 4.1 and 4.2.

### 4.3.1 Solution strategy

In the branch-and-bound tree, each node  $\nu$  represents a subproblem that is defined by (i) a set  $T$  of selected discs (i.e. active radars which must be ON) in the solution, that is  $T = \{i : y_i = 1; i = \{1, \dots, q\}\} \subseteq Q$ , (ii) a set of discs that cannot be in the solution (i.e. radars that must be OFF) and (iii) a set of discs that are not yet decided upon (i.e., radars that are not yet fixed to ON or OFF). If a radar is OFF, we consider the corresponding disc *deleted* from the set of available discs (radars) for that specific subproblem. Let  $Q(\nu)$  be the set of available discs at node (subproblem)  $\nu$ . Then the generic subproblem may be formulated as follows:

$$\begin{aligned}
 (P(T, \nu)) \quad & \min \quad \sum_{i \in Q(\nu) \setminus T} (f_i y_i + b_i x_i^2) + \sum_{i \in T} (f_i + b_i x_i^2) \\
 & \text{s.t.} \quad x_i \leq y_i \quad \text{for all } i \in Q(\nu) \setminus T \quad (c1_\nu) \\
 & \quad \quad \sum_{i \in Q(\nu)} x_i = 1 \quad (c2_\nu) \\
 & \quad \quad x_i \geq 0 \quad \text{for all } i \in Q(\nu) \quad (c3_\nu) \\
 & \quad \quad y_i \in \{0, 1\} \quad \text{for all } i \in Q(\nu) \setminus T \quad (c4_\nu)
 \end{aligned}$$

Again, similarly to what was done in Section 3.1 for problem  $LRP$ , relaxing constraints  $(c1_\nu)$  in a Lagrangean fashion, using multipliers  $\kappa_i$  with  $\kappa_i = 0$  for all  $i \in T$ , we obtain problem  $LRP(T, \nu)$ :

$$\min \left\{ \sum_{i \in Q(\nu)} (b_i x_i^2 + \kappa_i x_i) + \sum_{i \in Q(\nu) \setminus T} (f_i - \kappa_i) y_i + \sum_{i \in T} f_i : (c2_\nu), (c3_\nu) \text{ and } (c4_\nu) \right\}$$

which, in turn, is equivalent to

$$\min \left\{ \sum_{i \in Q(\nu)} (b_i x_i^2 + \kappa_i x_i) : (c2_\nu), (c3_\nu) \right\} + \sum_{i \in Q(\nu) \setminus T, f_i < \kappa_i} (f_i - \kappa_i) + \sum_{i \in T} f_i$$

Neglecting the last two constant summations, we have a problem in the  $x$  variables which is a special instance of  $LRP'$  defined in Section 3.1. Thus, a lower bound can be computed at each node by solving the Lagrangean dual of the corresponding problem  $LRP(T, \nu)$ , by means of the procedure summarized in Figure 2.

An upper bound at the *root node* is provided by the heuristic in Figure 3.

### 4.3.2 Branching strategy

At node  $\nu$  of the enumeration tree, we branch on a binary variable  $y_i$ ,  $i \in Q(\nu) \setminus T$ , splitting subproblem  $\nu$  into two new subproblems  $\nu'$  and  $\nu''$ . Disc  $i$  is selected in  $\nu'$  (i.e.  $y_i = 1$  and  $T := T \cup \{i\}$ ) and it is deleted in  $\nu''$  (i.e.  $y_i = 0$  and  $Q(\nu'') := Q(\nu) \setminus \{i\}$ ).

Let  $\kappa_i^*$  be the optimal values for the multipliers in the solution of the Lagrangean dual, and  $x_i^*, y_i^*$ ,  $i \in Q(\nu) \setminus T$  the optimal variable values obtained for  $LRP(T, \nu)$ . The branching rule is to branch on a variable  $y_i$ , such that  $y_i^* = 0$  and  $x_i^* > 0$ . If such a variable does not exist (i.e.,  $x^*$  and  $y^*$  are feasible for the subproblem  $\nu$ ) then branch on variable  $y_i^*$ , such that  $y_i^* = 1$ ,  $x_i^* < 1$  and  $\kappa_i^* > 0$ . If such a variable does not exist then  $\kappa_i^*(y_i^* - x_i^*) = 0$ , for all  $i \in Q(\nu) \setminus T$  and, therefore, the  $x^*$  and  $y^*$  are (feasible and) *optimal* for subproblem  $\nu$ . The corresponding node in the enumeration tree is then fathomed.

## 5 Computational experiments

The design of the computational experiments is described in the next subsection while the computational results are discussed in Section 5.2. All the results reported in this section refer to tests performed on a 3.00 GHz Pentium IV, 1024 MB RAM, running Windows XP. The algorithms have been coded in C++. See [8] for more details.

### 5.1 Design of experiments

Any instance of Problem  $P$  is characterized by a pair of vectors with  $q$  components  $(b, f)$ , representing discs' variable and fixed costs.

We say that a disc  $i$  *dominates* disc  $j$  if  $(b_i \leq b_j)$ ,  $(f_i \leq f_j)$ , and  $(b_i, f_i) \neq (b_j, f_j)$ . In our experiments no disc pair exists such that one is dominated by the other, since there is no sense in considering dominated disc types in  $Q$ .

Therefore, we impose the following cost relations:

$$b_1 \leq b_2 \leq \dots \leq b_q \quad \text{and} \quad f_1 \geq f_2 \geq \dots \geq f_q.$$

We start with a special class of instances (“*base class*”) having the following properties:

- $(b_i \neq b_j)$  and  $(f_i \neq f_j)$ , for all  $1 \leq i < j \leq q$ .
- On the average,  $b_{i+1} \approx b_i + 1$  for all  $1 \leq i < q$ .
- $b_i = f_{q-i+1}$ , for all  $1 \leq i \leq q$  (to exclude dominated cases).

We generate all the instances used in the experiments by suitable modifications of a randomly generated base class instance. In particular, any instance is identified by the four integers  $(q, s, t, u)$ , where

$q$ : the number of available discs which determines the size of the instance.

$s$ : amplification factor by which  $b$  of the base class instance is multiplied. For instances with this parameter, on the average,  $b \approx \{s, 2s, \dots, qs\}$ .

$t$ : the parameter that characterizes the vector of setup costs  $f$ , which is obtained using  $t \geq 1$  as a multiplication factor of the  $b$  vector determined as above: on the average,  $f_{q-i+1} \approx tb_i$  for instances with this parameter.

$u$ : the parameter that identifies the configuration for the test instance where a suitable subset of the cost coefficients  $b_i$ , or the setup costs  $f_i$ , or both, have the same value. 10 configurations were defined and  $u$  was labeled  $0, 1, \dots, 9$  where  $u = 0$  defines the base class where  $b_i$  and  $f_i$  all have different values. For example  $u = 2$  defines the class where the  $f_i$  for the selected discs in the optimal base class solution are set to the maximum  $f$  value in the base class. Parameter  $u$  attempts to make systematic changes with respect to  $f_i$  and  $b_i$  values in various instances. While some other  $u$  labels are described later in this paper, see [8] for more details on the other  $u$  labels.

As an example, the class  $(50, 1, 100, 0)$  refers to instances with 50 discs, all different types (since  $u = 0$ ), cost coefficients  $b_i$  as in the base class, and setup costs  $f_i$  amplified by a factor  $t = 100$ .

A set of preliminary tests were performed to determine the largest instances that our algorithm is able to solve optimally, in order to design our experiments. Results are reported in Table 1. All the instances of this preliminary test-set belong to the class  $(q, 1, 1, 0)$ . The branch-and-bound algorithm solved instances up to  $q = 400$  in less than 16 hours. No instance with  $q = 500$  was solved within the same time limit. Almost all the instances with  $q$  up to 200 are solved within one hour.

$q$	150	200	350	400	500
CPU time	1217.97 s.	2544.76 s.	$\sim 3.5$ hr.	$\sim 15.0$ hr.	$> 16$ h

Table 1: Preliminary test-set results

Based on these preliminary results, we planned our experiments with the following sizes:  $q \in \{10, 25, 50, 100, 200, 350, 400\}$ . Maximum running time was set to 1 hour (CPU time), except for the case  $q = 400$  where there was no timeout requirements. For each class except the  $q = 400$  case, 10 random instances were generated and the following *average* quantities were tracked:

- CPU time.
- Number of nodes in the enumeration tree.
- Depth of the enumeration tree.
- Upper bound at the root node.
- Optimal solution value, if any. (A minus “-” symbol is shown when the optimum is not reached within the time limit.)
- Best lower bound available after 1 hour.
- Percentage gap  $(UB - LB)/UB$ .

## 5.2 Results and analysis

Table 2 summarizes the results of the experiments. Each row shows, in order, the quantities of the above list, for one class of instances. Class name is reported in the first column where an asterisk “\*” denotes that, in at least one instance of the class, the algorithm did not reach the optimal solution value within the time limit (i.e. CPU time greater than 3600 sec.). The table also gives the initial “gap” between the UB and the LB at the root node, computed as  $(UB - LB)/UB$ . The results for classes with  $u = 3, 4, 6, 8, 9$  are not reported in Table 2 for the sake of brevity. For these classes, the performance of the algorithm is indeed comparable or even better than those reported.

First, we highlight the excellent performance of the heuristic: *in all the experiments* the value found by this procedure ( $UB_{root}$ ) equals the optimal value (opt.) found by the branch-and-bound algorithm. A few comments are in order:

instance ( $q, s, t, u$ )	CPU time (s.)	nodes' #	depth	$UB_{root}$	opt.	$LB$	gap <sup>†</sup>
10,10,1,0	1.547	29	10	77.368	77.368	44.8017	42.01%
10,10,1,1	1.469	27	10	80	80	46.696	41.63%
10,10,1,5	1.562	37	10	110	110	75.526	31.64%
10,1,100,0	0.063	0	0	506	506	506	0%
10,1,100,1	0.062	0	0	110	110	110	0%
10,1,100,5	0.047	0	0	506	506	506	0%
100,10,1,0	464.81	987	100	345.96	345.96	135.31	60.89%
100,10,1,1	464.25	1009	100	348.35	348.35	135.74	61.03%
100,10,1,5	932.47	1955	96	496.22	496.22	172.64	65.21%
100,1,100,0	31.937	135	67	200	200	187.44	6.28%
100,1,100,1	32.125	131	65	200	200	187.5	6.25%
100,1,100,5	33.047	131	65	596	596	584.92	1.86%
200,10,1,0	2586.01	2737	200	539.14	539.14	189.31	64.89%
200,10,1,1	2866.91	2527	200	540.8	540.8	189.66	64.93%
200,10,1,5*	> 3600	> 2597	$\geq 195$	737.98	–	227.93	69.12%
200,1,100,0	208.7	341	169	300	300	241.54	19.49%
200,1,100,1	189.281	361	174	300	300	241.67	19.45%
200,1,100,5	206.38	337	167	696	696	639.7	8.09%
350,10,1,0*	> 3600	> 2021	$\geq 345$	775.7	–	234.2	69.81%
350,10,1,1	959.97	1827	345	776.91	776.91	234.74	69.79%
350,10,1,5*	> 3600	$\gg 1289$	$\geq 344$	1022.49	–	285.4	72.09%
350,1,100,0	985.38	719	350	450	450	301.56	32.99%
350,1,100,1	959.97	703	350	450	450	298.51	33.67%
350,1,100,5	964.44	673	328	846	846	700.2	17.23%
400,1,1,0	36705.2	17373	400	84.71	84.71	24.28	71.34%

Table 2: Experiments results. († Gap is computed as the initial  $(UB - LB)/UB$ , where  $LB$  is the (initial) lower bound at the root node.)



1. Not surprisingly, branch-and-bound computational time is strictly related to the enumeration tree size: Table 3 shows strong positive correlation of CPU time with the number of nodes in the enumeration tree.
2. For a given number of discs  $q$ , we note that the CPU time does depend on the particular cost configuration, that is, on the particular pair  $(s, t)$ . The most difficult instances have  $t = 1$  (i.e.,  $b_i \approx f_{q-i+1}$ ); and, viceversa, the larger the  $t$  in comparison to  $s$ , the faster the computation.
3. Classes with  $u = 2$  and  $u = 5$  are the hardest. Parameters are chosen so that finding the set of selected discs becomes more difficult. For any instance  $I$  of the base class, we build the corresponding  $u = 5$  instance  $I'$  as follows. Let  $S$  be the set of selected discs in the optimal solution of  $I$ . The costs in  $I'$  are:  $b'_i = b_i$  if  $i \notin S$ ,  $b'_i = \max\{b_1, \dots, b_q\}$  if  $i \in S$ . Analogously  $f'_i = f_i$  if  $i \notin S$ ,  $f'_i = \max\{f_1, \dots, f_q\}$  if  $i \in S$ .  
The  $u = 2$  class is designed similarly but with  $b'_i = b_i$ , for all  $i = 1, \dots, q$ . (Experimental results for the  $u = 2$  class were similar to those of the  $u = 5$  class; therefore Table 2 reports only the latter.)
4. Figure 4 shows the CPU time cumulative distribution. The histogram was obtained empirically over 90 instances (with  $q = 25$ ): 80% of the instances are solved in a time smaller than 2.6 s. while the largest CPU time is an order of magnitude higher (17.2 s.).

$q$	10	25	50	100
$\rho_{nodes}$	0.982	0.998	0.999	0.999

Table 3: Correlation coefficient between CPU time and number of nodes of the enumeration tree.

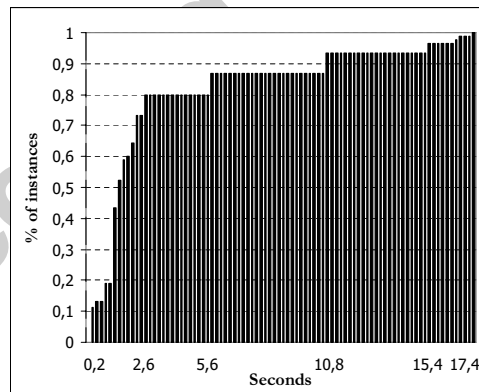


Figure 4: CPU time cumulative distribution ( $q = 25$ , time in seconds).

We noted previously that fixed costs are related to the choice of the subset  $S$  and they heavily affect the computational effort required by an instance. An evidence of this fact is illustrated in Table 4 where the results of the experiments with  $q, s, t = 10, 1, 1$  are compared for three values of  $u$  ( $u = 0, 1, 3$ ). Note that the  $u = 0$  class is the base class.

instance ( $q, s, t, u$ )	$S^*$	Opt. radii $x_i, i \in S^*$
10,1,1,0	{9,10}	{0.526, 0.474}
10,1,1,1	{9,10}	{0.5, 0.5}
10,1,1,3	{7,8,9}	{0.377, 0.330, 0.293}

Table 4: Set  $S^*$  (optimal set) response to  $f_i$  and  $b_i$  variations for  $q = 10$  and  $u = \{0, 1, 3\}$ .

Given an instance  $I$  of the base class, we build the corresponding  $u = 1$  ( $u = 3$  respectively) instance  $I'$  as follows. Let  $S$  be the set of selected discs in the optimal solution of  $I$ . The costs in  $I'$  are:  $b'_i = \max\{b_1, \dots, b_q\}$  ( $f'_i = \min\{f_1, \dots, f_q\}$  respectively) for  $i \in S$ . All the other parameters remain equal to those in the base class.

When the  $b_i$  are varied (compare classes with  $u = 0$  and  $u = 1$  in Table 4) the optimal solutions are slightly different. However, when the  $f_i$  vary (compare classes with  $u = 0$  and  $u = 3$ ) the two optimal solutions drastically differ from each other: there are three discs selected instead of two.

## 6 Conclusions and future work

In this paper we introduced and addressed the problem of covering a single line segment with radar sensors having a circular field of view. When the sensors are required to have identical radius, a simple polynomial search solves for optimal radius and number of sensors. When the sensors are modeled with variable diameter discs the problems becomes hard. We provided an exact solution algorithm which is based on a Lagrangean relaxation and a subgradient algorithm to find a lower bound (see Figure 2). A feasible solution is provided by the heuristic summarized in Figure 3. These bounds were exploited to design a branch-and-bound algorithm. Extensive computational testing, based on approximately 400 experiments, showed that the developed heuristic performs very well; the upper bounds were equal to the optimal solution whenever the latter was known (see Table 2). The experiments also show that setup costs play a crucial role both in computational effort and attainment of the optimal solution set.

Several directions for future work are being pursued. The two-arc network and planar tree network cases are being investigated (both for single and multiple discs types, for fixed and variable radius). Since discs can be located both on the arcs and on the plane, these are mixed network-planar problems and the development of locational models and algorithms is indeed very challenging.

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