White Noise Quantum time Shifts

LUIGI ACCARDI
Centro Vito Volterra, Università di Roma Tor Vergata
via Columbia 2, 00133 Roma, Italy
e-mail: accardi@volterra.mat.uniroma2.it

ABDESSATAR BARHOUMI
Department of Mathematics
High School of Sciences and Technology of Hammam Sousse
University of Sousse, Tunisia
Rue Lamine Abassi 4011 Hammam Sousse
e-mail: abdessatar.barhoumi@ipein.rnu.tn

HABIB OUERDIANE
Department of Mathematics, Faculty of sciences of Tunis
University of Tunis El-Manar, 1060 Tunis, Tunisia
e-mail: habib.ouerdiane@fst.rnu.tn

HABIB REBEI
Department of Mathematics
Higher Institute of Applied Sciences and Technology of Gabes
University of Gabes, Tunisia
e-mail: habib.rebei@ipein.rnu.tn

December 7, 2011

Abstract

In the present paper we extend the notion of quantum time shift, and the related results obtained in [9], from representations of current algebras of the Heisenberg Lie algebra to representations of current algebras of the Oscillator Lie algebra.
This produces quantum extensions of a class of classical Lévy processes much wider than the usual Brownian motion. In particular this class processes includes the Meixner processes and, by an approximation procedure, we construct quantum extensions of all classical Lévy processes with a Lévy measure with finite variance. Finally we compute the explicit form of the action, on the Weyl operators of the initial space, of the generators of the quantum Markov processes canonically associated to the

above class of Lévy processes. The emergence of the Meixner classes in connection with the renormalized second order white noise, is now well known. The fact that they also emerge from first order noise in a simple and canonical way, comes somehow as a surprise.

Keywords: Markovian semigroup, Lévy process, Oscillator algebra, Quantum shift, Weyl algebra, White noise, Wiener process.
MSC (2000): primary 60J65; secondary 60J45, 60H40.

1 Introduction and Notations

The usual time shift v_t° in Wiener space is the unique endomorphism of the associated algebra of measurable functions given by the map

$$v_t^\circ(W_s) := W_{s+t}.$$

The time shift u_t° on the corresponding increment process is the unique endomorphism of the associated algebra of measurable functions given by the map

$$u_t^\circ\,(W_s - W_r) := W_{s+t} - W_{r+t}.$$

Denoting j_t the restriction of the Wiener time shift on the time zero algebra, we see that v_t° is uniquely determined by the pair (j_t, u_t°) through the identity

$$v_t^\circ\,(W_s) := W_{s+t} = W_t + (W_{s+t} - W_t) = j_t\,(W_0) + u_t^\circ\,(W_s - W_0)$$

Taking exponentials we find, for $z \in \mathbb{C}$

$$v_t^\circ\left(e^{zW_s}\right) = j_t\left(e^{zW_0}\right) u_t^\circ\left(e^{z(W_s-W_0)}\right) = (j_t \otimes u_t^\circ)\left(e^{zW_0} \otimes e^{z(W_s-W_0)}\right)$$

By continuity and the endomorphism property v_t° is uniquely determined by the above relations. On the other hand, in the quantum formulation of the classical Wiener process, u_t° is the white noise time shift, and the increment $W_t - W_0$ is the momentum

$$P_{\chi_{(0,t]}} = W_t - W_0.$$

Therefore the usual time shift v_t° in Wiener space is the unique endomorphism of the associated algebra of measurable functions satisfying

$$v_t^\circ(W_s) = j_t(W_0) + u_t^\circ\left(P_{\chi_{(0,s]}}\right) = W_0 + P_{\chi_{(0,t]}} + P_{\chi_{(t,s+t]}} = W_0 + P_{\chi_{(0,s+t]}}$$

or, in exponential formulation:

$$v_t^\circ(e^{zW_s}) = j_t\left(e^{zW_0}\right)\, u_t^\circ\left(e^{zP_{\chi_{(0,s]}}}\right) = (j_t \otimes u_t^\circ)\,\left(e^{zW_0} \otimes e^{zP_{\chi_{(0,s]}}}\right) = (j_t \otimes u_t^\circ)\, e^{z(W_0+P_{\chi_{(0,t]}})}$$

Denoting

$$X_{(s,t]} = P_{\chi_{(s,t]}}, \tag{1}$$

we see that

$$X_{(0,t]} \,\hat{\in}\, 1_{\mathcal{H}_0} \otimes \mathcal{B}\Big(\Gamma(L^2((0,t])\Big)$$

in this case we say that $(s,t] \mapsto X_{(s,t]}$ is a pure noise operator process localized on the interval $(0,t]$. This process has the following properties:

- u°-covariant

$$u_r^\circ(X_{(s,t]}) = X_{(s+r,t+r]};$$

- additive

$$X_{(r,s]} + X_{(u,t]} = X_{(r,s]\cup(u,t]} \quad \text{if} \quad (u,t)\cap(r,s) = \emptyset;$$

- normal

$$[X_{(0,t]}, X^+_{(0,t]}] = 0;$$

- classical

$$[X_{(r,s]}, X_{(u,t]}] = 0 \quad \text{if} \quad (u,t]\cap(r,s] = \emptyset,$$

where, here and in the following, $[\,\cdot\,,\,\cdot\,]$ denotes the commutator.
Finally the operator process $\{(W_0 + X_{(0,t]}), \Phi\}$ is isomorphic to the Wiener process so that, in particular, $\{(X_{(s,t]}), \Phi\}$ is isomorphic to the stationary independent increment process associated to the Wiener process, i.e. the white noise process.
The choice (1) is not the only one leading to this conclusion. For example the choice

$$X_{(s,t]} = Q_{\chi_{(s,t]}} \tag{2}$$

leads to the same conclusion. This leads to the following definition.

Definition 1 *An operator valued measure $(X_{(s,t]})$ is called a classical time shift for the Wiener process if the operator process $\{(W_0 + X_{(s,t]}), \Phi\}$ is isomorphic to the Wiener process.*

Now choose $\mathcal{H}_0 := L^2(\mathbb{R})$ with the Schrödinger representation

$$[a_0\ ,\ a_0^+] = 1; \tag{3}$$

$$a_0 + a_0^+ = q_0 \qquad \text{(multiplication by the coordinate } x)$$

A natural quantum generalization of the above definition is the following.

Definition 2 *A quantum time shift for the quantum Brownian motion is defined by a pair of operator valued measures $\left(X^+_{(s,t]}\right)$, $\left(X^-_{(s,t]}\right)$ such that the operator process*

$$\left\{\left(a_0 + X^-_{(s,t]}\right), \left(a_0^+ + X^+_{(s,t]}\right), \Phi\right\}$$

is isomorphic to the quantum Wiener process.

A weaker notion is the following.

Definition 3 *Denote*

$$E_{0]} := \imath_0 \otimes \langle \Phi, (\,.\,)\Phi\rangle\ :\ \mathcal{B}(\mathcal{H}_0)\otimes\mathcal{B} \to \mathcal{B}(\mathcal{H}_0) \cong \mathcal{B}(\mathcal{H}_0)\otimes 1$$

and define j_t on the Weyl operators on $\mathcal{H}_0$ by:

$$j_t(W_0(z)) := j_t(\exp i(z a_0^+ + z^+ a_0)) = \tag{4}$$

$$= e^{i(za_0^+ + z^+ a_0) + i(zX^+_{(0,t]} + z^+ X^-_{(0,t]})} = W_0(z) e^{i(zX^+_{[0,t]} + z^+ X^-_{(0,t]})}$$

The pair of operator valued measures $(X^+_{(s,t]})$, $(X^-_{(s,t]})$ *is called a quantum time shift for the quantum Brownian motion if the map* $W_0(z) \mapsto j_t(W_0(z))$ *is a* $*$*-homomorphism and the* 1*-parameter family*

$$P_0^t := E_{0]}(j_t(a_0)), \quad a_0 \in \mathcal{B}(\mathcal{H}_0)$$

is a quantum Markov semigroup whose restriction to the algebra $L^\infty(q_0)$ *is the usual heat semigroup.*

If the operator process $X^+_{[0,t]} = X^-_{(0,t]} = X_{(s,t]}$ is either $Q_{\chi_{(s,t]}}$ (see (2)) or $P_{\chi_{(s,t]}}$ (see (1)) then (4) takes the form

$$j_t(W_0(z)) = W_0(z) e^{i2Re(z) X_{[0,t]}}.$$

and j_t can be extended to a $*$-homomorphism from $W(\mathcal{H}_0)$ to $W(\mathcal{H}_0 \otimes \mathcal{B}(\Gamma))$. In this sense we get a quantum extension of the classical time shift.
If $X^+_{[0,t]} \neq X^-_{(0,t]}$ then we get a truly quantum time shifts.
If f is a smooth function and $f(q_0)$ is multiplication by f in $L^2(\mathbb{R})$, then with the identification $p = \frac{1}{i}\frac{\partial}{\partial x}$ one has

$$[p, f(q_0)] = \frac{1}{i} M_{(\frac{\partial}{\partial x} f)}$$

hence

$$-[p, \cdot\,]^2 (f(q_0)) = -[p, [p, \ f(q_0)\ ]\,] = M_{(\frac{\partial^2}{\partial x^2} f)} \tag{5}$$

which gives the right answer when we restrict our attention to the classical Wiener process. Denoting p the momentum operator in the initial space one has

$$p = \frac{1}{i\sqrt{2}} (a_0 - a_0^+)$$

This implies that

$$\frac{1}{2} [p, \cdot\,]^2 = \frac{1}{2} [a_0 - a_0^+, \cdot\,]^2 = \Big[\frac{1}{\sqrt{2}} \Big([a_0, \cdot\,] + [a_0^+, \cdot\,] \Big) \Big]^2 \tag{6}$$

From (3) we deduce that

$$[a_0, \cdot\,] = \frac{\partial}{\partial a_0^+}, \qquad [a_0^+, \cdot\,] = -\frac{\partial}{\partial a_0}$$

therefore, since

$$W_0(z) = \exp i(za_0^+ + z^+ a_0)$$

we find

$$\frac{\partial}{\partial a_0^+} W_0(z) = (iz) W_0(z), \qquad \frac{\partial}{\partial a_0} W_0(z) = (iz^+) W_0(z)$$

hence

$$\Big(\frac{\partial}{\partial a_0^+}+\frac{\partial}{\partial a_0}\Big)W_0(z)=i(2\Re z)W_0(z)$$

and therefore

$$\Big[\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial a_0^+}+\frac{\partial}{\partial a_0}\Big)\Big]^2W_0(z)=-(2Rez)^2W_0(z)$$

This implies that the generator

$$L:=\Big[\frac{1}{\sqrt{2}}\Big(\frac{\partial}{\partial a_0^+}+\frac{\partial}{\partial a_0}\Big)\Big]^2=\frac{1}{2}[p,\,\cdot\,]^2$$

generates the quantum Markov semigroup on the Weyl algebra given by

$$P_0^t=\exp tL.$$

Given the identity (5) the semigroup P_0^t is a quantum extension, on the whole Weyl algebra, of the classical heat semigroup.

1.1 Notations

Let us denote

– $\Gamma(L^2(\mathbb{R})$ the boson Fock space over the one-particle space $L^2(\mathbb{R})$;

– $\mathcal{E}=\{\psi(f):f\in L^2(\mathbb{R})\}$ the set of exponential vectors in $\Gamma(L^2(\mathbb{R})$;

– $\Phi=\psi(0)$ the vacuum state in $\Gamma(L^2(\mathbb{R})$;

– $\Gamma(\chi_{[0,t]})$ the orthogonal projector defined by

$$\Gamma(\chi_{[0,t]})\psi(f)=\psi(\chi_{[0,t]}f);$$

– $\Phi_{t]}:=\Gamma(\chi_{[0,t]})\Phi$; $\quad\Phi_{[t}:=\Gamma(\chi_{[t,\infty)})\Phi$;

– the Weyl operators $W(f)$, $f\in L^2(\mathbb{R})$, characterized by the property

$$W(f)\psi(g)=e^{-\frac{\|f\|^2}{2}-\langle f,g\rangle}\psi(f+g).$$

The operators $W(f)$ are unitary operators on $\mathcal{H}$ satisfying the CCR

$$W(f)W(g)=e^{i\Im(\langle f,g\rangle)}W(f+g);$$

– the annihilation, creation and number (or gauge or conservation) fields A , A^+ , N defined on $\mathcal{E}$ by the relations:

$$A(f)\psi(g)=\langle f,g\rangle\psi(g)$$

$$A^+(f)\psi(g)=\frac{d}{dt}|_{t=0}\psi(g+tf)$$

$$N_t\psi(g)=\frac{d}{ds}|_{s=0}\psi(e^{s\chi_{[0,t]}}g)$$

– $f_{t]} = \chi_{[0,t]} f$; $f_{[t} = \chi_{[t,\infty]} f$;

– $\mathcal{H}_0$ a complex Hilbert space , called the initial space;

– $\mathcal{H} = \mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R}))$;

– $\mathcal{H}_{t]} = \mathcal{H}_0 \otimes \Gamma(L^2([0,t]) \otimes \Phi_{[t}$;

– $\mathcal{B} = \mathcal{B}(\mathcal{H}) = \mathcal{B}\Big(\mathcal{H}_0 \otimes \Gamma(L^2(\mathbb{R})\Big) = \mathcal{B}\Big(\mathcal{H}_0\Big) \otimes \mathcal{B}\Big(\Gamma(L^2(\mathbb{R})\Big)$;

– $\mathcal{B}_{t]} = \mathcal{B}(\mathcal{H}_0 \otimes \Gamma(L^2([0,t])) \otimes 1_{[t} = \mathcal{B}(\mathcal{H}_0) \otimes \mathcal{B}(\Gamma(L^2([0,t])) \otimes 1_{[t}$;

– $\mathcal{B}_{[t} = \mathcal{B}(1_{\mathcal{H}_0} \otimes 1_{t]} \otimes \Gamma(L^2([t,\infty)) = \mathcal{B}(1_{\mathcal{H}_0}) \otimes 1_{t]} \otimes \mathcal{B}(\Gamma(L^2([t,\infty))$.

2 Shift on the Lie algebra

2.1 Current algebras

Let $\mathcal{L}$ be a complex $*$-Lie algebra. Let

$$\{X^+_\alpha,\, X^-_\alpha,\, X^0_\beta, \quad \alpha \in I,\ \beta \in I_0\}$$

where I, I_0 are disjoint sets, be set of generators of $\mathcal{L}$ satisfying the following conditions:

$$(X^0_\beta)^* = X^0_\beta\,, \quad \forall\, \beta \in I_0\,; \quad (X^+_\alpha)^* = X^0_\alpha\,, \quad \forall\, \alpha \in I$$

We assume that there is a single central element, denoted E or 1, among the generators and that it is of X^0-type (i.e. self-adjoint).
We denote $C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',\varepsilon'')$ the structure constants of $\mathcal{L}$ with respect to the generators (X^ε_α), i.e., with $\alpha \in I$, $\beta \in I_0$, $\varepsilon, \varepsilon', \varepsilon'' \in \{+,-,0\}$, and assuming summation over repeated indices:

$$[X^\varepsilon_\alpha, X^{\varepsilon'}_\beta] = C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',\varepsilon'')X^{\varepsilon''}_\gamma =$$

$$= \sum_{\gamma\in I_0} C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',0)X^0_\gamma + \sum_{\gamma\in I} C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',+)X^+_\gamma + \sum_{\gamma\in I} C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',-)X^-_\gamma \qquad (7)$$

In the following we will consider only locally finite Lie algebra, i.e., those such that, for any pair α, $\beta \in I \cup I_0$ only a finite number of structure constants $C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',\varepsilon'')$ is different from zero.
To the set of generators $\{X^\varepsilon_\alpha\,;\ \alpha \in I \cup I_0,\ \varepsilon = 0,+,-\}$ one associates the set of skew adjoint generators defined by:

$$\{X^+_\alpha - X^-_\alpha,\ i(X^+_\alpha + X^-_\alpha),\ iX^0_\beta\,;\ \alpha \in I,\ \beta \in I_0\}$$

The real vector subspace $\mathcal{L}^{sk}$ of $\mathcal{L}$, generated by this set, coincides with the real sub-Lie algebra consisting of all the skew-adjoint elements of $\mathcal{L}$. Denoting by $\mathcal{L}^{sa}$ the real vector subspace consisting of the self-adjoint elements of $\mathcal{L}$, one has the relation $\mathcal{L}^{sk} = i\mathcal{L}^{sa}$.

Definition 4 *Let be given a $*$-Lie algebra $\mathcal{L}$ with a canonical set of generators*

$$\{X_\alpha^\varepsilon\,;\ \varepsilon \in \{+,-,0\},\ \alpha \in I \cup I_0\}$$

with Lie-brackets as in (7). Let $(S,\mathcal{B})$ be a measurable space and $\mathcal{C} \subset L^\infty_{\mathbb{C}}(\mathcal{S},\mathcal{B})$ be a $*$-sub-algebra. The current algebra over S of $\{\mathcal{L}, X_\alpha^\varepsilon\}$ is the $*$-Lie algebra $\mathcal{L}(S,\mathcal{C})$ defined as the vector space obtained by algebraic linear span of the family

$$\{X_\alpha^\varepsilon(f)\,;\ f \in \mathcal{C},\ \varepsilon \in \{+,-,0\},\ \alpha \in I_0 \cup I\}$$

such that:

- *the generators are independent in the sense that*

$$\sum_{\varepsilon,\,\alpha} X_\alpha^\varepsilon(f_{\varepsilon,\,\alpha}) = 0 \iff f_{\varepsilon,\,\alpha} = 0\,,\ \forall \varepsilon,\, \alpha$$

- *the map $f \mapsto X_\alpha^\varepsilon(f)$ is linear for $\varepsilon = +,0$ and anti-linear for $\varepsilon = -$*
- *the Lie-brackets are defined by*

$$[X_\alpha^\varepsilon(f), X_\beta^{\varepsilon'}(g)] = C^\gamma_{\alpha,\beta}(\varepsilon,\varepsilon',\varepsilon'')X_\gamma^{\varepsilon''}(f^\varepsilon g^{\varepsilon'})$$

where, for a test function f, we use the notation

$$\begin{cases} f^\varepsilon = f & \text{if } \varepsilon = +,0 \\ f^\varepsilon = \overline{f} & \text{if } \varepsilon = - \end{cases}$$

- *the involution is defined by*

$$(X_\alpha^\varepsilon(f))^* := X_\alpha^{\varepsilon^*}(\bar{f})$$

with $+^* := -\,,\ -^* := +\,,\ 0^* := 0$.

Definition 5 *A unitary representation of a real Lie algebra $\mathcal{L}$ is a triple $\{\mathcal{H},\pi,\mathcal{D}\}$ where $\mathcal{H}$ is a complex Hilbert space, $\mathcal{D} \subseteq \mathcal{H}$ is a total subset of $\mathcal{H}$ which is a core for each $\pi(a)$, $a \in \mathcal{L}$;*

$$\pi : \mathcal{L} \to \mathcal{L}^a(\mathcal{D})\ (\text{adjointable linear maps from lin-span}(\mathcal{D}) \to \mathcal{H})$$

$\pi(a)^+$, i.e., the $\mathcal{H}$-adjoint of $\pi(a)$, is defined on $\mathcal{D}$ and $\pi(a)^+ = -\pi(a)$;

$$\pi([a,b]) = [\pi(a),\pi(b)]\,,\quad \forall\, a,\, b \in \mathcal{L}$$

where the identity is meant weakly on $\mathcal{D}$.

Definition 6 *Let $\mathcal{L}$ be a real or complex $*$-Lie algebra with center $\mathfrak{C}(\mathcal{L})$. A central decomposition of $\mathcal{L}$ is a direct sum of vector spaces*

$$\mathcal{L} = \mathfrak{C}(\mathcal{L}) \oplus \mathcal{L}_0$$

where $\mathcal{L}_0$ is a sub-Lie algebra of $\mathcal{L}$.

Let $\mathcal{L}$ and $\mathcal{G}$ be real or complex $*$-Lie algebras and $\Phi : \mathcal{L} \to \mathcal{G}$ be a $*$-isomorphism of Lie algebras. Clearly if $\mathcal{L}$ has a central decomposition

$$\mathcal{L} = \mathfrak{C}(\mathcal{L}) \oplus \mathcal{L}_0$$

then $\mathcal{G}$ has the central decomposition

$$\mathcal{G} = \Phi(\mathfrak{C}(\mathcal{L})) \oplus \Phi(\mathcal{L}_0).$$

2.2 Lie algebra shifts

Let be given a complex $*$-Lie algebra $\mathcal{L}_S$ with scalar center $\mathfrak{C}(\mathcal{L}_S) = \mathbb{C}1_S$, a skew-adjoint sub-algebra $\mathcal{L}_S^{sk}$ of $\mathcal{L}_S$ with central decomposition

$$\mathcal{L}_S^{sk} = i\mathbb{R}1_S \oplus \mathcal{L}_{0,S}^{sk}$$

and a space of test function $\mathcal{C}$. Let $\mathcal{L}_S(\mathbb{R}^d, \mathcal{C})$ denotes the current algebra of $\mathcal{L}_S$ over $\mathbb{R}^d$ with a scalar center $\mathfrak{C}(\mathcal{L}_S(\mathbb{R}^d, \mathcal{C}))$. We assume the existence of a unitary representation of the skew-adjoint sub-algebra $\mathcal{L}_S^{sk} \subset \mathcal{L}_S$ on some Hilbert space $\mathcal{H}_S$, and a unitary representation of the skew-adjoint current sub-algebra $\mathcal{L}_S^{sk}(\mathbb{R}^d, \mathcal{C}) \subset \mathcal{L}_S(\mathbb{R}^d, \mathcal{C})$ on a Hilbert space $\mathcal{H}_N$.
In the following all the above Lie algebras are identified with their images under the corresponding unitary representations and these are omitted from the notations.

Definition 7 *In the above notations, a Lie algebra shift is a family of unitary homomorphisms of $*$-Lie algebras*

$$\hat{j}_I : \mathcal{L}_S^{sk} \to \mathcal{L}_S^{sk} \otimes 1 + 1_0 \otimes \mathcal{L}_S^{sk}(I, \mathcal{C}_I)$$

parameterized by the Borel subsets of $\mathbb{R}^d$ and with the following structure:

$$\hat{j}_I(X) := T_I^S(X) \otimes 1 + 1_0 \otimes T_I(X)$$

with the property that the exponential map exist and the map j_I°, defined by

$$j_I^\circ(e^{iX}) := e^{i\hat{j}_I(X)}, \quad X \in \mathcal{L}_{0,S}^{sa} \tag{8}$$

extends to a $*$-homomorphism of the Von Neumann algebra generated by the set

$$\{e^{iX} \,;\, X \in \mathcal{L}_S^{sa}\}$$

and for each $X \in \mathcal{L}_S^{sa}$, the map $t \in \mathbb{R} \mapsto e^{itX} \in \mathcal{L}_S^{sa}$ is strongly continuous.

Remark 1 *Remark*

1. *For $I \subset \mathbb{R}^d$, the test function space $\mathcal{C}_I$ denotes the space of test function in $\mathcal{C}$ with support in I.*

2. *The maps* $T_I^S : \mathcal{L}_S^{sk} \to \mathcal{L}_S^{sk}$ *and* $T_I : \mathcal{L}_S^{sk} \to \mathcal{L}_S^{sk}(I, \mathcal{C}_I)$ *in the above definition must be* $*$*-Lie algebra homomorphisms.*

3. *The maps* T_I^S *and* T_I *can be extended, by complex linearity, to* $*$*-Lie algebra morphism of the whole Lie algebra* $\mathcal{L}_S$*. The corresponding extension of the Lie algebra shift* $\hat{j}_I$ *will be still denoted* $\hat{j}_I$*.*

4. *If it is given a family* $\{X_\alpha\,;\ \alpha \in F\}$*, of generators of* $\mathcal{L}_S$*, and* $\{X_\alpha(\varphi)\,;\ \alpha \in F,\ \varphi \in \mathcal{C}_I\}$ *denotes the corresponding family of generators of* $\mathcal{L}_S(I, \mathcal{C}_I)$*, the action on the generators of the Lie algebra time shift is given by:*

$$\hat{j}_I(X_\alpha) := T_\alpha^\gamma X_\gamma \otimes 1 + 1_0 \otimes \sum_\beta X_\beta(\chi_I \varphi_{\alpha,\beta})$$

with $\varphi_{\alpha,\beta} \in \mathcal{C}$ *and* (T_α^γ) *is a finite matrix. In fact the set* F *of the index* α *is finite.*

5. *We want that the map* $\hat{j}_I$ *to be injective then, the matrix* (T_α^γ) *must be invertible.*

6. *To prevent confusions, if* $\mathcal{L}$ *is the Lie algebra of a Lie group* G *and the elements of* $\mathcal{L}$ *are realized as operators acting on some Hilbert space, in the following, for any element* $Y \in \mathcal{L}^{sk}$*, we will denote* $\exp(Y) \in G$ *the exponential of* Y *in the sense of Lie algebra theory and* e^Y *the exponential of* Y *in the given representation, defined by the spectral theorem or by the exponential series on some domain.*

In the following we will produce several examples of unitary homomorphism of $*$-Lie algebras $\hat{j}_I$ such that the map j_I°, defined by (8), does not extend to a Von Neumann algebra homomorphism.

Example 1 *Example (**Heisenberg Lie algebra**)*
Let $\mathcal{L}_S = \{a^+,\, a^-,\, 1_S\}$ *be the Heisenberg Lie algebra with the commutation relation* $[a^-, a^+] = 1_S$ *where* 1_S *is the central element. The map* T_I^S *is given by:*

$$T_I^S a^+ = T_+^+ a^+ + T_+^- a^- + T_0 1_S$$

$$T_I^S a^- = \overline{T_+^-} a^+ + \overline{T_+^+} a^- + \overline{T_0} 1_S$$

$$T_I^S 1_S = (|T_+^+|^2 - |T_+^-|^2) 1_S$$

where the complex numbers $T_\pm^\pm$ *and* T_0 *may depend on* $I \subseteq \mathbb{C}$*.*

The case $T_+^- = T_0 = 0$ and $T_+^+ = 1$, which corresponds to the case

$$T_I^S(X) = X\,, \quad \forall\, X \in \mathcal{L}_S \tag{9}$$

has been studied in the paper [9]. In this paper we will consider the oscillator algebra $\mathcal{L}_{osc} := \{a^+,\, a,\, a^+a,\, 1\}$ and the class of shifts satisfying the same condition (9).

3 Time shift on the oscillator algebra

3.1 The oscillator algebra

In the present section we recall some known facts on the operators associated to the Fock representation of the CCR and we use the notations of subsection 1.1.

Lemma 1 *Lemma (see [9] and [3]) Let $\mathcal{H}$ be an Hilbert space and $u \in \mathcal{H}$. Denoting $A^+(u)$, $A(u)$, respectively, the creation and annihilation operators acting on the Fock space $\Gamma(\mathcal{H})$. Then, the operators $e^{A^+(u)}$, $e^{A(u)}$ are well defined by the exponential series which converges strongly on the maximal algebraic domain. Moreover, for any self-adjoint operator T, on $\mathcal{H}$, the operator $\Lambda(T)$ is uniquely defined by the identity*

$$e^{it\Lambda(T)} = \Gamma(e^{itT}) \,, \quad t \in \mathbb{R}$$

and $e^{i\Lambda(T)}$ maps the maximal algebraic domain into it self.

Let $\mathcal{H} = L^2(\mathbb{R})$ be the Hilbert space of square integrable complex valued functions. A real valued function $\psi \in \mathcal{K} := L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ will be considered as multiplication operator on the space $\mathcal{H}$, which is bounded and self-adjoint operator.

Definition 8 *We define a generalized Weyl operator on the Fock space $\Gamma(\mathcal{H})$ to be the unitary operator given by*

$$W(\phi, \psi) = e^{i(A^+(\phi)+A(\phi)+\Lambda(\psi))} \,, \quad \psi \in \mathcal{K} \,,\ \phi \in \mathcal{H} \tag{10}$$

The norm closure, in $\Gamma(\mathcal{H})$, of the algebra generated by them, denoted $\mathcal{W}_g(\mathcal{H})$, will be called the oscillator algebra over $\mathcal{H}$.

Lemma 2 *Lemma (see [7]) Let $\mathcal{H}$ be an Hilbert space, u, $v \in \mathcal{H}$ and $X \in \mathcal{B}(\mathcal{H})$. Then we have the relations:*

$$e^{A^+(u)+\Lambda(X)+A(v)+\alpha} = e^{A^+(\tilde{u})} e^{\Lambda(X)} e^{A(\tilde{v})} e^{\tilde{\alpha}} \tag{11}$$

$$e^{\Lambda(X)} e^{A^+(u)} e^{-\Lambda(X)} = e^{A^+(e^X \cdot u)} \tag{12}$$

$$e^{A(u)} e^{A^+(v)} = e^{\langle u,v\rangle} e^{A^+(v)} e^{A(u)} \tag{13}$$

where

$$\tilde{u} = \sum_{n=1}^{\infty} \frac{X^{n-1}}{n!} u = e_1(X)u \qquad , \qquad \tilde{v} = \sum_{n=1}^{\infty} \frac{(X^*)^{n-1}}{n!} v = e_1(X^*)v$$

$$\tilde{\alpha} = \alpha + \sum_{n=2}^{\infty} \frac{1}{n!} \langle v, X^{n-2}u\rangle = \alpha + \langle v, e_2(X)u\rangle$$

with

$$e_1(x) := \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \qquad , \qquad e_2(x) := \sum_{n=2}^{\infty} \frac{x^{n-2}}{n!}$$

Theorem 1 *In the notation (10), for all $\phi_j \in \mathcal{H}$ and real valued function $\psi_j \in \mathcal{K}$ with norm*

$$\|\psi_j\| < \pi \qquad , \qquad j = 1, 2$$

the generalized Weyl CCR holds

$$W(\phi_1, \psi_1)W(\phi_2, \psi_2) = e^{i\gamma} W(\phi, \psi)$$

where ψ, ϕ and γ are given by the relations:

$$\psi = \psi_1 + \psi_2 \tag{14}$$

$$e_1(i\psi)\phi = e_1(i\psi_1)\phi_1 + e^{i\psi_1} e_1(i\psi_2)\phi_2 \tag{15}$$

$$\langle \phi, e_2(i\psi)\phi \rangle - i\gamma = \langle \phi_1, e_2(i\psi_1)\phi_1 \rangle + \langle \phi_2, e_2(i\psi_2)\phi_2 \rangle + \langle e_1(-i\psi_1)\phi_1, e_1(i\psi_2)\phi_2 \rangle$$

Proof: We start by the fact that the function

$$x \mapsto e_{-1}(x) = \frac{1}{e_1(x)}$$

is analytic with convergence radius equal to 2π.
Note that the norm of ψ as multiplication operator $\psi \equiv M_\psi$ is equal to $\|\psi\|_\infty$.
It follow that for all operator $\psi \in \mathcal{K}$ with norm $\|\psi\| < 2\pi$, we have $e_1(\psi)$ is invertible in $\mathcal{K}$ with inverse equal to $e_{-1}(\psi)$.
Using formula (11) in Lemma 2, for $j = 1$, 2, we can write

$$W(\phi_j, \psi_j) = e^{A^+(i\phi_j) + \Lambda(i\psi_j) + A(-i\phi_j)} = e^{A^+(\tilde{\phi}_j)} e^{\Lambda(i\psi_j)} e^{A(\tilde{\varphi}_j)} e^{\tilde{\alpha}_j}$$

where

$$\tilde{\phi}_j = e_1(i\psi_j) i\phi_j \, , \quad \tilde{\varphi}_j = -e_1(-i\psi_j) i\phi_j \, , \quad \tilde{\alpha}_j = -\langle \phi_j, e_2(i\psi_j)\phi_j \rangle$$

Using the identities (12) and (13) in the Lemma (**??**), we deduce

$$\begin{aligned}
W(\phi_1, \psi_1)W(\phi_2, \psi_2) &= e^{A^+(\tilde{\phi}_1)} e^{\Lambda(i\psi_1)} e^{A(\tilde{\varphi}_1)} e^{\tilde{\alpha}_1} e^{A^+(\tilde{\phi}_2)} e^{\Lambda(i\psi_2)} e^{A(\tilde{\varphi}_2)} e^{\tilde{\alpha}_2} \\
&= e^{A^+(\tilde{\phi}_1 + e^{i\psi_1}\tilde{\phi}_2)} e^{\Lambda(i\psi_1)} e^{\Lambda(i\psi_2)} e^{A(\tilde{\varphi}_2 + e^{-i\psi_2}\tilde{\varphi}_1)} e^{\tilde{\alpha}_1 + \tilde{\alpha}_2 + \langle \tilde{\varphi}_1, \tilde{\phi}_2 \rangle} \\
&= e^{A^+(\tilde{\phi})} e^{\Lambda(i\psi_1)} e^{\Lambda(i\psi_2)} e^{A(\tilde{\varphi})} e^{\tilde{\gamma}} \\
&= e^{A^+(\tilde{\phi})} e^{\Lambda(i\psi)} e^{A(\tilde{\varphi})} e^{\tilde{\gamma}}
\end{aligned}$$

where $\psi = \psi_1 + \psi_2$ and

$$\tilde{\phi} := \tilde{\phi}_1 + e^{i\psi_1}\tilde{\phi}_2 \, , \quad \tilde{\varphi} := \tilde{\varphi}_2 + e^{-i\psi_2}\tilde{\varphi}_1 \, , \quad \tilde{\gamma} := \tilde{\alpha}_1 + \tilde{\alpha}_2 + \langle \tilde{\varphi}_1, \tilde{\phi}_2 \rangle$$

The condition $\|\psi_j\| < \pi$, $j = 1$, 2, implies that $\|\psi\| = \|\psi_1 + \psi_2\| < 2\pi$ and this implies that $e_1(i\psi)$ is invertible. Therefore there exists $\phi \in \mathcal{H}$ such that the vector

$$\tilde{\phi} = \tilde{\phi}_1 + e^{i\psi_1}\tilde{\phi}_2 = e_1(i\psi_1)i\phi_1 + e^{i\psi_1} e_1(i\psi_2)i\phi_2$$

can be written in the form

$$\tilde{\phi} = e_1(i\psi)i\phi$$

We look for a $\varphi \in \mathcal{H}$ satisfying

$$\tilde{\varphi} = e_1(-i\psi)i\varphi$$

Using the relation

$$e_1(-\psi) = e^{-\psi}e_1(\psi)\,, \quad \forall\, \psi \in \mathcal{K}$$

we obtain

$$\begin{aligned}
\tilde{\varphi} &:= \tilde{\varphi}_2 + e^{-i\psi_2}\tilde{\varphi}_1 \\
&= -\left[e_1(-i\psi_2)i\phi_2 + e^{-i\psi_2}e_1(-i\psi_1)i\phi_1\right] \\
&= -e^{-i\psi_2}\left[e_1(i\psi_2)i\phi_2 + e_1(-i\psi_1)i\phi_1\right] \\
&= -e^{-i\psi_2}e^{-i\psi_1}\left[e^{i\psi_1}e_1(i\psi_2)i\phi_2 + e_1(i\psi_1)i\phi_1\right] \\
&= -(e^{i\psi_1}e^{i\psi_2})^{-1}\left[e^{i\psi_1}e_1(i\psi_2)i\phi_2 + e_1(i\psi_1)i\phi_1\right] \\
&= -e^{-i\psi}e_1(i\psi)i\phi \\
&= -e_1(-i\psi)i\phi.
\end{aligned}$$

Since e_1 is invertible, this gives $\varphi = -\phi$.
The above definitions of ϕ, φ and ψ give

$$\begin{aligned}
W(\phi_1,\psi_1)W(\phi_2,\psi_2) &= e^{A^+(\tilde{\phi})}e^{\Lambda(i\psi)}e^{A(\tilde{\varphi})}e^{\tilde{\gamma}} \\
&= e^{A^+(i\phi)+\Lambda(i\psi)+A(i\varphi)+i\gamma} \\
&= e^{i(A^+(\phi)+A(\phi)+\Lambda(\psi))+i\gamma} \\
&= e^{i\gamma}W(\phi,\psi)
\end{aligned}$$

with γ being a scalar such that $\tilde{\gamma} = i\gamma + \langle -i\phi, e_2(i\psi)i\phi\rangle$.
The relation $\tilde{\gamma} = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \langle\tilde{\varphi}_1, \tilde{\phi}_2\rangle$ gives the result. ■

Lemma 3 *Lemma Let $\mathcal{W} := \{W(\phi,\psi)\,,\ \phi \in \mathcal{H}, \psi \in \mathcal{K}\}$ be the set of generalized Weyl operators. Then $\mathcal{W}$ is a self-adjoint linearly independent set.*

Proof: We have

$$[W(\phi,\psi)]^* = e^{-i(A^+(\phi)+A(\phi)+\Lambda(\overline{\psi}))} = e^{i(A^+(-\phi)+A(-\phi)+\Lambda(-\psi))} = W(-\phi,-\psi) \in \mathcal{W}.$$

This gives that $\mathcal{W}$ is self-adjoint set.
Let $\lambda_j\,,\ j = 1,\cdots,n$ such that $\sum_{j=1}^{n}\lambda_j W(\phi_j,\psi_j) = 0$. By the formula (11) we obtain

$$\left\langle e(tu), \sum_{j=1}^{n}\lambda_j e^{A^+(\tilde{\phi}_j)}e^{\Lambda(i\psi_j)}e^{A(\tilde{\varphi}_j)}e^{\tilde{\gamma}_j}e(sv)\right\rangle = 0\,, \quad \forall\, \phi \in \mathcal{H},\ \psi \in \mathcal{K}.$$

This gives

$$\sum_j \lambda_j e^{\tilde{\gamma}_j} e^{t\left\langle u,\tilde{\phi}_j\right\rangle + s\langle\tilde{\varphi}_j,v\rangle + st\left\langle u,e^{i\psi_j}v\right\rangle} = 0.$$

Denote $\mu_j = \lambda_j e^{\tilde{\gamma}_j}$, $a_j = \langle u,\tilde{\phi}_j\rangle$, $b_j = \langle\tilde{\varphi}_j, v\rangle$ and $c_j = \left\langle u, e^{i\psi_j}v\right\rangle$. By using the property:

$$\text{for all } j,\, k = 1,\cdots,n\,,\ \ (a_j,b_j,c_j) = (a_k,b_k,c_k) \Longrightarrow (\phi_j,\psi_j) = (\phi_k,\psi_k),$$

and the independence of the set

$$\mathrm{B} := \{\kappa_{(a,b,c)} : \mathbb{R}^2 \ni (s,t) \longmapsto e^{at+bs+cst} \in \mathbb{C}\,, \quad (a,b,c) \in \mathbb{C}^3\},$$

we can conclude the statement. ■

3.2 Lie algebra morphisms of the oscillator algebra

Definition 9 *The infinite dimensional oscillator algebra is the complex $*$-Lie algebra $\mathcal{L}_{osc}$ with linearly independent generators $a_0(\phi)$, $a_0^+(\phi)$, $n_0(\phi)$, 1_0 (central element) and relations:*

$$[a_0(\phi), a_0^+(\psi)] = \langle\phi,\psi\rangle 1_0,$$
$$[n_0(\phi), a_0^+(\psi)] = a_0^+(\phi\psi)\,, \quad [n_0(\phi), a_0(\psi)] = -a_0(\overline{\phi}\psi),$$
$$[n_0(\phi), n_0(\psi)] = [a_0(\phi), a_0(\psi)] = [a_0^+(\phi), a_0^+(\psi)] = 0,$$
$$a_0^*(\phi) = a_0^+(\phi)\,, \quad n_0^*(\phi) = n_0(\overline{\phi}).$$

The test function algebra is chosen to be the space

$$\mathcal{K}_S = L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$$

and the localization is defined by the bounded intervals in $\mathbb{R}$.
The maps $\phi \mapsto a_0^+(\phi)$, $n_0(\phi)$ are linear in ϕ while $\phi \mapsto a_0(\phi)$ is anti-linear.

We consider the boson Fock representation of the oscillator algebra. We will study the class of Lie algebra shifts on the oscillator algebra whose action on the generators has the following form:

$$\hat{j}_t(a_0^+(\phi)) = a_0^+(\phi) + X_t^{(1)}(T_1\phi) = a_0^+(\phi)\otimes 1 + 1_0 \otimes X_t^{(1)}(T_1\phi) \qquad (16)$$

$$\hat{j}_t(a_0(\phi)) = a_0(\phi) + X_t^{(2)}(T_2\phi) = a_0(\phi)\otimes 1 + 1_0 \otimes X_t^{(2)}(T_2\phi) \qquad (17)$$

$$\hat{j}_t(n_0(\phi)) = n_0(\phi) + X_t^{(3)}(T_3\phi) = n_0(\phi)\otimes 1 + 1_0 \otimes X_t^{(3)}(T_3\phi) \qquad (18)$$

$$\hat{j}_t(1_0) = 1_0 + R_t = 1_0 \otimes 1 + 1_0 \otimes R_t \qquad (19)$$

where the $X_t^{(i)}$, $i = 1,2,3$, are given by

$$X_t^{(i)}(\phi) = \alpha_i A_t^+(\phi) + \beta_i A_t(\phi) + \gamma_i N_t(\phi) \qquad (20)$$

α_i, β_i, $\gamma_i \in \mathbb{C}$, R_t is a process acting on $\Gamma(L^2(\mathbb{R}), \mathcal{K}_S)$ and T_i, $i = 1,2,3$, are real linear operators acting on $\mathcal{K}_S$. To exclude trivial cases we assume that there exists $i \in \{1,2,3\}$, such that $(\alpha_i,\beta_i,\gamma_i) \neq (0,0,0)$ and $T_i \neq 0$.

Theorem 2 *Theorem A real linear map $\hat{j}_t$, of the form defined by (16)-(20), defines a 1-parameter family of homomorphism of $*$-Lie algebras if and only if it belongs to one of the following five classes:*

Class I.

$$\hat{j}_t^{(1)}(a_0^+(\phi)) = a_0^+(\phi)$$
$$\hat{j}_t^{(1)}(a_0(\phi)) = a_0(\phi)$$
$$\hat{j}_t^{(1)}(n_0(\phi)) = n_0(\phi) + A_t^+(T\phi) + A_t(T\phi)$$
$$\hat{j}_t^{(1)}(1_0) = \mathbf{1} = 1_0 + 1$$

with relations

$$T\bar{\phi} = T\phi\,, \quad \langle T\phi, T\psi\rangle = \langle T\psi, T\phi\rangle \ , \quad \forall\, \psi,\, \phi \in \mathcal{K}_S.$$

Class II.

$$\hat{j}_t^{(2)}(a_0^+(\phi)) = a_0^+(\phi)$$
$$\hat{j}_t^{(2)}(a_0(\phi)) = a_0(\phi)$$
$$\hat{j}_t^{(2)}(n_0(\phi)) = n_0(\phi) + N_t(T\phi)$$
$$\hat{j}_t^{(2)}(1_0) = \mathbf{1}$$

with the following relation

$$\overline{T\phi} = T\overline{\phi}\,, \quad \forall\, \phi \in \mathcal{K}_S.$$

Class III.

$$\hat{j}_t^{(3)}(a_0^+(\phi)) = a_0^+(\phi)$$
$$\hat{j}_t^{(3)}(a_0(\phi)) = a_0(\phi)$$
$$\hat{j}_t^{(3)}(n_0(\phi)) = n_0(\phi) + \delta A_t^+(T\phi) + \overline{\delta} A_t(T\phi) + N_t(T\phi)$$
$$\hat{j}_t^{(3)}(1_0) = \mathbf{1}$$

with the following relation

$$\overline{T\phi} = T\overline{\phi} = T\phi\,, \quad \forall\, \phi \in \mathcal{K}_S.$$

Class IV.

$$\hat{j}_t^{(4)}(a_0^+(\phi)) = a_0^+(\phi) + \lambda A_t^+(T\phi)$$
$$\hat{j}_t^{(4)}(a_0(\phi)) = a_0(\phi) + \overline{\lambda} A_t(T\phi)$$
$$\hat{j}_t^{(4)}(n_0(\phi)) = n_0(\phi) + \lambda_0 N_t(T\phi)$$
$$\hat{j}_t^{(4)}(1_0) = 1_0 + c|\lambda|^2 t\, 1$$

with relations

$$T\overline{\phi} = \frac{\overline{\lambda}_0}{\lambda_0}\,\overline{T\phi}\,, \quad \langle T\phi, T\psi\rangle = c\langle \phi, \psi\rangle\,, \quad T(\phi\psi) = \lambda_0 T\phi T\psi\,,\ \ \forall\, \psi,\, \phi \in \mathcal{K}_S. \tag{21}$$

Class V.

$$\hat{j}_t^{(5)}(a_0^+(\phi)) = a_0^+(\phi) + \overline{\lambda} A_t(T\phi)$$
$$\hat{j}_t^{(5)}(a_0(\phi)) = a_0(\phi) + \lambda A_t^+(T\phi)$$
$$\hat{j}_t^{(5)}(n_0(\phi)) = n_0(\phi) + \lambda_0 N_t(\overline{T\phi})$$
$$\hat{j}_t^{(5)}(1_0) = 1_0 - c|\lambda|^2 t\, 1$$

with relations

$$T\overline{\phi} = \frac{\lambda_0}{\overline{\lambda}_0}\overline{T\phi}\,, \quad \langle T\phi, T\psi\rangle = c\langle \psi, \phi\rangle\,, \quad T(\phi\psi) = -\overline{\lambda}_0 T\phi T\psi\,, \ \ \forall\, \psi, \phi \in \mathcal{K}_S. \tag{22}$$

Here $\lambda,\ \delta,\ \lambda_0 \in \mathbb{C}\,,\ \delta \neq 0,\ \lambda_0 \neq 0,\ c > 0$ and T is an $\mathbb{R}$-linear operator on $\mathcal{K}_S$.

Example 2 *Example The following examples show that none of the above five classes is empty.*

Class I. *We have*

$$T\bar{\phi} = T\phi\,, \quad \langle T\phi, T\varphi\rangle = \langle T\varphi, T\phi\rangle\, \forall\, \phi,\ \varphi \in \mathcal{K}_S.$$

Then $T(i\Im(\phi)) = 0$ for all $\phi \in \mathcal{K}_S$. This gives $T\phi = T(\Re(\phi))$ and we can choose as example $T(\phi) = M_\psi \Re(\phi) = \psi\Re(\phi)$, where ψ is a real valued function in $\mathcal{K}_S$.

Class II. *The condition $\overline{T\phi} = T\bar{\phi}$, $\forall \phi \in \mathcal{K}$ implies that $T\phi$ is a real valued function if ϕ is it and it's purely complex valued function if ϕ is it. We can choose T as a multiplication operator $T = M_\psi$ where ψ is a real valued function in $\mathcal{K}_S$.*

Class III. *Here we can choose T as in the first example.*

Class IV. *Here T verify the conditions (21) and we can choose the operator T_α defined by:*

$$T_\alpha(\phi)(s) = \frac{1}{\lambda_0}\phi\left(\frac{s}{c|\lambda_0|^2} + \alpha\right)\,, \quad \alpha \in \mathbb{R}.$$

Class V. *We can show that if an operator T verify the relations (21), then the operator $-\overline{T}$ verify the relations (22). It follow that the operator T_α can be replaced by $-\overline{T_\alpha}$.*

Proof of theorem 2. Since we want the $\hat{j}_t$ to be $*$-map, the process R_t must be self-adjoint. Moreover, the relations

$$[\hat{j}_t(a_0^+(\phi)]^* = \hat{j}_t([a_0^+(\phi)]^*) = \hat{j}_t(a_0(\phi))$$
$$[\hat{j}_t(a_0(\phi)]^* = \hat{j}_t([a_0(\phi)]^*) = \hat{j}_t(a_0^+(\phi))$$

give

$$\alpha_1 T_1\phi = \overline{\beta_2} T_2\phi\,, \quad \alpha_2 T_2\phi = \overline{\beta_1} T_1\phi\,, \quad \gamma_2 T_2\phi = \overline{\gamma_1}\overline{T_1\phi}. \tag{23}$$

On the other hand, from the relation

$$[\hat{j}_t(n_0(\phi))]^* = \hat{j}_t([n_0(\phi)]^*) = \hat{j}_t(n_0(\overline{\phi}))$$

one has

$$\alpha_3 T_3\overline{\phi} = \overline{\beta_3} T_3\phi, \tag{24}$$

$$\overline{\gamma_3}\overline{T_3\phi} = \gamma_3 T_3\overline{\phi}. \tag{25}$$

For any $\phi, \psi \in \mathcal{K}_S$, the commutation relations give

$$\begin{aligned}
&[\hat{j}_t(a_0(\psi)), \hat{j}_t(a_0^+(\phi))] = [a_0(\psi)), a_0^+(\phi)] \\
&+[\alpha_2 A_t^+(T_2\psi) + \beta_2 A_t(T_2\psi) + \gamma_2 N_t(T_2\psi), \alpha_1 A_t^+(T_1\phi) + \beta_1 A_t(T_1\phi) + \gamma_1 N_t(T_1\phi)] \\
&\quad = \langle\psi, \phi\rangle\, 1_0 + [\alpha_2 A_t^+(T_2\psi), \beta_1 A_t(T_1\phi)] + [\alpha_2 A_t^+(T_2\psi), \gamma_1 N_t(T_1\phi)] \\
&\quad +[\beta_2 A_t(T_2\psi), \alpha_1 A_t^+(T_1\phi)] + [\beta_2 A_t(T_2\psi), \gamma_1 N_t(T_1\phi)] \\
&\quad +[\gamma_2 N_t(T_2\psi), \beta_1 A_t(T_1\phi)] + [\gamma_2 N_t(T_2\psi), \alpha_1 A_t^+(T_1\phi)] \\
&\quad = \langle\psi, \phi\rangle\, 1_0 + (\beta_2\alpha_1 \langle T_2\psi, T_1\phi\rangle - \alpha_2\beta_1 \langle T_1\phi, T_2\psi\rangle)\, t1 \\
&\quad +A_t^+((\gamma_2\alpha_1 - \gamma_1\alpha_2) T_1\phi T_2\psi) + A_t(\overline{\gamma_1\beta_2} T_2\psi \overline{T_1\phi} - \overline{\gamma_2\beta_1} T_1\phi \overline{T_2\psi}).
\end{aligned}$$

On the other hand, we have

$$\hat{j}_t([a_0(\psi), a_0^+(\phi)]) = \hat{j}_t(\langle\psi, \phi\rangle\, 1_0) = \langle\psi, \phi\rangle\,(1_0 + R_t).$$

Therefore, the commutation relations are preserved if and only if for any ϕ, $\psi \in \mathcal{K}_S$,

$$\begin{aligned}
\langle\psi, \phi\rangle\, R_t &= (\beta_2\alpha_1 \langle T_2\psi, T_1\phi\rangle - \alpha_2\beta_1 \langle T_1\phi, T_2\psi\rangle) t1 \\
&\quad +A_t^+((\gamma_2\alpha_1 - \gamma_1\alpha_2) T_1\phi T_2\psi) + A_t(\overline{\gamma_1\beta_2} T_2\psi \overline{T_1\phi} - \overline{\gamma_2\beta_1} T_1\phi \overline{T_2\psi}).
\end{aligned}$$

This gives

$$\langle\psi, \phi\rangle\, R_t = (\beta_2\alpha_1 \langle T_2\psi, T_1\phi\rangle - \alpha_2\beta_1 \langle T_1\phi, T_2\psi\rangle)\, t1, \tag{26}$$

$$\gamma_2\alpha_1 T_1\phi T_2\psi = \gamma_1\alpha_2 T_1\phi T_2\psi\,, \quad \overline{\gamma_1\beta_2} T_2\psi \overline{T_1\phi} = \overline{\gamma_2\beta_1} T_1\phi \overline{T_2\psi}.$$

In a similar way, the identities

$$[\hat{j}_t(n_0(\phi)), \hat{j}_t(a_0^+(\psi))] = \hat{j}_t\left([n_0(\phi), a_0^+(\psi)]\right)$$

$$[\hat{j}_t(n_0(\phi)), \hat{j}_t(a_0(\psi))] = \hat{j}_t\left([n_0(\phi), a_0(\psi)]\right)$$

give, for any ϕ, $\psi \in \mathcal{K}_S$,

$$\beta_3\alpha_1 \langle T_3\phi, T_1\psi\rangle = \beta_1\alpha_3 \langle T_1\psi, T_3\phi\rangle\,,$$

$$\alpha_1 T_1\phi\psi = (\gamma_3\alpha_1 - \gamma_1\alpha_3)T_1\psi T_3\phi, \tag{27}$$

$$\overline{\beta_1}T_1\phi\psi = \overline{\gamma_1\beta_3}\ \overline{T_1\psi}T_3\phi - \overline{\gamma_3\beta_1}\ \overline{T_3\phi}T_1\psi, \tag{28}$$

$$\gamma_1 T_1\psi\phi = 0, \tag{29}$$

$$\beta_3\alpha_2\langle T_3\phi, T_2\psi\rangle = \beta_2\alpha_3\langle T_2\psi, T_3\phi\rangle = \alpha_2 T_2\overline{\phi}\psi = (\gamma_2\alpha_3 - \gamma_3\alpha_2)T_2\psi T_3\phi, \tag{30}$$

$$\overline{\beta_2}T_2\overline{\phi}\psi = \overline{\gamma_3\beta_2}\ \overline{T_3\phi}T_2\psi - \overline{\gamma_2\beta_3}\ \overline{T_2\psi}T_3\phi,$$

$$\gamma_2 T_2\overline{\psi}\phi = 0. \tag{31}$$

From the commutation relation

$$[\hat{j}_t(n_0(\phi)), \hat{j}_t(n_0(\psi))] = 0,$$

we deduce that

$$|\alpha_3|^2\left(\langle T_3\phi, T_3\psi\rangle - \langle T_3\psi, T_3\phi\rangle\right) = 0\,, \quad \phi,\, \psi \in \mathcal{K}_S. \tag{32}$$

Observe that the conditions ($T_1 = 0$, $T_2 \neq 0$), ($T_1 \neq 0$, $T_2 = 0$) and ($T_1 = T_2 = 0$) lead to the same conclusion. In fact, in such case we have

$$X_t^{(1)}(T_1\phi) = X_t^{(2)}(T_2\phi) = 0\,, \quad \forall\, \phi \in \mathcal{K}_S.$$

It follow that $R_t = 0$ and T_3 verify only the conditions (24), (25) and (32). Then we must discuss separately the case $T_1 = 0 = T_2$ and the case $T_1 \neq 0 \neq T_2$.
Step 1. $T_1 = 0 = T_2$.
From (26) we have $R_t = 0$. Since we want $\hat{j}_t$ to be a non trivial map then we must assume that $(\alpha_3, \beta_3, \gamma_3) \neq (0, 0, 0)$ and $T_3 \neq 0$.
We distinguish two cases:

Case 1. $\gamma_3 = 0$ and $T := \alpha_3 T_3 \neq 0$.
Taking ϕ real in (24), we obtain $\alpha_3 = \overline{\beta}_3$ and $T\overline{\phi} = T\phi$ for all $\phi \in \mathcal{K}_S$. Moreover the condition (32) implies that

$$\langle T\phi, T\psi\rangle = \langle T\psi, T\phi\rangle\ , \quad \phi,\, \psi \in \mathcal{K}_S$$

i.e., the restriction of the scalar product of $\mathcal{K}_S$ on the range of T is real valued. In this case $\hat{j}_t = \hat{j}_t^{(1)}$ belongs to the **Class I.**

Case 2. $\gamma_3 \neq 0$ and $T := \gamma_3 T_3 \neq 0$.
In this case, the equation (25) implies that

$$\overline{T\phi} = T\overline{\phi}\,, \quad \forall\, \phi \in \mathcal{K}_S.$$

Taking ϕ real valued function in the equation (25), we deduce that $\beta_3 = \bar{\alpha}_3$. Denoting $\delta = \alpha_3/\gamma_3$, the equation (24) implies that

$$\delta T\bar{\phi} = \delta T\phi\,, \quad \forall\, \phi \in \mathcal{K}_S.$$

This gives two cases :

Case 2.1 $\alpha_3 = 0$.

In this case $\delta = 0$ and then $\hat{j}_t = \hat{j}_t^{(2)}$ belongs to the **Class II.**

Case 2.2 $\alpha_3 \neq 0$.

In this case $\delta \neq 0$ and then $T\bar{\phi} = T\phi$, for all $\phi \in \mathcal{K}_S$. In this case $\hat{j}_t = \hat{j}_t^{(3)}$ belongs to the **Class III.**

Step 2. $T_1 \neq 0 \neq T_2$.
Eqs. (29) and (31) imply that $\gamma_1 = 0$ and $\gamma_2 = 0$, from which (26) becomes

$$\begin{aligned} \langle \psi, \phi \rangle R_t &= (\beta_2 \alpha_1 \langle T_2\psi, T_1\phi \rangle - \alpha_2 \beta_1 \langle T_1\phi, T_2\psi \rangle)\, t1 \\ &= \left(|\alpha_1|^2 \langle T_1\psi, T_1\phi \rangle - |\beta_1|^2 \langle T_1\psi, T_1\phi \rangle \right) t1. \end{aligned} \tag{33}$$

where, in the last equalities, (23) has been taken into account.

Fact: $\alpha_2 \neq 0$ is equivalent to $\alpha_1 = 0$.
Indeed, if $\alpha_2 \neq 0$ then second Eq. (23) implies that $\beta_1 \neq 0$ and $T_2 = (\overline{\beta_1}/\alpha_2)T_1 = c_2 T$ with $c_2 := \overline{\beta_1}/\alpha_2 \neq 0$ and $T := T_1 \neq 0$. Moreover (30) becomes

$$T_2\overline{\phi}\psi = -\gamma_3 T_2\psi T_3\phi \quad \text{or equivalently} \quad T\overline{\phi}\psi = -\gamma_3 T\psi T_3\phi \tag{34}$$

and this implies $\gamma_3 \neq 0$ and $T_3 \neq 0$.
If $\alpha_1 \neq 0$ (or equivalently, because of (23), $\beta_2 \neq 0$), then (34) becomes

$$T\overline{\phi}\psi = -\gamma_3 T\psi T_3\phi\,, \quad \forall\, \phi,\, \psi \in \mathcal{K}_S, \tag{35}$$

and (27) implies

$$T\phi\psi = \gamma_3 T\psi T_3\phi \tag{36}$$

Then, from (36) we have

$$\frac{T_3\phi}{T\phi} = \frac{T_3\psi}{T\psi}$$

which gives

$$T_3\phi = aT\phi\,, \quad \forall\, \phi \in \mathcal{K}_S \tag{37}$$

for some $a \in \mathbb{C}\backslash\{0\}$. Combining Eqs. (35), (36) and (37), we deduce that $T\phi = 0$ for each real valued ϕ. Thus T can be non zero only on purely imaginary valued functions. Since the product of two such functions is real valued and since $\gamma_3 \neq 0$, (36) implies that, for all real valued function $\phi \in \mathcal{K}$,

$$\gamma_3 (Ti\phi)^2 = -T\phi^2 = 0.$$

Hence $T\phi$ is also null for all purely complex valued function $\phi \in \mathcal{K}$. This leads to $T = 0$ which is in contradiction with our assumption. We conclude that the condition $\alpha_2 \neq 0$ implies the condition $\alpha_1 = 0$. The converse statement can be obtained by slight modification.
In view of the above fact, we distinguish two cases:

Case 3. $\alpha_2 \neq 0$ and $\alpha_1 = 0$, so that $\beta_1 \neq 0$ and $\beta_2 = 0$.

Eqs. (28) and (25) give, respectively,

$$T\phi\psi = -\overline{\gamma_3}\overline{T_3\phi}T\psi \quad \text{and} \quad \overline{T_3\phi} = \frac{\gamma_3}{\overline{\gamma_3}}T_3\overline{\phi}.$$

It follows that $T_3\phi = c_3\overline{T\phi}$ and therefore $T\phi\psi = -\overline{\lambda_0}T\phi T\psi$, $\overline{\lambda_0}T\overline{\phi} = \lambda_0\overline{T\phi}$ with $\lambda_0 = c_3\gamma_3$, or equivalently

$$T\phi\psi = -\overline{\lambda_0}T\phi T\psi\,, \quad T\overline{\phi} = \frac{\lambda_0}{\overline{\lambda_0}}\overline{T\phi}. \tag{38}$$

Hence, (33) becomes $R_t = -ct|\beta_1|^2$ with $c := \frac{\langle T\phi_0, T\psi_0\rangle}{\langle\psi_0,\phi_0\rangle}$ being a positif real number independent of ϕ_0, $\psi_0 \in \mathcal{K}_S$. On the other hand, combining Eqs. (24) and (38), we deduce

$$\alpha_3 c_3 \left\langle T\psi, \overline{T\phi}\right\rangle = 0$$

and then

$$\alpha_3 c_3 \frac{\overline{\lambda_0}}{\lambda_0}\left\langle T\psi, T\overline{\phi}\right\rangle = 0.$$

Taking $\overline{\phi}$ instead of ϕ in the above equation, we conclude that $\alpha_3 = 0$, and therefore from (24) we have $\beta_3 = 0$.

In conclusion, setting $\lambda := \bar{\beta}_1$, the map $\hat{j}_t$ takes the form:

$$\begin{aligned}
\hat{j}_t^{(4)}(a_0^+(\phi)) &= a_0^+(\phi) + \overline{\lambda}A_t(T\phi),\\
\hat{j}_t^{(4)}(a_0(\phi)) &= a_0(\phi) + \lambda A_t^+(T\phi),\\
\hat{j}_t^{(4)}(n_0(\phi)) &= n_0(\phi) + \lambda_0 N_t(\overline{T\phi}),\\
\hat{j}_t^{(4)}(1_0) &= 1_0 - c|\lambda|^2 t\,1,
\end{aligned}$$

with relations

$$T\overline{\phi} = \frac{\lambda_0}{\overline{\lambda_0}}\,\overline{T\phi}\,, \quad \langle T\phi, T\psi\rangle = c\langle\psi,\phi\rangle\,, \quad T\phi\psi = -\overline{\lambda}_0 T\phi T\psi.$$

Case 4. $\alpha_2 = 0$ and $\alpha_1 \neq 0$, so that $\beta_1 = 0$ and $\beta_2 \neq 0$.

A similar calculus as in the **Case 3.**, we obtain

$$\begin{aligned}
\hat{j}_t^{(3)}(a_0^+(\phi)) &= a_0^+(\phi) + \lambda A_t^+(T\phi),\\
\hat{j}_t^{(3)}(a_0(\phi)) &= a_0(\phi) + \overline{\lambda}A_t(T\phi),\\
\hat{j}_t^{(3)}(n_0(\phi)) &= n_0(\phi) + \lambda_0 N_t(T\phi)\\
\hat{j}_t^{(3)}(1_0) &= 1_0 + c|\lambda|^2 t\,1,
\end{aligned}$$

with relations

$$T\overline{\phi} = \frac{\overline{\lambda}_0}{\lambda_0}\,\overline{T\phi}\,, \quad \langle T\phi, T\psi\rangle = c\langle\phi,\psi\rangle\,, \quad T\phi\psi = \lambda_0 T\phi T\psi.$$

■

Theorem 3 *Theorem Let $\hat{j}_t$ be a one of the classes defined in Theorem 2. Define the map j_t° on the group operators by*

$$j_t^\circ(W(\psi,\varphi) = j_t^\circ(e^{i(a_0^+(\varphi)+a_0(\varphi)+n_0(\psi))}) := e^{i\hat{j}_t(a_0^+(\varphi)+a_0(\varphi)+n_0(\psi))}.$$

Then j_t° can be extended to a $*$-homomorphism still denoted j_t° of the W^*-algebra generated by the group operators of the oscillator algebra if and only if $\hat{j}_t$ is one of the first three classes in Theorem 2; i.e., only the maps $\hat{j}_t^{(1)}$, $\hat{j}_t^{(2)}$ and $\hat{j}_t^{(3)}$ are effectively a Lie algebra time shifts.

Proof: The $*$-map property is clear. For simplicity, we denote

$$W_j = W_0(\phi_j,\psi_j)\,, \quad j = 1,\,2$$

Because of Proposition 3, it is sufficient to verify the identity :

$$j_t^\circ(W_1W_2) = j_t^\circ(W_1)j_t^\circ(W_2) \tag{39}$$

for the five classes in the Theorem 2. By Theorem 1 we have

$$W_1W_2 = e^{i\gamma}W_0(\phi,\psi) = e^{i\gamma}e^{i(a_0^+(\phi)+a_0(\phi)+n_0(\psi))}.$$

Class I. By direct computation we have

$$\begin{aligned} j_t^\circ(W_1W_2) = j_t^\circ(e^{i\gamma}W_0(\phi,\psi)) &= e^{i\gamma}W_0(\phi,\psi)\otimes e^{i(A_t^+(T\psi)+A_t(T\psi))} \\ &= W_1W_2\otimes e^{i(A_t^+(T\psi)+A_t(T\psi))} \end{aligned}$$

On the other hand, we have

$$j_t^\circ(W_1)j_t^\circ(W_2) = W_1W_2\otimes e^{i(A_t^+(T\psi_1)+A_t(T\psi_1))}e^{i(A_t^+(T\psi_2)+A_t(T\psi_2))}$$

But the processes $A_t^+(T\psi_1)+A_t(T\psi_1)$ and $A_t^+(T\psi_2)+A_t(T\psi_2)$ commute. Then we obtain

$$\begin{aligned} j_t^\circ(W_1)j_t(W_2) &= W_1W_2\otimes e^{i(A_t^+(T\psi_1)+A_t(T\psi_1)+A_t^+(T\psi_2)+A_t(T\psi_2))} \\ &= W_1W_2\otimes e^{i(A_t^+(T\psi)+A_t(T\psi))} = j_t^\circ(W_1W_2) \end{aligned}$$

In conclusion, equality (39) is verified by **Class I.**
Class II. and **Class III.** In this cases we have

$$j_t^\circ(W_1W_2) \;=\; e^{i\gamma}j_t^\circ(W(\phi,\psi)) = e^{i\gamma}W(\phi,\psi)\otimes e^{i(A_t^+(\delta T\psi)+A_t(\delta T\psi)+N_t(T\psi))}$$

Where **Class II.** corresponds to $\delta = 0$. On the other hand we have

$$\begin{aligned} j_t^\circ(W_1)j_t^\circ(W_2) &= (W_1\otimes e^{i(A_t^+(\delta T\psi_1)+A_t(\delta T\psi_1)+N_t(T\psi_1))}) \\ &\times\; (W_1\otimes e^{i(A_t^+(\delta T\psi_2)+A_t(\delta T\psi_2)+N_t(T\psi_2))}) \\ &= W_1W_2\otimes e^{i(A_t^+(\delta T\psi_1)+A_t(\delta T\psi_1)+N_t(T\psi_1))} \\ &\times\; e^{i(A_t^+(\delta T\psi_2)+A_t(\delta T\psi_2)+N_t(T\psi_2))}. \end{aligned}$$

But it is not difficult to verify that the processes $A_t^+(\delta T\psi_1) + A_t(\delta T\psi_1) + N_t(T\psi_1)$ and $A_t^+(\delta T\psi_2) + A_t(\delta T\psi_2) + N_t(T\psi_2)$ commute in the two cases. Thus we obtain

$$\begin{aligned} j_t^\circ(W_1)j_t^\circ(W_2) &= W_1W_2 \otimes e^{i(A_t^+(\delta T(\psi_1+\psi_2))+A_t(\delta T(\psi_1+\psi_2))+N_t(T(\psi_1+\psi_2)))} \\ &= W_1W_2 \otimes e^{i(A_t^+(\delta T\psi)+A_t(\delta T\psi)+N_t(T\psi))} \\ &= j_t^\circ(W_1W_2). \end{aligned}$$

In conclusion, both **Class II.** and **Class III.** verify the equality (39).

Class IV. For this class, we begin by proving that the map T of Theorem 2 is a linear bounded operator of $\mathcal{K}_S$. In fact the condition $T(\phi\psi) = \lambda_0 T\phi T\psi$ for all ψ, $\phi \in \mathcal{K}_S$, gives $T(i\phi)T\psi = T(i\psi)T\phi$ for all ψ, $\phi \in \mathcal{K}_S$. Then there is a complex number α such that $T(i\phi) = \alpha T\phi$ for all $\phi \in \mathcal{K}_S$. But we have

$$\langle T\phi, T\psi\rangle = c\,\langle \phi, \psi\rangle\ , \quad \forall\, \psi,\ \phi \in \mathcal{K}_S$$

which implies

$$\langle T\phi, T(i\psi)\rangle = ci\,\langle \phi, \psi\rangle\ , \quad \forall\, \psi,\ \phi \in \mathcal{K}_S$$

On the other hand, we have

$$\langle T\phi, T(i\psi)\rangle = c\,\langle T\phi, \alpha T\psi\rangle = \alpha\,\langle \phi, \psi\rangle\ , \quad \forall\, \psi,\ \phi \in \mathcal{K}_S$$

this gives $\alpha = i$, and then

$$T((a+ib)\phi) = T(a\phi) + T(ib\phi) = aT\phi + ibT\phi = (a+ib)T\phi\,, \quad \forall\, \phi \in \mathcal{K}_S\ ,\ a, b \in \mathbb{R}$$

This proves the linearity of T. The equation

$$\langle T\phi, T\psi\rangle = c\,\langle \phi, \psi\rangle\ , \quad \forall\, \psi,\ \phi \in \mathcal{K}_S$$

proves that T is bounded.

In the next step we calculate $j_t^\circ(W_1W_2)$. In the present class we have

$$\begin{aligned} j_t^\circ(W_1W_2) &= e^{i\gamma} j_t^\circ(W_0(\phi,\psi)) = e^{i\gamma} W_0(\phi,\psi) \otimes e^{i(A_t^+(\lambda T\phi)+A_t(\lambda T\phi)+N_t(\lambda_0 T\psi))} \\ &= W_1W_2 \otimes W_0(\phi_t, \psi_t) \end{aligned} \tag{40}$$

with

$$\phi_t = \chi_{[0,t]\otimes}\lambda T\phi \quad \text{ and } \quad \psi_t = \chi_{[0,t]\otimes}\lambda_0 T\psi. \tag{41}$$

On the other hand we have

$$j_t^\circ(W_j) = W_j \otimes W_0(\phi_t^{(j)}, \psi_t^{(j)})\,, \quad j = 1,\ 2$$

where $\phi_t^{(j)}$ and $\psi_t^{(j)}$ are given as in (41). This gives

$$j_t^\circ(W_1)j_t^\circ(W_2) = W_1W_2 \otimes W_0(\phi_t^{(1)}, \psi_t^{(1)})W_0(\phi_t^{(2)}, \psi_t^{(2)}).$$

Using Lemma 1, we obtain

$$j_t^\circ(W_1)j_t^\circ(W_2) = W_1W_2 \otimes e^{i\gamma_t}W_0(\phi_t', \psi_t') \tag{42}$$

with ϕ_t' and ψ_t' given respectively as in Eqs.(15) and (14). Note that

$$\psi_t' = \psi_t^{(1)} + \psi_t^{(2)} = \chi_{[0,t]} \otimes \lambda_0 T\psi_1 + \chi_{[0,t]} \otimes \lambda_0 T\psi_2 = \chi_{[0,t]} \otimes \lambda_0 T\psi = \psi_t$$

and ϕ_t' is given by

$$e_1(i\psi_t')i\phi_t' = e_1(i\psi_t)i\phi_t' = e_1(i\psi_t^{(1)})i\phi_t^{(1)} + e^{i\psi_t^{(1)}}e_1(i\psi_t^{(2)})i\phi_t^{(2)}. \tag{43}$$

Taking the properties of the operator T into account, one can deduce the following:

$$\begin{aligned}
&e_1(i\psi_t^{(1)})i\phi_t^{(1)} + e^{i\psi_t^{(1)}}e_1(i\psi_t^{(2)})i\phi_t^{(2)} \\
&\quad= \chi_{[0,t]} \otimes \big[e_1(i\lambda_0 T\psi_1)i\lambda T\phi_1 + e^{\lambda_0 Ti\psi_1}e_1(\lambda_0 Ti\psi_2)\lambda Ti\phi_2\big] \\
&\quad= \chi_{[0,t]} \otimes \lambda\Big[\sum_{n\geq 1}\frac{1}{n!}(\lambda_0 Ti\psi_1)^{n-1}Ti\phi_1 + e^{\lambda_0 Ti\psi_1}\sum_{n\geq 1}\frac{1}{n!}(\lambda_0 Ti\psi_2)^{n-1}Ti\phi_2\Big] \\
&\quad= \chi_{[0,t]} \otimes \lambda\Big[\sum_{n\geq 1}\frac{1}{n!}T((i\psi_1)^{n-1}i\phi_1) + e^{\lambda_0 Ti\psi_1}\sum_{n\geq 1}\frac{1}{n!}T(i\psi_2)^{n-1}i\phi_2)\Big] \\
&\quad= \chi_{[0,t]} \otimes \lambda\Big[T(e_1(i\psi_1)i\phi_1) + e^{\lambda_0 Ti\psi_1}T(e_1(i\psi_2)i\phi_2)\Big] \\
&\quad= \chi_{[0,t]} \otimes \lambda\Big[T(e_1(i\psi_1)i\phi_1) + \sum_{n\geq 0}\frac{1}{n!}(\lambda_0 Ti\psi_1)^n T(e_1(i\psi_2)i\phi_2)\Big] \\
&\quad= \chi_{[0,t]} \otimes \lambda\Big[T(e_1(i\psi_1)i\phi_1) + T(e^{i\psi_1}e_1(i\psi_2)i\phi_2)\Big] = \chi_{[0,t]} \otimes \lambda T(e_1(i\psi)i\phi) \\
&\quad= \chi_{[0,t]} \otimes \lambda\frac{1}{n!}T((i\psi)^{n-1}i\phi) = \chi_{[0,t]} \otimes \frac{1}{n!}(i\lambda_0 T\psi)^{n-1}i\lambda T\phi \\
&\quad= \chi_{[0,t]} \otimes \frac{1}{n!}(i\psi_t)^{n-1}i\phi_t = e_1(i\psi_t)i\phi_t.
\end{aligned}$$

Combining the last result and (43), one can deduce $\phi_t' = \phi_t$ and from (42), we deduce

$$j_t^\circ(W_1)j_t^\circ(W_2) = e^{i\gamma_t}W_0(\phi_t, \psi_t). \tag{44}$$

Comparing (44) and (40) we deduce that condition (39) is satisfied if and only if the real $\gamma_t = 0$ for all $t \geq 0$. Notice that γ_t can be rewritten as

$$\begin{aligned}
\langle\phi_t, e_2(i\psi_t)\phi_t\rangle - i\gamma_t &= \Big\langle\phi_t^{(1)}, e_2(i\psi_t^{(1)})\phi_t^{(1)}\Big\rangle + \Big\langle\phi_t^{(2)}, e_2(i\psi_t^{(2)})\phi_t^{(2)}\Big\rangle + \\
&\quad+ \Big\langle e_1(-i\psi_t^{(1)})\phi_t^{(1)}, e_1(i\psi_t^{(2)})\phi_t^{(2)}\Big\rangle
\end{aligned}$$

Similarly we obtain

$$i\gamma_t - ct|\lambda|^2 \langle \phi, e_2(i\psi)\phi\rangle$$
$$= ct|\lambda|^2 [\gamma - \langle \phi_1, e_2(i\psi_1)\phi_1\rangle - \langle \phi_2, e_2(i\psi_2)\phi_2\rangle - \langle e_1(-i\psi_1)\phi_1, e_1(i\psi_2)\phi_2\rangle]$$

or equivalently

$$i\gamma_t - ct|\lambda|^2 \langle \phi, e_2(i\psi)\phi\rangle = ct|\lambda|^2 (\gamma - \langle \phi, e_2(i\psi)\phi\rangle) .$$

This implies $\gamma_t = ct|\lambda|^2\gamma$ which is null if and only if $c|\lambda|^2 = 0$. This corresponds to the trivial case. In conclusion, the **Class IV.** does'nt verify the identity (39).

Class V. The statement can be verified in a similar way as for **Class IV.** ■

Remark 2 *In all cases the Lie algebra time shift on the oscillator algebra is of the form*

$$\hat{j}_t(x) = x \otimes 1 + 1_0 \otimes X_{[0,t]}(x) = x + X_{[0,t]}(x)$$

where $X_{[0,t]}(x)$ is an independent increment process. The above Theorem shows that the Lie algebra $*$-homomorphism is a Lie algebra time shift if and only if the constant $c|\lambda|^2 = 0$, in which case $X_{[0,t]}(x)$ is a classical process. Therefore the Lie algebra time shifts are in fact shifts along classical processes.

3.3 The associated semigroup

In this subsection $\hat{j}_t$ is the stochastic process described in Theorem 3, Classes **I.**, **II.** and **III.** Define

$$j_t : \mathcal{A}_S \otimes 1_{[0,t]} \otimes \mathcal{B}_{[t} \to \mathcal{A} , \quad t \geq 0$$

as the unique $*-$homomorphism characterized by

$$j_t(x \otimes 1) = j_t^\circ(x) , \quad j_t(1 \otimes a_{[t}) = j_t^\circ(1) \otimes a_{[t} , \quad x \in \mathcal{A}_S , \ a_{[t} \in \mathcal{B}_{[t}. \qquad (45)$$

Each j_t can be extended in an obvious way to the algebraic linear span of elements $x \otimes Z_{[t}$ where $x \in \mathcal{A}_S$, and $Z_{[t}$ is an operator on $\mathcal{H}_{[s}$, by the same action described as in (45).

Lemma 4 *Lemma j_t° is a Markovian cocycle.*

Proof: We will check the property:

$$j_{t+s}^\circ(e^x) = j_t(u_t^\circ(j_s^\circ(e^x)))$$

We have

$$
\begin{aligned}
j_t(u_t^\circ(j_s^\circ(e^x))) &= j_t(u_t^\circ(e^x \otimes e^{X_{[0,s]}})) \\
&= j_t(e^x \otimes \Gamma(\theta_t^*) e^{X_{[0,s]}} \Gamma(\theta_t)) = j_t(e^x \otimes e^{X_{[t,t+s]}}) \\
&= j_t((e^x \otimes 1)\,(1 \otimes e^{X_{[t,t+s]}})) = j_t(e^x \otimes 1)\, j_t(1 \otimes e^{X_{[t,t+s]}}) \\
&= j_t^\circ(e^x)\,(1 \otimes e^{X_{[t,t+s]}}) = (e^x \otimes e^{X_{[0,t]}})\,(1 \otimes e^{X_{[t,t+s]}}) \\
&= e^x \otimes e^{X_{[0,t]}} \otimes e^{X_{[t,t+s]}}) = e^x \otimes e^{X_{[0,t]}+X_{[t,t+s]}} \\
&= e^x \otimes e^{X_{[0,t+s]}} = j_{t+s}^\circ(e^x).
\end{aligned}
$$

By the quantum Feynman-Kac formula [1], the Markovian cocycle (j_t°) defines a Markovian semigroup on $\mathcal{B}(\mathcal{H}_S)$ given by $P^t = E_{0]} \circ j_t^\circ$. ∎

Theorem 4 *Theorem Let $P^t = e^{tL}$ be the Markovian semigroup, with generator L, defined via Feynman-Kac formula as above. Then the generalized Weyl operators are eigenvectors of L. More precisely, denote $L^{(I)}$, $L^{(II)}$ and $L^{(III)}$, the generator of P^t associated to the classes **I.**, **II.** and **III.**, respectively. For any test functions $\phi \in L^2(\mathbb{R})$, $\psi \in \mathcal{K}_S$, one has*

$$L^{(I)}(W(\phi,\psi)) = \frac{1}{2}|T\psi|^2 W(\phi,\psi)$$

$$L^{(II)}(W(\phi,\psi)) = 0$$

$$L^{(III)}(W(\phi,\psi)) = |\delta|^2 \varepsilon (e^{T\psi} - T\psi - 1) W(\phi,\psi)$$

Proof: For $x = x(\phi,\psi) = a_0^+(\phi) + a_0(\phi) + n_0(\psi) \in \mathcal{L}_{osc}$, we have

$$P^t(e^x) = E_{0]}(e^x \otimes e^{X_{[0,t]}}) = \left\langle \Phi, e^{X_{[0,t]}} \Phi \right\rangle e^x$$

Then if we write $e^x = W(\phi,\psi)$, we obtain

$$P^t(W(\phi,\psi)) = \left\langle \Phi, e^{X_{[0,t]}} \Phi \right\rangle W(\phi,\psi)$$

Classe I. In this class we have $X_{[0,t]} = A_t^+(T\psi) + A_t(T\psi)$. Therefore, we get

$$\left\langle \Phi, e^{X_{[0,t]}} \Phi \right\rangle = e^{\frac{1}{2}t|T\psi|^2}$$

Classe II. Here we have $X_{[0,t]} = N_t(T\psi)$. Then

$$\left\langle \Phi, e^{X_{[0,t]}} \Phi \right\rangle = 1$$

from which the desired statement follows.

Classe III. In this class we read

$$X_{[0,t]} = \delta A_t^+(T\psi) + N_t(T\psi) + \overline{\delta} A_t(T\psi)$$

Then

$$\left\langle \Phi, e^{X_{[0,t]}}\Phi \right\rangle = e^{t|\delta|^2 \left\langle \overline{T\psi}, e_2(T\psi)T\psi \right\rangle} =$$

$$= t|\delta|^2 \int (T\psi)^2(x) e_2(T\psi)(x)dx = t|\delta|^2 \int (e^{T\psi(x)} - T\psi(x) - 1)dx =$$

$$= t|\delta|^2 \varepsilon(e^{T\psi} - T\psi - 1)$$

This completes the proof of the theorem. ∎

Remark 3 *In view of the above Theorem, in the first class the generator $L^{(I)}$ is nothing but the quantum Laplacian, denoted Δ^Q, obtained in Ref. [9], while the third class corresponds to Poisson processes.*

4 Lie algebra time shift and IIP

In this section we identify the classical stochastic processes corresponding to the Lie algebra time shifts studied in the previous sections.
One of the basic tenets of quantum probability is the fact that, if $(A_t)_{t\in I}$ is a self-adjoint family of commuting operators acting on a some Fock space $\Gamma(H)$ with Vacuum Φ, then, under additional analytical conditions which are automatically satisfied in the cases we are considering, there exists a classical stochastic process $(\tilde{Y}_t)_{t\in I}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that, for all bounded complex valued Borel functions $\varphi_1, \varphi_2, \cdots, \varphi_n$, $n \in \mathbb{N}$, one has

$$\mathbb{E}\left(\varphi_1(\tilde{Y}_{t_1})\,\varphi_2(\tilde{Y}_{t_2})\,\cdots\,\varphi_n(\tilde{Y}_{t_n})\right) = \langle \Phi, \varphi_1(A_{t_1})\varphi_2(A_{t_2})\cdots\varphi_n(A_{t_n})\Phi\rangle$$

In particular, the characteristic function of $(\tilde{Y}_t)_{t\in I}$ is given by

$$\mathbb{E}(e^{iz\tilde{Y}_t}) = \left\langle \Phi, e^{izA_t}\Phi \right\rangle$$

We will apply the above general statement to the case in which

$$A_t = \hat{j}_t(x) = x \otimes 1 + 1_0 \otimes X_t(x) \qquad ; \qquad t \geq 0$$

where x belongs to a self-adjoint commutative sub-algebra $\mathcal{L}_a^{sa}$ of the oscillator algebra $\mathcal{L}_0$ acting on the Fock space $\Gamma(\mathcal{K}_S)$ and the process $(X_t(x))_{t\geq 0}$ acts on the noise space $\Gamma(L^2(\mathbb{R}_+, \mathcal{H}))$. Then $\hat{j}_t(\mathcal{L}_a^{sa})$ is a commutative self-adjoint family of operators acting on the Fock space

$$\Gamma(\mathcal{K}_S) \otimes \Gamma(L^2(\mathbb{R}^+, \mathcal{H})) = \Gamma(\mathcal{K}_S \oplus L^2(\mathbb{R}^+, \mathcal{H}))$$

Then the process $(\hat{j}_t(x) = x + X_t(x))_{t\geq 0}$, with respect to the vacuum vector, can be identified to an independent increment operator process with initial operator $x = x \otimes 1$ and its characteristic function is given by

$$\mathbb{E}(e^{iz\hat{j}_t(x)}) = \mathbb{E}(e^{izx + izX_t(x)}) = \mathbb{E}(e^{izx})\mathbb{E}(e^{izX_t(x)}) =: e^{iz\Psi_t(x)}$$

where, in obvious notations, the cumulant function $\Psi_t(z)$ is given by

$$\Psi_t(z) = \psi_0(z) + \psi_t(z)$$

We begin by finding the self-adjoint commutative sub-algebras $\mathcal{L}_a^{sa}$ of the oscillator algebra. To this goal it is convenient to introduce the notion of complex Hilbert space with real structure.

Let $\mathcal{H}$ be a complex Hilbert space. A real structure of $\mathcal{H}$ is determined by a real Hilbert subspace $\mathcal{H}_r$ of $\mathcal{H}$, with real valued scalar product, characterized by the condition

$$\langle f, ig \rangle_{\mathcal{H}} = i \langle f, g \rangle_{\mathcal{H}_r} , \quad f,\, g \in \mathcal{H}_r$$

and an identification

$$\mathcal{H} = \mathcal{H}_r \oplus i\mathcal{H}_r$$

where the sum is direct, but not orthogonal.

In the following lemma, we take $\mathcal{H} = L^2(\mathbb{R})$.

Lemma 5 *Let $\mathcal{L}_{osc}$ be the oscillator algebra as in definition 9.*
Then the self-adjoint commutative sub-algebras of $\mathcal{L}_{osc}$ have the form:

$$\mathcal{L}_\beta := \{a_0^+(\phi) + a_0(\phi) + n_0(\beta\phi) + \alpha 1_0 , \quad \phi \in \mathcal{H} ,\ \Im(\phi) = 0 ,\ \alpha \in \mathbb{R}\}$$

for some non zero real valued function β.

Proof: Let be given

$$x(\phi, \varphi, \psi, \alpha) = a_0^+(\phi) + a_0(\varphi) + n_0(\psi) + \alpha 1_0 \in \mathcal{L}_{osc}.$$

Then $x(\phi, \varphi, \psi, \alpha)$ is self-adjoint if and only if

$$a_0^+(\phi) + a_0(\varphi) + n_0(\psi) + \alpha 1_0 = a_0(\phi) + a_0^+(\varphi) + n_0(\overline{\psi}) + \overline{\alpha} 1_0.$$

This gives $\varphi = \phi$, $\Im(\psi) = 0$ and $\alpha \in \mathbb{R}$. Denote

$$x(\phi, \psi, \alpha) = a_0^+(\phi) + a_0(\phi) + n_0(\psi) + \alpha 1_0.$$

We want to find the spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ of test functions ϕ and ψ, respectively, such that

$$[x(\phi, \psi, \alpha), x(\phi', \psi', \alpha')] = 0 , \quad \forall\, \phi,\, \phi' \in \mathcal{H}_\phi ,\ \forall\, \psi,\, \psi' \in \mathcal{H}_\psi.$$

This last identity is equivalent to

$$2i\Im(\langle \phi, \phi' \rangle) 1_0 + a_0^+(\psi\phi' - \phi'\psi) + a_0(\phi\psi' - \psi'\phi) = 0,$$

or equivalently

$$\Im(\langle \phi, \phi' \rangle) = 0 \quad \text{and} \quad \frac{\psi}{\phi} = \frac{\psi'}{\phi'} , \quad \forall\, \phi,\, \phi' \in \mathcal{H}_\phi ,\ \forall\, \psi,\, \psi' \in \mathcal{H}_\psi.$$

Hence $\mathcal{H}_\phi = \mathcal{H}_r$ and $\psi = \beta\phi$ for some $\beta \in \mathcal{H}$ with ψ being a real valued function. But the function β is the same for all functions ϕ, so β must be a real valued function, and then $\mathcal{H}_\psi = \beta\mathcal{H}_\phi$.
Remark. We distinguish two cases:
(I) $\beta \neq 0$. In this case the functions ϕ must be real valued and

$$x(\phi,\psi,\alpha) = a_0^+(\phi) + a_0(\phi) + n_0(\beta\phi) + \alpha 1_0 \,, \quad \phi \in \mathcal{H}_r \,,\ \Im(\phi) = 0 \,,\ \alpha \in \mathbb{R}$$

(II) $\beta = 0$. In this case

$$x(\phi,\psi,\alpha) = a_0^+(\phi) + a_0(\phi) + \alpha 1_0 \,, \quad \phi \in \mathcal{H}_r \,,\ \alpha \in \mathbb{R}$$

Proposition 1 *Let $\hat{j}_t$ be the stochastic process as class **I.** Then, for all*

$$x = x_\beta(\phi,\alpha) := a_0^+(\phi) + a_0(\phi) + n_0(\beta\phi) + \alpha 1_0,$$

the stochastic process $\hat{j}_t(x)$ is gaussian with mean zero and variance $|T(\beta\phi)|^2 t$ (i.e. a classical BM).

Proof: Let $\hat{j}_t(x) = x + X_t(x) \,,\ x \in \mathcal{L}_\beta$. Taking $x = x_\beta(\phi,\alpha) = a_0^+(\phi) + a_0(\phi) + n_0(\beta\phi) + \alpha 1_0$, we have

$$izX_t(x) = A_t^+(izT\beta\phi) + A_t(-izT\beta\phi) \,, \quad \forall\, z \in \mathbb{R}$$

This gives

$$\begin{aligned} e^{izX_t(x)} &= e^{A_t^+(izT(\beta\phi))} e^{A_t(-izT(\beta\phi))} e^{-\frac{1}{2}[A_t^+(izT(\beta\phi)),A_t(-izT(\beta\phi))]} \\ &= e^{A_t^+(izT(\beta\phi))} e^{A_t(-izT(\beta\phi))} e^{-\frac{1}{2}z^2 t|T(\beta\phi)|^2}. \end{aligned}$$

Therefore

$$\mathbb{E}(e^{izX_t}) = \left\langle \Phi, e^{izX_t(x)}\Phi \right\rangle = e^{-\frac{1}{2}z^2 t|T(\beta\phi)|^2}.$$

Hence the characteristic exponent is given by $\psi(z) = -\frac{1}{2}z^2|T(\beta\phi)|^2$. ■

Proposition 2 *Proposition Let $\hat{j}_t(x_\beta(\phi))$ be the stochastic process as in Theorem 3, classes **II.** and **III.** Then the characteristic exponent of $\hat{j}_t$ is given by*

$$\psi(z) = |\delta|^2 \int_{\mathbb{R}} (e^{iz\varphi(x)} - iz\varphi(x) - 1)dx$$

with

$$\varphi := T(\beta\phi)$$

Proof: In the conditions and notations of Theorem 2 we have

$$izX_t(x) = A_t^+(iz\delta T(\beta\phi)) + A_t(-iz\delta T(\beta\phi)) + N_t(izT(\beta\phi))$$

Therefore

$$e^{izX_t(x)} = e^{A_t^+(\phi_1)} e^{N_t(\psi)} e^{A_t(\phi_2)} e^{\alpha_t}$$

with

$$\alpha_t = \langle -iz\delta T(\beta\phi), e_2(izT(\beta\phi))iz\delta T(\beta\phi)\rangle\, t = |\delta|^2 t\varepsilon \left(e^{izT\beta\phi} - izT\beta\phi - 1\right)$$

where, for any integrable function f, we denote

$$\varepsilon(f) = \int_{\mathbb{R}} f(x)dx$$

Then we get

$$\mathbb{E}(e^{izX_t}) = \left\langle \Phi, e^{izX_t(x)}\Phi \right\rangle = e^{|\delta|^2 t\varepsilon(e^{izT(\beta\phi)} - izT(\beta\phi) - 1)}$$

Hence the characteristic exponent is given by

$$\psi(z) = |\delta|^2 \varepsilon(e^{izT(\beta\phi)} - izT(\beta\phi) - 1)$$

■

Remark 4 *We define the measure* ν *by*

$$\varepsilon(e^{iz\varphi} - iz\varphi - 1) = \int_{\mathbb{R}} (e^{iz\varphi(x)} - iz\varphi(x) - 1)dx =$$

$$= \int_{\mathbb{R}} (e^{izu} - izu - 1)d\varphi^{-1}(u) = \int_{\mathbb{R}} (e^{izu} - izu - 1)\nu(du)$$

i.e. ν *is the* φ*–image of the Lebesgue measure* λ *given by*

$$\nu(A) := \lambda(\varphi^{-1}(A))$$

Replacing the measure ν *by* $\nu/|\delta|^2$ *we can assume that*

$$\psi(z) = \int_{\mathbb{R}} (e^{iz\varphi(x)} - iz\varphi(x) - 1)dx = \int_{\mathbb{R}} (e^{izu} - izu - 1)\nu(du) \qquad (46)$$

We want to choose the function $\varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *so that the measure* ν *is a Lévy measure of some classical Lévy process.*
Then (see [21]) the function φ *must be chosen so that the measure* ν *satisfies the three following conditions:*

(C1) $\nu(\{0\}) = 0$

(C2) $\int_{\mathbb{R}} (|x|^2 \wedge 1)\nu(dx) < \infty$

(C3) $\int_{|x|>1} |x|\nu(dx) < \infty$

In the following to illustrate our development we shall discuss the class of Meixner processes, where their associated Lévy measure verify the conditions (C1), (C2) and (C3). In particular, according to the above discussion, we shall explicit the function $\varphi \in \mathcal{K} = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ giving the characteristic exponents of their associated processes $\hat{j}_t^{(3)}$.

Example 3 *Example(The Gamma process)*
It is well Known [21] that the Lévy measure of the Gamma Process of order α $\Gamma(c, \alpha)$ *is*

$$\nu(dx) = c\frac{e^{-\alpha x}}{x} 1_{(0,\infty)}(x)dx$$

Then, according to (46), the function φ *should satisfy the equation*

$$d\varphi^{-1}(x) = c\frac{e^{-\alpha x}}{x} 1_{(0,\infty)}(x)dx \tag{47}$$

If φ *is a diffeomorphism from* $Supp(\varphi)$ *onto the image* $\varphi(Supp(\varphi))$, *then (47) becomes*

$$\frac{1}{\varphi'(x)} = c\frac{e^{-\alpha\varphi(x)}}{\varphi(x)} 1_{(0,\infty)}(\varphi(x))$$

Thus φ *should be positive on its support and verify the differential equation*

$$y' = \frac{1}{c} y e^{\alpha y}$$

The solutions of this equation are given by the relation

$$\varrho(-\alpha y) = \frac{x}{c} + k\,, \quad k \in \mathbb{R}, \tag{48}$$

where ϱ *is the function defined on* $\mathbb{R}^*$ *by* $\varrho(x) = \displaystyle\int_{-\infty}^{x} \frac{e^t}{t} dt$.
Notice that the restriction of ϱ *on the interval* $(-\infty, 0)$ *is strictly decreasing* C^1*-diffeomorphism from* $(-\infty, 0)$ *to* $(-\infty, 0)$.
Combining (48) with the condition

$$y(x) = \varphi(x) > 0 \qquad ; \qquad \forall x \in Supp(\varphi)$$

we get $\frac{x}{c} + k < 0$ *for all* $x \in Supp(\varphi)$. *Hence* $Supp(\varphi) \subset (\infty, -ck)$ *and we have*

$$\varphi(x) = \frac{-1}{\alpha} \varrho^{-1}(\frac{x}{c} + k)\,, \quad x \in Supp(\varphi).$$

But we want the function φ *to be in* $L^\infty(\mathbb{R})$, *then* $|\varphi(x)| \leq M$ *for some positive real number* M*; or equivalently*

$$x \leq c_0 = c(Ei(-\alpha M) - k) < -ck$$

Therefore, a possible solution is

$$\varphi(x) = \frac{-1}{\alpha} \varrho^{-1}(\frac{x}{c} + k) 1_{(\infty, c_0)}(x)$$

Next we will show that $\varphi \in L^2(\mathbb{R})$.
We have

$$\begin{aligned}\int \varphi(x)^2 dx &= \frac{1}{\alpha^2}\int_{-\infty}^{c_0} (\varrho^{-1}(\frac{x}{c}+k))^2 dx \\ &= \frac{c}{\alpha^2}\int_{-\infty}^{\frac{c_0}{c}+k} (\varrho^{-1}(u))^2 du \\ &= \frac{c}{\alpha^2}\int_{0}^{\varrho^{-1}(\frac{c_0}{c}+k)} z^2\varrho'(z) dz \\ &= \frac{c}{\alpha^2}\int_{0}^{-\alpha\varphi(c_0)} z e^z dz < \infty.\end{aligned}$$

From (48) and the positivity of φ, we should have $\varphi(c_0) = +\infty$ and then $c_0 = -ck$. Consequently, φ can not be a solution in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. To avoid this constraint, we consider an approximating sequence (c_n) of c_0 and then a sequence of solutions $\varphi_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ defined by

$$\varphi_n(x) = \frac{-1}{\alpha}\varrho^{-1}\left(\frac{x}{c}+k\right) 1_{(\infty,c_n)}(x).$$

It is easy to verify that the sequence φ_n converge in $L^2(\mathbb{R})$ to the function

$$\varphi_0 : x \longmapsto \frac{-1}{\alpha}\varrho^{-1}\left(\frac{x}{c}+k\right) 1_{(\infty,c_0)}(x)$$

for which the corresponding measure ν is the Gamma Lévy measure.

Example 4 *Example(The Meixner process)*
The Lévy measure of the Meixner process $M(\alpha,\beta,\delta)$ is given by

$$\nu(dx) = \frac{\delta e^{\frac{\beta x}{\alpha}}}{x sh(\frac{\pi x}{\alpha})} dx\,, \quad \alpha,\, \delta > 0\,,\, -\pi < \beta < \pi.$$

Then with the same assumptions as in the Gamma case, we easily verify that the corresponding function ϕ satisfies the differential equation

$$y' = \frac{1}{2\delta} y(e^{\frac{\pi-\beta}{\alpha}x} - e^{-\frac{\pi+\beta}{\alpha}x}).$$

This equation has not a solution in the space $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$. However, it is possible to verify that the sequence $\varphi_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ defined by

$$\varphi_n(x) = \frac{\alpha}{\pi-\beta}\varrho^{-1}\left(\frac{x}{2\delta}+k\right) 1_{(\infty,-2\delta k-\frac{1}{n})}(x) + \frac{\alpha}{\pi+\beta}\varrho^{-1}\left(\frac{-x}{2\delta}-k\right) 1_{(-2\delta k+\frac{1}{n},\infty)}(x),$$

converges in $L^2(\mathbb{R})$ to $\varphi_0 \in L^\infty(\mathbb{R})$, and the corresponding measure ν is the Meixner Lévy measure.

Example 5 *Example(The Negative binomial process)*
The Lévy measure of the negative binomial process, with parameters $c > 0$ and $0 < p < 1$, is the measure defined on $\mathbb{N}$ by

$$\nu(\{k\}) = \frac{c(1-p)^k}{k}, \quad k \in \mathbb{N}.$$

The corresponding function φ must verify the equation

$$\nu(\{k\}) = \lambda(\varphi^{-1}(\{k\})) \tag{49}$$

where λ is the Lebesgue measure on $\mathbb{R}$. Then if we denote

$$A_k = \varphi^{-1}(\{k\}) \in \mathcal{B}(\mathbb{R}),$$

the function φ is expressed by

$$\varphi(x) = \sum_{k=1}^{\infty} k 1_{A_k}$$

Observe that the condition (49) implies $\lambda(A_k) = \frac{c(1-p)^k}{k}$, from which it's clear that the function φ is in $L^2(\mathbb{R})$, but does not lies in $L^\infty(\mathbb{R})$. Then, as in the previous examples, we consider an approximating sequence of functions $\varphi_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ converging in $L^2(\mathbb{R})$ to φ. Such sequence can be defined by

$$\varphi_n(x) = \sum_{k=1}^{n} k 1_{A_k}$$

The above discussion suggests that it is possible to obtain a larger class of classical Lévy processes as limits of sequences associated to the class $\hat{j}_t^{(3)}$. The following theorem shows that this conjecture is true and gives an explicit description of this class.

Theorem 5 *Theorem Let ν be a non singular Lévy measure satisfying the four conditions (C1), (C2), (C3) and*

$$\int_{\mathbb{R}} x^2 \nu(dx) < \infty \tag{50}$$

Then there exist a sequence $\varphi_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ converging in $L^2(\mathbb{R})$ to a function φ such that ν is the φ-image of the Lebesgue measure λ.

Proof:
Step I. Suppose in this step that the measure ν has a positive continuous density f. We will choose the function φ to be a C^1-diffeomorphism on it's support I_φ. The condition $\nu(A) = \lambda(\varphi^{-1}(A))$ gives $\frac{1}{\varphi'(x)} = f(\varphi(x))$ and then φ is a solution of the differential equation

$$y' = \frac{1}{f(y)}, \quad x \in I_\varphi. \tag{51}$$

This implies that the φ-image of the support of φ is in the support I_f of f. Denoting $(a, b) = I_\varphi$ and $(\alpha, \beta) = I_f$, we obtain

$$\varphi((a, b)) \subset (\alpha, \beta)\,, \quad a,\, b,\, \alpha, \beta \in \overline{\mathbb{R}}.$$

The solution of (51) is expressed by

$$\varphi(x) = F^{-1}(x + k)\,, \quad x \in (a, b)$$

where F is a primitive of f and k is a real number. Since F is strictly increasing continuous function, F^{-1} is well defined on (α, β).
We have

$$\begin{aligned}\int_{-\infty}^{+\infty} \varphi(x)^2 dx &= \int_a^b (F^{-1}(x + k))^2 dx \\ &= \int_{a+k}^{b+k} (F^{-1}(t))^2 dt \\ &= \int_{\varphi(a)}^{\varphi(b)} u^2 f(u) du < \infty,\end{aligned}$$

which proves that φ is in $L^2(\mathbb{R})$ but it is not in $L^\infty(\mathbb{R})$.
We have $\varphi((a, b)) \subset (\alpha, \beta)$. So, if the function f has a finite support, then we can choose φ as a solution of our problem, and if not, we can choose a sequence of intervals (a_n, b_n) converging to (a, b) so that $\varphi_n(\cdot) = F^{-1}(\cdot + k)1_(a_n, b_n)$ provides a solution of oue problem.
Step II. If the Lévy measure ν has a negative density f we can replace ν by $-\nu$ and the result is still true.
Step III. If ν is a non singular and non discrete measure then it's a finite sum of a disjoint supports measures of the type as in the steps **I.** and **II.**, the result is still true.
Step IV. Suppose that ν is a discrete measure. Then we can assume that it has a support in $\mathbb{Z}$. Denoting $A_k = \varphi^{-1}(\{k\}) \in \mathcal{B}(\mathbb{R})$, the function φ is expressed by

$$\varphi(x) = \sum_{k=-\infty}^{+\infty} k 1_{A_k}.$$

It's clear that φ is in $L^2(\mathbb{R})$. In fact we have

$$\int_{-\infty}^{+\infty} (\varphi(x))^2 \lambda(dx) = \sum_{k=-\infty}^{+\infty} k^2 \lambda(\varphi^{-1}(\{k\})) = \sum_{k=-\infty}^{+\infty} k^2 \nu(A_k) < \infty,$$

where the condition (50) is taken inti account. On the other hand,since φ is not necessarily in $L^\infty(\mathbb{R})$, we can consider a sequence of functions $\varphi_n \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ converging in $L^2(\mathbb{R})$ to φ. Such sequence can be defined by

$$\varphi_n(x) = \sum_{-n}^{k=n} k 1_{A_k}.$$

Step V. Every non singular measure is a finite sum of measures as in steps **III.** and **IV.** Hence the result can be obtained by simple embedding arguments. ■

References

[1] L. Accardi, *On the Quantum Feynman-Kac Formula*, Rendiconti del Seminario Mathematico e Fisico di Milano, Vol. XLVIII (1978).

[2] L. Accardi, *A note on Meyer's note* , Quantum Probability and Applications III, L. Accardi, W. Waldenfelds (eds.), Oberwolfach 1987, Lecture Notes in Mathematics Vol. 1303, Springer (1988), 1–5.

[3] L. Accardi, W. Wided, H. Ouerdiane.

[4] L. Accardi, G. Amosov and U. Franz, *Second quantized automorphisms of the renormalized square of white noise (RSWN) algebra*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. Vol. 7, no. 2 (2004), 183–194.

[5] L. Accardi and A. Mohari *On the Structure of Classical and Quantum Flows*, J. Funct. Anal. 135 (1996), 421–455.

[6] L. Accardi, Y.G. Lu and I. Volovich, *A white noise approach to stochastic calculus*, Acta App. Math. 63 (2000), 3–25.

[7] L. Accardi, U. Franz and M. Skeide, *Renormalized squares of white noise and other non-Gaussian as Lévy processes on real Lie algebras*, Comm. Math. Phys. 228 (2002), 123–150.

[8] L. Accardi, *Meixner classes and the square of white noise*, Contemp. Math. 317 (2003), 1–13.

[9] L. Accardi, A. Barhoumi and H. Ouerdiane, *A quantum approach to Laplace operators*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. Vol. 9, no. 2 (2006), 215–248.

[10] W. Arveson, *Noncommutative Dynamics and E-Semigroups*, Springer Monographs in Mathematics Series, 2003.

[11] E. B. Dynkin, *An application of flows to time shift and time reversal in stochastic processes*,Trans. Amer. Math. Soc. Vol. 287, no. 2 (1985), 613–619.

[12] F. Fagnola, *On the realisation of classical Markov processes as quantum flows in Fock space*, Probability theory and mathematical statistics (Vilnius, 1993), 253–275, TEV, Vilnius, 1994.

[13] U. Franz, *Lévy process on real Lie algebras*, Recent developments in stochastic analysis and related topics, 166–181, World Sci. Publ., Hackensack, NJ, 2004.

[14] P. A. Meyer, *A note on Shifts and Cocycles*, Quantum Probability and Applications III, L. Accardi, W. Waldenfelds (eds.), Oberwolfach 1987, Lecture Notes in Mathematics Vol. 1303, Springer (1988), 361–374.

[15] P. A. Meyer, *Quantum Probability for Probabilists*, Lectur Notes in Math., Vol. 1538, Springer-Verlag, 1995, 2nd edn.

[16] H. Nakazato, *Positive elements in the enveloping algebra of some Lie algebra*, Bull. Fac. Sci. Technol. Hirosaki Univ. Vol. 9, no. 2 (2007), 95–99. , Hiroshi Positive elements in the enveloping algebra of some Lie algebra.

[17] P. Śniady, *Quadratic bosonic and free white noises*, Comm. Math. Phys. Vol. 211, no. 3 (2000), 615–628.

[18] K. R. Parthasarathy, *An introduction to quantum stochastic calculus*, Birkhaüser Verlag, Basel 1992.

[19] A.M. Veršik, I.M. Gel'fand and M.I. Graev, *Representations of the group SL(2,**R**), where **R** is a ring of functions*, Uspehi Mat. Nauk 28 (1973), 371–384. English translation: Russian Math. Surveys Vol. 28, no. 5 (1973), 87–132.

[20] M.H. Lee, *Lie algebras of formal power series*, Rev. Mat. Complut. Vol. 20, no. 2 (2007), 463–481.

[21] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, 1999.