Interacting Fock space characterization of probability measures

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Abstract

In this paper we characterize the probability measures, on $\mathbb{R}^d$, with square summable support, in terms of their associated preservation operators and the commutators of the annihilation and creation operators.

1 Introduction

A program of expressing properties of a probability measure on $\mathbb{R}^d$, having finite moments of any order, in terms of their annihilation, creation, and preservation operators, was initiated in [1]. There, it was proved that a probability measure is polynomially symmetric if and only if all of its preservation operators vanish. The notion of "polynomially symmetry" is a weak form of the notion of "symmetry" from the classic Measure Theory, in the sense that a probability measure μ, on $\mathbb{R}^d$, is called *symmetric* if, for any Borel subset B of $\mathbb{R}^d$, $\mu(B) = \mu(-B)$, where $-B := \{-x \mid x \in B\}$, while μ is called *polynomially*

[1] *2000 Mathematics Subject Classifications:* 60A10, 05E35, 47B25.

Key words and phrases: probability measure, annihilation operator, creation operator, preservation operator, commutator, square summable support, Hilbert-Schmidt operator.

*Partially supported by the NSF grant DMS 0400526

symmetric if for any monomial $x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$, such that $i_1 + i_2 + \cdots + i_d$ is odd, we have $\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d} \mu(dx) = 0$, where $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$.

It was also proved in [1], that a probability measure μ on $\mathbb{R}^d$, having finite moments of any order, is polynomially factorisable, if and only if, for all $1 \leq i < j \leq d$, any operator from the set $\{a^-(i), a^0(i), a^+(i)\}$ commutes with any operator from the set $\{a^-(j), a^0(j), a^+(j)\}$, where, for any $k \in \{1, 2, \ldots, d\}$, $a^-(k)$, $a^0(k)$, and $a^+(k)$, denote the annihilation, preservation, and creation operators of index k, respectively. Again the notion of "polynomial factorisability" is a weak form of the notion of "product measure" from Measure Theory, since it does not necessarily mean that μ is a product measure of d probability measures $\mu_1, \mu_2, \ldots, \mu_d$ on $\mathbb{R}$, but only the fact that, for any monomial $x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d}$, $\int_{\mathbb{R}^d} x_1^{i_1} x_2^{i_2} \cdots x_d^{i_d} \mu(dx) = \int_{\mathbb{R}^d} x_1^{i_1} \mu(dx) \int_{\mathbb{R}^d} x_2^{i_2} \mu(dx) \cdots \int_{\mathbb{R}^d} x_d^{i_d} \mu(dx)$.

In [2], it was proved that two probability measures μ and ν, on $\mathbb{R}^d$, having finite moments of any order, have the same moments, if and only if they have the same preservation operators and the same commutators between the annihilation and creation operators. The domain of these operators is understood to be the space of all polynomial functions of d real variables $x_1, x_2, \ldots, x_d$, with complex coefficients. Thus the whole information about the moments of a probability measure is contained in two families of operators, namely the preservation operators and the commutators between the annihilation and creation operators. Hence, rather than considering the annihilation and creation operators separately, we can study properties of probability measures, having finite moments of any order, by looking at the joint action of these operators, expressed in terms of their commutators.

In this paper we continue the program started in [1], in the spirit of [2], by characterizing the probability measures, on $\mathbb{R}^d$, with square summable support, in terms of their preservation operators and the commutators between the annihilation and creation operators. We regard the result, from this paper, as an example of the interesting applications of quantum probabilistic, more precisely interacting Fock space, techniques, to the classical probability theory. We have included a minimal background about the notions of annihilation, preservation, and creation operators in section 2. The definition of the probability measures with square summable support and the main result of this paper are presented in section 3.

2 Background

Let μ be a probability measure defined on the Borel sigma field $\mathcal{B}$ of $\mathbb{R}^d$, where d is a fixed positive integer. Throughout this paper, we assume that μ has finite moments of any order, which means that for any $i \in \{1, 2, \ldots, d\}$ and any $p > 0$, $\int_{\mathbb{R}^d} |x_i|^p \mu(dx) < \infty$, where x_i denotes the i^{th} coordinate of the $d-$dimensional vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. For any non-negative integer n, we denote by F_n, the space of all polynomial functions $p(x_1, x_2, \ldots, x_d)$, of d real variables $x_1, x_2, \ldots, x_d$, with complex coefficients, and of total degree less than or equal to n. In F_n, two polynomials p and q, that are equal $\mu-$a.s. ("a.s." means "almost surely"), are considered to be the same, for all $n \geq 0$. Since μ has finite moments of any order, we have:

$$\mathbb{C} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset L^2(\mathbb{R}^d, \mu).$$

For all $n \geq 0$, F_n is a closed subspace of $L^2(\mathbb{R}^d, \mu)$, since F_n is a finite dimensional vector space. Let $G_0 := F_0 = \mathbb{C}$ and, for all $n \geq 1$, let $G_n := F_n \ominus F_{n-1}$, i.e., G_n is the orthogonal complement of F_{n-1} into F_n. This orthogonal complement is computed with respect to the inner product $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}\mu(dx)$, for $f,\, g \in L^2(\mathbb{R}^d, \mu)$. We define now the Hilbert space

$$\mathcal{H} := \oplus_{n=0}^{\infty} G_n \subset L^2(\mathbb{R}^d, \mu).$$

The Hilbert space $\mathcal{H}$ can be understood in two ways: either as the orthogonal sum of the countable family of finite dimensional Hilbert spaces $\{G_n\}_{n \geq 0}$ or as the closure of the space F, of all polynomial functions of d real variables, with complex coefficients, in the space $L^2(\mathbb{R}^d, \mu)$. We would like to mention again that, in F, two polynomial functions that are equal $\mu-$a.s., are considered to be identical. We also define $F_{-1} := \{0\}$ and $G_{-1} := \{0\}$, where $\{0\}$ denotes the null space.

For any $i \in \{1, 2, \ldots, d\}$, we denote the multiplication operator by the variable x_i, by X_i. The domain of this operator is considered to be the space F described above. Thus, if $p(x_1, x_2, \ldots, x_d)$ is a polynomial function, we have

$$X_i p(x_1, x_2, \ldots, x_d) = x_i p(x_1, x_2, \ldots, x_d). \tag{2.1}$$

We can see that, for any $i \in \{1, 2, \ldots, d\}$, X_i maps F into F, and since F is dense in $\mathcal{H}$, X_i is a densely defined linear operator on the Hilbert space $\mathcal{H}$. Let us also observe that X_i maps F_n into F_{n+1}, for all $1 \leq i \leq d$ and $n \geq 0$.

If $f,\, g \in L^2(\mathbb{R}^d, \mu)$, such that $\langle f, g \rangle = 0$, we say that f and g are orthogonal and denote this fact by $f \perp g$.

For all $n \geq 0$, let P_n denote the orthogonal projection of $\mathcal{H}$ onto G_n. If k and n are two non-negative integers such that $k \geq n + 2$, then since P_n maps

$\mathcal{H}$ onto G_n, $G_n \subset F_n$, and X_i maps F_n into F_{n+1}, we can see that $X_i P_n$ maps $\mathcal{H}$ into F_{n+1}. Since $n+1 < k$, we have $G_k \perp F_{n+1}$, and because P_k projects all polynomial functions into G_k, we conclude that:

$$P_k X_i P_n \quad = \quad 0, \tag{2.2}$$

for all $1 \leq i \leq d$ and all $k \geq n+2$. Taking the adjoint in both sides of the equality (2.2), we obtain:

$$P_n X_i P_k \quad = \quad 0, \tag{2.3}$$

for all $1 \leq i \leq d$ and all $k \geq n+2$. Thus, we conclude that, for all r and s non-negative integers, such that $|r-s| \geq 2$, and for all $1 \leq i \leq d$, we have:

$$P_r X_i P_s \quad = 0. \tag{2.4}$$

Let I be the identity operator of $\mathcal{H}$. Since $I = \sum_{n \geq 0} P_n$, it follows from (2.4) that, for all $1 \leq i \leq d$,

$$\begin{aligned} X_i &= I X_i I \\ &= \left(\sum_{k=0}^{\infty} P_k\right) X_i \left(\sum_{n=0}^{\infty} P_n\right) \\ &= \sum_{|k-n| \leq 1} P_k X_i P_n \\ &= \sum_{n=1}^{\infty} P_{n-1} X_i P_n + \sum_{n=0}^{\infty} P_n X_i P_n + \sum_{n=0}^{\infty} P_{n+1} X_i P_n. \end{aligned} \tag{2.5}$$

For all $i \in \{1, 2, \ldots, d\}$, we define the following three operators:

$$a^-(i) \quad = \quad \sum_{n=1}^{\infty} P_{n-1} X_i P_n, \tag{2.6}$$

$$a^0(i) \quad = \quad \sum_{n=0}^{\infty} P_n X_i P_n, \tag{2.7}$$

and

$$a^+(i) \quad = \quad \sum_{n=0}^{\infty} P_{n+1} X_i P_n. \tag{2.8}$$

Let us observe that, for any $n \geq 0$, the restrictions of these three operators to the space G_n, are:

$$a^-(i)|_{G_n} \quad = \quad P_{n-1} X_i P_n : G_n \to G_{n-1}, \tag{2.9}$$

$$a^0(i)|_{G_n} \quad = \quad P_n X_i P_n : G_n \to G_n, \tag{2.10}$$

and

$$a^+(i)|_{G_n} \quad = \quad P_{n+1} X_i P_n : G_n \to G_{n+1}. \tag{2.11}$$

We call $a^-(i)$, $a^0(i)$, and $a^+(i)$ the *annihilation*, *preservation* (*neutral*), and *creation* operators of index i, respectively. We can now rewrite the formula (2.5) as:

$$X_i \quad = \quad a^-(i) + a^0(i) + a^+(i), \tag{2.12}$$

for all $i \in \{1, 2, \ldots, d\}$. The domain of the operators X_i, $a^-(i)$, $a^0(i)$, and $a^+(i)$, involved in formula (2.12), is considered to be the space F.

For any two linear operators A and B densely defined on the same Hilbert space H, we define the commutator $[A, B]$, as:

$$[A, B] \quad := \quad AB - BA.$$

It is clear that, if K is a subspace of H, such that K is contained in both domains of A and B, $AK \subset K$, and $BK \subset K$, then K is contained in the domain of the commutator $[A, B]$.

Since $F_n = G_0 \oplus G_1 \oplus \cdots \oplus G_n$, using (2.9), (2.10), and (2.11), we conclude that the space F_n is invariant under the action of the operators $a^0(i)$ and $[a^-(j), a^+(k)]$, i.e., $a^0(i)F_n \subset F_n$ and $[a^-(j), a^+(k)]F_n \subset F_n$, for all $n \geq 0$ and all $i, j, k \in \{1, 2, \ldots, d\}$. We denote by $a^0(i)|_{F_n}$ and $[a^-(j), a^+(k)]|_{F_n}$ the restrictions of these operators to the finite dimensional space F_n.

3 Probability measures with square summable support

In this section, we will present the main result of this paper.

Definition 3.1 *A probability measure μ on $\mathbb{R}^d$ is said to have a* square summable support *if*

$$\mu = \sum_{n=1}^{\infty} p_n \delta_{x^{(n)}}, \tag{3.13}$$

for some sequence $\{p_n\}_{n \geq 1}$, of non-negative real numbers, such that

$$\sum_{n=1}^{\infty} p_n = 1,$$

and some sequence $\{x^{(n)}\}_{n\geq 1}$, *of vectors in* $\mathbb{R}^d$, *such that*

$$\sum_{n=1}^{\infty} |x^{(n)}|^2 < \infty, \tag{3.14}$$

where $|\cdot|$ *denotes the euclidian norm of* $\mathbb{R}^d$ *and* δ_x *the Dirac delta measure at* x, *for any point* x *in* $\mathbb{R}^d$.

The following lemma will be useful in proving the main result of the paper.

Lemma 3.2 *For any* $i \in \{1, 2, \ldots, d\}$, *and any* $n \geq 0$,

$$Tr([a^-(i), a^+(i)]|_{F_n}) \ = \| \ a^+(i)|_{G_n} \ \|_{HS}^2 = \| \ a^-(i)|_{G_{n+1}} \ \|_{HS}^2, \tag{3.15}$$

where $Tr([a^-(i), a^+(i)]|_{F_n})$ *denotes the trace of the restriction of* $[a^-(i), a^+(i)]$ *to the space* F_n, *and* $\| \ a^+(i)|_{G_n} \ \|_{HS}$ *and* $\| \ a^-(i)|_{G_{n+1}} \ \|_{HS}$ *the Hilbert-Schmidt norms of the restrictions of* $a^+(i)$ *to* G_n *and* $a^-(i)$ *to* G_{n+1}, *respectively.*

Proof. Let $i \in \{1, 2, \ldots, d\}$ and $n \geq 0$ be fixed. For all $k \geq 0$, let $\{e_u^{(k)}\}_{1\leq u\leq r_k}$, be an orthonormal basis of the space G_k. For all $1 \leq u \leq r_k$, since $e_u^{(k)} \in G_k$, we have:

$$\begin{aligned} a^+(i)e_u^{(k)} &= P_{k+1}X_i e_u^{(k)} \\ &= \sum_{v=1}^{r_{k+1}} \langle X_i e_u^{(k)}, e_v^{(k+1)}\rangle e_v^{(k+1)} \end{aligned}$$

and

$$\begin{aligned} a^-(i)e_u^{(k)} &= P_{k-1}X_i e_u^{(k)} \\ &= \sum_{w=1}^{r_{k-1}} \langle X_i e_u^{(k)}, e_w^{(k-1)}\rangle e_w^{(k-1)}. \end{aligned}$$

Thus, for all $k \geq 0$, we have:

$$\begin{aligned} & \sum_{u=1}^{r_k} \langle [a^-(i), a^+(i)]e_u^{(k)}, e_u^{(k)}\rangle \\ = & \sum_{u=1}^{r_k} \langle a^-(i)a^+(i)e_u^{(k)}, e_u^{(k)}\rangle - \sum_{u=1}^{r_k} \langle a^+(i)a^-(i)e_u^{(k)}, e_u^{(k)}\rangle \\ = & \sum_{u=1}^{r_k}\sum_{v=1}^{r_{k+1}} \langle X_i e_u^{(k)}, e_v^{(k+1)}\rangle \langle a^-(i)e_v^{(k+1)}, e_u^{(k)}\rangle \\ - & \sum_{u=1}^{r_k}\sum_{w=1}^{r_{k-1}} \langle X_i e_u^{(k)}, e_w^{(k-1)}\rangle \langle a^+(i)e_w^{(k-1)}, e_u^{(k)}\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u=1}^{r_k}\sum_{v=1}^{r_{k+1}} \langle X_i e_u^{(k)}, e_v^{(k+1)}\rangle \langle X_i e_v^{(k+1)}, e_u^{(k)}\rangle \\
&- \sum_{u=1}^{r_k}\sum_{w=1}^{r_{k-1}} \langle X_i e_u^{(k)}, e_w^{(k-1)}\rangle \langle X_i e_w^{(k-1)}, e_u^{(k)}\rangle \\
&= \sum_{u=1}^{r_k}\sum_{v=1}^{r_{k+1}} \langle X_i e_u^{(k)}, e_v^{(k+1)}\rangle \langle e_v^{(k+1)}, X_i e_u^{(k)}\rangle \\
&- \sum_{u=1}^{r_k}\sum_{w=1}^{r_{k-1}} \langle e_u^{(k)}, X_i e_w^{(k-1)}\rangle \langle X_i e_w^{(k-1)}, e_u^{(k)}\rangle \\
&= \sum_{u=1}^{r_k}\sum_{v=1}^{r_{k+1}} \mid \langle X_i e_u^{(k)}, e_v^{(k+1)}\rangle \mid^2 - \sum_{w=1}^{r_{k-1}}\sum_{u=1}^{r_k} \mid \langle X_i e_w^{(k-1)}, e_u^{(k)}\rangle \mid^2 . \quad (3.16)
\end{aligned}
$$

Summing in formula (3.16), from $k = 0$ to $k = n$, and using the fact that, for $k = 0$, $\sum_{w=1}^{r_{k-1}}\sum_{u=1}^{r_k} \mid \langle X_i e_w^{(k-1)}, e_u^{(k)}\rangle \mid^2 = 0$ (since $G_{-1} = \{0\}$), we obtain:

$$
\begin{aligned}
Tr([a^-(i), a^+(i)]|_{F_n}) &= \sum_{k=0}^{n}\sum_{u=1}^{r_k} \langle [a^-(i), a^+(i)] e_u^{(k)}, e_u^{(k)}\rangle \\
&= \sum_{u=1}^{r_n}\sum_{v=1}^{r_{n+1}} \mid \langle X_i e_u^{(n)}, e_v^{(n+1)}\rangle \mid^2 \qquad (3.17) \\
&= \sum_{u=1}^{r_n}\sum_{v=1}^{r_{n+1}} \mid \langle a^+(i) e_u^{(n)}, e_v^{(n+1)}\rangle \mid^2 \\
&= \sum_{u=1}^{r_n} \parallel a^+(i) e_u^{(n)} \parallel^2 \\
&= \parallel a^+(i)|_{G_n} \parallel_{HS}^2 .
\end{aligned}
$$

It follows also from (3.17) that:

$$
\begin{aligned}
Tr([a^-(i), a^+(i)]|_{F_n}) &= \sum_{u=1}^{r_n}\sum_{v=1}^{r_{n+1}} \mid \langle X_i e_u^{(n)}, e_v^{(n+1)}\rangle \mid^2 \\
&= \sum_{v=1}^{r_{n+1}}\sum_{u=1}^{r_n} \mid \langle X_i e_v^{(n+1)}, e_u^{(n)}\rangle \mid^2 \\
&= \sum_{v=1}^{r_{n+1}}\sum_{u=1}^{r_n} \mid \langle a^-(i) e_v^{(n+1)}, e_u^{(n)}\rangle \mid^2 \\
&= \sum_{v=1}^{r_{n+1}} \parallel a^-(i) e_v^{(n+1)} \parallel^2 \\
&= \parallel a^-(i)|_{G_{n+1}} \parallel_{HS}^2 .
\end{aligned}
$$

Hence the lemma is proved. □

The following theorem characterizes the probability measures, with a square summable support, in terms of their preservation and commutators between the annihilation and creation operators.

Theorem 3.3 *A probability measure μ on $\mathbb{R}^d$ has a square summable support if and only if it has finite moments of any order and, for all $i \in \{1, 2, \ldots, d\}$, the sequence $\{Tr((a^0(i)|_{F_n})^2)\}_{n\geq 0}$ is bounded and*

$$\sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}) < \infty. \tag{3.18}$$

Proof. Part 1: Necessity

Let us assume that μ has a square summable support. Then

$$\mu = \sum_{n=1}^{\infty} p_n \delta_{x^{(n)}},$$

with $\sum_{n=1}^{\infty} |x^{(n)}|^2 < \infty$.

Let $R^2 := \sum_{n=1}^{\infty} |x^{(n)}|^2 < \infty$. It is clear that μ is a discrete measure with compact support contained in the ball $B[0, R] := \{x \in \mathbb{R}^d \mid |x| \leq R\}$. Since μ has compact support, it has finite moments of any order. From the compactness of the support of μ it also follows that the space F, of all polynomial functions of d variables: x_1, x_2, …, x_d, is dense in $L^2(\mathbb{R}^d, \mu)$. Thus $\mathcal{H} = \oplus_{n=0}^{\infty} G_n = L^2(\mathbb{R}^d, \mu)$. Moreover, for all $i \in \{1, 2, \ldots, d\}$, the operator X_i, of multiplication by the variable x_i, is a bounded operator from $L^2(\mathbb{R}^d, \mu)$ to $L^2(\mathbb{R}^d, \mu)$.

Since $\mu = \sum_{n=1}^{\infty} p_n \delta_{x^{(n)}}$, $\{e_n\}_{n\geq 1}$ is an orthonormal basis for $L^2(\mathbb{R}^d, \mu)$, where $e_n := \frac{1}{\sqrt{p_n}} 1_{\{x^{(n)}\}}$, for all $n \geq 1$, such that $p_n > 0$ (it is possible that the measure μ has a finite support, in which case, all the p_n's are zero, except finitely many of them). For all $n \geq 1$ and $i \in \{1, 2, \ldots, d\}$, we denote the i^{th} component of the vector $x^{(n)}$ by $x_i^{(n)}$. We also denote the norm of the space $L^2(\mathbb{R}^d, \mu)$ by $\|\cdot\|$. For all $i \in \{1, 2, \ldots, d\}$, we have:

$$\begin{aligned}
\|X_i\|_{HS}^2 &= \sum_{n\geq 1} \|X_i e_n\|^2 \\
&= \sum_{n\geq 1} \|x_i^{(n)} e_n\|^2 \\
&= \sum_{n\geq 1} (x_i^{(n)})^2 \|e_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\geq 1}(x_i^{(n)})^2\\
&\leq \sum_{n\geq 1}|x^{(n)}|^2\\
&< \infty.
\end{aligned}
$$

Thus X_i is a Hilbert-Schmidt operator, for all $i \in \{1, 2, \ldots, d\}$.

For each $n \geq 0$, let $\{e_u^{(n)}\}_{1\leq u\leq r_n}$ be an orthonormal basis for G_n. Let

$$
U = \{e_u^{(0)}\}_{1\leq u\leq r_0}\bigcup\{e_u^{(1)}\}_{1\leq u\leq r_1}\bigcup\{e_u^{(2)}\}_{1\leq u\leq r_2}\bigcup\cdots.
$$

Then U is an orthonormal basis for $\mathcal{H}$. For any $i \in \{1, 2, \ldots, d\}$, any $n \geq 0$, and any $u \in \{1, 2, \ldots, r_n\}$, since $a^+(i)e_u^{(n)} \in G_{n+1}$, $a^0(i)e_u^{(n)} \in G_n$, $a^-(i)e_u^{(n)} \in G_{n-1}$, and the spaces G_{n+1}, G_n, and G_{n-1} are orthogonal, we have:

$$
\begin{aligned}
&\|X_i\|_{HS}^2\\
= \;& \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\|X_ie_u^{(n)}\|^2\\
= \;& \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\|a^+(i)e_u^{(n)} + a^0(i)e_u^{(n)} + a^-(i)e_u^{(n)}\|^2\\
= \;& \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\left(\|a^+(i)e_u^{(n)}\|^2 + \|a^0(i)e_u^{(n)}\|^2 + \|a^-(i)e_u^{(n)}\|^2\right)\\
= \;& \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\|a^+(i)e_u^{(n)}\|^2 + \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\|a^0(i)e_u^{(n)}\|^2 + \sum_{n=0}^{\infty}\sum_{u=1}^{r_n}\|a^-(i)e_u^{(n)}\|^2\\
= \;& \|a^+(i)\|_{HS}^2 + \|a^0(i)\|_{HS}^2 + \|a^-(i)\|_{HS}^2.
\end{aligned}
$$

Since $\|X_i\|_{HS} < \infty$, we get $\|a^+(i)\|_{HS} < \infty$ and $\|a^0(i)\|_{HS} < \infty$, for all $i \in \{1, 2, \ldots, d\}$.

Now, let us observe that $a^0(i)|_{F_n}$ is self-adjoint with respect to inner product $\langle\cdot,\cdot\rangle$ of the space $L^2(\mathbb{R}^d, \mu)$, for all $n \geq 0$. Therefore, we can see that:

$$
\begin{aligned}
\|a^0(i)\|_{HS}^2 &= \sup_{n\geq 0}\|a^0(i)|_{F_n}\|_{HS}^2\\
&= \sup_{n\geq 0}\sum_{k=0}^{n}\sum_{u=1}^{r_k}\langle a^0(i)e_u^{(k)}, a^0(i)e_u^{(k)}\rangle\\
&= \sup_{n\geq 0}\sum_{k=0}^{n}\sum_{u=1}^{r_k}\left\langle (a^0(i))^2e_u^{(k)}, e_u^{(k)}\right\rangle\\
&= \sup_{n\geq 0} Tr\left((a^0(i)|_{F_n})^2\right).
\end{aligned}
$$

This implies that the sequence $\{Tr((a^0(i)|_{F_n})^2)\}_{n\geq 0}$ is bounded, for all $i \in \{1, 2, \ldots, d\}$.

On the other hand, from Lemma 3.2, we know that,

$$\|a^+(i)|_{G_n}\|_{HS}^2 = Tr([a^-(i), a^+(i)]|_{F_n}),$$

for all $i \in \{1, 2, \ldots, d\}$. Thus

$$\begin{aligned}\|a^+(i)\|_{HS}^2 &= \sum_{n=0}^{\infty} \|a^+(i)|_{G_n}\|_{HS}^2 \\ &= \sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}).\end{aligned}$$

Hence $\sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}) < \infty$, for all $i \in \{1, 2, \ldots, d\}$.

Part 2: Sufficiency

Let us suppose that μ is a probability measure on $\mathbb{R}^d$, with finite moments of any order, such that, for all $i \in \{1, 2, \ldots, d\}$,

$$\sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}) < \infty$$

and the sequence $\{Tr((a^0(i)|_{F_n})^2)\}_{n\geq 0}$ is bounded.

We have seen before that

$$\|a^+(i)\|_{HS}^2 = \sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}).$$

It also follows from Lemma 3.2, that

$$\|a^-(i)\|_{HS}^2 = \sum_{n=0}^{\infty} Tr([a^-(i), a^+(i)]|_{F_n}).$$

Thus $a^+(i)$ and $a^-(i)$ are Hilbert-Schmidt operators from the Hilbert space $\mathcal{H}$ to itself, for all $i \in \{1, 2, \ldots, d\}$.

We have also seen before that the fact that the sequence

$$\left\{Tr\left((a^0(i)|_{F_n})^2\right)\right\}_{n\geq 0}$$

is bounded is equivalent to the fact that $a^0(i)$ is a Hilbert-Schmidt operator from $\mathcal{H}$ to $\mathcal{H}$. Thus, it follows, as before, that

$$\|X_i\|_{HS}^2 = \|a^+(i)\|_{HS}^2 + \|a^0(i)\|_{HS}^2 + \|a^-(i)\|_{HS}^2 < \infty.$$

Hence the multiplication operator X_i is a Hilbert-Schmidt operator from $\mathcal{H}$ to $\mathcal{H}$, for all $i \in \{1, 2, \ldots, d\}$. Being a Hilbert-Schmidt operator, X_i is also a bounded operator on $\mathcal{H}$, for all $i \in \{1, 2, \ldots, d\}$. Let $R_i := \|X_i\|_{\mathcal{H},\mathcal{H}}$ be the operator norm of X_i on $\mathcal{H}$. Hence for any polynomial function g of d variables, we have $\|X_i g\| \leq R_i \|g\|$. We denote by $E[\cdot]$ the expectation with respect to μ.

Let $\epsilon > 0$ be fixed and let $B_i = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid |x_i| \geq R_i + \epsilon\}$. Then for all $n \geq 1$,

$$\begin{aligned}
(R_i + \epsilon)^{2n} \mu(B_i) &\leq E[x_i^{2n} 1_{B_i}] \\
&\leq E[x_i^{2n} 1] \\
&= \|X_i^n 1\|^2 \\
&\leq \left(\|X_i^n\|_{\mathcal{H},\mathcal{H}} \cdot \|1\|\right)^2 \\
&\leq \left(\|X_i\|^n_{\mathcal{H},\mathcal{H}} \cdot 1\right)^2 \\
&= R_i^{2n}.
\end{aligned}$$

Thus we obtain $\mu(B_i) \leq R_i^{2n}/(R_i + \epsilon)^{2n}$, for all $n \geq 1$, and letting $n \to \infty$, we conclude that $\mu(B_i) = 0$, for all $\epsilon > 0$. Hence the support of μ is contained in the set

$$C_i := \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid |x_i| \leq R_i\},$$

for all $i \in \{1, 2, \ldots, d\}$. Therefore, μ has compact support contained in the set $\cap_{i=1}^d C_i$. Since μ has compact support, the space F of all polynomial functions is dense in $L^2(\mathbb{R}^d, \mu)$ and thus $\mathcal{H} = L^2(\mathbb{R}^d, \mu)$. Therefore, X_i is Hilbert-Schmidt and, in particular, bounded from $L^2(\mathbb{R}^d, \mu)$ into $L^2(\mathbb{R}^d, \mu)$. X_i is also a self-adjoint operator on $L^2(\mathbb{R}^d, \mu)$, for all $i \in \{1, 2, \ldots, d\}$.

From the general form of the self-adjoint Hilbert-Schmidt operators on a Hilbert space, we know that the spectrum of X_i is discrete and coincides with the point spectrum. That means, for all $i \in \{1, 2, \ldots, d\}$, there exist a sequence of real numbers $\{\lambda_n^{(i)}\}_{n \geq 1}$ and an orthonormal basis $\{f_n^{(i)}\}_{n \geq 1}$ for $L^2(\mathbb{R}^d, \mu)$, such that, for all $h \in L^2(\mathbb{R}^d, \mu)$,

$$X_i h = \sum_{n=1}^{\infty} \lambda_n^{(i)} \langle h, f_n^{(i)} \rangle f_n^{(i)}. \tag{3.19}$$

Moreover,

$$\sum_{n=1}^{\infty} (\lambda_n^{(i)})^2 = \|X_i\|_{HS}^2 < \infty. \tag{3.20}$$

For all $n \geq 1$, we have $X_i f_n^{(i)} = \lambda_n^{(i)} f_n^{(i)}$. This means $(x_i - \lambda_n^{(i)}) f_n^{(i)} = 0$, μ-a.s.. Since $\|f_n^{(i)}\| = 1$, we know that $f_n^{(i)}$ cannot be equal to zero μ-a.s.. Thus the hyperplane $\pi_n^{(i)} := \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^d \mid x_i = \lambda_i^{(n)}\}$ has a positive probability, i.e., $\mu(\pi_n^{(i)}) > 0$. On the complement of this hyperplane $f_n^{(i)}(x) = 0$, μ-a.s.. This means that $f_n^{(i)} 1_{\left(\pi_n^{(i)}\right)^c} = 0$, μ-a.s., where $1_{\left(\pi_n^{(i)}\right)^c}$ denotes the characteristic function of the complement of $\pi_n^{(i)}$. Let $g_n^{(i)} := f_n^{(i)} 1_{\pi_n^{(i)}}$. Then

$$\begin{aligned} f_n^{(i)} &= f_n^{(i)} 1_{\pi_n^{(i)}} + f_n^{(i)} 1_{\left(\pi_n^{(i)}\right)^c} \\ &= g_n^{(i)} + 0 \\ &= g_n^{(i)}, \quad \mu - \text{a.s..} \end{aligned}$$

Thus, we can replace the orthonormal basis $\{f_n^{(i)}\}$ by $\{g_n^{(i)}\}$, in Equation (3.19), to obtain the equality:

$$X_i h = \sum_{n=1}^{\infty} \lambda_n^{(i)} \langle h, g_n^{(i)} \rangle g_n^{(i)},$$

for all $h \in L^2(\mathbb{R}^d, \mu)$, where, for all $n \geq 1$, the support of $g_n^{(i)}$ is contained in the hyperplane $\pi_n^{(i)}$. Since $\{g_n^{(i)}\}_{n \geq 1}$ is an orthonormal basis for $L^2(\mathbb{R}^d, \mu)$, we have:

$$\begin{aligned} \mu\left(\left[\bigcup_{n=1}^{\infty} \pi_n^{(i)}\right]^c\right) &= \|1_{\left[\cup_{n=1}^{\infty} \pi_n^{(i)}\right]^c}\|^2 \\ &= \sum_{n=1}^{\infty} \left\langle 1_{\left[\cup_{n=1}^{\infty} \pi_n^{(i)}\right]^c}, g_n^{(i)} \right\rangle^2 \\ &= 0. \end{aligned}$$

Hence, for all $i \in \{1, 2, \ldots, d\}$, the support of μ is contained in the union of the hyperplanes $\pi_n^{(i)}$, for $n \geq 1$.

If λ is an eigenvalue of X_i, and $\lambda \neq 0$, then the eigenspace corresponding to λ is finite dimensional, because of the condition $\sum_{n=1}^{\infty} (\lambda_n^{(i)})^2 < \infty$. That means, if $\lambda \neq 0$, then the set $\{n \in \mathbb{N} \mid \lambda_n^{(i)} = \lambda\}$ is finite.

Let $i \in \{1,\ 2,\ \ldots,\ d\}$ and $\lambda = \lambda_n^{(i)} \neq 0$, for some $n \geq 0$, be fixed. If k denotes the multiplicity of λ, as an eigenvalue of X_i, we conclude that, for any sequence $\{B_l\}_{l \geq 1}$, of disjoint Borel subsets of the hyperplane $\pi := \{(x_1, x_2, \ldots, x_d) \mid x_i = \lambda\}$, there are at most k sets B_{l_1}, B_{l_2}, $\ldots$, such that $\mu(B_{l_1}) > 0$, $\mu(B_{l_2}) > 0$, $\ldots$, since the characteristic functions $1_{B_{l_1}}$, $1_{B_{l_2}}$, $\ldots$ are non-zero orthogonal eigenvectors of the multiplication operator X_i, corresponding to the same eigenvalue λ.

For all $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the family of cubes, of π, of the form

$$K_{n,r} = \pi \cap \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d \mid \frac{r_1}{2^n} \le x_1 < \frac{r_1+1}{2^n}, \ldots, \frac{r_d}{2^n} \le x_d < \frac{r_d+1}{2^d} \right\},$$

where $r = (r_1, \ldots, r_d) \in \mathbb{Z}^d$. It is clear that for all $r \neq s$, $K_{n,r} \cap K_{n,s} = \emptyset$. Since $\{K_{n,r}\}_{r \in \mathbb{Z}^d}$ is a partition of π composed of mutually disjoint Borel subsets, we conclude that at most k of the sets $\{K_{n,r}\}_{r \in \mathbb{Z}^d}$ have a positive probability measure μ. For all $n \in \mathbb{N}$, let t_n be the cardinality of the set $A_n := \{r \in \mathbb{Z}^d \mid \mu(K_{n,r}) > 0\}$. Then, for each $n \in \mathbb{N}$, t_n is a natural number less than or equal to k. Let us observe that, since each cube $K_{n,r}$, from $\mathcal{C}_n$, can be written as a finite union of cubes $K_{n+1,s}$, from $\mathcal{C}_{n+1}$, for each $r \in A_n$, there exists at least one cube $K_{n+1,s_r} \in \mathcal{C}_{n+1}$, such that $K_{n+1,s_r} \subset K_{n,r}$ and $\mu(K_{n+1,s_r}) > 0$. Thus $s_r \in A_{n+1}$. For each $r \in A_n$, we choose one s_r and fix it. If $r_1, r_2 \in A_n$, such that $r_1 \neq r_2$, we have $K_{n,r_1} \cap K_{n,r_2} = \emptyset$, and since $K_{n+1,s_{r_1}} \subset K_{n,r_1}$ and $K_{n+1,s_{r_2}} \subset K_{n,r_2}$, we conclude that $K_{n+1,s_{r_1}} \cap K_{n+1,s_{r_2}} = \emptyset$. Thus $s_{r_1} \neq s_{r_2}$ and so, the mapping $r \mapsto s_r$ is a one-to-one function from A_n to A_{n+1}. Hence the cardinality of A_n does not exceed the cardinality of A_{n+1}, or equivalently $t_n \le t_{n+1}$, for all $n \in \mathbb{N}$. Therefore, $t_1 \le t_2 \le t_3 \le \cdots \le k$. Since $\{t_n\}_{n \ge 1}$ is a bounded non-decreasing sequence of natural numbers, we conclude that it must be stationary, i.e., there exists $n_0 \in \mathbb{N}$, such that $t_{n_0} = t_{n_0+1} = t_{n_0+2} = \cdots$. From the fact that, for each $n \ge n_0$, $t_n = t_{n+1}$, it follows that, for each $r \in A_n$, there exists a unique $s_r \in A_{n+1}$, such that $K_{n+1,s_r} \subset K_{n,r}$. This uniqueness property implies that $\mu(K_{n+1,s_r}) = \mu(K_{n,r})$. Let $A_{n_0} = \{r_1, r_2, \ldots, r_{t_{n_0}}\}$. For any $j \in \{1, 2, \ldots, t_{n_0}\}$, we can construct a decreasing sequence of cubes $\{K_j^{(n)}\}_{n \ge n_0}$, in the following way: $K_j^{(n_0)} := K_{n_0,r_j}$, $K_j^{(n_0+1)}$ is the unique cube from $\mathcal{C}_{n_0+1}$, that is contained is K_{n_0,r_j} and has a positive probability measure μ, $K_j^{(n_0+2)}$ is the unique cube from $\mathcal{C}_{n_0+2}$ that is contained in $K_j^{(n_0+1)}$ and has a positive probability measure μ, and so on. Thus, we obtain a decreasing sequence of cubes: $K_j^{(n_0)} \supset K_j^{(n_0+1)} \supset K_j^{(n_0+2)} \supset \cdots$ such that $\mu(K_j^{(n_0)}) = \mu(K_j^{(n_0+1)}) = \mu(K_j^{(n_0+2)}) = \cdots > 0$. Since the diameter of the cube $K_j^{(n)}$ (i.e., the supremum of the distances between any two points of the cube) tends to 0, as $n \to \infty$, we know that the intersection of all these cubes is either the empty set or a set that contains only one point. By the monotone convergence theorem, we have: $\mu(\cap_{n \ge n_0} K_j^{(n)}) = \lim_{n \to \infty} \mu(K_j^{(n)}) = \mu(K_j^{(n_0)}) > 0$. Thus $\cap_{n \ge n_0} K_j^{(n)} \neq \emptyset$. Consequently, for all $j \in \{1, 2, \ldots, t_{n_0}\}$, there exists $x^{(j)} \in \pi$, such that $\cap_{n \ge n_0} K_j^{(n)} = \{x^{(j)}\}$ and $\mu(\{x^{(j)}\}) = \mu(K_j^{(n_0)}) > 0$. Hence, we have:

$$\begin{aligned} \mu(\pi) &= \mu(\cup_{r \in \mathbb{Z}^d} K_{n_0,r}) \\ &= \sum_{j=1}^{t_{n_0}} \mu(K_j^{(n_0)}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{t_{n_0}} \mu(\{x^{(j)}\}) \\
&= \mu(\{x^{(1)}, x^{(2)}, \ldots, x^{(t_{n_0})}\}).
\end{aligned}
$$

This implies that the restriction of the probability measure μ to the Borel subsets of the hyperplane π is a finite combination of Dirac delta measures. Therefore, for each $\lambda_n^{(i)} \neq 0$, there exist finitely many points $y_{1,n}^{(i)}$, $y_{2,n}^{(i)}$, $\ldots$, $y_{s_{n,i},n}^{(i)}$ in $\pi_n^{(i)}$, such that, for any Borel subset C of $\pi_n^{(i)}$,

$$
\mu(C) = \sum_{u=1}^{s_{n,i}} p_{u,n}^{(i)} \delta_{y_{u,n}^{(i)}}(C), \tag{3.21}
$$

where $p_{u,n}^{(i)} := \mu(\{y_{u,n}^{(i)}\}) > 0$, for all $u \in \{1, 2, \ldots, s_{n,i}\}$. The number of these points, $s_{n,i}$, coincides with the multiplicity of the eigenvalue $\lambda_n^{(i)}$. Hence

$$
\begin{aligned}
\|X_i\|_{HS}^2 &= \sum_{n=1}^{\infty} (\lambda_n^{(i)})^2 \\
&= \xi_i,
\end{aligned}
$$

where ξ_i denotes the sum of the squares of the i^{th} coordinates of $y_{u,n}^{(i)}$, for $n \geq 1$ and $1 \leq u \leq s_{n,i}$. The only eigenvalue of X_i that might have an infinite dimensional eigenspace is $\lambda = 0$, eventually. Thus, at this moment, we do not know the behavior of the probability measure μ on the Borel subsets of the hyperplane $\{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$. We may call such a hyperplane a "bad" hyperplane. We should not forget though, that our conclusion, regarding the fact that μ is a finite combination of delta measures, on each hyperplane of equation $x_i = \lambda$, for $\lambda \neq 0$, is true for all $i \in \{1, 2, \ldots, d\}$. This means that we know the behavior of μ everywhere, except on the intersection of all the bad hyperplanes. Fortunately, we have

$$
\bigcap_{i=1}^{d} \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \,|\, x_i = 0\} = \{(0, 0, \ldots, 0)\}.
$$

Hence besides the set

$$
D := \bigcup_{i=1}^{d} \bigcup_{n=1}^{\infty} \bigcup_{u=1}^{s_{n,i}} \{y_{u,n}^{(i)}\}, \tag{3.22}
$$

the support of μ might contain eventually only one more point, namely 0, the zero vector of $\mathbb{R}^d$. There are many repetitions among the singleton sets $\{y_{u,n}^{(i)}\}$, that participate in the unions from the right-hand side of (3.22). For example, if a point $y_{u,n}^{(i)}$ has all the coordinates different from zero, then $1_{\{y_{u,n}^{(i)}\}}$ is a non-zero eigenvector, corresponding to a non-zero eigenvalue, for each of the multiplication operators X_j, $1 \leq j \leq d$. However, if a point $y_{u,n}^{(i)}$ is different from all the points $y_{v,m}^{(j)}$, for a fixed j and all values of m and v, then the j^{th} coordinate of $y_{u,n}^{(i)}$ is zero. Thus, when we compute the sum of the squares of the j^{th} coordinates of all the points from the support of μ, the point $y_{u,n}^{(i)}$ does not contribute with anything. This fact is very important in proving the square summability of the support of μ. Let us rewrite the set $\cup_{i=1}^{d} \cup_{n=1}^{\infty} \cup_{u=1}^{s_{n,i}} \{y_{u,n}^{(i)}\}$ as $\{x^{(n)}\}_{n \geq 1}$, where $x^{(k)} \neq x^{(l)}$, for all $k \neq l$, and N could be a finite positive integer or ∞. Then,

$$\mu \quad = \quad p_0\delta_0 + \sum_{n=1}^{N} p_n \delta_{x^{(n)}}, \tag{3.23}$$

where, for all $n \geq 0$, $p_n \geq 0$ (if 0 is not in the spectrum of μ, then $p_0 = 0$), and $\sum_{n=0}^{N} p_n = 1$. Thus, we have:

$$\begin{aligned} \sum_{n=1}^{N} |x^{(n)}|^2 &= \sum_{i=1}^{d} \sum_{n=1}^{N} |x_i^{(n)}|^2 \\ &= \sum_{i=1}^{d} \xi_i \\ &= \sum_{i=1}^{d} \|X_i\|_{HS}^2 \\ &< \infty. \end{aligned}$$

This proves that μ has a square summable support. □

If $d = 1$ and V_n denotes the space of all polynomial functions, of one real variable, with complex coefficients, of degree at most n, then, since the algebraic codimension V_n into V_{n+1} is 1, we conclude that the dimension of G_n is at most 1, for all $n \geq 0$. In fact the dimension of G_n is equal to 1, for all $n \geq 0$, if and only if the support of the measure μ is an infinite set, in which case $F_n = V_n$, for all $n \geq 0$ (we should remember that F_n is the space V_n factorized to the equivalence relation given by the $\mu-$almost sure equality). In that case, since the dimension of G_n is 1, there exists a unique polynomial f_n in G_n that has the leading coefficient equal to 1, for all $n \geq 0$. Since we have only one multiplication operator X_1, one annihilation operator $a^+(1)$, one preservation operator $a^0(1)$, and one annihilation operator $a^-(1)$, we can denote them simply by X, a^+, a^0, and a^-, respectively. Also, sice $f_n \in G_n$ and $a^- : G_n \to G_{n-1}$, there exists a unique real number ω_n, such that $a^- f_n = \omega_n f_{n-1}$, for all $n \geq 1$ (for $n = 0$, sice $G_{-1} = \{0\}$, we can define $\omega_0 := 0$ and $f_{-1} := 0$). Similarly, there exists a unique real number α_n, such that $a^0 f_n = \alpha_n f_n$, for all $n \geq 0$. Since both f_{n+1} and $X f_n$ have the leading coefficient equal to 1, we conclude that $a^+ f_n = f_{n+1}$, for all $n \geq 0$. Thus, since $X = a^+ + a^0 + a^-$, we obtain that, for all $n \geq 0$,

$$X f_n = f_{n+1} + \alpha_n f_n + \omega_n f_{n-1}. \tag{3.24}$$

The sequences $\{\alpha_n\}_{n \geq 0}$ and $\{\omega_n\}_{n \geq 1}$, are called the *Szegö-Jacobi parameters* of μ. It is easy to see that $[a^-, a^+] f_k = (\omega_{k+1} - \omega_k) f_k$, for all $k \geq 0$, and thus since $\omega_0 = 0$, if one considers the algebraic base $\{f_k\}_{0 \leq k \leq n}$ (or the normalized orthogonal base $\{(1/ \parallel f_k \parallel) f_k\}_{0 \leq k \leq n}$) of F_n, then

$$\begin{aligned} Tr([a^-, a^+]|_{F_n}) &= \sum_{k=0}^{n} (\omega_{k+1} - \omega_k) \\ &= \omega_{n+1}, \end{aligned}$$

for all $n \geq 0$. Similarly, since $(a^0)^2 f_k = \alpha_k^2 f_k$, for all $k \geq 0$, we conclude that

$$Tr((a^0|_{F_n})^2) = \sum_{k=0}^{n} \alpha_k^2,$$

for all $n \geq 0$. If the support of μ is a finite set, then we can still make sense of the formula (3.24), by defining $f_n := 0$, $\alpha_n := 0$, and $\omega_n := 0$, for n large enough. Thus, from Theorem 3.3, we obtain the following corollary:

Corollary 3.4 *Let μ be a probability measure on $\mathbb{R}$ having finite moments of any order. Then μ has a square summable support if and only if both series $\sum_{n=0}^{\infty} \alpha_n^2$ and $\sum_{n=1}^{\infty} \omega_n$ are convergent, where $\{\alpha_n\}_{n \geq 0}$ and $\{\omega_n\}_{n \geq 1}$ denote the Szegö-Jacobi parameters of μ.*

Acknowledgements. Part of this work was done during the visit of H.-H. Kuo to the Vito Volterra Center from April 1 to July 31, 2005. He is grateful for the financial support, from the "Commissione Per Gli Scambi Culturali Fra L'Italia e gli Stati Uniti" (Italian Fulbright Commission), for a Fulbright Lecturing grant.

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