SOME MORE PROBLEMS ABOUT ORDERINGS OF ULTRAFILTERS

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Abstract. We discuss the connection between various orders on the class of all the ultrafilters and certain compactness properties of abstract logics and of topological spaces. We present a model theoretical characterization of Comfort order. We introduce a new order motivated by considerations in abstract model theory. For each of the above orders, we show that if E is a (λ, λ)-regular ultrafilter, and D is not (λ, λ)-regular, then $E \not\leq D$. Many problems are stated.

We refer to [C2, CK, CN, E, G3, G4, GS, L4, M] for unexplained notions.

Many orderings on the class of all ultrafilters have been introduced. All of these orderings can be viewed from many different points of view, and lead to equivalent formulations of some notions, either in purely ultrafilter theoretical terms, in topological terms, or in model theoretical terms. We discuss some of these connections, introduce still another order motivated by abstract model theoretical considerations, and state some further problems.

Throughout, let D be an ultrafilter over some set I, and E be an ultrafilter over some set J.

We first recall the definition of the classical Rudin-Keisler order.

Definition 1. For D and E ultrafilters, the *Rudin-Keisler* (pre-)order is defined as follows.

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$E \leq_{RK} D$ if and only if there is a function $f : I \to J$ such that, for every $Y \subseteq J$, it happens that $Y \in E$ if and only if $f^{-1}(Y) \in D$.

Notice that, in the above situation, the ultrafilter structure of E is completely determined by f and by the ultrafilter structure of D. If $E \leq_{RK} D$, we sometimes will say that E is a *quotient* of D. The Rudin-Keisler order can be given several equivalent reformulations.

Theorem 2. *For every pair of ultrafilters D and E, the following are equivalent.*

(1) $E \leq_{RK} D$.
(2) *for every model $\mathfrak{A}$, we have that $\prod_E \mathfrak{A}$ is elementarily embeddable in $\prod_D \mathfrak{A}$.*
(3) *Every D-pseudocompact topological space is E-pseudocompact.*
(4) *Every D-pseudocompact Tychonoff topological space is E-pseudocompact.*

Proof. (1) $\Leftrightarrow$ (2) is [CK, Exercise 4.3.41].
(1) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial.
(4) $\Rightarrow$ (1) is immediate from [G4, Lemma 1.4]. □

The notion of D-compactness has played a very important role in the study of compactness properties of topological spaces, particularly in connection with products. The paper [GS] is a milestone in the field. The next definition is due to W. Comfort, and appears in [G2, G3].

Definition 3. The *Comfort (pre-)order* is defined as follows.
$E \leq_C D$ if and only if every D-compact topological space is E-compact.

It can be shown that the Comfort order turns out to be the same if we restrict ourselves to Tychonoff spaces. Moreover, Garcia-Ferreira [G2] shows that the Rudin-Keisler order is strictly finer than the Comfort order.

We now give a model theoretical characterization of Comfort order.

Definition 4. If D is an ultrafilter, let us say that a class K of models of the same type is *D-closed* if and only if K is closed under isomorphism, K is closed under elementary substructures, and any ultraproduct by D of members of K still belongs to K.

Proposition 5. *Suppose that D is an ultrafilter over I, and E is an ultrafilter over J. Then the following are equivalent.*

(1) $E \leq_C D$.
(2) *Every D-closed class of models is E-closed.*

(3) *For every model* $\mathfrak{A}$*, the smallest* D*-closed class containing* $\mathfrak{A}$ *is* E*-closed.*
(4) *For every model* $\mathfrak{A}$ *with* $|A| = \sup\{|I|, |J|\}$*, the smallest* D*-closed class containing* $\mathfrak{A}$ *is* E*-closed.*

Proof. Without loss of generality, we can suppose that E and D are (not necessarily uniform) over the same cardinal α.

Garcia-Ferreira [G2, Theorem 2.3] and [G3, Theorem 3.3] proved that $E \leq_C D$ if and only if $E \in \beta_D(\alpha)$, where $\beta_D(\alpha)$ is the D-compactification of the discrete space α, that is, the smallest D-compact subspace of $\beta(\alpha)$ containing α, where $\beta(\alpha)$ denotes the Stone-Čech compactification of α.

(1) ⇒ (2) Suppose that $E \in \beta_D(\alpha)$. Since $\beta_D(\alpha)$ is the smallest D-compact subspace of $\beta(\alpha)$ containing α, it follows that every element of $\beta_D(\alpha)$ can be iteratively constructed starting by α, and taking D-limits of ultrafilters already known to be in $\beta_D(\alpha)$ (see [G3] for details). It is well-known, already from [F], that the D-limit of certain ultrafilters $(E_i)_{i\in I}$ corresponds to a quotient of the sum $\sum_D E_i$ (see [L4] for the definition). Model theoretically, if $D' = \sum_D E_i$, then, for every model $\mathfrak{A}$, $\prod_{D'} \mathfrak{A}$ is isomorphic to $\prod_D \prod_{E_i} \mathfrak{A}$. This implies that every class of models which is both D-closed and E_i-closed for every $i \in I$ is also D'-closed.

Moreover, if $D'' \leq_R D'$, then, for every model $\mathfrak{A}$, $\prod_{D''} \mathfrak{A}$ is elementarily embeddable in $\prod_{D'} \mathfrak{A}$, by Theorem 2 (1) ⇒ (2). Thus, every class of models which is D'-closed is also D''-closed.

By iterating the above arguments, and by the above description of $\beta_D(\alpha)$, we get that if $E \in \beta_D(\alpha)$, then every D-closed class of models is E-closed.

(2) ⇒ (3) ⇒ (4) are trivial.

(4) ⇒ (1) By the above arguments, for every model $\mathfrak{A}$, the smallest D-closed class containing $\mathfrak{A}$ is the class of all isomorphic copies of the models of the form $\prod_{E'} \mathfrak{A}$, for $E' \in \beta_D(\alpha)$.

Let $\mathfrak{A}$ be the complete model of cardinality α, and let K be the smallest D-closed class containing $\mathfrak{A}$. By assumption, K is E-closed; in particular, $\prod_E \mathfrak{A} \in K$. By the above remark, $\prod_E \mathfrak{A} \in K$ is isomorphic to $\prod_{E'} \mathfrak{A}$, for some $E' \in \beta_D(\alpha)$. Now, since $|A| = \alpha$, both E and E' are over α, and $\mathfrak{A}$ is a complete structure, then the ultrafilter structures of E and E' can be recovered from the structures, respectively, of $\prod_E \mathfrak{A}$ and of $\prod_{E'} \mathfrak{A}$. Since these latter models are isomorphic, E and E' are isomorphic, thus $E \in \beta_D(\alpha)$, since $E' \in \beta_D(\alpha)$.

□

Problem 6. What does the Comfort order become when we restrict ourselves to special classes of topological spaces?

In more detail, if T is a class of topological spaces, let us define the *Comfort (pre-)order relative to* T as follows.

$E \leq_{T,C} D$ if and only if every D-compact topological space belonging to T is also E-compact.

Does $E \leq_{T,C} D$ coincide with $E \leq_C D$, when T is the class of Hausdorff normal topological spaces?

Does $E \leq_{T,C} D$ coincide with $E \leq_C D$, when T is the class of topological groups?

In the particular case of ultrafilters over ω, the order $\leq_{T,C}$ has been introduced by Garcia-Ferreira [G1]. He also asked whether $E \leq_{T,C} D$ coincides with $E \leq_C D$, when T is the class of topological groups, and gave a partial affirmative answer.

We now introduce an analogue of the Comfort order, an analogue which refers to compactness properties of abstract logics.

By a *logic* we mean an extension of first-order logic satisfying certain regularity properties (see [E] for more details). Examples of logics in the present sense are logics allowing infinitary conjunctions and disjunctions (infinitary logics), or logics obtained by adding new quantifiers (e. g., cardinality logics). As far as the present note is concerned, the exact closure properties a logic is required to satisfy are those listed at the beginning of [C2, Section 2].

Makowsky and Shelah [MS] defined the notion of an ultrafilter *related* to a logic, and found many applications of this notion (see [M] for a survey). Essentially, an ultrafilter is related to a logic if and only if a version of Łoś Theorem holds for that ultrafilter and that logic. Later [C1, C2] gave an improved definition, and extended Makowsky and Shelah's results to a more general setting. We shall usually say that a logic $\mathcal{L}$ is *D-compact*, in place of saying that D is related to $\mathcal{L}$.

Definition 7. We define as follows the Caicedo-Makowsky-Shelah (pre-)order.

For ultrafilters D and E, we write $E \leq_{CMS} D$ to mean that every D-compact logic is E-compact.

The next proposition, asserting that $\leq_C$ is finer than $E \leq_{CMS} D$, is an immediate corollary of results from [C2].

Proposition 8. *For every pair of ultrafilters D and E, if $E \leq_C D$, then $E \leq_{CMS} D$.*

Proof. The proposition follows immediately from [C2, Lemma 2.3], which asserts that D-compactness of some logic $\mathcal{L}$ is equivalent to

D-compactness of certain topological spaces $E_\sigma(\mathcal{L})$ (whose definition depends only on $\mathcal{L}$, and not on D). □

Problems 9. Does $E \leq_C D$ coincide with $E \leq_{CMS} D$?

Does $E \leq_{T,C} D$ coincide with $E \leq_{CMS} D$, for T the class of topological groups? for T the class of Hausdorff normal topological spaces?

More generally, study the (pre-)order $E \leq_{CMS} D$, and characterize it in topological and set theoretical terms.

Notice that, by [C2, Lemma 2.3], $E \leq_{T,C} D$ coincides with $E \leq_{CMS} D$, when T is the class of the topological spaces of the form $E_\sigma(\mathcal{L})$, as defined in [C2].

We now show that (λ, λ)-regularity constitute a "dividing line" for each of the above orderings.

Proposition 10. *Suppose that λ is an infinite cardinal, D is not (λ, λ)-regular, and E is (λ, λ)-regular. Then the following holds.*

(1) *$E \not\leq_C D$.*
(2) *More generally, $E \not\leq_{T,C} D$, where T is the class of Hausdorff normal topological spaces.*
(3) *$E \not\leq_{T,C} D$, where T is the class of Tychonoff topological groups.*
(4) *$E \not\leq_{CMS} D$.*

Proof. (1) follows from either (2), (3) or (4).

(2) [L2, Proposition 1] and [L3, Corollary 2] constructed a Hausdorff normal topological space X such that, for every ultrafilter F, X is F-compact if and only if F is not (λ, λ)-regular. Thus, X is D-compact, but not E-compact, hence $E \leq_{T,C} D$ fails.

(3) is similar, using [L3, Proposition 3].

(4) First suppose that λ is regular. Let ω_α be a cardinal such that $\operatorname{cf} \omega_\alpha = \lambda$, and $\omega_\beta^{|I|} < \omega_\alpha$, for every $\beta < \alpha$. Then classical methods (see, e. g., [L1]) imply that $\mathcal{L}_{\omega,\omega}(Q_\alpha)$ is D-compact. This logic is not E-compact, since no elementary extension of $\prod_E \langle \lambda, \leq \rangle$ can be $\mathcal{L}_{\omega,\omega}(Q_\alpha)$-equivalent to $\prod_E \langle \lambda, \leq \rangle$.

The case when λ is singular is treated in a similar way, by using a logic generated by two cardinality quantifiers, since an ultrafilter is (λ, λ)-regular if and only if it is either $(\operatorname{cf} \lambda, \operatorname{cf} \lambda)$-regular or (λ^+, λ^+)-regular [L4]. □

Proposition 10 strongly suggests the hypothesis that the study of compactness properties both of logics and of (products of) topological spaces actually deals with properties of the Comfort and related orders, and that problems about (transfer of) compactness are best stated as problems about these orders. Indeed, results stated in terms of D-compactness are more powerful than results stated in terms of $[\lambda, \lambda]$-compactness. In fact, older results by the author are immediate consequences of Proposition 10.

Corollary 11. *For every infinite cardinals λ and μ, the following are equivalent.*

(1) *There exists a (λ, λ)-regular not (μ, μ)-regular ultrafilter.*
(2) *There exists a productively $[\lambda, \lambda]$-compact not productively $[\mu, \mu]$-compact topological space.*
(3) *There exists a productively $[\lambda, \lambda]$-compact not productively $[\mu, \mu]$-compact Tychonoff topological group.*
(4) *There exists a productively $[\lambda, \lambda]$-compact not productively $[\mu, \mu]$-compact Hausdorff normal topological space.*
(5) *There exists a $[\lambda, \lambda]$-compact not $[\mu, \mu]$-compact logic.*

Thus, a more detailed study of the orders $E \leq_{T,C} D$ and $E \leq_{CMS} D$ will probably shed more light to problems connected with compactness of logics and topological spaces.

It will be probably useful also to consider the possibility of dealing with more than two ultrafilters at a time.

Problem 12. Study the following relations (which are not pre-orders).

For families $(E_k)_{k\in K}$ and $(D_h)_{h\in H}$ of ultrafilters, let $(E_k)_{k\in K} \leq_C (D_h)_{h\in H}$ mean that every topological space which is D_h-compact, for every $h \in H$, is E_k-compact, for some $k \in K$.

The relation $(E_k)_{k\in K} \leq_{T,C} (D_h)_{h\in H}$, for T a class of topological spaces is defined similarly.

Similarly, let $(E_k)_{k\in K} \leq_{CMS} (D_h)_{h\in H}$ mean that every logic which is D_h-compact, for every $h \in H$, is E_k-compact, for some $k \in K$.

Proposition 13. *Let $\leq$ be any one of the following relations: $\leq_C$, $\leq_{CMS}$, or $\leq_{T,C}$, where T is a class of topological spaces closed under taking Frechet disjoint unions in the sense of* [L3, Definition 7]*.*

Then $(E_k)_{k\in K} \leq (D_h)_{h\in H}$ if and only if there is some $k \in K$ such that $E_k \leq (D_h)_{h\in H}$.

Proof. The if-condition is trivial.

For the converse, suppose by contradiction that, for every $k \in K$, we have $E_k \not\leq_{T,C} (D_h)_{h\in H}$. Thus, for every $k \in K$, there is a topological space $X_\kappa \in T$ which is not E_k-compact, but which is D_h-compact, for each $h \in H$. Then the Frechet disjoint union of the X_κ's witnesses the failure of $(E_k)_{k\in K} \leq_{T,C} (D_h)_{h\in H}$, by [L3, Proposition 8].

The argument for $\leq_{CMS}$ is similar, by taking a union of logics. □

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