Extending the Set of Quadratic Exponential Vectors*

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Abstract

We extend the square of white noise algebra over the step functions on $\mathbb{R}$ to the test function space $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and we show that in the Fock representation the exponential vectors exist for all test functions bounded by $\frac{1}{2}$.

1 Introduction

Modulo minor variations in the choice of the test function space, the square of white noise (SWN) algebra has been introduced by Accardi, Lu and Volovich [ALV99] as follows. Let $\mathfrak{L} = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $c > 0$ a constant. Then the ***SWN algebra*** $\mathcal{A}$ over $\mathfrak{L}$ is the unital $*$–algebra generated by symbols B_f, N_f $(f \in \mathfrak{L})$ and the commutation relations

$$[B_f, B_g^*] = 2c\langle f, g\rangle + 4N_{\overline{f}g}, \qquad [N_f, B_g^*] = 2B_{fg}^*,$$

$(f, g \in \mathfrak{L})$ and all other commutators 0. Note that by the first relation, $N_f^* = N_{\overline{f}}$.

A ***Fock representation*** of $\mathcal{A}$ is a representation ($*$, of course) π of $\mathcal{A}$ on a pre-Hilbert space H with a unit vector $\Phi \in H$, fulfilling $\mathcal{A}\Phi = H$ and $\pi(B_f)\Phi = \pi(N_f)\Phi = 0$ for all $f \in \mathfrak{L}$. From the commutation relations it follows that a Fock representation is unique up to unitary equivalence. Existence of a Fock representation has been established by different proofs in [ALV99, AS00a, Sni00, AFS02] for $d = 1$. They extend easily to general $d \in \mathbb{N}$. Henceforth, we speak about **the** Fock representation. The Fock representation would be faithful, if we require also that the N_f depend linearly on f. By abuse of notation, we identify $\mathcal{A}$ with its image $\pi(\mathcal{A})$ omitting, henceforth, π.

The ***exponential vector*** $\psi(f)$ to an element $f \in \mathfrak{L}$ is defined as

$$\psi(f) := \sum_{m=0}^{\infty} \frac{B_f^{*m}\Phi}{m!}$$

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whenever the series exists. In Accardi and Skeide [AS00b] is has been shown for $d = 1$ that $\psi(\sigma I\!I_{[0,t]})$ exists for $|\sigma| < \frac{1}{2}$ and that $\langle\psi(\sigma I\!I_{[0,t]}), \psi(\rho I\!I_{[0,t]})\rangle = e^{-\frac{ct}{2}\ln(1-4\overline{\sigma}\rho)}$. As noted in [AS00b], this extends to arbitrary step functions f, g on $\mathbb{R}$ with $\|f\|_\infty < \frac{1}{2}$, with inner product

$$\langle\psi(f), \psi(g)\rangle \;=\; e^{-\frac{c}{2}\int \ln(1-4\overline{f(t)}g(t))\,dt}.\text{[1]} \tag{$*$}$$

Our scope is to extend the set of exponential vectors and the formula in $(*)$ for their inner product to test functions $f \in \mathcal{L}$ with $\|f\|_\infty < \frac{1}{2}$.

In the "29th Quantum Probabililty Conference" in October 2008 in Hammamet, Tunisia, Dhahri explained that the extension can be done for exponential vectors to all elements f in $\mathcal{L}$ with $\|f\|_\infty < \frac{1}{2}$. This a part of the work Accardi and Dhahri [AD08] (in preparation) on the *second quantization functor* for the square of white noise. Here we give a simple proof of this partial result.

2 The result

2.1 Theorem. *The exponential vector $\psi(f)$ exists for every $f \in \mathcal{L}$ with $\|f\|_\infty < \frac{1}{2}$ and the inner product of two such exponential vectors is given by* $(*)$*.*

Proof. (i) We show that the right-hand side of $(*)$ exists. Indeed, by Taylor expansion we have $|\ln(1 + x)| \leq M_\delta\,|x|$ for $|x| \leq 1 - \delta$ for every $\delta \in (0, 1)$, where M_δ may depend on δ but not on x. Choose $\delta = 1 - 4\,\|f\|_\infty\,\|g\|_\infty \in (0, 1)$. Then

$$\left|\ln(1 - 4\overline{f(t)}g(t))\right| \;\leq\; M_\delta\left|4\overline{f(t)}g(t)\right|.$$

Since $\left|\overline{f(t)}g(t)\right|$ is integrable, so is $\ln(1 - 4\overline{f(t)}g(t))$.

(ii) The function $x \mapsto \ln x$ is increasing on the whole half line $(0, \infty)$. It follows that also the function $x \mapsto -\ln(1 - x)$ is increasing on $(-1, 1)$. We conclude that $\frac{1}{2} > |f| \geq |g|$ implies $-\ln(1 - 4\,|f(t)|^2) \geq -\ln(1 - 4\,|g(t)|^2)$. Choose for f an L^2–approximating sequence of step functions $(f_n)_{n\in\mathbb{N}}$ in such a way that $|f| \geq |f_n|$ for all $n \in \mathbb{N}$. By the *dominated convergence theorem*, $\lim_{n\to\infty} e^{-\frac{c}{2}\int \ln(1-4|f_n(t)|^2)\,dt} = e^{-\frac{c}{2}\int \ln(1-4|f(t)|^2)\,dt}$.

(iii) In precisely the same way as in [AS00b], one shows that $(*)$ is true for all step functions strictly bounded by $\frac{1}{2}$. It follows that $\lim_{n\to\infty} \|\psi(f_n)\|^2 = e^{-\frac{c}{2}\int \ln(1-4|f(t)|^2)\,dt}$.

(iv) Since $\langle B_f^{*m}\Phi, B_f^{*m}\Phi\rangle$ is a polinomial (of degree m) in $\langle f, f\rangle$, it depends continuously in

[1]The *correlation kernel* on the right-hand side coincides, modulo scaling, with the correlation kernel in Boukas' representation [Bou91] of Feinsilver's *finite difference algebra* [Fei87]. In [AS00b], this observation gave rise to the discovery of an intimate relation between the SWN algebra and the finite difference algebra.

L^2–norm on f. So, for every $M \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$\Big\langle \sum_{m=0}^{M} \frac{B_f^{*\,m}\Phi}{m!}, \sum_{m=0}^{M} \frac{B_f^{*\,m}\Phi}{m!} \Big\rangle \;\leq\; \Big\langle \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!}, \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!} \Big\rangle + 1$$
$$\leq\; \Big\langle \sum_{m=0}^{\infty} \frac{B_{f_n}^{*\,m}\Phi}{m!}, \sum_{m=0}^{\infty} \frac{B_{f_n}^{*\,m}\Phi}{m!} \Big\rangle + 1 \;=\; \|\psi(f_n)\|^2 + 1 \;\leq\; e^{-\frac{c}{2}\int \ln(1-4|f(t)|^2)\,dt} + 1.$$

By the theorem on exchange of limits under domination, it follows that

$$\lim_{M\to\infty} \Big\langle \sum_{m=0}^{M} \frac{B_f^{*\,m}\Phi}{m!}, \sum_{m=0}^{M} \frac{B_f^{*\,m}\Phi}{m!} \Big\rangle \;=\; \lim_{M\to\infty}\lim_{n\to\infty} \Big\langle \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!}, \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!} \Big\rangle$$
$$=\; \lim_{n\to\infty}\lim_{M\to\infty} \Big\langle \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!}, \sum_{m=0}^{M} \frac{B_{f_n}^{*\,m}\Phi}{m!} \Big\rangle \;=\; \lim_{n\to\infty} \|\psi(f_n)\|^2 \;=\; e^{-\frac{c}{2}\int \ln(1-4|f(t)|^2)\,dt}.$$

From this we conlcude that $\psi(f)$ exists and that $\|\psi(f)\|^2 = e^{-\frac{c}{2}\int \ln(1-4|f(t)|^2)\,dt}$.

(v) Doing the same sort of computation for the difference $\psi(f) - \psi(f_n)$, it follows that $\lim_{n\to\infty} \psi(f_n) = \psi(f)$. Approximating also g by a sequence of step functions g_n with $|g| \geq |g_n|$, we find $\lim_{n\to\infty}\langle \psi(f_n), \psi(g_n)\rangle = \langle \psi(f), \psi(g)\rangle$ (continuity of the inner product), and

$$\lim_{n\to\infty} e^{-\frac{c}{2}\int \ln(1-4\overline{f_n(t)}g_n(t))\,dt} \;=\; e^{-\frac{c}{2}\int \ln(1-4\overline{f(t)}g(t))\,dt}$$

(once more, by dominated convergence for $|\overline{f_n}g_n| \leq |\overline{f}g|$ on the other side. This shows $(*)$ for all f, g as specified. ■

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