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A Quantum Distance for Noncommutative Measure Spaces and an Application to Quantum Field Theory

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Abstract

In the first part of this dissertation, we study a pointed version of Rieffel's quantum Gromov–Hausdorff topology for compact quantum metric spaces (i.e, order–unit spaces with a Lipschitz–like seminorm inducing a distance on the space of positive normalized linear functionals which metrizes the w^*–topology). In particular, in analogy with Gromov's notion of metric tangent cone at a point of an (abstract) proper metric space, we propose a similar construction for (compact) quantum metric spaces, based on a suitable procedure of rescaling the Lipschitz seminorm on a given quantum metric space. As a result, we get a quantum analogue of the Gromov tangent cone, which extends the classical (say, commutative) construction. As a case study, we apply this procedure to the two–dimensional noncommutative torus, and we obtain what we call a noncommutative solenoid.

In the second part, we introduce a quantum distance on the set of dual Lip–von Neumann algebras (i.e., vN algebras with a dual Lip–norm which metrizes the w^*–topology on bounded subset). As for the other GH distances (classical or quantum), this dual quantum Gromov–Hausdorff (pseudo–)distance turns out to be a true distance on the (Lip–)isometry classes of Lip–vN algebras. We give also a precompactness criterion, relating the limit of a (strongly) uniform sequence of Lip–vN algebras to the (restricted) ultraproduct, over an ultrafilter, of the same sequence. As an application, we apply this construction to the study of the Buchholz–Verch scaling limit theory of a local net of (algebras of) observables in the algebraic quantum field theory framework, showing that the two approaches lead to the same result for the (real scalar) free field model.

Nella prima parte della Tesi, presenteremo una versione "puntata" della topologia di Gromov–Hausdorff quantistica introdotta da Rieffel per spazi metrici quantistici compatti (cioè, spazi con unità d'ordine e una seminorma Lipschitz che metrizza la topologia *–debole sullo spazio dei funzionali positivi normalizzati). In particolare, proporremo una nozione di cono tangente quantistico di uno spazio metrico quantistico, come analogo noncommutativo del cono tangente di Gromov in un punto di uno spazio metrico ordinario, basata su una opportuna procedura di riscalamento della seminorma Lipschitz definita su uno spazio metrico quantistico. Tale costruzione estende effettivamente la corrispondente costruzione valida per spazi metrici ordinari. Infine, a titolo di esempio, descriveremo il cono tangente quantistico del toro noncommutativo bidimensionale.

Nella seconda parte, invece, introdurremo una particolare distanza quantistica sull'insieme delle algebre di von Neumann Lip–normate (cioè, dotate di una ulteriore norma che metrizza la topologia debole sui sottoinsiemi limitati nella norma C^*). Come avviene per le distanze *a la* Gromov–Hausdorff, anche questa distanza GH duale è una pseudo–distanza, e diviene una

vera distanza solo sulle classi di equivalenza isometrica (rispetto alla norma Lip) delle algebre di von Neumann Lip–normate. Inoltre, dimostreremo un criterio di precompatteza per famiglie di algebre di vN Lip–normate (fortemente) uniformemente limitate, utilizzando la nozione di ultraprodotto (ristretto) di algebre di vN Lip–normate. Infine, nell'ambito del'approccio algebrico alla teoria quantistica dei campi, applicheremo tale costruzione allo studio del limite di scala (cioè, quando si fanno tendere a un punto le regioni dello spaziotempo su cui sono definiti gli osservabili della teoria) di una rete locale di algebre di vN (le algebre degli osservabili), confrontando l'approccio tramite ultraprodotti (e con la convergenza nella distanza quantistica) con la costruzione delle algebre "limite di scala" di Buchholz e Verch, mostrando che nel caso del campo libero bosonico le due procedure forniscono lo stesso risultato.

Introduction

The main principle of *Noncommutative Geometry* is the duality between spaces and the functions over them. For specific classes of spaces (for example, algebraic varieties, locally compact Hausdorff spaces, manifolds), we study appropriate (usually, commutative algebras of) functions (algebraic, continuous, smooth). When this duality is generalized to noncommutative algebras of the same type, we think that these algebras represent "noncommutative spaces". Then, one can study the corresponding "noncommutative geometry", mainly using geometric ideas in the framework of noncommutative algebras, and sometimes revealing back deep facts about the ordinary spaces from the algebra level. Specifically, when we consider only the Borel structure of a topological space and the duality between (σ–finite) measure spaces and the algebras of $\mathbb{C}$–valued measurable functions over them, we look at abstract W^*–algebras (or, at concrete von Neumann algebras) as "noncommutative measure spaces". Instead, when one considers the topological structure of the space and the duality between locally compact Hausdorff spaces and the algebras of continuous $\mathbb{C}$–valued functions over them vanishing at infinity, which are identified with commutative C^*–algebras by the Gel'fand representation theorem [50], one studies general C^*–algebras as "noncommutative locally compact Hausdorff spaces". If one looks at spaces with a richer structure, say differentiable manifolds, one studies "noncommutative differentiable manifolds", and this is called "noncommutative differential geometry". If one goes further into Riemannian manifolds, one studies "noncommutative Riemannian manifolds", and the corresponding "noncommutative Riemannian geometry" is what A. Connes founded by using the concept of *spectral triples* [17; 18; 19]. The main ingredient of a spectral triple is a Dirac operator D. On the one hand, it captures the differential structure by setting $df := i[D, f]$. On the other hand, it enables us to recover the Lipschitz seminorm L, which is usually defined as

$$L(f) := \sup\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y\}, \tag{1}$$

where ρ is the geodesic metric on the Riemannian manifold, rather than by $L(f) = \|[D, f]\|$, and then one recovers the metric ρ by

$$\rho(x, y) = \sup\{|f(x) - f(y)| : L(f) \leq 1\}. \tag{2}$$

Following this method of identifying metrics with seminorms [17; 18], M. Rieffel introduced the notion of *compact quantum metric spaces* [59; 60; 63], identifying a noncommutative metric on a compact noncommutative space as a seminorm on the corresponding unital C^*–algebra, which has the property that the distance defined on the state space of the algebra via (2) induces the w^*–topology. Roughly speaking, this means that the noncommutative metric on the noncommutative space induces the given topology. Often it is quite hard to show that certain natural Dirac operators define noncommutative metrics [62].

Motivated by questions in string theory, Rieffel also introduced the notion of *quantum Gromov–Hausdorff distance* for compact quantum metric spaces [61; 63]. This is defined as a modified ordinary Gromov–Hausdorff distance for the state spaces. One of the benefits of this quantum distance is that there is also a quantum version of the *Gromov completeness and compactness theorems*, which asserts that the set of isometry classes endowed with the Gromov–Hausdorff distance is complete, and provides a criterion to say when a subset of this complete space is precompact. Also, this quantum distance extends the ordinary Gromov–Hausdorff distance in the sense that the map which sends each compact metric space to the associated quantum compact metric space is a homeomorphism (though not an isometry) from the space of isometry classes of compact metric spaces to a closed subspace of the space of isometry classes of quantum compact metric spaces. Since this construction does not involve the multiplicative structure of the algebras but only needs the state spaces, Rieffel set up everything on more general spaces, namely *order–unit spaces*, based on Kadison's representation theory [2; 40]. As a consequence, the quantum Gromov–Hausdorff distance can not distinguish the multiplicative structure of the algebras, i.e. non–isomorphic C^*–algebras might have distance zero.

Let us mention that there are also other approaches to noncommutative metrics. In [70; 71; 72], N. Weaver defined noncommutative metric spaces as von Neumann algebras with W^*–derivations. For many of the examples which he considers, the seminorms induced by the W^*—derivations do give the w^*–topology on the state spaces [60]. Moreover, D. Kerr [43] has defined a *matricial quantum distance*, which is able to distinguish the multiplicative structures of algebras, and there is also some kind of quantum complete and compactness theorem.

As Connes [17; 18; 19] and Rieffel showed us [59; 60], the natural way to specify a metric on a C^*–algebra $\mathcal{A}$ is by means of an analogue for the Lipschitz seminorm on functions, that is, a densely defined seminorm on $\mathcal{A}$ which induces, by duality, an ordinary metric on the state space $S(\mathcal{A})$ of the C^*–algebra. If this metric topologizes the w^*–topology on the state space, one gets a compact quantum metric space (cQMS, for brevity), and the related notions of quantum Gromov–Hausdorff distance and quantum Gromov–Hausdorff topology [63]. In the ordinary setting, however, one may also define a non–compact version of this topology, at least for pointed proper metric spaces, i.e. boundedly compact spaces with a basepoint, the so-called *pointed Gromov–Hausdorff distance* [30; 10; 14]. For example, the tangent space at a point of a (finite–dimensional) Riemannian manifold can be obtained as a pointed Gromov–Hausdorff limit of the sequence consisting of the same manifold endowed with suitably rescaled metrics [10; 14].

At this point, a natural question arises: is it possible to define a *non–compact* quantum metric space by means of a quantum version of the pointed Gromov–Hausdorff distance? The first problem that one faces is the lack of an appropriate notion of non–compactness for quantum metric spaces. In fact, the w^*–compactness of the state space of a C^*–algebra is due to the presence of a unit. So, one may be tempted to consider non–unital C^*–algebras, in which case w^*–compactness of the state space is lost. This was done indeed by Latrémolière in [45], where he introduced the notion of *bounded Lip(schitz)–(semi)norm*, in order to deal also with non–unital C^*–algebras. It turns out then that the metric, induced by a bounded Lip–norm on the state space of a non–unital C^*–algebra, is a complete metric, and the corresponding induced topology is again the w^*–topology. In this setting, the basic structure of a quantum metric space, that is, the order–unit space consisting of the self–adjoint part of the C^*–algebra, might be replaced by the notion of approximate order–unit space [75], as in part is suggested by Latrémolière's construction, though not explicitly used in [45]. On the other hand, one might try to extend Rieffel's notion of quantum Gromov–Hausdorff convergence, in analogy with the classical setting, where the pointed Gromov–Hausdorff topology extends the classical Gromov–Hausdorff topology to non necessarily compact spaces.

This is one of the objective we aimed at. A first difference with the classical situation is the absence, in the noncommutative setting, of a natural notion of points of a noncommutative metric space. So far, a reasonable candidate for the role of "noncommutative points" around which one may "blow up" the space, is what we call a "shrinking family of subspaces" (see Section 1.4.2), that is, a family of nested (order–unit) subspaces of a given compact quantum metric space, whose radii tend to zero (so that their intersection reduces to the order–unit alone). So, once such a family has been selected, we may rescale correspondingly the radius of each subspace by a suitable factor, in such a way that the resulting family still has a uniform bound on the size of the radii, for this is one of the necessary conditions for a sequence of cQMS' to converge (see Theorem 1.2.18). Then, if the rescaled sequence of cQMS' is precompact in the quantum Gromov–Hausdorff topology, one may think at each limit point as "quantum tangent space" in the "quantum metric tangent cone" of the original QMS. As we shall see, when applied to the ordinary setting, this quantum procedure gives back the ordinary Gromov tangent cone of a pointed metric space at a fixed basepoint. It is worth noting that this "quantum construction", even in the commutative setting, actually produces more general objects than the ordinary one, depending on which family of subspaces one selects in the ambient (quantum) metric spaces.

As a case study, we apply these ideas to the case of the two–dimensional Noncommutative Torus [58; 18; 23]. As we shall see, the resulting quantum tangent cone is non–empty, and contains at least what we call a *Noncommutative Solenoid*, for it appears as a natural noncommutative generalization of the (a–adic) two–dimensional solenoid group [38]. We ought to say that, as the classical group is compact, also its noncommutative counterpart turns out to be, at a first sight, a compact object, since the underlying C^*–algebra is unital. However, as the classical solenoid contains a continuous (though not bi–continuous) dense image of (the additive group of) the real numbers, one might imagine that, by considering a different topology on its state space, which comes out quite naturally from the limiting process, a noncommutative plane "hides" inside this noncommutative solenoid.

In this approach, as Rieffel himself pointed out, the metric spaces are called quantum spaces, rather than noncommutative, because it is the state space of the C^*–algebra (or of its selfadjoint part considered as order–unit space), which carries the metric structure, and in quantum physics, which was one of the inspiring motivation for his work on this subject (cf. the discussion after Definition 2.2 in [63]), the states are the principal objects. So, one could instead reverse the setting, and assume that the space carrying the Lipschitz seminorm can be thought of as a "pre–dual", which induces a metric on its dual similarly to Rieffel's prescription.

With this view in mind, we introduce the notion of *rigged Lip-space*, that is, essentially, a *Lip–space* [32] (i.e., a Banach space endowed with a *Lip-norm*, which is a densely defined norm with the property that the unit ball [in the Banach norm] becomes compact in the corresponding induced topology) endowed with a further norm p smaller than the Banach norm, and the notion of *dual rigged Lip–space*, which is a *dual Lip–space* (i.e., a dual Banach space whose dual norm metrizes the w^*–topology on bounded subsets) endowed with a further norm p' greater than the dual Banach norm (see Section 2.2). In this setting, it becomes quite natural to consider as reference objects, instead of C^*–algebras, von Neumann algebras, precisely because in this case one has a (unique) Banach space predual ([67; 69; 49]) and, when it is separable (or, equivalently, when the Hilbert space on which acts the von Neumann algebra is separable), the Banach dual, that is, the von Neumann algebra itself, has indeed the property that the w^*–topology on bounded subsets is metrizable. Correspondingly, we introduce the notion of *dual Lip–von Neumann algebra*, whose bounded subsets then becomes the (compact) metric spaces for which we introduce, in analogy with Rieffel's construction, a Gromov–Hausdorff–type of distance.

As we shall notice, a notion of Hausdorff distance between von Neumann subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$ has been introduced by Haagerup and Winslow in [35; 36], where it is called the *Effros–Maréchal distance*, for it goes back essentially to an idea of O. Maréchal [46], and it is shown to metrize the Effros topology [22] on the family of closed unit balls of the von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$. So, as the Gromov–Hausdorff distance between ordinary (compact) metric space can be regarded as an extension of the usual Hausdorff distance between closed subsets of a fixed compact metric space to abstract (compact) metric spaces, not necessarily lying in the same ambient space, similarly we define a (pseudo–)distance which indeed becomes a distance on the space of Lip–isometry equivalence classes of (dual) Lip–von Neumann algebras. In other words, if the distance between two (dual) Lip–von Neumann algebras is zero, then the two algebras are (Lip–)$*$–isomorphic. Then, we also study the problem of finding reasonable conditions under which a sequence of (dual) Lip–von Neumann algebras convergences, and, as we shall see, the notion of *restricted ultraproduct*, introduced by D. Guido and T. Isola in [32] for various classes of Lip–spaces, will be very useful in this setting as well. In fact, we will show that a *uniform family* of *rigged von Neumann algebras* (i.e., Lip–von Neumann algebras whose Lip–norm satisfies a further property; see Definition 2.4.7) is precompact in the topology induced by this *dual quantum Gromov–Hausdorff distance* between von Neumann algebras.

As an example, we apply this construction to the problem of studying the scaling limit of the local algebras of observables in the framework of the algebraic approach to theory of quantum fields, the so–called *Algebraic Quantum Field Theory* [33]. We refer principally to the theory of renormalization in the algebraic setting, as proposed by D. Buchholz and R. Verch in [12; 13], in which the problem of the behavior, at small scales, of the observables of a quantum theory, is carefully analysed. We propose here an approach based on the "metric–ultraproduct methods" for (rigged) von Neumann algebras so far developed. This approach, studied in collaboration with D. Guido and H. Bostelmann [8], appears to be in some sense more general than that of Buchholz and Verch, for the scaling limit of a theory will depend, in our setting, also on the choice of the Lip–norm on the local algebras of the theory, so that it seems that one may produce much more limits for a given theory than by the Buchholz–Verch procedure. Though this feature may be regarded as some kind of mathematical richness of our construction, from the physical point of view one might expect that the small scales behavior of a given physical theory should be somehow unique. Finally, we will show that, on the level of von Neumann (local) algebras, in the case of the (real scalar) free field model, the two procedures, that is, the ultraproduct construction and the Buchholz–Verch construction [13], give compatible results, in the sense that the Buchholz–Verch representation of the scaling limit net in the vacuum sector embeds as a subrepresentation of the net obtained by the ultraproduct construction.

This dissertation is organized as follows.

In Chapter 1, we review the definitions and the basic properties of ordinary Gromov–Hausdorff distance and Rieffel's quantum distance, and propose a quantum version of the pointed Gromov–Hausdorff distance between proper metric spaces. We also give, as a case study, an example of this pointed–like convergence for the noncommutative 2–dimensional torus.

In Chapter 2, we review the definitions and the basic properties of Lip–spaces, dual Lip–

spaces and the related notion of ultraproducts and restricted ultraproducts, over an ultrafilter, of families of such spaces. We introduce also the concept of (dual) Lip– and rigged von Neumann algebra, and study the restricted ultraproduct of uniform families of (dual) Lip– and rigged von Neumann algebras.

In Chapter 3, we introduce the notion of dual quantum Gromov–Hausdorff distance for Lip–von Neumann algebras, and study the relation between the convergence of a sequence with respect to this distance and the corresponding ultraproduct.

Finally, in Chapter 4 we apply the ultraproduct methods to the study of the Buchholz–Verch scaling limit theory for local algebras of observables in the algebraic quantum field theory framework, and we show that the two approaches lead to compatible results for the (real scalar) free field model, in the sense specified therein.

Chapter 1

Gromov–Hausdorff Distance for Ordinary and Quantum Metric Spaces

In the first two sections of this chapter, we introduce the Gromov–Hausdorff distance for ordinary metric spaces, due to Gromov [29; 30], and its quantum version for compact quantum metric spaces, due to Rieffel [63; 59; 60]. In the third section, we recall also the pointed version of the Gromov–Hausdorff convergence, and introduce a quantum version of it. As we shall notice, in this construction the non–compactness will emerge as a peculiar feature of the distance, which indeed is non–bounded, though at the level of the C^*–algebras involved, the limit object of a pointed quantum Gromov–Hausdorff sequence will be a unital C^*–algebra (or, to be more precise, a dense set in the selfadjoint part of the algebra). This limit algebra (which we will call simply a quantum metric space) will be still endowed with a (Lipschitz) seminorm, inducing a metric on the state space whose topology is no longer compact, in analogy with the classical situation, where the resulting objects of the pointed Gromov–Hausdorff limiting process are usually non–compact.

1.1 Gromov–Hausdorff Distance for Compact Metric Spaces

In this section we will recall briefly the basic definitions and properties of the Gromov–Hausdorff distance between (ordinary) compact metric spaces. The reader is referred to [30; 14] for more details.

1.1.1 Hausdorff Distance

Let (X, ρ) be a metric space, i.e. X is a set and ρ is a metric on it. For any subset $Y \subseteq X$ and $r > 0$, let

$$\mathcal{N}_r(Y) \equiv \mathcal{N}(Y, r) := \{x \in X : \rho(x, y) < r \text{ for some } y \in Y\} \tag{1.1}$$

be the *r–neighborhood* of Y, i.e the set of points with distance less than r from Y. When $Y = \{x\}$, $\mathcal{N}_r(\{x\})$ is simply the open ball of radius r centered at x, and we shall write it as $B_r(x)$ or $B(x, r)$.

1.1.1 Definition. *Let Y and Z be (nonempty) compact subsets of a metric space (X,ρ). The* Hausdorff distance *between Y and Z is defined as*

$$\mathrm{dist}_H^{\rho}(Y,Z) := \inf\{r>0 : Y \subseteq \mathcal{N}_r(Z), Z \subseteq \mathcal{N}_r(Y)\}. \tag{1.2}$$

The Hausdorff distance measures the distance between Y and Z inside X. We will also use the notation $\mathrm{dist}_H^X(Y,Z)$, when no confusion arises about the metric on X.

Let $(\mathcal{SUB}(X),\ \mathrm{dist}_H^{\rho})$ be the set of (nonempty) compact subsets of X, equipped with the Hausdorff distance. The basic properties of the Hausdorff distance are summarized in the following

1.1.2 Proposition. [10; 14] *Let (X,ρ) be a metric space. Then*

(1) dist_H^{ρ} *is a metric on* $\mathcal{SUB}(X)$;

(2) $\mathcal{SUB}(X)$ *is complete if and only if X is complete*;

(3) $\mathcal{SUB}(X)$ *is compact if and only if X is compact.*

1.1.2 Gromov–Hausdorff Distance

The Gromov–Hausdorff distance was first introduced by Gromov in [29]. Let X and Y be compact metric spaces, and let h_X, h_Y be isometric embeddings of X and Y into some metric space Z.

1.1.3 Definition. *The* Gromov–Hausdorff distance *between the compact metric spaces X and Y is defined as*

$$\mathrm{dist}_{GH}^{Z}(X,Y) := \inf\{\mathrm{dist}_H^{Z}(h_X(X),h_Y(Y)) : h_X, h_Y \text{ are isometric embeddings in } Z\}. \tag{1.3}$$

In the previous Definition, Z is any metric space in which X and Y can be isometrically embedded. However, as shown by Gromov [29], it is possible to reduce this large class of metric spaces to the disjoint union $X \amalg Y$. A distance ρ on $X \amalg Y$ is called *admissible* if the inclusions $X, Y \hookrightarrow X \amalg Y$ are isometric embeddings. It turns out that

$$\mathrm{dist}_{GH}^{\rho}(X,Y) = \inf\{\mathrm{dist}_H^{\rho}(h_X(X),h_Y(Y)) : \rho \text{ is admissible on } X \amalg Y\}. \tag{1.4}$$

Indeed, we have the following

1.1.4 Lemma. *The two definitions above are equivalent, namely* $\mathrm{dist}_{GH}^{Z}(X,Y) = \mathrm{dist}_{GH}^{\rho}(X,Y)$.

Proof. Let $\delta \equiv \mathrm{dist}_{GH}^{\rho}(X,Y)$, and let $\rho_{X\amalg Y}$ be an admissible metric on $X \amalg Y$. Then, we have $\mathrm{dist}_{GH}^{Z}(X,Y) \le \rho_{X\amalg Y}(h_X(X),h_Y(Y))$, where $h_X : X \to X \amalg Y$ and $h_Y : Y \to X \amalg Y$ are the canonical embeddings. Since this inequality holds for any admissible metric, we have $\mathrm{dist}_{GH}^{Z}(X,Y) \le \delta$.
Conversely, suppose that $h_X : X \to Z$ and $h_Y : Y \to Z$ are isometric embeddings of X and Y into a metric space Z. If the images $h_X(X)$, $h_Y(Y)$ in Z are disjoint, then $h_X(X) \cup h_Y(Y)$ is a faithful representation of $X \amalg Y$, and the restriction of ρ_Z to this disjoint union gives a metric ρ on $X \amalg Y$, with $\rho_Z(h_X(X),h_Y(Y)) = \rho(X,Y) \ge \delta$. Now, if the images $h_X(X)$, $h_Y(Y)$ are not disjoint, we can separate them by the trick of crossing Z with a small interval $I_\varepsilon = [0,\varepsilon]$. Put the product metric

$$\rho((x,t),(y,s)) := \sqrt{\rho_Z(x,y)^2 + |t-s|^2}$$

on $Z \times I_\varepsilon$, and then embed X by $h_0(x) = (h_X(x), 0)$ and Y by $h_\varepsilon(y) = (h_Y(y), \varepsilon)$. Their images are now disjoint and we have $\rho(h_0(X), h_\varepsilon(Y)) = \sqrt{\rho_Z(h_X(X), h_Y(Y))^2 + \varepsilon^2}$. Therefore, $\delta \leq \sqrt{\rho_Z(h_X(X), h_Y(Y))^2 + \varepsilon^2}$, and, taking $\varepsilon \to 0$, we get $\delta \leq \rho_Z(h_X(X), h_Y(Y))$ for any isometric embeddings h_X, h_Y, and so $\delta \leq \mathrm{dist}^Z_{GH}(X, Y)$. ∎

1.1.5 Notation. In view of the previous Lemma, in the following we will denote simply by $\mathrm{dist}_{GH}(X, Y)$ the Gromov–Hausdorff distance between two (compact) metric spaces X and Y. Moreover, let us denote by $\mathrm{diam}(X)$ the diameter of the compact metric space (X, ρ), defined as $\mathrm{diam}(X) := \max\{\rho(x, y) : x, y \in X\}$, and by $r_X = \frac{\mathrm{diam}(X)}{2}$ the radius of (X, ρ).

The basic properties of the Gromov–Hausdorff distance are summarized in the following

1.1.6 Proposition. [30; 10; 14] *Let X and Y be compact metric spaces. Then,*

(1) $\mathrm{dist}_{GH}(X, Y) = 0$ *if and only if X and Y are isometric*;

(2) $\mathrm{dist}_{GH}(X, Y)$ *defines a metric on the set $\mathcal{CM}$ of isometry classes of compact metric spaces*;

(3) $|r_X - r_Y| \leq \mathrm{dist}_{GH}(X, Y) \leq \max(r_X, r_Y)$.

1.1.7 Definition. *For a compact metric space (X, ρ) and any $\varepsilon > 0$, the* covering number *$Cov_\rho(X, \varepsilon)$ (also denoted by $n_\varepsilon(X)$) is defined as the smallest number of open balls of radius ε whose union covers X.*

A remarkable property of the Gromov–Hausdorff distance is given by the following

1.1.8 Theorem (Gromov's Completeness and Compactness Theorem). [29] *The space $(\mathcal{CM}, \mathrm{dist}_{GH})$ is a complete metric space. A subset $\mathcal{S} \subseteq \mathcal{CM}$ is totally bounded if and only if*

(1) *there is a constant D such that $diam(X, \rho) \leq D$ for all $(X, \rho) \in \mathcal{S}$;*

(2) *for any $\varepsilon > 0$, there is a constant $K_\varepsilon > 0$ such that $Cov_\rho(X, \varepsilon) \leq K_\varepsilon$ for all $(X, \rho) \in \mathcal{S}$.*

Gromov's proof is based on Proposition 1.1.2 and on the following

1.1.9 Proposition. [29] *If a subset $\mathcal{S} \subseteq \mathcal{CM}$ satisfies the two conditions in Theorem 1.1.8, then there is a compact metric space (Z, ρ) such that each $X \in \mathcal{S}$ can be isometrically embedded into Z.*

In practical situations, it is usually very hard to compute the precise value of the Gromov–Hausdorff distance for two given compact metric spaces. In most cases, a crucial tool for estimating the Gromov–Hausdorff distance is given by the next Proposition, which follows directly from Definition 1.1.3.

1.1.10 Proposition. *For two compact subsets X and Y of a metric space (Z, ρ), we have:*

$$\mathrm{dist}_{GH}(X, Y) \leq \mathrm{dist}^\rho_H(X, Y). \tag{1.5}$$

1.1.11 Remark. Assume that the sequence $\{X_n\}$ of compact metric space converges to the (compact metric) space X in the Gromov–Hausdorff topology, Then, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for any $n \geq n_\varepsilon$, one can find an admissible metric ρ_n on the disjoint union $X_n \amalg X$ such that $\mathrm{dist}^{\rho_n}_H(X_n, X) < \varepsilon$. When such a sequence of metrics $\{\rho_n\}$ is chosen and fixed, it makes sense to speak about convergence of a sequence of points $\{x_n\}$, with $x_n \in X_n$, to a point $x \in X$, for this just means that $\lim_{n\to\infty} \rho_n(\iota_n(x_n), \iota(x)) = 0$, with ι_n (resp., ι) the canonical inclusion of X_n (resp., X) in $X_n \amalg X$.

1.2 Rieffel's Quantum Gromov–Hausdoff Distance

The reader is referred to [59; 60; 61; 63] for more details.

1.2.1 Order–unit Spaces

As Rieffel pointed out, the right framework for the quantum metric spaces is that of order–unit spaces. Typically, these arise as real–linear subspaces of the vector space of selfadjoint operators on a Hilbert space, containing the identity operator (the order–unit). In fact, any order–unit space can be realized in this way. The (real–linear) space of all selfadjoint elements of a unital C^*–algebra is a very useful example of order–unit space. There is an abstract characterization of order–unit spaces due to Kadison [40; 2].

1.2.1 Definition. *An* order–unit space *is a real partially ordered vector space A with a distinguished element e (the* order–unit*) which satisfies:*

1) (order–unit property) *for each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leq re$;*

2) (Archimedean property) *if $a \in A$ and if $a \leq re$ for all $r \in \mathbb{R}$ with $r > 0$, then $a \leq 0$.*

On an order–unit space (A, e), we can define a norm by

$$\|a\| := \inf\{r \in \mathbb{R} : -re \leq a \leq re\}. \tag{1.6}$$

Then, A becomes a normed vector space, and so we can consider its dual A', consisting of the bounded linear functionals on A, equipped with the dual norm $\|\cdot\|'$. By a *state* of an order–unit space (A, e) we mean a $\mu \in A'$ such that $\mu(e) = 1 = \|\mu\|'$. (Notice that the states are automatically positive [2].) We denote by $S(A)$ the collection of all the states of A, and call it the state space of A. It is a bounded closed convex subset of A', and so it is compact for the w^*–topology on A'. Each $a \in A$ defines a continuous affine function on $S(A)$ by $\hat{a}(\mu) = \mu(a)$, and e becomes the constant function 1 on $S(A)$. By Kadison's Representation Theorem [40; 2], this representation $\hat{a} \mapsto a$ is an isometric order isomorphism of A onto a dense subspace of the space $\mathrm{Af}_{\mathbb{R}}(S(A))$ of all affine $\mathbb{R}$–valued continuous functions, equipped with the supremum norm and the usual order on functions. In particular, the image of A is the whole of $\mathrm{Af}_{\mathbb{R}}(S(A))$ if, and only if, A is complete in the order–unit norm (cf. [2], Theorem II.1.8). Conversely, for any compact convex subset $\mathfrak{X}$ of a Hausdorff topological real vector space V, every linear subspace $A \subseteq \mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$ containing the constant function 1 is an order–unit space [2]. Therefore, we can view order–unit spaces as dense linear subspaces of $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$ containing 1, where $\mathfrak{X}$ is any compact convex subset of a Hausdorff topological real vector space.

An important notion is that of morphisms between order–unit spaces. Let A and B be order–unit spaces. By a *morphism* $\varphi : A \to B$, we mean a linear positive map preserving the order–units, hence norm–continuous (cf. [2], Proposition II.1.3). Then the dual map $\varphi' : B' \to A'$, restricted to $S(B)$, sending $\nu \in S(B)$ to $\varphi'(\nu) \in S(A)$, defines a continuous (for the w^*–topologies) affine map $S(\varphi)$ from $S(B)$ to $S(A)$. The notion of isomorphisms between order–unit spaces follows directly from that of morphisms.

In particular, when ϕ is surjective, $S(\varphi)$ is injective. We call such a pair (φ, B) a quotient of A. Since both $S(A)$ and $S(B)$ are compact Hausdorff spaces, $S(\varphi)$ is a homeomorphism from $S(B)$ onto its image, and we can identify $S(B)$ with its image $S(\varphi)(S(B))$, regarded as a closed convex subset of $S(A)$. Then, the morphism $\varphi : A \to B$ becomes the restriction of vectors in $A \subseteq \mathrm{Af}_{\mathbb{R}}(S(A))$, which are affine functions on $S(A)$, to $S(\varphi)(S(B))$. In fact, the converse is also true.

1.2.2 Proposition. [63] *Let A be an order–unit space. There is a natural bijection between isomorphism classes of quotients of A and closed convex subsets of $S(A)$.*

1.2.2 Compact Quantum Metric Spaces

As the metric on an ordinary compact metric space is determined by the Lipschitz seminorm it defines on (continuous) functions[1], Rieffel suggested that for "noncommutative spaces", that is, C^*–algebras, the way to specify a "metric" is by means of a seminorm which plays the role of a Lipschitz seminorm.

For an order–unit space (A, e) and a seminorm L on A, we may define an ordinary metric ρ_L on the state space $S(A)$ by

$$\rho_L(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}. \tag{1.7}$$

(Notice that, in the absence of further hypotheses, $\rho_L(\mu, \nu)$ may take the value $+\infty$.) This is a generalization of the Monge–Kantorovič metric on the space of probability measures on an ordinary compact metric space [41; 42], and, within the context of Dirac operators, it was introduced into Noncommutative Geometry by Connes in [17; 18]. In [60], these ideas were extended to consider Lipschitz seminorms on order–unit spaces. Since C^*–algebras are linear spaces over the complex numbers, while order–unit spaces are over the real numbers, we should point out that, for a C^*–algebra $\mathcal{A}$, we require a Lipschitz seminorm L to satisfy $L(a^*) = L(a)$, for any $a \in \mathcal{A}$. Under this condition, then it suffices to take the above supremum just over selfadjoint elements of A when defining ρ_L.

1.2.3 Definition. *Let (A, e) be an order–unit space. By a* Lipschitz seminorm *on A, we mean a densely defined, lower semicontinuous seminorm L on A satisfying:*

1) *for $a \in A$ we have $L(a) = 0$ if, and only if, $a \in \mathbb{R}e$;*

We call L a Lip–seminorm *if it satisfies further:*

2) *the topology on $S(A)$, induced by the metric ρ_L, is the w^*–topology.*

1.2.4 Notation. Since in the following chapter we will define Lip–norms in a more general setting, which are really norms and not only seminorms, we prefer to use here the term Lip–seminorm, in contrast with the standard terminology introduced by Rieffel.

[1] Let (X, ρ) be a compact space X. The Lipschitz seminorm L_ρ determined by ρ is defined on functions f on X by

$$L_\rho(f) = \sup\{|f(x) - f(y)|/\rho(x, y) : x \neq y\}.$$

(Notice that it may take the value $+\infty$.) Then, one can recover ρ from L_ρ by

$$\rho(x, y) = \sup\{|f(x) - f(y)| : L_\rho(f) \leq 1\}.$$

1.2.5 Definition. *By a* compact Quantum Metric Space (cQMS), *we mean a pair* (A, L) *consisting of an order–unit space* A *with a Lip–seminorm* L *defined on it. The* diameter $diam(A)$, *the* radius r_A, *and the* covering number $Cov(A, \varepsilon)$ *of* (A, L) *are defined to be those of* $(S(A), \rho_L)$.

(Let us notice that, if the seminorm L is not lower semicontinuous, we can always replace it by the largest lower semicontinuous seminorm smaller than L, as this will give the same metric on $S(A)$ (cf. [60], Theorem 4.2).)

A useful tool will be the following

1.2.6 Lemma (Comparison Lemma). [59] *Let* (A, e) *be an order–unit space,* L *a desely defined (Lipschitz) seminorm on* A*, and* B *a subspace of* A*, endowed with a seminorm* M*. Let* ρ_M *and* ρ_L *denote the corresponding metrics on* $S(A)$ *as in* (1.7) *(possibly taking the value* $+\infty$*). Assume that, on* B*,*

$$M \geq L,$$

in the sense that $M(a) \geq L(a)$ *for all* $a \in B$*. Then,*

$$\rho_M \leq \rho_L,$$

in the sense that $\rho_M(\mu, \nu) \leq \rho_L(\mu, \nu)$ *for all* $\mu, \nu \in S(A)$*. Thus,*

(i) *if* ρ_L *is finite, so is* ρ_M*;*

(ii) *if* ρ_L *is bounded, so is* ρ_M*;*

(iii) *if the* ρ_L*–topology on* $S(A)$ *agrees with the* w^**–topology, then so does the* ρ_M*–topology.*

Proof. If $a \in B$ and $M(a) \leq 1$, then $L(a) \leq 1$. Thus, the supremum defining ρ_M is taken over a smaller set than that for ρ_L, hence $\rho_M \leq \rho_L$. The claims (i) and (ii) are then obvious, and (iii) follows from the fact that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism (cf., for instance, [50], Proposition 1.6.8). ∎

As a consequence, we get also the following

1.2.7 Proposition. [59] *Let* (A, e) *be an order–unit space, and* L *a desely defined seminorm on* A*. Let* t *be a strictly positive real number, and set* $M := tL$ *on* $\mathcal{L} := \{a \in A : L(a) < \infty\}$*. Then,* $\rho_M = t^{-1}\rho_L$*. Thus,*

(i) *if* ρ_L *is finite, so is* ρ_M*;*

(ii) *if* ρ_L *is bounded, so is* ρ_M*;*

(iii) *if the* ρ_L*–topology on* $S(A)$ *agrees with the* w^**–topology, then so does the* ρ_M*–topology.*

Proof. Clearly,

$$\rho_M(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : M(a) \leq 1\} = t \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\} = \rho_L(\mu, \nu).$$

Then, the proof is the same as in the previous Lemma. ∎

1.2.8 Definition. *By a* C^*–algebraic Compact Quantum Metric Space *we mean a pair* $(\mathcal{A}, L)$ *consisting of a unital* C^**–algebra* $\mathcal{A}$ *and a (possibly* $+\infty$*–valued) seminorm* L *on* $\mathcal{A}$ *satisfying the* reality condition

$$L(a) = L(a^*), \tag{1.8}$$

and such that the linear subspace $\{a \in \mathcal{A} : L(a) < \infty\}$ *is dense in* $\mathcal{A}$*, contains* $e_{\mathcal{A}}$ *(the identity of* $\mathcal{A}$*), and* L *restricted to the (order–unit) space*

$$A := \{a \in \mathcal{A} : L(a) < \infty\} \cap \mathcal{A}_{sa} \tag{1.9}$$

is a Lip–seminorm. ($\mathcal{A}_{sa}$ *denotes the set of all selfadjoint elements of* $\mathcal{A}$*.) We call* $(A, L|_A)$ *the* associated Compact Quantum Metric Space. *The* diameter*,* radius *and* covering number *of* $(\mathcal{A}, L)$ *are defined to be the same as those of* $(A, L|_A)$*.*

1.2.9 Example. For any compact metric space (X, ρ), define the Lipschitz seminorm L_ρ (which may take the value $+\infty$) on $C_{\mathbb{C}}(X)$, the set of all $\mathbb{C}$–valued continuous functions over X, by

$$L_\rho(f) := \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y\right\}. \tag{1.10}$$

Clearly, L_ρ satisfies the reality condition (1.8) and, moreover, the *Leibniz rule*

$$L_\rho(f \cdot g) \leq L_\rho(f)\|g\| + \|f\|L_\rho(g), \quad f, g \in C_{\mathbb{C}}(X). \tag{1.11}$$

Let us define

$$A_{(X,\rho)} := \{f \in C_{\mathbb{R}}(X) : L_\rho(f) < \infty\}. \tag{1.12}$$

Then, $(A_{(X,\rho)}, L_\rho|_{A_{(X,\rho)}})$ is a closed compact quantum metric space, and $(C_{\mathbb{C}}(X), L_\rho)$ is a C^*–algebraic compact quantum metric space. It is called the *associated compact quantum metric space* of (X, ρ). Notice that $S(A_{(X,\rho)})$ is the set of probability measures on X, and the induced metric on $S(A_{(X,\rho)})$ is the Monge–Kantorovič metric [41; 42].

We can restrict a quantum metric space to its closed convex subsets by the following

1.2.10 Proposition. [63] *Let* A *be an order–unit space, and let* B *be a quotient of* A *with the quotient map* $\psi : A \to B$*. Let* L *be a Lipschitz seminorm on* A*, and let* L_B *be the corresponding quotient seminorm on* B*, defined as*

$$L_B(b) := \inf\{L(a) : \psi(a) = b\}. \tag{1.13}$$

Then, $S(\psi)$ *is an isometry for the corresponding metrics* ρ_L *and* ρ_{L_B}*. If* L *is actually a Lip–seminorm, then so is* L_B*. If furthermore* L *is closed, then so is* L_B*, and we have* $\psi(\mathcal{L}_1(A)) = \mathcal{L}_1(B)$ *(where* $\mathcal{L}_1(A) := \{a \in A : L(a) \leq 1\}$*).*

As shown by Rieffel [59; 62], a wide class of cQMS's consists of C^*–algebras with metrics arising from ergodic actions of compact groups, i.e. actions for which the only invariant elements are the scalar multiple of the identity. So, let G be a compact group with a length function, i.e. a continuous real–valued function ℓ on G, such that

$$\begin{aligned} \ell(xy) &\leq \ell(x) + \ell(y) \quad \forall x, y \in G, && (1.14)\\ \ell(x^{-1}) &= \ell(x) \quad \forall x \in G, && (1.15)\\ \ell(x) &= 0 \quad \text{if, and only if,} \quad x = e_G, && (1.16) \end{aligned}$$

where e_G is the identity of G. Let $\mathcal{A}$ be a unital C^*–algebra, and let α be a strongly continuous action of G on $\mathcal{A}$ by automorphisms. Define a (possibly $+\infty$–valued) seminorm L on $\mathcal{A}$ by

$$L^G(a) := \sup\left\{\frac{\|\alpha_x(a) - a\|}{\ell(x)} : x \in G, x \neq e_G\right\} \tag{1.17}$$

Then, clearly, L satisfies the reality condition (1.8) and the Leibniz rule (1.11). It can be shown [59] that the set of elements in $\mathcal{A}$ where L is finite, is always a dense $*$–subalgebra of $\mathcal{A}$. Hence, the set $A = \{\mathcal{A}_{sa} : L(a) < 1\}$ is a dense subspace of $\mathcal{A}_{sa}$ containing the identity $e_{\mathcal{A}}$ of $\mathcal{A}$, and the state space of A can be identified with that of $\mathcal{A}$. Furthermore, when the action α is ergodic, i.e. the only α–invariant elements are the scalar multiples of the identity, $(A, L|_A)$ is a closed compact quantum metric space and $(\mathcal{A}, L)$ is a C^*–algebraic compact quantum metric space, so that L should be thought of as a (compact) metric on the quantum space $\mathcal{A}$. (Notice that to any length function ℓ on G there corresponds a left invariant metric ρ on G, given by $\rho(x, y) = \ell(x^{-1}y)$. Conversely, if ρ is a metric on the compact group G, then $\tilde{\rho}(x, y) := \int_G \rho(zx, zy)dz$, with dz the Haar measure, is a left–invariant metric. Hence, G has a length function if, and only if, it is metrizable.)

We have then the following

1.2.11 Theorem. [59] *Let α be an ergodic action of a compact group G on a unital C^*–algebra $\mathcal{A}$. Let ℓ be a length function on G, L^G be the corresponding seminorm, and set $A^G := \{a \in \mathcal{A}_{sa} : L^G(a) < \infty\}$. Moreover, let ρ_G be the corresponding metric on the state space $S(\mathcal{A})$ of $\mathcal{A}$. Then the ρ_G–topology on $S(\mathcal{A})$ agrees with the w^*–topology (and ρ_G is bounded by $2\int_G \ell(x)dx$, where dx is the (normalized) Haar measure on G).*

An even more interesting situation is when G is a Lie group. So, let G be a connected (compact) Lie group, $\mathfrak{g}$ its Lie algebra, and fix a norm $\|\cdot\|$ on $\mathfrak{g}$. For any action α of G on a Banach space B, we let B^1 denote the space of α–differentiable elements of B. Thus, if $b \in B^1$, then, for each $X \in \mathfrak{g}$, there exists a $d_X b \in B$, such that

$$\lim_{t\to 0}(\alpha_{\exp(tX)}(b) - b)/t = d_X b. \tag{1.18}$$

Clearly, $X \mapsto d_X b$ is a linear map from $\mathfrak{g}$ into B, and we denote it by db (cf. [9]). Since $\mathfrak{g}$ and B both have norms, the operator norm $\|db\|$ of db is defined (and finite), namely

$$\|db\| := \sup\{\|d_X b\| : X \in \mathfrak{g}, \|X\| = 1\} < +\infty. \tag{1.19}$$

Then, a standard smoothing argument shows that B^1 is dense in B (cf. [9]).

Suppose now that $\mathcal{A}$ is a C^*–algebra and α is an action by automorphisms of $\mathcal{A}$. We set $A^{\mathfrak{g}} \equiv A^1$ and $L^{\mathfrak{g}}(a) := \|da\|$. It is easily verified that $A^{\mathfrak{g}}$ is a $*$–subalgebra of $\mathcal{A}$. Since G is connected, $L^{\mathfrak{g}}(a) = 0$ if, and only if, a is α–invariant. Let us prove the following

1.2.12 Theorem. [59] *Let G be a compact connected Lie group and fix a norm on $\mathfrak{g}$. Let α be an ergodic action of G on a unital C^*–algebra $\mathcal{A}$. Set $A^{\mathfrak{g}} = A^1$ and $L^{\mathfrak{g}}(a) = \|da\|$, as above, and let $\rho_{\mathfrak{g}}$ denote the corresponding metric on the state space $S(\mathcal{A})$ of $\mathcal{A}$. Then the $\rho_{\mathfrak{g}}$–topology on $S(\mathcal{A})$ agrees with the w^*–topology.*

Proof. Let $\langle\cdot,\cdot\rangle_e$ be an inner product on $\mathfrak{g}$, with e the unit of G. Its corresponding norm $\|\cdot\|_e$ is clearly equivalent to the given norm (as $\mathfrak{g}$. is a finite–dimensional vector space), and so, by the Comparison Lemma 1.2.6, it suffices to deal with the norm from the inner product. We can left–translate this inner product over G to obtain a left–invariant Riemannian metric on G, and then a corresponding left–invariant ordinary metric on G, as follows: we define $\langle X, Y\rangle_p := \langle (dL_{p^{-1}})_p(X), (dL_{p^{-1}})_p(Y)\rangle_e$, $p \in G$, $X, Y \in T_pG$ (the tangent space at p), where $L_p : G \to G$, $L_p(q) := pq$ is the left–translation. Since L_p depends differentiably on p, we get a Riemannian metric, which is clearly left–invariant, i.e. it satisfies $\langle X, Y\rangle_p := \langle (dL_p)_q(X), (dL_p)_q(Y)\rangle_{L_p(q)}$, for all $p, q \in G$ and $X, Y \in T_pG$. Then, let $\gamma : [\alpha, \beta] \to G$ be a (smooth) curve segment, with $\gamma(\alpha) = p$, $\gamma(\beta) = q$. The arc length of γ is defined by

$$L(\gamma) := \int_\alpha^\beta (\langle \dot\gamma(t), \dot\gamma(t)\rangle_{\gamma(t)})^{1/2} dt = \int_\alpha^\beta \|\dot\gamma(t)\|_{\gamma(t)} dt,$$

and the distance from p to q is then defined as

$$\rho(p, q) := \inf_\gamma L(\gamma),$$

where the infimum has to be taken over all the arc segments joining p and q. We see that, by construction, ρ is left–invariant. Now, let $\ell(x)$ denote the corresponding distance from x to e, i.e. $\ell(x) := \rho(x, e)$. Then, ℓ is a continuous length function on G satisfying conditions (1.15–1.16). Indeed, we have clearly $\ell(x) = 0$ if, and only if, $x = e$, $\ell(xy) = \rho(xy, e) \le \rho(xy, x) + \rho(x, e) = \ell(x) + \ell(y)$ (by left–invariance of ρ, and the fact that it is a metric), and finally $\ell(x^{-1}) = \rho(x^{-1}, e) = \rho(xx^{-1}, x) = \ell(x)$. Then, the elements of A^1 are Lipschitz for ℓ. In fact, let $a \in A^1$ and let $c : [0, 1] \to G$ be a smooth path from e to a point $x \in G$. Then, the function $\phi : [0, 1] \to A^1$, defined as $\phi(t) = \alpha_{c(t)}(a)$, is differentiable, and thus we have

$$\|\alpha_x(a) - a\| = \|\int_0^1 \dot\phi(t) dt\| \le \int_0^1 \|\alpha_{c(t)}(d_{\dot c(t)} a)\| dt \le \|da\| \int_0^1 \|\dot c(t)\|_e dt = L(c).$$

Thus, taking the infimum on the r.h.s. over all paths from x to e, we obtain

$$\|\alpha_x(a) - a\| \le \|da\|\ell(x).$$

(Notice that the above argument works for any norm on $\mathfrak{g}$.) Then, if we let A^G and L^G be defined just in terms of ℓ as above, we see that $A^{\mathfrak{g}} \subseteq A^G$ and $L^G \le L^{\mathfrak{g}}$. Hence, we can apply the Comparison Lemma 1.2.6 to obtain the desired conclusion. ∎

Now, let us choose a norm $\|\cdot\|$ on $\mathfrak{g}$ coming form an Ad–invariant inner product, where, for $p \in G$, the adjoint representation $Ad(p) : \mathfrak{g} \to \mathfrak{g}$ is defined as the differential $Ad(p) := d(R_{p^{-1}} L_p)$; for example, we may take the negative of the Killing form, that is $\langle X, Y\rangle := -tr(Ad(X)Ad(Y))$ (see, for instance, [37]). Then, ℓ will satisfy the extra condition $\ell(yxy^{-1}) = \ell(x)$, and, as a consequence, A^1 will be α–invariant. Indeed, the Ad–invariance of a (left–invariant) inner product implies also its right–invariance, namely

$$\begin{aligned} \langle X, Y\rangle_p &= \langle (dR_{p^{-1}})_p(X), (dR_{p^{-1}})_p(Y)\rangle_e \\ &= \langle ((dR_{p^{-1}})_p \circ (dL_p)_{p^{-1}})(X), (dR_{p^{-1}})_p \circ (dL_p)_{p^{-1}})(Y)\rangle_e \\ &= \langle Ad(p)(X), Ad(p)Y\rangle_e = \langle X, Y\rangle_p. \end{aligned}$$

Hence, the corresponding metric will be left– and right–invariant, that is, $\ell(yxy^{-1}) = \rho(yxy^{-1}, e) = \ell(x) = \rho(x, e)$. Moreover, we have also the α–invariance of the seminorm, namely

$$\begin{aligned} L^G(\alpha_y(a)) &= \sup\left\{ \frac{\|\alpha_y(\alpha_{y^{-1}xy}(a) - a)\|}{\ell(x)} : x \neq e_G \right\} \\ &= \sup\left\{ \frac{\|\alpha_x(a) - a\|}{\ell(yxy^{-1})} : x \neq e_G \right\} = L^G(a). \end{aligned}$$

Then, for any $X \in \mathfrak{g}$ and t in a sufficiently small neighborhood of $0 \in \mathbb{R}$, we have $\ell(\exp(tX)) = |t|\|X\|^2$, and thus, if $\|X\| = 1$, we get

$$\|(\alpha_{\exp(tX)}(a) - a)/t\| = \|(\alpha_{\exp(tX)}(a) - a)\|/\ell(\exp(tX)).$$

From this we see that $\|d_X a\| \leq L^G(a)$ for all X with $\|X\| = 1$. Thus, we have established the following

1.2.13 Proposition. *For $L^{\mathfrak{g}}$ defined, as above, through a norm on $\mathfrak{g}$, and for L^G defined in terms of the corresponding length function on G, we have $L^{\mathfrak{g}} = L^G$ and $A^{\mathfrak{g}} = A^G$.*

1.2.3 Quantum Gromov–Hausdorff Distance

Let (A, L_A) and (B, L_B) be compact quantum metric spaces. The dual object for the disjoint union $X \amalg Y$ is the direct sum $A \oplus B$ of vector spaces, with (e_A, e_B) as order–unit, and with the natural order structure. $A \oplus B$ is thus an order–unit space, and the natural projections from $A \oplus B$ to A and B are surjective order–unit space morphisms. Correspondingly, we call a Lip–seminorm L on $A \oplus B$ *admissible* if it induces L_A and L_B as in Proposition 1.2.10, and we denote by $\mathcal{L}(L_A, L_B)$ the set of admissible Lip–seminorms on $A \oplus B$. Then, a natural notion of quantum distance, introduced by Rieffel, is the following:

1.2.14 Definition. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. We define the* quantum Gromov–Hausdorff distance *between them as*

$$\mathrm{dist}_q(A, B) := \inf\{\mathrm{dist}_H^L(S(A), S(B)) : L \in \mathcal{L}(L_A, L_B)\} \tag{1.20}$$

It is evident that $\mathrm{dist}_q(A, B)$ is symmetric in A and B. As for the triangle inequality, one has the following

1.2.15 Theorem. [63] *Let (A, L_A), (B, L_B) and (C, L_C) be quantum metric spaces. Then,*

$$\mathrm{dist}_q(A, C) \leq \mathrm{dist}_q(A, B) + \mathrm{dist}_q(B, C). \tag{1.21}$$

For a compact quantum metric space (A, L), we denote by A^c the (norm) closure of A and by L^c the extension of L to A^c (which may take now the value $+\infty$).

1.2.16 Definition. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. By an* isometry *from (A, L_A) to (B, L_B), we mean an order isomorphism φ from A^c onto B^c, such that $L_A^c = L_B^c \circ \varphi$.*

[2] Indeed, the map $\exp \equiv \exp_e : T_eG := \mathfrak{g} \to G$ is a local diffeomorphism (in the sense that there exists an $\varepsilon > 0$ such that $\exp : B_\varepsilon(0) \subset T_eG \to G$ is a diffeomorphism of the ε–ball $B_\varepsilon(0)$ onto an open subset of G), and, moreover, it is also an isometry for the inner product in $\mathfrak{g}$ and the corresponding metric on G, and this follows from the Gauss' Lemma (see, for instance, Proposition 2.9 in [24]).

The quantum version of Proposition 1.1.6 is the following

1.2.17 Theorem. [63] *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. Then:*

(1) $\mathrm{dist}_q(A, B) = 0$ *if, and only if,* (A, L_A) *and* (B, L_B) *are isometric;*

(2) *the isometries from (A, L_A) to (B, L_B) are in natural bijective correspondence with the affine isometries from $(S(B), \rho_{L_B})$ onto $(S(A), \rho_{L_A})$ through the map $\varphi \mapsto S(\varphi)$;*

(3) dist_q *defines a metric on the set $\mathcal{QCM}$ of isometry classes of compact quantum metric spaces;*

(4) $|r_A - r_B| \leq \mathrm{dist}_q(A, B) \leq \mathrm{diam}(A) + \mathrm{diam}(B)$.

Theorems 1.2.17(1)–(2) and 1.2.15 tell us that dist_q is indeed a distance between the state spaces of compact quantum metric spaces, equipped with the induced metrics and the convex structures.

There is also a quantum version of Gromov's Completeness and Compactness Theorem, due to Rieffel.

1.2.18 Theorem (Rieffel's Quantum Completeness and Compactness Theorem). [63] *The space $(\mathcal{QCM}, \mathrm{dist}_q)$ is a complete metric space. A subset $\mathcal{S} \subseteq \mathcal{QCM}$ is totally bounded if and only if*

(1) *there is a constant D such that $diam(A, L) \leq D$ for all $(A, L) \in \mathcal{S}$;*

(2) *for any $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that $Cov(A, \varepsilon) \leq K_\varepsilon$ for all $(A, L) \in \mathcal{S}$.*

We have a quantum version of Proposition 1.1.10 as well:

1.2.19 Proposition. [63] *Let (A, L_A) be a compact quantum metric space, and let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be compact convex subsets of $S(A)$. Let (B_j, L_j) for $j = 1, 2$ be the corresponding quotients. Then*

$$\mathrm{dist}_q(B_1, B_2) \leq \mathrm{dist}_H^{\rho_{L_A}}(\mathfrak{X}_1, \mathfrak{X}_2). \tag{1.22}$$

The quantum distance dist_q extends indeed the ordinary Gromov–Hausdorff distance dist_{GH}.

1.2.20 Proposition. [63] *The Gromov–Hausdorff distance between two compact metric spaces is greater than or equal to the quantum distance between the associated compact quantum metric spaces. The map from $\mathcal{CM}$ to $\mathcal{QCM}$ sending each compact metric space to its associated compact quantum metric space is a homeomorphism from $\mathcal{CM}$ onto a closed subspace of $\mathcal{QCM}$.*

1.2.21 Remark. In general, the Gromov–Hausdorff distance between two compact metric spaces is not equal to the quantum distance between the associated compact quantum metric spaces (see, for a concrete example, Appendix 1 in [63]). This happens because one admits Lip–seminorms, which need not to come from metrics on the disjoint union of the two spaces.

1.2.22 Example (Quantum Tori). Fix $n \geq 2$, and let Θ be the space of all skew symmetric $n \times n$ matrices. For $\theta \in \Theta$, let $\mathcal{A}_\theta$ be the corresponding quantum torus [57; 58]. It is defined as follows. Let σ_θ denote the skew bicharacter on $\mathbb{Z}^n$ defined by

$$\sigma_\theta(p, q) := e^{2\pi i p \cdot \theta q}. \tag{1.23}$$

Equip $C_c(\mathbb{Z}^n)$, the space of $\mathbb{C}$–valued functions on $\mathbb{Z}^n$ of finite support, with the product consisting of convolution twisted by σ_θ. That is, for $f, g \in C_c(\mathbb{Z}^n)$, we define

$$(f * g)(p) := \sum_{q \in \mathbb{Z}^n} f(q)g(p-q)\sigma_\theta(p,q). \tag{1.24}$$

Equip $C_c(\mathbb{Z}^n)$ also with the involution $f^*(p) = \overline{f(-p)}$, and the norm of $\ell^1(\mathbb{Z}^n)$, so that $C_c(\mathbb{Z}^n)$ becomes a normed $*$–algebra. Let π_θ denote the $*$–representation of $C_c(\mathbb{Z}^n)$ on the Hilbert space $\ell^2(\mathbb{Z}^n)$, given by

$$\pi_\theta(f)\xi := (f * \xi)(p) := \sum_{q \in \mathbb{Z}^n} f(q)\xi(p-q)\sigma_\theta(p,q), \quad \xi \in \ell^2(\mathbb{Z}^n). \tag{1.25}$$

Moreover, let $\|\cdot\|_\theta$ be the C^*–norm on $C_c(\mathbb{Z}^n)$ given by $\|f\|_\theta := \|\pi_\theta(f)\|$. Then, $\mathcal{A}_\theta$ is defined to be the completion of $C_c(\mathbb{Z}^n)$ for this norm. (In this way, the elements of $\mathcal{A}_\theta$ can be thought of as some kind of functions on $\mathbb{Z}^n$.) The n.-torus $\mathbb{T}^n$ has a canonical ergodic action α_θ on $\mathcal{A}_\theta$. If we denote the duality between $\mathbb{Z}^n$ and $\mathbb{T}^n$ by $\langle p, x\rangle$ for $x \in \mathbb{T}^n$ and $p \in \mathbb{Z}^n$, then α_θ is determined by

$$(\alpha_{\theta,x}(f))(p) = \langle p, x\rangle f(p). \tag{1.26}$$

Now, fix a length function on $G = \mathbb{T}^n$, let L_θ be the seminorm defined by (1.17) and set, as above, $A_\theta = \{a \in (\mathcal{A}_\theta)_{sa} : L_\theta < \infty\}$. Then, one has the following

1.2.23 Proposition. *L_θ is a Lip–seminorm and (A_θ, L_θ) is a (compact) quantum metric space.*

Proof. Since the action α_θ of the compact group $\mathbb{T}^n$ is clearly ergodic, it suffices to apply Theorem 1.2.11. ∎

Moreover, we will use intensively the following fundamental result of Rieffel (see [63], Theorem 9.2):

1.2.24 Proposition. [63] *For any $\varepsilon > 0$ there exists a $\delta > 0$ such that, if $\|\theta - \psi\| < \delta$, then* $\mathrm{dist}_q(A_\theta, A_\psi) < \varepsilon$.

1.2.25 Remark. As we shall see, for $n = 2$, the quantum torus, also known as *(ir)rational rotation algebra*, depending on whether θ is in $\mathbb{Q}$ or in $\mathbb{R}\backslash\mathbb{Q}$ (see, for instance, [23] for more details), can be equivalently defined by a universality property, namely, it is the universal C^*–algebras generated by two unitaries U and V satisfying the commutation relation

$$UV = e^{2\pi i\theta}VU. \tag{1.27}$$

Indeed, let $\mathcal{H} = L^2(\mathbb{R}/\mathbb{Z})$, and consider two unitary operators on $\mathcal{H}$, the operator $U := M_{z(t)}$ of multiplication by the unimodular function $z(t) = e^{2\pi it}$ and the operator V of rotation by θ, that is

$$Uf(t) = z(t)f(t) = e^{2\pi i\theta}f(t) \quad \text{and} \quad Vf(t) = f(t-\theta). \tag{1.28}$$

A simple calculation then yields

$$\begin{aligned} VUf(t) &= (Uf)(t-\theta) = z(t-\theta)f(t-\theta) \\ &= e^{-2\pi i\theta}z(t)(Vf)(t) = e^{-2\pi i\theta}UVf(t). \end{aligned}$$

Hence,

$$UV = e^{2\pi i\theta} VU.$$

We recall that a C^*–algebra $\mathcal{A}_\theta$ is universal for the relation (1.27), provided that it is generated by two unitaries $\widetilde{U}$ and $\widetilde{V}$ satisfying (1.27) and, whenever $\mathfrak{A} = C^*(U,V)$ is another C^*–algebra satisfying (1.27), there is a ($*$–)homomorphism of $\mathcal{A}_\theta$ onto $\mathfrak{A}$, which carries $\widetilde{U}$ to U and $\widetilde{V}$ to V. Since we know that there are unitaries satisfying the relation (the two unitaries in (1.28)), we may consider the collection of all irreducible pair of unitaries (U_α, V_α) in $\mathcal{B}(\mathcal{H})$ satisfying (1.27). Then, consider the operators

$$\widetilde{U} = \sum_\alpha^\oplus U_\alpha \quad \text{and} \quad \widetilde{V} = \sum_\alpha^\oplus V_\alpha, \tag{1.29}$$

and set $\mathcal{A}_\theta := C^*(\widetilde{U}, \widetilde{V})$. In order to see that $\mathcal{A}_\theta$ is universal, let $\mathfrak{A} := C^*(U,V)$ be another C^*–algebra satisfying (1.27). To verify that there is a well defined homomorphism $\varphi : \mathcal{A}_\theta \to \mathfrak{A}$ such that $\varphi(\widetilde{U}) = U$ and $\varphi(\widetilde{V}) = V$, it suffices to show that

$$\|p(U,V,U^*,V^*)\| \leq \|p(\widetilde{U}, \widetilde{V}, \widetilde{U}^*, \widetilde{V}^*)\|$$

for every noncommutative polynomial in four variables. So, fix a polynomial p, and let $A = p(U,V,U^*,V^*)$. By the GNS construction, there exists an irreducible representation π of $\mathfrak{A}$ such that $\|\pi(A)\| = \|A\|$. Consider now the pair of unitaries $U' := \pi(U)$ and $V' := \pi(V)$. Then (U', V') is an irreducible pair of unitaries satisfying (1.27). Hence, by construction, we see that

$$\|p(\widetilde{U}, \widetilde{V}, \widetilde{U}^*, \widetilde{V}^*)\| \geq \|p(U', V', U'^*, V'^*)\| = \|p(U,V,U^*,V^*)\|.$$

Therefore, φ is well defined and contractive from the $*$–algebra generated by $\widetilde{U}$ and $\widetilde{V}$ into $\mathfrak{A}$. Thus, it extends by continuity to a homomorphism of $\mathcal{A}_\theta$ onto $\mathfrak{A}$.

As in the following we will use some "ultratechniques", in the next section we introduce some basic notions about ultrafilters and ultraproducts.

1.3 Ultrafilters and Ultraproducts

In this section, we briefly recall the definition and the principal properties of ultrafilters. (The reader is referred to [68; 44] for a detailed exposition.)

1.3.1 Definition. *Let X be a set and $\mathcal{U}$ a non–empty family of subsets of X. $\mathcal{U}$ is a* ultrafilter *if the following properties are satisfied:*

1) $\emptyset \notin \mathcal{U}$,

2) $A, B \in \mathcal{U}$ *implies* $A \cap B \in \mathcal{U}$,

3) $A \in \mathcal{U}$ *implies* $B \in \mathcal{U}$, $\forall B \supseteq A$,

4) *for each* $A \subseteq X$, *one has either* $A \in \mathcal{U}$ *or* $X \backslash A \in \mathcal{U}$.

The ultrafilter is principal *(or* fixed*, or* trivial*) if it satisfies the further property:*

5) *the filter contains a least element, i.e. there exists a set* $B \subset X$*, called the* basis *for* $\mathcal{U}$*, such that* $\mathcal{U}$ *is the collection of all the supersets of* B*.*

A free *(*non–principal*) ultrafilter is an ultrafilter which is not principal.*

1.3.2 Lemma. *Let* $\mathcal{U}$ *be an ultrafilter on a set* X*. Then,*

1. *if* $\mathcal{U}$ *is principal, its basis is a singleton;*

2. *if* $\mathcal{U}$ *is non–principal, it does not contain finite sets.*

Proof. 1. Let B be the basis for $\mathcal{U}$. If B is not a singleton, we can take a non trivial partition of B. Then, one, and only one, of the sets of this partition belongs to $\mathcal{U}$, contradicting the minimality of B. 2. Assuming the contrary, let $A \in \mathcal{U}$ be a finite set, and let $a \in A$. Then, only one of the two sets $\{a\}$ and $A\backslash\{a\}$ belongs to $\mathcal{U}$. If $\{a\}$ does, $\mathcal{U}$ should be principal with basis $\{a\}$. If $A\backslash\{a\} \in \mathcal{U}$, one can repeat the argument, ending up with a singleton. In both cases, one gets a contradiction, hence the claim. ∎

1.3.3 Definition. *Let* $\{x_i\}_{i\in\mathbb{N}}$ *be a family of real numbers and* $\mathcal{U}$ *an ultrafilter on the index set* $\mathbb{N}$*. We say that* $\lim_{\mathcal{U}} x_\alpha = x$*, if the set* $\{i \in \mathbb{N} : |x_\alpha - x| < \varepsilon\}$ *is in* $\mathcal{U}$ *for any* $\varepsilon > 0$*.*

1.3.4 Example. We want to show that a convergent sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ converges over any ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Let us consider separately the two cases, namely $\mathcal{U}$ principal and $\mathcal{U}$ free. If $\mathcal{U}$ is principal, let $\{n_0\}$ be its basis. By definition, $A_\varepsilon = \{n \in \mathbb{N} : |x_n - x_{n_0}| < \varepsilon\}$ contains n_0 for any $\varepsilon > 0$. Thus, $A_\varepsilon \in \mathcal{U}$ (by property 3) of the Definition), and consequently $\lim_{\mathcal{U}} x_n = x_{n_0}$. If, instead, $\mathcal{U}$ is non–principal, let x be the usual limit of the sequence (i.e. $x = \lim_{n\to\infty} x_n$). Given $\varepsilon > 0$, one should have either $A_\varepsilon \in \mathcal{U}$ or $\mathbb{N}\backslash A_\varepsilon \in \mathcal{U}$ (by property 4)). Since $\mathbb{N}\backslash A_\varepsilon$ is evidently finite, by the previous Lemma one can conclude that $A_\varepsilon \in \mathcal{U}$ for any $\varepsilon > 0$.

1.3.5 Remark. A more refined version of the previous argument shows that every bounded net $\{x_i\}_{i\in\mathbb{I}}$ is convergent over any given ultrafilter $\mathcal{U}$ on $\mathbb{I}$. Indeed, one can use a Bolzano–Weierstrass argument as follows: let $\{x_i\}_{i\in\mathbb{I}} \subseteq [-M, M]$, and set $R_1 = [-M, 0]$, $R_2 = (0, M]$ and $F_\alpha = \{i \in \mathbb{I} : x_i \in R_\alpha\}$, $\alpha = 1, 2$. One, and only one, among F_1 and F_2 belongs to $\mathcal{U}$. (If it happens that F_2 is in $\mathcal{U}$, we change R_2 with its closure $\overline{R_2}$ and thus we obtain a subset of $\mathbb{I}$ which contains F_2, and so is still in $\mathcal{U}$). Proceeding in this way, we get a sequence of closed sets R_n, whose diameters halves at each step and containing infinitely many elements of the net. Then $\bigcap R_n$ is a singleton $\{x\}$, and then it follows that $\lim_{\mathcal{U}} x_i = x$.

More generally, we want to link ultrafilters with the concept of convergence in topological spaces.

1.3.6 Definition. *Let* $(X, \mathcal{T})$ *be a Hausdorff topological space,* $\{x_i\}_{i\in\mathbb{I}} \subseteq X$*, with* $\mathbb{I}$ *an index set, and let* $\mathcal{U}$ *be an ultrafilter on* $\mathbb{I}$*. Then, we say that* $\lim_{\mathcal{U}} x_i (\cong \mathcal{T} - \lim_{\mathcal{U}} x_i) = x_0$*, if for every neighbourhood* N *of* x_0 *we have* $\{i \in \mathbb{I} : x_i \in N\} \in \mathcal{U}$*.*

Notice that limits over $\mathcal{U}$ are unique, and if $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and $\{x_i\}$ is a bounded sequence in $\mathbb{R}$, then

$$\liminf_n x_n \le \lim_{\mathcal{U}} x_n \le \limsup_n x_n.$$

Moreover, if C is a closed subset of X and $\{x_i\}_{i\in\mathbb{I}} \subseteq C$, then $\lim_{\mathcal{U}} x_i$ belongs to C, whenever it exists.

1.3.7 Remark. Let X be a metric space. If $\mathcal{U}$ is an ultrafilter over $\mathbb{N}$ and $\lim_{\mathcal{U}} x_n = x$, with $\{x_n\} \subset X$, then there exists a subsequence of $\{x_n\}$ which converges to x. Indeed, if we set $U_k := \{k \in \mathbb{N} : d(x_n, x) < 1/k\}$, then $U_k \in \mathcal{U}$. For each k, let n_k be the smallest element in U_k. The subsequence $\{x_{n_k}\}$ clearly converges to x.

The next theorem characterizes compactness by use of ultrafilters.

1.3.8 Theorem. [1] *Let K be a Hausdorff topological space. Then K is compact if, and only if, $\lim_{\mathcal{U}} x_i$ exists for all $\{x_i\}_{i\in\mathbb{I}} \subseteq K$ and any ultrafilter $\mathcal{U}$ over $\mathbb{I}$.*

Proof. Assume that K is compact. Let $\{x_i\}_{i\in\mathbb{I}}$ be a subset of K and $\mathcal{U}$ an ultrafilter over $\mathbb{I}$. Suppose that $\{x_i\}_{i\in\mathbb{I}}$ does not converge to any $x \in K$. Then, each x in K has a neighborhood V_x such that $\{i \in \mathbb{I} : x_i \in V_x\} \notin K$. Since $K \subset \bigcup_{x\in K} V_x$ and K is compact, there exist $V_{y_1}, V_{y_2}, \ldots, V_{y_n}$ such that $K \subset \bigcup_{j=1}^n V_{y_j}$, which implies that $\mathbb{I} = \bigcup_{j=1}^n I_j$, where $I_j := \{i \in \mathbb{I} : x_i \in V_{y_j}\}$. Now, if we show that some $I_k \in \mathcal{U}$, we get a contradiction, thus proving the implication. So, assume, to the contrary, that $I_j \notin K$ for $j = 1, \ldots, n$. Then, by property 4) in Definition 1.3.1, we have $\mathbb{I}\backslash I_j \in \mathcal{U}$, and thus $\bigcap_{j=1}^n (\mathbb{I}\backslash I_j) = \emptyset \in K$, which is a contradiction.

Conversely, suppose that any subset $\{x_i\}_{i\in\mathbb{I}}$ of K is convergent over any ultrafilter $\mathcal{U}$ on $\mathbb{I}$. Let $\{F_\alpha\}_{\alpha\in\Gamma}$ be a family of closed subsets of K which has the finite intersection property (i.e., any finite subfamily of $\{F_\alpha\}_{\alpha\in\Gamma}$ has non..empty intersection). We will prove that $\bigcap_\alpha F_\alpha \neq \emptyset$, which will imply that K is indeed compact. So, consider the set $\mathbb{I} := \{A \subset \Gamma : A \text{ is finite}\}$ and let $x_A \in \bigcap_\alpha F_\alpha$. Set $\mathcal{B} := \{[A, \infty) := \{B \in \mathbb{I} : A \subset B\} : A \in 2^{\mathbb{I}}\}$. Since $[A, \infty) \cap [A', \infty) = [A \cup A', \infty)$, $\mathcal{B}$ is stable under intersection, and we may define a filter on $\mathbb{I}$ by setting $\mathcal{F}(\mathcal{B}) := \{A \subset \mathbb{I} : \text{ there is a } B \in \mathcal{B} \text{ such that } B \subset A\}$. Then, $\mathcal{F}(\mathcal{B})$ is a proper filter (i.e., $\mathcal{F}(\mathcal{B}) \neq 2^{\mathbb{I}}$) because $\emptyset \neq \mathcal{B}$. Let $\mathcal{U}$ be some ultrafilter on $\mathbb{I}$ which extends $\mathcal{F}(\mathcal{B})$. We can now use the assumption on K that every subset is convergent over any ultrafilter, and deduce that $\lim_{\mathcal{U}} x_A = x$ exists.
In order to complete the proof, we have to show that x is an element of every F_α. Suppose, to the contrary, that $x \notin F_{\alpha'}$ for some $\alpha' \in \Gamma$. Then, x has a neighborhood V_x such that $V_x \cap F_{\alpha'} = \emptyset$. Since $\lim_{\mathcal{U}} x_A = x$, we have $I_x := \{A \in \mathbb{I} : x_A \in V_x\} \in \mathcal{U}$. We also know that $I_x \cap [\{\alpha'\}, \infty) \in \mathcal{U}$, because $[\{\alpha'\}, \infty) \in \mathcal{B} \subset \mathcal{F}(\mathcal{B}) \subset \mathcal{U}$. Now, for any $A \in I_x \cap [\{\alpha'\}, \infty) \in \mathcal{U}$, we have both $x_A \in V_x$ and $x_A \in \bigcap_{\gamma\in A} F_\gamma \subset F_{\alpha'}$, which contradicts the assumption that $V_x \cap F_{\alpha'} = \emptyset$. Hence, $I_x \cap [\{\alpha'\}, \infty) = \emptyset \in \mathcal{U}$. But this is also a contradiction, and the proof is now complete. ∎

1.3.9 Remark. As seen in Theorem 1.2.18, the space $(\mathcal{QCM}, \mathrm{dist}_q)$ of isometry equivalence classes of compact quantum metric spaces is itself a complete metric space. Since it is also separable (cf. Theorem 13.15 in [63]), $(\mathcal{QCM}, \mathrm{dist}_q)$ is a Hausdorff space. Therefore, if $\mathcal{S}$ is any totally bounded subset in $\mathcal{QCM}$, its closure $\overline{\mathcal{S}}$ is compact, and we can apply the previous Theorem to deduce that any sequence $(A_n, L_n)_{n\in\mathbb{N}} \subset \overline{\mathcal{S}}$ converges to $\lim_{\mathcal{U}}(A_n, L_n)$ for any ultrafilter $\mathcal{U}$ over $\mathbb{N}$.

When the space is a linear topological vector space, the convergence over an ultrafilter shares the same properties with the usual convergence. In particular, we have the following

1.3.10 Proposition. [44; 68] *Let X be a linear topological vector space, and $\mathcal{U}$ an ultrafilter over an index set $\mathbb{I}$.*

(i) *Suppose that $\{x_i\}_{i\in\mathbb{I}}$ and $\{y_i\}_{i\in\mathbb{I}}$ are two subsets of X such that $\lim_{\mathcal{U}} x_i$ and $\lim_{\mathcal{U}} y_i$ exist. Then*

$$\lim_{\mathcal{U}} x_i + y_i = \lim_{\mathcal{U}} x_i + \lim_{\mathcal{U}} y_i, \quad \text{and} \quad \lim_{\mathcal{U}} \alpha x_i = \alpha \lim_{\mathcal{U}} x_i$$

for any scalar $\alpha \in \mathbb{R}$.

(ii) *If* X *is a Banach lattice and* $\{x_i\}_{i\in\mathbb{I}}$ *is a subset of positive elements of* X*, then* $\lim_{\mathcal{U}} x_i$ *is also positive.*

We can now introduce the notion of ultrapower of a Banach space. So, let X be a Banach space and $\mathcal{U}$ an ultrafilter over an index set $\mathbb{I}$. We define

$$\ell^\infty(X) := \{\{x_i\}_{i\in\mathbb{I}} : \|\{x_i\}\| := \sup_{i\in\mathbb{I}} \|x_i\| < \infty\}. \tag{1.30}$$

Then,

$$N_{\mathcal{U}}(X) := \{\{x_i\}_{i\in\mathbb{I}} \in \ell^\infty(X) : \lim_{\mathcal{U}} \|x_i\| = 0\} \tag{1.31}$$

is a closed linear subspace of $\ell^\infty(X)$ (see Proposition 1.3.15 below).

1.3.11 Definition. *The Banach space ultrapower of* X *over* $\mathcal{U}$ *is defined to be the Banach space quotient*

$$\ell^\infty(X,\mathcal{U}) := \ell^\infty(X)/N_{\mathcal{U}}(X), \tag{1.32}$$

with elements denoted by $[x_i]_{\mathcal{U}}$*, where* $\{x_i\}$ *is a representative of the equivalence class. The quotient norm is canonically given by*

$$\|[x_i]_{\mathcal{U}}\| = \inf\{\|\{x_i + y_i\}\| : \{y_i\} \in N_{\mathcal{U}}(X)\}. \tag{1.33}$$

1.3.12 Proposition. [1] *The quotient norm on* $\ell^\infty(X,\mathcal{U})$ *satisfies*

$$\|[x_i]_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\| \quad \text{for any } [x_i]_{\mathcal{U}} \in \ell^\infty(X,\mathcal{U}). \tag{1.34}$$

Proof. Let $x_{\mathcal{U}} \equiv [x_i]_{\mathcal{U}}$ be in $\ell^\infty(X,\mathcal{U})$. Then, $x_{\mathcal{U}} = \{\{x_i + y_i\} : \{y_i\} \in N_{\mathcal{U}}(X)\}$, and thus

$$\lim_{\mathcal{U}} \|x_i + y_i\| = \lim_{\mathcal{U}} \|x_i\| \le \|\{x_i + y_i\}\|,$$

for any $\{y_i\} \in N_{\mathcal{U}}(X)$, which implies that $\lim_{\mathcal{U}} \|x_i\| \le \|x_{\mathcal{U}}\|$. As for the reverse inequality, consider the set

$$I_\varepsilon := \{i \in \mathbb{I} : \|x_i\| \le \lim_{\mathcal{U}} \|x_i\| + \varepsilon\}, \quad \varepsilon > 0.$$

By the very definition of limit over $\mathcal{U}$, we have $I_\varepsilon \in \mathcal{U}$. Now, define $\{y_i\}$ by setting $y_i = -x_i$ if $i \notin I_\varepsilon$ and $y_i = 0$ otherwise. Then, $\lim_{\mathcal{U}} \|y_i\| = 0$, and so $\{x_i + y_i\}$ is a representative of $x_{\mathcal{U}}$. But $\|\{x_i + y_i\}\| = \sup_{i\in I_\varepsilon} \|x_i\|$, which implies that

$$\|\{x_i + y_i\}\| \le \lim_{\mathcal{U}} \|x_i\| + \varepsilon.$$

Hence, $\|x_{\mathcal{U}}\| \le \|\{x_i + y_i\}\| \le \lim_{\mathcal{U}} \|x_i\| + \varepsilon$. By the arbitrariness of ε, the claim follows. ∎

1.3.13 Remark. The map $\pi_{\mathcal{U}} : X \to \ell^\infty(X,\mathcal{U})$, defined by

$$\pi_{\mathcal{U}}(x) := [x_i] := [x_i]_{\mathcal{U}}, \quad \text{where } x_i = x, \text{ for all } i \in \mathbb{I}, \tag{1.35}$$

is an isometric embedding of X into $\ell^\infty(X,\mathcal{U})$. Using the map $\pi_{\mathcal{U}}$, one may identify X with $\pi_{\mathcal{U}}(X)$, so that X can be regarded as a subspace of $\ell^\infty(X,\mathcal{U})$.

1.3.14 Example (Ultrapowers of a Hilbert space). It is known that a Banach space X is a Hilbert space if, and only if, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in X$. Let $\ell^\infty(X,\mathcal{U})$ be an ultrapower of X and let $[x_i]$, and $[y_i]$ be two elements in $\ell^\infty(X,\mathcal{U})$. Then, we have

$$\|[x_i]+[y_i]\|^2 = \|[x_i+y_i]\|^2 = \lim_{\mathcal{U}} \|x_i+y_i\|^2$$

and

$$\|[x_i]-[y_i]\|^2 = \|[x_i-y_i]\|^2 = \lim_{\mathcal{U}} \|x_i-y_i\|^2$$

Since

$$\lim_{\mathcal{U}} \|x_i+y_i\|^2 + \lim_{\mathcal{U}} \|x_i-y_i\|^2 = \lim_{\mathcal{U}} (\|x_i+y_i\|^2 + \|x_i-y_i\|^2)$$

and using the Hilbert structure of X, we get

$$\lim_{\mathcal{U}} \|x_i+y_i\|^2 + \lim_{\mathcal{U}} \|x_i-y_i\|^2 = \lim_{\mathcal{U}} (2\|x_i\|^2 + 2\|y_i\|^2)$$

Hence,

$$\|[x_i]+[y_i]\|^2 + \|[x_i]-[y_i]\|^2 = 2\,\|[x_i]\|^2 + \|[y_i]\|^2\,,$$

which implies that $\ell^\infty(X,\mathcal{U})$ is a Hilbert space.

The ultraproduct of a family of Banach spaces is defined in a similar manner. In fact, let $\{X_i\}_{i\in\mathbb{I}}$ be a family of Banach spaces and $\mathcal{U}$ an ultrafilter over the index set $\mathbb{I}$. We define

$$\ell^\infty(X_i) := \{\{x_i\}_{i\in\mathbb{I}} : \|x_i\| := \sup_{i\in\mathbb{I}} \|x_i\| < \infty\}. \tag{1.36}$$

Then, as above, the subspace

$$N(X_i,\mathcal{U}) := \{\{x_i\}_{i\in\mathbb{I}} \in \ell^\infty(X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\} \tag{1.37}$$

is a closed linear subspace of $\ell^\infty(X_i)$. Indeed, we have the following

1.3.15 Proposition. [1] *$N(X_i,\mathcal{U})$ is a closed subspace of $\ell^\infty(X_i)$.*

Proof. By construction, $N(X_i,\mathcal{U})$ is clearly a subspace of $\ell^\infty(X_i)$. In order to show that it is closed, Take a Cauchy sequence $\{\{x_i^n\}_{i\in\mathbb{I}}\}_{n\in\mathbb{N}}$ in $\ell^\infty(X_i)$ with $\{x_i^n\}_{i\in\mathbb{I}} \in N(X_i,\mathcal{U})$ for each $n \in \mathbb{N}$. Since $\ell^\infty(X_i)$ is complete, we know that $\{\{x_i^n\}_{i\in\mathbb{I}}\}_{n\in\mathbb{N}}$ converges to some $\{x_i\} \in \ell^\infty(X_i)$. Hence, let $\varepsilon > 0$ be given, and consider the nonempty set $J_\varepsilon := \{n : \|\{x_i^n\} - \{x_i\}\| \le \varepsilon\}$. For any $n \in J_\varepsilon$ and $i \in \mathbb{I}$, we have $\|x_i^n - x_i\| \le \varepsilon$. Then, $\lim_{\mathcal{U}} \|x_i\| \le \lim_{\mathcal{U}} \|x_i^n\| + \varepsilon$. Since $\{x_i^n\} \in N(X_i,\mathcal{U})$, we have $\lim_{\mathcal{U}} \|x_i\| \le \varepsilon$, which completes the proof. ∎

1.3.16 Definition. *The ultraproduct of the family of Banach spaces $\{X_i\}_{i\in\mathbb{I}}$ is defined to be the Banach space quotient*

$$\ell^\infty(X_i,\mathcal{U}) := \ell^\infty(X_i)/N(X_i,\mathcal{U}), \tag{1.38}$$

with elements denoted by $[x_i]_{\mathcal{U}}$, where $\{x_i\}$ is a representative of the equivalence class. In view of Proposition 1.3.12, the quotient norm is given by

$$\|[x_i]_{\mathcal{U}}\| := \lim_{\mathcal{U}} \|x_i\|. \tag{1.39}$$

1.3.17 Example (Ultralimits of sequences of pointed metric spaces). Recall that a pointed metric space (X, x, ρ) is a metric space (X, ρ) with a distinguished point (the *basepoint*) $x \in X$. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $(X_i, x_i, \rho_i)_{i\in\mathbb{N}}$ be a sequence of (proper) metric spaces with basepoint x_i. Consider the set

$$X_\infty := \{\{y_i\}_{i\in\mathbb{N}} : \rho_i(x_i, y_i) < \infty\}. \tag{1.40}$$

Since $\{\rho_i(x_i, y_i)\}_{i\in\mathbb{N}}$ is a bounded sequence, we may define $\widetilde{\rho}_\mathcal{U} : X_\infty \times X_\infty \to \mathbb{R}$ by $\widetilde{\rho}_\mathcal{U}(y, z) := \lim_\mathcal{U} \rho_i(y_i, z_i)$. Then, $\widetilde{\rho}_\mathcal{U}$ is a pseudo–distance. We define the ultralimit of the sequence (X_i, x_i, ρ_i) to be the quotient metric space $(X_\mathcal{U}, x_\mathcal{U}, \rho_\mathcal{U})$, where $x_\mathcal{U} := [x_i]_\mathcal{U}$ denotes the element corresponding to $\{x_i\} \in X_\infty$ in the quotient.

1.3.18 Lemma. *If $\{(X_i, x_i, \rho_i)\}_{i\in\mathbb{N}}$ is a sequence of pointed metric spaces, then $(X_\mathcal{U}, x_\mathcal{U}, \rho_\mathcal{U})$ is complete.*

Proof. Let $y^n_\mathcal{U}$ be a Cauchy sequence in $X_\mathcal{U}$, where $y^n_\mathcal{U} = [y^n_i]_\mathcal{U}$. Let $U_0 = \mathbb{N}$, and define inductively the family of subsets U_k by

$$U_k := \{n \geq k : |\rho_n(y^j_n, y^i_n) - \rho_\mathcal{U}(y^j_\mathcal{U}, y^i_\mathcal{U})| < 1/2^k,\ 1 \leq i, j \leq k\}.$$

Then, $U_k \in \mathcal{U}$, $U_{k+1} \subseteq U_k$ and $\bigcup_{k\geq 0} U_k \backslash U_{k+1} = \mathbb{N}$. For $n \in U_k \backslash U_{k+1}$, let us define $z_n = y^k_n$. Then, $z_\mathcal{U} = [y^k_n]_\mathcal{U}$, since the sequence $\{\rho_n(x_n, z_n)\}$ is bounded, and thus $y^j_\mathcal{U} \to z_\mathcal{U}$ as $j \to \infty$. Indeed, we have

$$|\rho_n(y^j_n, z_n) - \rho_\mathcal{U}(y^j_\mathcal{U}, z_\mathcal{U})| < \frac{1}{2^k}.$$

Hence, since $j \to \infty$ implies $k \to \infty$, and $\lim_{j\to\infty} \rho_n(y^j_n, z_n) = \lim_{j\to\infty} \rho_n(y^j_n, y^k_n) = 0$, we get $\rho_\mathcal{U}(y^j_\mathcal{U}, z_\mathcal{U}) \to 0$ for $j \to \infty$, as claimed. ∎

Let us notice that one of the main benefits of the ultralimit formulation is to avoid the messy process of passing to subsequences, sub–subsequences, and so on, by making once and for all some choice in advance. For instance, the ultralimit construction naturally gives an embedding between two limits in each other, provided that an embedding is given between elements with the same index. In fact, let $\{(Y_i, d_i)\}_{i\in\mathbb{N}}$, $\{(X_i, \rho_i)\}_{i\in\mathbb{N}}$ be two sequences of proper metric spaces, and let let $\iota_i : Y_i \to X_i$, $i \in \mathbb{N}$ be an isometric embedding of Y_i into X_i. Fix a basepoint $y_i \in Y_i$ for each $i \in \mathbb{N}$, set $x_i \cong \iota_i(y_i)$, and consider the two sequences $\{(Y_i, y_i, d_i)\}_{i\in\mathbb{N}}$, $\{(X_i, x_i, \rho_i)\}_{i\in\mathbb{N}}$ of pointed metric spaces. Then, we have the following

1.3.19 Lemma. *For any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the ultralimit $(Y_\mathcal{U}, y_\mathcal{U}, d_\mathcal{U})$ of the sequence $\{(Y_i, y_i, d_i)\}_{i\in\mathbb{N}}$ isometrically embeds into the ultralimit $(X_\mathcal{U}, x_\mathcal{U}, \rho_\mathcal{U})$ of $\{(X_i, x_i, \rho_i)\}_{i\in\mathbb{N}}$.*

Proof. Indeed, as $\iota_i : Y_i \to X_i$ are isometric embeddings for all $i \in \mathbb{N}$, the family $\{\iota_i\}$ induces an isometric embedding $\iota_\mathcal{U} : Y_\mathcal{U} \to X_\mathcal{U}$ by setting $\iota_\mathcal{U}(z_\mathcal{U}) := [\iota_i(z_i)]_\mathcal{U}$ for any sequence $\{z_i\} \in Y_\infty$. In fact, for any two sequences $\{z_i\}, \{\widetilde{z}_i\}$ in the same equivalence class $z_\mathcal{U}$, one has $\lim_\mathcal{U} d_i(z_i, \widetilde{z}_i) = \lim_\mathcal{U}(\iota_i(z_i), \iota_i(\widetilde{z}_i)) = 0$, which implies $d_\mathcal{U}(z_\mathcal{U}, \widetilde{z}_\mathcal{U}) = \rho_\mathcal{U}(\iota_\mathcal{U}(z_\mathcal{U}), \iota_\mathcal{U}(\widetilde{z}_\mathcal{U})$. In particular, $x_\mathcal{U} = \iota_\mathcal{U}(y_\mathcal{U})$, and the claim follows. ∎

1.4 Towards a Quantum Metric Tangent Space

1.4.1 The pointed Gromov–Hausdorff convergence for (proper) metric spaces

Let us briefly recall the basic definitions of pointed-Gromov–Hausdorff convergence and of tangent sets of a metric space. The reader is referred to [30; 10; 14] for more details.

1.4.1 Notation. If (X, ρ) is a metric space, we shall denote by $B(x, r)$ the open ball $\{y \in X : d(x, y) < r\}$, by $\overline{B}(x, r)$ the closed ball $\{y \in X : d(x, y) \leq r\}$, and by $\overline{B(x, r)}$ the closure of $B(x, r)$. Moreover, we let $\mathcal{N}_\varepsilon(E) := \{x \in X : \inf_{y \in E} d(x, y) < \varepsilon\}$ be the ε–neighborhood of the subset E in X.

In the case of non–compact, proper (i.e., boundedly compact) metric spaces, one is lead to consider the pointed Gromov–Hausdorff topology as a good substitute for the Gromov–Hausdorff topology, which is defined only for family of compact spaces. So, let (X, x) be a pointed metric space. There are several equivalent manners to define this topology.

1.4.2 Definition. [29] *A neighbourhood base for the pointed Gromov–Hausdorff topology consists of the sets $\mathcal{N}_\varepsilon(X, x)$, $\varepsilon \in (0, \frac{1}{2})$, where*

$$\mathcal{N}_\varepsilon(X, x) := \{(Y, y) : \tilde{d}_H((X, x), (Y, y)) < \varepsilon\}, \tag{1.41}$$

and $\tilde{d}_H((X, x), (Y, y))$ is defined as the infimum of the $\varepsilon > 0$ for which there is an admissible metric ρ on the disjoint union $X \amalg Y$ of X and Y, such that $\rho(x, y) < \varepsilon$, $\overline{B_X}(x, \frac{1}{\varepsilon}) \subset B_\varepsilon(Y)$, and $\overline{B_Y}(y, \frac{1}{\varepsilon}) \subset B_\varepsilon(X)$.

As pointed out by Gromov in [29], the function $\tilde{d}_H$ is not properly a distance (Gromov calls it a "modified Hausdorff distance"): it satisfies the triangle inequality provided that at least two of the three "distances" involved are small enough (say $\leq 1/2$). Nevertheless, the family of sets $\mathcal{N}_\varepsilon(X, x)$ do define a neighbourhood base.

Equivalently,

1.4.3 Definition. [29] *A neighbourhood base for the pointed Gromov–Hausdorff topology consists of the sets $\mathcal{N}_{R,\varepsilon}(X, x)$, with $R > 0$ and $\varepsilon \in (0, 1)$, where*

$$\mathcal{N}_{R,\varepsilon}(X, x) := \{(Y, y) : d_R((X, x), (Y, y)) < \varepsilon\}, \tag{1.42}$$

and $d_R((X, x), (Y, y))$ is defined as the infimum of the $\varepsilon > 0$ such that there are isometric embeddings h_X, h_Y of X and Y into a metric space (Z, ρ) for which $\rho(h_X(x), h_Y(y)) < \varepsilon$, $h_X(\overline{B_X}(x, R)) \subset \mathcal{N}_\varepsilon(h_Y(Y))$ and $h_Y(\overline{B_Y}(y, R)) \subset \mathcal{N}_\varepsilon(h_X(X))$.

On the isometry classes of proper metric spaces, the pointed Gromov–Hausdorff topology is a Hausdorff topology, and, since it is separable, it is determined by its converging sequences. Indeed, it is also equivalently defined by the following

1.4.4 Proposition. [29] *(X_n, x_n) converges to (X, x) in the pointed Gromov–Hausdorff topology if, and only if, for any $R > 0$ there exists a positive infinitesimal sequence ε_n such that, for any $\eta > 0$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, there are isometric embeddings h_n, h of $\overline{B_{X_n}}(x_n, R + \varepsilon_n)$ and $\overline{B_X}(x, R)$ into a metric space (Z_n, ρ_n) for which $\rho_n(h_n(x_n), h(x)) < \eta$, $h_n(\overline{B_{X_n}}(x_n, R + \varepsilon_n)) \subset \mathcal{N}_\eta(h(\overline{B_X}(x, R)))$ and $h_n(\overline{B_X}(x, R)) \subset \mathcal{N}_\eta(h(\overline{B_{X_n}}(x_n, R + \varepsilon_n)))$.*

We may rephrase the notion of pointed Gromov–Hausdorff convergence in terms of ultralimits. Indeed, we have the following

1.4.5 Theorem. *If the family $(X_i, x_i, \rho_i)_{i\in\mathbb{N}}$ of pointed metric spaces is precompact in the pointed Gromov–Hausdorff topology (i.e., for any $r > 0$, the family $(\overline{B_i}(x_i, r), \rho_i)_{i\in\mathbb{N}}$ is precompact in the Gromov–Hausdorff topology), then $(X_\mathcal{U}, x_\mathcal{U}, \rho_\mathcal{U})$ is a limit point of the sequence $(X_i, x_i, \rho_i)_{i\in\mathbb{N}}$ for any free ultrafilter $\mathcal{U}$ over $\mathbb{N}$.*

Proof. Let $\mathcal{U}$ be a free ultrafilter. Given $\varepsilon > 0$ and $r > 0$, then, by total boundedness, there exists an $N(\varepsilon, r) \in \mathbb{N}$ such that we can find a family $\{y_i^j\}_{j=1}^{N(\varepsilon,r)} \subset \overline{B_i}(x_i, r)$ of $N(\varepsilon, r)$ elements, which is an ε–net in $\overline{B_i}(x_i, r)$ for any i. The $N(\varepsilon, r)$ sequences $\{y_i^j\}_{i\in\mathbb{N}}$ for $1 \le j \le N(\varepsilon, r)$ give us $N(\varepsilon, r)$ elements $y_\mathcal{U}^j \in X_\mathcal{U}$. If $z_\mathcal{U} \in \overline{B_\mathcal{U}}(x_\mathcal{U}, r)$, then we can find an element $U_r \in \mathcal{U}$ such that $\rho_i(x_i, z_i) < r$ for any $i \in U_r$. Moreover, for $\varepsilon > 0$ given, there exists an $U_\varepsilon \in \mathcal{U}$ such that $|\rho_\mathcal{U}(y_\mathcal{U}^j, z_\mathcal{U}) - \rho_i(y_i^j, z_i)| < \varepsilon$ for $i \in U_\varepsilon$, which implies that $\rho_\mathcal{U}(y_\mathcal{U}^j, z_\mathcal{U}) < 2\varepsilon$ for some $1 \le j \le N(\varepsilon, r)$. Hence, $\overline{B_\mathcal{U}}(x_\mathcal{U}, r)$ is totally bounded, and for all $\varepsilon > 0$ there is a 2ε–net in $\overline{B_\mathcal{U}}(x_\mathcal{U}, r)$ which is a Gromov–Hausdorff limit point of ε–nets in the balls $\overline{B_i}(x_i, r)$. It follows then that (X_i, x_i, ρ_i) converges over $\mathcal{U}$ to $(X_\mathcal{U}, x_\mathcal{U}, \rho_\mathcal{U})$ in the pointed Gromov–Hausdorff topology. ∎

In particular, the conclusion of the previous Theorem clearly holds for any precompact sequence of compact metric spaces.

1.4.6 Remark. The ultralimit formulation of the (pointed) Gromov–Hausdorff convergence allows us to find an embedding between two limits in each other, provided that an embedding is given between elements with the same index (see Lemma 1.3.19 above). In this way, the GH–limits of any precompact sequence of compact subspaces $Y_i \subset X_i$ (containing the basepoint x_i) canonically embed into $X_\mathcal{U}$. As a consequence, if (X_∞, x_∞) is the pointed Gromov–Hausdorff limit of the sequence of proper spaces (X_n, x_n), then, for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultralimit $(X_\mathcal{U}, x_\mathcal{U})$ is isometric to (X_∞, x_∞).

1.4.7 Lemma. *Let $\{X_n\}$ be a sequence of compact metric spaces, and let $Y_n \subseteq X_n$ be a closed subspace for each $n \in \mathbb{N}$. If $\{X_n\}$ converges to X in the Gromov–Hausdorff topology, then there exists a subsequence of $\{Y_n\}$ Gromov–Hausdorff converging to a closed subset $Y \subseteq X$.*

Proof. In fact, since $Y_n \subseteq X_n$, the sequence $\{Y_n\}$ is precompact in the Gromov–Hausdorff topology. Therefore, it converges to its ultralimit $Y_\mathcal{U}$ for any ultrafilter $\mathcal{U}$. Let $X_\mathcal{U}$ be the ultralimit of $\{X_n\}$ over the same ultrafilter. Now, $X_\mathcal{U}$ is isometric to X (call $\varphi_\mathcal{U} : X \to X_\mathcal{U}$ this isometry), and we have $Y_\mathcal{U} \subseteq X_\mathcal{U}$ by Lemma 1.3.19 (since $\iota_\mathcal{U}(y_\mathcal{U}) = [\iota_i(y_i)]_\mathcal{U} = [y_i]_\mathcal{U} = y_\mathcal{U}$ for any $y_\mathcal{U} \in Y_\mathcal{U}$). Let $Y := \varphi_\mathcal{U}^{-1}|_{Y_\mathcal{U}}(Y_\mathcal{U})$ be the corresponding isometric copy of $Y_\mathcal{U}$ in X. Then, $Y \subseteq X$ and we can find a subsequence of $\{Y_n\}$ converging to Y in the Gromov–Hausdorff topology. ∎

Then, from the previous characterization, one can easily prove the following

1.4.8 Proposition. *If $\{(X_n, x_n)\}$ converges to (X, x) in the pointed Gromov–Hausdorff topology, then, for any ultrafilter $\mathcal{U}$ over $\mathbb{N}$, we have $B_\mathcal{U} = \mathrm{GH} - \lim_\mathcal{U} \overline{B_{X_n}}(x_n, R)$, with $B_X(x, R) \subseteq B_\mathcal{U} \subseteq \overline{B_X}(x, R)$.*

Proof. Indeed, let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. Then, by Theorem 1.4.5 above, the pointed Gromov–Hausdorff limit (X, x) coincides with (i.e., is isometric to) the ultralimit $(X_{\mathcal{U}}, x_{\mathcal{U}})$ over $\mathcal{U}$ of the sequence $\{(X_n, x_n)\}$, and we have

$$\overline{B_X}(x, R) = \{[y_n]_{\mathcal{U}} : \lim_{\mathcal{U}} \rho_n(x_n, y_n) \leq R\},$$

while $\overline{B_X(x, R)}$ coincides with the closure of of the open ball

$$B_X(x, R) = \{[y_n]_{\mathcal{U}} : \lim_{\mathcal{U}} \rho_n(x_n, y_n) < R\} = \{[y_n]_{\mathcal{U}} : \rho_n(x_n, y_n) < R, \, \forall n \in \mathbb{N}\},$$

where the second equality follows from the fact that, since $\lim_{\mathcal{U}} \rho_n(x_n, y_n) < R$ implies $\rho_n(x_n, y_n) < R$ eventually, one may always replace the elements of a sequence $\{y_n\} \in [y_n]_{\mathcal{U}}$ with distance from the basepoint greater than R, with points at distance strictly smaller than R, and the equivalence class does not change. Finally, we have

$$B_{\mathcal{U}} = \mathrm{GH} - \lim_{\mathcal{U}} \overline{B_{X_n}(x_n, R)} = \{[y_n]_{\mathcal{U}} : \rho_n(x_n, y_n) \leq R, \, \forall n \in \mathbb{N}\},$$

and thus, $\overline{B_X(x, R)} \subseteq B_{\mathcal{U}} \subseteq \overline{B_X}(x, R)$, as claimed. Let us notice that $B_{\mathcal{U}}$ clearly depends on $\mathcal{U}$, as well as the embeddings. ∎

Tangent sets of abstract metric spaces at a point have been defined by Gromov (cf. [30; 10; 14]), as a natural generalization of the notion of tangent cone at a convex set, e.g. in $\mathbb{R}^n$.

1.4.9 Definition (Tangent Cone). *Let (X, ρ) be a (proper) metric space, and $x \in X$. A* tangent set *of X at x is any limit point, for $t \to \infty$, of $(X, x, t\rho)$ in the pointed Gromov–Hausdorff topology, where $t\rho$ denotes the rescaled distance by the parameter t. We write also tX for $(X, x, t\rho)$ when the metric and x are clear from the context. We shall denote by $\mathcal{T}_x X$, and call it the* tangent cone *of X at x, the family of tangent sets of X at x. A* tangent ball *of X at x is any ball centered in x of some tangent set $T \in \mathcal{T}_x X$.*

Recall that $Cov_\rho(X, r)$ denotes the minimum number of open balls of radius r necessary to cover a subset $E \subset X$. Then, as an application of the Gromov compactness criterion (cf. Theorem 1.1.8), one can easily prove the following

1.4.10 Proposition. *Let (X, x) be such that*

$$\limsup_{r \to 0} Cov_\rho(\overline{B_X}(x, r), \lambda r) < \infty \quad \forall \lambda > 0. \tag{1.43}$$

Then, $\mathcal{T}_x X$ is not empty. Indeed, given any sequence $t_n \to +\infty$, there exists a subsequence t_{n_k} for which $(X, x, t_{n_k}\rho)$ converges to a unique proper space in the pointed Gromov–Hausdorff topology.

Proof. Since $Cov_\rho(\overline{B_X}(x, r), \lambda r) = Cov_\rho(\overline{B_{\lambda X}}(x, \lambda r), r)$, the claim follows from the Gromov compactness criterion for compact metric spaces (see Theorem 1.1.8). ∎

Let us recall that a pointed metric space (X, x, ρ) is called a *cone*, if it is invariant under rescaling, i.e. if $(X, x, t\rho)$ is isometric to (X, x, ρ), as pointed spaces, for any $t > 0$.

1.4.11 Remark. We list some basic properties of tangent sets.
(i) A tangent set cannot be empty, since it necessarily contains the basepoint x. However, it may happen that $\mathcal{T}_x X$ is empty, namely that $(X, x, t\rho)$ has no limit points.
(ii) If X is a (Riemannian n-dimensional) manifold, the tangent set at x is unique, and coincides with the ordinary tangent space (see next example; cf., also, [30; 10; 14]).
(iii) $\mathcal{T}_x X$ is dilation invariant in the following sense: if (T, ρ_T) is a tangent set of X at x, given by the converging sequence $(X, x, t_n\rho)$, and $\alpha > 0$, then $(X, x, \alpha t_n \rho)$ converges to $(T, \alpha\rho_T)$. As a consequence, if $\mathcal{T}_x X$ consists of a unique set, such set is a cone in the usual sense. For this reason, one refers to the set $\mathcal{T}_x X$ with the name of *Gromov tangent cone*.
(iv) If all the metric spaces X_n are subsets of the same proper metric space Z, the pointed Gromov–Hausdorff convergence may be replaced by the Attouch–Wets convergence[3]. Let us note that in this case we do not need to specify a point in Z.
(v) If the ambient space Z is dilation invariant, e.g. $Z = \mathbb{R}^n$, then the dilations of a given subset are still subsets of Z. Hence, the tangent sets can be defined as Attouch–Wets limits, and are subsets of Z. Even if the two topologies do not coincide, the families of tangent sets at a given point do.

One may say that the tangent cone is a local notion, since it grasps the behavior of a set in a small neighborhood of a point. On the opposite side, let us mention the asymptotic cones, which do the same, but "near infinity".

1.4.12 Definition (Asymptotic Cone). *Let (X, ρ) be a (proper) metric space, and $x \in X$. An* asymptotic cone *of X, or of X at infinity, is any limit point, for $t \to 0$, if one exists, of $(X, x, t\rho)$ in the pointed Gromov–Hausdorff topology.*

An example: the tangent cone to a Riemannian manifold

As mentioned in Remark (ii) above, the tangent cone to a Riemannian manifold (M, g) at a point m_0 is then the usual tangent space at m_0 with the Euclidean distance function ρ_E defined by the Riemannian metric g, and the dilation is the usual homothety of a vector space. To prove it, we need some preparation. (We will follow [47], Chapter 8.)

1.4.13 Lemma (Approximate isometry criterion). *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, and suppose that A is a subset of X and $f : A \to Y$ is a map, not necessarily continuous, such that*

$$|\rho_Y(f(a_1), f(a_2)) - \rho_X(a_1, a_2)| \leq \delta$$

for all $a_1, a_2 \in A$. Suppose that every point of X lies within ε_X of A and every point of Y lies within ε_Y of the image $f(A)$. Then,

$$\mathrm{dist}_{GH}(X, Y) \leq \max(\varepsilon_X, \varepsilon_Y) + \delta/2.$$

Proof. We shall define a compatible metric on the disjoint union $X \amalg Y$. So, for $x \in X$ and $y \in Y$, set

$$\rho(z, y) := \inf_{a \in A}\{\rho_X(x, a) + \rho_Y(f(a), y)\} + \delta/2.$$

[3] Given a metric space (X, d), the Attouch–Wets topology on the family $CL(X)$ of closed sets of X is the topology that $CL(X)$ inherits from $C(X, \mathbb{R})$, the algebra of $\mathbb{R}$–valued continuous functions equipped with the topology of uniform convergence on bounded subsets of X, under the identification $A \leftrightarrow d(\cdot, A)$, where $\rho(x, A) = \inf_{y \in A} \rho(x, y)$.

The constant $\delta/2$ is required by the triangle inequality. In order to show that ρ defines an admissible metric, we must verify the triangle inequality. Let us check that $\rho(x_1, x_2) \leq \rho(x_1, y) + \rho(y, x_2)$ for $x_1, x_2 \in X$, $y \in Y$ (the other cases are proven similarly). By the definition of ρ,

$$\begin{aligned} \rho(x_1, y) + \rho(y, x_2) &= \inf_{a_1}\{\rho_X(x, a_1) + \rho_Y(f(a_1), y)\} + \delta/2 \\ &\quad + \inf_{a_2}\{\rho_Y(y, f(a_2)) + \rho_X(a_2, x_2)\} + \delta/2. \end{aligned}$$

On the other hand,

$$\rho(x_1, x_2) = \rho_X(x_1, x_2) \leq \rho_X(x_1, a_1) + \rho_X(a_1, a_2) + \rho_X(a_2, x_2),$$

and, by the assumption of the lemma,

$$\begin{aligned} \rho_X(a_1, a_2) &\leq \rho_Y(f(a_1), f(a_2)) + \delta \\ &\leq \rho_Y(f(a_1), y) + \rho_Y((y, f(a_2)) + \delta, \end{aligned}$$

$$\rho(x_1, x_2) \leq \rho_X(x_1, a_1) + \rho_Y(f(a_1), y) + \rho_Y((y, f(a_2)) + \rho_X(a_2, x_2) + \delta.$$

By taking the infimum over all $a_1, a_2 \in A$, we get $\rho(x_1, x_2) \leq \rho(x_1, y) + \rho(y, x_2)$.
By assumption, for any $y \in Y$, there is an $a \in A$ with $\rho_Y(f(a), y) < \varepsilon_Y$. Therefore $\rho(a, y) \leq \rho_Y(f(a), y) + \delta/2 < \varepsilon_Y + \delta/2$, which implies that Y is contained in the δ_1–neighborhood $\mathcal{N}(X, \delta_1)$ for the metric ρ, with $\delta_1 \equiv \varepsilon_Y + \delta/2$. A similar argument shows that $X \subset \mathcal{N}(Y, \delta_2)$ with $\delta_2 \equiv \varepsilon_X + \delta/2$. Thus, we have $\mathrm{dist}_{GH}(X, Y) \leq \max(\delta_1, \delta_2) = \max(\varepsilon_X, \varepsilon_Y) + \delta/2$. ∎

1.4.14 Definition. *A map F satisfying the assumption of Lemma 1.4.13 will be called an* approximate isometry *between X and Y, or a $(\delta, \varepsilon_X, \varepsilon_Y)$*–isometry.

We say that a metric space (X, ρ) has the *continuous expansion property* at $x_0 \in X$ if $\mathcal{N}_h(B(r, x_0)) = B(r + h, x_0)$ for all $r, h > 0$. The continuous expansion property holds for Riemannian spaces, and indeed for any length space, but can fail for discrete metric spaces (cf. [30]).

1.4.15 Proposition. *Let (X, x_0) be a pointed space with a metric ρ_0 that admits dilations δ_t, $t > 0$. Suppose that ρ is another metric defined in a ρ_0–neighborhood U of x_0 and that the estimate*

$$|\rho(\delta_t(\cdot), \delta_t(\cdot)) - \rho_0(\delta_t(\cdot), \delta_t(\cdot))| \equiv |\delta_t^*(\rho - \rho_0)| = o(t) \tag{1.44}$$

holds uniformly on U as $t \to 0$. If both ρ_0 and ρ have the continuous expansion property, then the Gromov tangent cone of (X, x_0, ρ) at x_0 is (X, x_0, ρ_0).

Proof. We write $B_t(r)$ for the ball of radius r about x_0 w.r.t. the metric $(1/t)\rho$, and $B_0(r)$ for the ball of radius r about x_0 w.r.t. ρ_0. We will show that $B_t(r)$ Gromov–Hausdorff converges to $B_0(r)$, using the approximate isometry criterion of Lemma 1.4.13 above.
The estimate in (1.44) means that

$$|\rho(\delta_t q_1, d_t q_2) - \rho_0(\delta_t q_1, \delta_t q_2)| \leq f(t)$$

for all q_1, q_2 in the neighborhood U and some function f such that $f(t)/t \to 0$ as $t \to 0$. Since the δ_t are ρ_0–dilations, $\rho_0(\delta_t q_1, \delta_t q_2) = t\rho_0(q_1, q_2)$, so the estimate can be rewritten as

$$|\frac{1}{t}\rho(\delta_t q_1, \delta_t q_2) - \rho_0(\delta_t q_1, \delta_t q_2)| \leq h(t)$$

where $h(t) = f(t)/t$. This last estimate asserts that δ_t is an $h(t)$–approximate isometry between $B_0(r)$ and $B_t(r)$. The same estimate, applied with $q_1 = x_0$, shows that $\delta_t(B_0(r - h(t))) \subset B_t(r)$ and $B_t(r - f(t)) \subset \delta_t(B_0(r))$. Consequently, if we take the domain A of δ_t to be $\delta_t^{-1}(B_t(r)) \cap B_0(r)$, then we find that $B_0(t - h(t)) \subset A$ and $B_t(r - h(t)) \subset \delta_t(A)$. The continuous expansion property, i.e. $\mathcal{N}_h(B(r)) = B(r + h)$ now implies that the dilation δ_t is a $(h(t), h(t), h(t))$–approximate isometry, and consequently that $\mathrm{dist}_{GH}(B_t(r), B_0(r)) \leq 3h(t)/2$. Since $h(t) \to 0$ as $t \to 0$, the claim then follows. ∎

Finally, we prove the following

1.4.16 Theorem (Riemannian Tangent Cone). *Let (M, g) be an n–dimensional Riemannian manifold. Then, the tangent cone at a point $m_0 \in M$ is the usual tangent space at m_0 with the Euclidean distance function defined by the Riemannian metric.*

Proof. Let $(x_1, \dots, x_n)$ be normal coordinates for a neighborhood U of the point $m_0 \in M$. Then, the estimate $ds^2 = \sum_i (dx_i)^2 + O(|x|^2)$ (i.e., $g_{ij} = \delta_{ij} + O(|x|^2)$) relates the Riemannian metric to the Euclidean metric $\sum_i (dx_i)^2$ associated to the normal coordinates (see, for instance, Lemma 13 in [51]). Let ρ_E be the Euclidean distance function associated to the Euclidean metric tensor δ_{ij}, and let ρ_g be the Riemannian distance function associated to g_{ij}. Let $\delta_t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$ be the Euclidean dilation. The estimate implies that $|\delta_t^*(\rho_g - \rho_E)| = O(t^2)$ uniformly near m_0. This estimate is stronger than that assumed above in Proposition 1.4.15. (Recall that $f(t) = O(t^2)$ means that there is a constant $c > 0$ such that $|f(t)|/t^2 \leq c$.) Hence, Proposition 1.4.15 implies that the Gromov tangent cone to a Riemannian manifold is its standard manifold tangent space, endowed with the Euclidean structure. ∎

1.4.2 From the commutative to the noncommutative setting

In this section, we will illustrate the passage from the ordinary pointed Gromov–Hausdorff topology to a (possible) quantum version of it. As a first step towards the noncommutative setting, we shall rephrase the pointed Gromov–Hausdorff construction in quantum language. We want to show that the above notion of pointed Gromov–Hausdorff convergence of a sequence of proper metric spaces, can be rephrased in terms of the *quantum* Gromv–Hausdorff convergence of a suitable family of (order–unit) spaces of functions. In fact, as the pointed Gromov–Hausdorff limit is essentially an inductive limit over the family of closed balls given by Gromov–Hausdorff limits of sequences of closed balls with "almost" fixed radius (cf. Proposition 1.4.4), we will show that the limit space can be recovered also by taking a (suitable) inductive family of quantum metric spaces, each of which is the quantum Gromov–Hausdorff limit of the sequence of the spaces of Lipschitz functions defined on (quotients of) the original closed balls. So, let us see how to do this.

First of all, we have to "dualize" the construction, for in the quantum setting the main object is no longer the metric space itself but a suitable set of functions on it. Therefore, since we want to end up with an inductive family of quantum metric spaces, we have to pass from an injective family of (subsets of) metric spaces to a projective one, in such a way that the resulting family of quantum metric spaces (the spaces of Lipschitz functions) will be (isometrically) injective w.r.t. the Lip–seminorms. To this aim, let (X, x, ρ) be a pointed proper metric space, and let $\overline{B}(x, r)$ be the closed ball with center x and radius $r > 0$. We define a new distance on it as follows: let

$C(x,r)$, $r>0$, be the complement in X of the open ball $B(x,r)$, i.e. $C(x,r) := X\backslash B(x,r)$, and let $\rho^\bullet$ be the metric on $\overline{B}(x,r)$ given by

$$\rho^\bullet(x_1,x_2) := \min\left(\rho(x_1,x_2), \rho(x_1, C(x,r)) + \rho(x_2, C(x,r))\right), \tag{1.45}$$

where $\rho(y, C(x,r)) = \inf\{\rho(y,z) : z \in C(x,r)\}$. Thus, $\rho^\bullet$ is the quotient metric w.r.t. the equivalence relation defined by: $y \sim z$ if $y = z$ or $y, z \in C(x,r)$ (see, for instance, [72], Proposition 1.4.4). Let us denote by $\overline{B}^\bullet(x,r)$ the quotient of $\overline{B}(x,r)$ w.r.t. this metric. Then, for $x_1, x_2 \in \overline{B}(x, r/2)$, we have clearly $\rho^\bullet(x_1,x_2) = \rho(x_1,x_2)$. Hence, the (identity) map from $\overline{B}(x,r/2)$ into $\overline{B}^\bullet(x,r/2)$ sending x to itself is an isometry w.r.t. ρ on $\overline{B}(x,r/2)$ and to $\rho^\bullet$ on $\overline{B}^\bullet(x,r/2)$.

1.4.17 Remark. Notice that $\overline{B}^\bullet(x,r)$, $r>0$, coincides with the one–point compactification of the open ball $B(x,r)$ if, and only if, $C(x,r) \neq \emptyset$. In fact, if $C(x,r)$ is not empty, we can identify it with the point at infinity of $B(x,r)$.

Now, we consider the unital C^*–algebra of (complex–valued) bounded continuous functions on X, which are constant on $C(x,r)$, $r>0$, that is,

$$C(\overline{B}^\bullet(x,r)) := \{f \in C_b(X) : f(x)|_{C(x,r)} \equiv \mathit{const}\}, \tag{1.46}$$

where $C_b(X)$ is the C^*–algebra of (complex–valued) bounded continuous functions on X, and we endow $C(\overline{B}^\bullet(x,r))$ with the restriction of the usual (global) Lipschitz seminorm

$$L(f) := \sup\{|f(x) - f(y)|/\rho(x,y) : x \neq y\}.$$

The corresponding subspace of real–valued Lipschitz functions

$$\mathrm{Lip}(\overline{B}^\bullet(r)) := \mathrm{Lip}(\overline{B}^\bullet(x,r), L) \tag{1.47}$$

then becomes an order–unit space with a seminorm, and since the metric induced by L on the state space, given by

$$\rho_L(\mu,\nu) = \sup\{|\mu(f) - \nu(f)| : f \in \mathrm{Lip}(\overline{B}^\bullet(r)),\ L(f) \leq 1\},$$

metrizes the w^*–topology, we get an injective sequence of (compact C^*–algebraic) quantum metric spaces, that is,

$$\mathrm{Lip}(\overline{B}^\bullet(r_1)) \hookrightarrow \mathrm{Lip}(\overline{B}^\bullet(r_2)), \quad r_2 \geq r_1. \tag{1.48}$$

Identifying pure states with points via the Gel'fand representation theorem [50], we have then the following

1.4.18 Proposition. *The metric ρ_L coincides with the quotient metric $\rho^\bullet$ on $\overline{B}^\bullet(x,r)$ for any $r>0$.*

Proof. Indeed, let us define $f^y(w) := \rho^\bullet(w,y)$. Then, since we already know that $\rho^\bullet$ is a distance, it satisfies the triangle inequality, thus we have

$$\begin{aligned} L_\rho(f^y) &= \sup\left\{\frac{|f^y(w) - f^y(z)|}{\rho(w,z)} : w \neq z\right\} = \sup\left\{\frac{|\rho^\bullet(w,y) - \rho^\bullet(z,y)|}{\rho(w,z)} : w \neq z\right\} \\ &\leq \sup\left\{\frac{\rho^\bullet(w,z)}{\rho(w,z)} : w \neq z\right\} \leq 1, \end{aligned}$$

and, taking w and z such that $\rho^\bullet(w,z) = \rho(w,z)$, we see that $L_\rho(f^y) = 1$. Thus,

$$\rho_L(w,z) = \sup\{|g(w) - g(z)| : L_\rho(g) \le 1\} \ge |f^z(w) - f^z(z)| = \rho^\bullet(w,z).$$

As for the reverse inequality, we have clearly $\rho_L \le \rho$ (on X). So, if $\rho(w,z) \le r/2$, then $\rho(w,z) = \rho^\bullet(w,z)$ and thus $\rho_L \le \rho^\bullet$. If $\rho(w,z) > r/2$, since the functions in $\mathrm{Lip}(\overline{B}^\bullet(r))$ are constant outside $\overline{B}(r)$, we have, for $y \in C(x,r)$,

$$\rho_L(w,y) = \sup\{|f(w)| : f|_{C(x,r)} \equiv 0, L_\rho(f) \le 1\} = \rho(w, C(x,r)),$$

and thus

$$\rho_L(w,z) \le \rho_L(w,y) + \rho_L(y,z) \le \rho(w, C(x,r)) + \rho(z, C(x,r)).$$

Therefore, $\rho_L \le \rho^\bullet$, and thus $\rho_L = \rho^\bullet$ on $\overline{B}^\bullet(x,r)$, as claimed. ∎

Now, let (X,x,ρ) be a pointed proper metric space, and let $(X,x,\lambda\rho)$, $\lambda > 0$, be the corresponding family of (pointed proper metric) spaces with rescaled metrics. We set $\rho_\lambda := \lambda\rho$, and denote X_λ the space X with this metric. We want to show that the limit points in the metric tangent cone $\mathcal{T}_x X$ of X at x can be recovered "functionally" by a "quantum" procedure. To this end, we replace the continuous parameter λ with an increasing (and divergent) sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of positive numbers, and denote by X_n the space X with the metric $\rho_n := \lambda_n\rho$. We have the following

1.4.19 Proposition. *Suppose that the family $\{(X,x,\rho_n)\}_{n\in\mathbb{N}}$ of (rescaled) metric spaces is precompact in the pointed Gromov–Hausdorff topology. Then, with the same assumptions and notation as above, the sequence $\mathrm{Lip}_n(\overline{B}^\bullet(r))$ is precompact in the quantum Gromov–Hausdorff topology for any $r > 0$.*

Proof. In fact, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and let $(X_\mathcal{U}, x, \rho_\mathcal{U})$ be the pGH–limit over $\mathcal{U}$ of the sequence $\{(X_n, x, \rho_n)\}_{n\in\mathbb{N}}$. From Proposition 1.4.4 then it follows that, for any $r > 0$, there exist a positive infinitesimal sequence $\{\varepsilon_n\}$, a $\delta > 0$ and an element $U_\delta \in \mathcal{U}$ such that, for all $n \in U_\delta$, one has (to simplify notation, we omit the isometric embeddings):

$$(\overline{B_n}(x, r+\varepsilon_n), \rho_n) \subset \mathcal{N}_\delta(\overline{B_\mathcal{U}}(x,r), \rho_\mathcal{U}).$$

Thus, in particular, we have

$$(\overline{B_n}(x,r), \rho_n) \subset \mathcal{N}_\delta(\overline{B_\mathcal{U}}(x,r), \rho_\mathcal{U})$$

which implies, for the covering numbers,

$$n_{2\delta}(\overline{B_n}(x,r), \rho_n) \le n_\delta(\overline{B_\mathcal{U}}(x,r), \rho_\mathcal{U}), \quad n \in U_\delta.$$

Moreover, since $\rho_n^\bullet(y,z) \le \rho_n(y,z)$ for all $y, z \in \overline{B_n}^\bullet(x,r)$, we have also

$$n_{2\delta}(\overline{B_n}^\bullet(x,r), \rho_n^\bullet) \le n_{2\delta}(\overline{B_n}(x,r), \rho_n).$$

Therefore, the sequence $\{(\overline{B_n}^\bullet(x,r), \rho_n^\bullet)\}_{n\in\mathbb{N}}$ is precompact in the Gromov–Hausdorff topology. In view of Theorem 1.2.20, this implies that $\{\mathrm{Lip}_n(\overline{B}^\bullet(r))\}_{n\in\mathbb{N}}$ is precompact in the quantum Gromov–Hausdorff topology, as claimed. ∎

Now, by the previous Proposition, we see that, if the sequence $\{(X_n, x, \rho_n)\}_{n\in\mathbb{N}}$ converges in the pointed Gromov–Hausdorff topology to $(X_{\mathcal{U}}, x, \rho_{\mathcal{U}})$ over the ultrafilter $\mathcal{U}$ on $\mathbb{N}$, then, for any $r > 0$, the corresponding sequence of quantum metric spaces $\{\mathrm{Lip}_n(\overline{B}^{\bullet}(x,r))\}_{n\in\mathbb{N}}$ will converge to $\mathrm{Lip}_{\mathcal{U}}(\overline{B}^{\bullet}(x,r))$ (over $\mathcal{U}$) in the *quantum* Gromov–Hausdorff topology. In this way, we will get an injective system of quantum metric spaces, that is,

$$\mathrm{Lip}_{\mathcal{U}}(\overline{B}^{\bullet}(r_1)) \hookrightarrow \mathrm{Lip}_{\mathcal{U}}(\overline{B}^{\bullet}(r_2)), \quad r_2 \geq r_1. \tag{1.49}$$

or, equivalently, considering the associated C^*–algebraic quantum metric spaces,

$$C_{\mathcal{U}}(\overline{B}^{\bullet}(x,r_1)) \hookrightarrow C_{\mathcal{U}}(\overline{B}^{\bullet}(x,r_2)), \quad r_2 \geq r_1, \tag{1.50}$$

where $C_{\mathcal{U}}(\overline{B}^{\bullet}(x,r)) := (C(\overline{B}^{\bullet}_{\mathcal{U}}(x,r)), L_{\mathcal{U}})$, $r > 0$.

Let $(X_{\mathcal{U}}, x, \rho_{\mathcal{U}})$ be the pGH–limit over $\mathcal{U}$ of the (precompact) sequence $\{(X, x, \rho_n)\}_{n\in\mathbb{N}}$. Then, the C^*–inductive limit $C(\overline{X}^{\bullet}_{\mathcal{U}})$ of the family $\{C(\overline{B}^{\bullet}_{\mathcal{U}}(x,r))\}_{r>0}$, for $r \nearrow +\infty$, can be regarded as a quantum version of the (pointed) metric space $(X_{\mathcal{U}}, x, \rho_{\mathcal{U}})$. (Notice that now $(\overline{X}^{\bullet}_{\mathcal{U}}, \rho_{L_{\mathcal{U}}})$ is the projective limit of the compact metric spaces $(\overline{B}^{\bullet}_{\mathcal{U}}(x,r), \rho_{L_{\mathcal{U}}})$, and the metric $\rho_{L_{\mathcal{U}}}$ on $\overline{X}^{\bullet}_{\mathcal{U}}$ is the one induced by the (limit) Lipschitz seminorm $L_{\mathcal{U}}$.) Indeed, we have the following

1.4.20 Theorem. *Let (X, x, ρ) be an (ordinary) pointed metric space, and let $\mathcal{U}$ be a free ultrafilter over $\mathbb{N}$. With the same assumptions and notations as above, let $(X_{\mathcal{U}}, x, \rho_{\mathcal{U}}) \in \mathcal{T}_x X$ be the limit over $\mathcal{U}$ of the sequence $\{(X, x, \rho_n)\}_{n\in\mathbb{N}}$ in the pointed Gromov–Hausdorff topology, and let $(\mathrm{Lip}_{\mathbb{R}}(\overline{X}^{\bullet}_{\mathcal{U}}), L_{\mathcal{U}})$ be the corresponding quantum limit. Then, the (proper) metric space $(X_{\mathcal{U}}, x, \rho_{\mathcal{U}})$ is isometrically homeomorphic to the subspace $(X_\infty, x, \rho_{L_{\mathcal{U}}})$ of $(\overline{X}^{\bullet}_{\mathcal{U}}, x, \rho_{L_{\mathcal{U}}})$ defined as:*

$$X_\infty := \{y \in \overline{X}^{\bullet}_{\mathcal{U}} : \rho_{L_{\mathcal{U}}}(x,y) < \infty\} \tag{1.51}$$

where $\rho_{L_{\mathcal{U}}}$ is the metric on $\overline{X}^{\bullet}_{\mathcal{U}}$ induced by the (limit) Lipschitz seminorm $L_{\mathcal{U}}$.

Proof. In fact, by Proposition 1.4.18, we have, for any $r > 0$,

$$\overline{B}^{\bullet}(x,r)_{\mathcal{U}} := \{y \in \overline{B}^{\bullet}_{\mathcal{U}}(x,2r) : \rho_{L_{\mathcal{U}}}(x,y) \leq r\} = \{y \in \overline{B}^{\bullet}_{\mathcal{U}}(x,2r) : \rho^{\bullet}_{\mathcal{U}}(x,y) \leq r\} \simeq \overline{B_{\mathcal{U}}}(x,r),$$

where $\rho^{\bullet}_{\mathcal{U}}$ is the (quotient) metric on $\overline{B_{\mathcal{U}}}(x,2r)$ defined by

$$\rho^{\bullet}_{\mathcal{U}}(x_1,x_2) := \min\left(\rho_{\mathcal{U}}(x_1,x_2), \rho_{\mathcal{U}}(x_1, C_{\mathcal{U}}(x,2r)) + \rho_{\mathcal{U}}(x_2, C_{\mathcal{U}}(x,2r))\right),$$

$C_{\mathcal{U}}(x,2r) := X_{\mathcal{U}} \backslash B_{\mathcal{U}}(x,2r)$, and $\overline{B}^{\bullet}_{\mathcal{U}}(x,2r)$ is the quotient of $\overline{B_{\mathcal{U}}}(x,2r)$ w.r.t. this metric (cf. Proposition 1.4.18). Hence,

$$(\overline{B}^{\bullet}(x,r)_{\mathcal{U}}, \rho_{L_{\mathcal{U}}}) \simeq (\overline{B_{\mathcal{U}}}(x,r), \rho_{\mathcal{U}}).$$

Moreover, since the family $(\overline{B_{\mathcal{U}}}(x,r), \rho_{\mathcal{U}})$ is inductive, by taking the limit of $(\overline{B}^{\bullet}(x,r)_{\mathcal{U}}, \rho_{L_{\mathcal{U}}})$ for $r \nearrow +\infty$, we obtain a metric space X_∞ which is an isometric copy of the metric tangent space, $X_{\mathcal{U}}$, of X at x. ∎

Therefore, we see that the quantum metric space $(\mathrm{Lip}_{\mathbb{R}}(\overline{X}^{\bullet}_{\mathcal{U}}), L_{\mathcal{U}})$, obtained by means of this "quantum procedure", gives us precisely the same "metric information" about the metric tangent cone of the original space X at x, that we would get by the classical pointed Gromov–Hausdorff construction. (Let us observe that $\overline{X}^{\bullet}_{\mathcal{U}}$ can be regarded as the one–point (metric) compactification of X_∞, in which the point at infinity has infinite distance from all other points in X_∞.)

1.4.21 Remark. One might guess why we do not consider, in the construction above, the inverse system of compact quantum metric spaces, as it comes out naturally when passing from a direct system of (compact) spaces to the corresponding C^*–algebras of continuous functions, that is

$$\mathrm{Lip}_{\mathcal{U}}(\overline{B}(x,r_1)) \leftarrow \mathrm{Lip}_{\mathcal{U}}(\overline{B}(x,r_2)), \quad r_2 \geq r_1$$

where now $\mathrm{Lip}_{\mathcal{U}}(\overline{B}(x,r_1)) := \{f \in C(\overline{B}(x,r_1)) : L_{\mathcal{U}}(f) < \infty\}$, $C(\overline{B}(x,r_1))$ is the unital C^*–algebra of continuous functions on the compact set $\overline{B}(x,r_1)$, and the projection is the process of restricting a function. In this case, each space, regarded as a quantum metric space, would be a quantum metric subspace, for the quotient Lip–seminorm, of the successive one in the family. So, we might consider the corresponding sequence of C^*–algebraic cQMS's:

$$(C(\overline{B}(x,r_1)), L_{\mathcal{U}}) \leftarrow (C(\overline{B}(x,r_2)), L_{\mathcal{U}}).$$

Then, taking the inverse limit, we would get the commutative pro–C^*–algebra [52; 53] of all continuous (bounded and unbouonded) functions on the (proper metric) space $X_{\mathcal{U}}$, which is defined as follows: since $X_{\mathcal{U}}$ is a proper metric space, i.e. a locally compact, separable Hausdorff space, it is, in particular, a countably compactly generated Hausdorff space, that is, $X_{\mathcal{U}}$ is the union of an increasing sequence of compact subsets: $X_{\mathcal{U}} = \cup_n \mathcal{C}_n$. For each compact subset $\mathcal{C}_n \subset X_{\mathcal{U}}$, let $C(\mathcal{C}_n)$ denote the C^*–algebra of (complex–valued) continuous functions on $\mathcal{C}$. Since $\mathcal{C}_m \subseteq \mathcal{C}_n$, $m \leq n$, we have correspondingly a family of (surjective) $*$–homomorphism π^n_m from $C(\mathcal{C}_n)$ onto $C(\mathcal{C}_m)$, given by restriction of functions on $\mathcal{C}_n$ to $\mathcal{C}_m$. In this way, the family $\{(C(\mathcal{C}_m), \pi^n_m)\}_{m\in\mathbb{N}}$ will be a projective (inverse) system of (commutative) unital C^*–algebras. The corresponding projective limit is then a (commutative, unital) pro–C^*–algebra. In particular, since the family is countable, it is a (commutative, unital) σ–C^*–algebra, and coincide with the ($*$–)algebra of all continuous (bounded or unbounded) functions on $X_{\mathcal{U}}$. In general, when the space X is locally compact, one can canonically recover from the σ–C^*–algebra $C(X)$ the (σ–unital) C^*–algebra $C_0(X)$ of continuous functions vanishing at infinity [3]. Applying this construction to the inverse system above, one then gets as inverse limit the σ–C^*–algebra $C(X_{\mathcal{U}})$ on the tangent cone at $x \in X$.
Let us notice, however, that this construction is meaningful only when a (σ–unital) C^*–algebra has a rich ideal structure (as in the case of $C_0(X)$, where ideals are given by the sets of functions vanishing outside a given compact set). Since in the noncommutative realm, one often deals with C^*–algebras which are simple, i.e. without non–trivial ideals (as the quantum torus below), this approach would not produce any interesting result. This is one of the reasons why we prefer to "dualize" the setting, and consider instead direct systems of C^*–algebraic quantum metric spaces.

The final step will be now to pass to the full non–commutative setting, by taking as starting object a compact (C^*–algebraic) quantum metric space. We want to reproduce the above construction in a general noncommutative framework. As seen before, when dealing with the pointed Gromov–Hausdorff convergence of ordinary metric spaces, non–compact metric spaces naturally enter on the scene, so that a question arises: which should be the correct notion, if any, of non–compact quantum metric space? A first step towards the extension to the non–compact case was proposed by Latrémolière in his paper [45]. A natural way to remove the compactness property is to consider, as starting point, a C^*–algebra without a unit, so that its state space is no more w^*-compact (but not even w^*–locally compact). Let us recall Latrémolière's construction: starting with a separable C*-algebra $\mathcal{A}$ and a seminorm L defined on a norm–dense subset dom(L) of the set $\mathcal{A}_{sa}$ of selfadjoint elements of $\mathcal{A}$, he first defines the *bounded–Lipschitz distance* d_L, dual to L, on the state space $S(\mathcal{A})$ of $\mathcal{A}$ by setting, for all $\phi, \psi \in S(\mathcal{A})$:

$$d_L(\phi, \psi) = \sup\{|\phi(a) - \psi(a)| : a \in \mathcal{B}_L\},$$

where $\mathcal{B}_L \equiv \mathcal{B}_{1,1} := \{a \in \mathrm{dom}(L) : L(a) \leq 1, \|a\| \leq 1\}$, and then he proves that the metric d_L metrizes the w^*–topology on $S(\mathcal{A})$ if, and only if, the set $\mathcal{B}_L$ is totally bounded for the weak–uniform topology on $\mathcal{A}$. Correspondingly, he calls L (a densley defined seminorm on the set of selfadjoint elements of a separable C^*–algebra $\mathcal{A}$) a *quasi-Lip–seminorm* (a *quasi-Lip–norm* in [45], Definition 2.8) if its dual bounded–Lipschitz distance d_L induces (the restriction of) the w^*–topology on the state space $S(\mathcal{A})$ of $\mathcal{A}$. Then, he gets as a result (cfr. Proposition 2.11 of [45]) that $(S(\mathcal{A}), d_L)$ is a complete path–metric space if there exists an approximate unit $\{e_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}_{sa}$ such that, for all $n \in \mathbb{N}$, one has $L(e_n) \leq 1$ and $\|e_n\| \leq 1$.

Our approach is slightly different: we do not require the C^*–algebras to be necessarily non–unital. Instead, the non–compactness will emerge as a consequence of the limiting process, as in the situation described in the preceding paragraph. So, let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be an injective family of (unital) C^*–algebras, each endowed with a (densely defined Lipschitz) seminorm L_n such that, if $\varphi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ is the injective morphism, then $L_{n+1}(\varphi_n(a)) = L_n(a)$, $\forall a \in \mathcal{A}_n$. (In this case, we will say that the seminorms L_n are *compatible*.) Given such a compatible family of seminorms, we can inductively define a Lipschitz seminorm L on the (unital) C^*–inductive limit $\mathcal{A}_\infty$ of the family. Now, if L_n is a Lip–seminorm for any n, then the distance induced by (the restriction) L_n (of L to each $\mathcal{A}_n$) metrizes the (restriction of the) w^*–topology on each state space $S(\mathcal{A}_n)$. We shall call *limit–Lip–seminorm* a Lipschitz seminorm L with the above property of inducing by restriction the w^*–topology on the state space $S(\mathcal{A}_n)$ for each n.

1.4.22 Definition (Limit–Lip–seminorm). *Let $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ be an injective family of C^*–algebras and let $\mathcal{A}_\infty = \varinjlim \mathcal{A}_n$ be its C^*–inductive limit. Let L_∞ be a Lipschitz seminorm on $\mathcal{A}_\infty$. We call L_∞ a* limit–Lip–seminorm *if the restriction L_n of L_∞ to each $\mathcal{A}_n$ is a Lip–seminorm, i.e. if the topology induced by the distance $\rho_{L_n}(\mu, \nu) := \sup\{|\mu(a) - \nu(a)| : a \in (\mathcal{A}_n)_{sa}, L_n(a) \leq 1\}$ is the restriction of the w^*–topology on $S(\mathcal{A}_\infty)$ to $S(\mathcal{A}_n)$, and* $\mathrm{diam}(A_n, L_n) \leq \mathrm{diam}(A_{n+1}, L_{n+1})$ *for each $n \in \mathbb{N}$.*

An order–unit space (A, e) with a limit–Lip–seminorm L defined on it, will be called simply a Quantum Metric Space, compact or non–compact depending on whether L is itself a Lip-seminorm or not.

Now, let (A, L) be a (C^*–algebraic) quantum metric space, that is, A is the selfadjoint part of a separable unital C^*–algebra $\mathcal{A}$, and L is a Lip–seminorm defined on it. We assume, moreover, that we are given a nested family[4] of (quantum metric) spaces in (A, L), each isometrically embedded in the previous one, that is,

$$(A, L) \equiv (A_1, L) \supset (A_2, L) \supset (A_3, L) \supset \cdots \supset (A_k, L) \supset \dots, \tag{1.52}$$

and such that $\mathrm{diam}(A_k, L) \equiv 2R_k \to 0$ as $k \to \infty$. (In analogy with the ordinary situation, where a point in a metric space can be regarded as the limit of a nested family of balls centered at that point, whose radii go to zero, we might think at such a family of nested quantum metric spaces

[4]By a *nested family* $\{\mathcal{A}_n\}$ of C^*–subalgebras of a C^*–algebra $\mathcal{A}$, we mean the following: $\mathcal{A} \equiv \mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \cdots \supset \mathcal{A}_n \supset \cdots$, and $\cap_n \mathcal{A}_n = \mathbb{C}I$.

as the noncommutative counterpart of an ordinary point. But, for the moment, we will not yield to a temptation of calling it a "noncommutative point". We shall call such a family simply a *shrinking family* of quantum metric spaces.) As done before in the commutative setting, given a quantum metric space (A, L), we introduce the family of its rescaled copies (A, tL), $t > 0$. So, we have a fixed (order–unit) space A, but its state space will be endowed with a family of rescaled metrics, i.e. $\rho_{tL}(\mu, \nu) = t^{-1}\rho_L(\mu, \nu)$ (cf. Proposition 1.2.7). Now, let $\{t_n\}_{n\in\mathbb{N}}$, be a decreasing sequence of positive real numbers converging to zero. Correspondingly, for each n, we get a nested family of subspaces in (A, t_nL) given by

$$(A, t_nL) \equiv (A_1, t_nL) \supset (A_2, t_nL) \supset (A_3, t_nL) \supset \cdots \supset (A_k, t_nL) \supset \dots \tag{1.53}$$

with $\mathrm{diam}(A_k, t_nL) = 2t_n^{-1}R_k$. (We shall take care in a while of the behavior of this numerical sequence.) So, we can build up the following double sequence of compact quantum metric spaces:

$$\begin{array}{llllll}
(A, L) & & & & & \\
(A_1, t_1L) & \subset \ (A, t_1L) & & & & \\
(A_2, t_2L) & \subset \ (A_1, t_2L) & \subset \ (A, t_2L) & & & \\
\ \ \vdots & & & & & \\
(A_n, t_nL) & \subset (A_{n-1}, t_nL) & \subset (A_{n-2}, t_nL) & \subset \dots & \subset (A, t_nL) & \\
\ \ \vdots & & & & &
\end{array} \tag{1.54}$$

where the inclusions along each row are isometries into the respective images, and the quantum metric spaces along each column (let $k = 1, 2, \dots$ denote the column index) have diameters given by $\mathrm{diam}(A_k, t_nL) = 2t_n^{-1}R_k$. Let us notice now that, in order to prevent from pathological situation, we shall require that the sequence of radii behaves asymptotically like an exponential, in the sense that there exists an $n_0 \in \mathbb{N}$ and a constant $c > 0$, such that, for all $n > n_0$, one has

$$n \le c \log \frac{1}{R_n} \le n + 1. \tag{1.55}$$

When this happens, then obviously $a^{n+1} \le R_n \le a^n$, with $a \equiv e^{-\frac{1}{c}} (< 1)$, and so we can choose as "dilation parameter" precisely $t = a$ and set $t_k := a^k$. In this way, we have a control on the asymptotics of the sequence of the radii of each subfamily, that is, we have

$$2a^{n-k+1} \le \mathrm{diam}(A_n, t_kL) = 2t_k^{-1}R_n \le 2a^{n-k}. \tag{1.56}$$

(Let us notice that this can always be achieved. Indeed, let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Then, given two divergent sequence of (positive) real numbers $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$, one of which, say $\{\beta_n\}$, is exponential ($\beta_n = b^n$, $b > 1$), one can find an element U in $\mathcal{U}$ such that, for all $n \in U$, $b^n \le \alpha_n \le b^{n+1}$, that is, $n \le (\log b)^{-1} \log \alpha_n \le n + 1$.)

Suppose now that the first vertical sequence $\{(A_n, t_nL)\}$ is precompact in the quantum Gromov–Hausdorff topology. According to the quantum Gromov compactness theorem (see Theorem 1.2.18), this means the following:

1) the diameters have a uniform bound, i.e. there exists a constant $D > 0$ such that $\mathrm{diam}(A_n, t_nL) \le D$;

2) for any $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that $Cov((A_n, t_n L), \varepsilon) \leq K_\varepsilon$ for all $n \in \mathbb{N}$. In this case, thanks to the bounds in (1.56), we see that all the vertical sequences $(A_{n-k}, t_n L)$ have diameters bounded from below[5] and from above, so that condition 1) above is satisfied. As for the precompactness condition 2), let us observe that in the original statement of the quantum Gromov compactness theorem (cf. Theorem 13.5 in [63]), actually it suffices a weaker condition, namely
2') for any $\varepsilon > 0$, there exists a function $G : \mathbb{R}_+ \to \mathbb{N}$ such that $\mathrm{Fin}_L((A_n, t_n L), \varepsilon) \leq G(\varepsilon)$ for all $n \in \mathbb{N}$, where, for a cQMS (A, L_A), $\mathrm{Fin}_L((A, L_A), \varepsilon)$ is defined as the smallest integer k such that there is a cQMS (B, L_B) such that $\mathrm{dist}_q(A, B) < \varepsilon$ and $\mathrm{dim}B \leq k$ ($\mathrm{dim}B$ is the vector–space dimension of B).

1.4.23 Lemma. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. Then,*

$$\mathrm{dist}_q((A, L_A), (B, L_B)) = b \quad \Longrightarrow \quad \mathrm{dist}_q((A, \alpha L_A), (B, \alpha L_B)) = \alpha^{-1} b \tag{1.57}$$

Proof. Indeed, by a simple rescaling argument, we have

$$\mathcal{L}(\alpha L_A, \alpha L_B) = \{\alpha L : L \in \mathcal{L}(L_A, L_B)\},$$

hence, $\mathrm{dist}_H^{\rho_{\alpha L}}(S(A), S(B)) = \mathrm{dist}_H^{\alpha^{-1}\rho_L}(S(A), S(B)) = \alpha^{-1}\, \mathrm{dist}_H^{\rho_L}(S(A), S(B))$, from which the claim follows. ∎

As a consequence, one has also the following relation:

$$\mathrm{Fin}_L((A, \alpha L_A), \varepsilon) = \mathrm{Fin}_L((A, L_A), \frac{\varepsilon}{\alpha}), \tag{1.58}$$

and thus, in condition 2') above, it suffices to take the function $G_\alpha(\varepsilon) := G(\frac{\varepsilon}{\alpha})$.

Therefore, we see that, under the assumption (1.56) of an exponential asymptotic behavior of the sequence of radii, all the vertical sequences in (1.54) will be precompact, so we can take the limit of each of them over any given ultrafilter $\mathcal{U}$. In this way, we will end up with the following injective family of (C^*–algebraic) quantum metric spaces:

$$(A^1_{\mathcal{U}}, L_{\mathcal{U}}) \hookrightarrow (A^2_{\mathcal{U}}, L_{\mathcal{U}}) \hookrightarrow (A^3_{\mathcal{U}}, L_{\mathcal{U}}) \hookrightarrow \cdots \hookrightarrow (A^k_{\mathcal{U}}, L_{\mathcal{U}}) \cdots \tag{1.59}$$

Finally, we take the injective limit $(A_{\mathcal{U}}, L_{\mathcal{U}}) := \varinjlim (A^k_{\mathcal{U}}, L_{\mathcal{U}})$ of this family, and we call it a *pointed quantum Gromov–Hausdorff limit* of $\{(A, t_n L)\}_{n\in\mathbb{N}}$ w.r.t. the family $\{(A_n, L)\}_{n\in\mathbb{N}}$. We call the set of all such limit points w.r.t. any shrinking family of subspaces the *quantum Tangent Cone* of (A, L).

Summing up, we can state the following

1.4.24 Theorem. *Let (A, L) be a C^*–algebraic compact quantum metric space, let $\{A_n\}_{n\in\mathbb{N}}$ be a shrinking family of subspaces in A, and, for each $n \in \mathbb{N}$, denote by R_n the radius of the cQMS (A_n, L). Suppose that*

(1) *the radii R_n satisfy the following condition: there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$, there holds*

$$n \leq c \log \frac{1}{R_n} \leq n + 1; \tag{1.60}$$

[5] A uniform bound from below assures that any eventual limit point is non–trivial, that is, it does not consist only of the multiples of the identity.

(2) *for a suitable sequence* $\{t_n\}_{n\in\mathbb{N}}$ *of positive real numbers, the sequence* $(A_n, t_nL)_{n\in\mathbb{N}}$ *is precompact in the quantum Gromov–Hausdorff topology.*

Then, for any ultrafilter $\mathcal{U}$ *on* $\mathbb{N}$*, the family* $\{(A, t_nL)\}_{n\in\mathbb{N}}$ *converges to* $(A, L_{\mathcal{U}})$ *over* $\mathcal{U}$ *in the pointed quantum Gromov–Hausdorff sense, as described above.*

Proof. As seen, it suffices to take $t_n = e^{-n/c}$ and then to apply the procedure illustrated above. ∎

Therefore, we have sufficient conditions for a quantum metric space to admit (limit) points in its quantum Tangent Cone in the sense specified above, and Theorem 1.4.20 shows that the notion of quantum Tangent Cone extends the classical definition. It is worth noting that this "quantum construction", even in the commutative setting, actually produces more general objects than the ordinary ones, depending on which family of subspaces one selects in the ambient (quantum) metric spaces. This will be evident in the following section, where we will show an example of quantum Tangent Cone to a proper quantum metric space, the quantum torus. Indeed, as we shall see, when considering the ordinary torus as a particular case of the quantum torus, the procedure illustrated above will produce a solenoid, due to a specific choice of subspaces that we will consider.

1.4.25 Remark. Let (A_∞, L_∞) be the injective limit of a sequence (A_n, L_n) of (compact) C^*–algebraic quantum metric space such that, for each n, $\mathrm{diam}(A_n, L_n) \leq \mathrm{diam}(A_{n+1}, L_{n+1})$ and $\lim_{n\to\infty} \mathrm{diam}(A_n, L_n) = +\infty$. We know that, since the family $\{A_n\}_{n\in\mathbb{N}}$ is injective, the state space $S(A_\infty)$ of the limit algebra A_∞ will be the projective limit of the state spaces $S(A_n)$. Nevertheless, suppose that, for each n, we have also isometric inclusions ι_n of $S(A_n)$ into $S(A_\infty)$. Then, given $\mu \in S(A_m)$, let $\overline{B}_m(\mu, r) = \{\nu \in S(A_m) : \rho_{L_m}(\mu, \nu) \leq r\}$ be the w^*–closed r–ball in $S(A_m)$ centered at μ. Since ι_m is an isometry, the set $\iota_m(\overline{B}_m(\mu, r))$ will be the w^*–closed r–ball in $S(A_\infty)$ centered at $\iota_m(\mu)$. As $\mathrm{diam}(A_n, L_n) \leq \mathrm{diam}(A_{n+1}, L_{n+1})$, we can construct a system of w^*–closed (in the relative topology) neighborhoods of $\iota_m(\mu) \in S(A_\infty)$, with non–decreasing radii. Indeed, if we choose a suitable increasing sequence of positive numbers $\{r_n\}$ and take the union over this family, we get, inside $S(A_\infty)$, a metric space which is no longer compact. This is precisely the quantum version of the metric space X_∞ considered in Theorem 1.4.20.

1.5 A case study: the Quantum Torus

1.5.1 The Quantum Torus

We already introduced quantum (or noncommutative) n–tori in Example 1.2.22, but here we specialize to the case $n = 2$. (The reader is referred to [23] for a detailed introduction to this topic.) As seen in Remark 1.2.25, given $\theta \in \mathbb{R}$, we define the 2-dimensional quantum torus $\mathcal{A}_\theta$ as the universal C^*–algebra generated by two unitaries U, V, which satisfy the following commutation relations:

$$U^m V^n = e^{2\pi i(mn\theta)} V^n U^m. \tag{1.61}$$

Let $\mathcal{S}(\mathbb{Z}^2)$ denote the space of Schwartz functions over $\mathbb{Z}^2$. Then, the complex $*$–algebra generated by U and V with coefficients in $\mathcal{S}(\mathbb{Z}^2)$, namely

$$\mathbb{A}_\theta := *\text{–alg}\Big\{ \sum_{m,n\in\mathbb{Z}} a_{m,n} e^{imP} e^{inQ} : a_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \Big\}, \tag{1.62}$$

is a dense $*$–subalgebra of $\mathcal{A}_\theta$.
There is a natural action γ of the (ordinary) torus $\mathbb{T}^2$ on $\mathbb{A}_\theta$, defined on the generators by:

$$\gamma_{(s_1,s_2)}(e^{imP}e^{inQ}) = e^{2\pi i(s_1 m + s_2 n)}e^{imP}e^{inQ}, \tag{1.63}$$

and then extended by linearity to the whole $\mathbb{A}_\theta$.
Let A_θ denote the selfadjoint part of $\mathbb{A}_\theta$, i.e. $A_\theta := (\mathbb{A}_\theta)_{sa}$. We define a seminorm on A_θ by:

$$L_\theta(a) := \sup\Big\{ \frac{\|\gamma_{(s_1,s_2)}(a) - a\|}{\ell(s_1,s_2)} : (s_1,s_2) \neq (0,0) \Big\}, \tag{1.64}$$

where $\ell(\cdot,\cdot)$ is the length function on $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$, induced from the Euclidean one on $\mathbb{R}^2$ (that is, the Euclidean distance from the origin).

1.5.1 Remark. Let us denote by δ_1, δ_2 the standard basic derivations on $\mathbb{A}_\theta$, defined on the generators by:

$$\delta_1(U^m) = 2\pi i m U^m, \quad \delta_1(V^n) = 0, \quad \delta_2(U^m) = 0, \quad \delta_2(V^n) = 2\pi i n V^n. \tag{1.65}$$

(δ_1 and δ_2 can be thought of as the generators of the automorphisms group $\{\gamma_{(t_1,t_2)} : (t_1,t_2) \in \mathbb{T}^2\}$.) Then, since

$$(\alpha_1\delta_1 + \alpha_2\delta_2)(a) = \lim_{t\to 0^+} \frac{\gamma_{(\exp(t\alpha_1\delta_1), \exp(t\alpha_2\delta_2))}(a) - a}{t},$$

by Proposition 1.2.13, the seminorm L_θ (1.64) can be equivalently computed as

$$L_\theta(a) = \sup\{\|(\alpha_1\delta_1 + \alpha_2\delta_2)(a)\| : \alpha_1^2 + \alpha_2^2 = 1\}. \tag{1.66}$$

Moreover, $\mathbb{A}_\theta$ is exactly the $*$–algebra of smooth elements w.r.t. the action γ.

The seminorm L_θ is actually a Lip–seminorm, and (A_θ, L_θ) is a (compact) quantum metric space (cf. Proposition 1.2.23)

1.5.2 Shrinking Families of Subalgebras

For any $p \in \mathbb{N}$, p prime, we may consider the following C^*–subalgebras of $\mathcal{A}_\theta$:

$$\begin{aligned} \mathcal{A}_\theta^{(p^l,p^l)} \equiv \mathcal{A}_\theta^{p^l} &:= C^*\Big\{ \sum_{m,n\in p^l\mathbb{Z}} a_{m,n} U^{mp^l} V^{np^l} : a_{m,n} \in \mathcal{S}(p^l\mathbb{Z}^2) \Big\} \\ &= C^*\Big\{ \sum_{m,n\in\mathbb{Z}} a'_{m,n} (U^{p^l})^m (V^{p^l})^n : a'_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \Big\}, \end{aligned} \tag{1.67}$$

and the corresponding (dense) $*$–subalgebras of smooth elements:

$$\mathbb{A}_\theta^{(p^l,p^l)} \equiv \mathbb{A}_\theta^{p^l} := *\text{–alg}\Big\{ \sum_{m,n\in\mathbb{Z}} a'_{m,n} U^{mp^l} V^{np^l} : a'_{m,n} \in \mathcal{S}(\mathbb{Z}^2) \Big\}. \tag{1.68}$$

For notational convenience, we will consider only the case $p = 2$. Then, the family $(A_\theta^{2^l}, L_\theta)$, where $A_\theta^{2^l} = (\mathbb{A}_\theta^{2^l})_{sa}$, will be the shrinking family, which we will use to construct the quantum Tangent Cone.
We have the following identification:

1.5.2 Proposition. *We have* $(A_\theta^{2^l}, L_\theta) \cong (A_{4^l\theta}, 2^l L_{4^l\theta})$ *as (compact) quantum metric spaces.*

Proof. Let $\mathcal{A}_{4^l\theta}$ be the universal C^*–algebra generated by two unitaries $\tilde{U}$, $\tilde{V}$ satisfying the commutation relations:

$$\tilde{U}^m\tilde{V}^n = e^{2\pi i(4^l mn\theta)}\tilde{V}^n\tilde{U}^m.$$

As $\tilde{U}$, $\tilde{V}$ and U^{2^l}, V^{2^l} satisfy the same commutation relations, by the universality property, for each integer l there exists a $*$–homomorphism σ_l between the two C^*–algebras $\mathcal{A}_{4^l\theta}$ and $\mathcal{A}_\theta^{2^l}$. Since, for any $\psi \in \mathbb{R}$, the C^*–algebras $\mathcal{A}_\psi$ is known to be simple (cf. Theorem VI.1.4 in [23]), σ_l is an isomorphism for any l. Let us denote by γ and γ' the action of $\mathbb{T}^2$ on $\mathcal{A}_\theta^{2^l}$ and on $\mathcal{A}_{4^l\theta}$, respectively. Then, we have

$$\sigma_l(\gamma_{(s_1,s_2)}(U^{m2^l}V^{n2^l})) = \gamma'_{(2^l s_1, 2^l s_2)}(\tilde{U}^m\tilde{V}^n) = \gamma'_{(2^l s_1, 2^l s_2)}(\sigma_l(U^{m2^l}V^{n2^l})),$$

and a simple computation gives us the following relation between the basic derivations:

$$2^l\delta_j(\sigma_l(a)) = \sigma_l(\delta_j(a)), \quad a \in A_\theta^{2^l} \quad j = 1, 2.$$

Hence, for every $a \in A_\theta^{2^l}$, we get

$$\begin{aligned} L_\theta(a) &= \sup\{\|(\alpha_1\delta_1 + \alpha_2\delta_2)(a)\| : \alpha_1^2 + \alpha_2^2 = 1\} \\ &= \sup\{\|2^l(\alpha_1\delta_1 + \alpha_2\delta_2)(\sigma_l(a))\| : \alpha_1^2 + \alpha_2^2 = 1\} = 2^l L_{4^l\theta}(\sigma_l(a)), \end{aligned}$$

proving the thesis. ∎

By the previous Proposition, we see that the sequence of radii of the family $\{(A_\theta^{2^l}, L_\theta)\}_{l\in\mathbb{N}}$ satisfies the condition given in Theorem 1.4.24, since the radius of $(A_{4^l\theta}, 2^l L_{4^l\theta})$, for each l, is bounded by $2^{-l}\int_{\mathbb{T}^2}\ell(r,s)d\mu$ from above (cf. Theorem 1.2.12), hence the corresponding sequence of diameters tends to zero.

1.5.3 Remark. We stress once again that the choice of the sequence $\{2^l\}_{l\in\mathbb{N}}$ is purely a convenience. The same conclusions hold for any other choice $\{p^l\}_{l\in\mathbb{N}}$, with p prime, or, more generally, one could take any sequence of integers $\{m_l\}_{l\in\mathbb{N}}$ such that m_l divides m_{l+1} for any l.

1.5.3 The Quantum Tangent Cone

In view of the definition of shrinking family of subspaces (cf. Section 1.4.2), the starting point will be the following double sequence:

$$\begin{array}{lllll} (A_\theta, L_\theta) & & & & \\ (A_\theta^2, 2^{-1}L_\theta) & \subset (A_\theta, 2^{-1}L_\theta) & & & \\ (A_\theta^{2^2}, 2^{-2}L_\theta) & \subset (A_\theta^2, 2^{-2}L_\theta) & \subset (A_\theta, 2^{-2}L_\theta) & & \\ \vdots & & & & \\ (A_\theta^{2^l}, 2^{-l}L_\theta) & \subset (A_\theta^{2^{l-1}}, 2^{-l}L_\theta) & \subset & \cdots & \subset (A_\theta, 2^{-l}L_\theta) \\ \vdots & & & & \end{array} \tag{1.69}$$

where along each column all the QMS's have the same (uniform) bound on the diameter, namely $2^k \int_{\mathbb{T}^2} \ell(r,s)d\mu$ ($k = 1, 2, 3, \ldots$ denotes the column index), while the inclusions along each row are isometric inclusions.

By Proposition 1.5.2, the previous double sequence can be rewritten as

$$\begin{array}{llll}
(A_\theta, L_\theta) & & & \\
(A_{4\theta}, L_{4\theta}) & \hookrightarrow (A_\theta, 2^{-1}L_\theta) & & \\
(A_{4^2\theta}, L_{4^2\theta}) & \hookrightarrow (A_{4\theta}, 2^{-1}L_{4\theta}) & \hookrightarrow (A_\theta, 2^{-2}L_\theta) & \\
\vdots & & & \\
(A_{4^l\theta}, L_{4^l\theta}) & \hookrightarrow (A_{4^{l-1}\theta}, 2^{-1}L_{4^{l-1}\theta}) & \hookrightarrow \quad \cdots & \hookrightarrow (A_\theta, 2^{-l}L_\theta) \\
\vdots & & &
\end{array} \tag{1.70}$$

where at each step the isometric inclusion is given by:

$$(A_{\tau/4^l}, 2^{-l}L_{\tau/4^l}) \cong (A^2_{\tau/4^{l+1}}, 2^{-l-1}L_{\tau/4^{l+1}}) \subset (A_{\tau/4^{l+1}}, 2^{-l-1}L_{\tau/4^{l+1}}). \tag{1.71}$$

Now, let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$, and let $\tau \in [0,1)$ be the limit (mod1), over $\mathcal{U}$, of the sequence $\{4^l\theta\}_{l\in\mathbb{N}}$. Then, by Proposition 1.2.24, each vertical sequence (quantum Gromov–Hausdorff) converges to the corresponding quantum torus $(A_{\tau/4^k}, 2^{-k}L_{\tau/4^k})$, $k = 0, 1, 2, \ldots$. Thus, taking the limit, over $\mathcal{U}$, of each column, we get, finally, the following injective family of quantum metric spaces:

$$(A_\tau, L_\tau) \hookrightarrow (A_{\tau/4}, 2^{-1}L_{\tau/4}) \hookrightarrow \cdots \hookrightarrow (A_{\tau/4^l}, 2^{-l}L_{\tau/4^l}) \hookrightarrow \cdots \tag{1.72}$$

The next step will be to take the injective limit of this sequence. To this aim, we first represent the C^*–algebra $\mathcal{A}_\tau$ on a suitable Hilbert space. So, for any $\tau \in \mathbb{R}$, let $(\pi, \mathcal{H} \equiv L^2(\mathcal{A}_\tau, \tau_0))$ be the GNS representation associated to the faithful tracial state τ_0 given by

$$\tau_0(A) = \tau_0(\sum_{n,m\in\mathbb{Z}} a_{m,n}U^mV^n) := a_{0,0}, \quad A \in \mathcal{A}_\theta. \tag{1.73}$$

Since τ_0 is faithful, we shall identify each algebra with its image in $\mathcal{B}(\mathcal{H})$. Thus, we can find two selfadjoint operators P and Q acting on $\mathcal{H}$ such that the corresponding unitaries $U = e^{iP}$, $V = e^{iQ}$ satisfy the commutation relation defining the quantum torus $\mathcal{A}_\tau$. Then, for each $l \in \mathbb{N}$, we define $\mathcal{A}_\tau^{2^{-l}}$ as the universal C^*–algebra generated by the two unitaries $\bar{U} = e^{iP'}$, $\bar{V} = e^{iQ'}$, with $P' := 2^{-l}P$ and $Q' := 2^{-l}Q$, subject to the commutation relations:

$$e^{imP'}e^{inQ'} = e^{2\pi i(4^{-l}mn\tau)}e^{inQ'}e^{imP'}, \quad m, n \in \mathbb{Z}. \tag{1.74}$$

As before, we shall consider the dense $*$–subalgebras of smooth elements, given by

$$\mathbb{A}_\tau^{2^{-l}} := *\text{–alg}\Big\{ \sum_{m,n\in\mathbb{Z}} a'_{m,n}e^{imP'}e^{inQ'} : a'_{m,n} \in \mathcal{S}(\mathbb{Z}^2)\Big\}. \tag{1.75}$$

Then, since $e^{imP'}e^{inQ'} = e^{im2^{-l}P}e^{in2^{-l}Q}$, we can write $\mathbb{A}_\tau^{2^{-l}}$ equivalently as

$$\mathbb{A}_\tau^{2^{-l}} = *\text{–alg}\Big\{ \sum_{p,q\in\mathbb{Z}[2^{-l}]} a_{p,q} e^{ipP}e^{iqQ} : a_{p,q} \in \mathcal{S}(\mathbb{Z}^2[2^{-l}])\Big\}, \tag{1.76}$$

where $\mathbb{Z}^2[2^{-l}] = \{(p,q) := (\frac{m}{2^l}, \frac{n}{2^l}) : m, n \in \mathbb{Z}\}$, and $a_{p,q} = a_{m/2^l, n/2^l} := a'_{m,n}$. Thus, taking the norm closure, we get

$$\begin{aligned} \mathcal{A}_\tau^{2^{-l}} &= C^*\Big\{ \sum_{m,n\in\mathbb{Z}} a'_{m,n} e^{imP'}e^{inQ'} : a_{m,n} \in \mathcal{S}(\mathbb{Z}^2)\Big\} \\ &= C^*\Big\{ \sum_{p,q\in\mathbb{Z}[2^{-l}]} a_{p,q} e^{ipP}e^{iqQ} : a_{p,q} \in \mathcal{S}(\mathbb{Z}^2[2^{-l}])\Big\}. \end{aligned} \tag{1.77}$$

Hence, $\mathcal{A}_\tau^{2^{-l}}$ can be also viewed as the universal C^*–algebra generated by the two unitaries $e^{im2^{-l}P}$, $e^{in2^{-l}Q}$ satisfying the commutation relations:

$$e^{im2^{-l}P}e^{in2^{-l}Q} = e^{2\pi i(\frac{mn}{4^l}\tau)}e^{in2^{-l}Q}e^{im2^{-l}P}, \quad m, n \in \mathbb{Z}. \tag{1.78}$$

From the action of the torus $\mathbb{T}^2 \cong (\mathbb{R}/2^l\mathbb{Z})^2$ on $\mathcal{A}_\tau^{2^{-l}}$, we define, as before, a seminorm on $A_\tau^{2^{-l}} \equiv \mathbb{A}_\tau^{2^{-l}}$ as:

$$L_\tau(a) := \sup\Big\{ \frac{\|\gamma'_{(s_1,s_2)}(a) - a\|}{\ell'(s_1,s_2)} : (s_1,s_2) \neq (0,0)\Big\}, \tag{1.79}$$

where the length function ℓ' on $(\mathbb{R}/2^l\mathbb{Z})^2$, induced from that on $\mathbb{R}^2$, satisfies $2^l\ell(s_1,s_2) = \ell'(2^ls_1, 2^ls_2)$.

We can state the analogue of Proposition 1.5.2.

1.5.4 Proposition. *We have $(A_{\tau/4^l}, 2^{-l}L_{\tau/4^l}) \cong (A_\tau^{2^{-l}}, L_\tau)$ as (compact) quantum metric spaces.*

Proof. We proceed as in the proof of Proposition 1.5.2. So, let $\bar{U}$, $\bar{V}$ denote the generators of $\mathcal{A}_{\tau/4^l}$. Since the unitaries $\bar{U}$, $\bar{V}$ and $e^{i2^{-l}P}$, $e^{i2^{-l}Q}$ satisfy the same commutation relations, from the universality property it follows that, for each l, there exists a $*$–homomorphism σ_l from $\mathcal{A}_{\tau/4^l}$ onto $\mathcal{A}_\tau^{2^{-l}}$. Since $\mathcal{A}_{\tau/4^l}$ is simple ([23], Theorem VI.1.4), σ_l is an isomorphism for any l.
Now, let us denote by γ and γ' the action of $\mathbb{T}^2$ on $\mathcal{A}_{\tau/4^l}$ and on $\mathcal{A}_\tau^{2^{-l}}$, respectively. Then, we have

$$\sigma_l(\gamma_{(s_1,s_2)}(\bar{U}^m\bar{V}^n)) = \gamma'_{(2^ls_1,2^ls_2)}(e^{im2^{-l}P}e^{in2^{-l}Q}) = \gamma'_{(2^ls_1,2^ls_2)}(\sigma_l(\bar{U}^m\bar{V}^n)),$$

and, with a simple computation, we get:

$$2^l\delta_j(\sigma_l(a)) = \sigma_l(\delta_j(a)), \quad a \in A_{\tau/4^l} \quad j = 1, 2.$$

Hence, for every $a \in A_{\tau/4^l}$, we obtain

$$\begin{aligned} L_\tau(\sigma_l(a)) &= \sup\{\|(\alpha_1\delta_1 + \alpha_2\delta_2)(\sigma_l(a))\| : \alpha_1^2 + \alpha_2^2 = 1\} \\ &= \sup\{\|2^{-l}(\alpha_1\delta_1 + \alpha_2\delta_2)(\sigma_l(a))\| : \alpha_1^2 + \alpha_2^2 = 1\} = 2^{-l}L_{\tau/4^l}(a), \end{aligned}$$

and the thesis follows. ∎

The sequence (1.72) of (isometric) inclusions now can be written as

$$(A_\tau, L_\tau) \hookrightarrow (A_\tau^{2^{-1}}, L_\tau^{2^{-1}}) \hookrightarrow (A_\tau^{2^{-2}}, L_\tau^{2^{-2}}) \hookrightarrow \cdots \hookrightarrow (A_\tau^{2^{-l}}, L_\tau^{2^{-l}}) \hookrightarrow \cdots \tag{1.80}$$

What do we get as C^*–inductive limit of the sequence of C^*–algebras $\{\mathcal{A}_\tau^{2^{-l}}\}_{l\in\mathbb{N}}$? Clearly, we get a C^*–algebra, which we denote by $\mathcal{A}_\tau^\infty$, whose dense $*$–subalgebra of smooth elements is given by

$$\begin{aligned} \mathbb{A}_\tau^\infty &= \bigcup_{l\in\mathbb{N}} \mathbb{A}_\tau^{2^{-l}} = *\text{–alg}\Big\{ \bigcup_{l\in\mathbb{N}} \Big\{ \sum_{p,q\in\mathbb{Z}[2^{-l}]} a_{p,q} e^{ipP} e^{iqQ} : a_{p,q} \in \mathcal{S}(\mathbb{Z}^2[2^{-l}]) \Big\} \Big\} \\ &= *\text{–alg}\Big\{ \sum_{u,v\in\mathbb{Z}[2^{-\infty}]} a_{u,v} e^{iuP} e^{ivQ} : a_{u,v} \in \mathcal{S}(\mathbb{Z}^2[2^{-\infty}]) \Big\}, \end{aligned} \tag{1.81}$$

where $\mathbb{Z}[2^{-\infty}] = \{2^{-n}m : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is the (additive) group of 2–adic numbers. Notice that $A_\tau^\infty \equiv (\mathbb{A}_\tau^\infty)_{sa}$ is endowed with a seminorm L_τ, defined by restriction to each subalgebra $A_\tau^{2^{-l}}$. L_τ is densely defined, and vanishes by construction only on the multiples of the identity, hence it is a limit–Lip–seminorm and A_τ^∞ is a quantum metric space (cf. Definition 1.4.22).

Finally, we can conclude saying that:
For each limit point $\tau \in [0,1)$ of the sequence $\{4^l\theta\}_{l\in\mathbb{N}}$, L_τ^∞ is a limit–Lip–seminorm and $(A_\tau^\infty, L_\tau^\infty)$ belongs to the Quantum Tangent Cone of (A_θ, L_θ) induced by the family $(A_\theta^{2^l}, L_\theta)$.

Let us see now which kind of limits we get by the above procedure. We must distinguish the following cases.

1) Let $\theta \in \mathbb{Q}$, with $\theta = \frac{m}{n}$, $(m,n) = 1$, and $m < n$. If n is a power of 2, then $\lim_{l\to\infty} 4^l\theta = 0(\text{mod}1)$ and the sequence (1.80) becomes

$$(C^\infty(\mathbb{T}^2), L_0) \hookrightarrow (C^\infty(\mathbb{T}_1^2), L_0) \hookrightarrow (C^\infty(\mathbb{T}_2^2), L_0) \cdots \hookrightarrow (C^\infty(\mathbb{T}_l^2), L_0) \cdots, \tag{1.82}$$

where

$$C^\infty(\mathbb{T}_l^2) = *\text{–alg}\Big\{ f(x,y) = \sum_{p,q\in\mathbb{Z}[2^{-l}]} a_{p,q} e^{ipx} e^{iqy} : a_{p,q} \in \mathcal{S}(\mathbb{Z}^2[2^{-l}]) \Big\}, \tag{1.83}$$

$$L_0(f) = \sup\Big\{ \Big\| \sum_{p,q\in\mathbb{Z}[2^{-l}]} (\alpha_1 p + \alpha_2 q) a_{p,q} e^{ipx} e^{iqy} \Big\| : \alpha_1^2 + \alpha_2^2 = 1 \Big\}, \tag{1.84}$$

and $\mathbb{T}_l^2$ denotes the 2^l–fold covering of the torus $\mathbb{T}^2 (= \mathbb{T}_0^2)$, with projection map given by the process of dividing by 2, i.e. $\mathbb{T}_{l+1}^2 \ni (x,y) \mapsto (x/2, y/2) \in \mathbb{T}_l^2$. The pointed quantum Gromov–Hausdorff limit is then $(C^\infty(\mathcal{S}_2^2), L_0)$, where $C^\infty(\mathcal{S}_2^2)$ is the algebra of smooth functions on the projective limit of the family $\{\mathbb{T}_l^2\}_{l\in\mathbb{N}_0}$, which is, by definition, the 2–adic (2–dimensional) solenoid group[6] $\mathcal{S}_2^2$, i.e. the (Pontryagin) dual of $\mathbb{Z}^2[2^{-\infty}]$. So, we see that the quantum tangent cone to the ordinary torus contains more objects than the ordinary tangent space (i.e., the 2–dimensional Euclidean space $\mathbb{R}^2$).

[6]It can be equivalently defined as follows (see, for instance, [38], Definition 10.12): let $\mathbf{a} := (a_0, a_1, a_2, \dots)$ be any sequence of integers all greater than 1, set $\mathbf{u} := (1, 0, 0, \dots)$, and consider the additive (locally compact) group $\mathbb{R} \times \triangle_{\mathbf{a}}$, where $\triangle_{\mathbf{a}}$ is the additive group of $\mathbf{a}$–adic integers. Let $N_{\mathbf{u}}$ be the subgroup of $\mathbb{R} \times \triangle_{\mathbf{a}}$ given by $\{(n, n\mathbf{u})\}_{n=-\infty}^{+\infty}$. The (one–dimensional) $\mathbf{a}$–adic solenoid $\mathcal{S}_{\mathbf{a}}$ is then defined as the quotient group $\mathbb{R} \times \triangle_{\mathbf{a}}/N_{\mathbf{u}}$.

2) Let $\theta \in \mathbb{Q}$, with $\theta = \frac{m}{n}$, $(m,n) = 1$, $m < n$, and $n = 4^k h$, where 4^k is the maximal power of 4 dividing n. Then, the sequence $\{4^{l-k}\frac{m}{h}(\mathrm{mod}1)\}_{l\in\mathbb{N}}$ has at most h limit points among the solutions of the congruence $(4^{l'}m \equiv h(\mathrm{mod}h))/h$, i.e. in the set $\{\frac{1}{h}, \frac{2}{h}, \ldots, 1-\frac{1}{h}\}$, and the sequence (1.80) becomes

$$(A_q, L_q) \hookrightarrow (A_q^{2^{-1}}, L_q) \hookrightarrow (A_q^{2^{-2}}, L_q) \hookrightarrow \cdots \hookrightarrow (A_q^{2^{-l}}, L_q) \hookrightarrow \cdots \tag{1.85}$$

with $q \in \{\frac{1}{h}, \frac{2}{h}, \ldots, 1-\frac{1}{h}\}$. We shall call the pointed quantum Gromov–Hausdorff limit (A_q^∞, L_q) the *Noncommutative* (or *Quantum*) *Solenoid* with rational parameter q.

3) Finally, when $\theta \in \mathbb{R}\backslash\mathbb{Q}$, then, for any $\phi \in [0,1)$, there exits a subsequence, $\{4^{l_k}\theta(\mathrm{mod}1)\}_{k\in\mathbb{N}}$, of $\{4^l\theta(\mathrm{mod}1)\}_{l\in\mathbb{N}}$ converging to ϕ, and Proposition 1.2.24 tells us that

$$\lim_{k\to\infty} 4^{l_k}\theta = \phi(\mathrm{mod}1) \Rightarrow \lim_{\mathrm{qGH}}(A_{4^{l_k}\theta}, L_{4^{l_k}\theta}) = (A_\phi, L_\phi) \tag{1.86}$$

For a given subsequence $\{4^{l_k}\theta(\mathrm{mod}1)\}_{k\in\mathbb{N}}$, the family (1.80) becomes:

$$(A_\phi, L_\phi) \hookrightarrow (A_\phi^{2^{-1}}, L_\phi) \hookrightarrow (A_\phi^{2^{-2}}, L_\phi) \hookrightarrow \cdots \hookrightarrow (A_\phi^{2^{-l}}, L_\phi) \hookrightarrow \cdots \tag{1.87}$$

and we shall call the pointed quantum Gromov–Hausdorff limit (A_ϕ^∞, L_ϕ) the *Noncommutative* (or *Quantum*) *Solenoid* with irrational parameter ϕ.

1.5.4 Quantum Subspaces of the Quantum Torus

We recall (see Definition 1.2.16) that a morphism $\varphi : A \to B$ between two compact quantum metric spaces (A, L_A) and (B, L_B) is a linear positive map preserving the order–units and the Lip–seminorms (i.e. $L_B = L_A \circ \varphi$), and that, if φ is surjective, the dual map $S(\varphi) : S(B) \to S(A)$ between the corresponding state spaces is an injective (affine) map. In this case, the image $(\varphi(A), L_B)$ of (A, L_A) in (B, L_B) is called a *quantum metric subspace* of (B, L_B). In this section, we show that $(A_\theta^{2^l}, L_\theta)$ is actually a quantum metric subspace of (A_θ, L_θ).

Let us recall that a *Conditional Expectation* of a C^*–algebra onto a subalgebra is a positive, unital idempotent linear map[7]. We will show that the (2–dimensional) quantum torus admits conditional expectations onto its subalgebras. First, consider the two following automorphisms of $\mathcal{A}_\theta$: for any λ, μ on the unit circle ($|\lambda| = |\mu| = 1$), let $\rho_{\lambda,\mu}$ be the endomorphism of $\mathcal{A}_\theta$ given by

$$\rho_{\lambda,\mu}(U) = \lambda U, \qquad \rho_{\lambda,\mu}(V) = \mu V, \tag{1.88}$$

and let $\sigma := \rho_{\bar\lambda,\bar\mu}\rho_{\lambda,\mu}$. Since $\sigma(U) = U$ and $\sigma(V) = V$, we have $\sigma = id$. Thus, $\rho_{\lambda,\mu}$ is an automorphism.

For each fixed A in $\mathcal{A}_\theta$, the map from $\mathbb{T}^2$ to $\mathcal{A}_\theta$ given by $f(\lambda,\mu) = \rho_{\lambda,\mu}(A)$ is norm continuous[8]. We define two maps of $\mathcal{A}_\theta$ into itself by the formulae[9]

$$\Phi_{1,0}(A) = \int_0^1 \rho_{1,e^{2\pi i t}}(A)dt, \qquad \Phi_{0,1}(A) = \int_0^1 \rho_{e^{2\pi i t},1}(A)dt. \tag{1.89}$$

[7]Recall that a map Φ is *contractive* if $\|\Phi\| \le 1$, *idempotent* if $\Phi^2 = \Phi$, and a positive map is *faithful* if $A \ge 0$ and $\Phi(A) = 0$ implies that $A = 0$.

[8]To verify this, notice that it is true for all non-commuting plynomials in U, V, U^*, V^*; but these are dense and the automorphisms are contractive, so that the rest follows by a simple approximation argument.

[9]These integrals make sense as Riemann integrals since the integrand is a norm continuous function.

1.5.5 Theorem. [23] $\Phi_{1,0}$ *is positive contractive idempotent and faithful, and maps $\mathcal{A}_\theta$ onto $C^*(U)$. Moreover,*

$$\Phi_{1,0}(f(U)Ag(U)) = f(U)\Phi_{1,0}(A)g(U) \tag{1.90}$$

for all $f, g \in C(\mathbb{T})$. For any finite linear combination of $\{U^m V^n : m, n \in \mathbb{Z}\}$,

$$\Phi_{1,0}(\sum_{m,n} a_{m,n} U^m V^n) = \sum_m a_{m,0} U^m. \tag{1.91}$$

Finally, for every A in A_θ,

$$\Phi_{1,0}(A) = \lim_{n\to\infty} \frac{1}{2n+1} \sum_{j=-n}^{n} U^j A U^{-j}. \tag{1.92}$$

The corresponding result for Φ_2 obviously holds as well. Combining the two results, we obtain

1.5.6 Corollary. *The map $\tau := \Phi_{1,0} \circ \Phi_{0,1} = \Phi_{0,1} \circ \Phi_{1,0}$ is the unique faithful unital scalar–valued trace on $\mathcal{A}_\theta$, and is defined as*

$$\tau(A) = \tau(\sum_{n,m\in\mathbb{Z}} a_{m,n} U^m V^n) := a_{0,0}. \tag{1.93}$$

Next, we show that $\mathcal{A}_\theta$ has conditional expectations onto each of its subalgebras $\mathcal{A}_\theta^{(2^h,2^k)}$. Pick any A in $\mathcal{A}_\theta$, and decompose it into the "even" and "odd" parts w.r.t. U in the following way:

$$A = \sum_{m,n\in\mathbb{Z}} a_{m,n} U^m V^n = \sum_{m,n\in\mathbb{Z}} a_{2m,n} U^{2m} V^n + \sum_{m,n\in\mathbb{Z}} a_{2m+1,n} U^{2m+1} V^n. \tag{1.94}$$

Let $\mathcal{A}_\theta^{2,1}$ be the subalgebra generated by U^2, V. Any element $A \in \mathcal{A}_\theta$ can be written as

$$A = x + yU, \tag{1.95}$$

where

$$x = \sum_{m,n\in\mathbb{Z}} a_{2m,n} U^{2m} V^n, \qquad y = \sum_{m,n\in\mathbb{Z}} a_{2m+1,n} e^{-2\pi i n\theta} U^{2m} V^n. \tag{1.96}$$

We define a linear map $E_{2,0}$ from $\mathcal{A}_\theta$ onto $\mathcal{A}_\theta^{2,1}$ by $E_{2,0}(A) \equiv E_{2,0}(x + yU) := x$. Then, we have $E_{2,0}(I) = I$ and

$$\begin{aligned} E_{2,0}(A^*A) &= E_{2,0}((x+yU)^*(x+yU)) = E_{2,0}(x^*x + U^*y^*yU + x^*yU + U^*y^*x) \\ &= x^*x + U^*y^*yU \geq x^*x = E_{2,0}(A^*)E_{2,0}(A), \end{aligned}$$

showing that $E_{2,0}$ is a positive map. (Notice that we can always write U^*y^*x as $\tilde{y}^*\tilde{x}U$, with $\tilde{y}^*$

given by the following computation:

$$\begin{aligned}
U^*y^*x &= U^*(\sum_{h,k\in\mathbb{Z}} a_{2h+1,k}e^{-2\pi ik\theta}U^{2h}V^k)^*(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m}V^n)\\
&= U^*(\sum_{h,k\in\mathbb{Z}} \overline{a_{2h+1,k}}e^{2\pi ik\theta}V^{-k}U^{-2h})^*(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m}V^n)\\
&= (\sum_{h,k\in\mathbb{Z}} \overline{a_{2h+1,k}}V^{-k}U^{-2h-1})^*(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m}V^n)\\
&= (\sum_{h,k\in\mathbb{Z}} \overline{a_{2h+1,k}}e^{-2\pi ik(2h+1)\theta}U^{-2h-1}V^{-k})(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m}V^n)\\
&= (\sum_{h,k\in\mathbb{Z}} \overline{a_{-2h-1,-k}}e^{-2\pi ik(2h+1)\theta}U^{2h+1}V^{k})(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m}V^n)\\
&= (\sum_{h,k\in\mathbb{Z}} \overline{a_{-2h-1,-k}}e^{-2\pi ik(2h+1)\theta}U^{2h}V^{k})(\sum_{m,n\in\mathbb{Z}} a_{2m,n}U^{2m+1}V^n)\\
&= (\sum_{h,k\in\mathbb{Z}} \overline{a_{-2h-1,-k}}e^{-2\pi ik(2h+1)\theta}U^{2h}V^{k})(\sum_{m,n\in\mathbb{Z}} a_{2m,n}e^{-2\pi in\theta}U^{2m}V^n)U\\
&= \tilde{y}^*\tilde{x}U.)
\end{aligned}$$

Let $(\pi_\tau, \mathcal{H}_\tau \equiv L^2(\mathcal{A}_\theta, \tau))$ be the GNS representation associated to the tracial state τ (1.93). Since τ is faithful, we shall identify $\mathcal{A}_\theta$ with $\pi_\tau(\mathcal{A}_\theta) \subset \mathcal{B}(\mathcal{H}_\tau)$. By the previous considerations, we can decompose the Hilbert space $L^2(\mathcal{A}_\theta, \tau)$ into the direct sum of the even and odd subspaces w.r.t. U:

$$\mathcal{H}_\tau = \mathcal{H}_\tau^e \oplus \mathcal{H}_\tau^o, \tag{1.97}$$

where $\mathcal{H}_\tau^e = L^2(\mathcal{A}_\theta, \tau)_e$ and $\mathcal{H}_\tau^o = L^2(\mathcal{A}_\theta, \tau)_o$. In fact, given $A \in \mathcal{H}_\tau^e$ and $B \in \mathcal{H}_\tau^o$, we have

$$\begin{aligned}
(A,B)_{\mathcal{H}_\tau} &= \tau(A^*B) = \tau((\sum a_{2m,n}U^{2m}V^n)^*(\sum b_{2p+1,q}U^{2p+1}V^q))\\
&= \tau((\sum \overline{a_{-2m,-n}}e^{2\pi i2mn\theta}U^{2m}V^n)(\sum b_{2p+1,q}U^{2p+1}V^q))\\
&= \tau(\sum \overline{a_{-2m,-n}}e^{2\pi i2mn\theta}b_{2p+1,q}U^{2m}V^nU^{2p+1}V^q)\\
&= \tau(\sum \overline{a_{-2m,-n}}e^{2\pi i2mn\theta}e^{-2\pi i2mn\theta}b_{2p+1,q}U^{2m+2p+1}V^{n+q})\\
&= \tau(\sum \overline{a_{-2m,-n}}b_{2p+1,q}U^{2m+2p+1}V^{n+q})\\
&= \tau(\sum c_{2h+1,k}U^{2h+1}V^k) = 0,
\end{aligned}$$

where $c_{2h+1,k} = \sum_{m,n} \overline{a_{-2m,-n}}b_{2h+1-2m,k-n}$.

Consequently, the C^*–algebra $\pi_\tau(\mathcal{A}_\theta)(\cong \mathcal{A}_\theta)$ becomes a $\mathbb{Z}_2$-graded algebra:

$$A = A_+ \oplus A_- \tag{1.98}$$

$$A_+ : \mathcal{H}_\tau^e \to \mathcal{H}_\tau^e, \quad A_+ : \mathcal{H}_\tau^o \to \mathcal{H}_\tau^o \tag{1.99}$$

$$A_- : \mathcal{H}_\tau^e \to \mathcal{H}_\tau^o, \quad A_- : \mathcal{H}_\tau^o \to \mathcal{H}_\tau^e. \tag{1.100}$$

If we write A in matrix notation,

$$A = \begin{pmatrix} a_e & a_o \\ a_o & a_e \end{pmatrix}, \tag{1.101}$$

$$A_+ = \begin{pmatrix} a_e & 0 \\ 0 & a_e \end{pmatrix}, \quad A_- = \begin{pmatrix} 0 & a_o \\ a_o & 0 \end{pmatrix}, \tag{1.102}$$

we get, for the norm,

$$\begin{aligned}
\|A\| &= \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=1} \left\| \begin{pmatrix} a_e & a_o \\ a_o & a_e \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\| \\
&= \sup_{\|x\|=1} \left\| \begin{pmatrix} a_e x_1 + a_o x_2 \\ a_o x_1 + a_e x_2 \end{pmatrix} \right\| \\
&= \sup_{\|x\|=1} \left(\|a_e x_1 + a_o x_2\|^2 + \|a_o x_1 + a_e x_2\|^2 \right)^{\frac{1}{2}} \\
&\geq \sup_{\|x_1\|=1} \left(\|a_e x_1\|^2 + \|a_o x_1\|^2 \right)^{\frac{1}{2}} \geq \sup_{\|x\|=1} \|a_e x\| = \|A_+\|.
\end{aligned}$$

Since the composition of two conditional expectations is again a conditional expectation, we obtain for each subalgebra of the form $\mathcal{A}_\theta^{2^l} (= \mathcal{A}_\theta^{2^l, 2^l})$, a conditional expectation E_l from $\mathcal{A}_\theta$ onto $\mathcal{A}_\theta^{2^l}$, which, by duality, means that the corresponding order–unit subspaces are quantum metric subspaces, once they are given the quotient seminorms.

So, we have to compare the two seminorms on $A_\theta^{2,1}$, the one given by restriction of that from A_θ, and the quotient seminorn on $A_\theta^{2,1}$ given by

$$L_2(x) = \inf\{L(A) : y \in A_\theta^{2,1}\}, \ A \in A_\theta, \quad A = x + yU, \quad x, y \in A_\theta^{2,1}. \tag{1.103}$$

On the one hand, we have

$$L_{2,1}(x) = \inf_{y \in A_\theta^{2,1}} L(A) \leq L(A)_{y=0} = L(x). \tag{1.104}$$

On the other hand, by the previous computations, we have

$$\|(\alpha_1 \delta_1 + \alpha_2 \delta_2)(A)\| \geq \|(\alpha_1 \delta_1 + \alpha_2 \delta_2)(x)\|, \ \forall \alpha_1, \alpha_2 \in \mathbb{C}, \tag{1.105}$$

and thus,

$$L(A) = \sup_{\alpha_1^2 + \alpha_2^2 = 1} \|(\alpha_1 \delta_1 + \alpha_2 \delta_2)(A)\| \geq \|(\alpha_1 \delta_1 + \alpha_2 \delta_2)(x)\|, \tag{1.106}$$

which implies, for every $y \in A_\theta^{2,1}$,

$$L(A) = L(x + yU) \geq \sup_{\alpha_1^2 + \alpha_2^2 = 1} \|(\alpha_1 \delta_1 + \alpha_2 \delta_2)(x)\| = L(x), \tag{1.107}$$

and, in particular,

$$L_{2,1}(x) = \inf_{y \in A_\theta^{2,1}} L(A) \geq L(x). \tag{1.108}$$

Finally, we get the desired equality between the two seminorms on $A_\theta^{2,1}$, namely

$$L_{2,1}(x) = L(x). \tag{1.109}$$

The above construction evidently holds true, if we pass from $A_\theta^{2,1}$ to $A_\theta^{2,2} = A_\theta^2$, and, by iteration, we get the following

1.5.7 Proposition. *For each positive integer l, $(A_\theta^{2^l}, L_\theta)$ is a quantum metric subspace of (A_θ, L_θ).*

We see that, by the previous Proposition, each QMS in the double sequence (1.80) is actually a quantm metric subspace of the successive one in each row. So, as discussed in Remark 1.4.25, given a state on the quantum solenoid $A_{(\cdot)}^\infty$, for instance the trace τ defined above in (1.93), one may consider the ball $\overline{B}^\infty(\tau, r) := \iota_m(\overline{B}_m(\tau, r))$ ($\iota_l : S(A_\tau^{2^{-l}}) \to S(A_\tau^\infty)$ is the dual map of the projection $\pi_l : A_\tau^\infty \to A_\tau^{2^{-l}}$ given by the conditional expectation), and then the space obtained as union over all (admissible) $r > 0$, namely

$$X_\infty := \bigcup_{r>0} \overline{B}^\infty(\tau, r).$$

Thus, for instance, for $A_0^\infty = C^\infty(\mathcal{S}_2^2)$, we obtain the (isometric) copy of $\mathbb{R}^2$ embedded in the 2–adic solenoid $\mathcal{S}_2^2$ (see Theorem 10.13 in [38]). Therefore, we may say that $C^\infty(\mathcal{S}_2^2)$ is some kind of “compactification” of $C^\infty(\mathbb{R}^2)$, in the sense that the weak closure in $\mathcal{B}(L^2(\mathbb{R}))$ of the GNS representations of the corresponding C^*–algebras of continuous functions coincide, as we shall see in the next section.

1.5.5 The Weyl Unitaries and the Quantum Plane

The Quantum Plane

(The reader is referred to [73] for more details.)

Let us begin recalling the definition of the Quantum Plane.

1.5.8 Definition. *Let $\theta \geq 0$. For $\boldsymbol{t} \equiv (t_1, t_2) \in \mathbb{R}^2$, we define a unitary operator $W_\theta(\boldsymbol{t})$ on $L^2(\mathbb{R}^2)$ by*

$$(W_\theta(\boldsymbol{t})g)(x_1, x_2) := e^{i(x_1t_1+x_2t_2)} g(x_1 + \frac{1}{2}\theta t_2, x_2 - \frac{1}{2}\theta t_1). \tag{1.110}$$

For $f, g \in \mathcal{S}(\mathbb{R}^2)$, we define the twisted product $f \times_\theta g \in \mathcal{S}(\mathbb{R}^2)$ by

$$(f \times_\theta g)(x_1, x_2) := \frac{1}{2\pi} \int_{\mathbb{R}^2} dt_1\, dt_2\, \widetilde{f}(t_1, t_2)(W_\theta(\boldsymbol{t})g)(x_1, x_2), \tag{1.111}$$

with $\widetilde{f}$ the Fourier transform of f. Finally, for $f \in \mathcal{S}(\mathbb{R}^2)$, we define the twisted multiplication operator $W_\theta(f)$ on $L^2(\mathbb{R}^2)$ by

$$(W_\theta(f))(g) := (f \times_\theta g). \tag{1.112}$$

The unitaries $W_\theta(\mathbf{s})$, $\mathbf{s} \in \mathbb{R}^2$ satisfy the Weyl Commutation Relations, i.e.

$$W_\theta(\mathbf{s})W_\theta(\mathbf{t}) = e^{1\theta(s_2t_1 - s_1t_2)} W_\theta(\mathbf{s} + \mathbf{t}). \tag{1.113}$$

One has then the following

1.5.9 Proposition. [73]

(i) *Let* $f, g \in \mathcal{S}(\mathbb{R}^2)$ *and* $a, b \in \mathbb{C}$. *Then,* $aW_\theta(f) + bW_\tau(g) = W_\theta(af + bg)$.

(ii) *Let* $f, g, h \in \mathcal{S}(\mathbb{R}^2)$. *Then,* $(f \times_\theta g) \times_\theta h = f \times_\theta (g \times_\theta h)$. *Thus,* $W_\theta(f)W_\theta(g) = W_\tau(f \times_\theta g)$.

(iii) $W_\theta(\boldsymbol{t})^* = W_\theta(-\boldsymbol{t})$ *for all* $\boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2$. *For* $f \in \mathcal{S}(\mathbb{R}^2)$, *one has* $W_\theta(f)^* = W_\theta(g)$, *where* $g(x_1, x_2) := f(-x_1, -x_2)$.

1.5.10 Definition. *The* Quantum Plane *is the* C^**–algebra generated by the operators* $W_\theta(f)$ *for* $f \in \mathcal{S}(\mathbb{R}^2)$.

The Weyl Commutation Relations

We can reinterpret the commutation relations defining the noncommutative torus as a discrete form of the Weyl commutation relations. Indeed, let $W_\theta(\mathbf{m})$ be the unitary defined as

$$W_\theta(\mathbf{m}) := e^{-i\pi\theta m_1 m_2} U^{m_1} V^{m_2} \tag{1.114}$$

where $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$. The operators $W_\theta(\mathbf{m})$ then satisfies, as above, the Canonical Commutation Relations (CCR) in Weyl form:

$$W_\theta(\mathbf{m})W_\theta(\mathbf{n}) = e^{i\pi\sigma_\theta(\mathbf{m},\mathbf{n})} W_\theta(\mathbf{m} + \mathbf{n}) = e^{2i\pi\sigma_\theta(\mathbf{m},\mathbf{n})} W_\theta(\mathbf{n})W_\theta(\mathbf{m}) \tag{1.115}$$

with σ_θ the symplectic form

$$\sigma_\theta(\mathbf{m}, \mathbf{n}) = \theta(m_1 n_2 - m_2 n_1). \tag{1.116}$$

(Notice that $W_\theta(\mathbf{m})^* = W_\theta(-\mathbf{m})$ and that products like $\prod_k W_\theta(\mathbf{m}_k)$ are reducible to a single $W_\theta(\sum_k \mathbf{m}_k)$ multiplied by a phase.) Thus, we can rewrite the noncommutative torus as

$$\mathcal{A}_\theta = C^*\Big\{ \sum_{\mathbf{m}\in\mathbb{Z}^2} a_{\mathbf{m}} W_\theta(\mathbf{m}) : a_{\mathbf{m}} \in \mathcal{S}(\mathbb{Z}^2) \Big\}. \tag{1.117}$$

Moreover, for each $f \in C^\infty(\mathbb{T}^2)$, let $\sum_{\mathbf{m}\in\mathbb{Z}^2} a_{\mathbf{m}} e^{2i\pi(\mathbf{m},\mathbf{x})}$ be its Fourier expansion and define

$$W_\theta(f) := \sum_{\mathbf{m}\in\mathbb{Z}^2} a_{\mathbf{m}} W_\theta(\mathbf{m}). \tag{1.118}$$

Then,

$$\mathcal{A}_\theta = C^*\{W_\theta(f) : f \in C^\infty(\mathbb{T}^2)\}. \tag{1.119}$$

(We recall that $\mathcal{A}_0 \cong C(\mathbb{T}^2)$.)

Consider now the quantum solenoid $\mathcal{A}_\tau^\infty$ in the Weyl form, namely

$$\begin{aligned} \mathcal{A}_\tau^\infty &= C^*\Big\{ \sum_{\mathbf{u}\in\mathbb{Z}^2[2^{-\infty}]} a_{\mathbf{u}} W_\tau(\mathbf{u}) : a_{\mathbf{u}} \in \mathcal{S}(\mathbb{Z}^2[2^{-\infty}]) \Big\} \\ &= C^*\Big\{ \bigcup_{l\in\mathbb{N}} \Big\{ \sum_{\mathbf{q}\in\mathbb{Z}^2[2^{-l}]} a_{\mathbf{q}} W_\tau(\mathbf{q}) : a_{\mathbf{q}} \in \mathcal{S}(\mathbb{Z}^2[2^{-l}]) \Big\} \Big\}, \end{aligned} \tag{1.120}$$

and rewrite it as

$$\mathcal{A}_\tau^\infty = C^*\Big\{ \bigcup_{l\in\mathbb{N}_0} \Big\{ W_\tau(f) : f \in C^\infty(\mathbb{T}_l^2) \Big\} \Big\} \tag{1.121}$$

where $\mathbb{T}_l^2$ is the 2^l–fold covering of $\mathbb{T}^2$. Then, we have evidently

$$\mathcal{A}_\tau^\infty = C^*\{W_\tau(f) : f \in C^\infty(\mathcal{S}_2^2)\} \tag{1.122}$$

Therefore, each element in the quantum tangent cone to the quantum torus can be regarded as lying in this quantum plane, in the sense that there exists a representation, the tracial representation, in which the corresponding weak closures are the same.

In fact, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, let $W_\mathbf{x}, W_\mathbf{y}$ be Weyl unitaries satisfying the following commutation relations:

$$W_\mathbf{x} W_\mathbf{y} = e^{2i\pi\sigma_\tau(\mathbf{x},\mathbf{y})} W_\mathbf{y} W_\mathbf{x}, \tag{1.123}$$
$$\sigma_\tau(\mathbf{x},\mathbf{y}) := \tau(x_1y_2 - x_2y_1), \tag{1.124}$$

and denote by $\mathcal{W}_\tau$ the corresponding (Weyl) C^*–algebra:

$$\mathcal{W}_\tau := C^*\{W_\mathbf{x} : \mathbf{x} \in \mathbb{R}^2\}. \tag{1.125}$$

Clearly, $\mathcal{A}_\tau^\infty \subset \mathcal{W}_\tau$, but $\mathcal{W}_\tau$ is not the (τ–)quantum plane we are interested in. So, for $f \in \mathcal{S}(\mathbb{R}^2)$, let us define first the operators

$$W_\tau(f) := \int_{\mathbb{R}^2} \widetilde{f}(\mathbf{x}) W_\mathbf{x} \mathrm{d}\mathbf{x}, \tag{1.126}$$

($\widetilde{f}$ denotes the Fourier transform of f) and then the "true" τ–quantum plane as

$$\mathcal{P}_\tau := C^*\{W_\tau(f) : f \in \mathcal{S}(\mathbb{R}^2)\}. \tag{1.127}$$

Now, we want to represent $\mathcal{A}_\tau$ and $\mathcal{P}_\tau$ on $L^2(\mathbb{R})$ in such a way that:

$$\pi_1(\mathcal{A}_\tau^\infty)'' = \pi_2(\mathcal{P}_\tau)''. \tag{1.128}$$

(We denote by π_2 the GNS representation induced by the tracial state τ on $\mathcal{P}_\tau$, defined as $\tau(W_\tau(f)) := \widetilde{f}(\mathbf{0})$.) Since clearly $\pi_2(\mathcal{P}_\tau)'' = \pi_2(\mathcal{W}_\tau)''$, it suffices to show that

$$\pi_1(\mathcal{A}_\tau^\infty)'' = \pi_2(\mathcal{W}_\tau)''. \tag{1.129}$$

To this aim, let $\mathcal{H} = L^2(\mathbb{R})$, and let U_α, V_β be two strongly continuous groups of unitaries on $\mathcal{H}$, given by

$$(U_\alpha f)(t) = f(t+\alpha), \quad (V_\beta f)(t) = e^{i\tau(t\cdot\beta)} f(t), \quad \alpha, \beta, t \in \mathbb{R}^2,\ f \in \mathcal{S}(\mathbb{R}) \tag{1.130}$$

with $(t \cdot \beta)$ the usual scalar product in $\mathbb{R}^2$. Then, U_α, V_β satisfy the following relations:

$$U_\alpha U_{\alpha'} = U_{\alpha+\alpha'}, \quad V_\beta V_{\beta'} = V_{\beta+\beta'}, \quad U_\alpha V_\beta = e^{i\tau(\alpha\cdot\beta)} V_\beta U_\alpha. \tag{1.131}$$

The Weyl unitaries $W_\mathbf{x}$, $\mathbf{x} = (\alpha, \beta) \in \mathbb{R}^2$, are then given by

$$W_\mathbf{x} = e^{-\frac{i}{2}\tau(\alpha\cdot\beta)} U_\alpha V_\beta, \tag{1.132}$$

and satisfy the commutation relations:

$$W_\mathbf{x} W_\mathbf{y} = e^{\frac{i}{2}\sigma_\tau(x,y)} W_{\mathbf{x}+\mathbf{y}} \tag{1.133}$$

with $\sigma_\tau(\mathbf{x},\mathbf{y}) = \tau(x_1y_2 - x_2y_1)$, $x = (x_1, x_2)$, $y = (y_1, y_2)$. For $f \in \mathcal{S}(\mathbb{R})$, we define the action of $W_\mathbf{x}$ on f by

$$(W_\mathbf{x} f)(t) = e^{2i\pi\tau(\frac{1}{2}x_1x_2+tx_2)} f(t + x_2). \tag{1.134}$$

Given $f, g \in \mathcal{S}(\mathbb{R})$, let us consider

$$(W_\mathbf{x} f, g)_\mathcal{H} = \int_\mathbb{R} e^{2i\pi\tau(\frac{1}{2}x_1x_2+tx_2)} f(t + x_1)\overline{g(t)}\mathrm{d}t; \tag{1.135}$$

then, setting $t = y - \frac{x_1}{2}$, we obtain

$$(W_\mathbf{x} f, g)_\mathcal{H} = \int_\mathbb{R} e^{2i\pi\tau(yx_2)} f(y + \frac{x_1}{2})\overline{g(y - \frac{x_1}{2})}\mathrm{d}y =: V_{(f,g)}(x_1, x_2). \tag{1.136}$$

The bilinear map $V_{(\cdot,\cdot)} : \mathcal{S}(\mathbb{R}) \times \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}^2)$ extends to a (bounded) bilinear map from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^2(\mathbb{R}^2)$ (see, for instance, Proposition 2.4 and Corollary 3.5 of [74]). Since we can approximate any $x \in \mathbb{R}$ by a sequence of dyadic rationals $\{\frac{m_k}{2^{n_k}}\}_{k\in\mathbb{N}}$, we finally get

$$w - \lim_{k\to\infty} W_\tau(\mathbf{q}_k) = W_\mathbf{x}. \tag{1.137}$$

Hence,

$$\pi_2(\mathcal{W}_\tau)'' \subset \pi_1(\mathcal{A}_\tau^\infty)'', \tag{1.138}$$

and, as $\mathcal{A}_\tau^\infty \subset \mathcal{W}_\tau$ by construction, we obtain the reverse inclusion, hence the equality.

Hence, we have proven the following

1.5.11 Proposition. *Let π_1 and π_2 be the tracial representation on $L^2(\mathbb{R})$ of the quantum solenoid $\mathcal{A}_\tau^\infty$ (1.122) and of the quantum plane $\mathcal{P}_\tau$ (1.127), respectively. Then, we have*

$$\pi_1(\mathcal{A}_\tau^\infty)'' = \pi_2(\mathcal{W}_\tau)''. \tag{1.139}$$

Chapter 2

Lip–von Neumann Algebras and Ultraproducts

In the first two sections of this chapter, we will introduce the notion of Lip–spaces and Rigged Lip–spaces, along with the corresponding ultraproduct construction, while the last section is dedicated to the definition of Lip–von Neumann Algebras and their ultraproducts.

2.1 (Dual) Lip–spaces

We recall some basic facts about Lip–spaces (see [32] for more details).

2.1.1 Definition. *We call* Lip–space *a triple* $(X, \|\cdot\|, L)$*, where:*

(i) $(X, \|\cdot\|)$ *is a Banach space,*

(ii) $L : X \to [0, +\infty]$ *is finite on a dense vector subspace* X_0*, where it is a norm,*

(iii) *the unit ball w.r.t.* L*,* $\{x \in X : L(x) \leq 1\}$*, is compact in* $(X, \|\cdot\|)$*.*

We call Lip–norm *a norm* L *satisfying properties* (ii) *and* (iii) *above.*

We call *radius* of the Lip–space $(X, \|\cdot\|, L)$, and denote it by r_X, the maximum of $\|\cdot\|$ on the unit ball w.r.t. $L(\cdot)$, hence

$$\|x\| \leq r_X L(x), \qquad x \in X. \tag{2.1}$$

This is the analogue of the radius of a compact quantum metric space (cf. Definition 1.2.5 and Proposition 2.2 in [60]).

2.1.2 Remark. Notice that every quantum metric space (A, L_A) may be viewed also as a Lip–space, simply by setting, for any a in the domain of L_A,

$$L(a) := \max(L_A(a), \frac{1}{r_A}\|a\|).$$

The Rieffel's Lip–seminorm L_A can be then recovered as $L_A(a) = \inf_{\lambda\in\mathbb{R}} L(a - \lambda e_A)$ (cf. [32], Proposition 2.2).

2.1.3 Proposition. *Let $(X, \|\cdot\|, L)$ be a Lip–space. Then, the dual norm*

$$L'(\xi) := \sup_{x \in X} \frac{|\langle \xi, x \rangle|}{L(x)} \tag{2.2}$$

induces the w^*–topology on the bounded subsets of X', the Banach space dual of $(X, \|\cdot\|)$, and the radius r_X is also equal to the radius, in the L'–norm, of the unit ball of $(X', \|\cdot\|')$

Proof. First observe that $L'(\cdot)$, which is obviously a seminorm, is indeed a norm. In fact, if $L'(\xi) = 0$, then ξ vanishes on X_0, which is dense, i.e. $\xi = 0$.

Now, we consider the identity map ι from the closed unit ball B'_1 of X', endowed with the w^*–topology, to the same set endowed with the topology induced by L'. Given $r > 0$, let $\{x_i : i = 1, \ldots, n\}$ be an $r/2$–net in $\{x \in X : L(x) \leq 1\}$. Then, if $\|\xi\|' \leq 1$ and $L(x) \leq 1$,

$$|\langle \xi, x \rangle| \leq \max_{i=1,\ldots,n} |\langle \xi, x_i \rangle| + r/2.$$

Therefore, the w^*–open set in B'_1

$$U = \{\|\xi\|' \leq 1 : \max_{i=1,\ldots,n} |\langle \xi, x_i \rangle| < r/2\},$$

is contained in the L'–open set in B'_1

$$V = \{\|\xi\|' \leq 1 : L'(\xi) < r\},$$

showing that ι is continuous. Since the domain is compact and the range is Hausdorff, ι is a homeomorphism.

Finally, the radius of the unit ball of X' in the L'–norm is given by

$$\sup_{\|\xi\|' \leq 1} L'(\xi) = \sup_{\xi \neq 0, x \neq 0} \frac{|\langle \xi, x \rangle|}{L(x)\|\xi\|'} = \sup_{x \neq 0} \frac{\|x\|}{L(x)} \sup_{\xi \neq 0} \frac{|\langle \xi, x \rangle|}{\|\xi\|'\|x\|} = r_X.$$

∎

2.1.4 Definition. *A* Dual Lip–space (DLS) *is a dual Banach space X' with a dual Lip–norm L' which metrizes the w^*–topology on bounded sets.*

2.1.5 Proposition. *$(X, \|\cdot\|, L)$ is a Lip–space if, and only if, $(X', \|\cdot\|', L')$ is a dual Lip–space.*

Proof. If $(X, \|\cdot\|, L)$ is a Lip–space, then $(X', \|\cdot\|', L')$ is a dual Lip–space by the Proposition 2.1.3. Conversely, suppose that $(X', \|\cdot\|', L')$ is a dual Lip–space. Then, identifying X with its (isometric) image in the (Banach) bidual X'' of X, we may consider the set $\{x \in X : L(x) \leq 1\}$ as a family of w^*–continuous functions on the w^*–compact set $\{\xi \in X' : \|\xi\|' \leq 1\} \subseteq \{\xi \in X' : L'(\xi) \leq r_X\}$. Since $|\langle \xi, x \rangle| \leq L(x)L'(\xi)$, we see that the family $\{x \in X : L(x) \leq 1\}$ is equibounded by r_X on $\{\xi \in X' : \|\xi\|' \leq 1\}$. Moreover, as $|\langle \xi_1, x \rangle - \langle \xi_2, x \rangle| \leq L'(\xi_1 - \xi_2)$, and L' induces the w^*–topology on the bounded subsets of X', then the family $\{x \in X : L(x) \leq 1\}$ is also w^*–equicontinuous. Therefore, by the Ascoli–Arzelà Theorem [66], $\{x \in X : L(x) \leq 1\}$ is compact in the sup–norm $\|\cdot\|_\infty$, which coincides, on $\{\xi \in X' : \|\xi\|' \leq 1\}$, with the original (Banach) norm $\|\cdot\|$.

∎

For the applications, we need the fundamental notion of uniformity for families of Lip–spaces.

2.1.6 Definition. *A family $\mathcal{F}$ of Lip–spaces is called* uniform *if for all $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that, for any $(X, \|\cdot\|, L)$ in $\mathcal{F}$, $\{x \in X : L(x) \le 1\}$ can be covered by n_ε $\|\cdot\|$–balls of radius ε.*

The next Lemma tells us that a uniform family of Lip–spaces is also uniformly bounded.

2.1.7 Lemma. [32] *If $\mathcal{F}$ is a uniform family of Lip–spaces, then there is $R > 0$ such that $\|x\| \le RL(x)$ for any $(X, \|\cdot\|, L)$ in $\mathcal{F}$, $x \in X$.*

Proof. Let $(X, \|\cdot\|, L)$ be a Lip–space such that $\{x \in X : L(x) \le 1\}$ can be covered by n balls of radius 1, and let $x_0 \in X$, $L(x_0) = 1$. Since the set $\{tx_0 : t \in [0, 1]\}$ is contained in $\{x \in X : L(x) \le 1\}$, it is covered by at most n balls of radius 1, hence its length is majorised by $2n$, and thus $R \le 2n$. ∎

2.1.8 Notation. Let X be a normed (linear) space, and let X_0 be a subset of X. We say that X_0 is *ε–dense* in X if, for any $x \in X$, one can find an $x_0 \in X_0$ such that $\|x - x_0\| < \varepsilon$. When the set X_0 is discrete, we shall call it an ε–net. When the space involved carries more than one norm, we will always specify the norm w.r.t. which a set will be dense in some other set.

We need two technical Lemmas.

2.1.9 Lemma. *Let (X, ρ) be a metric space, $B_X(x, r)$ the open r–ball with center x, $n_r(\Omega)$ the least number of open balls of radius r which cover $\Omega \subset X$, and $\nu_r(\Omega)$ the largest number of disjoint open balls of radius r centered in Ω. Then, one has*

$$n_r(\Omega) \ge \nu_r(\Omega) \ge n_{2r}(\Omega).$$

Proof. For the first inequality, let $B_X(x_i, r)$, $i = 1, \dots, \nu_r(\Omega)$, be disjoint balls with centres in Ω. Then, any r–ball of a covering of Ω may contain at most one of the x_i's. Indeed, if $x_i, x_j \in B_X(x, r)$, then $\{x\} \subset B_X(x_i, r) \cap B_X(x_j, r) \ne \emptyset$, so that $x_i = x_j$.
As for the second inequality, we need to prove it only when ν_r is finite. So, let us assume that $\{B_X(x_i, r)\}_{i=1}^{\nu_r(\Omega)}$ are disjoint balls centered in Ω, and observe that, for any $y \in \Omega$,

$$\rho(y, \bigcup_{i=1}^{\nu_r(\Omega)} B_X(x_i, r)) := \inf\{\rho(y, z) : z \in \bigcup_{i=1}^{\nu_r(\Omega)} B_X(x_i, r)\} < r,$$

otherwise $B_X(y, r)$ would be disjoint from $\bigcup_{i=1}^{\nu_r(\Omega)} B_X(x_i, r))$, contradicting the maximality of $\nu_r(\Omega)$. Thus, for all $y \in \Omega$, there is a j such that $\rho(y, B_X(x_j, r)) < r$, that is,

$$\Omega \subset \bigcup_{i=1}^{\nu_r(\Omega)} B_X(x_i, r)),$$

which implies the thesis. ∎

2.1.10 Lemma. *Let $(V, \|\cdot\|)$ be an n–dimensional normed space. Then, the ball of radius R can be covered by $(2R/\varepsilon)^n$ balls of radius ε.*

Proof. Let us recall that, denoting by $n_\varepsilon(\Omega)$ the minimum number of balls of radius ε covering Ω, and by $\nu_\varepsilon(\Omega)$ the maximum number of disjoint balls of radius ε contained in Ω, by the previous Lemma, one gets $n_\varepsilon(\Omega) \leq \nu_{\varepsilon/2}(\Omega)$. Then, denoting by vol the Lebesgue measure and by B_r the ball of radius r w.r.t. the given norm, we get $\text{vol}(B_R) \geq \nu_\varepsilon(B_R)\text{vol}(B_\varepsilon)$, and $\text{vol}(B_R) = (R/\varepsilon)^n\text{vol}(B_\varepsilon)$, hence $n_\varepsilon(B_R) \leq (2R/\varepsilon)^n$. ∎

Finally, we have a useful characterization of uniformity in terms of finite approximability, given by the following

2.1.11 Proposition. *A family $\mathcal{F}$ of Lip–spaces is uniform if, and only if, there exists a constant R as in Lemma 2.1.7, and, for any $\varepsilon > 0$, there is $N_\varepsilon \in \mathbb{N}$ such that any Lip–space X in $\mathcal{F}$ has a subspace V with $\dim V \leq N_\varepsilon$, such that $\{x \in V : L(x) \leq 1\}$ is ε–dense in $\{x \in X : L(x) \leq 1\}$.*

Proof. Let $\mathcal{F}$ be uniform. The constant R exists by Lemma 2.1.7. Then, choose a covering of $\{x \in X : L(x) \leq 1\}$ by N_ε norm–balls of radius ε, and consider the vector space V generated by their centers. Its dimension is clearly majorised by N_ε.
As for the converse direction, take $\varepsilon \leq 1$. The elements in $\{x \in V : L(x) \leq 1\}$ are contained in $\{x \in V : \|x\| \leq R\}$, hence any covering of the norm–ball of V of radius R with balls of radius ε gives a covering of the Lip–norm unit ball in X with balls of radius 2ε. By Lemma 2.1.10, one can realise the former covering with $(2R/\varepsilon)^{N_\varepsilon}$ balls, hence the implication is proved. ∎

2.2 (Dual) Rigged Lip–spaces

2.2.1 Definition. *A* Rigged Lip–space (RLS) *is a Banach space X with a densely defined Lip–norm L such that $\{x \in X : L(x) \leq 1\}$ is norm compact, and a further norm p, with $p(x) \leq \|x\|$.*

A Dual Rigged Lip–space (DRLS) *is a dual Banach space X' with a dual Lip–norm L' which metrizes the w^*–topology on bounded sets, and a (possibly infinite) norm p', with $p'(x') \geq \|x'\|$, such that $\{x' \in X'_1 : p'(x) < \infty\}$ is w^*–dense in X'_1.*

We see that a rigged Lip–space is a Lip–space endowed with a further norm p smaller than the Banach norm, while a dual rigged Lip–space is a dual Banach space endowed with a further norm p' greater than the dual Banach norm. As we shall see in the following, in the W^*–algebraic setting, where the dual Banach space will be a W^*–algebra (or a von Neumann algebra), this notion is somehow dual to that of Rieffel, which is essentially a C^*–algebraic construction, for if the Banach space of normal linear functionals on W^*–algebra is a rigged Lip–space, then its dual, i.e. the W^*–algebra itself, or better, its (norm) unit ball, will be the compact metric space playing the role of the state space of a quantum metric space *a la* Rieffel.

2.2.2 Proposition. *Let X be a Banach space, X' be its dual Banach space. (X', L', p') is a dual rigged Lip–space (DRLS) if and only if (X, L, p) is a rigged Lip–space (RLS).*

Proof. The fact that (X, L) is a Lip–space if, and only if, (X', L') is a dual Lip–space follows from Proposition 2.1.5. Therefore, we only have to check that $\{x' \in X'_1 : p'(x) < \infty\}$ is w^*–dense in X'_1 if, and only if, p is a norm on X. Assume that p is a norm on X. Since X is a Lip–space, then, by Proposition 2.1.11, for any n we may find a finite subset $\mathcal{F}_n$ of $\{x \in X : L(x) \leq 1\}$ such that $\cup_{x \in \mathcal{F}_n} B(x, 1/n) \supset \{x \in X : L(x) \leq 1\}$. Therefore, if V_n is the vector space generated by $\mathcal{F}_k$, $k \leq n$, then $\{V_n\}_{n \in \mathbb{N}}$ is an increasing sequence of finite–dimensional spaces whose union V_∞ is dense in X. For a given $x' \in X'$, with $\|x'\| \leq 1$, $x'|_{V_n}$ is p–bounded, since V_n is finite–dimensional and p is a norm, hence, by the Hahn–Banach Theorem, we may find an extension x'_n to the whole X such that $\|x'_n\| \leq 1$ and $p'(x'_n) < \infty$. We want to show that $x'_n \to x'$ in the w^*–topology. Indeed, for any $x \in X$, let $x_n \in V_n$ be a sequence approximating x in norm. We have then

$$|\langle (x' - x'_n), x\rangle| = |\langle (x' - x'_n), (x - x_n)\rangle| \leq 2\|x - x_n\| \to 0.$$

Conversely, if p is only a seminorm on X, namely there exists an $x \in X$, with $\|x\| = 1$ and $p(x) = 0$, we get, for any $x' \in X'$ with $p'(x') < \infty$, $p'(x') \geq |\langle x', \lambda x\rangle| = |\lambda|\ |\langle x', x\rangle|$ for any λ, hence $\langle x', x\rangle = 0$. Therefore, if we choose $x' \in X'$ such that $\langle x', x\rangle = 1$, and pick a sequence $\{x'_n\}$ w^*–converging to x', then we get $\langle x'_n, x\rangle \to 1$, namely x'_n has eventually infinite p'–norm. ∎

2.3 Ultraproducts of Lip–spaces

(The reader is referred to Section 1.3 for basic definitions and properties of ultrafilters and ultrapowers for families of (metric or normed) spaces.)

2.3.1 Restricted Ultraproducts of (Dual) Lip–spaces

Given a sequence $\{(X_n, \|\cdot\|, L_n)\}_{n\in\mathbb{N}}$ of Lip–spaces, let $\ell^\infty(X_n,\mathcal{U})$ be the Banach ultraproduct of the sequence $\{(X_n, \|\cdot\|)\}_{n\in\mathbb{N}}$ of Banach spaces (see Definition 1.3.16), and denote by $\pi_\mathcal{U}$ the projection from $\ell^\infty(X_n)$ onto $\ell^\infty(X_n,\mathcal{U})$.

2.3.1 Definition. *Given a sequence $\{(X_n, \|\cdot\|, L_n)\}_{n\in\mathbb{N}}$ of Lip–spaces, we call* restricted ultraproduct *of the sequence, and denote it by $\ell^\infty_R(X_n,\mathcal{U})$, or simply by $X_\mathcal{U}$, the norm closure of the image under $\pi_\mathcal{U}$ of the subspace $\ell^\infty_R(X_n)$, defined as $\ell^\infty_R(X_n) := \{\{x_n\} \in \ell^\infty(X_n) : L(\{x_n\}) := \sup_n L_n(x_n) < +\infty\}$.*

The quotient norm $\|\cdot\|_\mathcal{U}$ of the equivalence class $x_\mathcal{U}$ of a sequence $\{x_n\}$ is defined as

$$\|x_\mathcal{U}\|_\mathcal{U} := \inf_{[y_n]=x_\mathcal{U}} \sup_n \|y_n\|, \tag{2.3}$$

and, by Proposition 1.3.12, we see that $\|x_\mathcal{U}\|_\mathcal{U} = \lim_\mathcal{U} \|x_n\|$ (see [1], 2.3).

Analogously, the quotient norm $L_\mathcal{U}(x_\mathcal{U})$ of $x_\mathcal{U}$ is defined as

$$L_\mathcal{U}(x_\mathcal{U}) := \inf_{[y_n]=x_\mathcal{U}} \sup_n L_n(y_n). \tag{2.4}$$

This, in particular, implies that $L_\mathcal{U}(x_\mathcal{U}) \leq \lim_\mathcal{U} L_n(x_n)$. In fact, for any $\varepsilon > 0$, there exists an element U of the ultrafilter such that, for any $n \in U$, $L_n(x_n) \leq \lim_\mathcal{U} L_m(x_m) + \varepsilon$. Then, if we define $y_n = x_n$ for $n \in U$ and $y_n = 0$ for $n \notin U$, since $[y_n] = [x_n]$, the result follows. Moreover, it can be shown that the infimum in (2.4) is a minimum (see [32], Lemma 2.9), i.e. there exists a sequence $\{\tilde{x}_n\}$ in the same equivalence class of $\{x_n\}$, such that

$$L_\mathcal{U}(x_\mathcal{U}) = \lim_\mathcal{U} L_n(\tilde{x}_n) = \sup_n L_n(\tilde{x}_n). \tag{2.5}$$

In particular, it follows that, for any element $x_\mathcal{U} \in \ell^\infty_R(X_n,\mathcal{U})$,

$$L_\mathcal{U}(x_\mathcal{U}) = \min_{[x_n]=x_\mathcal{U}} \lim_\mathcal{U} L_n(x_n). \tag{2.6}$$

2.3.2 Proposition. [32] *Given a uniform sequence $(X_n, \|\cdot\|, L_n)$ of Lip–spaces, the restricted ultraproduct $\ell^\infty_R(X_n,\mathcal{U})$, endowed with the quotient norms $\|\cdot\|_\mathcal{U}$, $L_\mathcal{U}$, is a Lip–space. Moreover, the radius $r_\mathcal{U}$ for $\ell^\infty_R(X_n,\mathcal{U})$ is equal to $\lim_\mathcal{U} r_n$, where r_n is the radius of X_n.*

Let $\{(X'_n, \|\cdot\|', L'_n)\}_{n\in\mathbb{N}}$ be a sequence of dual Lip–spaces, and let $\ell^\infty(X'_n,\mathcal{U})$ be the (Banach) ultraproduct of the sequence $\{(X'_n, \|\cdot\|')\}_{n\in\mathbb{N}}$ (see Definition 1.3.16).

The relation between $\ell^\infty(X'_n,\mathcal{U})$ and $\ell^\infty_R(X_n,\mathcal{U})'$ is given in the following

2.3.3 Proposition. [32] *Let $\{\xi_n : \xi_n \in X'_n\}$ be a uniformly bounded sequence, and consider the functional $\xi_\mathcal{U}$ on $\ell^\infty(X_n,\mathcal{U})$ given by $\xi_\mathcal{U}(x_\mathcal{U}) := \lim_\mathcal{U} \xi_n(x_n)$, with $[x_n] = x_\mathcal{U} \in \ell^\infty_R(X_n,\mathcal{U})$. Then, $\xi_\mathcal{U}$ is well–defined, $\xi_\mathcal{U} \in \ell^\infty_R(X_n,\mathcal{U})'$, and*

$$L'_\mathcal{U}(\xi_\mathcal{U}) = \lim_\mathcal{U} L'_n(\xi_n). \tag{2.7}$$

Proof. Let $M > 0$ be such that $\|\xi_n\|' \le M$, $n \in \mathbb{N}$. We first prove that $\xi_\mathcal{U}$ is well–defined and bounded. Indeed, if $[x'_n] = [x_n] \in \ell^\infty_R(X_n,\mathcal{U})$, then $\lim_\mathcal{U} |\xi_n(x'_n) - \xi_n(x_n)| \le M \lim_\mathcal{U} \|x'_n - x_n\| = 0$. Moreover, $|\xi_\mathcal{U}(x_\mathcal{U})| \le M \lim_\mathcal{U} \|x_n\| = M\|x_\mathcal{U}\|$, so that $\|\xi_\mathcal{U}\|'_\mathcal{U} \le M$. Finally,

$$\begin{aligned}
\lim_\mathcal{U} L'_n(\xi_n) &= \lim_\mathcal{U} \sup_{x_n\in X_n} \frac{|\xi_n(x_n)|}{L_n(x_n)} \\
&= \sup_{\{x_n\}\in\ell^\infty_R(X_n)} \lim_\mathcal{U} \frac{|\xi_n(x_n)|}{L_n(x_n)} \\
&= \sup_{\{x_n\}\in\ell^\infty_R(X_n)} \frac{\lim_\mathcal{U} |\xi_n(x_n)|}{\lim_\mathcal{U} L_n(x_n)} = \sup_{x_\mathcal{U}\in\ell^\infty_R(X_n,\mathcal{U})} \sup_{[x_n]=x_\mathcal{U}} \frac{|\xi_\mathcal{U}(x_\mathcal{U})|}{\lim_\mathcal{U} L_n(x_n)} \\
&= \sup_{x_\mathcal{U}\in\ell^\infty_R(X_n,\mathcal{U})} \frac{|\xi_\mathcal{U}(x_\mathcal{U})|}{L_\mathcal{U}(x_\mathcal{U})} = L'_\mathcal{U}(\xi_\mathcal{U}),
\end{aligned}$$

where in the last but one equality we used (2.6). Note also that, in that equality, the set of allowed elements in the supremum on the right is tacitly assumed not to contain $x_\mathcal{U} = 0$, while the set of allowed elements in the supremum on the left might also contain $x_\mathcal{U} = 0$, since in some examples one may find sequences $\{x_n\}$ such that $[x_n] = 0$ but $\lim_\mathcal{U} L_n(x_n) > 0$. However, for such sequences the numerator $|\xi_\mathcal{U}(x_\mathcal{U})|$ is zero, therefore the supremum does not change. ∎

Now, let $\ell^\infty(X'_n) := \{\{x'_n\} : x'_n \in X'_n, \|\{x'_n\}\| := \sup_n \|x'_n\| < \infty\}$, the Banach space of (uniformly) bounded sequences, on which we set $L'(\{\xi_n\}) := \sup_n L'_n(\xi_n)$. Let us consider the subspace

$$K'_{L',\mathcal{U}} := \{\{\xi_n\} \in \ell^\infty(X'_n) : \lim_\mathcal{U} L'_n(\xi_n) = 0\}, \tag{2.8}$$

and the corresponding quotient map

$$\pi'_\mathcal{U} : \ell^\infty(X'_n) \to \ell^\infty(X'_n)/K'_{L',\mathcal{U}}. \tag{2.9}$$

Let us notice that, in absence of further hypotheses, $K'_{L',\mathcal{U}}$ is not complete in the Banach norm, but it becomes a Banach subspace as soon as we assume that the sequence $\{X_n\}$ is uniform. Indeed, in this case we have $L'_n(\mu_n - \nu_n) \le r_n\|\mu_n - \nu_n\| \le R\|\mu_n - \nu_n\|$, where r_n are the radii of X_n (cf. Proposition 2.1.3).

2.3.4 Definition. *We define* $X'_{\mathcal{U}}$ *as the image* $\pi'_{\mathcal{U}}(\ell^\infty(X'_n))$ *of* $\ell^\infty(X'_n)$ *in* $\ell^\infty(X'_n)/K'_{L',\mathcal{U}}$*, with the quotient norms* $\|\cdot\|'_{\mathcal{U}}$ *and* $L'_{\mathcal{U}}$*, and call it the* dual restricted ultraproduct *of the family* $\{(X'_n, \|\cdot\|', L'_n)\}_{n\in\mathbb{N}}$.

The pairing between $\ell^\infty(X'_n)$ and $\ell^\infty(X_n)$, given by $\langle\{\xi_n\},\{x_n\}\rangle = \lim_{\mathcal{U}} \xi_n(x_n)$, gives rise to a pairing between $\ell^\infty(X'_n,\mathcal{U})$ and $\ell^\infty(X_n,\mathcal{U})$ (see [68], Lemma 1, p. 77), hence to an isometric map $\ell^\infty(X'_n,\mathcal{U}) \to \ell^\infty(X_n,\mathcal{U})'$. By composing this isometric map with the projection map from $\ell^\infty(X_n,\mathcal{U})'$ to $\ell^\infty_R(X_n,\mathcal{U})'$ (i.e., the dual of the inclusion map of $\ell^\infty_R(X_n,\mathcal{U})$ into $\ell^\infty(X_n,\mathcal{U})$), one obtains a contraction $\pi : \ell^\infty(X'_n,\mathcal{U}) \to \ell^\infty_R(X_n,\mathcal{U})'$. The fact that π is surjective is essentially the content of the following

2.3.5 Theorem. [32] *Given a uniform sequence* $(X_n, \|\cdot\|, L_n)$ *of Lip–spaces, the ultraproduct* $\ell^\infty(X'_n,\mathcal{U})$ *of the dual spaces projects on the dual* $\ell^\infty_R(X_n,\mathcal{U})'$ *of the restricted ultraproduct. Moreover, given a sequence* $\{\xi_n\}$ *in* $\ell^\infty(X'_n)$*, the element* $\xi_{\mathcal{U}}$ *in* $\ell^\infty(X'_n,\mathcal{U})$ *gives the null functional on* $\ell^\infty_R(X_n,\mathcal{U})$ *if, and only if,* $\lim_{\mathcal{U}} L'_n(\xi_n) = 0$.

As a consequence, we see that the kernel of the map $\pi : \ell^\infty(X'_n,\mathcal{U}) \to \ell^\infty_R(X_n,\mathcal{U})'$ is (isometrically) isomorphic to $K'_{L,\mathcal{U}}$ for the respective (dual) Lip–norms. With the same notation as above, we then have the following

2.3.6 Corollary. *The dual restricted ultraproduct* $X'_{\mathcal{U}}$ *of the family* $\{X'_n\}$ *is (isometrically) isomorphic to the Banach dual* $(X_{\mathcal{U}})' \equiv \ell^\infty_R(X_n,\mathcal{U})'$ *of the restricted ultraproduct* $\ell^\infty_R(X_n,\mathcal{U})$ *of the family* $\{X_n\}$.

2.3.2 Restricted Ultraproduct of (Dual) Rigged Lip–spaces

Given a sequence $\{X_n\}$ of RLS, let $\ell^\infty(X_n) = \{\{x_n\} : x_n \in X_n, \|\{x_n\}\| := \sup_n \|x_n\| < \infty\}$ be the Banach space of bounded sequences, on which we set $p(\{x_n\}) := \sup_n p_n(x_n)$. We consider the subspace $\ell^\infty_R(X_n) = \{\{x_n\} \in \ell^\infty(X_n) : L(\{x_n\}) := \sup_n L_n(x_n) < \infty\}$ and, for a given ultrafilter $\mathcal{U}$, the subspace $K_{\mathcal{U}} = \{\{x_n\} \in \ell^\infty(X_n) : \lim_{\mathcal{U}} \|x_n\| = 0\}$. $K_{\mathcal{U}}$ is a Banach subspace of $\ell^\infty(X_n)$, therefore we may consider the projection $\pi_{\mathcal{U}} : \ell^\infty(X_n) \to \ell^\infty(X_n)/K_{\mathcal{U}}$ to the quotient space. Let us observe that such quotient is a Banach space, with the quotient Banach norm.

2.3.7 Definition. *We define* $X_{\mathcal{U}}$ *as the norm closure of* $\pi_{\mathcal{U}}(\ell^\infty_R(X_n))$ *in* $\ell^\infty(X_n)/K_{\mathcal{U}}$*, with the quotient norms* $p_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}}$*, and* $L_{\mathcal{U}}$*. We call it the* restrictred ultraproduct *of RLS spaces. Let us observe that, in general,* $p_{\mathcal{U}}$ *is only a seminorm.*

Similarly, given a sequence $\{X'_n\}$ of DRLS, let $\ell^\infty(X'_n) = \{\{x'_n\} : x'_n \in X'_n, \|\{x'_n\}\| := \sup_n \|x'_n\| < \infty\}$ be the Banach space of bounded sequences, on which we set $L'(\{x'_n\}) := \sup_n L'_n(x'_n)$. We consider the subspace $\ell^\infty_{p'}(X'_n) := \{\{x'_n\} \in \ell^\infty(X'_n) : p'(\{x'_n\}) := \sup_n p'_n(x'_n) < \infty\}$ and, for a given ultrafilter $\mathcal{U}$, the subspace $K'_{L',\mathcal{U}} = \{\{x'_n\} \in \ell^\infty(X'_n) : \lim_{\mathcal{U}} L'_n(x'_n) = 0\}$, and the quotient map $\pi'_{\mathcal{U}} : \ell^\infty(X'_n) \to \ell^\infty(X'_n)/K'_{L',\mathcal{U}}$.

2.3.8 Definition. *We define* $X'_{\mathcal{U}}$ *as the image* $\pi'_{\mathcal{U}}(\ell^\infty(X'_n))$ *of* $\ell^\infty(X'_n)$ *in* $\ell^\infty(X'_n)/K'_{L',\mathcal{U}}$*, with the quotient norms* $p'_{\mathcal{U}}, \|\cdot\|_{\mathcal{U}}$*, and* $L'_{\mathcal{U}}$*. We call it the* dual restrictred ultraproduct *of DRLS spaces.*

As for Lip–spaces, we need the concept of uniformity, given in the following

2.3.9 Definition. *A sequence X_n of RLS's is* uniform *if the sequence $\{x \in X_n : L(x) \leq 1\}$ is uniformly totally bounded, and, for any sequence $\{x'_n\} \in \ell^\infty(X'_n)$, $\|\{x'_n\}\| \leq 1$, and any $\varepsilon > 0$, we may find $\{y'_n\} \in \ell^\infty_{p'}(X'_n)$, $\|\{y'_n\}\| \leq 1$, such that $L'(\{x'_n - y'_n\}) \leq \varepsilon$.*

Then, we have the following

2.3.10 Theorem. *Assume the sequence X_n is uniform. Then $X_\mathcal{U}$ is a RLS, $X'_\mathcal{U}$ is a DRLS, and $X'_\mathcal{U}$ is the dual of $X_\mathcal{U}$.*

Proof. We already know that, by Proposition 2.3.2, $X_\mathcal{U}$ is a Lip–space, $X'_\mathcal{U}$ is a dual Lip–space, and, by Corollary 2.3.6, $X'_\mathcal{U}$ is the dual of $X_\mathcal{U}$, so we have only to check the rigged structure. In view of Proposition 2.2.2, it is enough to show that $\{x' \in X'_{\mathcal{U},1} : p'_\mathcal{U}(x) < \infty\}$ is w^*–dense in $X'_{\mathcal{U},1}$. By definition, for any $\{x'_n\} \in \ell^\infty(X'_n)$, $\|\{x'_n\}\| \leq 1$, and any $\varepsilon > 0$, we may find $\{y'_n\} \in \ell^\infty_{p'}(X'_n)$, $\|\{y'_n\}\| \leq 1$ such that $L'(\{x'_n - y'_n\}) \leq \varepsilon$, namely $\{x'_n\}$ can be approximated in L'—norm, hence the claim follows. ∎

2.4 Lip–von Neumann Algebras and Ultraproducts

In the following, we will consider concrete von Neumann algebras, but all the results are valid for abstract W^*–algebras as well, as we do not make any reference to the representing Hilbert space.

2.4.1 Definition (Lip–von Neumann Algebra). *A Lip–von Neumann algebra (LvNA) is a von Neumann algebra M with a dual Lip–norm L', which metrizes the w^*–topology on bounded subsets, i.e. such that $\{x \in M : \|x\| \leq 1\}$ is w^*–compact in the topology induced by L'. Equivalently, M_* has a densely defined norm L such that $\{\omega \in M_* : L(\omega) \leq 1\}$ is norm compact.*

2.4.2 Example (Commutative Lip–von Neumann algebras). Let us recall that every commutative von Neumann algebra on a separable Hilbert space is isometrically $*$–isomorphic to $L^\infty(X)$ for some Radon integral on a compact, second countable Hausdorff space X. The predual of $L^\infty(X)$ will be then $L^1(X)$, and so, in order to get a (dual) Lip–norm L' on $L^\infty(X)$, it suffices to construct a densely defined norm L on $L^1(X)$ such that the set $\{f \in L^1(X) : L(f) \leq 1\}$ is norm compact. So, in particular, if we are given a compact linear map $T : C(X) \to L^1(X)$, by setting $L_T(f) := \|T(f)\|_1$, we get immediately a dual Lip–norm L' on $L^\infty(X)$ by setting

$$L'_T(g) := \sup\{|\langle f, g\rangle| : L_T(f) \leq 1\}. \tag{2.10}$$

For example, let $k : X \times X \to \mathbb{C}$ be a continuous function. Then, since X is compact, $k \in L^2(X \times X, \mu \times \mu)$, and the corresponding linear map $T_k : C(X) \to L^1(X)$, given by

$$(T_k f)(s) := \int k(s,t) f(t) d\mu(t), \quad f \in C(X), \tag{2.11}$$

is compact for the L^1–norm topology (see, for instance, Lemma 13.4 in [39]). Hence, we have a plenty of dual Lip–norms on any given commutative von Neumann algebra acting on a separable Hilbert space.

2.4.3 Definition. *The LvNA's (M, L'_M) and (N, L'_N) are said to be* Lip–isometric *if there is an isometric $*$–isomorphism between them, namely a $*$–isomorphism $\varphi : M \to N$, such that*

$$L'_N(\varphi(a)) = L'_M(a), \quad \textit{for any} \quad a \in M. \tag{2.12}$$

Let $\{(M_n, L'_n)\}_{n\in\mathbb{N}}$ be a sequence of Lip–von Neumann algebras, with corresponding preduals (M_{n*}, L_n), and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. As (M_{n*}, L_n) is in particular a Lip–space, we may consider the space $\ell^\infty(M_{n*}) = \{\{\omega_n\} : \omega_n \in M_{n*}, \|\{\omega_n\}\| = \sup_n \|\omega_n\| < \infty\}$, the subspace $\ell^\infty_R(M_{n*}) = \{\{\omega_n\} \in \ell^\infty(M_{n*}) : L(\{\omega_n\}) = \sup_n L_n(\omega_n) < \infty\}$, $K_\mathcal{U} = \{\{\omega_n\} \in \ell^\infty(M_{n*}) : \lim_\mathcal{U} \|\omega_n\| = 0\}$, and the quotient projection $\pi_\mathcal{U} : \ell^\infty(M_{n*}) \to \ell^\infty(M_{n*})/K_\mathcal{U}$. In view of Definition 2.3.1, the restricted ultraproduct $M_{\mathcal{U}*}$ of the family $\{M_{n*}\}$ is the norm closure of the image $\pi_\mathcal{U}(\ell^\infty_R(M_{n*}))$ of $\ell^\infty_R(M_{n*})$ in $\ell^\infty(M_{n*})/K_\mathcal{U}$, with the quotient norms $\|\cdot\|_\mathcal{U}$ and $L_\mathcal{U}$.

Then, since (M_n, L'_n) is a dual Lip–space, in view of Definition 2.3.4, we can construct the dual restricted ultraproduct $M_\mathcal{U}$ of the family $\{(M_n, L'_n)\}$ as the image $\pi'_\mathcal{U}(\ell^\infty(M_n,\mathcal{U}))$ of $\ell^\infty(M_n,\mathcal{U})$ in $\ell^\infty(M_n,\mathcal{U})/K'_{L,\mathcal{U}}$, with the quotient norms $\|\cdot\|_\mathcal{U}$ and $L'_\mathcal{U}$, where $\ell^\infty(M_n,\mathcal{U})$ is the (Banach) ultraproduct of the sequence $\{M_n\}$, on which we set $L'(\{a_n\}) = \sup_n L'_n(a_n)$, $K'_{L',\mathcal{U}} = \{\{a_n\} \in \ell^\infty(M_n,\mathcal{U}) : \lim_\mathcal{U} L'(a_n) = 0\}$ (which is not an ideal!), and $\pi'_\mathcal{U} : \ell^\infty(M_n,\mathcal{U}) \to \ell^\infty(M_n,\mathcal{U})/K'_{L',\mathcal{U}}$ is the quotient projection.

If we want that the (restricted) ultraproduct of a family of LvNA's is itself a LvNA, we need a further condition on the Lip–norm. So, we introduce a "rigged structure" for Lip–von Neumann algebras, given in the following

2.4.4 Definition (Rigged von Neumann Algebra). *A* Rigged von Neumann Algebra (RvNA) *is a Lip–von Neumann Algebra M with a dual Lip–norm L' such that $\{a \in M : \|a\| \le 1\}$ is L'–compact, and setting*

$$N(a) := \sup_{L'(b)\le 1} \max(L'(ab), L'(ba)), \tag{2.13}$$

the set $\{a \in M_1 : N(a) < \infty\}$ is w^*–dense in M_1.

2.4.5 Lemma. *Let (M, L', N) be a RvNA. Then,*

(i) $L'(ab) \le L'(a)N(b)$, $L'(ba) \le L'(b)N(a)$.

(ii) $N(ab) \le N(a)N(b)$, $N(a^*) = N(a)$, hence $M_N := \{a \in M : N(a) < \infty\}$ is a $*$-algebra.

If we set

$$p'(a) := \max(\|a\|, N(a)), \tag{2.14}$$

(M, L', p') is dual rigged Lip–space.

Proof. Property (i) follows immediately by the Definition. As for (ii), $L'(a_1a_2b) \le N(a_1)L'(a_2b)$, hence

$$\sup_{L'(b)\le 1} L'(a_1a_2b) \le N(a_1) \sup_{L'(b)\le 1} L'(a_2b) \le N(a_1)N(a_2). \tag{2.15}$$

Analogously, $\sup_{L'(b)\le 1} L'(ba_1a_2) \le N(a_1)N(a_2)$. As a consequence, $N(ab) \le N(a)N(b)$. Since L' is a dual Lip–norm, $L'(a^*) = L'(a)$, hence $N(a^*) = N(a)$ follows. The last statement is now obvious. ∎

We introduce now two types of uniformity, one for families of Lip–von Neumann algebras (we call it *weak uniformity*), the other for families of rigged von Neumann algebras (*strong uniformity*).

2.4.6 Definition (Weak Uniformity). *A family (M_i, L'_i), $i \in \mathbb{I}$, of Lip–von Neumann algebras is* weakly uniform *if it is uniformly totally bounded, i.e., if for any $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that, for any $i \in \mathbb{I}$, the unit ball $\{x \in M_i : L'_i(x) \le 1\}$ can be covered by n_ε L'_i–balls of radius ε.*

2.4.7 Definition (Strong Uniformity). *A family* (M_i, L'_i), $i \in \mathbb{I}$, *of rigged von Neumann algebras is* strongly uniform *if*

(a) Uniform compactness: *The family* $(M_i, L'_i)_{i\in\mathbb{I}}$ *is uniformly totally bounded.*

(b) Uniform normalizer condition: $\forall\varepsilon > 0$, $\exists K > 0 : \forall i \in \mathbb{I}$, $a \in (M_i)_1$ $\exists b \in (M_i)_1$: $N_i(b) \le K$ *and* $L'_i(a - b) \le \varepsilon$.

2.4.8 Remark. Since, by Lemma 2.4.5, each (M_i, L'_i, p'_i), with $p'_i(a) = \max(\|a\|, N_i(a))$, is a dual rigged Lip–space, in view of Definition 2.3.9 we see that the notion of strong uniformity for families of RvNA's coincides with the notion of uniformity for families of dual rigged Lip–spaces.

We end this chapter with the following fundamental

2.4.9 Theorem. *Let* $\{M_n\}_{n\in\mathbb{N}}$ *be a (strongly) uniform sequence of RvNA,* $\mathcal{U}$ *a free ultrafilter, and define*

$$\ell^\infty_{p'}(M_n) := \left\{ \{a_n\} \in \ell^\infty(M_n) : p'(\{a_n\}) := \sup_n p'_n(a_n) < \infty \right\}.$$

Then,

(i) $\mathcal{A}_\mathcal{U} := \pi'_\mathcal{U}(\ell^\infty_{p'}(M_n)^{-\|\cdot\|})$ *is a* C^**-algebra.*

(ii) Let $\mathcal{A}''_\mathcal{U}$ *be the weak closure of* $\mathcal{A}_\mathcal{U}$ *in the direct sum of the GNS representations associated with all states in* $(M_\mathcal{U})_*$*. Then* $(\mathcal{A}''_\mathcal{U})_* = (M_\mathcal{U})_*$*, hence* $\mathcal{A}''_\mathcal{U}$ *is isomorphic with* $M_\mathcal{U}$ *as a Banach space.*

(iii) $M_\mathcal{U}$ *is a LvNA.*

Proof. (i). By Lemma 2.4.5 above, if $\|\{a_n\}\| < \infty$, $N(\{a_n\}) := \sup_n N_n(a_n) < \infty$, and $\lim_\mathcal{U} L'_n(b_n) = 0$, then

$$\lim_\mathcal{U} L'(a_n b_n) \le \lim_\mathcal{U} N_n(a_n) L'_n(b_n) \le N(\{a_n\}) \lim_\mathcal{U} L'_n(b_n) = 0,$$

and analogously, $\lim_\mathcal{U} L'_n(b_n a_n) = 0$. This shows that the space $K'_{L',\mathcal{U}}$ is a bimodule for $\ell^\infty_{p'}(M_n)$, which is a $*$–algebra by Lemma 2.4.5 (ii). Notice that this result extends to the norm closure of $\ell^\infty_{p'}(M_n)$. Indeed, if $\{a_n\}$ belongs to the norm closure and $\|\{a_n\}\| = \sup_n \|a_n\| \le K$, then, for any $\varepsilon > 0$, we can find a $\{b_n\} \in \ell^\infty_{p'}(M_n)$, such that $\|\{a_n - b_n\}\| < \varepsilon$. Therefore, if $\lim_\mathcal{U} L'_n(c_n) = 0$ and $r_\mathcal{U} := \sup_n r_n < \infty$, then

$$\begin{aligned}\lim_\mathcal{U} L'_n(a_n c_n) &\le \lim_\mathcal{U} L'_n((a_n - b_n)c_n) + \lim_\mathcal{U} L'_n(b_n c_n)\\ &\le r_\mathcal{U}\|\{a_n - b_n\}\|\|\{c_n\}\| \le r_\mathcal{U}\|\{c_n\}\|\varepsilon.\end{aligned}$$

The result then follows by the arbitrariness of ε.

(ii). Set $\mathcal{S}_\mathcal{U} := \{\underline{\omega} := [\omega_n] \in M_{\mathcal{U}*} : \omega_n \in (M_{n*})_{1,+}\}$, and observe that $\mathcal{S}_\mathcal{U}$ is a closed convex subset of $M_{\mathcal{U}*}$ whose linear span is the whole $M_{\mathcal{U}*}$. Then, consider the embedding ι of $\mathcal{S}_\mathcal{U}$ into the set of states of $\mathcal{A}_\mathcal{U}$. The embedding ι is indeed isometric, since the unit ball of $\mathcal{A}_\mathcal{U}$ is w^*–dense in $(M_\mathcal{U})_1$, where $M_\mathcal{U} = \pi'_\mathcal{U}(\ell^\infty(M_n))$. Then, consider the representation

$$\pi_\mathcal{U} := \bigoplus_{\underline{\omega}\in\mathcal{S}_\mathcal{U}} \pi_{\underline{\omega}},$$

where $(\pi_{\underline{\omega}}, \mathcal{H}_{\underline{\omega}}, \xi_{\underline{\omega}})$ is the GNS representation associated to $\underline{\omega}$. Clearly, $\iota(\mathcal{S}_{\mathcal{U}})$ consists of vector states for $\pi_{\mathcal{U}}$.

For any $\underline{\omega} \in \mathcal{S}_{\mathcal{U}}$, a dense set of vectors in $\mathcal{H}_{\underline{\omega}}$ is given by $\underline{a}\xi_{\underline{\omega}}$, with $\underline{a} := [a_n]$, $\{a_n\} \in \ell^\infty_{p'}(M_n)$, so the corresponding states are given by $\langle \underline{\omega}_a, \underline{b} \rangle = \langle \underline{\omega}, \underline{a}^* \underline{b} \underline{a} \rangle$. Let us observe that

$$
\begin{aligned}
L(\underline{\omega}_a) &= \sup \frac{\langle \underline{\omega}, \underline{a}^* \underline{ba} \rangle}{L'(\underline{b})} = \sup \frac{\langle \underline{\omega}, \underline{a}^* \underline{ba} \rangle}{L'(\underline{a}^* \underline{ba})} \frac{L'(\underline{a}^* \underline{ba})}{L'(\underline{b})} \\
&\leq L(\underline{\omega}) N(\underline{a})^2,
\end{aligned}
$$

namely $\underline{\omega}_a \in \mathcal{S}_{\mathcal{U}}$. This shows that all vector states of $\pi_{\mathcal{U}}$ are given by norm–limits of convex combinations of states in $\mathcal{S}_{\mathcal{U}}$, namely are represented by elements of $\mathcal{S}_{\mathcal{U}}$. On the other hand, normal states are given by converging series of vector states, hence $\iota(\mathcal{S}_{\mathcal{U}})$ contains all normal states for $\pi_{\mathcal{U}}$, which implies the thesis.

(ii). By part (ii), $M_{\mathcal{U}}$ is a von Neumann algebra, and, by Corollary 2.3.6, it is a dual Lip–space, hence a Lip-von Neumann algebra. ∎

Chapter 3

The Dual Quantum Gromov–Hausdorff Distance dist_{qGH^*}.

3.1 Effros–Maréchal Topology

Let $\mathcal{H}$ be a (fixed) Hilbert space, and let $\mathrm{vN}(\mathcal{H})$ be the set of von Neumann algebras acting on $\mathcal{H}$. We can endow the space $\mathrm{vN}(\mathcal{H})$ with a certain natural topological structure. The definition of this topology goes back to the works of Effros [22] and Maréchal [46]. There are at least three different – but actually equivalent – ways to describe the Effros–Maréchal topology, as shown by Haagerup and Winslow in the two papers [35], [36]. The first one is the original definition due to Maréchal:

3.1.1 Definition. *The* Effros–Maréchal topology *is the weakest topology on vN($\mathcal{H}$) in which the maps*

$$vN(\mathcal{H}) \ni M \mapsto \|\varphi|_M\|$$

are continuous on vN($\mathcal{H}$) for every $\varphi \in \mathcal{B}(\mathcal{H})_*$.

If $\mathcal{H}$ is separable, then $\mathrm{vN}(\mathcal{H})$ is a Polish space in this topology [46], hence metrizable.

The second and third definitions need some preliminary notions to be introduced. First, we recall how to define a topology on the family of closed subsets of a compact Hausdorff space (cf. [21]).

3.1.2 Notation. We recall that, if X is a set and $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ is a net in X based on directed set $\mathbb{A}$ and Y is a subset of X, we say that $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ is *frequently* in Y if, for every $\alpha \in \mathbb{A}$ there exists some $\beta \geq \alpha$, $\beta \in \mathbb{A}$, such that x_β is in Y.
We say that $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ is *eventually* in Y if there exists a $\gamma \in \mathbb{A}$ such that x_β is in Y for any $\beta \geq \gamma$.

3.1.3 Definition. *Let X be a compact Hausdorff space, and let $c(X)$ the set of closed subsets of X. For $x \in X$, denote by $\omega(x)$ the set of open neighborhoods of x. Let $\{C_\alpha\} \subseteq c(X)$ be a net, and define*

$$\underline{\lim} C_\alpha := \{x \in X : \forall U \in \omega(x), U \cap C_\alpha \neq \emptyset \text{ eventually}\} \tag{3.1}$$
$$\overline{\lim} C_\alpha := \{x \in X : \forall U \in \omega(x), U \cap C_\alpha \neq \emptyset \text{ frequently}\}. \tag{3.2}$$

Then, it can be shown [21] that there is a unique topology on $c(X)$, called the *convergence topology*, in which convergence is given by

$$C_\alpha \underset{ct}{\longrightarrow} C \quad \Longleftrightarrow \quad \underline{\lim} C_\alpha = C = \overline{\lim} C_\alpha$$

for a net $\{C_\alpha\} \subseteq c(X)$ and $C \in c(X)$.

Let $c_0(X)$ be the set of non empty closed subsets of the compact metric space (X, ρ), and let

$$\mathrm{dist}_H(C_1, C_2) = \max\left(\sup_{x \in C_1} \left\{ \inf_{y \in C_2} \rho(x,y) \right\}, \sup_{x \in C_2} \left\{ \inf_{x \in C_1} d(x,y) \right\} \right) \tag{3.3}$$

be the Hausdorff distance on $c_0(X)$ induced by ρ.

3.1.4 Theorem. [21] *Let (X, ρ) be a compact metric space and let $\{C_n\}_{n \in \mathbb{N}} \subset c_0(X)$ be a sequence of closed subsets. Then,*

$$C_n \underset{ct}{\longrightarrow} C \quad \Longleftrightarrow \quad \lim_{n \to \infty} \mathrm{dist}_H(C_n, C) = 0. \tag{3.4}$$

As unit balls in von Neumann algebras are weak operator (*wo*–)compact, one may define inferior and superior limits in $\mathrm{vN}(\mathcal{H})$ (or, more in general, in the set $\mathrm{SA}(M)$ of all von Neumann subalgebras of a given M, if $M \neq \mathcal{B}(\mathcal{H})$), using the above concepts on unit balls. So, we have the second definition of the Effros–Maréchal Topology:

3.1.5 Definition. *Let $\{N_\alpha\} \subseteq SA(M)$ be a net. The* Effros-Maréchal topology *is described by the following notion of convergence*

$$N_\alpha \to N \quad \Longleftrightarrow \quad \underline{\lim}(N_\alpha)_1 = N_1 = \overline{\lim}(N_\alpha)_1,$$

where the subscript 1 denotes the (closed) unit ball.

Let $x \in \mathcal{B}(\mathcal{H})$, and denote by $so^*(x)$ the set of (open) neighborhoods of x w.r.t. the strong* operator topology.

3.1.6 Definition. *Let $\{N_\alpha\} \subseteq SA(M)$ be a net. We define*

$$\liminf N_\alpha := \{x \in M : \forall U \in so^*(x), U \cap N_\alpha \neq \emptyset \text{ eventually}\} \tag{3.5}$$

3.1.7 Theorem. [35] *Let $\{N_\alpha\} \subseteq SA(M)$ be a net. Then* $\liminf N_\alpha \in SA(M)$.

Let $\mathcal{U}(M)$ denote the unitary group of M.

3.1.8 Theorem. [35] *Let $\{M_\alpha\} \subseteq SA(M)$ be a net. Then, we have*

$$\mathcal{U}(\liminf N_\alpha) = \underline{\lim}(N_\alpha)_1 \cap \mathcal{U}(M), \tag{3.6}$$

and

$$\liminf N_\alpha = \bigvee \{n \in SA(M) | N_1 \subseteq \underline{\lim}(N_\alpha)_1\}, \tag{3.7}$$

where the unit balls are equipped with the (compact) wo-topology, and $\vee$ denotes the usual supremum in $SA(M)$. In particular, if $\underline{\lim}(N_\alpha)_1 = N_1$ for some $N \in SA(M)$, then $\liminf N_\alpha = N$.

This theorem tells us that $\liminf N_\alpha$ is the largest element in $\mathrm{SA}(M)$ whose unit ball is contained in $\underline{\lim}(N_\alpha)_1$. This motivates the following

3.1.9 Definition. *For a net $\{N_\alpha\} \subseteq SA(M)$, we define*

$$\limsup N_\alpha := (\overline{\lim}(N_\alpha)_1)'',$$

i.e. $\limsup N_\alpha$ *is the smallest element of SA(M) whose unit ball contains* $\overline{\lim}(N_\alpha)_1$.

Finally, we give the third definition of the Effros-Maréchal topology:

3.1.10 Definition. *Let $\{N_\alpha\} \subseteq SA(M)$ be a net. The* Effros–Maréchal topology *is described by the following notion of convergence*

$$N_\alpha \to N \quad \Longleftrightarrow \quad \liminf N_\alpha = N = \limsup N_\alpha.$$

The equivalence of these three definitions is the content of the following

3.1.11 Theorem. [35] *Let $\{N_\alpha\} \subseteq SA(M)$ be a net, and $N \in SA(M)$. Then the following statements are equivalent:*

(i) $\liminf N_\alpha = N = \limsup N_\alpha$,

(ii) $\underline{\lim}(N_\alpha)_1 = N_1 = \overline{\lim}(N_\alpha)_1$,

(iii) $\|\varphi|_{N_\alpha}\| \to \|\varphi|_N\|$ *for all* $\varphi \in M_*$.

Hence, if M is separable, the topology on SA(M) defined by (i)–(iii) *is Polish by Corollary 2 of [46].*

Proof. (i) $\Rightarrow$ (ii). Assuming (i), by Theorem 3.1.8 and Definition 3.1.9 we get:

$$\begin{aligned} N_1 &= (\liminf N_\alpha)_1 = \underline{\lim}(N_\alpha)_1 && (3.8)\\ &\subseteq \overline{\lim}(N_\alpha)_1 \subseteq (\limsup N_\alpha)_1 = N_1. && (3.9)\end{aligned}$$

(ii) $\Rightarrow$ (i). Assuming (ii), by Theorem 3.1.8 we get

$$N \subseteq \liminf N_\alpha \subseteq \limsup N_\alpha = (\overline{\lim}(N_\alpha)_1)'' = N'' = N.$$

(ii) $\Leftrightarrow$ (iii). Let $\mathrm{CS}(M)$ be the set of all *wo*–closed, convex, balanced subsets of M_1. The convergence topology on $\mathrm{CS}(M)$ is compact (cf. [21]), and it makes the functions

$$E \mapsto \left(\sup_{x\in E} |\varphi(x)|\right)_{\varphi\in M_*}$$

continuous on $\mathrm{CS}(M)$ (in the product topology for the range); as it is injective by the Hahn–Banach Theorem, it is a homeomorphism. Restricting to $\{N_1 : N \in \mathrm{SA}(M)\}$, we get the claim. ∎

We specialize now to the separable case. By the above considerations, we know that the Effros–Maréchal topology is metrizable, second countable and complete, i.e. SA(M) is a Polish space. We want to construct a metric on SA(M) which induces the Effros–Maréchal topology. To this aim, take any distance ρ on M inducing the *wo*–topology on bounded subsets of M (which coincides with the σ–weak topology on bounded sets).[1] The corresponding Hausdorff distance between unit balls of von Neumann algebras in SA(M) will be then the desired metric, that is,

$$\begin{aligned}\mathrm{dist}_{EM}(N_\alpha, N_\beta) &:= \mathrm{dist}_H((N_\alpha)_1, (N_\beta)_1) \\ &= \max\left(\sup_{x\in(N_\alpha)_1}\left\{\inf_{y\in(N_\beta)_1}\rho(x,y)\right\}, \sup_{x\in(N_\beta)_1}\left\{\inf_{x\in(N_\beta)_1}\rho(x,y)\right\}\right)\end{aligned} \tag{3.11}$$

Thus, in view of Theorem 3.1.4, we have the following

3.1.12 Theorem. *Assume that M_* is separable, and let $\{N_n\} \subseteq SA(M)$ be a sequence. Then. for every free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the following statements are equivalent:*

(i) *$N_n \to N$ over $\mathcal{U}$ in the Effros-Maréchal topology;*

(ii) $\lim_{\mathcal{U}} \mathrm{dist}_{EM}(N_n, N) = 0$.

3.2 The Distance dist_{qGH^*}

As seen in the previous section, given a (separable) Hilbert space $\mathcal{H}$ and two von Neumann subalgebras of $\mathcal{B}(\mathcal{H})$, it is possible to define a Hausdorff–like distance between them. As in the case of ordinary (compact) metric spaces, one may proceed from the Hausdorff distance between closed subsets of a (concrete) metric space to the Gromov–Hausdorff distance, which is a pseudo–distance between (abstract) metric spaces. This pseudo–distance then becomes a true distance on the space of isometry equivalence classes of compact metric spaces. This is indeed one of the ideas which inspired our construction.

Let M, N be two Lip–von Neumann algebras with dual Lip–norms L'_M, L'_N. We want to introduce a Gromov–Hausdorff–type notion of distance between them. In order to get the distance–zero property (i.e., the property that, when two LvNA's are at distance zero, then they are isometrically Lip–isomorphic), we need to consider not only the original algebras M and N, but also the 2×2–matrix algebras $\mathcal{M}_2(M)$ and $\mathcal{M}_2(N)$ with entries in M and N, respectively. We introduce some notation.

3.2.1 Notation. For a LvNA M, we still denote by L'_M the dual Lip–norm on $\mathcal{M}_2(M)$ induced by that on M as follows:

$$L'_M((a_{ij})) := \max_{i,j=1,2}(L'_M(a_{ij})), \qquad (a_{ij}) \in \mathcal{M}_2(M). \tag{3.12}$$

Notice that L'_M gives back the original Lip–norm, when restricted to the copy of M diagonally embedded in $\mathcal{M}_2(M)$. Moreover, we will denote by X_M the positive part of the unit ball $\mathcal{M}_2(M)_{1,+}$ of $\mathcal{M}_2(M)$.

[1]For example, one could take the following distance, which is known to satisfy the requirement (see, for instance, [69]):

$$\rho(x,y) := \sum_n \frac{1}{2^n}|(\xi_n, (x-y)\xi_n)|, \quad x,y \in M_1, \tag{3.10}$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ is a dense set of vectors in the unit ball of $\mathcal{H}$.

Let $\mathcal{L}'(M,N)$ denote the set of all (dual) Lip–norms $L' = L'_{M\oplus N}$ on the direct sum $M \oplus N$, such that $L'_{M\oplus\{0\}} = L'_M$ and $L'_{\{0\}\oplus N} = L'_N$.

3.2.2 Definition. *Let (M, L'_M) and (N, L'_N) be Lip–von Neumann algebras. We define the* dual quantum Gromov–Hausdorff distance *between them, by setting*

$$\mathrm{dist}_{qGH^*}(M,N) := \inf\{\mathrm{dist}_H^{L'}(X_M, X_N) : L' \in \mathcal{L}'(M,N)\}, \tag{3.13}$$

where $X_M := \mathcal{M}_2(M)_{1,+}$ and $X_N := \mathcal{M}_2(N)_{1,+}$.

We need to show that dist_{qGH^*} is a metric. It is clearly symmetric in M and N.

3.2.3 Theorem (Triangle Inequality). *Let (M_1, L'_1), (M_2, L'_2), (M_3, L'_3) be Lip–von Neumann algebras. Then*

$$\mathrm{dist}_{qGH^*}(M_1, M_3) \le \mathrm{dist}_{qGH^*}(M_1, M_2) + \mathrm{dist}_{qGH^*}(M_2, M_3). \tag{3.14}$$

Proof. Let $1 \ge \varepsilon > 0$ be given. Then, we can find an $L'_{12} \in \mathcal{L}'(M_1, M_2)$ such that

$$\mathrm{dist}_H^{L'_{12}}(X_{M_1}, X_{M_2}) \le \mathrm{dist}_{qGH^*}(M_1, M_2) + \varepsilon/2.$$

Similarly, we can find $L'_{23} \in \mathcal{L}'(M_2, M_3)$ such that

$$\mathrm{dist}_H^{L'_{23}}(X_{M_2}, X_{M_3}) \le \mathrm{dist}_{qGH^*}(M_2, M_3) + \varepsilon/2.$$

We define

$$L'_{13}(x_1, x_3) := \inf_{x_2\in M_2}(L'_{12}(x_1, x_2) + L'_{23}(x_2, x_3)).$$

We shall prove that it is a seminorm, whose restrictions to M_1 and M_3 are L'_1 and L'_3, respectively. Indeed, the positive homogeneity is clear, and we have

$$\begin{aligned} & L'_{13}(x_1+y_1, x_3+y_3) = \inf_{x_2\in M_2}(L'_{12}(x_1+y_1, x_2) + L'_{23}(x_2, x_3+y_3)) \\ = & \inf_{x_2+y_2\in M_2}(L'_{12}(x_1+y_1, x_2+y_2) + L'_{23}(x_2+y_2, x_3+y_3)) \\ = & \inf_{x_2,y_2\in M_2}(L'_{12}(x_1+y_1, x_2+y_2) + L'_{23}(x_2+y_2, x_3+y_3)) \\ \le & \inf_{x_2,y_2\in M_2}(L'_{12}(x_1, x_2) + L'_{12}(y_1, y_2) + L'_{23}(x_2, x_3) + L'_{23}(y_2, y_3)) \\ = & \inf_{x_2\in M_2}(L'_{12}(x_1, x_2) + L'_{23}(x_2, x_3)) + \inf_{y_2\in M_2}(L'_{12}(y_1, y_2) + L'_{23}(y_2, y_3)) \\ = & L'_{13}(x_1, x_3) + L'_{13}(y_1, y_3). \end{aligned}$$

Then, let us check the restriction requirement: since $L'_{23}(x_2, 0) = L'_2(x_2) = L'_{12}(0, x_2)$, we have

$$\begin{aligned} L'_{13}(x_1, 0) &= \inf_{x_2\in M_2}(L'_{12}(x_1, x_2) + L'_{23}(x_2, 0)) \\ &= \inf_{x_2\in M_2}(L'_{12}(x_1, x_2) + L'_2(x_2)) \le L'_{12}(x_1, 0) = L'_1(x_1), \end{aligned}$$

and, since $L'_{12}(x_1, x_2) = L'_{12}((x_1, 0) + (0, x_2)) \ge |L'_1(x_1) - L'_2(x_2)|$,

$$L'_{13}(x_1, 0) \ge \inf_{x_2\in M_2}(|L'_1(x_1) - L'_2(x_2)| + L'_2(x_2)) = L'_1(x_1),$$

and so $L'_{13}(x_1, 0) = L'_1(x_1)$. Similarly, $L'_{13}(0, x_3) = L'_3(x_3)$. Now, in order to get a norm, we simply define

$$L'_{13,\delta}(x_1, x_3) := (1-\delta)L'_{13}(x_1, x_3) + \delta L_1(x_1) + \delta L_3(x_3), \quad 0 < \delta \leq 1.$$

Clearly, the restrictions of $L'_{13,\delta}$ to M_1 and M_3 are still L'_1 and L'_3. Finally, we have to show that the unit ball $(M_1 \oplus M_3)_1$ is $L'_{13,\delta}$–compact. But any L'–norm on $M_1 \oplus M_3$, which restricts to the given Lip–norms, satisfies this requirement, since one has $L'(x_1, x_3) \leq L'_1(x_1) + L'_3(x_3)$. Therefore, $L'_{13,\delta}$, $0 < \delta \leq 1$, is a Lip–norm on $M_1 \oplus M_3$.

Now, suppose that we have $\mathrm{dist}_H^{L'_{12}}(X_{M_1}, X_{M_2}) = d_{12}$ and $\mathrm{dist}_H^{L'_{23}}(X_{M_2}, X_{M_3}) = d_{23}$. By definition of Hausdorff distance, for any given $x_1 \in X_{M_1}$, we can find an $x_2 \in X_{M_2}$ – call it $f(x_1)$ – such that $L'_{12}(x_1, x_2) = L'_{12}(x_1, f(x_1)) \leq d_{12}$, and, analogously, for any $x_2 \in X_{M_2}$, a corresponding $x_3 \in X_{M_3}$ – call it $g(x_2)$ – with $L'_{23}(x_2, x_3) = L'_{23}(x_2, g(x_2)) \leq d_{23}$. In other words, for any given $x_1 \in X_{M_1}$, we can find an $x_3 = g(f(x_1)) \in X_{M_3}$, such that

$$L'_{13,\delta}(x_1, g(f(x_1))) \leq (1-\delta)(L'_{12}(x_1, f(x_1)) + L'_{23}(f(x_1), g(f(x_1)))) + 2\delta \leq d_{12} + d_{23} + 2\delta.$$

Similarly, for any given $x_3 \in X_{M_3}$, we can find an $x_1 = h(k(x_3)) \in X_{M_1}$, such that

$$L'_{13,\delta}(h(k(x_3)), x_3) \leq (1-\delta)(L'_{12}(h(k(x_3)), k(x_3)) + L'_{23}(k(x_3), h(k(x_3)))) + 2\delta \leq d_{12} + d_{23} + 2\delta.$$

Since this holds for any x_1 in X_{M_1} and x_3 in X_{M_3}, taking $\delta = \varepsilon$, we obtain

$$\begin{aligned} \mathrm{dist}_H^{L'_{13,\varepsilon}}(X_{M_1}, X_{M_3}) &\leq \mathrm{dist}_H^{L'_{12}}(X_{M_1}, X_{M_2}) + \mathrm{dist}_H^{L'_{23}}(X_{M_2}, X_{M_3}) + 2\varepsilon \\ &\leq \mathrm{dist}_{qGH^*}(M_1, M_2) + \mathrm{dist}_{qGH^*}(M_2, M_3) + 3\varepsilon. \end{aligned}$$

Therefore, taking the infimum on the l.h.s., we obtain

$$\mathrm{dist}_{qGH^*}(M_1, M_3) \leq \mathrm{dist}_{qGH^*}(M_1, M_2) + \mathrm{dist}_{qGH^*}(M_2, M_3) + 3\varepsilon,$$

and so, by the arbitrariness of ε, the thesis follows. ∎

We may characterize the radius of a LvNA by the following

3.2.4 Proposition. *Let (M, L'_M) be a Lip–von Neumann algebra, and let R be its radius (as dual Lip–space). Then, we have*

$$R = \mathrm{dist}_{qGH^*}(M, \{0\}), \tag{3.15}$$

where $(\{0\}, L'_0)$ is the trivial LvNA with only one element.

Proof. By definition, one has

$$R = \sup_{x \in M} \frac{L'_M(x)}{\|x\|} = \sup_{\|x\| \leq 1} L'_M(x).$$

Since $L'_M : M \to \mathbb{R}_+$ is w^*–continuous and the unit ball M_1 is w^*–compact, the above supremum is actually a maximum, i.e. there exists an $x_0 \in M_1$ such that $R = L'_M(x_0)$. Now, we have clearly $\mathrm{dist}_{qGH^*}(M, \{0\}) \leq R$. Indeed, for any $L' \in \mathcal{L}'(M, \{0\})$, we have $L'(x \oplus 0) = L'_M(x)$, and thus

$$\mathrm{dist}_{qGH^*}(M, \{0\}) = \mathrm{dist}_H^{L'}(X_M, \{0\}) \leq R.$$

On the other hand, one has $\{0\} \subset \mathcal{N}_r(X_M, L')$ for any $r > 0$ and, since $L'(x \oplus 0) \leq L'_M(x_0) = R$ for any $x \in X_M$, $X_M \subset \mathcal{N}_r(\{0\}, L')$ for any $r > R$[2]. Recalling that

$$\mathrm{dist}_H^{L'}(X_M, \{0\}) = \inf\{r > 0 : \{0\} \subset \mathcal{N}_r(X_M, L') \,\mathrm{and}\, X_M \subset \mathcal{N}_r(\{0\}, L')\},$$

we see that $\mathrm{dist}_H^{L'}(X_M, \{0\}) \geq R$. Hence, $\mathrm{dist}_{qGH^*}(M, \{0\}) = R$, and the proof is complete. ∎

As a consequence, we have (cf. Theorem 1.2.17 (4)):

3.2.5 Lemma. *Let (M, L'_M), (N, L'_N) be Lip–von Neumann algebras, and let R_M, R_N be the respective radii. Then,*

$$|R_M - R_N| \leq \mathrm{dist}_{qGH^*}(M, N) \leq R_M + R_N. \tag{3.16}$$

Proof. Indeed, let $d = \mathrm{dist}_{qGH^*}(M, N)$. Given $\varepsilon > 0$, we can find $L' \in \mathcal{L}'(M, N)$ such that $\mathrm{dist}_H^{L'}(X_M, X_N) < d + \varepsilon$. Then, for any $x \in X_M$, there is an $y \in X_N$ such that

$$L'_M(x) \leq L'(x \oplus y) + L'_N(y) < d + \varepsilon + R_N.$$

Since ε is arbitrary, it follows that

$$R_M \leq d + R_N.$$

Reversing the roles of M and N, we obtain also

$$R_N \leq d + R_M,$$

and the first inequality is proven. As for the second one, it follows evidently by

$$L'(x \oplus y) \leq L'_M(x) + L'_N(y) \leq R_M + R_N, \quad x \in X_M, \, y \in X_N,$$

and the proof is now complete. ∎

Finally, we want to show that, if two Lip–von Neumann algebras have distance dist_{qGH^*} equal to zero, then they are isometrically $*$–isomorphic, i.e. Lip–isomorphic (cf. Definition 2.4.3). The following proof is inspired by Rieffel's proof of the same property for the quantum Gromov–Hausdorff distance between compact quantum metric space (cf. Section 1.2.3).

In order to prove this distance–zero property, we must allow (Lip)–seminorms as well. In fact, let us denote by $\widetilde{\mathcal{L}}'(M, N)$ the set of lifts of all the seminorms on $M \oplus N$ which restrict to the original Lip–norms L'_M and L'_N on each direct summand M and N in $M \oplus N$.

3.2.6 Lemma. *If (M, L'_M) is a Lip–von Neumann algebra, then also $(\mathcal{M}_2(M), L'_M)$ is a LvNA.*

Proof. Indeed, we have $\mathcal{M}_2(M)_* \cong \mathcal{M}_2(M_*)$. Therefore, if the unit ball M_1 of M is w^*–compact in the topology induced by L'_M, then $\mathcal{M}_2(M_1)$ will be w^*–compact in the topology induced by the lift of L'_M to $\mathcal{M}_2(M)$. Since $\mathcal{M}_2(M)_1$ is a w^*–closed subset of $\mathcal{M}_2(M_1)$, it follows that also $\mathcal{M}_2(M)_1$ is w^*–compact in the topology induced by (the lift of) L'_M. ∎

[2]We recall that, for a given $L' \in \mathcal{L}(M, N)$, we denote by $\mathcal{N}_r(X_M, L')$ the set

$$\{z \in \mathcal{M}_2(M) \oplus \mathcal{M}_2(N) : \exists x \in X_M \,\mathrm{s.t.}\, L'(z - (x \oplus 0)) < r\},$$

and analogously for $\mathcal{N}_r(X_N, L')$.

3.2.7 Lemma. *The family $\widetilde{\mathcal{L}}'(M,N)$ of seminorms on $X_M \oplus X_N$ is uniformly (w^*–)equicontinuous.*

Proof. For any $\varepsilon > 0$, and any given $x_0 \in X_M$, $y_0 \in X_N$, let

$$\begin{aligned} \mathcal{N}(x_0, \varepsilon/2) &= \{x \in X_M : L'_M(x - x_0) < \varepsilon/2\}, \\ \mathcal{N}(y_0, \varepsilon/2) &= \{y \in X_N : L'_N(y - y_0) < \varepsilon/2\}, \end{aligned}$$

so that $\mathcal{N}(x_0, \varepsilon/2) \oplus \mathcal{N}(y_0, \varepsilon/2)$ is a (w^*–)neighborhood of $x_0 \oplus y_0$. If $x \oplus y \in \mathcal{N}(x_0, \varepsilon/2) \oplus \mathcal{N}(y_0, \varepsilon/2)$, then, for any $\widetilde{L}' \in \widetilde{\mathcal{L}}'(M,N)$, we have

$$\begin{aligned} |\widetilde{L}'(x \oplus y) - \widetilde{L}'(x_0 \oplus y_0)| &\le |\widetilde{L}'(x \oplus y) - \widetilde{L}'(x \oplus y_0)| + \\ &\qquad |\widetilde{L}'(x \oplus y_0) - \widetilde{L}'(x_0 \oplus y_0)| \\ &\le L'_N(y - y_0) + L'_M(x - x_0) < \varepsilon, \end{aligned}$$

hence, $\widetilde{L}'$ is uniformly (w^*–)equicontinuous. ∎

3.2.8 Lemma. *Let $\{L'_n\}_{n\in\mathbb{N}}$ be a uniform sequence in $\mathcal{L}'(M,N) \subset \widetilde{\mathcal{L}}'(M,N)$ and let $\widetilde{L}'_0$ be its limit. Then, $\widetilde{L}'_0 \in \widetilde{\mathcal{L}}'(M,N)$.*

Proof. Notice that the limit of a (convergent) uniform sequence of seminorms is a seminorm. Indeed, fix $\varepsilon > 0$, and let $n_\varepsilon \in \mathbb{N}$ be such that $|L'_n(x) - \widetilde{L}'_0(x)| \le \varepsilon$ for all $n \ge n_\varepsilon$. Then,

$$\widetilde{L}'_0(x+y) \le L'_n(x+y) + \varepsilon \le L'_n(x) + L'_n(y) + \varepsilon \le \widetilde{L}'_0(x) + \widetilde{L}'_0(y) + 3\varepsilon,$$

and, for $\alpha \in \mathbb{R}$,

$$\begin{aligned} \widetilde{L}'_0(\alpha x) \le L'_n(\alpha x) + \varepsilon = |\alpha| L'_n(x) + \varepsilon \le |\alpha| \widetilde{L}'_0(x) + 2\varepsilon \\ \widetilde{L}'_0(\alpha x) \ge L'_n(\alpha x) - \varepsilon = |\alpha| L'_n(x) - \varepsilon \le |\alpha| \widetilde{L}'_0(x) - 2\varepsilon. \end{aligned}$$

By arbitrariness of ε, we see that $\widetilde{L}'_0$ is a seminorm. Since the restriction requirement clearly holds also for $\widetilde{L}'_0$, it follows that $\widetilde{L}'_0 \in \widetilde{\mathcal{L}}'(M,N)$, as claimed. ∎

By the previous Lemma, we see that all the conditions in the definition of a Lip–norm are closed conditions, except the norm–zero condition (i.e., the fact that $L'(x) = 0 \Rightarrow x = 0$). It is precisely for this reason that one drops this requirement and allows seminorms.

3.2.9 Lemma. *Let $\widetilde{L}' \in \widetilde{\mathcal{L}}'(M,N)$. For each $x \in X_M$ there is at most one $y \in X_N$ such that $\widetilde{L}'(x \oplus -y) = 0$, and similarly for each $y \in X_N$.*

Proof. If $\widetilde{L}'(x \oplus -y) = 0 = \widetilde{L}'(x \oplus -y')$, then

$$L'_N(y - y') = \widetilde{L}'(0 \oplus y - 0 \oplus y') \le \widetilde{L}'(-x \oplus y) + \widetilde{L}'(x \oplus -y') = 0,$$

so that $y' = y$. ∎

Now, we can prove the following

3.2.10 Theorem. *Let (M, L'_M) and (N, L'_N) be Lip–von Neumann algebras. If*

$$\mathrm{dist}_{qGH^*}(M,N) = 0,$$

then there is an isometric $*$–isomorphism between (M, L'_M) and (N, L'_N), i.e. they are Lip–isometric.

Proof. If $\mathrm{dist}_{qGH^*}(M,N)=0$, then there is a sequence $\{L'_n\}$ of Lip–norms on $M\oplus N$, whose restrictions to M and N are L'_M and L'_N respectively, such that

$$\mathrm{dist}_H^{L'_n}(X_M,X_N)<\frac{1}{n}.$$

Clearly, $L'_n\in\widetilde{\mathcal{L}}'(M,N)$ for each $n\in\mathbb{N}$, and the sequence $\{L'_n\}$ on $X_M\oplus X_N$ is uniformly bounded by $2\max(R_M,R_N)$ (cf. Lemma 3.2.5), where R_M (resp., R_N) is the radius of (M,L'_M) (resp., (N,L'_N)). Since it is also (w^*–)equicontinuous by Lemma 3.2.7, we can apply the Ascoli–Arzelà Theorem [66] to conclude that it admits a uniformly convergent subsequence. For simplicity, we still denote this subsequence by $\{L'_n\}$. Let $\widetilde{L}'_0$ be its limit. Since $\widetilde{\mathcal{L}}'(M,N)$ is (uniformly) closed, $\widetilde{L}'_0$ must be a seminorm, and it realizes the distance zero, since, given $\varepsilon>0$, we can find an n_ε such that for all $n\geq n_\varepsilon$, we have $|\widetilde{L}'_0(x\oplus -y)-\widetilde{L}'_n(x\oplus -y)|<\varepsilon/2$, and thus

$$\begin{aligned}\widetilde{L}'_0(x\oplus -y) &\leq |\widetilde{L}'_0(x\oplus -y)-\widetilde{L}'_n(x\oplus -y)|+\widetilde{L}'_n(x\oplus -y)\\ &\leq \varepsilon/2+\widetilde{L}'_n(x\oplus -y).\end{aligned}$$

Hence, for any $x\in X_M$, if we take the infimum over all $y\in X_N$, we obtain, for n sufficiently large,

$$\inf_{y\in X_N}\widetilde{L}'_0(x\oplus -y)\leq\varepsilon/2+\inf_{y\in X_N}\widetilde{L}'_n(x\oplus -y)<\varepsilon/2+\varepsilon/2=\varepsilon.$$

Thus, $\widetilde{L}'_0$ determines an isometry φ from X_M onto X_N, by the condition that, for each $x\in X_M$, there is at most one $y\in X_N$ with $\widetilde{L}'_0(x\oplus -y)=0$ (cf. Lemma 3.2.9). Set $y=\varphi(x)$. We want to show that φ is an affine map. To this aim, let $x_1,x_2\in X_M$ and let y_1,y_2 be the corresponding elements in X_N for which $\widetilde{L}'_0(x_i\oplus -y_i)$, $i=1,2$. Then, for any $t\in[0,1]$, we have

$$\begin{aligned}&\widetilde{L}'_0(tx_1+(1-t)x_2\oplus -(ty_1+(1-t)y_2))\\ &=\widetilde{L}'_0(t(x_1\oplus -y_1)+(1-t)(x_2\oplus -y_2))\\ &\leq t\widetilde{L}'_0(x_1\oplus -y_1)+(1-t)\widetilde{L}'_0(x_2\oplus -y_2)=0,\end{aligned}$$

and thus

$$\varphi(tx_1+(1-t)x_2)=ty_1+(1-t)y_2=t\varphi(x_1)+(1-t)\varphi(x_2),$$

showing that φ is affine.

Now, since φ is an affine bijective map from $\mathcal{M}_2(M)_{1,+}=\{x\in\mathcal{M}_2(M):0\leq x\leq I_{\mathcal{M}_2(M)}\}$ onto $\mathcal{M}_2(N)_{1,+}=\{y\in\mathcal{M}_2(N):0\leq y\leq I_{\mathcal{M}_2(N)}\}$, it is automatically positive and unital, namely

$$\begin{aligned}&0\leq x_1\leq x_2\leq I_{\mathcal{M}_2(M)}\Rightarrow 0\leq x_2-x_1\leq I_{\mathcal{M}_2(M)}\Rightarrow\\ &0\leq\varphi(x_2-x_1)\leq I_{\mathcal{M}_2(N)}\Rightarrow\varphi(x_1)\leq\varphi(x_2),\end{aligned}$$

and

$$\begin{aligned}&\varphi(I_{\mathcal{M}_2(M)})\geq y\Rightarrow\varphi(I_{\mathcal{M}_2(M)})\geq I_{\mathcal{M}_2(N)}\\ &\varphi(x)\leq I_{\mathcal{M}_2(N)}\Rightarrow\varphi(I_{\mathcal{M}_2(M)})\leq I_{\mathcal{M}_2(N)},\end{aligned}$$

i.e. $\varphi(I_{\mathcal{M}_2(M)})=I_{\mathcal{M}_2(N)}$. Evidently, φ extends to a (bijective) positive linear map from $\mathcal{M}_2(M)$ onto $\mathcal{M}_2(N)$. It remains to show that φ is of the form $\mathrm{id}_2\otimes\phi$, with ϕ a (bijective) positive linear

map between M and N.
So, let $a \in M_{1,+}$ and consider

$$\varphi\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\right) = \left(\begin{array}{cc} \varphi_{11}\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\right) & 0 \\ 0 & 0 \end{array}\right),$$

$$\varphi\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right)\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & \varphi_{22}\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right)\right) \end{array}\right).$$

Since we have

$$\widetilde{L}'((a_{ij}) \oplus (\varphi_{ij}(a_{ij}))) = \max_{ij}(\widetilde{L}'(a_{ij} \oplus \varphi_{ij}((a_{ij}))) = 0,$$

it then follows that

$$\varphi_{11}\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\right) = \varphi_{22}\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right)\right),$$

and so we can conclude that there exists a (bijective) positive linear map ϕ between $M_{1,+}$ and $N_{1,+}$, such that

$$\varphi\left(\left(\begin{array}{cc} a & 0 \\ 0 & 0 \end{array}\right)\right) = \left(\begin{array}{cc} \phi(a) & 0 \\ 0 & 0 \end{array}\right)$$

and

$$\varphi\left(\left(\begin{array}{cc} 0 & 0 \\ 0 & a \end{array}\right)\right) = \left(\begin{array}{cc} 0 & 0 \\ 0 & \phi(a) \end{array}\right).$$

Concerning the off–diagonal elements, one can always find $\lambda, \mu \in \mathbb{R}$ such that, given $b \in M_{1,+}$, one has

$$\left(\begin{array}{cc} \lambda I_M & \mu b \\ \mu b & \lambda I_M \end{array}\right) \in \mathcal{M}_2(M)_{1,+}.$$

Then, reasoning as above, we obtain

$$\varphi\left(\left(\begin{array}{cc} \lambda I_M & \mu b \\ \mu b & \lambda I_M \end{array}\right)\right) = \left(\begin{array}{cc} \lambda I_N & \mu\phi(b) \\ \mu\phi(b) & \lambda I_N \end{array}\right).$$

Reversing the roles of M and N, we see that also $\varphi^{-1} : \mathcal{M}_2(N) \to \mathcal{M}_2(M)$ is a unital, 2–positive (bijective linear) map. Thus, φ extends to a unital 2–order isomorphism between M and N, which, by a result due to Choi [15], is automatically a $*$–isomorphism. ∎

Let us observe that we might define the distance dist_{qGH^*} in the following equivalent way. Given two Lip–von Neumann algebras (M, L'_M), (N, L'_N), we consider all the Lip–von Neumann algebras (R, L'_R) such that there exist positive isometric (both for the C^*–norms and the Lip–norms) embeddings

$$\begin{array}{ll} h_M : \mathcal{M}_2(M) \to \mathcal{M}_2(R), & L'_R(h_M(\cdot)) = L'_M(\cdot), \\ h_N : \mathcal{M}_2(N) \to \mathcal{M}_2(R), & L'_R(h_N(\cdot)) = L'_N(\cdot), \end{array}$$

and we denote by $\mathcal{L}'_R \equiv \mathcal{L}'_R(M, N)$ the set of all such triples (R, h_M, h_N). We then define

$$\mathrm{dist}^R_{qGH^*}(M, N) := \inf\{\mathrm{dist}^R_H(h_M(X_M), h_N(X_N)) : (R, h_M, h_N) \in \mathcal{L}'_R\}. \quad (3.17)$$

3.2.11 Proposition. *For any pair of Lip–von Neumann algebras* (M, L'_M), (N, L'_N), *we have:*

$$\mathrm{dist}_{qGH^*}(M,N) = \mathrm{dist}^R_{qGH^*}(M,N). \tag{3.18}$$

Proof. Clearly, $\mathrm{dist}_{qGH^*}(M,N) \geq \mathrm{dist}^R_{qGH^*}(M,N)$, since $R = M \oplus N$ is just a particular choice, and, on the r.h.s., we take the infimum over all such choices. For the reverse inequality, let $1 \geq \varepsilon > 0$ and $(R, h_M, h_N) \in \mathcal{L}'_R$ be given. We will construct a (Lip–)norm $L' \in \mathcal{L}'(R,R)$ such that the two copies of X_R are ε–close to each other, i.e. $L'(x \oplus -x) \leq \varepsilon L'_R(x)$ for any $x \in R$. In fact, setting

$$L'(x \oplus y) := \max(L'_R(x+y), \varepsilon L'_R(x), \varepsilon L'_R(y)), \quad 0 < \varepsilon \leq 1, \tag{3.19}$$

then L' is clearly a norm which satisfies the requirement, and restricts to L'_R on each summand. (Indeed, $L'(x \oplus 0) = \max(L'_R(x), \varepsilon L'_R(x)) = L'_R(x)$, and similarly $L'(0 \oplus y) = L'_R(y)$.) We define $L'_{M\oplus N}$ on $X_R \oplus X_R$ as follows:

$$L'_{M\oplus N} := \begin{cases} L' & \text{on} \quad h_M(X_M) \oplus h_N(X_N) \\ 0 & \text{on} \quad X_R \oplus X_R \backslash h_M(X_M) \oplus h_N(X_N) \end{cases}$$

Then, since $L'_{M\oplus N} \in \mathcal{L}'(h_M(M), h_N(N))$ implies $L'_{M\oplus N} \circ (h_M \oplus h_N) \in \mathcal{L}'(M,N)$, and $L' \geq L'_{M\oplus N}$, we have

$$\begin{aligned}
& \mathrm{dist}_{qGH^*}(M,N) \leq \mathrm{dist}_H^{L'_{M\oplus N}}(h_M(X_M), h_N(X_N)) \\
\leq\ & \mathrm{dist}_H^{R\oplus R}(h_M(X_M) \oplus \{0\}, \{0\} \oplus h_N(X_N)) \\
\leq\ & \mathrm{dist}_H^{R\oplus R}(h_M(X_M) \oplus \{0\}, \{0\} \oplus h_M(X_M)) + \mathrm{dist}_H^{R\oplus R}(\{0\} \oplus h_M(X_M), \{0\} \oplus h_N(X_N)) \\
=\ & \mathrm{dist}_H^{R\oplus R}(h_M(X_M) \oplus \{0\}, \{0\} \oplus h_M(X_M)) + \mathrm{dist}_H^{R}(h_M(X_M), h_N(X_N)) \\
\leq\ & r\varepsilon + \mathrm{dist}_H^{R}(h_M(X_M), h_N(X_N)),
\end{aligned}$$

where r is the radius of R. By the arbitrariness of ε, the thesis follows. ∎

3.2.12 Theorem. dist_{qGH^*} *is a metric on the space of Lip–isomorphism equivalence classes of Lip–von Neumann algebras.*

Proof. By Theorem 3.2.10, we already know that, if $\mathrm{dist}_{qGH^*}(M,N) = 0$, then M and N are Lip–isomorphic.
We show now the reverse implication. Let $\varphi : M \to N$ be a Lip–isomorphism from (M, L'_M) onto (N, L'_N). We set $R := N \oplus N$, $h_M := (\mathrm{id}_2 \otimes \varphi) \oplus 0$, $h_N := 0 \oplus (\mathrm{id}_2 \otimes \iota)$, where ι is the identity map on N, and we define the following (Lip–)norm on R:

$$L'_{R,\varepsilon}(\varphi(x) \oplus \iota(y)) := \max(L'_N(\varphi(x) - y), \varepsilon L'_M(x), \varepsilon L'_N(y)).$$

where $\varepsilon \in (0, 1]$. Notice that $L'_{R,\varepsilon}(h_M(x)) = L'_M(x)$ for any $x \in \mathcal{M}_2(M)$, and $L'_{R,\varepsilon}(h_N(y)) = L'_N(y)$ for any $y \in \mathcal{M}_2(N)$. Then, by the previous Proposition, we have

$$\mathrm{dist}_{qGH^*}(M,N) \leq \mathrm{dist}_H^R(h_M(X_M), h_N(X_N)).$$

Moreover, we have also, by construction,

$$h_M(X_M) \subset \mathcal{N}_\varepsilon(h_N(X_N), L'_{R,\varepsilon}), \quad h_N(X_N) \subset \mathcal{N}_\varepsilon(h_M(X_M), L'_{R,\varepsilon})$$

Hence, $\mathrm{dist}_H^R(h_M(X_M), h_N(X_N)) < \varepsilon$, and, by the arbitrariness of ε, we get the claim. ∎

3.2.13 Remark. Let us notice that the distance dist_{qGH^*} does not appear to be complete, essentially because we do not have an estimate for the Lip–norm $L'(xy)$ of products of elements, much like in the Rieffel's setting (see [32]). Also in this case, it should be possible to develop a theory for dual operator systems (see, for instance, [48; 54] for a definition), and show that a Cauchy sequence of LvNA's is always converging to a dual operator system. But this will be possibly object of further work.

3.3 dist_{qGH^*} and Ultraproducts

In this section we study the relation between the distance dist_{qGH^*} and the ultraproduct construction. To this aim, we need a suitable notion of finite–dimensional approximation for a uniform family of LvNA's. In view of the fact that the distance between LvNA's is actually a distance between the positive part of the unit ball of the (2×2–)matrices with entries in the algebras, the natural setting in which this finite–dimensional approximability can be expressed is that of (finite–dimensional) Lip–operator subsystems, that is, dual Lip–spaces with a matrix ordered structure. So, let us recall the definition of (abstract) operator systems.

3.3.1 Definition (Operator Systems). *An operator system V is a complex vector space with a conjugate linear involution $^* : v \in V \to v^* \in V$ (we will call such an X a $*$–*vector space*), satisfying*

(i) *V is matrix ordered, that is,*

  (i') *for any $p \in \mathbb{N}$, there is a proper cone $\mathcal{M}_p(V)_+ \subset \mathcal{M}_p(V)_{sa}$, where $\mathcal{M}_p(V)_{sa} := \{(v_{ij}) \in \mathcal{M}_p(v) : (v_{ij})^* := (v^*_{ji}) = (v_{ij})\}$,*

  (i") *for any $p, q \in \mathbb{N}$, $(a_{ij}) \in \mathcal{M}_{qp}(\mathbb{C})$, $(a_{ij})^* \mathcal{M}_q(V)_+ (a_{ij}) \subset \mathcal{M}_p(V)_+$;*

(ii) *V has a matrix order–unit, i.e. there is an element $e \in V_{sa}$ such that, with $e^p := diag(e, \ldots, e) \in \mathcal{M}_p(V)_+$, for any $v \in \mathcal{M}_p(V)_{sa}$, there is an $r > 0$ such that $v + re^p \in \mathcal{M}_p(V)_+$;*

(iii) *the matrix order–unit e is Archimedean, i.e. if $v \in \mathcal{M}_p(V)$ is such that $v + re^p \in \mathcal{M}_p(V)_+$, for all $r > 0$, then $v \in \mathcal{M}_p(V)_+$.*

Given two operator systems V and W, we say that a linear map $\varphi : V \to W$ is *n–positive* if the map $\mathrm{id}_n \otimes \varphi : \mathcal{M}_n \otimes V \to \mathcal{M}_n \otimes W$ is positive, and if $\mathrm{id}_n \otimes \varphi$ is positive for all $n \in \mathbb{N}$, then we say that φ is *completely positive.* A completely positive (resp. unital completely positive) linear map will be referred to as a *c.p.* (resp. *u.c.p.*) map. If $\varphi : V \to W$ is a unital m–positive map with m–positive inverse for $m = 1, \ldots, n$, then φ is a *unital n–order isomorphism*, and if φ is u.c.p. with c.p. inverse then φ is a *unital complete order isomorphism.* (The reader is referred to [48] for more details on operator systems and operator spaces.)

Operator systems are characterized concretely by the following

3.3.2 Theorem. [16] *If V is an operator system, then there exists a Hilbert space $\mathcal{H}$, an operator system $W \subseteq \mathcal{B}(\mathcal{H})$, and a complete order isomorphism $\varphi : V \to W$ with $\varphi(e) = I_{\mathcal{H}}$. Conversely, every (concrete) $*$–vector subspace of $\mathcal{B}(\mathcal{H})$ containing $I_{\mathcal{H}}$ is an operator system.*

In particular, since every unital C^*–algebra is an operator system, any unital $*$–vector subspace of a C^*–algebra is naturally an operator system.

3.3.3 Definition. *By a* Lip–operator system, *we mean an operator system concretely given as a unital $*$–vector subspace of a Lip–von Neumann algebra.*

3.3.4 Remark. We ought to stress that this notion of *Lip–operator system* and that of Lip–normed operator system introduced by Kerr in [43] are actually different, as a consequence of the fact that the (dual) Lip–norm on the operator system is precisely the restriction of the (dual) Lip–norm on the ambient von Neumann algebra.

3.3.5 Lemma. *If a family $\mathcal{F}$ of Lip–operator systems is uniformly totally bounded, then also the corresponding family $\mathcal{M}_2(\mathcal{F}) := \{\mathcal{M}_2(V) : V \in \mathcal{F}\}$ of 2×2 matrices is uniformly totally bounded.*

Proof. Indeed, let $V \in \mathcal{F}$ be a Lip–operator system. Then, given $\varepsilon > 0$, there is $n_\varepsilon \in \mathbb{N}$ such that, for any $V \in \mathcal{F}$, the unit ball $\{x \in V : \|x\| \leq 1\}$ can be covered by n_ε L'–balls of radius ε. Let us denote by $\{x_i\}_{i=1}^{n_\varepsilon}$ the respective centers. Then, the n_ε^4 L'–balls of radius ε and centers in $\mathcal{M}_2(\{x_i : i = 1, \ldots, n_\varepsilon\})$, i.e. the 2 by 2 matrices with entries in $\{x_i : i = 1, \ldots, n_\varepsilon\}$, will cover $\mathcal{M}_2(V_1)$. Since $\mathcal{M}_2(V)_1 \subset \mathcal{M}_2(V_1)$, we can find n_ε^4 elements $\{\tilde{x}_j\}_{j=1}^{n_\varepsilon^4}$ in $\mathcal{M}_2(V)_1$ such that the corresponding L'–balls of radius 2ε centered in them will cover $\mathcal{M}_2(V)_1$. ∎

Let $\mathcal{F}$ be a (uniform) family of Lip–operator systems. Then,

3.3.6 Lemma. *If the family $\mathcal{M}_2(\mathcal{F}) := \{\mathcal{M}_2(V) : V \in \mathcal{F}\}$ is uniformly totally bounded, then also the corresponding family $\mathcal{M}_2(\widetilde{\mathcal{F}})_+ := \{\mathcal{M}_2(V)_+ : V \in \mathcal{F}\}$ of 2×2 positive matrices is uniformly totally bounded.*

Proof. Indeed, let $\{x_i\}$, $i = 1, \ldots, n_\varepsilon$, be an ε–net in $\mathcal{M}_2(V)_1$, $V \in \mathcal{F}$, and let $B(x_i, \varepsilon)$ be the (open L'–)ball with center x_i and radius ε. Thus, we have $\bigcup_{i=1}^{n_\varepsilon} B(x_i, \varepsilon) \supseteq \mathcal{M}_2(V)_1$, and, since $\mathcal{M}_2(V)_{1,+}$ is a subset of $\mathcal{M}_2(V)_1$, we can find $j_1, j_2, \ldots, j_{n'_\varepsilon} \in \{1, \ldots, n_\varepsilon\}$ such that $B(x_{j_k}, \varepsilon) \cap \mathcal{M}_2(V)_{1,+} \neq \emptyset$. As a consequence, for each j_k, $k = 1 \ldots n'_\varepsilon$, we can find an $\tilde{x}_{j_k}$ in $B(x_{j_k}, \varepsilon) \cap \mathcal{M}_2(V)_{1,+}$, $k = 1, \ldots, n'_\varepsilon$, such that

$$\bigcup_{k=1}^{n'_\varepsilon} B(\tilde{x}_{j_k}, 2\varepsilon) \supset \mathcal{M}_2(V)_{1,+}.$$

As $n'_\varepsilon \leq n_\varepsilon$ and the family $\mathcal{M}_2(\mathcal{F})$ is uniform, the claim follows. ∎

The next one is simply Lemma 2.1.7 rephrased in terms of LvNA's.

3.3.7 Lemma. *If a familiy $\mathcal{F}$ of Lip–von Neumann algebras is uniformly totally bounded, then there exists $R > 0$ such that $L'(x) \leq R\|x\|$ for any $(M, L') \in \mathcal{F}$, $x \in M$.*

Proof. Let $n \in \mathbb{N}$ be such that, for any $(M, L') \in \mathcal{F}$, M_1 can be covered by n L'–balls of radius 1, and let $x_0 \in M$, $\|x_0\| = 1$. Since the set $\{tx_0 : t \in [0, 1]\}$ is contained in M_1, it is covered by at most n balls of L'–radius 1, so its length is majorised by $2n$, i.e. $R \leq 2n$. ∎

Notice that, by Lemma 3.3.5, the previous Lemma holds as well for the corresponding family $\{(\mathcal{M}_2(M), L') : M \in \mathcal{F}\}$ of 2×2 matrix algebras, with the same constant R. (In fact, we have $L'((x_{ij})) = \max_{ij} L'(x_{ij}) = L'(x_{i_0 j_0}) \leq R\|x_{i_0 j_0}\| \leq R\|(x_{ij})\|$.)

3.3.8 Proposition. *A familiy $\mathcal{F}$ of Lip–von Neumann algebras is uniformly totally bounded if, and only if, there exists an $R > 0$ as in Lemma 3.3.7, and, for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that any $(M, L') \in \mathcal{F}$ has an operator subsystem V with $\dim V \leq N_\varepsilon$ and $\{x \in V : \|x\| \leq 1\}$ is ε–dense in M_1 in the L'–norm.*

Proof. Suppose $\mathcal{F}$ is uniformly totally bounded, and let $(M, L') \in \mathcal{F}$. If we choose a covering of M_1 by n_ε balls of L'–radius $\varepsilon/2$ and consider the operator system V generated by their centers, then evidently $\dim V \leq 2n_\varepsilon + 1 \equiv N_\varepsilon$ and $\{x \in V : \|x\| \leq 1\}$ is ε–dense in M_1 in the L'–norm.

Viceversa, since the elements of $\{x \in V : \|x\| \leq 1\}$ are contained in $\{x \in V : L'(x) \leq R\}$, any covering of the latter with $L'-$balls of radius $\varepsilon/2$ gives a covering of M_1 with L'–balls of radius ε, and, by Lemma 2.1.10, such a covering of $\{x \in V : L'(x) \leq R\}$ can be realised with $(4R/\varepsilon)^{N_\varepsilon}$ balls. ∎

We can now extend the definition of the distance dist_{qGH^*} to include *Lip–operator systems* as well. In fact, given two operator subsystems V_i of the LvNA's M_i, $i = 1, 2$, respectively, we simply define the distance between them as the restriction of the corresponding distance between the respective ambient algebras, that is

$$\mathrm{dist}_{qGH^*}(V_1, V_2) := \inf\{\mathrm{dist}_H^{L'}(Y_1, Y_2) : L' \in \mathcal{L}'(M_1, M_2)\}, \tag{3.20}$$

where we set $Y_i := \mathcal{M}_2(V_i)_{1,+} \subset \mathcal{M}_2(M_i)_{1,+}$, $i = 1, 2$. Let us notice that, however, when restricted to Lip–operator systems, dist_{qGH^*} is no longer a metric, but only a pseudo–metric, for $\mathrm{dist}_{qGH^*}(V_1, V_2) = 0$ does not imply, in general, that V_1 and V_2 are (Lip–)isomorphic as operator systems. They are just 2–order isomorphic.

3.3.9 Lemma. *Let V be an operator subsystem of the LvNA M, and let $\varepsilon \in (0, 1]$. If $\mathcal{M}_2(V)_{1,+}$ is ε–dense in $\mathcal{M}_2(M)_{1,+}$ in the L'–norm, then* $\mathrm{dist}_{qGH^*}(V, M) < \varepsilon$.

Proof. Indeed, given $\varepsilon \in (0, 1]$, if we set $L'(v \oplus w) := \max(L'(v - w), \varepsilon L'(v), \varepsilon L'(w))$, $v \in V$, $w \in M$, then $L' \in \mathcal{L}'(M, M)$, and, by density, for any $x \in X_M$, we can find an $y \in Y_V$ such that $L'(x \oplus y) < \varepsilon$, hence $X_M \subset \mathcal{N}_\varepsilon(Y_V, L')$. Viceversa, since $Y_V \subset X_M$, we get immediately $Y_V \subset \mathcal{N}_\varepsilon(X_M, L')$ for any $\varepsilon > 0$. Thus, we have $\mathrm{dist}_H^{L'}(Y_V, X_M) < \varepsilon$, and the thesis follows. ∎

3.3.10 Lemma. *Let (V_n, L'_n) be a uniform sequence of dual Lip–spaces, and, for any ultrafilter $\mathcal{U}$, let $(V_\mathcal{U}, L'_\mathcal{U})$ be its (dual restricted) ultraproduct. Then, for any $x \in (V_\mathcal{U})_1$, there exists a sequence $\{\tilde{x}_n\}_{n\in\mathbb{N}}$ in $\ell^\infty(V_n)$ realising it such that $\tilde{x}_n \in (V_n)_1$ for any n.*

Proof. Given $x \in (V_\mathcal{U})_1$, we may choose sequences $\{x_n^k\}$ realising it such that $\|x_n^k\| \leq \|x\|(1 + \frac{1}{k})$. Then, we set

$$\begin{aligned} U_k &:= \left\{ n \geq k : L'_n(x_n^i - x_n^j) \leq \frac{1}{i}, \quad i \leq j \leq k \right\}, \\ U_0 &:= \mathbb{N}, \end{aligned}$$

and observe that $U_k \in \mathcal{U}$, $U_{k+1} \subseteq U_k$, and $\bigcup_{k\geq 0} U_k/U_{k+1} = \mathbb{N}$. Now, we define

$$\tilde{x}_n = \frac{k}{k+1} x_n^k, \quad n \in U_k/U_{k+1},$$

and thus $\|\tilde{x}_n\| \leq \|x\|$. It remains to show that $\pi'_{\mathcal{U}}(\{\tilde{x}_n\}) = x$. Indeed, if $n \in U_i$, then there exists $k \geq i$ such that $n \in U_k \backslash U_{k+1}$, and thus

$$\begin{aligned} L'_n(\tilde{x}_n - x^i_n) &\leq L'_n(\tilde{x}_n - x^k_n) + L'_n(x^k_n - x^i_n) \leq \frac{1}{k+1} L'_n(x^k_n) + \frac{1}{i} \\ &\leq \frac{R}{k+1}\|x^k_n\| + \frac{1}{i} \leq (R+1)\frac{1}{i}, \end{aligned}$$

where R is as in Lemma 3.3.7. Since n is eventually in U_i w.r.t. $\mathcal{U}$, we get

$$L'_{\mathcal{U}}(\pi'_{\mathcal{U}}(\{\tilde{x}_n\}) - x) = \lim_{\mathcal{U}} L'_n(\tilde{x}_n - x^i_n) \leq (R+1)\frac{1}{i},$$

and by arbitrariness of i, $\pi'_{\mathcal{U}}(\{\tilde{x}_n\}) = x$. ∎

3.3.11 Remark. Let us notice that the sequence constructed in the previous Lemma also realises the minimum in the formula defining the quotient norm $\|x\| = \inf\{\sup_n \|y_n\| : \pi'_{\mathcal{U}}(\{y_n\}) = x\}$, for this implies that $\|x\| \leq \lim_{\mathcal{U}} \|y_n\|$. In fact, given $\varepsilon > 0$, there exists an element U of the ultrafilter such that, for any $n \in U$, $\|y_n\| \leq \lim_{\mathcal{U}} \|y_m\| + \varepsilon$. Then, we may define $x_n = y_n$ for $n \in U$ and $x_n = 0$ for $n \notin U$, so that $\pi'_{\mathcal{U}}(\{x_n\}) = \pi'_{\mathcal{U}}(\{y_n\})$. Therefore, if we choose y_n as in the Lemma above, we get

$$\|x\| = \lim_{\mathcal{U}} \|\tilde{x}_n\| = \sup_n \|\tilde{x}_n\|. \tag{3.21}$$

3.3.12 Remark. Let $(V_{\mathcal{U}}, L'_{\mathcal{U}})$ be the (restricted) ultraproduct of a uniform sequence $\{(V_n, L'_n)\}$ of N–dimensional dual Lip–spaces. Assume that the (restricted) ultraproduct $V_{\mathcal{U}}$ has the same dimension N. (Let us observe that, if $\{V_n\}$ is a sequence of finite–dimensional vector space with $\sup_n \dim V_n \leq D$, then the dimension of $\ell^\infty(V_n)$ clearly satisfies the same bound. Since the restricted ultraproduct is a quotient of $\ell^\infty(V_n)$, its dimension can not increase, and thus $\dim V_{\mathcal{U}} \leq D$.).

Let $\{e^{(i)}\}_{i=1}^N$, $N = \dim V_{\mathcal{U}}$, be a (vector) basis for $V_{\mathcal{U}}$, with $\|e^{(i)}\| = 1$, $i = 1, \ldots, N$. By Lemma 3.3.10, we can always choose a sequence $\{\tilde{e}^{(i)}_n\}_{n\in\mathbb{N}}$ realising $e^{(i)}$, such that $\|\tilde{e}^{(i)}_n\| \leq 1$ and the vectors $\tilde{e}^{(i)}_n$, $i = 1, \ldots, N$, are linearly independent for n eventually in some element of the ultrafilter. Indeed, let $U_{ld} \subset \mathbb{N}$ be such that, for any $m \in U_{ld}$, $\sum_{i=1}^N \alpha^{(i)}_m \tilde{e}^{(i)}_m = 0$ with $\alpha^{(i)}_m$, $i = 1, \ldots, N$, not all zero. We want to show that $U_{ld} \notin \mathcal{U}$. Assume that $\max_i(\alpha^{(i)}_m) = 1$, and let $\alpha^{(i)} = \lim_{\mathcal{U}} \alpha^{(i)}_m$, so that $\max_i(\alpha^{(i)}) = 1$ as well. Then, since $L'_m(\sum_{i=1}^N \alpha^{(i)}_m \tilde{e}^{(i)}_m) = 0$, we get $\lim_{\mathcal{U}} L'_m(\sum_{i=1}^N \alpha^{(i)}_m \tilde{e}^{(i)}_m) = 0$, hence $L'_{\mathcal{U}}(\sum_{i=1}^N \alpha^{(i)} e^{(i)}) = 0$, which implies $\sum_{i=1}^N \alpha^{(i)} e^{(i)} = 0$. But the elements $e^{(i)}$, $i = 1, \ldots, N$, are linearly indipendent, and $\max_i(\alpha^{(i)}) = 1$. Therefore, $U_{ld} \notin \mathcal{U}$, as claimed. It follows that $\tilde{e}^{(i)}_n$, $i = 1, \ldots, N$, are linearly independent for n eventually in $U_{li} := \mathbb{N} \backslash U_{ld} \in \mathcal{U}$.

Moreover, since $L'_{\mathcal{U}}(e^{(i)}) \neq 0$, we may suppose that $L'_n(\tilde{e}^{(i)}_n) \neq 0$ for n eventually in some $U_i \in \mathcal{U}$, and then take the intersection $U_0 := \cap_{i=1}^N U_i$. Now, if we define

$$e^{(i)}_n := \frac{L'_{\mathcal{U}}(e^{(i)})}{L'_n(\tilde{e}^{(i)}_n)} \tilde{e}^{(i)}_n,$$

then we get a new system of linearly independent elements in V_n, such that $\|e^{(i)}_n\| \leq 2$, and $L'_n(e^{(i)}_n) = L'_{\mathcal{U}}(e^{(i)}) \equiv k_i \leq R$, with R as in Lemma 3.3.8. Indeed, consider the following subset of $\mathbb{N}$:

$$U_i^{(r)} := \left\{ n \in \mathbb{N} : |L'_n(e_n^{(i)}) - L'_{\mathcal{U}}(e^{(i)})| \le r L'_{\mathcal{U}}(e^{(i)}) \right\}.$$

Then, $U_i^{(r)} \in \mathcal{U}$, and taking $r \le \frac{1}{2}$, we get $L'_n(e_n^{(i)}) \ge (1-r) L'_{\mathcal{U}}(e^{(i)})$, hence

$$\frac{L'_{\mathcal{U}}(e^{(i)})}{L'_n(e_n^{(i)})} \le \frac{1}{1-r} \le 2,$$

for any n in $U_0^{(r)} := \left(\cap_{i=1}^M U_i^{(r)} \right) \cap U_0 \cap U_{li} \in \mathcal{U}$.

3.3.13 Lemma. *Let $(V_{\mathcal{U}}, L'_{\mathcal{U}})$ be the (dual) restricted ultraproduct of a family $\{(V_n, L'_n)\}_{n\in\mathbb{N}}$ of finite–dimensional dual Lip–spaces. Then, there exist a linear map $T_n : V_{\mathcal{U}} \to V_n$ and, given $\varepsilon > 0$, an element $U_\varepsilon \in \mathcal{U}$ such that, for any $n \in U_\varepsilon$, we have*

$$|L'_n(T_n(x)) - L'_{\mathcal{U}}(x)| \le \varepsilon L'_{\mathcal{U}}(x) \tag{3.22}$$

for any $x \in V_{\mathcal{U}}$.

Proof. Let $(V_{\mathcal{U}}, L'_{\mathcal{U}})$ be the (dual restricted) ultraproduct of $\{(V_n, L'_n)\}_{n\in\mathbb{N}}$, and let $\{e^{(i)}\}_{i=1}^M$, $M = \dim V_{\mathcal{U}}$, be a (vector) basis for $V_{\mathcal{U}}$, with $\|e^{(i)}\| = 1$ for $i = 1, \ldots, M$. By Lemma 3.3.10 and the Remark above, we can always choose the sequence $\{e_n^{(i)}\}_{n\in\mathbb{N}}$ realising $e^{(i)}$, such that $\|e_n^{(i)}\| \le 2$ and $L'_n(e_n^{(i)}) = L'_{\mathcal{U}}(e^{(i)}) \equiv k_i \le R$ for each $i = 1, \ldots, M$ (and n eventually in some element of the ultrafilter). Consider now the vector subspace $\widetilde{V}_n = \text{l.s.}\{e_n^{(i)} : i = 1, \ldots, M\} \subseteq V_n$, and define the map $T_n : V_{\mathcal{U}} \to \widetilde{V}_n$ by its action on the basis:

$$T_n(e^{(i)}) = e_n^{(i)}, \quad i = 1, \ldots, M.$$

Then, T_n is linear and

$$\sup\{L'_n(T_n(x)) : L'_{\mathcal{U}}(x) \le 1\} \le K$$

where

$$K := \max \left\{ \sum_{i=1}^M k_i |\lambda_i| : L'_{\mathcal{U}}(\sum_{i=1}^M \lambda_i e^{(i)}) = 1 \right\}.$$

We have clearly $\pi'_{\mathcal{U}}(\{T_n(x)\}) = x$. In fact, for any $x = \sum_{j=1}^M \lambda_j e^{(j)}$, we have

$$\begin{aligned}
L'_{\mathcal{U}}(x) &= L'_{\mathcal{U}}(\sum_{j=1}^M \lambda_i e^{(j)}) = \lim_{\mathcal{U}} L'_n(\sum_{j=1}^M \lambda_i e_n^{(j)}) \\
&= \lim_{\mathcal{U}} L'_n(\sum_{j=1}^M \lambda_i T_n(e^{(j)})) = \lim_{\mathcal{U}} L'_n(T_n(\sum_{j=1}^M \lambda_i e^{(j)})) \\
&= \lim_{\mathcal{U}} L'_n(T_n(x)),
\end{aligned}$$

Now, let δ be a positive number and let $\{y^{(1)}, y^{(2)}, \ldots, y^{(m)}\}$ be a finite δ–net in the unit sphere of $V_{\mathcal{U}}$ (i.e. for any x with $L'_{\mathcal{U}}(x) = 1$, there is a $y^{(j)}$, $L'_{\mathcal{U}}(y^{(j)}) = 1$, such that $L'_{\mathcal{U}}(x - y^{(j)}) \le \delta$), and set

$$U_1^{(\delta)} := \bigcap_{k=1}^m U_{y^{(k)}},$$

where, for $x \in V_{\mathcal{U}}$, we set

$$U_x := \left\{ n \in \mathbb{N} : |L'_n(T_n(x)) - L'_{\mathcal{U}}(x)| \leq \frac{\varepsilon}{2} L'_{\mathcal{U}}(x) \right\} \in \mathcal{U}.$$

Then, for $n \in U_1^{(\delta)}$ and $x \in V_{\mathcal{U}}$ with $L'_{\mathcal{U}}(x) = 1$, we have

$$\begin{aligned}
& |L'_n(T_n(x)) - L'_{\mathcal{U}}(x)| \\
\leq \; & \min_{k=1,\dots,m} \left(L'_n(T_n(x - y^{(k)})) + L'_{\mathcal{U}}(x - y^{(k)}) + |L'_n(T_n(y^{(k)})) - L'_{\mathcal{U}}(y^{(k)})| \right) \\
\leq \; & \min_{k=1,\dots,m} \left((K+1) L'_{\mathcal{U}}(x - y^{(k)}) + |L'_n(T_n(y^{(k)})) - L'_{\mathcal{U}}(y^{(k)})| \right) \\
\leq \; & (K+1)\delta + \frac{\varepsilon}{2}.
\end{aligned}$$

Now, taking $\delta = \frac{\varepsilon}{2(K+1)}$ and $U_\varepsilon = U_0^{(\delta)} \cap U_1^{(\delta)}$, with $U_0^{(\delta)}$ as in the previous Remark, we get

$$|L'_n(T_n(x)) - L'_{\mathcal{U}}(x)| \leq \varepsilon, \qquad n \in U_\varepsilon,$$

as claimed. ∎

Given a pair of Lip–operator subsystems (V_1, L'_1), (V_2, L'_2) in the LvNA's (M_1, L'_1), (M_2, L'_2) respectively, we consider all the Lip–operator systems (V, L'_V) such that there exist two linear embeddings

$$\phi_1 : V \hookrightarrow V_1, \quad \phi_2 : V \hookrightarrow V_2,$$

with the property that, for any $v \in V$,

$$L'_V(v) = L'_1(\phi_1(v)) = L'_2(\phi_2(v)),$$

and we define new norms on V by

$$\|v\|_1 := \|\phi_1(v)\|, \; \|v\|_2 := \|\phi_2(v)\|.$$

Let us denote by $\mathcal{V}_{12}$ the set of all triples (V, ϕ_1, ϕ_2) with the properties above (notice that $\mathcal{V}_{12}$ contains at least the "identity triple", i.e. $(\mathbb{C}, \iota_1, \iota_2)$), and define, for any $x \in V_1$, $y \in V_2$,

$$(\widetilde{L}')^V(x \oplus y) = \inf_{v, v' \in V} (L'_1(x - \phi_1(v)) + L'_2(\phi_2(v') - y) + L'_V(v - v')). \tag{3.23}$$

Then, we have the following

3.3.14 Lemma. *$(\widetilde{L}')^V$ is a (Lip–)seminorm on $V_1 \oplus V_2$ (viewed as a Lip–subspace of the LvNA $M_1 \oplus M_2$), which restricts to L'_1, L'_2 on V_1, V_2 respectively.*

Proof. First, we claim that $(\widetilde{L}')^V$ is a seminorm. Let us check it. We have

$$(\widetilde{L}')^V((x_1 + x_2) \oplus (y_1 + y_2)) \leq (\widetilde{L}')^V(x_1 \oplus y_1) + (\widetilde{L}')^V(x_2 \oplus y_2).$$

Indeed,

$$
\begin{aligned}
& (\widetilde{L}')^V((x_1+x_2)\oplus(y_1+y_2)) \\
= \; & \inf_{v,v'\in V}\big(L'_1((x_1+x_2)-\phi_1(v))+L'_2(\phi_2(v')-(y_1+y_2))+L'_V(v-v')\big) \\
= \; & \inf_{v_1+v_2,v'_1+v'_2\in V}\big(L'_1((x_1+x_2)-\phi_1(v_1+v_2)) \\
& \qquad + L'_2(\phi_2(v'_1+v'_2)-(y_1+y_2)))+L'_V((v_1+v_2)-(v'_1+v'_2))\big) \\
= \; & \inf_{v_1,v'_1,v_2,v'_2\in V}\big(L'_1((x_1+x_2)-(\phi_1(v_1)+\phi_1(v_2))) \\
& \qquad + L'_2(\phi_2(v'_1)+\phi_2(v'_2)-(y_1+y_2)))+L'_V((v_1-v'_1)+(v_2-v'_2))\big) \\
\le \; & \inf_{v_1,v'_1\in V}\big(L'_1(x_1-\phi_1(v_1)+L'_2(\phi_2(v'_1)-y_1))+L'_V(v_1-v'_1)\big) \\
& \qquad + \inf_{v_2,v'_2\in V}\big(L'_1(x_2-\phi_1(v_2))+L'_2(\phi_2(v'_2)-y_2)+L'_V(v_2-v'_2)\big) \\
\le \; & (\widetilde{L}')^V(x_1\oplus y_1)+(\widetilde{L}')^V(x_2\oplus y_2).
\end{aligned}
$$

Next, $(\widetilde{L}')^V(x\oplus y)=0$ if, and only if, there exists $v\in V$ such that $\phi_1(v)=x$ and $\phi_2(v)=y$, i.e. $x=\phi_1\circ\phi_2^{-1}(y)$.
Moreover, $(\widetilde{L}')^V$ restricts to L'_1 and L'_2. In fact, we have

$$(\widetilde{L}')^V(x\oplus 0)\le L'_1(x),$$

and

$$
\begin{aligned}
(\widetilde{L}')^V(x\oplus 0) &= \inf_{v,v'\in V}\big(L'_1(x-\phi_1(v))+L'_2(\phi_2(v'))+L'_V(v-v')\big) \\
&\ge \inf_{v,v'\in V}\big(|L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))+|L'_V(v)-L'_V(v')|\big).
\end{aligned}
$$

We must distinguish the various cases:

(1) if $L'_V(v)\ge L'_V(v')$, then

$$
\begin{aligned}
& |L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))+|L'_V(v)-L'_V(v')| \\
&= |L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))+L'_V(v)-L'_V(v') \\
&= |L'_1(x)-L'_1(\phi_1(v))|+L'_V(v),
\end{aligned}
$$

and thus

$$
\begin{aligned}
& \inf_{v,v'\in V}(|L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))+L'_V(v)-L'_V(v')) \\
&= \inf_{v\in V}(|L'_1(x)-L'_1(\phi_1(v))|+L'_V(v))=L'_1(x);
\end{aligned}
$$

(2) if $L'_V(v')\ge L'_V(v)$, then

$$
\begin{aligned}
& |L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))+|L'_V(v)-L'_V(v')| \\
&= |L'_1(x)-L'_1(\phi_1(v))|+L'_2(\phi_2(v'))-L'_V(v)+L'_V(v') \\
&= |L'_1(x)-L'_1(\phi_1(v))|+2L'_2(\phi_2(v'))-L'_V(v) \\
&= \begin{cases} L'_1(x)-2L'_1(\phi_1(v))+2L'_2(\phi_2(v')) & \text{if } L'_1(x)\ge L'_1(\phi_1(v)) \\ 2L'_2(\phi_2(v'))-L'_1(x) & \text{if } L'_1(x)\le L'_1(\phi_1(v)) \end{cases}
\end{aligned}
$$

and thus

$$\inf_{v,v'\in V}(L_1'(x) - 2L_1'(\phi_1(v)) + 2L_2'(\phi_2(v')) = L_1'(x)$$
$$\inf_{v'\in V}(2L_2'(\phi_2(v')) - L_1'(x)) = L_1'(x).$$

In all cases, we get

$$(\widetilde{L}')^V(x\oplus 0) \geq L_1'(x),$$

hence

$$(\widetilde{L}')^V(x\oplus 0) = L_1'(x).$$

Exactly in the same way, one can show that $(\widetilde{L}')^V(0\oplus y) = L_2'(y)$, and the proof is complete. ∎

Let us notice that, in order to get a (Lip–)norm and not only a seminorm, it suffices to define, for $\varepsilon \in (0,1]$,

$$(L')^V := \max\left((\widetilde{L}')^V(x\oplus y), \varepsilon L_1'(x), \varepsilon L_2'(y)\right). \tag{3.24}$$

3.3.15 Remark. Obviously, the same conclusion of the previous Lemma remain valid if we pass from the Lip–operator subsystems (V_1, L_1'), (V_2, L_2') in the LvNA's (M_1, L_1'), (M_2, L_2'), to the corresponding Lip–operator subsystems $(\mathcal{M}_2(V_1), L_1')$, $(\mathcal{M}_2(V_2), L_2')$ in the LvNA's $(\mathcal{M}_2(M_1), L_1')$, $(\mathcal{M}_2(M_2), L_2')$ respectively, and we consider all the Lip–operator systems $(\mathcal{M}_2(V), L_V')$ with the linear embeddings

$$\mathrm{id}_2\otimes\phi_1 : \mathcal{M}_2(V) \hookrightarrow \mathcal{M}_2(V_1), \quad \mathrm{id}_2\otimes\phi_2 : \mathcal{M}_2(V) \hookrightarrow \mathcal{M}_2(V_2).$$

3.3.16 Theorem. *Let $\{(V_n, L_n')\}$ be a uniformly totally bounded sequence of finite–dimensional (dual) Lip–operator systems. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and let $(V_\mathcal{U}, L_\mathcal{U}')$ be the restricted ultraproduct of $\{(V_n, L_n')\}$. Then, given $\varepsilon > 0$, there exists an element $U_\varepsilon \in \mathcal{U}$ such that, for all $n \in U_\varepsilon$,*

$$\mathrm{dist}_{qGH^*}(V_n, V_\mathcal{U}) < \varepsilon.$$

Proof. So, let $(V_\mathcal{U}, L_\mathcal{U}')$ be the restricted ultraproduct of the sequence $\{(V_n, L_n')\}_{n\in\mathbb{N}}$, and let $\varepsilon > 0$ be given. Since we want to show that there exists an element $U_\varepsilon \in \mathcal{U}$ such that, for all $n \in U_\varepsilon$, $\mathrm{dist}_{qGH^*}(V_n, V_\mathcal{U}) < \varepsilon$, we have to estimate the Hausdorff distances between (Y_n, L_n') and $(Y_\mathcal{U}, L_\mathcal{U}')$, where $Y_n = \mathcal{M}_2(V_n)_{1,+}$ and $Y_\mathcal{U} = \mathcal{M}_2(V_\mathcal{U})_{1,+}$.

We assume, for the moment, that $\dim V_n = \dim V_\mathcal{U} = d$. Let $\{e^{(i)}\}_{i=1}^d$ and $\{e_n^{(i)}\}_{i=1}^d$, be two (vector) basis for $V_\mathcal{U}$ and V_n respectively (constructed as in Lemma 3.3.13), and assume that $(0 <)L_n'(e_n^{(i)}) = L_\mathcal{U}'(e^{(i)}) = k_i (\leq R)$ for any n in some $U_1 \in \mathcal{U}$ and $i = 1, \ldots, d$ (cf. Remark 3.3.12). For $n \in U_1$, we endow the spaces V_n with the ℓ^1–norm w.r.t. the given basis $\{e_n^{(i)}\}$, namely

$$\ell_n'(\sum_{i=1}^d \lambda_i e_n^{(i)}) := \sum_{i=1}^d k_i|\lambda_i|, \tag{3.25}$$

and we denote by $\hat{V}_n$ the space V_n with this new (Lip–)norm. Fix $\varepsilon' < 1$, and let T_n and $U_2^{(\varepsilon')}$ be as in Lemma 3.3.13. Then, for any $x \in V_n$, $n \in U_1 \cap U_2^{(\varepsilon')}$, we have

$$\begin{aligned} L_n'(x) &= L_n'(T_n\circ T_n^{-1}(x)) = L_n'(T_n(x_\mathcal{U})) \geq (1-\varepsilon')L_\mathcal{U}'(x_\mathcal{U}) \\ &\geq c(1-\varepsilon')\ell_\mathcal{U}'(x_\mathcal{U}) = c(1-\varepsilon')\sum_{i=1}^d k_i|\lambda_i| = c(1-\varepsilon')\ell_n'(x), \end{aligned}$$

where $c = \inf\{L'_{\mathcal{U}}(y_{\mathcal{U}}) : \ell'_{\mathcal{U}}(y_{\mathcal{U}}) = 1\} > 0$. Therefore, we get

$$\|x\| \geq \frac{1}{R} L'_n(x) \geq \frac{c}{R}(1-\varepsilon')\ell'_n(x).$$

Consider now the family $\{(\hat{V}_n, \|\cdot\|, \ell'_n)\}$, $n \in U_1 \cap U_2^{(\varepsilon')}$. It is uniform, since the $\hat{V}_n$'s are finite–dimensional and the radii

$$R'_n = \sup_{x \neq 0} \frac{\ell'_n(x)}{\|x\|} \leq \frac{R}{c(1-\varepsilon')}$$

are uniformly bounded. Moreover,

$$\lim_{\mathcal{U}} \ell'_n(x_n) = 0 \quad \Longleftrightarrow \quad \lim_{\mathcal{U}} L'_n(x_n) = 0,$$

since, for any $x \in V_n (\equiv \hat{V}_n$ as vector space), $x = \sum_{i=1}^d \lambda_i e_n^{(i)}$, and

$$c(1-\varepsilon')\ell'_n(x) \leq L'_n(x) \leq \ell'_n(x).$$

Hence, as vector space, $\hat{V}_{\mathcal{U}} = V_{\mathcal{U}}$.

Let us notice that, if $X \subseteq \mathcal{M}_2(M)$ and $Y \subseteq \mathcal{M}_2(N)$ are Lip–operator systems and $L'_1, L'_2 \in \mathcal{L}'(M,N)$ are Lip–norms, then $L'_1(x \oplus y) \geq L'_2(x \oplus y)$ implies $\mathrm{dist}_H^{L'_1}(X,Y) \geq \mathrm{dist}_H^{L'_2}(X,Y)$. From this we see that the Hausdorff distance between $(\hat{V}_n, \ell'_n)$ and $(\hat{V}_m, \ell'_m)$ will be greater than the Hausdorff distance between (V_n, L'_n) and (V_m, L'_m).

Now, take $V = \mathbb{C}^d$ with the standard (orthonormal) basis $\{f^{(i)}\}$. We define the ℓ'–norm of an element $z = \sum_{i=1}^d \alpha_i f^{(i)}$ as $\ell'_V(z) := \sum_{i=1}^d |\alpha_i| k_i$, where $k_i = L'_n(e_n^{(i)})$, the maps $\phi_{n(\mathcal{U})} : V \to V_{n(\mathcal{U})}$ by

$$\phi_n(f^{(i)}) = e_n^{(i)},\ \phi_{\mathcal{U}}(f^{(i)}) = e^{(i)},\ i = 1, \ldots, d,$$

and the induced norms as

$$\begin{aligned}
\|v\|_n &:= \|\phi_n(v)\|, \quad \ell'_n(\sum_{i=1}^d \lambda_i e_n^{(i)}) := \sum_{i=1}^d k_i |\lambda_i|, \\
\|v\|_{\mathcal{U}} &:= \|\phi_{\mathcal{U}}(v)\|, \quad \ell'_{\mathcal{U}}(\sum_{i=1}^d \lambda_i e^{(i)}) := \sum_{i=1}^d k_i |\lambda_i|,
\end{aligned}$$

We have then

$$\begin{aligned}
\ell'_n(\phi_n(\sum_{i=1}^d \lambda_i f^{(i)})) &= \ell'_n(\sum_{i=1}^d \lambda_i \phi_n(f^{(i)})) = \ell'_n(\sum_{i=1}^d \lambda_i e_n^{(i)}) \\
&= \sum_{i=1}^d |\lambda_i| k_i = \ell'_V(\sum_{i=1}^d \lambda_i f^{(i)}),
\end{aligned}$$

and similarly,

$$\ell'_{\mathcal{U}}(\phi_{\mathcal{U}}(\sum_{i=1}^d \lambda_i f^{(i)})) = \ell'_V(\sum_{i=1}^d \lambda_i f^{(i)})$$

i.e. $\ell'_n(\phi_n(v)) = \ell'_{\mathcal{U}}(\phi_{\mathcal{U}}(v)) = \ell'_V(v)$ for each $v \in V$, $n \in U_1 \cap U_2^{(\varepsilon')}$. Then, it follows that, for any $x \in \hat{V}_n$ and $y \in \hat{V}_{\mathcal{U}}$,

$$(\widetilde{\ell}'_{n\mathcal{U}})^V(x \oplus y) = \inf_{v,v' \in V} (\ell'_n(x - \phi_n(v)) + \ell'_{\mathcal{U}}(\phi_{\mathcal{U}}(v') - y) + \ell'_V(v - v')) = 0,$$

and thus, if $\|x\| \leq 1$ and $\|y\| \leq 1$,

$$\begin{aligned}(\ell'_{n\mathcal{U}})^V(x \oplus y) &= \max\left((\widetilde{\ell}'_{n\mathcal{U}})^V(x \oplus y), \frac{1}{n}\ell'_n(x), \frac{1}{n}\ell'_{\mathcal{U}}(y)\right) \\ &= \max\left(\frac{1}{n}\ell'_n(x), \frac{1}{n}\ell'_{\mathcal{U}}(y)\right) \leq \frac{1}{n}\max\left(\frac{R}{c(1-\varepsilon')}, R\right) \equiv \frac{1}{n}\hat{R}.\end{aligned}$$

By setting $U_3^{(\varepsilon)} := \{n \in \mathbb{N} : n > \frac{3\hat{R}}{2\varepsilon}\} \in \mathcal{U}$, we see that, for all $n \in U_1 \cap U_2^{(\varepsilon')} \cap U_3^{(\varepsilon)}$, we get

$$\begin{aligned}\mathrm{dist}_{qGH^*}(V_n, V_{\mathcal{U}}) &\leq \mathrm{dist}_H^{L'_{n\mathcal{U}}}((Y_n, L'_n), (Y_{\mathcal{U}}, L'_{\mathcal{U}})) \\ &\leq \mathrm{dist}_H^{\ell'_{n\mathcal{U}}}((Y'_n, \ell'_n), (Y'_{\mathcal{U}}, \ell'_{\mathcal{U}})) < \frac{2}{3}\varepsilon.\end{aligned}$$

Lastly, we have to consider the case when $d = \dim V_{\mathcal{U}} < \dim V_n = d'$. This means that, in order to get a complete basis for the V_n's from that of $V_{\mathcal{U}}$ as before, we have to add further (linearly independent) elements. That is to say, when passing to the quotient, we "lose" one or more dimensions, in the sense that, for $m \in \{d+1, \ldots, d'\}$, $\lim_{\mathcal{U}} L'_n(e_n^{(m)} - \sum_{i=1}^d \lambda_i e_n^{(i)}) = 0$, with the λ_i's not all zero. We assume for simplicity that $\dim V_{\mathcal{U}} = \dim V_n - 1$, i.e. we are losing only one dimension. But then we can find an element $U_4^{(\varepsilon)}$ of the ultrafilter such that, if we write $v_n \in V_n$ as

$$\begin{aligned}v_n &= \sum_{i=1}^{d'} \mu_i e_n^{(i)} = \sum_{i=1}^{d'-1} \mu_i e_n^{(i)} + \mu_{d'} e_n^{(d')} \\ &= \sum_{i=1}^{d'-1} (\mu_i + \mu_{d'}\lambda_i) e_n^{(i)} + \mu_{d'}(e_n^{(d')} - \sum_{i=1}^{d'-1} \lambda_i e_n^{(i)}),\end{aligned}$$

then, for $n \in U_4^{(\varepsilon)}$,

$$L'_n(v_n - \sum_{i=1}^{d'-1} (\mu_i + \mu_{d'}\lambda_i) e_n^{(i)}) = L'_n(\mu_{d'}(e_n^{(d')} - \sum_{i=1}^{d'-1} \lambda_i e_n^{(i)})) \leq \frac{1}{3}\varepsilon,$$

that is to say, for all $n \in U_4^{(\varepsilon)}$ and $v_n \in (V_n)_1$, we have

$$\inf_{v \in (\widetilde{V}_n)_1} L'_n(v_n - v) \leq \frac{1}{3}\varepsilon,$$

where $\widetilde{V}_n$ is the subspaces of V_n spanned by $\{e_n^{(i)} : i \neq d'\}$. Hence, if we consider the sequence $\{\widetilde{V}_n\}$ instead of the original one, for all $n \in U_4^{(\varepsilon)}$ the "error" that we get is less than $\varepsilon/3$, and thus

$$\mathrm{dist}_{qGH^*}(\widetilde{V}_n, V_n) \leq \frac{1}{3}\varepsilon.$$

Finally, setting $U_\varepsilon := U_1 \cap U_2^{(\varepsilon')} \cap U_3^{(\varepsilon)} \cap U_4^{(\varepsilon)} \in \mathcal{U}$, for $n \in U_\varepsilon$ we obtain

$$\mathrm{dist}_{qGH^*}(V_n, V_{\mathcal{U}}) < \varepsilon,$$

and the proof is complete. ∎

Now we are ready to pass to the algebraic setting. So, let (M_n, L'_n) be a sequence of Lip–von Neumann algebras, with corresponding preduals (M_{n*}, L_n), and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. Recall that the (dual) restricted ultraproduct $M_{\mathcal{U}}$ is defined as the image $\pi'_{\mathcal{U}}(\ell^\infty(M_n))$ in the quotient $\ell^\infty(M_n)/K'_{L',\mathcal{U}}$, where

$$\begin{aligned} \ell^\infty(M_n) &= \{\{a_n\} : a_n \in M_n, \|\{a_n\}\| = \sup_n \|a_n\| < \infty\}, \\ K'_{L',\mathcal{U}} &= \{\{a_n\} \in \ell^\infty(M_n) : \lim_{\mathcal{U}} L'_n(a_n) = 0\} \end{aligned}$$

As a first result, we want to show that, if a sequence of LvNA's converges in the distance dist_{qGH^*}, then the limit is exactly the (dual) restricted ultraproduct over any given ultrafilter over $\mathbb{N}$.

3.3.17 Proposition. *Let $\{(M_n, L'_n)\}$ be a convergent sequence of Lip–von Neumann algebras in the distance* dist_{qGH^*}. *Then, $\{(M_n, L'_n)\}$ is weakly uniform.*

Proof. We shall apply Proposition 3.3.8. Since the sequence is convergent, there exists a $\Delta > 0$ such that $\mathrm{dist}_{qGH^*}(M_n, M) \leq \Delta$ for any $n \in \mathbb{N}$. Let (M, L'_M) be the qGH*–limit of the sequence. By Lemma 3.2.5, we have $|R_n - R| \leq \mathrm{dist}_{qGH^*}(M_n, M) \leq \Delta$, where R_n and R are the radii of (M_n, L'_n) and (M, L'_M) respectively. Hence, $\sup_n R_n$ is finite.

Now, by definition of qGH*–convergence, given $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$, such that for any $n > n_\varepsilon$, we can find a Lip–norm $L'_{nM} \in \mathcal{L}'(M_n, M)$ such that $\mathrm{dist}_H^{L'_{nM}}(X_n, X_M) < \varepsilon/16$, where $X_n := \mathcal{M}_2(M_n)_{1,+}$. This means that, for any $x \in X_M$, we can find a sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in X_n$, such that $L'_{nM}(x_n \oplus x) < \varepsilon/16$. By a standard argument, it follows that the same holds true for all $x \in \mathcal{M}_2(M)_1$. Indeed, since any $x \in \mathcal{M}_2(M)_1$ can be written in the form $x = z_1 - z_2 + i(z_3 - z_4)$ with $z_i \in X_M$, $i = 1, \ldots, 4$, we have

$$L'_{nM}(x_n \oplus x) = L'_{nM}((z_n)_1 \oplus z_1 - (z_n)_2 \oplus z_2 + i((z_n)_3 \oplus z_3 - (z_n)_4 \oplus z_4))) < \varepsilon/4,$$

where $(z_n)_i \in X_n$, $i = 1, \ldots, 4$, and $z_n = (z_n)_1 - (z_n)_2 + i((z_n)_3 - (z_n)_4) \in \mathcal{M}_2(M_n)_1$. Then, by taking the diagonal copies of $(M_n)_{1,+}$ and $(M)_{1,+}$ in X_n and X_M respectively, we see that also any $x \in M_1$ admits an approximating sequence $\{x_n\}_{n\in\mathbb{N}}$, $x_n \in (M_n)_1$, such that $L'_{nM}(x_n \oplus x) < \varepsilon/4$.

Let V be an $\varepsilon/2$–dense (for the L'_M–norm) finite–dimensional operator subsystem in M, take any vector basis $\{x_i\}_{i=1}^{d_\varepsilon} \subset (V)_1$ for V, and let $\{x_n^i\}_{n\in\mathbb{N}}$, $\{x_n^i\}_{i=1}^{d_\varepsilon} \subset (M_n)_1$ be a sequence approximating x_i, $i = 1, \ldots, d_\varepsilon$. We define V_n to be the operator subsystem in M_n generated by the set $\{x_n^i\}_{i=1}^{d_\varepsilon}$. We have clearly $\dim V_n \leq d_\varepsilon$. Moreover, we can find an n'_ε such that, for all $m > n'_\varepsilon$, one has $L'_{mM}(x_m^i \oplus x_i) < \varepsilon/(4K_\varepsilon)$, where

$$K_\varepsilon := \max\{\sum_{i=1}^{d_\varepsilon} |\alpha_i| : \|\sum_{i=1}^{d_\varepsilon} \alpha_i x_i\| \leq 1\} < \infty \quad (\varepsilon > 0).$$

In this way, given $y \in (V)_1$, there exists a $y_m \in (V_m)_1$ such that $L'_{mM}(y_m \oplus y) < \varepsilon/4$. We set $N_\varepsilon := \max(n_\varepsilon, n'_\varepsilon)$. Then, for any $n > N_\varepsilon$ and $x \in (M_n)_1$, we can find $y_n \in (V_n)_1$, and two elements $\tilde{x} \in M_1$, $y \in (V)_1$, such that

$$L'_{nM}(x \oplus \tilde{x}) < \varepsilon/4, \quad L'_{nM}(y_n \oplus y) < \varepsilon/4, \quad \text{and} \quad L'_M(\tilde{x} - y) < \varepsilon/2.$$

Hence, we get

$$\begin{aligned} L'_n(x - y_n) &= L'_{nM}((x - y_n) \oplus 0) = L'_{nM}((x - y_n) \oplus (\tilde{x} - y) - 0 \oplus (\tilde{x} - y)) \\ &\leq L'_{nM}(x \oplus \tilde{x} - y_n \oplus y) + L'_{nM}(0 \oplus (\tilde{x} - y)) \\ &\leq L'_{nM}(x \oplus \tilde{x}) + L'_{nM}(y_n \oplus y) + L'_M(\tilde{x} - y) < \varepsilon, \end{aligned}$$

which implies that $(V_n)_1$ is ε–dense in $(M_n)_1$, for all $n > N_\varepsilon$. Since a finite family of LvNA's is clearly uniformly totally bounded, we can conclude that the whole family $\{(M_n, L'_n)\}$ is uniformly totally bounded, and the proof is complete. ∎

3.3.18 Theorem. *Let (M_n, L'_n) be a sequence of Lip–von Neumann algebras converging to the LvNA (M, L') in the distance dist_{qGH^*}. Then, for any ultrafilter $\mathcal{U}$, we have $M = M_\mathcal{U}$ and $M_* = M_{\mathcal{U}*}$.*

Proof. Since the sequence (M_n, L'_n) is convergent, by the previous Proposition it is weakly uniform, thus we can consider the (dual) restricted ultraproduct $M_\mathcal{U}$ over any given ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Moreover, for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$, such that for any $n, m > n_\varepsilon$, we can find a Lip–norm $L'_{nm} \in \mathcal{L}'(M_n, M_m)$ such that $\mathrm{dist}_H^{L'_{nm}}(X_n, X_m) < \varepsilon/2$. Fix $n > n_\varepsilon$, and consider the sequence $\{M_n \oplus M_m\}_{m \in \mathbb{N}}$. For any ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the (dual) restricted ultraproduct of $\{M_n \oplus M_m\}_{m \in \mathbb{N}}$ naturally identifies with $M_n \oplus M_\mathcal{U}$ and we get a (dual) Lip–norm $L'_{n\mathcal{U}}$ on $M_n \oplus M_\mathcal{U}$ as $\lim_\mathcal{U} L'_{nm}$ (cf. Propositin 2.3.3), where, for $m > n_\varepsilon$, we take care to choose L'_{nm} as above. Set $X_n := \mathcal{M}_2(M_n)_{1,+}$, $X_\mathcal{U} := \mathcal{M}_2(M_\mathcal{U})_{1,+}$. Then, we have

$$\begin{aligned} \mathrm{dist}_{qGH^*}(M, M_\mathcal{U}) &\leq \mathrm{dist}_{qGH^*}(M, M_n) + \mathrm{dist}_{qGH^*}(M_n, M_\mathcal{U}) \\ &< \varepsilon/2 + \mathrm{dist}_{qGH^*}(M_n, M_\mathcal{U}). \end{aligned}$$

Now, in order to show that $\mathrm{dist}_{qGH^*}(M_n, M_\mathcal{U}) \leq \varepsilon/2$, it suffices to verify that $\mathrm{dist}_H^{L'_{n\mathcal{U}}}(X_n, X_\mathcal{U}) \leq \varepsilon/2$, where $L'_{n\mathcal{U}} = \lim_\mathcal{U} L'_{nm}$. By construction, for any $x \in X_n$, we can find an $y_m \in X_m$ $(m > n_\varepsilon)$ such that $L'_{nm}(x, y_m) < \varepsilon/2$. If we set $y_\mathcal{U} = \pi'_\mathcal{U}(\{y_m\})$, then obviously $y_\mathcal{U} \in X_\mathcal{U}$ and

$$L'_{n\mathcal{U}}(x \oplus y_\mathcal{U}) = \lim_{m \to \mathcal{U}} L'_{nm}(x \oplus y_m) \leq \varepsilon/2,$$

hence $X_n \subset \mathcal{N}_{\varepsilon/2}(X_\mathcal{U}, L'_{n\mathcal{U}})$. Viceversa, for any $y \in X_\mathcal{U}$, with $y = \pi'_\mathcal{U}(\{y_m\})$, $y_m \in X_m$, we can find, for $m > n_\varepsilon$, an $x_m \in X_n$ such that $L'_{nm}(x_m \oplus y_m) < \varepsilon/2$. If we take the ($w^*$–)limit $x = \lim_{m \to \mathcal{U}} x_m \in X_n$, then $\lim_{m \to \mathcal{U}} L'_n(x - x_m) = 0$, and thus

$$\begin{aligned} L'_{n\mathcal{U}}(x \oplus y) &= \lim_{m \to \mathcal{U}} L'_{nm}(x \oplus y_m) \\ &\leq \lim_{m \to \mathcal{U}} (L'_{nm}((x - x_m) \oplus 0) + L'_{nm}(x_m \oplus y_m)) \leq \varepsilon/2, \end{aligned}$$

showing that $X_\mathcal{U} \subset \mathcal{N}_{\varepsilon/2}(X_n, L'_{n\mathcal{U}})$, i.e. $\mathrm{dist}_H^{L'_{n\mathcal{U}}}(X_n, X_\mathcal{U}) \leq \varepsilon/2$, as claimed. Therefore,

$$\mathrm{dist}_{qGH^*}(M, M_\mathcal{U}) < \varepsilon$$

and, since ε was arbitrary, we get the thesis. ∎

Next, we want to prove some partial converse, namely we establish some precompactness conditions, in the topology induced by dist_{qGH^*}, for uniform sequences (M_n, L'_n) of (rigged or Lip–)von Neumann algebras.

3.3.19 Theorem. *Let (M_n, L'_n) be a (weakly) uniform sequence of Lip–von Neumann algebras. Then, for any ultrafilter $\mathcal{U}$ over $\mathbb{N}$, the restriction of (M_n, L'_n) to $\mathcal{U}$ is a Cauchy sequence w.r.t. the distance dist_{qGH^*}, that is, for any $\varepsilon > 0$ there exists an element $U_\varepsilon \in \mathcal{U}$ such that $\mathrm{dist}_{qGH^*}(M_m, M_n) < \varepsilon$ for $m, n \in U_\varepsilon$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and let $\varepsilon > 0$ be given. Set $\varepsilon' := \varepsilon/4$. Then, by uniformity (cf. Lemmas 3.3.5, 3.3.6), for each n, there exists an ε'–net $\{x_n^{(i)}\}_{i=1}^{N_{\varepsilon'}} \subset \mathcal{M}_2(M_n)_{1,+}$ w.r.t. the L'_n–norm. We may suppose that $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ is a sequence in $\ell^\infty(\mathcal{M}_2(M_n))$ such that $\|x_n^{(i)}\| \leq 1$, for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, N_{\varepsilon'}\}$ (cf. Lemma 3.3.10). Let $\mathcal{M}_2(V_n)$ be the Lip–operator system generated by the set $\{x_n^i\}_{i=1}^{N_{\varepsilon'}}$, where V_n is the smallest unital $*$–vector subspace in M_n containig the $N_{\varepsilon'}^4$ elements $(x_{hk})_n^i$, $h, k = 1, 2$, $i = 1, \ldots N_{\varepsilon'}$. Then, by Lemma 3.3.9, for all $m, n \in \mathbb{N}$, we have

$$\begin{aligned} \mathrm{dist}_{qGH^*}(M_m, M_n) &\leq \mathrm{dist}_{qGH^*}(M_m, V_m) + \mathrm{dist}_{qGH^*}(V_m, V_n) \\ &\quad + \mathrm{dist}_{qGH^*}(V_n, M_n) \\ &< 2\varepsilon' + \mathrm{dist}_{qGH^*}(V_m, V_n). \end{aligned}$$

Now, notice that $\{V_n, L'_n)\}_{n\in\mathbb{N}}$ is a uniform sequence of finite–dimensional (dual) Lip–operator systems, and thus, let $(V_\mathcal{U}, L'_\mathcal{U})$ be its (dual restricted) ultraproduct over $\mathcal{U}$. Then, by Theorem 3.3.16, we can find an element $\widetilde{U}_{\varepsilon'} \in \mathcal{U}$ such that, for any $h, k \in \widetilde{U}_{\varepsilon'}$, we have

$$\mathrm{dist}_{qGH^*}(V_h, V_k) < 2\varepsilon'.$$

Then, setting $U_\varepsilon \equiv \widetilde{U}_{\varepsilon'}$, we finally get

$$\mathrm{dist}_{qGH^*}(M_m, M_n) < \varepsilon$$

for all $m, n \in U_\varepsilon$, as claimed. ∎

For a weakly uniform family of LvNA's, without further hypothesis, we have the following

3.3.20 Theorem. *Let $\{(M_n, L'_n)\}_{n\in\mathbb{N}}$ be a weakly uniform sequence of Lip–von Neumann algebras, and let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$. Suppose that the restricted ultraproduct $M_\mathcal{U}$ over $\mathcal{U}$ is a LvNA. Then, the* qGH*$^*-$ *limit over $\mathcal{U}$ of $\{(M_n, L'_n)\}$ coincides with the ultraproduct $M_\mathcal{U}$. Hence, in particular, it is a Lip–von Neumann algebra.*

Proof. Indeed, let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and $(M_\mathcal{U}, L'_\mathcal{U})$ the restricted ultraproduct of the sequence $\{(M_n, L'_n)\}_{n\in\mathbb{N}}$. Since, by hypothesis, $M_\mathcal{U}$ is a Lip–von Neumann algebra, the positive part $(M_\mathcal{U})_{1,+}$ of the unit ball $M_\mathcal{U}$ is totally bounded in the Lip–norm $L'_\mathcal{U}$. Then, also $\mathcal{M}_2(M_\mathcal{U})_{1,+}$ is totally bounded. Therefore, for any given $\varepsilon > 0$, we can find an ε–net $\{x^{(i)}\}_{i=1}^{N_\varepsilon} \subset \mathcal{M}_2(M_\mathcal{U})_{1,+}$ w.r.t. the $L'_\mathcal{U}$–norm. (We may suppose, for simplicity, that $x^{(i)} \neq 0$ for all i.) Let $\{x_n^{(i)}\}_{n\in\mathbb{N}}$ be a sequence in $\ell^\infty(\mathcal{M}_2(M_n)_+)$[3] realising $x^{(i)}$ as in Lemma 3.3.10, i.e. such that $\|x_n^{(i)}\| \leq 1$, for

[3] In fact, let $x \in \mathcal{M}_2(M_\mathcal{U})_+$, and let $\{\tilde{x}_n\}$ be a sequence realising it. We may assume that all the $\tilde{x}_n$ are selfadjoint, and, since $x \geq 0$, there exists $U \in \mathcal{U}$ and a (bounded) sequence $\{\alpha_n\}$ of positive real numbers converging to zero such that $\tilde{x}_n \geq -\alpha_n I_n$ for all $n \in U$. Then, $\lim_\mathcal{U} L'_n(\tilde{x}_n + \alpha_n I_n) = L'_\mathcal{U}(x)$, and so we may define the sequence $\{x_n\} \in \ell^\infty(\mathcal{M}_2(M_n)_+)$ by setting $x_n = x_n + \alpha_n I_n$ for $n \in U$ and $x_n = 0$ for $n \notin U$.

all $n \in \mathbb{N}$ and $i \in \{1, \ldots, N_\varepsilon\}$. We want to show that, for n sufficiently large, the set $\{x_n^{(i)}\}_{i=1}^{N_\varepsilon}$ satisfies the same property for $\mathcal{M}_2(M_n)_{1,+}$ w.r.t. L'_n, with ε replaced by 2ε.
In fact, given a sequence $\{y_n\} \in \ell^\infty(\mathcal{M}_2(M_n))$ with $0 \le y_n \le I_n$ for all $n \in \mathbb{N}$ (I_n is the identity in $\mathcal{M}_2(M_n)$), one has $y_\mathcal{U} = \pi'_\mathcal{U}(\{y_n\}) \in \mathcal{M}_2(M_\mathcal{U})_{1,+}$, and so there is at least one $x^{(j)} \in \{x^{(i)}\}_{i=1}^{N_\varepsilon}$ with $L'_\mathcal{U}(y_\mathcal{U} - x^{(j)}) < \varepsilon$. Hence, we can find an element $U_1^{(\varepsilon)}$ of the ultrafilter $\mathcal{U}$ such that $L'_n(y_n - x_n^{(j)}) \le \lim_\mathcal{U} L'_n(y_n - x_n^{(j)}) + \varepsilon = L'_\mathcal{U}(y_\mathcal{U} - x^{(j)}) + \varepsilon < 2\varepsilon$ for all $n \in U_1^{(\varepsilon)}$.
Now, for each $n \in U_1^{(\varepsilon)}$, let $\mathcal{M}_2(V_n)$ and $\mathcal{M}_2(V_\mathcal{U})$ be the Lip–operator systems generated by the sets $\{x_n^i\}_{i=1}^{N_\varepsilon}$ and $\{x_i\}_{i=1}^{N_\varepsilon}$, respectively, where V_n and $V_\mathcal{U}$ are the smallest unital $*$–vector subspaces in M_n and $M_\mathcal{U}$ containing $(x_{hk})_n^i$ and $(x_{hk})^i$, $h, k = 1, 2$, $i = 1, \ldots N_\varepsilon$, respectively. Then, by Lemma 3.3.9, we have

$$\begin{aligned} \mathrm{dist}_{qGH^*}(M_n, M_\mathcal{U}) &\le \mathrm{dist}_{qGH^*}(M_n, V_n) + \mathrm{dist}_{qGH^*}(V_n, V_\mathcal{U}) \\ &\quad + \mathrm{dist}_{qGH^*}(V_\mathcal{U}, M_\mathcal{U}) \\ &< 3\varepsilon + \mathrm{dist}_{qGH^*}(V_n, V_\mathcal{U}),\ n \in U_1^{(\varepsilon)}. \end{aligned}$$

Since $(\mathcal{M}_2(V_\mathcal{U}), L'_\mathcal{U})$ is the (dual restricted) ultraproduct of the sequence $\{(\mathcal{M}_2(V_n), L'_n)\}_{n\in\mathbb{N}}$ of Lip–operator systems, by Theorem 3.3.16, we can find an element $U_2^{(\varepsilon)}$, such that, for all $n \in U_2^{(\varepsilon)}$, we have

$$\mathrm{dist}_{qGH^*}(V_n, V_\mathcal{U}) < \varepsilon.$$

Hence, for all $n \in U_1^{(\varepsilon)} \cap U_2^{(\varepsilon)}$, we get

$$\mathrm{dist}_{qGH^*}(M_n, M_\mathcal{U}) < 4\varepsilon,$$

and, by arbitrariness of ε, the thesis follows. ∎

As a consequence, for a strongly uniform family of RvNA's, we get:

3.3.21 Corollary. *If (M_n, L'_n) is a strongly uniform sequence of rigged von Neumann algebras, then* qGH**−limit* $\lim_{n\to\mathcal{U}} M_n$ *is a Lip–von Neumann algebra.*

Proof. By Theorem 2.4.9, the (restricted) ultraproduct $M_\mathcal{U}$ is a Lip–von Neumann algebra. Since, by the previous Theorem, the qGH*−limit of the sequence (M_n, L'_n) coincides with the restricted ultraproduct $(M_\mathcal{U}, L'_\mathcal{U})$, it is a Lip–von Neumann algebra as well. ∎

Chapter 4

Dual Quantum GH Distance, Ultraproducts and Quantum Fields.

For the reader's convenience, we shall recall the basic assumptions of Algebraic Quantum Field Theory (AQFT). (For a detailed introduction to this subject, the reader is referred to the monograph [33].)

1. (*Locality*) We assume that the local observables of the theory generate a local net over the Minkowski space $\mathbb{R}^4$, i.e. a map $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ from the set of open, bounded regions $\mathcal{O} \subset \mathbb{R}^4$ to unital C^*–algebras on a suitable Hilbert space $\mathcal{H}$, which preserves inclusions, that is,

$$\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2) \qquad \text{if} \quad \mathcal{O}_1 \subseteq \mathcal{O}_2. \tag{4.1}$$

The net $\{\mathcal{A}(\mathcal{O}) : \mathcal{O} \subset \mathbb{R}^4\}$ is supposed to satisfy the the principle of locality (or Einstein's causality), according to which observables in spacelike separated regions commute,

$$\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)' \qquad \text{if} \quad \mathcal{O}_1 \subseteq \mathcal{O}_2', \tag{4.2}$$

where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathcal{A}(\mathcal{O})'$ the set of operators in $\mathcal{B}(\mathcal{H})$ which commute with all operators in $\mathcal{A}(\mathcal{O})$. The C^*–algebra $\mathcal{A}$ given by the C^*–inductive limit by all the local algebras $\mathcal{A}(\mathcal{O})$ is called the *quasi–local* algebra, and is assumed to act irreducibly on $\mathcal{H}$.

2. (*Covariance*) There exists on $\mathcal{H}$ a strongly continuous unitary representation U of the Poincaré group $\mathcal{P}_+^\uparrow$, which induces automorphisms of the net as follows: for each $(\Lambda, x) \in \mathcal{P}_+^\uparrow$ there is an $\alpha_{(\Lambda,x)} \in \mathrm{Aut}(\mathcal{A})$ given by

$$\alpha_{(\Lambda,x)}(a) := U(\Lambda, x)aU(\Lambda, x)^{-1}, \quad a \in \mathcal{A}, \tag{4.3}$$

and, for any region $\mathcal{O}$,

$$\alpha_{(\Lambda,x)}(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\Lambda\mathcal{O} + x). \tag{4.4}$$

A reasonable extra-requirement will be that the operator-valued functions

$$(\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(a), \quad a \in \mathcal{A}, \tag{4.5}$$

are continuous in the norm topology. This assumption does not impose any essential restrictions of generality. In fact, by the continuity of the representation U and by the boundedness of the operators a, these funtions are always continuous in the strong operator topology, and thus by convolution with suitable test functions, one can always produce a local net satisfying this strengthened continuity condition, which is dense in the original net w.r.t. the strong operator (*so*–)topology. In other words, the subnet still contains the relevant information about the physical states. (In order to preserve uniqueness, one also requires that the local algebras $\mathcal{A}(\mathcal{O})$ are maximal, in the sense that any operator in the *so*–closure $\mathcal{R}(\mathcal{O})$ of $\mathcal{A}(\mathcal{O})$ satisfying the above condition, is already contained in $\mathcal{A}(\mathcal{O})$. Possibly by enlarging the local algebras, the net can be always assumed to comply with this maximality condition.)

3. (*Spectrum Condition*). The joint spectrum of the generators of the unitary representation of the spacetime translations is contained in the closed forward lightcone $\overline{V}_+$. Moreover, there is a unique (up to a phase) vector $\Omega \in \mathcal{H}$, representing the vacuum, which is invariant under the action of U, that is,

$$U(\Lambda, x)\Omega = \Omega, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow. \tag{4.6}$$

The vector state induced by Ω is called the *vacuum state*. Since it is characterized by the existence of a vacuum state, this particular representation of $\mathcal{A}$ is called the vacuum representation, and may be regarded as the defining representation of the theory. (Notice that all the other vector states in the original Hilbert space $\mathcal{H}$ induce by the GNS contruction, the identical representation of the quasi–local algebra $\mathcal{A}$.) The other states of physical interest correspond to (positive, linear and normalized) functionals ω on $\mathcal{A}$, which are locally normal w.r.t. the vacuum representation, i.e. such that the restrictions $\omega|_{\mathcal{A}(\mathcal{O})}$ of all these states to any local algebra can be represented by vectors $\Omega_{\mathcal{O}}$ in the vacuum Hilbert space $\mathcal{H}$ as

$$\omega(a) = (\Omega_{\mathcal{O}}, a\Omega_{\mathcal{O}}), \quad a \in \mathcal{A}(\mathcal{O}). \tag{4.7}$$

The algebraic approach to relativistic quantum field theory has proven to be an efficient setting for the structural analysis of properties of physical systems at the upper end of the spatio–temporal scale. Examples are the classification of the possible statistics and superselection structure of particles, collision theory and the clarification of the infrared properties of theories with long range forces [33]. However, at the lower end of the scale the algebraic point of view has been, for a long time, less successful. Basic phenomena such as the parton picture or the notion of asymptotic freedom did not fit appropriately in the algebraic setting, due to the absence, in this approach, of the analogue of the renormalization group [4], which allows one to transform a theory at given scale into the corresponding theories at other scales.

In the algebraic approach, quantum fields, which are a basic ingredient in the conventional approach to the renormalization group, are regarded as a kind of coordinatization of the local algebras and therefore do not appear explicitly in this setting. This is justified by the observation that different irreducible sets of field operators which are relatively local to each other yield the same scattering matrix [5]. Thus, the physical content of a theory does not depend on a particular choice of fields. The absence of quantum fields in the algebraic setting causes problems, however, if one wants to apply the ideas of the renormalization group. In the conventional framework of quantum field theory, the renormalization group transformations R_λ, $\lambda > 0$, act on the underlying quantum fields $\phi(x)$ by scaling the spacetime coordinates x, accompanied by a multiplicative renormalization, $R_\lambda : \phi(x) \to \phi_\lambda(x) := N_\lambda \phi(\lambda x)$. In this way, one maps the theory at the original scale, say $\lambda = 1$, onto the corresponding theory at scale λ whithout changing the value of the fundamental physical constants, i.e. the speed of light c and Planck's constant $\hbar$. Moreover, by the multiplicative renormalization factor N_λ, the scale of field strength is adjusted in such a way that the mean values and mean square fluctuations of the fields in some fixed reference state are of the same order of magnitude at small scales. Thus, the quantum fields are employed to identify at each scale λ a set of operators with a fixed physical interpretation, and these operators can then be used to compare the properties of the theory at different scales.

4.1 The Buchholz–Verch Scaling Limit Construction

In the papers [12; 13], Buchholz and Verch answered the question of how to implement the renormalization group in the AQFT setting, and provided a solution within a mathematical framework suitable for the structural analysis of local nets at small scales. Their approach is based on the following observations.

(i) According to the geometrical significance of the renormalization group, the transformations R_λ should map the given net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ at the original spatio–temporal scale $\lambda = 1$ onto the corresponding net $\mathcal{O} \to \mathcal{A}_\lambda(\mathcal{O}) := \mathcal{A}(\lambda\mathcal{O})$ at scale λ, namely

$$R_\lambda : \mathcal{A}(\mathcal{O}) \to \mathcal{A}_\lambda(\mathcal{O}) \tag{4.8}$$

for every region $\mathcal{O} \subset \mathbb{R}^4$. Since space and time are scaled in the same way, the value of the speed of light c is kept fixed under these maps.

(ii) The condition that $\hbar$ remains constant under renormalization group transformations can be expressed in the algebraic setting as follows. If one scales space and time by λ and does not want to change the unit of action, one has to rescale energy and momentum by λ^{-1}. The energy–momentum scale can be set by determining the energy and momentum which is transferred by the action of observables to physical states. Hence, if $\widetilde{\mathcal{A}}(\widetilde{\mathcal{O}})$ denotes the subspace of all (quasi–local) observables which, at the original scale $\lambda = 1$, can transfer energy–momentum contained in the set $\widetilde{\mathcal{O}} \subset \mathbb{R}^4$, and if $\widetilde{\mathcal{A}}_\lambda(\widetilde{\mathcal{O}}) := \widetilde{\mathcal{A}}(\lambda^{-1}\widetilde{\mathcal{O}})$ denotes the corresponding space at scale λ, then the transformations R_λ should induce a map

$$R_\lambda : \widetilde{\mathcal{A}}(\widetilde{\mathcal{O}}) \to \widetilde{\mathcal{A}}_\lambda(\widetilde{\mathcal{O}}) \tag{4.9}$$

for every $\widetilde{\mathcal{O}}$. (An analogous relation should hold for the angular momentum transfer.)

(iii) In the case of dilation invariant theories the transformations R_λ are expected to be isomorphisms, yet this will not be true in general since the algebraic relations between observables may depend on the scale. But since the transformations R_λ are designed to identify observables at different scales, they still ought to be continuous, bounded maps, uniformly in λ.

The above conditions subsume the physical constraints imposed on the renormalization group transformations R_λ, although they do not fix these maps. In fact, there exists an abundance of such maps for any given $\lambda > 0$. But all of these maps identify the same net at scale λ, they merely reshuffle the operators within the local algebras in different ways. Since the basic hypothesis of algebraic quantum field theory is that the physical information of a theory is contained in the net, it should thus not matter which map one picks for the short distance analysis of a theory. One may consider any one of them or all of them. Buchholz and Verch adopt the latter point of view, which can be conveniently expressed by introducing the concept of *scaling algebra*. Roughly speaking, the scaling algebra consists of operator–valued functions $\lambda : a \to R_\lambda(a)$, $\lambda > 0$, which are the orbits of the local observables a under the action of all admissible transformations R_λ. If the renormalization group transformations R_λ comply with the specific properties indicated above, then the scaling algebra still has the structure of a local net on which the Poincaré group acts in a continuous manner. Moreover, the renormalization group induces an additional symmetry of this net: the scaling transformations. The states of physical interest can be lifted to the scaling algebra, and the transformed states have, at arbitrarily small scales, limits which are vacuum states. As a result, if the underlying theory is invariant under dilations and satisfies the Haag–Swieca compactness criterion [34], then it is invariant under the action of the renormalization group and coincides with its scaling limit. (This is the case, for instance, of the massless free scalar field, see [13].)

We shall illustrate the basic features of the Buchholz–Verch construction. So let us assume that the net $(\mathcal{A}, \alpha_{\mathcal{P}_+^\uparrow})$ is defined at spatio–temporal scale $\lambda = 1$. Then, the Poincaré transformations at any other scale $\lambda > 0$ are given by

$$\alpha^{(\lambda)}_{(\Lambda,x)} := \alpha_{(\Lambda,\lambda x)}, \quad (\Lambda, x) \in \mathcal{P}_+^\uparrow \tag{4.10}$$

Notice that $(\mathcal{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow})$ defines again a local, Poincaré covariant net over the Minkowski space. Thus, one keeps the Minkowski space fixed – or, better, one keeps the causal structure of the Minkowski space fixed, and re-labels the spacetime regions $\mathcal{O} \mapsto \mathcal{O}_\lambda := \lambda\mathcal{O}$ – and interpret the properties of the underlying theory at small scales in terms of the modified theories (nets) $(\mathcal{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow})$. In general, the nets $(\mathcal{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow})$ will describe distinct theories for different values of λ (with different energy–momentum spectrum, collision cross sections, etc.). Within the algebraic setting, these differences find a formal expression in the fact that the corresponding nets are non–isomorphic. Conversely, any two local, Poincaré–covariant nets, which are isomorphic, are physically indistinguishable, and consequently represent the same theory. We recall the notion of net isomorphism in the following

4.1.1 Definition. *For $j = 1, 2$, let $(\mathcal{O} \to \mathcal{A}^{(j)}(\mathcal{O}), \alpha^{(j)}_{\mathcal{P}_+^\uparrow})$ be two local, Poincaré covariant nets on Minkowski space with C^*–inductive limits $\mathcal{A}^{(j)}$. The two nets are said to be isomorphic if there is an isomorphism $\phi : \mathcal{A}^{(1)} \to \mathcal{A}^{(2)}$, which preserves localization,*

$$\phi(\mathcal{A}^{(1)}(\mathcal{O})) = \mathcal{A}^{(2)}(\mathcal{O}), \quad \mathcal{O} \subset \mathbb{R}^4, \tag{4.11}$$

and intertwines the Poincaré transformations,

$$\phi \circ \alpha^{(1)}_{(\Lambda,x)} = \alpha^{(2)}_{(\Lambda,x)} \circ \phi, \quad (\Lambda, \lambda x) \in \mathcal{P}_+^\uparrow. \tag{4.12}$$

Any such isomorphism ϕ is called a net isomorphism. *A net isomorphism which maps a given net onto itself is called an* internal symmetry.

According to this definition, the nets $(\mathcal{A}_\lambda, \alpha^{(\lambda)}_{\mathcal{P}_+^\uparrow})$ are isomorphic for different values of λ if, and only if, dilations are a (geometrical) symmetry of the underlying theory. The physical content of the theory is then invariant under changes of the spatio–temporal scale. In general, in the interesting cases, the underlying theory does not possess such a symmetry. Consequently, one cannot exploit the notion of net–isomorphism. However, one can use the renormalization group transformations R_λ in order to compare the properties of the theory at different scales. Since what really matters is to have control on the phase–space properties of the orbits $\lambda \to R_\lambda(a)$, $\lambda > 0$, of local observables $a \in \mathcal{A}$ under renormalization group transformations, it suffices that these transformations do not change the fundamental physical units c and $\hbar$, which can be expressed by the uniform (w.r.t. λ) continuity in the following sense:

$$\sup_{\lambda>0} \|\alpha_{(\Lambda,\lambda x)}(R_\lambda(a)) - R_\lambda(a)\| \to 0 \quad \text{for} \quad (\Lambda, x) \to (1, 0). \tag{4.13}$$

One considers functions $\underline{a} : \mathbb{R}_+ \to \mathcal{A}$ from the domain $\mathbb{R}_+$ of the scaling parameter λ to the underlying algebra of observables, with the following algebraic structure: given two functions $\underline{a}$, $\underline{b}$ and $\mu_1, \mu_2 \in \mathbb{C}$, we set for $\lambda > 0$

$$\begin{aligned} (\mu_1 \underline{a} + \mu_2 \underline{b})_\lambda &:= \mu_1 a_\lambda + \mu_2 b_\lambda \\ (\underline{a} \cdot \underline{b})_\lambda &:= a_\lambda \cdot b_\lambda \\ (\underline{a}^*)_\lambda &:= a^*_\lambda. \end{aligned} \tag{4.14}$$

In this way, we get a unital $*$–algebra, with unit $(\underline{I})_\lambda = I$. Moreover, since we are only interested in uniformly bounded functions, it is natural to introduce the norm

$$\|\underline{a}\| := \sup_{\lambda>0} \|a_\lambda\|, \tag{4.15}$$

which is in fact a C^*–norm. The induced action of the Poincaré transformations on the functions is then given by

$$(\underline{\alpha}_{(\Lambda,x)}(\underline{a}))_\lambda := \alpha_{(\Lambda,\lambda x)}(a_\lambda) \tag{4.16}$$

It follows that the continuity requirement (4.13) can be expressed in the simple form

$$\sup_{\lambda>0} \|\underline{\alpha}_{(\Lambda,x)}(\underline{a}) - \underline{a}\| \to 0 \quad \text{for} \quad (\Lambda, x) \to (1, 0). \tag{4.17}$$

It remains only to impose on the functions the localization condition.

4.1.2 Definition. *Let $\mathcal{O} \subset \mathbb{R}^4$ be any open, bounded region. Then $\underline{\mathcal{A}}(\mathcal{O})$ denotes the set of all uniformly bounded functions $\underline{a}$ which are continuous with respect to Poincaré transformations in the sense of relation (4.17) and satisfy*

$$a_\lambda \in \mathcal{A}(\lambda\mathcal{O}), \quad \lambda > 0. \tag{4.18}$$

Since each $\mathcal{A}(\lambda\mathcal{O})$ is a C^*–algebra, it follows that $\underline{\mathcal{A}}(\mathcal{O})$ is a C^*–algebra as well: it is stable under the algebraic operations (4.14) and complete with respect to the C^*–norm (4.15). It is also evident from the definition that $\underline{\mathcal{A}}(\mathcal{O})$ is monotonous w.r.t. $\mathcal{O}$,

$$\underline{\mathcal{A}}(\mathcal{O}_1) \subset \underline{\mathcal{A}}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \tag{4.19}$$

Thus, the assignment $\mathcal{O} \to \underline{\mathcal{A}}(\mathcal{O})$ defines a net of C^*–algebras over the Minkowski space, which satisfies also locality and Poincaré covariance (see [12] for details).

4.1.3 Definition. *The local, covariant net* $(\underline{\mathcal{A}}, \underline{\alpha}_{\mathcal{P}_+^\uparrow})$ *is called the* scaling net *of the underlying theory. The* C^**–inductive limit of all the local algebras* $\underline{\mathcal{A}}(\mathcal{O})$ *is called* scaling algebra *and is denoted by* $\underline{\mathcal{A}}$.

The principal objective is then to study the properties of the physical states of the underlying theory at arbitrarily small scales. To this end, one lifts these states to the scaling algebra and study their behaviour under scaling transformations.

4.1.4 Definition. *Let* ω *be a state on the underlying global algebra* $\mathcal{A}$. *Its* canonical lift $\underline{\omega}$ *on the scaling algebra* $\underline{\mathcal{A}}$ *is defined by*

$$\underline{\omega}(\underline{a}) := \omega(a_{\lambda=1}), \quad \underline{a} \in \underline{\mathcal{A}}. \tag{4.20}$$

Conversely, given any state $\underline{\omega}$ on the scaling algebra $\underline{\mathcal{A}}$, let $(\underline{\pi}, \underline{\mathcal{H}}, \underline{\Omega}))$ be the corresponding GNS–representation, and denote by

$$\underline{\mathcal{A}}^{\underline{\pi}} := \underline{\mathcal{A}}/\ker(\underline{\pi}) \tag{4.21}$$

the quotient of $\underline{\mathcal{A}}$ w.r.t. the kernel $\ker(\underline{\pi})$ of $\underline{\pi}$. Then, $\underline{\mathcal{A}}^{\underline{\pi}} \simeq \underline{\pi}(\underline{\mathcal{A}})$. Let us denote by ψ the canonical isomorphism between these algebras, given by $\psi(\underline{A}^{\underline{\pi}}) = \underline{\pi}(\underline{A})$, where $\underline{A}^{\underline{\pi}}$ is the equivalence class of $\underline{A} \in \underline{\mathcal{A}}$ modulo $\ker(\underline{\pi})$. The projection of $\underline{\omega}$ to the quotient $\underline{A}^{\underline{\pi}}$ is then given by

$$\mathrm{proj}(\underline{\omega}) := (\underline{\Omega}, \psi(\cdot)\underline{\Omega}) \tag{4.22}$$

The physical interpretation of the states $\underline{\omega}$ will be based on their projections $\mathrm{proj}(\underline{\omega})$, regarded as states on the net

$$\mathcal{O} \to \underline{\mathcal{A}}^{\underline{\pi}}(\mathcal{O}) := \underline{\mathcal{A}}(\mathcal{O})/\ker(\underline{\pi}) \tag{4.23}$$

on the Minkowski space. These nets are again local. Moreover, if $\ker(\underline{\pi})$ is invariant under the Poincaré transformations $\underline{\alpha}_{\mathcal{P}_+^\uparrow}$, one can also define an automorphic action of the Poincaré group on $\underline{\mathcal{A}}^{\underline{\pi}}$, setting for $(\Lambda, x) \in \mathcal{P}_+^\uparrow$

$$\alpha^{\underline{\pi}}_{(\Lambda,x)}(\underline{a}^{\underline{\pi}}) := (\underline{\alpha}_{(\Lambda,x)}(\underline{a}))^{\underline{\pi}} \tag{4.24}$$

In this way, any suitable state $\underline{\omega}$ on $\underline{\mathcal{A}}$ determines a local, covariant net $(\underline{\mathcal{A}}^{\underline{\pi}}, \underline{\alpha}^{\underline{\pi}}_{\mathcal{P}_+^\uparrow})$, and a distinguished state $\mathrm{proj}(\underline{\omega})$ on $\underline{\mathcal{A}}^{\underline{\pi}}$.

Now, let us consider the family of states on $\underline{\mathcal{A}}$ given by

$$\underline{\omega}_\lambda := \underline{\omega} \circ \underline{\sigma}_\lambda, \tag{4.25}$$

where $\underline{\sigma}_\lambda : \underline{\mathcal{A}}(\mathcal{O}) \to \underline{\mathcal{A}}(\lambda\mathcal{O})$ are the scaling transformations, as a net directed towards $\lambda = 0$. The aim is to determine the properties of states at small scales with the help of the functions $\underline{a} \in \underline{\mathcal{A}}$. However, one finds that $\cap_{\lambda>0}\mathcal{A}(\lambda\mathcal{O})^- = \mathbb{C} \cdot I$, hence any function $\underline{a}$, such that a_λ converges in norm for $\lambda \searrow 0$, inevitably converges to a multiple of the identity. Consequently, such functions are not suitable to test the properties of states in the scaling limit, since they have the same limit in every state on $\mathcal{A}$. This is the reason why one does not assume from the outset that the elements of $\underline{\mathcal{A}}$ are continuous at $\lambda = 0$. As a consequence, the nets $\{\underline{\omega}_\lambda\}_{\lambda>0}$ are not convergent. This apparent difficulty can be handled, however, with the help of the Banach–Alaoglu Theorem [55], according to which every bounded set in the dual space of a Banach space is precompact in the w^*–topology. Applying this theorem to the family of states $\{\underline{\omega}_\lambda\}_{\lambda>0}$, one sees that this family contains (many) subnets which converge in the w^*–topology for $\lambda \searrow 0$.

4.1.5 Definition. *Let* ω *be a state on* $\mathcal{A}$ *and* $\underline{\omega}$ *its canonical lift on* $\underline{\mathcal{A}}$*. Each* w^**–limit point of the net* $\{\underline{\omega}_\lambda\}_{\lambda>0}$ *for* $\lambda \searrow 0$ *is called a* scaling limit state *of* $\underline{\omega}$*. The scaling limit states of* $\underline{\omega}$ *are denoted by* $\underline{\omega}_{0,\iota}$*,* $\iota \in \mathbb{I}$*, with* $\mathbb{I}$ *some index set, and the set of all scaling limit states of* $\underline{\omega}$ *is denoted by* $SL(\underline{\omega})$*.*

Recall that a *physical state* on the underlying thory $\mathcal{A}$ is defined to be a state which is locally normal w.r.t. the underlying vacuum representation. Then, the underlying theory is said to have a *unique scaling limit*, if all the scaling limit nets derived from physical states on $\mathcal{A}$, are isomorphic, and if there is a net–isomorphism which connect also the respective vacuum states, the theory is said to have a *unique vacuum structure* in the scaling limit.

4.2 The dist_{qGH^*}–Ultraproduct Construction

In this section, we shall illustrate the approach based on the ultraproduct construction. In this setting, the scaling net will be given by a limit point in the distance dist_{qGH^*}, of a suitable sequence of local von Neumann algebras through the ultraproduct construction. First, we have to analyse the requirements needed for a local net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ to be a (strongly) uniform family of rigged von Neumann algebras.

To this end, let $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ be a local net of von Neumann algebras acting in the vacuum representation, H the generator of time translations, φ a suitable postive unbounded function, and set

$$L'(a) := \|\varphi(H+1)^{-1} a \varphi(H+1)^{-1}\|. \tag{4.26}$$

In the following, we shall investigate the conditions under which the net $\mathcal{O} \to (\mathcal{R}(\mathcal{O}), L')$, where $\mathcal{O} \subset \mathbb{R}^4$ is a bounded region in Minkowski spacetime and L' is given by (4.26), is a local net of rigged von Neumann algebras.

We begin by showing first that L' is a Lip–norm. To this aim, we will use the following result by Buchholz and Porrmann [11].

4.2.1 Theorem. *Let* $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ *be the net of von Neumann algebras for the free scalar field (any mass) in dimension* $3+1$*. Then, for any bounded region* $\mathcal{O}$ *and any* $\beta > 0$*, the following map is compact:*

$$\mathcal{R}(\mathcal{O}) \ni a \mapsto e^{-\beta H} a e^{-\beta H} \in \mathcal{B}(\mathcal{H}). \tag{4.27}$$

As known, a linear map between Banach spaces is *compact* if the image K of the unit ball is totally bounded, namely if, for any $\varepsilon > 0$, the number of balls of radius ε needed to cover K is finite. We shall denote, as usual, by $n_\varepsilon(K)$ such (minimal) number.

4.2.2 Theorem. *Let* $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ *be a net of local observables, and assume that, for a given region* $\mathcal{O}$*, the map 4.27 is compact for* $\beta = 1$*, namely* $n_\varepsilon := n_\varepsilon(e^{-H}\mathcal{R}(\mathcal{O})_1 e^{-H})$ *is finite for any* $\varepsilon > 0$*. Choose a function* $\varphi \in C([0, +\infty))$ *with the following properties:*

- φ *is increasing;*
- $\varphi(0) = 1$*;*
- $\lim_{t\to+\infty} \varphi(t) = +\infty$*.*

Then, the following map is also compact:

$$\mathcal{R}(\mathcal{O}) \ni a \mapsto \varphi(H)^{-1} a \varphi(H)^{-1} \in \mathcal{B}(\mathcal{H}). \tag{4.28}$$

Proof. Choose $\varepsilon > 0$, let $T \equiv T_\varepsilon$ such that $\varphi(T) = 3/\varepsilon$, and set $c \equiv c_\varepsilon := \min_{t\in[0,T]} \varphi(t)e^{-t}$. Consider now the function $\psi(t) := \max(\varphi(t), ce^t)$, and observe that $\psi(t) = \varphi(t)$ for $t \leq T$. Therefore

$$0 \leq \sup_{t\geq 0}(\varphi(t)^{-1} - \psi(t)^{-1}) \leq \sup_{t\geq T} \varphi(t)^{-1} = \frac{\varepsilon}{3}.$$

Since $\psi(t) \geq ce^t$, the map $a \mapsto \psi(H)^{-1} a \psi(H)^{-1}$ is compact, thus we can find $x_1, \ldots, x_n \in \psi(H)^{-1}\mathcal{R}(\mathcal{O})_1\psi(H)^{-1}$, $n \equiv n_{\varepsilon/3}(\psi(H)^{-1}\mathcal{R}(\mathcal{O})_1\psi(H)^{-1})$, such that

$$\bigcup_{i=1}^{n} B(x_i, \varepsilon/3) \supset \psi(H)^{-1}\mathcal{R}(\mathcal{O})_1\psi(H)^{-1}.$$

In this way, for any $a \in \mathcal{R}(\mathcal{O})_1$, we get an $i \in \{1, \ldots, n\}$ such that

$$\|\psi(H)^{-1} a \psi(H)^{-1} - x_i\| < \varepsilon/3.$$

Then, we have

$$\begin{aligned} & \|\varphi(H)^{-1} a \varphi(H)^{-1} - x_i\| \\ \leq\ & \|\psi(H)^{-1} a \psi(H)^{-1} - x_i\| + \|\psi(H)^{-1} a \psi(H)^{-1} - \varphi(H)^{-1} a \varphi(H)^{-1}\| \\ \leq\ & \varepsilon/3 + \|(\psi(H)^{-1} - \varphi(H)^{-1}) a \psi(H)^{-1}\| + \|\varphi(H)^{-1} a (\psi(H)^{-1} - \varphi(H)^{-1})\| \\ \leq\ & \varepsilon/3 + 2\|\psi(t)^{-1} - \varphi(t)^{-1}\|_\infty \leq \varepsilon, \end{aligned}$$

which means that

$$n_\varepsilon(\varphi(H)^{-1}\mathcal{R}(\mathcal{O})_1\varphi(H)^{-1}) \leq n_\varepsilon(\psi(H)^{-1}\mathcal{R}(\mathcal{O})_1\psi(H)^{-1}),$$

which implies the thesis. ∎

4.2.3 Remark. For example, the functions $\varphi(t) = (1+t)^n$, $n \geq 1$ satisfy the hypothesis above.

For completeness, we give also an estimate on the order of compactness of the map (4.28).

4.2.4 Lemma. *Let X, Y be Banach spaces, $S, T : X \to Y$ linear maps such that T is compact and, for a suitable constant $c > 0$, $\|Sx\| \leq c\|Tx\|$ for any $x \in X$. Then*

$$n_\varepsilon(SX_1) \leq n_{\varepsilon/c}(TX_1).$$

Proof. By hypothesis, ST^{-1} is bounded on TX with norm $\leq c$, hence we may extend it to a bounded operator R from $\overline{TX}$ to Y, and thus $S = RT$. For a given $\varepsilon > 0$, choose $x_1, \ldots, x_n \in TX_1$, $n \equiv n_{\varepsilon/c}(TX_1)$, such that

$$\bigcup_{i=1}^{n} B(x_i, \varepsilon/c) \supset TX_1.$$

In this way, for any $x \in X_1$ we get an $i \in \{1, \ldots, n\}$ such that $\|Tx - x_i\| < \varepsilon/c$. Then $\|Sx - Rx_i\| \leq c\|Tx - x_i\| < \varepsilon$, namely

$$\bigcup_{i=1}^{n} B(Rx_i, \varepsilon) \supset SX_1,$$

which implies the thesis. ∎

4.2.5 Corollary. *Let $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ be a net of local observables, and assume that, for a given region $\mathcal{O}$, the map (4.27) is compact for $\beta = 1$. Then, for ε small enough,*

$$n_\varepsilon((H+1)^{-n}\mathcal{R}(\mathcal{O})_1(H+1)^{-n}) \le n_\delta(e^{-H}\mathcal{R}(\mathcal{O})_1 e^{-H}), \quad \delta = \frac{3}{\varepsilon}e^{-2(\sqrt[n]{3/\varepsilon}-1)}.$$

Proof. With the same notations as in the proof of Theorem 4.2.2, we have

$$\|\psi(H)^{-1}a\psi(H)^{-1}\| \le c^{-2}\|e^{-H}ae^{-H}\|,$$

and by the previous Lemma, we get

$$n_{\varepsilon/3}(\psi(H)^{-1}\mathcal{R}(\mathcal{O})_1\psi(H)^{-1}) \le n_\delta(e^{-H}\mathcal{R}(\mathcal{O})_1 e^{-H}),$$

with $\delta = \frac{\varepsilon}{3}c^2$. Now, choosing $\varphi(t) = (1+t)^n$, $n \ge 1$, and ε such that $e^t \ge (1+t)^n$ for $t \ge T \equiv \sqrt[n]{3/\varepsilon} - 1$, then, setting $c = (1+T)^n e^{-T}$, we get

$$n_\varepsilon((H+1)^{-n}\mathcal{R}(\mathcal{O})_1(H+1)^{-n}) \le n_\delta(e^{-H}\mathcal{R}(\mathcal{O})_1 e^{-H}), \quad \delta = \frac{3}{\varepsilon}e^{-2(\sqrt[n]{3/\varepsilon}-1)}.$$

∎

Now, let us recall the definition of (p–)nuclear maps.

4.2.6 Definition. *Let X and Y be Banach spaces, and let Θ be a linear map from X into Y.*

(i) *The map Θ is said to be* compact, *if the image of the unit ball of X through Θ is totally bounded in Y.*

(ii) *The map Θ is said to be* p–nuclear, $p \in R_+$, *if there exist functionals $e_i \in X^*$ and elements $y_i \in Y$, such that, in the sense of strong convergence,*

$$\Theta(\cdot) = \sum_i e_i(\cdot)y_i, \tag{4.29}$$

and

$$\|\Theta\|_p = \inf\left(\sum_i \|e_i\|^p\|y_i\|^p\right)^{1/p}, \tag{4.30}$$

where the infimum is taken w.r.t. all the possible choices of $e_i \in X^*$ and $y_i \in Y$ in the representation of Θ as in (4.29). The norm $\|\cdot\|$ in (4.30) is called the p–norm, but it is only a quasi–norm if $p < 1$. For $p = 1$ one obtains the nuclear maps.

In order to show that the norm (4.26) metrizes the w^*–topology on bounded subsets, we shall need a notion of uniformity, w.r.t. the (scaling) parameter, of the maps involved.

4.2.7 Definition. *Let X and Y be Banach spaces, and let Θ_r, $r \in \mathbb{R}_+(\equiv \{x \in \mathbb{R} : x > 0\})$, be linear maps from X into Y.*

(i) *The family of maps Θ_r is said to be* uniformly compact, *if the images of the unit ball of X through Θ_r are uniformly totally bounded in Y, i.e. for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that the covering numbers $n_{\varepsilon,r} := n_\varepsilon(\Theta_r(X_1))$ satisfy $n_{\varepsilon,r} \le N$ for all $r > 0$.*

(ii) *The family of maps* Θ_r *is said to be* uniformly (p)–nuclear, $p \in R_+$, *if* Θ_r *is nuclear for all* $r > 0$, *and there exists a continuous function* $F : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+$ *such that*

$$\|\Theta_r\|_p \leq F(r). \tag{4.31}$$

Let $\mathcal{O}_r$ be the standard double cone of radius $r > 0$ around the origin.

4.2.8 Assumption (Uniform nuclearity). *We shall assume that* $\forall r_0 > 0$ *there exists* $d > 0$ *such that* $\forall r \leq r_0$, *with* $r/\beta \leq d$, *the maps*

$$\Xi_{\beta,r} : \mathcal{R}(\mathcal{O}_r) \to \mathcal{B}(\mathcal{H}), \quad a \mapsto e^{-\beta H} a e^{-\beta H} \tag{4.32}$$

are nuclear, uniformly in r/β.

Actually, it suffices the weaker condition of uniform compactness.

4.2.9 Assumption (Uniform compactness). *The maps*

$$\begin{array}{rcl} \Theta_r : \mathcal{R}(\mathcal{O}_r) & \to & \mathcal{B}(\mathcal{H}), \\ a & \mapsto & (I + rH)^{-1} a (I + rH)^{-1} \end{array} \tag{4.33}$$

are uniformly compact w.r.t. r.

4.2.10 Theorem. *Uniform nuclearity implies uniform compactness.*

Proof. It follows from Corollary 4.2.5. ∎

Le us notice that uniform compactness seems to be a reasonable assumption, since it holds, for instance, in the (real scalar) free field case, as we shall see in the following section. For the moment, we only quote this result.

4.2.11 Theorem. *Uniform nuclearity holds for the real scalar free field of mass* $m \geq 0$ *in* $s \geq 3$ *spatial dimensions.*

4.2.12 Lemma. *Let* $A : X \to Y$ *a compact operator between Banach spaces, and assume that* Y *is separable and* $A^* : Y^* \to X^*$ *is injective. Then, setting* $L'(y') = \|A^* y'\|$, L' *is a norm on* Y^*, *inducing the* w^**–topology on* Y_1^*.

Proof. Since A^* is injective, L' is a norm. We now show that any w^*–compact set C is L'–closed. Indeed, a sequence $\{y'_n\} \subset C$ is L'–converging if and only if $A^* y'_n$ is norm–converging in X^*. So, let x' be its limit. We may assume, by possibly passing to a subsequence, that $\{y'_n\}$ is also w^*–converging to some $y' \in C$. Then, for any $x \in X$,

$$\langle x', x \rangle = \lim_n \langle A^* y'_n, x \rangle = \lim_n \langle y'_n, Ax \rangle = \langle y', Ax \rangle = \langle A^* y', x \rangle,$$

namely $A^* y'_n \to A^* y'$ in norm, or, equivalently, $y'_n \to y'$ in the L'–norm. As a consequence, the identity map from $\iota : (C, L') \to (C, w^*)$ is a continuous map.
Now observe that, since C is w^*–compact, it is bounded, and since A^* is compact, $A^* C$ is totally bounded, or, equivalently, C is L'–totally bounded. By the observation above, C is L'–compact, namely ι is a homeomorphism. ∎

The following Theorem tells us that $(\mathcal{R}(\mathcal{O}_r), L'_r)$, $r > 0$, is a RvNA w.r.t. the norm

$$N_r(a) = \max\left(\|(I + rH)a(I + rH)^{-1}\|, \|(I + rH)^{-1}a(I + rH)\|\right). \tag{4.34}$$

4.2.13 Theorem. *Assume that the net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ satisfies uniform compactness. Then, setting $L'_r(a) = \|(I + rH)^{-1}a(I + rH)^{-1}\|$, $(\mathcal{R}(\mathcal{O}_r), L'_r)$ is a RvNA, with $N_r(a) \leq \|a\| + r\|[H, a]\|$.*

Proof. Indeed, let $A_r : \mathcal{B}(\mathcal{H})_* \to \mathcal{R}(\mathcal{O}_r)_*$ be given by $A_r\phi = (I + rH)^{-1}\phi(I + rH)^{-1}|_{\mathcal{R}(\mathcal{O}_r)}$, and observe that $A_r^* : \mathcal{R}(\mathcal{O}_r) \to \mathcal{B}(\mathcal{H})$ is given by $A_r^*a = (I + rH)^{-1}a(I + rH)^{-1}$. By uniform compactness, A_r and A_r^* are compact, uniformly in r.
We now set $L'_r(a) = \|A_r^*a\| = \|(I + rH)^{-1}a(I + rH)^{-1}\|$. By Lemma 4.2.12, it is not difficult to see that $(\mathcal{R}(\mathcal{O}_r), L'_r)$ is a Lip–von Neumann algebra. Indeed,

$$N_r(a) = \max\left(\|(I + rH)a(I + rH)^{-1}\|, \|(I + rH)^{-1}a(I + rH)\|\right) \leq \|a\| + r\|[H, a]\|,$$

since we have

$$\begin{aligned} \|(I + rH)^{-1}a(I + rH)\| &= \|(I + rH)^{-1}(a + rHa - r[H, a])\| \\ &= \|a - (I + rH)^{-1}r\delta(a)\| \leq \|a\| + r\|(I + rH)^{-1}[H, a]\| \\ &\leq \|a\| + r\|[H, a]\|, \end{aligned}$$

and analogously $\|(I + rH)a(I + rH)^{-1}\| \leq \|a\| + r\|[H, a]\|$. Since H is the generators of a strongly continuous one–parameter group of automorphisms, the set of elements $a \in \mathcal{R}(\mathcal{O}_r)$ with $r\|[H, a]\| < \infty$ is w^*–dense [49; 9], and thus we get the claim. ∎

Finally, we want to show that the net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ form a (strongly) uniform family of RvNA's. Assuming that the uniform compactness condition holds true, in order to get the uniform normalizer condition, we shall make the following

4.2.14 Assumption (Uniform inner regularity). *For any family $a_\lambda \in \mathcal{R}(\mathcal{O}_\lambda)$, $\sup_\lambda \|a_\lambda\| \leq 1$, and for any $\varepsilon > 0$, there exist an $r < 1$ and a family $a'_\lambda \in \mathcal{R}(\mathcal{O}_{r\lambda})$, $\sup_\lambda \|a'_\lambda\| \leq 1$, such that*

$$L'_\lambda(a_\lambda - a'_\lambda) = \|(I + \lambda H)^{-1}(a_\lambda - a'_\lambda)(I + \lambda H)^{-1}\| < \varepsilon. \tag{4.35}$$

As we shall see, uniform inner regularity holds in the (real scalar) free field case.

4.2.15 Remark. Let us recall that a (local) net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ is said to be *inner regular*, if the following relation holds:

$$\mathcal{R}(\mathcal{O}) = \left(\cup_{\hat{\mathcal{O}} \subset\subset \mathcal{O}} \mathcal{R}(\hat{\mathcal{O}})\right)'', \tag{4.36}$$

where $\hat{\mathcal{O}} \subset\subset \mathcal{O}$ means that the closure of $\hat{\mathcal{O}}$ is contained in the interior of $\mathcal{O}$. (Notice that this conditions is fulfilled, for instance, in the free field case.) This implies, in particular, that each element in $\mathcal{R}(\mathcal{O})$ can be approximated in the weak–operator topology by elements from $\mathcal{R}(\hat{\mathcal{O}})$. Since the weak–operator topology coincides with the w^*–topology on bounded subsets, we see that uniform inner regularity is simply a strengthening of inner regularity (restricted to the unit balls of the local algebras).

Finally, we can show that under the uniform compactness and the uniform inner regularity assumptions, the rescaled family $(\mathcal{R}(\mathcal{O}_\lambda), L'_\lambda)$ of RvNA's is (strongly) uniform (cf. Definition 2.4.7).

4.2.16 Theorem. *Let $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ be a local net as above, satisfying uniform compactness and uniform inner regularity. Then, for any $\mathcal{O}$, the family $\mathcal{O}_\lambda \to \mathcal{R}(\mathcal{O}_\lambda)$ is a (strongly) uniform family of rigged von Neumann algebras. Therefore, for any ultrafilter $\mathcal{U}$, the Lip–vNA's $\mathcal{R}(\mathcal{O})_\mathcal{U} =$* qGH* $-\lim_\mathcal{U} \mathcal{R}(\mathcal{O}_\lambda)$ *form a net, which we call the* qGH*–scaling limit net.

Proof. By uniform inner regularity, any family $a_\lambda \in \mathcal{R}(\mathcal{O}_\lambda)$ may be uniformly L'–approximated by a family $a'_\lambda \in \mathcal{R}(\mathcal{O}_{r\lambda})$. The latter will be approximated by

$$a^\delta_\lambda := \int \frac{1}{\delta} g(\frac{t}{\delta}) \alpha_{\lambda t}(a'_\lambda) dt,$$

where g is a positive function with integral 1 and support contained in $[-1, 1]$. We have $\|a^\delta_\lambda\| \leq 1$, $N_\lambda(a^\delta_\lambda) \leq 1 + \frac{1}{\delta}\int |g'(t)|dt$ and $a^\delta_\lambda \in \mathcal{R}(\mathcal{O}_\lambda)$ as soon as $r + \delta \leq 1$. Now, we estimate $L'_\lambda(a'_\lambda - a^\delta_\lambda)$. First observe that, setting $a := a'_\lambda$, $a^\delta := a^\delta_\lambda$, $g_\delta(t) := \frac{1}{\delta} g(\frac{t}{\delta})$, and letting $E(x)$ be the spectral measure of H, we have

$$\begin{aligned}(I + \lambda H)^{-1} a^\delta (I + \lambda H)^{-1} &= \int g_\delta(t)(I + \lambda H)^{-1} e^{-i\lambda t H} a e^{i\lambda t H}(I + \lambda H)^{-1} dt \\ &= \int g_\delta(t) \left(\int (1 + \lambda x)^{-1} e^{-i\lambda t x} dE(x) \right) a \left(\int e^{i\lambda t y}(1 + \lambda y)^{-1} dE(y) \right) dt \\ &= \int g_\delta(t) \left(\int (1 + x)^{-1} e^{-itx} dE(x/\lambda) \right) a \left(\int e^{ity}(1 + y)^{-1} dE(y/\lambda) \right) dt.\end{aligned}$$

Then, we see that

$$\int (1 + x)^{-1} e^{-itx} dE(x/\lambda) = \int e^{-itx} \frac{1 + it(1 + x)}{(1 + x)^2} E(x/\lambda) dx.$$

As a consequence,

$$\begin{aligned}&(I + \lambda H)^{-1} a^\delta (I + \lambda H)^{-1} \\ &= \int dx\, dy\, \frac{E(x/\lambda) a E(y/\lambda)}{(1 + x)^2(1 + y)^2} \left(\widetilde{g_\delta}(x - y) + (x - y)\widetilde{g_\delta}'(x - y) + (1 + x)(1 + y)\widetilde{g_\delta}''(x - y) \right).\end{aligned}$$

Hence, we get

$$\begin{aligned}&\|(I + \lambda H)^{-1}(a^\delta - a)(I + \lambda H)^{-1}\| \\ &\leq \|a\| \left(\int_{\mathbb{R}^2_+} \frac{|\widetilde{g_\delta}(x - y) - 1|}{(1 + x)^2(1 + y)^2} dx\, dy + \int_{\mathbb{R}^2_+} \frac{|(x - y)\widetilde{g_\delta}'(x - y)|}{(1 + x)^2(1 + y)^2} dx\, dy + \int_{\mathbb{R}^2_+} \frac{|\widetilde{g_\delta}''(x - y)|}{(1 + x)(1 + y)} dx\, dy \right)\end{aligned}$$

Setting $\beta := x - y$, and assuming that g is an even function, the first integral may be rewritten as

$$2 \int_0^\infty d\beta |\widetilde{g_\delta}(\beta) - 1| h(\beta), \quad \text{where} \quad h(\beta) = \frac{1}{|\beta|^3} \left(\frac{|\beta|(|\beta| + 2)}{(|\beta| + 1)} - 2\log(|\beta| + 1) \right),$$

where $\int_0^\infty d\beta\, 2h(\beta) = 1$. Changing variable, the first integral may finally be rewritten as

$$2 \int_0^\infty d\beta |\widetilde{g}(\beta) - 1| \frac{1}{\delta} h(\frac{\beta}{\delta}) \to 0 \quad \text{when} \;\; \delta \to 0,$$

since $\frac{1}{\delta}h(\frac{\beta}{\delta})$ approximates the Dirac delta in 0, and $\widetilde{g}(0) = 1$. With the same calculations, we get, for the second integral,

$$2\int_0^\infty |\widetilde{g_\delta}'(\beta)|\beta h(\beta)d\beta = 2\int_0^\infty \delta|\widetilde{g}'(\delta\beta)|\beta h(\beta)d\beta,$$

and the latter converges to the average of $\beta h(\beta)$ on $\mathbb{R}$, which is 0. As for the third integral, we get, for a suitable bounded function $k(\beta)$,

$$2\int_0^\infty |\widetilde{g_\delta}''(\beta)|k(\beta)d\beta = 2\delta\int_0^\infty \delta|\widetilde{g}''(\delta\beta)|k(\beta)d\beta \le 2\delta\|\widetilde{g}''\|_1\|k\|_\infty \to 0.$$

In view of Theorem 4.2.13, the uniform normalizer condition is proven, and so the family is (strongly) uniform. Given a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \searrow 0$, and an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $\mathcal{R}(\mathcal{O})_\mathcal{U} = \mathrm{qGH}^* - \lim_\mathcal{U} \mathcal{R}(\mathcal{O}_{\lambda_n})$ be the ultraproduct of $\mathcal{R}(\mathcal{O}_{\lambda_n})$ over $\mathcal{U}$. Since the net structure clearly passes to the ultraproduct, the proof is complete. ∎

4.2.1 Relations with the Buchholz-Verch Construction

We want to analyse the relation between our construction and the Buchholz-Verch construction of the scaling algebra w.r.t. the renormalization group (scaling) transformations.

The scaling algebra

We define the qGH*–scaling algebra as follows: since $N_\lambda(a) \le \|a\| + \lambda\|[H,a]\|$,

$$\begin{aligned} \underline{\mathcal{A}}(\mathcal{O}) &:= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda N_\lambda(a_\lambda) < \infty\}^{-\|\cdot\|} \\ &= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \lambda\|[H,a]\| < \infty\}^{-\|\cdot\|} \\ &= \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \|\alpha_{\lambda t}(a_\lambda) - a_\lambda\| \to 0,\, t\in\mathbb{R},\, t\to 0\}. \end{aligned} \tag{4.37}$$

Instead, the Buchholz-Verch scaling algebra is the following (see Definition 4.1.2):

$$\underline{\mathcal{A}}(\mathcal{O})_{BV} = \{\{a_\lambda\} \in \ell^\infty(\mathcal{R}(\lambda\mathcal{O})) : \sup_\lambda \|\alpha_{\lambda x}(a_\lambda) - a_\lambda\| \to 0,\, x\in\mathbb{R}^4,\, x\to 0\}. \tag{4.38}$$

4.2.17 Remark. Since $\sup_\lambda \|\alpha_{\lambda x}(a_\lambda) - a_\lambda\| \to 0$ as $x \to 0$ implies $\sup_\lambda \|\alpha_{\lambda t}(a_\lambda) - a_\lambda\| \to 0$ as $t \to 0$, one has $\underline{\mathcal{A}}(\mathcal{O})_{BV} \subseteq \underline{\mathcal{A}}(\mathcal{O})$. Notice that the reverse inclusion needs not to hold in general. Nevertheless, on the level of von Neumann algebras the two constructions produce the same results, at least in the case of the (real scalar) free field. In fact, let π_λ, $\lambda > 0$, be the representation of $\underline{\mathcal{A}}(\mathcal{O})$ on $\mathcal{H}$ given by $\underline{a} \mapsto a_\lambda$, and let π_0 be the Buchholz–Verch limit representatioin (extended to $\underline{\mathcal{A}}(\mathcal{O})$). Then, we ask that the following holds true:

(a) for any $\lambda > 0$,
$$(\pi_\lambda\underline{\mathcal{A}}(\mathcal{O})_{BV})'' = \pi_\lambda\underline{\mathcal{A}}(\mathcal{O})'' = \mathcal{R}(\lambda\mathcal{O}); \tag{4.39}$$

(b)
$$(\pi_0\underline{\mathcal{A}}(\mathcal{O})_{BV})'' = \pi_0\underline{\mathcal{A}}(\mathcal{O})''. \tag{4.40}$$

As for (a), since elements $\underline{a}$ of $\underline{\mathcal{A}}(\mathcal{O})_{BV}$ can be obtained by smearing operators from the original net over the Poincaré group with suitable test functions of arbitrarily small support (cf. the discussion in [12; 13]), then, choosing a delta sequence as test functions, $\pi_\lambda(\underline{a})$ converges to the original operator from $\mathcal{R}(\mathcal{O})$ in the strong operator topology. If we assume that the the net $\mathcal{O} \to \mathcal{R}(\mathcal{O})$ is inner regular, namely

$$\mathcal{R}(\mathcal{O}) = \left(\cup_{\hat{\mathcal{O}}\subset\subset\mathcal{O}}\mathcal{R}(\hat{\mathcal{O}})\right)'',$$

(which is fulfilled in the free field case), then we obtain $(\pi_\lambda\underline{\mathcal{A}}(\mathcal{O})_{BV})'' = \mathcal{R}(\lambda\mathcal{O})$. Since the inclusion $(\pi_\lambda\underline{\mathcal{A}}(\mathcal{O})_{BV})'' \subset \pi_\lambda\underline{\mathcal{A}}(\mathcal{O})'' \subset \mathcal{R}(\lambda\mathcal{O})$ clearly holds, we get the equality.
As for (b), it is not evident whether it holds in general. However, for the only models where the scaling limit has been computed so far, i.e. the free scalar field in $2+1$, $3+1$ spacetime dimensions, it actually holds true. Indeed, in [13] it is shown that $\pi_0\underline{\mathcal{A}}(\mathcal{O})_{BV} = \mathcal{A}_0(\mathcal{O})$, where $\mathcal{O} \to \mathcal{A}_0(\mathcal{O})$ is the local net of the free massless scalar field, and the main argument in the proof of the inclusion $\pi_0\underline{\mathcal{A}}(\mathcal{O})_{BV} \supset \mathcal{A}_0(\mathcal{O})$ (see the proof of Lemma 3.3 in [13]), is compactness in the sense of Haag–Swieca [34], which involves the energy operator, i.e. the properties of the time translations group only. Hence, the same arguments can be applied to the possibly larger algebra $\pi_0\underline{\mathcal{A}}(\mathcal{O})$, so that one gets $(\pi_0\underline{\mathcal{A}}(\mathcal{O}))'' = \overline{\mathcal{A}_0(\mathcal{O})}$, the strong closure of $\mathcal{A}_0(\mathcal{O})$.
In conclusion, it is worth noting that, in order to get the same local net on the level of C^*–algebras as well, one should modify the definition of qGH*–limit to include a larger symmetry group rather than time translations only, as the strongly continuous unitary implementation of the symmetry group in the limit representation plays an important role in physics. However, while the Buchholz–Verch construction is essentially based on the C^*–algebraic nature of the limiting process, our construction involves instead the von Neumann algebraic aspects of the local theory. Therefore, so far, the problem of selecting, via the action of the symmetry group, a particular weakly dense C^*–subnet of the original (Poincarè covariant) net of local von Neumann algebras, does not appear to affect, in a relevant manner, the study of the small scales behavior of the theory. This is the reason why, basically for mathematical simplicity, in our analysis we restrict to the subgroup given by time–translations, assuming the (norm) continuity of the (operator–valued) functions $\lambda \mapsto a_\lambda$ only w.r.t. this subgroup.

The representation

The qGH*–representation is defined as follows:

$$\mathcal{R}(\mathcal{O})_{\mathcal{U}} = \bigoplus_{\underline{\omega}\in\underline{\mathcal{S}}_{\mathcal{U}}} \pi_{\underline{\omega}}(\underline{\mathcal{A}}(\mathcal{O}))'', \tag{4.41}$$

where $\underline{\mathcal{S}}_{\mathcal{U}} \ni \underline{\omega}$ if $\langle\underline{\omega}, \underline{a}\rangle = \lim_{\mathcal{U}}\langle\omega_\lambda, a_\lambda\rangle$, with $\{\omega_\lambda\}$ in $\mathcal{B}(\mathcal{H})_*$ and $L(\{\omega_\lambda\}) = \sup_\lambda L_\lambda(\omega_\lambda) < \infty$ (cf. Theorem 2.4.9), where, for any $\lambda > 0$,

$$\begin{aligned} L_\lambda(\omega_\lambda) &:= \sup\{\tfrac{|\langle\omega_\lambda, x\rangle|}{L'_\lambda(x)} : x \in \mathcal{R}(\mathcal{O}_\lambda)\} \\ &= \sup\{\tfrac{|\langle\omega_\lambda, (I+\lambda H)y(I+\lambda H)\rangle|}{\|y\|} : y \in (I+\lambda H)^{-1}\mathcal{R}(\mathcal{O}_\lambda)(I+\lambda H)^{-1}\}. \end{aligned}$$

Instead, the Buchholz-Verch representations are the following:

$$\mathcal{R}(\mathcal{O})_{\mathcal{U}} = \pi_{\underline{\omega}}(\underline{\mathcal{A}}(\mathcal{O})_{BV})'', \tag{4.42}$$

where $\langle \underline{\omega}, \underline{a} \rangle = \lim_{\mathcal{U}} \langle \omega, a_\lambda \rangle$, $\omega \in \mathcal{B}(\mathcal{H})_*$.

Since $\pi_{\underline{\omega}}(\underline{\mathcal{A}}(\mathcal{O})_{BV})'' = \pi_{\underline{\omega}}(\underline{\mathcal{A}}(\mathcal{O}))''$, we have the following

4.2.18 Corollary (to Theorem 4.2.16). *With the assumptions above, the qGH*–scaling limit net $\mathcal{O} \to \mathcal{R}(\mathcal{O})_{\mathcal{U}}$ is local, and any Buchholz–Verch scaling limit net of the original theory embeds as a subrepresentation in the qGH*–scaling limit net associated with some ultrafilter.*

4.3 Applications to the (real scalar) Free Field

In their second paper [13], Buchholz and Verch computed the scaling limit theories of the free scalar fields of any mass $m \geq 0$ in $s = 2, 3$ space dimension, which turns out to be the massless free scalar field in the same spacetime dimensions. In the following, we shall illustrate this result, but first we need to recall some notations and definitions. So, let us consider the Weyl algebra $\mathcal{W}^\circ$ over $\mathcal{D}(\mathbb{R}^s)$, $s = 2, 3$, the space of complex–valued test–functions with compact support in the configuration space $\mathbb{R}^s$, namely the $*$–algebra generated by the unitary operators $W(f)$, $f \in \mathcal{D}(\mathbb{R}^s)$ obeying the Weyl relations:

$$W(f)W(g) = e^{-\frac{i}{2}\sigma(f,g)} W(f+g), \quad f, g \in \mathcal{D}(\mathbb{R}^s), \tag{4.43}$$

where the symplectic form σ is given by

$$\sigma(f, g) := \mathrm{Im} \int d^s x \overline{f(\mathbf{x})} g(\mathbf{x}). \tag{4.44}$$

Then, the action of the spatial translations $\mathbb{R}^s$ on the Weyl operators is given by

$$\alpha_{\mathbf{x}}(W(f)) := W(\tau_{\mathbf{x}} f), \quad \mathbf{x} \in \mathbb{R}^s, \tag{4.45}$$

where $(\tau_{\mathbf{x}} f)(\mathbf{y}) := f(\mathbf{x} - \mathbf{y})$. For any given mass $m \geq 0$, we define the corresponding time translations by

$$\alpha_t^{(m)}(W(f)) := W(\tau_t^{(m)} f), \quad t \in \mathbb{R}. \tag{4.46}$$

We write $\widetilde{f}(\mathbf{p}) = (2\pi)^{-s/2} \int d\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{x}\cdot\mathbf{p}}$ for the Fourier transform of f, and splitting f into $f = f_R + i f_I$, with $f_R = \mathrm{Re} f$ and $f_I = \mathrm{Im} f$, we define

$$\begin{aligned}(\tau_t^{(m)} f)\widetilde{\ }(\mathbf{p}) \;&:=\; (\cos(t\omega_m(\mathbf{p})) + i\omega_m(\mathbf{p})^{-1}\sin(t\omega_m(\mathbf{p})))\widetilde{f_R}(\mathbf{p}) \\ &\quad + i(\cos(t\omega_m(\mathbf{p})) + i\omega_m(\mathbf{p})\sin(t\omega_m(\mathbf{p})))\widetilde{f_I}(\mathbf{p}),\end{aligned}$$

where $\omega_m(\mathbf{p}) := \sqrt{\mathbf{p}^2 + m^2}$. Notice that $(\tau_t^{(m)} f)$ has support in a ball of radius $r + |t|$, if f has support in a ball of radius r, hence $\mathcal{D}(\mathbb{R}^s)$ is stable under the action of $\tau_t^{(m)}$. By the formulas above, it is evident that the automorphisms $\alpha_{\mathbf{x}}$ and $\alpha_t^{(m)}$ commute for arbitrary $m \geq 0$, but the time translations corresponding to different values of m do not commute. We define also the action of lenght scale transformation (dilations) on $\mathcal{W}^\circ$ by

$$\sigma_\lambda(W(f)) := W(\delta_\lambda f), \quad \lambda > 0, \tag{4.47}$$

where

$$(\delta_\lambda f)(\mathbf{x}) := \lambda^{-\frac{s+1}{2}} f_R(\lambda^{-1}\mathbf{x}) + i\lambda^{-\frac{s-1}{2}} f_I(\lambda^{-1}\mathbf{x}).$$

It is evident that, if $\text{supp}(f) \subset \mathcal{O}$, then $\text{supp}(\delta_\lambda f) \subset \lambda\mathcal{O}$. (Indeed, by definition, for any $\mathbf{x} \notin \lambda\mathcal{O}$, then $\lambda^{-1}\mathbf{x} \notin \mathcal{O}$, hence $f_R(\lambda^{-1}\mathbf{x}) = f_I(\lambda^{-1}\mathbf{x}) = 0$.) Then, there holds the following relation between Poincaré–transformations and dilations:

$$\sigma_\lambda \circ \alpha^{(\lambda m)}_{\Lambda,x} = \alpha^{(m)}_{\Lambda,\lambda x} \circ \sigma_\lambda, \quad \lambda > 0. \tag{4.48}$$

Next, we define the vacuum states of mass $m \geq 0$ on $\mathcal{W}^\circ$ by

$$\omega^{(m)}(W(f)) := e^{-\frac{1}{2}\|f\|^2_m}, \tag{4.49}$$

where

$$\|f\|^2_m := \frac{1}{2}\int_{\mathbb{R}^s} d^s\mathbf{p}\left|\omega_m(\mathbf{p})^{-1/2}\widetilde{f_R}(\mathbf{p}) + i\omega_m(\mathbf{p})^{1/2}\widetilde{f}_I(\mathbf{p})\right|^2. \tag{4.50}$$

Then, $\omega^{(m)} \circ \alpha^{(m)}_{\Lambda,x} = \omega^{(m)}$ and $\omega^{(m)} \circ \sigma_\lambda = \omega^{(\lambda m)}$.
Now, we consider the GNS–representation $(\pi^{(0)}, \mathcal{H}^{(0)}, \Omega^{(0)})$ of $\mathcal{W}^\circ$ induced by the massless vacuum state $\omega^{(0)}$, and for each $m \geq 0$, we define a net $\mathcal{O} \mapsto \mathcal{R}^{(m)}(\mathcal{O})$ of von Neumann algebras on $\mathcal{H}^{(0)}$ by

$$\mathcal{R}^{(m)}(\Lambda\mathcal{O}_r + x) := \left\{\pi^{(0)}(\alpha^{(m)}_{\Lambda,x}(W(g))) : \text{supp}(g) \subset \mathcal{O}_r\right\}'', \tag{4.51}$$

where $\mathcal{O}_r$ is any double cone with base the open ball B_r in the time $t = 0$ plane. Due to the local normality of the different states $\omega^{(m)}$, $m \geq 0$, with respect to each other (see [20]), these nets are isomorphic to the nets generated by the free scalar field of mass m on the respective Fock spaces. Moreover, the automorphisms $\alpha^{(m)}_{\Lambda,x}$ extend to the local von Neumann algebras $\mathcal{R}^{(m)}(\mathcal{O})$ and act covariantly on the net, i.e.

$$\alpha^{(m)}_{\Lambda,x}\mathcal{R}^{(m)}(\mathcal{O}) = \mathcal{R}^{(m)}(\Lambda\mathcal{O} + x). \tag{4.52}$$

However, notice that for m different from the mass of the chosen standard state, the time translations $\alpha^{(m)}_t$ are not unitarly implemented in the underlying Hilbert space. We have, finally,

$$\sigma_\lambda(\mathcal{R}^{(0)}(\mathcal{O})) = \mathcal{R}^{(0)}(\lambda\mathcal{O}) = \left\{\pi^{(0)}(W(\delta_\lambda f)) : \text{supp}(f) \subset \mathcal{O}\right\}''. \tag{4.53}$$

As discussed in Section 4.1, in order to get a scaling algebra on which the renormalization group transformations act in a canonical manner, one has to pass from the local net of von Neumann algebras to a corresponding subnet of C^*–algebras consisting of operators which transform strongly continuously under the action of Poincaré transformations or, more generally, spacetime translation. For simplicity, one restricts to the latter case, and consider for fixed m the weakly dense subnet of $\mathcal{O} \to \mathcal{R}^{(m)}(\mathcal{O})$, given by

$$\mathcal{A}^{(m)}(\mathcal{O}) := \{a \in \mathcal{R}^{(m)}(\mathcal{O}) : \lim_{x\to 0}\|\alpha^{(m)}_x(a) - a\| = 0\}. \tag{4.54}$$

This net still transforms covariantly under the Poincaré transformations, and, in the case $m = 0$, also under the dilations σ_λ. Its C^*–inductive limit will be denoted by $\mathcal{A}^{(m)}$, and the various vacuum states extend to this algebra by local normality, due to the Eckmann and Fröhlich Theorem [20].

Then, one has the following

4.3.1 Theorem. [13] *Let $s = 2, 3$, $m \geq 0$, and let $\underline{\omega}_{0,\iota}^{(m)}$ be any scaling limit state of the theory $(\mathcal{A}^{(m)}, \alpha^{(m)}, \omega^{(m)})$ of a free scalar field of mass m in $(1+s)$–dimensional Minkowski–spacetime. Then, the associated scaling limit theory $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)}, \omega_{0,\iota}^{(m)})$ is net–isomorphic to the theory $(\mathcal{A}^{(0)}, \alpha^{(0)}, \omega^{(0)})$ of the massless free scalar field in the same spacetime dimensions, and the corresponding net–isomorphism connects $\omega_{0,\iota}^{(m)}$ and $\omega^{(0)}$.*

4.3.2 Remark. This result implies that, according to the classification in [12], these free field theories have a unique quantum scaling limit with a unique vacuum structure. A similar theorem holds for the scaling limit theories of the local nets if one imposes the continuity requirement (4.54) for the whole Poincaré group (cf. [6], Section 7.2).

In the following, we shall verify that the free field net complies with all the regularity assumption one needs in order to apply the ultraproduct construction, so that, by uniqueness, the ultraproduct also will coincide with the Buchholz–Verch scaling limit, though at the von Neumann algebraic level only, as specified in Remark 4.2.17.

4.3.1 Uniform Nuclearity

(This section is due to H. Bostelmann.)
We want to show uniform nuclearity of the map

$$\Xi_{\beta,\mathcal{O}} : \mathcal{R}^{(m)}(\mathcal{O}) \to \mathcal{B}(\mathcal{H}), \quad A \mapsto e^{-\beta H} A e^{-\beta H}, \tag{4.55}$$

in the case of a (real scalar) free field of mass $m \geq 0$ in $s \geq 3$ spatial dimensions. For simplicity, we restrict to standard double cones $\mathcal{O} = \mathcal{O}_r$ of radius r around the origin, and denote the corresponding map by $\Xi_{\beta,r}$.

The local algebras $\mathcal{R}^{(m)}(\mathcal{O}_r)$ are generated (via weak closure) by the Weyl algebras $\mathcal{W}^\circ(\mathcal{O}_r)$. In terms of the free field ϕ and its time derivative $\partial_0\phi$ in the time–0 plane, their elements (the Weyl operators) can be written as

$$W(f) = \exp i\big(\phi(\mathrm{Re}\, f) - \partial_0\phi(\mathrm{Im}\, f)\big), \quad f \in \mathcal{D}(\mathcal{B}_r). \tag{4.56}$$

Here $\mathcal{B}_r \subset \mathbb{R}^s$ is the ball of radius r around the origin. The single particle Hilbert space ("momentum space") will be denoted by $\mathcal{K}$, with energy operator $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$.

In order to prove nuclearity of $\Xi_{\beta,r}$, we will use methods as in [6, Section 7.3].

Here we aim at estimates for the nuclear norm of $\Xi_{\beta,r}$ that are uniform in β/r, valid for small values of r and β.

We will need some multi–index notation as in [6]. Given $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$, we consider multi–indexes $\nu = (\nu_1, \ldots, \nu_n) \in (\{0,1\} \times \mathbb{N}_0^s)^n$, namely each ν_j has the form $\nu_j = (\nu_{j0}, \nu_{j1}, \ldots, \nu_{js})$ with $\nu_{j0} \in \{0,1\}$, and $\nu_{jk} \in \mathbb{N}_0$ for $1 \leq k \leq s$. These indices will be used for labeling derivatives in configuration space, $\partial^{\nu_j} = \partial_0^{\nu_{j0}} \ldots \partial_s^{\nu_{js}}$. Correspondingly, we consider $p^{\nu_j} := \omega(\mathbf{p})^{\nu_{j0}} p_1^{\nu_{j1}} \ldots p_s^{\nu_{js}}$ as a function in momentum space. We set

$$\nu_j! = \prod_{k=0}^{s} \nu_{jk}!\,, \quad \nu! = \prod_{j=1}^{n} \nu_j!\,, \quad |\nu_j| = \sum_{k=0}^{s} \nu_{jk}\,, \quad |\nu| = \sum_{j=1}^{n} |\nu_j|. \tag{4.57}$$

As shown in [6, Lemma 7.6], the Weyl operators can be expanded in a series:

$$A = \sum_{n=0}^{\infty} \sum_{\nu} \sigma_{n,\nu}(A)\, \phi_{n,\nu} \tag{4.58}$$

for all $A \in \mathcal{W}^\circ(\mathcal{O}_r)$, in the sense of matrix elements between vectors of finite energy and finite particle number. Here the quadratic forms $\phi_{n,\nu}$ are defined as

$$\phi_{n,\nu} = :\prod_{j=1}^{n} \partial^{\nu_j}\phi : (0), \tag{4.59}$$

and the functionals $\sigma_{n,\nu} \in \mathcal{R}^{(m)}(\mathcal{O}_r)_*$ are given by

$$\sigma_{n,\nu}(A) = \frac{i^n(-1)^{\sum_j \nu_{j0}}}{n!\,\nu!}\big(\Omega\big|[\partial_0^{(1-\nu_{10})}\phi(h_{\nu_1}),[\ldots[\partial_0^{(1-\nu_{n0})}\phi(h_{\nu_n}),A]\ldots]\Omega\big). \tag{4.60}$$

We set $h_{\nu_j}(\mathbf{x}) = \prod_{k=1}^{s} x_k^{\nu_{jk}} h(\mathbf{x})$, where $h \in \mathcal{D}(\mathbb{R}^s)$ is a certain test function which is equal to 1 for $|\mathbf{x}| \leq r$. (The functionals $\sigma_{n,\nu}$ (at fixed r) are independent of the choice of h. They formally depend on r, but in a way that is compatible with restriction to smaller algebras.)

Our task is to extend (4.58) to a norm–convergent expansion of the map $\Xi_{\beta,r}$. To that end, we need estimates of the forms and functionals involved (cf. [6, Lemma 7.7]).

4.3.3 Lemma. *Given $s \geq 3$, $m \geq 0$, and $r_0 > 0$, there exists a constant c such that the following holds for any n, ν.*

$$\|e^{-\beta H}\phi_{n,\nu}e^{-\beta H}\| \leq c^n\,(n!)^{1/2}\,\nu!\,(2\sqrt{s}/\beta)^{|\nu|+n(s-1)/2} \qquad \text{for any } \beta > 0, \tag{a}$$

$$\|\sigma_{n,\nu}|_{\mathcal{R}^{(m)}(\mathcal{O}_r)}\| \leq c^n\,(n!)^{-1/2}(\nu!)^{-1}\,(3r)^{|\nu|+n(s-1)/2} \qquad \text{for any } r \leq r_0. \tag{b}$$

Proof. Part (b) is proven in [6, Lemma 7.7], while part (a) needs a slightly extended argument, using techniques from [11].

We first note that for any functions $f_1, \ldots, f_k \in \mathcal{K}$ which are in the domain of $\omega^{-1/2}$, one has "energy bounds" of the form

$$\begin{aligned}\|e^{-\beta H}a^*(f_1)\ldots a^*(f_k)\| &\leq \|e^{-\beta H/2}a^*(e^{-\beta\omega/2}f_1)\ldots a^*(e^{-\beta\omega/2}f_k)\| \\ &\leq \Big(\frac{k}{e\beta}\Big)^{k/2}\prod_{j=1}^{k}\|\omega^{-1/2}e^{-\beta\omega/2}f_j\|;\end{aligned} \tag{4.61}$$

(see [11, Lemma 3.3]). Due to the damping factor $e^{-\beta\omega/2}$, we can then extend this relation to functions f_j which do not necessarily decay at large momenta, but are polynomially bounded, in particular to polynomials in the momentum components.

Now, writing $\phi_{n,\nu}$ as a sum of 2^n creator–annihilator products, and applying (4.61) as well as its adjoint form, we obtain

$$\|e^{-\beta H}\phi_{n,\nu}e^{-\beta H}\| \leq \Big(\frac{4n}{e\beta}\Big)^{n/2}\prod_{j=1}^{n}\|\omega^{-1/2}e^{-\beta\omega/2}p^{\nu_j}\|. \tag{4.62}$$

For the single–particle space norms, one uses scaling arguments to obtain the estimate

$$\|\omega^{-1/2}e^{-\beta\omega/2}p^{\nu_j}\| \leq c_1\,\nu_j!\,\Big(\frac{2\sqrt{s}}{\beta}\Big)^{|\nu_j|+s/2-1} \quad \text{for all } \beta > 0, \tag{4.63}$$

where c_1 is a constant (depending on s). Note that the condition $s \geq 3$ enters here. Inserted into (4.62), this gives

$$\|e^{-\beta H}\phi_{n,\nu}e^{-\beta H}\| \leq \Big(\frac{2c_1^2 n}{e\sqrt{s}}\Big)^{n/2}\nu!\,\Big(\frac{2\sqrt{s}}{\beta}\Big)^{|\nu|+n(s-1)/2}. \tag{4.64}$$

This implies (a) by an application of Stirling's formula. ∎

We can now prove nuclearity by summing the norm estimates above.

4.3.4 Theorem. *Let $s \geq 3$, $m \geq 0$. For any $r_0 > 0$, there exists a constant d and a smooth function $F : [0, d] \to \mathbb{R}_+$ such that for any $r \leq r_0$ and $r/\beta \leq d$, the map $\Xi_{\beta,r}$ is nuclear, and its nuclear norm obeys the estimate*

$$\|\Xi_{\beta,r}\|_1 \leq F(r/\beta).$$

Proof. Due to (4.58), we know that

$$\Xi_{\beta,r} = \sum_{n=0}^{\infty} \sum_{\nu} \sigma_{n,\nu}(\cdot)\, e^{-\beta H} \phi_{n,\nu} e^{-\beta H} \tag{4.65}$$

on a weakly dense set of operators, and with convergence in a dense set of matrix elements. If we can prove norm–convergence of the series, then density arguments imply nuclearity of the map (along with an estimate for the nuclear norm).

Let $r_0 > 0$ be fixed in the following, and $r \leq r_0$. From Lemma 4.3.3, we obtain the estimate

$$\|\sigma_{n,\nu}|_{\mathcal{R}^{(m)}(\mathcal{O}_r)}\| \|e^{-\beta H} \phi_{n,\nu} e^{-\beta H}\| \leq c^{2n}\, z^{|\nu|+n(s-1)/2}, \quad \text{where } z = 6\sqrt{s}\frac{r}{\beta}. \tag{4.66}$$

We need to sum this over n and ν. We factorize the sum over multi–indexes ν into components, and obtain

$$\begin{aligned} &\sum_n \sum_\nu \|\sigma_{n,\nu}|_{\mathcal{R}^{(m)}(\mathcal{O}_r)}\| \|e^{-\beta H} \phi_{n,\nu} e^{-\beta H}\| \\ &\qquad \leq \sum_{n=0}^{\infty} c^{2n} z^{n(s-1)/2} \prod_{j=1}^{n} \sum_{\nu_{j0} \in \{0,1\}} z^{\nu_{j0}} \prod_{k=1}^{s} \sum_{\nu_{jk}=0}^{\infty} z^{\nu_{jk}}. \end{aligned} \tag{4.67}$$

Assuming $z < 1$, the sum over ν_{jk} converges as a geometric series. The sum over ν_{j0} is estimated by introducing a factor of 2. This yields for sufficiently small z,

$$\begin{aligned} &\sum_n \sum_\nu \|\sigma_{n,\nu}|_{\mathcal{R}^{(m)}(\mathcal{O}_r)}\| \|e^{-\beta H} \phi_{n,\nu} e^{-\beta H}\| \\ &\qquad \leq \sum_{n=0}^{\infty} \Big(\frac{2c^2 z^{(s-1)/2}}{(1-z)^s}\Big)^n = \Big(1 - \frac{2c^2 z^{(s-1)/2}}{(1-z)^s}\Big)^{-1}. \end{aligned} \tag{4.68}$$

That implies norm–convergence of the sum in (4.65), and, re-inserting $z = 6\sqrt{s}r/\beta$, gives the proposed estimate on $\|\Xi_{\beta,r}\|_1$.

∎

4.3.5 Remark. Notice that the argument is largely the same as in [6, Theorem 7.10]. It turns out from the proof that the estimate can be much more refined, if required. Indeed, since scale–independent norm estimates for the individual terms of the series are known, it should be possible to obtain direct estimates on the ε–content of the map. Finally, the restriction to a real scalar field is chosen only for simplicity. So, the same methods should apply to free theories with any finite number of fields, bosonic or fermionic, of any finite spin.

4.3.2 Uniform Inner Regularity

We want to show uniform inner regularity of the norm

$$L'_\lambda(a_\lambda) = \|(1+\lambda H)^{-1} a_\lambda (1+\lambda H)^{-1}\|, \quad a_\lambda \in \mathcal{R}^{(m)}(\mathcal{O}_\lambda) \tag{4.69}$$

in the case of the (real scalar) free field of mass $m \geq 0$ in $s \geq 2$ spatial dimensions. Namely, we want to show that, for any family $a_\lambda \in \mathcal{R}^{(m)}(\mathcal{O}_\lambda)$, with $\sup_\lambda \|a_\lambda\| \leq 1$, and for any $\varepsilon > 0$, there exist $r < 1$ and $a'_\lambda \in \mathcal{R}^{(m)}(\mathcal{O}_{r\lambda})$ such that

$$L'_\lambda(a_\lambda - a'_\lambda) = \|(I+\lambda H)^{-1}(a_\lambda - a'_\lambda)(I+\lambda H)^{-1}\| < \varepsilon.$$

Now, we know that, for any regular representation of the Weyl algebra, and in particular for the m–mass representation $\pi^{(m)} : \mathcal{W}^\circ(\mathbb{R}^s) \to \mathcal{B}(\mathcal{H}^{(m)})$, i.e. the GNS–representation induced by the (vacuum) state $\omega^{(m)}(W(f)) = e^{-\frac{1}{2}\|f\|_m^2}$, the map

$$\left(L^2(\mathbb{R}^s), \|\cdot\|_m\right) \ni f \mapsto \pi^{(m)}(W(f)) \in \pi^{(m)}(\mathcal{W}^\circ(\mathbb{R}^s))'' \tag{4.70}$$

is weak–operator continuous. Since the weak–operator topology coincides with the w^*–topology on bounded sets, and the map $a_\lambda \mapsto L'_\lambda(a_\lambda)$ is continuous in the w^*–topology by the nuclearity condition, it suffices to show that any $a_\lambda \in \mathcal{R}^{(m)}(\mathcal{O}_\lambda)_1$ can be uniformly approximated from the inside (i.e., with elements from $\mathcal{R}^{(m)}(\mathcal{O}_{r\lambda})$, $r < 1$) in the w^*–topology. (Notice that, since the support of function is, by definition, a closed set, the local algebras $\mathcal{R}^{(m)}(\mathcal{O})$ are "continuous from the inside", i.e. for each increasing family of open regions $\mathcal{O}_i$, with $\mathcal{O}_i \subset\subset \mathcal{O}_j$, $i < j$, and $\bigcup_i \mathcal{O}_i = \mathcal{O}$, we have $\mathcal{R}^{(m)}(\mathcal{O}) = \left(\bigcup_i \mathcal{R}^{(m)}(\mathcal{O}_i)\right)''$, that is to say, the net is inner regular (see Remark 4.2.15 above).)

As first, we consider the case $m = 0$. In this particular case, the dilation operator δ_λ acting on test functions is an isometry from $L^2(\mathbb{R}^s)$ onto itself, w.r.t. the norm

$$\|f\|_0^2 = \frac{1}{2}\int_{\mathbb{R}^s} d^s\mathbf{p} \left|\omega_0(\mathbf{p})^{-1/2}\widetilde{f_R}(\mathbf{p}) + i\omega_0(\mathbf{p})^{1/2}\widetilde{f_I}(\mathbf{p})\right|^2.$$

Thus, given $f \in \mathcal{S}(\mathbb{R}^s)$ with $\mathrm{supp}(f) \subset \mathcal{O}_\lambda$, we just take the function $\delta_r f$, which has support in $\mathcal{O}_{r\lambda}$, since

$$\|\delta_\lambda(f - \delta_r f)\|_0 = \|f - \delta_r f\|_0 \to 0 \quad \text{as } r \to 1.$$

Hence, given $\varepsilon > 0$, we can find an $r < 1$ such that

$$L'_\lambda(\pi^{(0)}(W(\delta_\lambda f)) - \pi^{(0)}(W(\delta_{r\lambda} f))) < \varepsilon.$$

Now, any $a_\lambda \in (\pi_0(\mathcal{W}(\mathcal{O}_\lambda))'')_1$ can be approximated in the w^*–topology by finite linear combinations of Weyl unitaries $\sum_{i=1}^N \alpha_i W(\delta_\lambda f_i)$, with $\sum_{i=1}^N |\alpha_i| \leq 1$. We then apply a standard $\varepsilon/2$–argument. Namely, given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that, for any $N \geq n_0$, we have

$$L'_\lambda\left(a_\lambda - \sum_{i=1}^N \alpha_i \pi^{(0)}(W(\delta_\lambda f_i))\right) < \frac{\varepsilon}{2},$$

and, with the same $\varepsilon > 0$ fixed, we can also find an $r_i < 1$ for each unitary such that

$$L'_\lambda(\pi^{(0)}(W(\delta_\lambda f_i)) - \pi^{(0)}(W(\delta_{r_i\lambda} f_i))) < \frac{\varepsilon}{2}.$$

Then, setting $r = \max(r_1, \dots, r_N) < 1$, $a'_\lambda = \sum_{i=1}^N \alpha_i \pi^{(0)}(W(\delta_{r_i\lambda} f_i))) \in (\pi^{(0)}(\mathcal{R}^{(0)}(\mathcal{O}_{r\lambda}))'')_1$, and taking $N \geq n_0$, we obtain

$$\begin{aligned} & L'_\lambda(a_\lambda - a'_\lambda) \\ \leq\ & L'_\lambda \left(a_\lambda - \sum_{i=1}^N \alpha_i \pi^{(0)}(W(\delta_\lambda f_i)) \right) + L'_\lambda \left(\sum_{i=1}^N \alpha_i \pi^{(0)}(W(\delta_\lambda f_i)) - \sum_{i=1}^N \alpha_i \pi^{(0)}(W(\delta_{r_i\lambda} f_i)) \right) \\ <\ & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We consider now the case $m > 0$. We recall the following estimates:

4.3.6 Lemma. [27] *Let $\mathcal{O} \subset \mathbb{R}^s$ be bounded. Then, for any $f \in \mathcal{S}(\mathbb{R}^s)$ real–valued, supp$(f) \subset \mathcal{O}$,*

$$\|f\|_{1/2,m} := \left(\int_{\mathbb{R}^s} d^s\mathbf{p}\, \omega_m(\mathbf{p}) |\widetilde{f}(\mathbf{p})|^2 \right)^{1/2} \leq c(m, \mathcal{O}) \|f\|_{1/2,0}, \tag{4.71}$$

$$\|f\|_{-1/2,m} := \left(\int_{\mathbb{R}^s} d^s\mathbf{p}\, \omega_m^{-1}(\mathbf{p}) |\widetilde{f}(\mathbf{p})|^2 \right)^{1/2} \leq \|f\|_{-1/2,0}. \tag{4.72}$$

Proof. Let $\chi_1(\mathbf{p})$ be the characteristic function of the unit ball in $\mathbb{R}^s$. Then, it is easy to see that

$$\omega_m(\mathbf{p}) = \sqrt{|\mathbf{p}|^2 + m^2} \leq \sqrt{1 + m^2}|\mathbf{p}| + m\chi_1(\mathbf{p}),$$

and thus

$$\begin{aligned} \|f\|^2_{1/2,m} &= \int_{\mathbb{R}^s} d^s\mathbf{p}\, \omega_m(\mathbf{p}) |\widetilde{f}(\mathbf{p})|^2 \\ &\leq \sqrt{1+m^2} \|f\|^2_{1/2,0} + \int_{|\mathbf{p}| \leq 1} d^s\mathbf{p}\, |\widetilde{f}(\mathbf{p})|^2. \end{aligned}$$

Now, let $\varphi \in \mathcal{D}(\mathbb{R}^s)$ be such that $\varphi \equiv 1$ on supp$f$. Then,

$$\begin{aligned} & \int_{|\mathbf{p}| \leq 1} d^s\mathbf{p}\, |\widetilde{f}(\mathbf{p})|^2 = \int d^s\mathbf{p}\, \chi_1(\mathbf{p}) \left| \int d^s\mathbf{k} \widetilde{f}(\mathbf{k}) \widetilde{\varphi}(\mathbf{p} - \mathbf{k}) \right|^2 \\ =\ & \int d^s\mathbf{p}\, \chi_1(\mathbf{p}) \left| \int d^s\mathbf{k}\, (|\mathbf{k}|^{1/2} \widetilde{f}(\mathbf{k}))(|\mathbf{k}|^{-1/2} \widetilde{\varphi}(\mathbf{p} - \mathbf{k}) \right|^2 \\ \leq\ & \int d^s\mathbf{p}\, \chi_1(\mathbf{p}) \left(\int d^s\mathbf{k}\, |\mathbf{k}| |\widetilde{f}(\mathbf{k})|^2 \right) \left(\int d^s\mathbf{k}\, |\mathbf{k}|^{-1} |\widetilde{\varphi}(\mathbf{p} - \mathbf{k})|^2 \right) \\ =\ & \|f\|_{1/2,0} \int d^s\mathbf{p}\, d^s\mathbf{k}\, \chi_1(\mathbf{p}) |\mathbf{k}|^{-1} |\widetilde{\varphi}(\mathbf{p} - \mathbf{k})|^2. \end{aligned}$$

Hence, we get $\|f\|_{1/2,m} \leq c(m, \mathcal{O}) \|f\|_{1/2,0}$, where

$$c(m, \mathcal{O}) = \sqrt[4]{1 + m^2} + \left(m \int d^s\mathbf{p}\, d^s\mathbf{k}\, \chi_1(\mathbf{p}) |\mathbf{k}|^{-1} |\widetilde{\varphi}(\mathbf{p} - \mathbf{k})|^2 \right)^{1/2}.$$

More easly, since $\omega_m^{-1}(\mathbf{p}) \leq |\mathbf{p}|^{-1}$, we obtain the second inequality:

$$\begin{aligned} & \int_{\mathbb{R}^s} d^s\mathbf{p}\, \omega_m^{-1}(\mathbf{p}) |\widetilde{f}(\mathbf{p})|^2 \\ \leq\ & \int_{\mathbb{R}^s} d^s\mathbf{p}\, |\mathbf{p}|^{-1} |\widetilde{f}(\mathbf{p})|^2 = \|f\|_{-1/2,0}. \end{aligned}$$

∎

Therefore, if $m > 0$, we see that

$$\begin{aligned}
& \|\delta_\lambda f - \delta_{r\lambda} f\|_m^2 \\
= \; & \frac{1}{2}\int_{\mathbb{R}^s} d^s\mathbf{p} \left|\omega_m(\mathbf{p})^{-1/2}\lambda^{\frac{s-1}{2}}(\widetilde{f_R}(\lambda\mathbf{p}) - r^{\frac{s-1}{2}}\widetilde{f_R}(r\lambda\mathbf{p})) + i\omega_m(\mathbf{p})^{1/2}\lambda^{\frac{s+1}{2}}(\widetilde{f_I}(\lambda\mathbf{p}) - r^{\frac{s+1}{2}}\widetilde{f_I}(r\lambda\mathbf{p}))\right|^2 \\
= \; & \frac{1}{2}\int_{\mathbb{R}^s} d^s\mathbf{p} \left|\omega_{\lambda m}(\mathbf{p})^{-1/2}(\widetilde{f_R}(\mathbf{p}) - r^{\frac{s-1}{2}}\widetilde{f_R}(r\mathbf{p})) + i\omega_{\lambda m}(\mathbf{p})^{1/2}(\widetilde{f_I}(\mathbf{p}) - r^{\frac{s+1}{2}}\widetilde{f_I}(r\mathbf{p}))\right|^2 \\
= \; & \|f_R - \delta_r f_R\|_{\lambda m}^2 + \|f_I - \delta_r f_I\|_{\lambda m}^2 \\
\leq \; & c(\lambda m, \mathcal{O}_\lambda)\|f_R - \delta_r f_R\|_0^2 + \|f_I - \delta_r f_I\|_0^2, \qquad (4.73)
\end{aligned}$$

where the last inequality clearly follows from the previous Lemma. Now, since we are interested in the limit $\lambda \to 0$, we can restrict λ to the interval $(0, 1)$, and we get finally

$$\begin{aligned}
\|\delta_\lambda f - \delta_{r\lambda} f\|_m^2 \; & \leq \left(\sup_{\lambda\in(0,1)} c(\lambda m, \mathcal{O}_\lambda)\right)\|f_R - \delta_r f_R\|_0^2 + \|f_I - \delta_r f_I\|_0^2 \\
& \leq c(m, \mathcal{O})\|f_R - \delta_r f_R\|_0^2 + \|f_I - \delta_r f_I\|_0^2,
\end{aligned}$$

whence, the family of operators $\{\delta_{r\lambda}\}_{r\in(0,1)}$ is (strongly) continuous, uniformly w.r.t. $\lambda \in (0, 1)$, and so we can apply the argument above. Thus, we can summarize in the following

4.3.7 Proposition. *For any $\mathcal{O} \subset \mathbb{R}^s$ and any $\lambda > 0$, the Lip–norm L'_λ (4.69) on the local algebras $\mathcal{R}^{(m)}(\mathcal{O}_\lambda)$ of the (real scalar) free field satisfies the uniform inner regularity in the vacuum representation, for any mass $m \geq 0$.*

Therefore, in view of Theorem 4.2.16, we see that, for any $m \geq 0$ and $s \geq 3$ spatial dimensions, the local qGH*–scaling limit net $\mathcal{O} \to \mathcal{R}^{(m)}(\mathcal{O})_{\mathcal{U}}$ exists for any ultrafilter $\mathcal{U}$. Moreover, by Corollary 4.2.18 and Theorem 4.3.1, for $m \geq 0$ and $s = 3$ spatial dimensions, the Buchholz–Verch scaling limit net in the vacuum sector embeds as a subrepresentation in the qGH*–scaling limit net associated with $\mathcal{U}$.

In fact, we represent all the (local) algebras involved on the same Hilbert space $\mathcal{H}$, which is the Hilbert space associated to the standard representation of the Weyl algebra $\mathcal{W}^\circ$ induced by the mass zero vacuum state $\omega^{(0)}$. Let $(\mathcal{A}_{0,\iota}^{(m)}, \alpha^{(m;0,\iota)}, \omega_{0,\iota}^{(m)})$ be the Buchholz–Verch scaling limit theory, and denote by $\omega_\lambda^{(m)}$ the restriction to $\mathcal{R}^{(m)}(\mathcal{O}_\lambda)$ of the vacuum state $\omega^{(m)}$. Let $\mathcal{U}$ be an ultrafilter over $\mathbb{N}$, and let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a suitable sequence of positive real numbers converging to zero. On the one hand, we have the Buchholz–Verch scaling limit state

$$\underline{\omega}_{0,\iota}^{(m)}(\underline{a}) = \lim_{\mathcal{U}} \underline{\omega}_{\lambda_n}^{(m)}(\underline{a}), \quad \underline{a} \in \underline{\mathcal{A}}(\mathcal{O})_{BV}. \tag{4.74}$$

On the other hand, since one has (cf. Lemma 3.2 (a) in [13])

$$\lim_{\mathcal{U}} \|(\omega^{(m)} - \omega^{(0)})|_{\mathcal{R}^{(0)}(\mathcal{O}_{\lambda_n})}\| = 0, \tag{4.75}$$

and $\mathcal{R}^{(m)}(\mathcal{O}_\lambda) \cong \mathcal{R}^{(0)}(\mathcal{O}_\lambda)$ by the Eckmann–Fröhlich Theorem [20], then we get as qGH*–scaling limit state exactly the equivalence class $\underline{\omega}^{(0)} \equiv [\omega_{\lambda_n}]$ of the mass zero vacuum state $\omega^{(0)}$, namely

$$\underline{\omega}^{(0)}(\underline{a}') = \lim_{\mathcal{U}} \omega_{\lambda_n}^{(m)}(a'_{\lambda_n}), \quad \underline{a}' \in \underline{\mathcal{A}}(\mathcal{O}). \tag{4.76}$$

(We recall that, since the norms L'_{λ_n} on the local algebras $\mathcal{R}^{(m)}(\mathcal{O}_{\lambda_n})$ are dual Lip–norms, we have

$$L(\{\omega^{(m)}_{\lambda_n}\}) = \sup_n L_{\lambda_n}(\omega^{(m)}_{\lambda_n}) < \infty,$$

where, for any $\lambda > 0$,

$$\begin{aligned} L_\lambda(\omega^{(m)}_\lambda) &:= \sup\{\tfrac{|\langle \omega^{(m)}_\lambda, x\rangle|}{L'_\lambda(x)} : x \in \mathcal{R}^{(m)}(\mathcal{O}_\lambda)\} \\ &= \sup\{\tfrac{|\langle \omega^{(m)}_\lambda, (I+\lambda H)y(I+\lambda H)\rangle|}{\|y\|} : y \in (I+\lambda H)^{-1}\mathcal{R}^{(m)}(\mathcal{O}_\lambda)(I+\lambda H)^{-1}\}. \end{aligned}$$

Indeed, as $\omega^{(m)}(\cdot) = (\Omega, \pi^{(m)}(\cdot)\Omega)$ is the vacuum state and $H\Omega = 0$, we have clearly

$$\omega^{(m)}_\lambda((I+\lambda H)y(I+\lambda H)) = (\Omega, (I+\lambda H)y(I+\lambda H)\Omega) = (\Omega, y\,\Omega) \leq \|y\|,$$

for any $y \in (I+\lambda H)^{-1}\mathcal{R}^{(m)}(\mathcal{O}_\lambda)(I+\lambda H)^{-1}$. Hence $\underline{\omega}^{(0)} \in \mathcal{S}_{\mathcal{U}}$ (cf. the proof of Theorem 2.4.9).)
Since $(\mathcal{A}^{(m)}_{0,\iota}, \alpha^{(m;0,\iota)}, \omega^{(m)}_{0,\iota})$ is net–isomorphic to the theory $(\mathcal{A}^{(0)}, \alpha^{(0)}, \omega^{(0)})$ of the massless free scalar field in the same spacetime dimensions, and the corresponding net–isomorphism connects $\omega^{(m)}_{0,\iota}$ and $\omega^{(0)}$, we finally obtain (cf. Section 4.2.1)

$$\pi_{\underline{\omega}^{(m)}_{0,\iota}}(\underline{\mathcal{A}}^{(m)}_{0,\iota}(\mathcal{O})) \cong \pi_{\underline{\omega}^{(0)}}(\underline{\mathcal{A}}^{(0)}(\mathcal{O})_{BV})'' = \pi_{\underline{\omega}^{(0)}}(\underline{\mathcal{A}}(\mathcal{O}))''. \tag{4.77}$$

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