2-C^*-Categories with non-simple units

Pasquale A. Zito

Università di Roma "Tor Vergata"
Tesi di Dottorato in Matematica

Ciclo XVI

Tutore John E. Roberts

Contents

Introduction

Tensor categories arise in many mathematical contexts, such as quantum field theory, topological field theory, the theory of topological invariants for knots and links, quantum groups . The various examples may be characterized by more specific properties such as the notions of duality or of spherical (resp. braided, symmetric, modular) category. These have in common the "tensor" structure, i.e. a $\otimes$ product defined on objects and on arrows and a particular object ι in the category, the unit with respect to this tensor product. C^* tensor categories are naturally related to the theory of operator algebras, and are at the heart of the structure of superselection theory in algebraic quantum field theory and duality for compact groups (see [5], [6]).

A 2-category may be viewed as a generalization of the notion of tensor category. We recover the notion of a tensor category as the particular case of a 2-category with only one object, where we can regard 1-arrows as the objects of the tensor category and 2-arrows as the arrows in the tensor category. The horizontal composition gives the tensor structure.

$2-C^*$-categories describe much of the structure in the theory of subfactors of von Neumann algebras . Many tools used in the past years which have proved to be fundamental in order to characterize subfactors can be described in the context of $2-C^*$-categories. Furthermore the language of categories makes easier the link with other fields of mathematics.

In a 2-category for each object A we have a particular 1-arrow $A \xleftarrow{\iota_A} A$, a unit with respect to the $\otimes$ composition. In the tensor category case this corresponds to the unit object. In most of the literature units are supposed to be simple, i.e. $(\iota_A, \iota_A) = \mathbb{K}$, the spaces of 2-arrows connecting the unit to itself is a field.

In this work we drop this hypothesis. More precisely, our object of study will be $2-C^*$-categories closed under conjugation, projections and direct sums.

The axioms for the $\otimes$, $\circ$ products imply that the spaces (ι_A, ι_A) are commutative C^* algebras, i.e. $(\iota_A, \iota_A) \cong C(\Omega_A)$, the continuous functions over a compact Hausdorff space Ω_A.

We show that for each couple $B \xleftarrow{\rho} A$, $B \xleftarrow{\sigma} A$ the space of 2-arrows (ρ, σ) connecting ρ to σ has the structure of a Hilbert bundle over the topological space Ω_A. This structure is derived by the conjugation relations, and depends on the choice of solutions up to continuous linear isomorphisms which respect the fibre structure. Furthermore, when $\rho = \sigma$, (ρ, ρ) has the structure of a C^*-algebra bundle. We discuss the existence of a class of "standard" solutions which, when they exist, are a natural choice stable under composition of arrows, direct sums and projections.

We describe how this bundle structure behaves with respect to categorical operations such as conjugation and the $\circ$ and $\otimes$ products.

The hypothesis of a non simple unit has been considered by some authors in related contexts:

[2] deals with the categorical structure arising from the extension of a C^* algebra with non trivial centre by a Hilbert C^* system;

[8] deals with the existence of minimal conditional expectations for the inclusion of two von Neumann algebras with discrete centres;

[21] deals with the structure of depth two inclusions of finite dimensional C^*-algebras.

[11] studies a notion of Jones index and its relation to conjugation in the context of the $2-C^*$-category of Hilbert C^* Bimodules;

[24] deals with crossed products of C^* algebras with nontrivial centres by endomorphism, C^* algebra bundles, group bundles.

Our work lives in a more abstract setting, and it may be viewed as a development of some of the results of [16] for the case $(\iota, \iota) \neq \mathbb{C}$.

The work is organized as follows:

- In the first part we recall the basic definitions and state some basic results concerning the structure of $2-C^*$-categories.
- In the second part we give some examples of objects where the category structure has an important role and at the same time review some of the above mentioned results which are related to our work.
- In the third part we develop further the structure of 2-C^*-categories. In particular the fibre structure appears.
- In the fourth part we define and study the class of "standard solutions " to the conjugate equations.
- In the fifth and sixth part make a few remarks in the context of Q-systems and Hopf algebra bundles.

1 Preliminaries

We recall here briefly some basic concepts from category theory (the classical reference is [17], see also the introduction of [13] for a quick review). We will be interested only in small categories and we will not need more general definitions which avoid the notion of "set" from the beginning.

Definition 1. *A* category $\mathcal{C}$ *consists of a set of objects* $\mathcal{C}_0$ *and a set of arrows* $\mathcal{C}_1$ *together with the following structure:*

- *a* source *map:* $\mathcal{C}_1 \to \mathcal{C}_0$ *assigning an object* $\mathrm{s}(f)$ *to each arrow* $f \in \mathcal{C}_1$
- *a* target *map:* $\mathcal{C}_1 \to \mathcal{C}_0$ *assigning an object* $\mathrm{t}(f)$ *to each arrow* $f \in \mathcal{C}_1$
- *an identity* 1 *map* $: \mathcal{C}_0 \to \mathcal{C}_1$ *assigning to each object* a *an arrow* 1_a *with* $\mathrm{s}(1_a) = \mathrm{t}(1_a) = a$.
- *a composition map* $\circ : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_1$ *assigning to each pair of arrows* f, g *s.t.* $\mathrm{s}(g) = \mathrm{t}(f)$ *a third arrow* $g \circ f$ *with* $\mathrm{s}(g \circ f) = \mathrm{s}(f)$ *and* $\mathrm{t}(g \circ f) = \mathrm{t}(g)$.
- *the composition* $\circ$ *is associative, i.e.* $h \circ (g \circ f) = (h \circ g) \circ f$ *whenever these compositions make sense.*
- *the identity map satisfies* $f \circ 1_a = f, \quad \forall f \quad s.t. \quad \mathrm{s}(f) = a$ *and* $1_a \circ g = g, \quad \forall g \quad s.t. \quad \mathrm{t}(g) = a$.

Remark 1.1. An alternative way of defining a category is that of a set $\mathcal{C}$ together with a set of pairs $< f, g >$ of elements in $\mathcal{C}$, which are said to be composable, and a composition map $\circ :< f, g > \to f \circ g$ assigning to each composable couple an element in $\mathcal{C}$. This map is supposed to be strictly associative. Elements u such that $f \circ u = f$ and $u \circ g = g$ for any f, g such that the compositions are defined are called units. For each $f \in \mathcal{C}$ there exist right and left units. Thus in this approach one deals only with arrows and the objects are identified with their units .

Definition 2. *A functor* $F : \mathcal{C} \to \mathcal{C}'$ *from the category* $\mathcal{C}$ *to the category* $\mathcal{C}'$ *consists of a map sending objects to objects and arrows to arrows such that:*

- *for any object* $A \in \mathcal{C}_0$ *we have* $F(1_A) = 1_{F(A)}$
- *for any arrow* $f \in \mathcal{C}_1$ *we have* $\mathrm{s}(F(f)) = F(\mathrm{s}(f))$ *and* $\mathrm{t}(F(f)) = F(\mathrm{t}(f))$.
- $F(g \circ f) = F(g) \circ F(f)$, *if* f *and* g *are composable.*

Definition 3. *Let* $F, G : \mathcal{C} \to \mathcal{C}'$ *be functors from the category* $\mathcal{C}$ *to the category* $\mathcal{C}'$. *A natural transformation* $\eta : F \to G$ *is a family of arrows* $\eta(A) : F(A) \to G(A)$ *in* $\mathcal{C}'$ *such that for any morphism* $f : A \to B$ *in* $\mathcal{C}$ *the following holds*

$$G(f) \circ \eta(A) = \eta(B) \circ F(f).$$

Definition 4. *A strict 2-category* $\mathcal{B}$ *consists of*

- *a set* $\mathcal{B}_0$ *, whose elements are called objects.*
- *a set* $\mathcal{B}_1$ *, whose elements are called 1-arrows. A source map* $\mathrm{s} : \mathcal{B}_1 \to \mathcal{B}_0$ *and a target map* $\mathrm{t} : \mathcal{B}_1 \to \mathcal{B}_0$ *assigning to each 1-arrow a source and a target object.*
- *an identity map* $\iota : \mathcal{B}_0 \to \mathcal{B}_1$ *assigning to each object* A *a 1-arrow* ι_A *such that* $\mathrm{s}(\iota_A) = \mathrm{t}(\iota_A) = A$ *.*
- *a composition map* $\otimes : \mathcal{B}_1 \times \mathcal{B}_1 \to \mathcal{B}_1$ *making* $(\mathcal{B}_0, \mathcal{B}_1)$ *into a category, i.e. for any* $\rho, \sigma \in \mathcal{B}_1$ *such that* $\mathrm{s}(\sigma) = \mathrm{t}(\rho)$, $\mathrm{s}(\sigma \otimes \rho) = \mathrm{s}(\rho)$ *and* $\mathrm{t}(\sigma \otimes \rho) = \mathrm{t}(\sigma)$ *hold.*
- *a set* $\mathcal{B}_2$ *, whose elements are called 2-arrows. A source map (we use the same symbols as above)* $\mathrm{s} : \mathcal{B}_2 \to \mathcal{B}_1$ *and a target map* $\mathrm{t} : \mathcal{B}_2 \to \mathcal{B}_1$ *assigning to each 2-arrow* $S \in \mathcal{B}_2$ *source and target 1-arrows. The maps* s *and* t *satisfy* $\mathrm{s}(\mathrm{s}(S)) = \mathrm{s}(\mathrm{t}(S))$ *and* $\mathrm{t}(\mathrm{t}(S)) = \mathrm{t}(\mathrm{s}(S))$ *for any* $S \in \mathcal{B}_2$ *.*
- *a composition map (the "vertical" composition)* $\circ : \mathcal{B}_2 \times \mathcal{B}_2 \to \mathcal{B}_2$ *defined for each pair of* S, T *for* $\mathcal{B}_2$ *such that* $\mathrm{s}(T) = \mathrm{t}(S)$ *.*
- *a unit map* $1 : \mathcal{B}_1 \to \mathcal{B}_2$ *assigning to each 1-arrow* ρ *a 2-arrow* 1_ρ *such that* $\mathrm{s}(1_\rho) = \rho = \mathrm{t}(1_\rho)$ *.*
- *a composition map (the "horizontal" composition, for which we use the same symbol as above)* $\otimes : \mathcal{B}_2 \times \mathcal{B}_2 \to \mathcal{B}_2$ *defined for each pair of 2-arrows* S, T *such that* $\mathrm{s}(\mathrm{s}(T)) = \mathrm{t}(\mathrm{s}(S))$ *and* $\mathrm{s}(\mathrm{t}(T)) = \mathrm{t}(\mathrm{t}(S))$ *.*
- *the following equation for elements of* $\mathcal{B}_2$ *holds, whenever the compositions make sense*
$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T').$$
- *for each* $A \in \mathcal{B}_0$ *the following equations hold, whenever the composition with elements* $S, T \in \mathcal{B}_2$ *make sense*
$$T \otimes 1_{\iota_A} = T, \quad 1_{\iota_A} \otimes S = S.$$

Remark 1.2. Concerning notation, we will give priority, when not specified, to the $\otimes$ product respect to the $\circ$ product, i.e. $S \otimes T \circ U$ has to be read $(S \otimes T) \circ U$ and not $S \otimes (T \circ U)$.

Note that for any pair of objects A, B, we have a category $HOM(A, B)$, with the set of 1-arrows with source A and target B as objects of $HOM(A, B)$ and the set of 2-arrows with the latter as source and target as arrows and $\circ$ as composition and 1 as unit map.

Note that each object A determines a unit 1-arrow ι_A, which in turn determines a unit 2-arrow 1_{ι_A}.

Lemma 5. *The set of 2-arrows* (ι_A, ι_A) *is a commutative monoid.*

Proof. In fact for any $w, z \in (\iota_A, \iota_A)$ we have $w \otimes z = (1_{\iota_A} \circ w) \otimes (z \circ 1_{\iota_A}) = (1_{\iota_A} \otimes z) \circ (w \otimes 1_{\iota_A}) = z \circ w = (z \otimes 1_{\iota_A}) \circ (1_{\iota_A} \otimes w) = (z \circ 1_{\iota_A}) \otimes (1_{\iota_A} \circ w) = z \otimes w$, thus $\otimes$ and $\circ$ in this case agree and define a commutative monoid with unit 1_{ι_A}.

Remark 1.3. As same as for categories, we can describe 2-categories in terms of 2-arrows. One considers a set with two operations $\otimes, \circ$ and units such that each operation gives the set the structure of a category. Furthermore one supposes that all $\otimes$-units are also $\circ$-units and that an associativity relation as above

$$(S \otimes T) \circ (S' \otimes T') = (S \circ S') \otimes (T \circ T')$$

holds for the two products.

A weak 2-category (the term bicategory is also used) is a 2-category as above, where the associativity and unit identities are replaced by natural isomorphisms satisfying pentagon and triangle axioms. As we will deal (almost) only with the strict case, will not state these relations explicitly.

A $2-C^*$-category is a 2-category for which the following hold:

- for each pair of 1-arrows ρ, σ the space (ρ, σ) is a complex Banach space.
- there is an antilinear involution $*$ acting on 2-arrows, i.e. $* : S \in (\rho, \sigma) \to S^* \in (\sigma, \rho)$.
- the Banach norm satisfies the C^*- condition $\|S^* \circ S\| = \|S\|^2$.
- for any 2-arrow $S \in (\rho, \sigma)$, $S^* \circ S$ is a positive element in (ρ, ρ).

Remark 1.4. The above axioms imply that for each 1-arrow ρ the space (ρ, ρ) is a unital C^*-algebra. For each unit 1-arrow ι_A the space (ι_A, ι_A) is a commutative unital C^*-algebra.

We assume our category to be closed under projections (the term "retract" is sometimes used as well). By this we mean the following: take a 1-arrow $A \xleftarrow{\rho} B$ and consider the space (ρ, ρ), which has also the structure of an algebra, as we have seen. Then for each projection $P \in (\rho, \rho)$ there exists a corresponding sub-1-arrow $A \xleftarrow{\rho_P} B$ and an isometry $W \in (\rho_P, \rho)$ such that $W^* \circ W = 1_{\rho_P}$ and $W \circ W^* = P$.

We assume our category is closed under direct sums. By this we mean that for each pair of 1-arrows ρ_1, ρ_2 there exists a 1-arrow ρ and isometries $W_1 \in (\rho_1, \rho)$, $W_2 \in (\rho_2, \rho)$ such that $W_1 \circ W_1^* + W_2 \circ W_2^* = 1_\rho$ and $W_i^* \circ W_j = 1_{\rho_i} \delta_{i,j}$. Consistently with the previous definition, ρ_1, ρ_2 are sub 1-arrows of ρ.

We will sometimes simply write $\rho_1 \oplus \rho_2$ for a direct sum. Analogously we may identify a projection $P \in (\rho, \rho)$ with the unit 1_{ρ_P} of its corresponding sub-1-arrow.

Thus if $W_1, W_2, \rho_1, \rho_2, \rho$ are as above and $T_1 \in (\rho_1, \rho_1)$, $T_2 \in (\rho_2, \rho_2)$, we will simply indicate by $T_1 \oplus T_2 \in (\rho_1 \oplus \rho_2)$ the 2-arrow $W_1 \circ T_1 \circ W_1^* + W_2 \circ T_2 \circ W_2^*$.

We assume that the category is closed under conjugation, that is, for each 1-arrow ρ going from A to B there exists another 1-arrow $\bar{\rho}$ from B to A and two 2-arrows $R \in (\iota_A, \bar{\rho}\rho)$ and $\bar{R} \in (\iota_B, \rho\bar{\rho})$ satisfying the following relations:

$$\bar{R}^* \otimes 1_\rho \circ 1_\rho \otimes R = 1_\rho; \quad R^* \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \bar{R} = 1_{\bar{\rho}}.$$

This property is symmetric, i.e. if $\bar{\rho}$ is a conjugate for ρ, then ρ is a conjugate for $\bar{\rho}$, as is easily seen by taking $R_{\bar{\rho}} := \bar{R}_\rho$, $\bar{R}_{\bar{\rho}} := R_\rho$ as solutions. R and $\overline{R}$ are fixed up to a choice of an invertible element in (ρ, ρ), i.e. if R' and $\overline{R}'$ is another solution, then there exists an invertible $A \in (\rho, \rho)$ such that $R' = (1_{\overline{\rho}} \otimes A) \circ R$ and $\overline{R}' = (A^{-1*} \otimes 1_\rho) \circ \overline{R}$. In fact, simply take $A = (\bar{R}^* \otimes 1_\rho) \circ (1_\rho \otimes R')$.

Conjugacy is determined up to isomorphism, i.e. given conjugate a 1-arrows ρ and $\bar{\rho}$ with solution $R, \bar{R}$, any other 1-arrow $\bar{\rho}'$ conjugate to ρ is isomorphic $\bar{\rho}$. In fact, let $R', \bar{R}'$ be solutions for ρ and $\bar{\rho}'$, then $(1_{\bar{\rho}} \otimes \bar{R}'^*) \circ (R \otimes 1_{\bar{\rho}'}) \in (\bar{\rho}', \bar{\rho})$ is invertible.

Given two pairs of conjugate 1-arrows $\rho_1, \bar{\rho}_1$ and $\rho_2, \bar{\rho}_2$ with solutions $R_1, \bar{R}_1$ and $R_2, \bar{R}_2$ respectively, one can check that their sum $R_1 \oplus R_2, \bar{R}_1 \oplus \bar{R}_2$ is a solution for the couple of conjugate 1-arrows $\rho_1 \oplus \rho_2$ and $\bar{\rho}_2 \oplus \bar{\rho}_2$. Analogously given two pairs $\rho, \bar{\rho}$ and $\sigma, \bar{\sigma}$ with solutions $R_\rho, \bar{R}_\rho$ and $R_\sigma, \bar{R}_\sigma$ respectively, such that the composition $\sigma \otimes \rho$ is defined, one can consider the product solution for $\sigma \otimes \rho, \bar{\rho} \otimes \bar{\sigma}$ defined as $R_{\sigma\otimes\rho} := (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho$, $\bar{R}_{\sigma\otimes\rho} := (1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}}) \circ \bar{R}_\sigma$.

The conjugate relations imply, among other things, Frobenius reciprocity, i.e. the spaces $(\rho, \sigma \otimes \eta) \cong (\rho \otimes \overline{\eta}, \sigma) \cong (\overline{\sigma} \otimes \rho, \eta) \cong (\overline{\eta} \otimes \overline{\sigma} \otimes \rho, \iota_A)$ are isomorphic.

We recall the definition of the $\bullet$ map introduced in [16].

Definition 6. *Given two 1-arrows ρ, σ , their conjugates $\overline{\rho}, \overline{\sigma}$, and a choice of solutions to the conjugation equations $R_\rho, \overline{R}_\rho$, $R_\sigma, \overline{R}_\sigma$, we define $\bullet : (\rho, \sigma) \to (\overline{\rho}, \overline{\sigma})$ by*

$$S^\bullet := (1_{\overline{\sigma}} \otimes \overline{R}_\rho^*) \circ (1_{\overline{\sigma}} \otimes S^* \otimes 1_{\overline{\rho}}) \circ (R_\sigma \otimes 1_{\overline{\rho}}), \ \forall S \in (\rho, \sigma)$$

and $\bullet : (\bar{\rho}, \bar{\sigma}) \to (\rho, \sigma)$ by

$$T^\bullet := (1_\sigma \otimes \bar{R}_{\bar{\rho}}^*) \circ (1_\sigma \otimes T^* \otimes 1_\rho) \circ (R_{\bar{\sigma}} \otimes 1_\rho), \ \forall T \in (\bar{\rho}, \bar{\sigma})$$

It is an antilinear isomorphism, and its square is the identity. When $\rho = \sigma$ it is an algebraic antilinear isomorphism and it satisfies $1_\rho^\bullet = 1_{\bar{\rho}}$. Notice that $\bullet$ depends on the choice of the solution $R, \bar{R}$, and that in general it does not commute with the $*$ operation.

$R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are positive elements of the commutative C^*- algebras (ι_A, ι_A) and (ι_B, ι_B) respectively, so they can be thought of as positive functions in $C(\Omega_A)$ and $C(\Omega_B)$, Ω_A and Ω_B the spectra of the two commutative algebras.

But we can say more, the functions $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are either zero or strictly positive on the connected components of Ω_A and Ω_B. Thus each

1-arrow ρ defines a projection in $C(\Omega_A)$, namely the projection on the components where $R^* \circ R$ is different from zero, and analogously $\bar{R}^* \circ \bar{R}$ with $C(\Omega_B)$. These projections do not depend on the choice of the solutions of the conjugate equations R and $\bar{R}$. In particular, if Ω_A and Ω_B are connected, then $R^* \circ R$ and $\overline{R}^* \circ \overline{R}$ are positive invertible functions.

These assertions are consequences of the following lemmas and propositions, most of which have been taken from [20] and [16].

Lemma 7. *Let $w \in (\iota_A, \iota_A)$, then the following conditions are equivalent*

a) $1_\rho \otimes w = 0$

b) $R^* \circ R \circ w = 0$

similarly if $z \in (\iota_B, \iota_B)$ the following conditions are equivalent

a′) $z \otimes 1_\rho = 0$

b′) $z \circ \overline{R}^* \circ \overline{R} = 0$

Proof. The implication $a) \to b)$ is obvious. Suppose, without loss of generality, that w is positive. Then $R^* \circ R \circ w = R^* \otimes w^{\frac{1}{2}} \circ R \otimes w^{\frac{1}{2}} = 0$, which implies $R \otimes w^{\frac{1}{2}} = 0$ by the C^*-property of the norm. But then $1_\rho \otimes w^{\frac{1}{2}} = \overline{R}^* \otimes 1_\rho \circ 1_\rho \otimes R \otimes w^{\frac{1}{2}} = 0$, thus $1_\rho \otimes w = 1_\rho \otimes w^{\frac{1}{2}} \otimes w^{\frac{1}{2}} = 0$. The proof of $a') \leftrightarrow b')$ is analogous. ∎

The maps $w \in (\iota_A, \iota_A) \to 1_\rho \otimes w \in Z(\rho, \rho)$ (the centre of (ρ, ρ)) and $z \in (\iota_B, \iota_B) \to z \otimes 1_\rho \in Z(\rho, \rho)$ are C^* homomorphisms into $Z(\rho, \rho)$. Denote by $S_l(\rho)$ and $S_r(\rho)$ the closed subspaces of Ω_A and Ω_B corresponding to the kernels of these maps.

Lemma 8. *$S_l(\rho)$ and $S_r(\rho)$ are the supports of $R^* \circ R$ and $\overline{R}^* \circ \overline{R}$ respectively.*

Proof. Suppose $u \subset \Omega_A$ is an open subset such that $(R^* \circ R)_{|_u} = 0$, then for any $w \in (\iota_A, \iota_A)$ with support in u we have $R^* \circ R \circ w = 0$. But this implies $1_\rho \otimes w = 0$ by lemma 7, thus $u \cap S_l(\rho) = \emptyset$ and $S_l(\rho) \subset suppR^* \circ R$.

If $\omega \notin S_l(\rho)$ then since $S_l(\rho)$ is closed we can find a $w \in (\iota_A, \iota_A)$ such that $\omega(w) \neq 0$ and $\omega'(w) = 0 \ \forall \omega' \in S_l(\rho)$. Thus $1_\rho \otimes w = 0$, so by lemma 7 $0 = \omega(R^* \circ R \circ w) = \omega(R^* \circ R)\omega(w)$ which implies $\omega(R^* \circ R) = 0$ and $suppR^* \circ R \subset S_l(\rho)$. The proof for $S_r(\rho)$ is analogous. ∎

Corollary 9. *The supports of $R^* \circ R$ and $\overline{R}^* \circ \overline{R}$ do not depend on the choice of R and $\overline{R}$.*

Lemma 10. *The following inequalities hold:*

$R \circ R^* \leq (R^* \circ R) \otimes 1_{\overline{\rho}\rho}$

$R \circ R^* \leq 1_{\overline{\rho}\rho} \otimes (R^* \circ R)$

$\overline{R} \circ \overline{R}^* \leq (\overline{R}^* \circ \overline{R}) \otimes 1_{\rho\overline{\rho}}$

$\overline{R} \circ \overline{R}^* \leq 1_{\rho\overline{\rho}} \otimes (\overline{R}^* \circ \overline{R})$.

Proof. Notice that $(R \circ R^*) \circ (R \circ R^*) = (R \circ R^*) \circ 1_{\overline{\rho}\rho} \otimes (R^* \circ R) = (R \circ R^*) \circ (R^* \circ R) \otimes 1_{\overline{\rho}\rho}$ where we are regarding $(R \circ R^*)$, $1_{\overline{\rho}\rho} \otimes (R^* \circ R)$, $(R^* \circ R) \otimes 1_{\overline{\rho}\rho}$

as positive elements of the algebra (ρ,ρ). In particular $1_{\overline{\rho}\rho}\otimes(R^*\circ R)$ and $(R^*\circ R)\otimes 1_{\overline{\rho}\rho}$ are elements of the centre $Z(\bar\rho\rho,\bar\rho\rho)$. Analogous relations hold for $\overline{R}\circ\overline{R}^*$.

Now, in general if we have a positive element X in a C^*-algebra A such that $X^2=XZ$, where Z is a positive element of the centre of A, we have $Z\geq X$. In fact, take a faithful representation (π,H) of the algebra A, take two generic vectors $\alpha\in H,\beta\in(X^{\frac{1}{2}}H)^\perp$

$$(X^{\frac{1}{2}}\alpha+\beta,X(X^{\frac{1}{2}}\alpha+\beta)) \;=\; (\alpha,X^2\alpha),$$
$$(X^{\frac{1}{2}}\alpha+\beta,Z(X^{\frac{1}{2}}\alpha+\beta)) \;=\; (\alpha,X^2\alpha)+(\beta,Z\beta).$$

Thus $Z\geq X$. ▮

Proposition 11. *For each positive $X\in(\rho\otimes\sigma,\rho\otimes\sigma)$ the following inequality holds:*

$$X\leq(\overline{R}^*\circ\overline{R})\otimes 1_\rho\otimes(R^*\otimes 1_\sigma\circ(1_{\overline{\rho}}\otimes X)\circ R\otimes 1_\sigma).$$

Proof

$$X=(1_\rho\otimes R^*\otimes 1_\sigma)\circ(\overline{R}\circ\overline{R}^*)\otimes X\circ(1_\rho\otimes R\otimes 1_\sigma)\leq$$

$$(1_\rho\otimes R^*\otimes 1_\sigma)\circ(\overline{R}^*\circ\overline{R})\otimes 1_{\rho\overline{\rho}}\otimes X\circ(1_\rho\otimes R\otimes 1_\sigma)$$

$$=(\overline{R}^*\circ\overline{R})\otimes 1_\rho\otimes(R^*\otimes 1_\sigma\circ(1_{\overline{\rho}}\otimes X)\circ R\otimes 1_\sigma)$$

where in the first line we have used the conjugation equations and in the second we have used the third inequality of the preceding lemma. ▮

Corollary 12. *The following inequality holds:*

$$(\overline{R}^*\circ\overline{R})\otimes 1_\rho\circ 1_\rho\otimes(R^*\circ R)\geq 1_\rho.$$

Corollary 13. *The following hold*

i) $1_\rho\otimes R^*\circ R\geq\frac{1}{||\overline{R}||^2}1_\rho$ *;* $\overline{R}^*\circ\overline{R}\otimes 1_\rho\geq\frac{1}{||R||^2}1_\rho$

ii) $(R^*\circ R)_{|_{S_l(\rho)}}\geq\frac{1}{||\overline{R}||^2}$ *;* $(\overline{R}^*\circ\overline{R})_{|_{S_r(\rho)}}\geq\frac{1}{||R||^2}$

iii) $S_l(\rho)$ *and* $S_r(\rho)$ *are open and closed.*

Lemma 14. *Let ρ,σ be 1-arrows from A to B and denote by $E_{S_l(\rho)}$ and $E_{S_l(\sigma)}$ the associated projections on $S_l(\rho)$, $S_l(\sigma)$ in (ι_A,ι_A). Suppose $E_{S_l(\rho)}E_{S_l(\sigma)}=0$, then $(\rho,\sigma)=0$.*

If $T\in(\rho,\sigma)$, then $T=1_\sigma\circ T\circ 1_\rho=(1_\sigma\otimes E_{S_l(\sigma)})\circ T\circ(1_\rho\otimes E_{S_l(\rho)})=1_\sigma\circ(T\otimes(E_{S_l(\sigma)}E_{S_l(\rho)})\circ 1_\rho=0$. ▮

An analogous assertion holds for right supports $S_r(\rho),S_r(\sigma)$.

Now consider two objects A,B and their corresponding identity 1-arrows ι_A,ι_B. As we have seen the algebras (ι_A,ι_A) and (ι_B,ι_B) are commutative C^*-algebras with spectra Ω_A and Ω_B respectively. Let P_A^i and P_B^j be the projections on the connected components of the two spectra. Then obviously $\sum_i P_A^i = 1_{\iota_A}$ and $\sum_j P_B^j = 1_{\iota_B}$.

To each projection P_A^i there will correspond a 1-arrow, which we will call ι_{A_i}. The same way to each P_B^j there will correspond a 1- arrow ι_{B_j}.

We would like to think of these 1-arrows as units corresponding to objects. In other words, we would like to "decompose" objects into sub-objects with corresponding connected spectrum. 1-arrows and 2-arrows should be decomposed accordingly . We must show that this can be done in a consistent manner.

Suppose we have a $2-C^*$-category $\mathcal{A}$ closed for direct sums, projections and conjugation. We define a new $2-C^*$-category $\mathcal{B}$ the following way (we will use the "only 2-arrows approach" mentioned above):

- for each object $A \in \mathcal{A}$ of the original category let $\{1_{\iota_{A_i}}\}$ be the set of central projections mentioned above,
- define as 2-arrows of the new category $\mathcal{B}$ all the elements of the form

  $$\{1_{\iota_{B_j}} \otimes S \otimes 1_{\iota_{A_i}}, \ \ \forall \ \ 1_{\iota_{B_j}}, 1_{\iota_{A_i}}, \ B \overset{\rho}{\leftarrow} A, \ B \overset{\sigma}{\leftarrow} A, \ S \in (\rho, \sigma)\}$$

  with the same $\otimes$ and $\circ$ operations of the original category.
- we set each projection $1_{\iota_{A_i}}$ to be a $\otimes$-unit (thus, also a $\circ$-unit),
- we define the $\circ$-units to be the set $\{1_{\iota_{B_j}} \otimes 1_\rho \otimes 1_{\iota_{A_i}}, \ \forall \ 1_{\iota_{A_i}}, \ 1_{\iota_{B_j}}, \ A \overset{\rho}{\leftarrow} B\}$

The units satisfy the necessary properties by definition and compatibility between the $\otimes$ and the $\circ$ products descends from the original one. The new 2-category is still closed under conjugation. In fact, let $B \overset{\rho}{\leftarrow} A$ and $A \overset{\bar\rho}{\leftarrow} B$ be two conjugate 1-arrows in the original category. Then each $1_{\iota_{B_j}} \otimes 1_\rho \otimes 1_{\iota_{A_i}}$ has as conjugate $1_{\iota_{A_i}} \otimes 1_{\bar\rho} \otimes 1_{\iota_{B_j}}$ with conjugate solutions $R_{i,j} := (1_{\bar\rho} \otimes 1_{\iota_{B_j}} \otimes 1_\rho) \circ R \circ 1_{\iota_{A_i}}$ and $\bar R_{i,j} := (1_\rho \otimes 1_{\iota_{A_i}} \otimes 1_{\bar\rho}) \circ \bar R \circ 1_{\iota_{B_j}}$.

We have *supp* $R_{i,j}^* \circ R_{i,j} = 1_{\iota_{A_i}}$ and *supp* $\bar R_{i,j}^* \circ \bar R^{i,j} = 1_{\iota_{B_j}}$.

Remark 1.5. The above construction is not a completion of a 2-C^*-category by subobjects in a usual sense. In fact, each unit 1_{ι_A} is sent to a finite set of units $1_{\iota_{A_i}}$.

So, formally speaking, our old $2-C^*$-category is not included in the new one, but it is easily recovered from it by considering linear combinations of the elements of $\mathcal{B}$.

In the sequel we will suppose that each object ($\otimes$-unit) has a connected spectrum, as this will ease the treatment of the subject.

2 Examples

2.1 Amplimorphisms between C^*-algebras and subfactors

Consider the $2-C^*$-category of amplimorphisms between unital C^*-algebras. The objects $M, N, \dots$ are unital C^*-algebras. The 1-arrows are amplimorphisms, for example $M \overset{\rho}{\leftarrow} N$ is $\rho : N \to M \otimes \mathsf{M}_n$, which assigns to each element $a \in N$ an $n \times n$ matrix with coefficients ${\rho_i}^j(a)$ in M. ρ is not supposed to be unital, thus ${\rho_i}^j(1)$ is in general a projection in $M \otimes \mathrm{M}_n$. Also note the dimension n of the matrix is not fixed and varies according to ρ. The $\otimes$ product is given by composition of amplimorphisms, i.e. for ${\rho_i}^j(.) : N \to M \otimes \mathsf{M}_n$ and ${\rho'}_{i'}^{j'}(.) : M \to P \otimes \mathrm{M}_{n'}$ $\rho' \otimes \rho : N \to P \otimes \mathrm{M}_{n' \times n}$ is defined by the expression

$$\rho' \otimes \rho\ {}_{i,i'}^{j,j'}(a) := {\rho'}_{i'}^{j'}({\rho_i}^j(a))\ \forall a \in N.$$

A 2-arrow $T \in (\rho, \sigma)$ connecting ${\rho_i}^j(.) : N \to M \times \mathrm{M}_n$ to ${\sigma_i}^{j'}(.) : N \to M \otimes \mathrm{M}_{n'}$ is an $n' \times n$ matrix with coefficients in M such that the following holds

$$\sum_k {T_i}^k {\rho_k}^j(a) = \sum_{k'} {\sigma_i}^{k'}(a) T_{k'}^j\ \forall a \in N$$

For each object N the corresponding unit 1-arrow is the identity morphism $\iota_N : N \to N$. For each 1-arrow (i.e. amplimorphism) ${\rho_i}^j$ the corresponding identity 2-arrow is the projection ${\rho_i}^j(1)$.

The $\circ$ product for 2-arrows is defined by matrix multiplication. i.e. for $\rho : N \to M \otimes \mathsf{M}_n$, $\sigma : N \to M \otimes \mathrm{M}_{n'}$, $\tau : N \to M \otimes \mathrm{M}_{n''}$, $S \in (\rho, \sigma)$ and $S' \in (\sigma, \tau)$, $S' \circ S$ is the $n'' \times n$ matrix $(S' \circ S)\ {}_{i''}^{j}$ with coefficients $\sum_{k'} {S'}_{i''}^{k'} S_{k'}^{j}$.

The $\otimes$ product for 2-arrows is defined as follows. Let $\rho : N \to M \otimes \mathsf{M}_n$, $\rho' : N \to M \otimes \mathrm{M}_{n'}$, $\sigma : M \to P \otimes \mathrm{M}_m$, $\sigma' : M \to P \otimes \mathrm{M}_{m'}$ and let $S \in (\rho, \rho')$, $T \in (\sigma, \sigma')$. Then $T_{i'}^{j'} \otimes {S_i}^j := \sum_{k'} T_{i'}^{k'} \sigma_{k'}^{j'}({T_i}^j)$.

Now consider two conjugate amplimorphism $\rho : N \to M \otimes \mathsf{M}_n$ and $\bar{\rho} : M \to N \otimes \mathrm{M}_m$. The 2-arrows $R \in (\iota_N, \bar{\rho} \otimes \rho)$ and $\bar{R} \in (\iota_M, \rho \otimes \bar{\rho})$ will be $n \times m$ column matrices of elements of N and M respectively (the same way, R^* and $\bar{R}^*$ will be $n \times m$ row matrices).

The conjugate equations

$$(\bar{R}^* \otimes 1_\rho) \circ (1_\rho \otimes R) = 1_\rho$$

$$(R^* \otimes 1_{\bar{\rho}}) \circ (1_{\bar{\rho}} \otimes \bar{R}) = 1_{\bar{\rho}}$$

in this case read

$$\sum_{m_1, n_2, n_1} \bar{R}^{*\ n_2, m_1} \rho_{n_1'}^{n_1}(1) \rho_{n_2}^{n_2'}(R_{m_1, n_1}) = \rho_{n_1'}^{n_2'}(1)$$

$$\sum_{n_1,m_2,m_1} R^{*\ m_2,n_1}\bar{\rho}_{m_1'}^{\ m_1}(1)\rho_{m_2}^{m_2'}(\bar{R}_{n_1,m_1}) = \bar{\rho}_{m_1'}^{m_2'}(1)$$

where $m_1, n_1', m_2, m_2' \in \{1, \dots m\}$ and $n_1, n_1', n_2, n_2' \in \{1, \dots n\}$.

$R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are positive elements of the centre of N and M respectively.

We also have a faithful conditional expectation from the algebra $(M \otimes \mathsf{M}_n)\rho_i^j(1)$ to the algebra N given by the expression

$$\sum_{m_1,m_1',n_1} R^{*m_1,n_1}\bar{\rho}_{m_1}^{m_1'}(b_{n_1}^{n_1'})R_{m_1',n_1'},\ \forall b_{n_1}^{n_1'} \in (M \otimes \mathsf{M}_n)\rho_i^j(1)$$

and an analogous conditional expectation from $(N \otimes \mathrm{M}_m)\bar{\rho}_i^j(1)$ to the algebra M given by the expression

$$\sum_{n_1,n_1',m_1} \bar{R}^{*n_1,m_1}\rho_{n_1}^{n_1'}(a_{m_1}^{m_1'})\bar{R}_{n_1',m_1'},\ \forall a_{m_1}^{m_1'} \in (N \otimes \mathrm{M}_m)\bar{\rho}_i^j(1)\ .$$

Conversely, suppose we have an inclusion of unital C^*-algebras $N \subset M$ with a faithful conditional expectation $\mathcal{E} : M \to N$ and a Pimsner-Popa basis, i.e. a finite set of elements $\lambda_i \in M$ such that the following holds

$$\sum_i \lambda_i \mathcal{E}(\lambda_i^* b) = b\ \forall b \in M.$$

Then, if we consider the inclusion morphism $\rho : N \to M$, a conjugate amplimorphism is given by $\bar{\rho}_i^{\ j}(b) := \mathcal{E}(\lambda_i b \lambda_j^*)$ with solutions $R_i := \mathcal{E}(\lambda_i^*)$, $\bar{R}_i := \lambda_i^*$.

If $N \subset M$ is an inclusion of von Neumann algebra subfactors, a Pimsner-Popa basis always exists. If the inclusion has finite index, the basis is finite (analogous results holds for the case of inclusions of simple C^*-algebras, see [9]). $R^* \circ R$ and $\bar{R}^* \circ \bar{R}$ are multiples of the identities in N and M. Choosing a standard solution (cfr. section 4), their product gives the value of the minimal index of the inclusion.

If the algebras are infinite, i.e. $M \otimes \mathsf{M}_n \simeq M$, then we can avoid the use of amplimorphisms, and it is sufficient to consider endomorphisms. This is the case of sectors (see [14]) for infinite factors.

An alternative example of a $2 - C^*$-category commonly used to describe the structure of subfactors is that of bimodules. It is equivalent to the above scheme.

A version of the Jones index theory for inclusions of C^*-algebras with conditional expectation is treated in [25].

2.2 Endomorphisms of a C^*-algebra

The C^*-category of unital endomorphisms of a C^*-algebra with non trivial centre $\mathcal{Z}$ is considered in [1] and [2]. The aim of these works is a generalization for the case of $\mathcal{Z} \neq \mathbf{C}$ of the following result, which was obtained as a step towards a theory of duality for compact groups (see [5] and [6]).

Proposition 15. *Let $\mathcal{A}$ be a unital C^*-algebra with trivial centre and let $\mathcal{T}$ be a symmetric tensor C^*-category of endomorphisms of $\mathcal{A}$, then there exists a C^*-algebra $\mathcal{F}$ and a compact group G of continuous automorphisms of $\mathcal{F}$, such that the following hold:*

- $\mathcal{A} \subset \mathcal{F}$ *and* $\mathcal{A}' \cap \mathcal{F} = \mathbf{C}I$,
- $\mathcal{F}^G = \mathcal{A}$,
- *there is a functorial map assigning to each element ρ of the category $\mathcal{T}$ an element H_ρ of the category of Hilbert spaces with support I in $\mathcal{F}$,*
- *$\mathcal{F}$ is generated by $\mathcal{A}$ and $\{H_\rho,\ \rho \in \mathcal{T}\}$,*
- *H_ρ induces ρ on $\mathcal{A}$,*
- $\epsilon(\rho, \rho') = \theta(H_\rho, H_{\rho'}),\ \rho, \rho' \in \mathcal{T}$.

By a Hilbert space in a C^*-algebra $\mathcal{F}$ with support I one means a linear space spanned by a set of partial isometries $\{\psi_i^\rho\}$ such that the following hold

- $\psi_i^{\rho *} \psi_j^\rho = \delta_{i,j} I$,
- $\sum_i \psi_i^\rho \psi_i^{\rho *} = I$.

These Hilbert spaces form a tensor C^*-category, with linear maps between them as arrows and each such space determines an endomorphism in $\mathcal{F}$ by the formula $a \in \mathcal{F} \to \sum_i \psi_i^\rho a \psi_i^{\rho *}$. By $\theta(H_\rho, H_{\rho'})$ one indicates the element $\sum_{i,j} \psi_i^{\rho'} \psi_j^\rho \psi_i^{\rho' *} \psi_j^{\rho *}$.

In [1] analogous results for the case of non trivial centres are obtained by demanding that the following conditions hold for the category $\mathcal{T}$. They state the existence of a sub-category which has many of the properties of the tensor C^*-category of Hilbert spaces (note that following statements are not independent):

P.1.1 $\mathcal{T}$ is closed w.r.t. direct sums $\alpha \oplus \beta$

P.1.2 $\mathcal{T}$ is closed w.r.t. subobjects $\beta < \alpha$,

P.1.3 $\mathcal{T}$ is closed w.r.t. complementary subobjects, i.e. if $\alpha \in \mathrm{Ob}\,\mathcal{T}$ and $\beta < \alpha$, then there is a subobject $\beta' < \alpha$ such that $\alpha = \beta \oplus \beta'$.

P.2 $\mathcal{T}$ contains a C*–subcategory $\mathcal{T}_{\mathbf{C}}$ with $\mathrm{Ob}\,\mathcal{T}_{\mathbf{C}} = \mathrm{Ob}\,\mathcal{T}$, where the arrows $(\alpha, \beta)_{\mathbf{C}} \subset (\alpha, \beta)$ satisfy the following properties:

P.2.1 $(\beta, \gamma)_{\mathbf{C}} \cdot (\alpha, \beta)_{\mathbf{C}} \subseteq (\alpha, \gamma)_{\mathbf{C}}$,

P.2.2 $1_\alpha \otimes (\beta, \gamma)_{\mathbf{C}} \subseteq (\alpha\beta, \alpha\gamma)_{\mathbf{C}}$,

P.2.3 $(\alpha, \beta)_{\mathbf{C}} \otimes 1_\gamma \subseteq (\alpha\gamma, \beta\gamma)_{\mathbf{C}}$,

P.2.4 $(\alpha, \beta)^*_{\mathbf{C}} \subseteq (\beta, \alpha)_{\mathbf{C}}$,

P.2.5 $(\iota, \iota)_{\mathbf{C}} = \mathbf{C}I$,

P.2.6 Any finite set of linearly independent elements $F_1, F_2, \ldots, F_n \in (\alpha, \beta)_{\mathbf{C}}$ (a complex Banach space), is linearly independent modulo $\alpha(\mathcal{Z})$ in (α, β), i.e. if $\sum_{j=1}^n \lambda_j F_j = 0$, $\lambda_j \in \mathbf{C}$, implies $\lambda_j = 0$, then also $\sum_{j=1}^n F_j \alpha(Z_j) = 0$ implies $\alpha(Z_j) = 0$. Moreover, $(\alpha, \beta)_{\mathbf{C}}$ generates (α, β), i.e. $(\alpha, \beta) = (\alpha, \beta)_{\mathbf{C}} \alpha(\mathcal{Z}) = \beta(\mathcal{Z})(\alpha, \beta)_{\mathbf{C}} \alpha(\mathcal{Z})$.

P.2.7 $\mathcal{T}_{\mathbf{C}}$ is closed w.r.t. direct sums, subobjects and complementary subobjects, i.e. now the required projections and isometries must lie in the corresponding intertwiner spaces $(\cdot, \cdot)_{\mathbf{C}}$.

P.3 There is a permutation structure on $\mathcal{T}_{\mathbf{C}}$

P.4 There is a conjugation structure on $\mathcal{T}_{\mathbf{C}}$.

Making these assumptions the authors obtain the following:

Theorem 16. *(i) Let $\mathcal{T}$ satisfy the postulates (P.1)–(P.4) above. Then there exists a Hilbert extension $\{\mathcal{F}, \mathcal{G}\}$ of $\mathcal{A}$ with $\mathcal{A}' \cap \mathcal{F} = \mathcal{Z}$ such that $\mathcal{T}$ is isomorphic to the category of all canonical endomorphisms of $\{\mathcal{F}, \mathcal{G}\}$.*

where the expression "Hilbert extension" is preferred to the more common "crossed product". Statements about uniqueness up to isomorphism of this constructions with respect to the choice of the subcategory $\mathcal{T}_{\mathbf{C}}$ are also given (we refer to the original work for precise definitions).

Remark 2.1. One major difficulty appearing when the algebra $\mathcal{A}$ has non trivial centre is to find a permutation structure for the tensor C^*-category $\mathcal{T}$, as in general an endomorphism $\rho \in \mathcal{T}$ acts non trivially on the centre $\mathcal{Z}$, i.e. $\rho(z) \neq z$ for $z \in \mathcal{Z}$, which would be in contrast with the usual axioms for a permutation symmetry. This is the main reason for the request of a (non full) subcategory $\mathcal{T}_{\mathbf{C}}$ on which a permutation symmetry is realized as in the classical case $\mathcal{Z} = \mathbf{C}$. Notice that requiring the existence of such a subcategory implies that the spaces of arrows (ρ, σ) are free $\mathcal{Z}$-bimodules.

Furthermore the action of the endomorphism on the centre of $\mathcal{Z}$ of the algebra is characterized by a commutative group called "the chain group", which turns out to be the dual of the centre of the group G. Conditions for the extension of the permutation symmetry to the whole $\mathcal{T}$ are discussed, as well as a central decomposition of the two algebras $\mathcal{A}$ and $\mathcal{F}$ (see [2] for details).

A more geometrically involved approach is undertaken in [24]. Here notions of "special endomorphism" and "weak permutation symmetry" for an endomorphism ρ of the algebra $\mathcal{A}$ are introduced, which generalize the notions given in

[5]. Such an object ρ determines a dimension $c_0(\rho)$ and a first Chern class $c_1(\rho)$ for vector bundles over the topological space X corresponding to the spectrum of $\mathcal{Z}^\rho$, the fixed points of the centre $\mathcal{Z}$ with respect to the action of the endomorphism ρ. It is then possible to embed the algebra $\mathcal{A}$ into a crossed product algebra of $\mathcal{A}$ by the endomorphism ρ, this construction depending on a choice $\mathcal{E}$ of a vector bundle over X with dimension c_0 and first Chern class $c_1(\rho)$. $\mathcal{A}$ is then recovered as the fixed points respect to the action of a closed subgroup of $\mathbf{SU}\mathcal{E}$, the continuous sections of special unitaries acting on the fibres of $\mathcal{E}$.

2.3 Hilbert C^*-modules

In [11] a notion of index and its relation to conjugation is studied in the context of 2-C^*-categories of bi-hilbertian bimodules over (non necessarily unital) C^*-algebras. In contrast to the case of von Neumann algebras factors, where most of the categorical structure appears with no additional assumptions, more care is needed here.

We briefly recall some definitions. Let A and B be C^*-algebras and X_A a right Hilbert C^*–module over A. By $\mathcal{L}(X_A)$ one denotes the C^*–algebra of A–module maps on X with an adjoint, by $\theta^r_{x,y}$ the rank one operator on X defined by $\theta^r_{x,y}(z) = x(y|z)_A$. The linear span of rank one operators is denoted by $FR(X_A)$ and called the ideal of finite rank operators. Its norm closure, $\mathcal{K}(X_A)$, is the C^*–algebra of compact operators, which is a closed ideal in $\mathcal{L}(X_A)$.

For a left Hilbert A–module X the definitions of rank one operators $\theta^l_{x,y}(z) = {}_A(z|y)x$, the spaces of finite rank operators $FR({}_AX)$, compact operators $\mathcal{K}({}_AX)$ and adjointable left A–module maps $\mathcal{L}({}_AX)$ are analogous.

An ${}_AX_B$ a bimodule over A and B is a right Hilbert A–B bimodule if , as a right B-module, it is endowed with a B-valued inner product making it into a right Hilbert B–module and for all $a \in A$ the map $\phi(a) : x \in X \mapsto ax \in X$ is adjointable, with adjoint $\phi(a)^* = \phi(a^*)$. The notion of left Hilbert A–B bimodule is defined in an analogous manner.

An A–B bimodule ${}_AX_B$ will be called bi-hilbertian if it is endowed with a right as well as a left Hilbert A–B C^*–bimodule structure in such a way that the two Banach space norms arising from the two inner products are equivalent.

For a bi-hilbertian bimodule X there exists a unique map $F : FR(X_B) \to A$, the additive extension of the form ${}_A(\cdot|\cdot)$ to the finite rank operators on X_B, F satisfying $F(T^*T) \geq 0$, $F(T^*) = F(T)^*$, $F(\phi(a)T) = aF(T)$, $F(T\phi(a)) = F(T)a$, for $a \in A$, $T \in FR(X_B)$.

In [11] the authors define a Hilbert bimodule ${}_AX_B$ to be of finite right numerical index if there exists $\lambda > 0$ such that for all $n \in \mathbb{N}$ and for all $x_1, \ldots, x_n \in X$,

$$\| \sum_1^n {}_A(x_i|x_i) \| \leq \lambda \| \sum_1^n \theta^r_{x_i,x_i} \|$$

(other equivalent conditions are given as well).

If X is a bi-Hilbertian A-B C^*–bimodule of finite right numerical index, then the additive extension F of the left inner product to $FR(X_B)$ can be extended uniquely to a norm continuous, positive, bilinear map $F : \mathcal{K}(X_B) \to A$. One has: $\|F\| = r - I[X]$ (the right numerical index). Furthermore one can extend uniquely the maps $\phi : A \to \mathcal{L}(X_B)$, $F : \mathcal{K}(X_B) \to A$ to normal positive maps $\phi'' : A'' \to \mathcal{K}(X_B)''$, $F'' : \mathcal{K}(X_B)'' \to A''$ between the corresponding enveloping von Neumann algebras. The right index is defined to be the element $F''(I)$ of A''. X is said to have finite right index when $F''(I) \in M(A)$. The left index is defined analogously, and a bimodule of both finite left and right (numerical) index is said to be finite (numerical) index . Let ${}_AX_B$ and ${}_BX_C$ be two bi-Hilbertian bimodules. One can consider the algebraic tensor product $X \odot_B Y$, which is an A-C bimodule. But it is in general not necessarily possible to define a norm on it, so as to have a bi-Hilbertian bimodule.

Proposition (2.13) in [11] states that if we consider bi-Hilbertian bimodules of finite numerical index the answer to the above problem is positive, and one can define univocally a bi-Hilbertian tensor product $X \otimes_B Y$.

Thus one can consider the following $2 - C^*$-category, with C^*-algebras as objects and finite numerical index bi-hilbertian bimodules ${}_AX_B$ as 1-arrows. As the space of 2-arrows connecting ${}_AX_B$ and ${}_AY_B$ one considers ${}_A\mathcal{L}_B({}_AX_B, {}_AY_B)$, the space of A–B-module adjointable maps.

The 1-arrow identity ι_A corresponding to the object A is ${}_AA_A$, i.e. A seen as a bimodule over itself , and $(\iota_A, \iota_A) = \mathcal{Z}M(A)$, the centre of the multiplier algebra.

One defines conjugation in a natural way: Let $X = {}_AX_B$ be an object of ${}_\mathcal{A}\mathcal{H}_\mathcal{A}$. An object $Y = {}_BY_A$ of ${}_\mathcal{A}\mathcal{F}_\mathcal{A}$ is called a conjugate of X if there exist intertwiners $R \in (\iota_B, Y \otimes_A X) \in_\mathcal{A} \mathcal{F}_\mathcal{A}$ and $\bar{R} \in (\iota_A, X \otimes_B Y) \in_\mathcal{A} \mathcal{F}_\mathcal{A}$ such that

$$\bar{R}^* \otimes I_X \circ I_X \otimes R = I_X; R^* \otimes I_Y \circ I_Y \otimes \bar{R} = I_Y.$$

It turns out that for a bi-hilbertian bimodule X the property of having a conjugate bimodule is equivalent to having finite index.

Finally, we quote the following result from (theorem 3.3 from [11]) which is the analogue in this context of our proposition (34)

3.3 Theorem *Let X be a bi-Hilbertian A-B C^*–bimodule of finite right index, and let Ω be the spectrum of $Z(M(A))$. Then for each $\omega \in \Omega$, the quotient C^*-algebra $Ł_\omega$ is finite dimensional, and*

$$dim(Ł_\omega) \leq [\lambda'^{-1}(r - Ind[X])(\omega)]^2,$$

where λ' is the best constant for which $\|{}_A(x|x)\| \geq \lambda' \|(x|x)_B\|$ and $[\mu]$ denotes the integral part of the real number μ. Furthermore the collection of epimorphisms $\pi_\omega : {}_AŁ(X_B) \to Ł_\omega$, $\omega \in \Omega$, defines a continuous bundle of C^*–algebras in the sense of [12].

2.4 Extensions of C^*-Categories.

In [23] a procedure for extending C^* categories by means of a C^*-algebra is given. This can be used to construct a category with non simple unit when (for example) extending by a commutative C^*-algebra.

Let $\mathcal{C}$ be a C^*-category with simple unit and $\mathcal{A}$ a C^*-algebra with identity 1. Given ρ, σ objects in the category $\mathcal{C}$ one defines $(\rho,\sigma)^{\mathcal{A}} := \mathcal{A} \otimes (\rho, \sigma)$, where $\otimes$ is the spatial tensor product. In a natural way one defines a composition law $(\sigma,\tau)_a^{\mathcal{A}} \times (\rho,\sigma)_a^{\mathcal{A}} \to (\rho,\tau)_a^{\mathcal{A}}$ and an involution $* : (\rho,\sigma)_a^{\mathcal{A}} \to (\sigma,\rho)_a^{\mathcal{A}}$ by extending naturally the one defined on $\mathcal{C}$. One then considers the category $\widetilde{\mathcal{C}}^{\mathcal{A}}$ having the same objects of $\mathcal{C}$ and arrows $(\rho,\sigma)^{\mathcal{A}}$ (which can be shown to be still a C^*-category) and closes under subobjects and obtains in this way a C^*-category $\mathcal{C}^{\mathcal{A}}$ whose objects are the projections $e \in (\rho,\rho)^{\mathcal{A}}$. This is called $\mathcal{C}^{\mathcal{A}}$, the *extension of* $\mathcal{C}$ *by* $\mathcal{A}$.

This procedure depends covariantly on the C^*-algebra and covariantly (contravariantly) on the category, corresponding to covariant (contravariant) C^*-functors on $\mathcal{C}$.

When the C^*-algebra $\mathcal{A}$ is commutative, i.e. $\mathcal{A} = C(\Omega)$ for some compact topological space Ω the above procedure turns out to preserve the (eventual) tensor structure of the category. We quote the following from [23], where $\mathcal{C}^{\Omega}$ indicates the extension given by the commutative C^* algebra $C(\Omega)$.

Proposition 17. *Let $\mathcal{C}$ be a C^*-category, Ω a compact Hausdorff space. Then*

(1) *Each C^*-functor $F : \mathcal{C}_1 \times \mathcal{C}_1 \to \mathcal{C}_2$ induces a C^*-functor $F^{\Omega} : \mathcal{C}_1^{\Omega} \times \mathcal{C}_1^{\Omega} \to \mathcal{C}_2^{\Omega}$.*

(2) *$\mathcal{C}^{\Omega}$ is a strict tensor C^*-category if $\mathcal{C}$ itself is so; the space of intertwiners of the identity object $\iota_\Omega := 1 \otimes 1_\iota$ in $\mathcal{C}^{\Omega}$ is $(\iota_\Omega, \iota_\Omega) \simeq \mathcal{C}(\Omega) \otimes (\iota,\iota)$, where ι is the identity object in $\mathcal{C}$.*

(3) *$\mathcal{C}^{\Omega}$ is symmetric if $\mathcal{C}$ is symmetric.*

(4) *If $\mathcal{C}$ has conjugates, then so does $\mathcal{C}^{\Omega}$.*

The following proposition, which we still quote from [23], emphasizes a geometrical interpretation of $\mathcal{C}^{\Omega}$ which has some analogies with our results.

Proposition 18. *Let $\mathcal{C}$ be a C^*-category closed under subobjects, Ω a compact Hausdorff space. Then*

(1) *Each object e in $\mathcal{C}^{\Omega}$ defines, via a fibre functor $\omega_* : \mathcal{C}^{\Omega} \to \mathcal{C}$, a family $\{e_\omega\}_{\omega\in\Omega}$ of objects in $\mathcal{C}$, called the fibres of e;*

(2) *The involution on $\mathcal{C}^{\Omega}$ is defined fibrewise, i.e. $* \circ \omega_* = \omega_* \circ *$ for each $\omega \in \Omega$;*

(3) *For each e, f objects in $\mathcal{C}^\Omega$, the space of arrows (e, f) defines a locally trivial continuous field of Banach spaces*

$$\{(e, f), \omega_* : (e, f) \to (e_\omega, f_\omega)\}, \tag{2.1}$$

over Ω, with fibres the spaces of intertwiners (e_ω, f_ω) of the fibres of e, f in $\mathcal{C}$.

(4) *If $\mathcal{C}$ is strict tensor the tensor product defined on $\mathcal{C}^\Omega$ induces on the spaces of arrows morphisms of continuous fields of Banach spaces.*

Remark 2.2. In this case the fibre bundles which arise are in a natural way locally trivial. This is not the general situation, as we will see.

2.5 Inclusions of von Neumann algebras

Faithful conditional expectations for inclusions of von Neumann algebras are related to solutions for the conjugation equations for amplimorphisms, as we have seen. Minimal conditional expectations correspond to "standard" solutions, which we will introduce in the sequel. Existence and uniqueness of minimal conditional expectations for inclusions of von Neumann algebras with discrete centres have been studied in [8], where it is shown that in the infinite discrete case uniqueness does not necessarily hold .

An irreducible subfactor is an inclusion of von Neumann algebra factors $N \subset M$ such that $N' \cap M = \mathbf{C}1$. Jones' basic construction gives an infinite tower of inclusions of factors $N \subset M \subset M_1 subset M_2 \dots$. Depth two means that the finite dimensional C^*-algebra $N' \cap M_2$ may be as well realized as the basic construction for the inclusion of finite algebras $\mathcal{C} = N' \cap M \subset N' \cap M_1$. In the categorical the setting if we call ρ the inclusion $N \subset M$ and $\bar{\rho} : M \to N$ the conjugate amplimorphism in the 2-C^*-category $\mathcal{C}$ with objects M, N and relative amplimorphisms and intertwiners as 1-arrows and 2-arrows, irreducible of depth two means that $(\rho, \rho) = \mathbf{C}1_\rho$ and, in the sub-2-category generated by ρ and $\bar{\rho}$, all irreducible 1-arrows from N to M are isomorphic to ρ. The finite dimensional algebras $N' \cap M_1$ and $M' \cap M_2$ turn out to be finite dimensional Hopf algebras in duality.

We quote the following result of [21] (where, among other things the notion of weak Hopf algebra can be found). It deals with a finite inclusion of C^*-algebras with finite centres. This corresponds to a 1-arrow ρ in a 2-C^*-category $\mathcal{C}$ with two objects M, N and with (ι_M, ι_M) and (ι_N, ι_N) finite dimensional C^*-algebras, i.e. Ω_M and Ω_N are discrete and finite sets.

Theorem 19. *Let $\mathcal{C}$ be a 2-C^*-category and $M \xleftarrow{\rho} N$ a 1-arrow satisfying the following assumptions:*

- *ρ has a conjugate $\bar{\rho}$;*
- *ρ is of depth two;*

- (ι_N, ι_N) *is a finite dimensional algebra.*

  Then $A := (\bar{\rho} \otimes \rho, \bar{\rho} \otimes \rho)$ *and* $B := (\rho \otimes \bar{\rho}, \rho \otimes \bar{\rho})$ *are finite dimensional algebras and there exists a bilinear form* $< . >: A \times B \to \mathbf{C}$ *such that* A *and* B *are weak Hopf algebras in duality.*

In section 5 we will take a step in a different direction, giving an analogous result in the case of connected (but in general greater than a single point) spectra Ω_M, Ω_N.

3 Bimodules and bundles.

We focus for a moment our attention on the spaces (ρ,σ). They have a structure of $(\iota_B,\iota_B)-(\iota_A,\iota_A)$ Hilbert bimodule given by the conjugation relations and the $\otimes$ product. In fact, given $z\in(\iota_B,\iota_B)$ and $w\in(\iota_A,\iota_A)$ and $T\in(\rho,\sigma)$ we can consider the tensor products $z\otimes T$ and $T\otimes w$ both still in (ρ,σ). Given $T,S\in(\rho,\sigma)$ we define the right (ι_A,ι_A) valued product $<T,S>^{(\rho,\sigma)}_{(\iota_A,\iota_A)}$ by $(R^*\circ R)^{-1}\circ R^*\circ(1_{\bar\rho}\otimes(S^*\circ T))\circ R$. The same way we define the left (ι_B,ι_B) valued product ${}_{(\iota_B,\iota_B)}<T,S>^{(\rho,\sigma)}$ by $(\bar R^*\circ\bar R)^{-1}\circ\bar R^*\circ((S^*\circ T)\otimes 1_{\bar\rho})\circ\bar R$. Both of these products are faithful.

When $\rho=\sigma$ the bimodule (ρ,ρ) has an obvious C^*- algebra structure by $\circ$ composition of 2-arrows. We call ρ indecomposable when the only projections in the centre $Z(\rho,\rho)$ of (ρ,ρ) are 1_ρ and 0.

Now we make the following

Assumption 1. *We assume that for each indecomposable 1-arrow $B\overset{\rho}{\leftarrow}A$ the following property holds:*

$$(\iota_B,\iota_B)\otimes 1_\rho = 1_\rho\otimes(\iota_A,\iota_A).$$

We don't claim that 2-arrows of this form exhaust all of $Z(\rho,\rho)$, but simply that for each $z\in(\iota_B,\iota_B)$ there exists $w\in(\iota_A,\iota_A)$ s.t. $z\otimes 1_\rho=1_\rho\otimes w$ (and vice versa). Notice that this property is closed for $\otimes$ products and sub-objects, thus it identifies a well defined sub-category.

We have the following

Proposition 20. *Each non zero indecomposable 1-arrow ρ from A to B gives an isomorphism of the two algebras (ι_A,ι_A) and (ι_B,ι_B). The isomorphism, which we shall denote $\theta_\rho:(\iota_A,\iota_A)\to(\iota_B,\iota_B)$, is independent of the choice of solutions of the conjugation equation and is given by the expression $\theta_\rho(w)={}_{(\iota_B,\iota_B)}<1_\rho\otimes w,1_\rho>^{(\rho,\rho)}$.*

Proof The first sentence is just a restatement of assumption 1: the maps $(\iota_B,\iota_B)\to(\iota_B,\iota_B)\otimes 1_\rho$ and $(\iota_A,\iota_A)\to 1_\rho\otimes(\iota_A,\iota_A)$ are injective (this follows from lemma 7 and the fact that $R^*\circ R$ and $\bar R^*\circ\bar R$ are invertible) . Assumption 1 tells us that they have the same images in $Z(\rho,\rho)$.

In order give the rest of the proof we first introduce a simple lemma

Lemma 21. *For ρ indecomposable one has $w\otimes 1_{\bar\rho\rho}=1_{\bar\rho\rho}\otimes w$ for all $w\in(\iota_A,\iota_A)$.*

As there exists $w'\in(\iota_A,\iota_A)$ s.t. $w\otimes 1_{\bar\rho\rho}=1_{\bar\rho\rho}\otimes w'$, we only have to prove that they are the same. But $w\otimes 1_{\bar\rho\rho}=1_{\bar\rho\rho}\otimes w'$ implies $w\otimes R=R\otimes w'$. And $w\otimes R=R\circ w=R\otimes w$ which implies $w'=w$, as we have shown that tensoring elements of (ι_A,ι_A) with R is an injective application.∎

We show that $\theta_\rho(w)\otimes 1_\rho=1_\rho\otimes w$. To do so we take the difference of the two elements and evaluate the product $<(1_\rho\otimes w-\theta_\rho(w)\otimes 1_\rho),(1_\rho\otimes w-\theta_\rho(w)\otimes$

$1_\rho) >^{(\rho,\rho)}_{(\iota_A,\iota_A)}$. Making use of the previous lemma shows that the product is zero, so the two objects must be equal as the (ι_A, ι_A) inner product is faithful. Also notice that the right hand side of $\theta_\rho(w) \otimes 1_\rho = 1_\rho \otimes w$ does not depend on the choice of solutions $R, \bar{R}$, thus the isomorphism θ_ρ must be independent as well. ∎

Remark 3.1.

In the same way we have an expression for $\theta_\rho^{-1} : (\iota_B, \iota_B) \to (\iota_A, \iota_A)$ with $\theta_\rho^{-1}(z) :=< z \otimes 1_\rho, 1_\rho >^{(\rho,\rho)}_{(\iota_A,\iota_A)}$. Also note that $\theta_\rho = \theta_{\bar{\rho}}^{-1}$ and $\theta_{\rho'} \circ \theta_\rho = \theta_{\rho' \otimes \rho}$ (when the composition of arrows is defined).

For the rest of this section we consider only indecomposable 1-arrows.

We recall the definition of a Banach bundle (see for ex. [4]).

Definition 22. *Let Ω be a compact Hausdorff topological space. A Banach bundle E over Ω is a family of Banach spaces $\{E_\omega,\ \|.\|^\omega\ ,\ \omega \in \Omega\}$ with a set $\Gamma \subset \prod_{\omega \in \Omega} E_\omega$ such that:*

i) Γ is a linear subspace of $\prod_{\omega\in\Omega} E_\omega$

ii) $\forall \omega \in \Omega\ \ \{S_{|_\omega}\ ,\ S \in \Gamma\}$ is dense in E_ω

iii) $\forall S \in \Gamma$ the norm function $\omega \to \|S_{|_\omega}\|^\omega$ is a continuous function on Ω

iv) Let $X \in \prod_{\omega\in\Omega} E_\omega$. If $\forall \omega \in \Omega$ and $\forall \epsilon > 0\ \exists S \in \Gamma$ such that $\|X_{|_\omega} - S_{|_\omega}\|^\omega < \epsilon$ in a neighbourhood of ω, then $X \in \Gamma$.

The elements of $\prod_{\omega\in\Omega_A} E_\omega$ are called sections and those of Γ continuous sections.

Analogously if the spaces E_ω have the structure of Hilbert spaces and the norm is given by the inner product, we will talk about Hilbert bundles. If the E_ω have the structure of C^*-algebras, and the space of continuous sections Γ is closed under multiplication and the $*$ operation, we will talk about a C^*-algebra bundle.

In a Banach bundle the fibre space might vary according to the base point. So it is a more general situation than that of a locally trivial bundle. The choice of a set $\Gamma \subset \prod_\omega E_\omega$ as the space of continuous sections is part of the initial data, as in general we have no local charts, with the implicit notion of continuity given by them.

Proposition 23. *Given two 1-arrows ρ, σ from objects A to B and a choice of the conjugation equations R_ρ , $\overline{R_\rho}$ for ρ, (ρ, σ) has the structure of a Hilbert bundle.*

Proof We evaluate the product $< S, T >^{(\rho,\sigma)}_{(\iota_A,\iota_A)}$ on each $\omega \in \Omega_A$ for any $S, T \in (\rho, \sigma)$. The procedure of the GNS construction gives us for each point ω a Hilbert space, which we shall denote by $(\rho, \sigma)_\omega$. We take $\prod_{\omega\in\Omega_A} (\rho, \sigma)_\omega$ as fibre bundle and the image of (ρ, σ) (which we will still denote by (ρ, σ)) as the module of continuous sections.

Note that for $S \in (\rho, \sigma)$ the topology given by $sup_{|_{\omega\in\Omega}} \|S_\omega\|^\omega$ is equivalent to the original one. In fact we have :

$sup_{|\omega\in\Omega_A} \|S_{|\omega}\|^\omega = \| < S,S >^{\frac{1}{2}}_{(\iota_A,\iota_A)} \| = \| < S,S >_{(\iota_A,\iota_A)} \|^{\frac{1}{2}}$
$\leq \|S^* S\|^{\frac{1}{2}} \| < 1_\rho, 1_\rho >_{(\iota_A,\iota_A)} \|^{\frac{1}{2}} = \|S^* S\|^{\frac{1}{2}}$
$\leq \| < S,S >_{(\iota_A,\iota_A)} (R^* R)^2 \theta_\rho^{-1}(\overline{R}^* \overline{R})\|^{\frac{1}{2}} = \| < S,S >^{\frac{1}{2}}_{(\iota_A,\iota_A)} (R^* R)\theta_\rho^{-1}(\overline{R}^* \overline{R})^{\frac{1}{2}}\|$
where we have used the expression of the definition of the inner product $<,>_{(\iota_A,\iota_A)}$, the monotonicity of the square root function and the inequality for $S^* S$ given by proposition 11.

So (ρ,σ) is closed even as a subspace of the Banach bundle.

The first three conditions are either very easy to prove or follow from the definitions. We prove only the last one. Suppose we have $X \in \prod_{\omega\in\Omega_A}(\rho,\sigma)_\omega$ satisfying condition $iv)$. Then, as Ω_A is compact, $\forall \epsilon > 0$ we can choose a finite family of elements $S^\alpha \in (\rho,\sigma)$ and a corresponding finite open covering $\{U_\alpha\}$ of Ω_A such that $\|X - S^\alpha\|^\omega_{|U_\alpha} \leq \epsilon$. Take a partition of unity f_α subordinate to the open covering. Then $\|X - \sum_\alpha f_\alpha S^\alpha\|^\omega \leq \epsilon$ because of convexity of the norm. Thus $X \in \overline{(\rho,\sigma)} = (\rho,\sigma)$. ∎

This gives the $(\iota_B,\iota_B) - (\iota_A,\iota_A)$ bimodule (ρ,σ) the structure of a Hilbert bundle over the compact topological space Ω_A. So we can think of each element T as a continuous section $T_{|\omega}$ in this Hilbert bundle. The right action of (ι_A,ι_A) is given by multiplying functions in $C(\Omega_A)$. The left action of (ι_B,ι_B) is given by the composition of the right action and the automorphism θ_ρ^{-1} into (ι_A,ι_A)[1].

But (ρ,σ) is not only a bimodule, as its elements can be regarded as operators between other spaces by $\circ$ composition of 2-arrows. For example the elements of (ρ,σ) can be regarded as operators from (η,ρ) to (η,σ). Suppose we have chosen solutions to the conjugation equations for ρ and for η, with the relative induced Hibert bundle structures. We have the following

Proposition 24. *For $T, T' \in (\rho,\sigma)$ such that $T_{|\omega} = T'_{|\omega}$, $P, P' \in (\eta,\rho)$ such that $P_{|\omega} = P'_{|\omega}$ then $(T\circ P)_{|\omega} = (T'\circ P)_{|\omega} = (T\circ P')_{|\omega} = (T'\circ P')_{|\omega}$.*

Proof $T_\omega = T'_\omega$ means $< T,S >^{(\rho,\sigma)}_{(\iota_A,\iota_A)\ |\omega} = < T',S >^{(\rho,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$ $\forall S \in (\rho,\sigma)$. Analogous relations hold for $P, P' \in (\eta,\rho)$.

We have $< T\circ P, Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |\omega} = < P, T^*\circ Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$
$= < P', T^*\circ Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |\omega} = < T\circ P', Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$, $\forall Q \in (\eta,\rho)$.

Thus $(T\circ P)_{|\omega} = (T\circ P')_{|\omega}$, and in the same way we have $(T'\circ P)_{|\omega} = (T'\circ P')_{|\omega}$.

Also we have $< T\circ P, Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$
$= (R^*_\eta \circ R_\eta)^{-1}(R^*_\rho\circ R_\rho) < T, Q\circ P^{*\bullet *\bullet *} >^{(\rho,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$
$= (R^*_\eta \circ R_\eta)^{-1}(R^*_\rho\circ R_\rho) < T', Q\circ P^{*\bullet *\bullet *} >^{(\rho,\sigma)}_{(\iota_A,\iota_A)\ |\omega}$

[1] This assertion relies on the hypothesis that ρ is indecomposable, the description in the general case is a little more tedious as it involves the action of a different automorphism for each connected component of the spectrum.

$=< T' \circ P, Q >^{(\eta,\sigma)}_{(\iota_A,\iota_A)\ |_\omega}$, $\forall Q \in (\eta,\rho)$, thus $(T\circ P)_{|_\omega} = (T'\circ P)_{|_\omega}$. ∎

Thus the $\circ$ composition preserves the fibre structure. The behaviour of the $\otimes$ product is a little more complicated. It is sufficient to describe what happens when tensoring with unit elements. Suppose we have $S, S' \in (C \xleftarrow{\rho'} B, C \xleftarrow{\sigma'} B)$ and $B \xleftarrow{\rho} A, D \xleftarrow{\rho''} C$. Then we have the following

Proposition 25. $(S\otimes 1_\rho)_{|_\omega} = (S'\otimes 1_\rho)_{|_\omega}$ *iff* $S_{|_{\theta_\rho^{-1*}(\omega)}} = S'_{|_{\theta_\rho^{-1*}(\omega)}}$.

$(1_{\rho''}\otimes S)_{|_\omega} = (1_{\rho''}\otimes S')_{|_\omega}$ *iff* $S_{|_\omega} = S'_{|_\omega}$.

Proof We fix solutions of the conjugation equations for ρ, ρ', ρ''. For the product of the $1-$arrows we take the product of the solutions. For example: $R_{\rho'\otimes\rho} := (1_{\overline{\rho}}\otimes R_{\rho'}\otimes 1_\rho)\circ R_\rho$, $\overline{R}_{\rho'\otimes\rho} := (1_{\rho'}\otimes \overline{R}_\rho \otimes 1_{\overline{\rho'}})\circ \overline{R}_{\rho'}$.

$(S\otimes 1_\rho)_{|_\omega} = (S'\otimes 1_\rho)_{|_\omega}$ if and only if

$0 \ = < (S-S')\otimes 1_\rho, (S-S')\otimes 1_\rho >^{(\rho'\otimes\rho\ ,\sigma'\otimes\rho)}_{(\iota_A,\iota_A)\ |_\omega}$

$= \ \theta_\rho^{-1}(< (S-S'), (S-S') >^{(\rho',\sigma')}_{(\iota_B,\iota_B)})_{|_\omega}$. Thus $S_{|_{\theta_\rho^{-1*}(\omega)}} = S'_{|_{\theta_\rho^{-1*}(\omega)}}$.

The proof of the second statement is analogous. ∎

Corollary 26. *For each* $\omega\in\Omega_A$ *the set* $I_\omega := \{S\in(\rho,\rho)$ *s.t.* $< S, S >_{(\iota_A,\iota_A)|_\omega} = 0\}$ *is a closed bilateral ideal of* (ρ,ρ).

Remark 3.2. $(\rho,\rho)/I_\omega$ is the pre-Hilbert space that gives rise to the fibre-Hilbert space $(\rho,\rho)_\omega$ when completed respect to the pre-scalar product norm. For each ω we can pursue the whole GNS construction and obtain a C^*-algebra $\pi_\omega(\rho,\rho)$ acting on this Hilbert space. The preceding corollary shows that $\pi_\omega(\rho,\rho)$ is the completion of the same pre-Hilbert space as above $(\rho,\rho)/I_\omega$ with respect to the C^*-norm given by the GNS construction.

We have the following

Proposition 27. (ρ,ρ) *has the structure of a* C^**-bundle*

Proof Proceed as in the beginning of proposition 23 and for each ω consider the GNS construction. We must show the continuity of this C^* norm with respect to the base point ω. For $A\in(\rho,\rho)/I_\omega$ we define

$$\|A\|_\omega^{C_1^*} := sup_{\tilde{y}\in(\rho,\rho)/I_\omega, \|\tilde{y}\|^{Hilbert}\leq 1}\|A\tilde{y}\|_\omega^{Hilbert},$$

where by $\|A\tilde{y}\|_\omega^{Hilbert}$ we mean $< Ay, Ay >_\omega^{\frac{1}{2}}$, for any $y\in(\rho,\rho)$ such that $y_{|_\omega} = \tilde{y}$. As this norm is defined as a *sup* over continuous functions, it is a priori only lower semicontinuous. As I_ω is a bilateral C^*-ideal, there is another candidate C^* norm, namely $\|A\|_\omega^{C_2^*} := inf_{y\in I_\omega}\|A-y\|$, the C^*-norm of the quotient C^*-algebra $(\rho,\rho)/I_\omega$. We show that these two norms are the same.

Lemma 28. $\|.\|_\omega^{C_2^*} = \|.\|_\omega^{C_1^*}$.

Take an approximate unit u_λ for I_ω. Then

$$inf_{y \in I_\omega} \|A - y\| = lim_{\lambda \to \infty} \|A(1 - u_\lambda)\| =$$

$$lim_{\lambda \to \infty}\ sup_{\phi \in \mathcal{S}(\rho,\rho)}\ \phi((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} =$$

$$lim_{\lambda \to \infty}\ sup_{\phi \in \mathcal{P}(\rho,\rho)}\ \phi((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}},$$

where $\mathcal{S}(\rho,\rho)$ and $\mathcal{P}(\rho,\rho)$ are the states and the pure states, respectively, of (ρ,ρ).

Notice that as $< .,. >$ is a faithful $C(\Omega_A)$-valued inner product, we can restrict ourselves evaluating the *sup* on pure states of the algebra (ρ,ρ) dominated by states of the form $< y, y >_{|_{\omega'}}$ for some $\omega' \in \Omega_A$ and some $y \in (\rho,\rho)$ such that $< y, y >_{|_{\omega'}} = 1$.

For each u_λ choose a sequence $\phi_n^\lambda \in \mathcal{P}(\rho,\rho)$ such that $lim_{n \to \infty}\ \phi_n^\lambda((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = \|A(1 - u_\lambda\|$. Then choose a diagonal sequence ϕ_λ such that $lim_{\lambda \to \infty}\ \phi_\lambda((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = lim_{\lambda \to \infty} \|A(1 - u_\lambda\|$. Let ϕ_0 be an accumulation point of this last sequence. Then $lim_{\lambda \to \infty}\ \phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}} = lim_{\lambda \to \infty}\ \|A(1 - u_\lambda)\|$. Suppose that ϕ_0 is dominated by a state of the kind $< y, .y >_{\omega'}$ for some $\omega' \neq \omega$. Then choosing a continuous function $g \in C(\Omega_A)$ such that $g(\omega') = 1$ and $g(\omega) = 0$ we have $lim_{\lambda \to \infty} \phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda)) = lim_{\lambda \to \infty} \phi_0(g(A(1 - u_\lambda))^* A(1 - u_\lambda))^{\frac{1}{2}}$. But the last term is zero, as $g \in I_\omega$. So any accumulation point must be dominated by a state of the form $< y, .y >_{|_\omega}$. But in ω each $u_\lambda = 0$, so $\phi_0((A(1 - u_\lambda))^* A(1 - u_\lambda)) = \phi_0(A^* A)$.

So we conclude

$$\|A\|_\omega^{C_2^*} = inf_{y \in I_\omega} \|A - y\| = \phi_0(A^* A)^{\frac{1}{2}} =$$

$$sup_{\tilde{y} \in (\rho,\rho)/I_\omega,\ \|\tilde{y}\|_\omega^{Hilbert} = 1} < A\tilde{y}, A\tilde{y} >_{|_\omega}^{\frac{1}{2}} = \|A\|_\omega^{C_1^*}.$$

Now we must show that $\|.\|^{C_2^*}$ is upper semicontinuous. Suppose $y^\omega \in I_\omega$ such that $\|A - y^\omega\| = \|A\|_\omega^{C_2^*} + \epsilon$. Then choose a neighbourhood $U_\omega \subset \Omega_A$ such that $\|y^\omega\|_{\omega'}^{Hilbert} \leq \epsilon\ \forall \omega' \in U_\omega$.

Then $\|A\|_{\omega'}^{C_2^*} = inf_{y^{\omega'} \in I_{\omega'}} \|A - y^{\omega'}\| \leq inf_{y^{\omega'} \in I_{\omega'}} (\|A - y^\omega\| + \|y^\omega - y^{\omega'}\|)$.

But $\|y^\omega - y^{\omega'}\| \leq const \| < (y^\omega - y^{\omega'}), (y^\omega - y^{\omega'}) > \|^{\frac{1}{2}}$, where the constant is independent of $y^\omega, y^{\omega'}$ and is given by lemma 11, and $inf_{y^{\omega'} \in I_{\omega'}} \| < (y^\omega - y^{\omega'}), (y^\omega - y^{\omega'}) > \|^{\frac{1}{2}} \leq \epsilon$. Thus $\|A\|_{\omega'}^{C_2^*} \leq \|A\|_\omega^{C_2^*} + \epsilon + const\epsilon$ for any $\omega' \in U_\omega$, which means that $\|.\|_\omega^{C_2^*}$ is upper semicontinuous.

Having proven continuity of the C^*-norm, the rest of the proof follows as in proposition 23. ∎

Remark 3.3. We have introduced two kinds of bundle structures for (ρ,ρ), the first one giving each fibre a scalar product norm, the second a C^* norm. As the inner (ι_A, ι_A) and (ι_B, ι_B) products depend on the choice of conjugation equation, the Hilbert bundle structure is defined only up to an isomorphism. In fact, consider (ρ,σ) with the bundle structure given by a solution $R, \bar{R}$ for ρ and a second bundle structure given by a second solution $(1_{\bar{\rho}} \otimes A) \circ R$, $(A^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$, where $A \in (\rho,\rho)$ is an invertible. Then the map $S \to S \circ A^{-1} \otimes (R^* \circ R)^{-1} \otimes (R^* \circ A^* \circ A \circ R)$ $\forall S \in (\rho,\sigma)$ is a unitary map between the two Hilbert bundles structures.

Even for $\rho \neq \sigma$ we can define for (ρ,σ) a Banach bundle structure where the fibre norm satisfies a C^* condition by defining $\|S_{|_\omega}\|_\omega := \|(S^* \circ S)_{|_\omega}\|_\omega^{C^*\ \frac{1}{2}}$. We will consider this structure by default, if not specified otherwise.

As we shall see in the sequel, the fibres are finite dimensional, so the two types of norms are equivalent. The latter bundle structure endows each Banach space fibre with its unique C^*-norm.

We summarize the situation as follows. Objects (say, A, B) connected by at least one non zero 1-arrow (say, ρ) have isomorphic centres, i.e. they determine a fixed topological space Ω, the spectrum of the centre. Objects corresponding to non isomorphic topological spaces have no nonzero 1-arrows connecting them. Thus the C^*- 2- category splits into disconnected components. The conjugation relations give explicit expressions of the isomorphisms, depending on the choice of the 1-arrow ρ, but not on the choice of the solutions to the conjugation equations. Each space of 2-arrows has a structure of a Hilbert bundle, which depends on the choice of solutions to the conjugation relations only up to isomorphism. Each element $S \in (\rho,\sigma)$ may be identified with a continuous section S_ω in the bundle over the topological space Ω.

Definition 29. *We define a $\circ$ and $\otimes$ product on each fibre space:*
$S_\omega \circ T_\omega := (S \circ T)_\omega$ *(for $S \in (\rho,\sigma)$,$T \in (\eta,\rho)$) ;*
$Q_{\theta_\rho^{-1*}(\omega)} \otimes S_\omega := (Q \otimes S)_\omega$ *(for $S \in (\rho,\sigma)$, $Q \in (\rho',\sigma')$).*

The consistency of these definitions is ensured by propositions 24 and 25.

Thus, we can think of the $\circ$ product as composing 2-arrows fibrewise and the $\otimes$ product acting on the fibre structure by "moving" the base points.

As a consequence we have

Corollary 30. *The maps*

$$S_{|_{\theta_\rho^{-1*}(\omega)}} \to S_{|_{\theta_\rho^{-1*}(\omega)}} \otimes 1_{\rho|_\omega};$$

$$S_{|_{\theta_\rho^{-1*}(\omega)}} \to 1_{\rho''|_{\theta_{\rho'}^{-1*} \circ \theta_\rho^{-1*}(\omega)}} \otimes S_{|_{\theta_\rho^{-1*}(\omega)}}$$

are injective.

Corollary 31. $R_{|_\omega}$ *and* $\overline{R}_{|_{\theta_\rho^{-1*}(\omega)}}$ *satisfy the conjugation relations:*

$$(\overline{R}^*_{|_{\theta_\rho^{-1*}(\omega)}} \otimes 1_{\rho|_\omega}) \circ (1_{\rho|_\omega} \otimes R_{|_\omega}) = 1_{\rho|_\omega};$$

$$(R^*_{|_\omega} \otimes 1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}}) \circ (1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes \overline{R}_{|_{\theta_\rho^{-1*}(\omega)}}) = 1_{\rho|_{\theta_\rho^{-1*}(\omega)}}.$$

Corollary 32. *The map $\bullet$ induces a conjugate linear isomorphism, which we will indicate with the same symbol, between the fibres* $\bullet : (\rho,\sigma)_{|_\omega} \to (\overline{\rho},\overline{\sigma})_{|_{\theta_\rho^{-1*}(\omega)}}$.

Corollary 33. *Let* $X_{|_\omega} \in (\rho,\rho)_{|_\omega}$ *be positive . Then the following inequality holds in* $(\rho,\rho)_{|_\omega}$ *:* $X_{|_\omega} \leq (\overline{R}^* \circ \overline{R})_{|_{\theta_\rho^{-1*}(\omega)}} \otimes 1_{\rho|_\omega} \otimes R^*_{|_\omega} \circ (1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes X_{|_\omega}) \circ R_{|_\omega}$.

These assertions are the "local" version of the ones already encountered in the introduction. For example corollary 33 is valid as long as $X_{|_\omega}$ is positive, independently of the value X in other points (thus, X need not be positive in (ρ,ρ)).

We also have the following

Proposition 34. *For each* (ρ,ρ) *and for each point of the base space* Ω *the associated* C^**-algebra fibre space* $(\rho,\rho)_\omega$ *is finite dimensional.*

Proof This essentially is the same proof as in [16].

Consider a set $\{X_{i|_\omega} \in (\rho,\rho)_\omega\}$ of positive elements of norm one with $\sum_i X_{i|_\omega} \leq 1_{\rho|_\omega}$. By the inequality in corollary 33 we have

$X_{i|_\omega} \ \leq \ (\overline{R}^* \circ \overline{R})_{|_{\theta_\rho^{-1*}(\omega)}} \otimes 1_{\rho|_\omega} \otimes (R^*_{|_\omega} \circ 1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes X_{i|_\omega} \circ R_{|_\omega})$

As the norm of each single $X_{i|_\omega}$ is one, we have

$1 = \|X_{i|_\omega}\| \leq \theta_\rho^{-1}(\overline{R}^* \circ \overline{R})_{|_\omega} \circ R^*_{|_\omega} \circ (1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes X_{i|_\omega}) \circ R_{|_\omega}$.

summing over i we have

$n \ \leq \ (\overline{R}^* \circ \overline{R})_{|_{\theta_\rho^{-1*}(\omega)}} \circ (R^*_{|_\omega} \circ 1_{\overline{\rho}|_{\theta_\rho^{-1*}(\omega)}} \otimes (\sum_i X_{i|_\omega}) \circ R_{|_\omega})$

$\leq \ (\overline{R}^* \circ \overline{R})_{|_{\theta_\rho^{-1*}(\omega)}} \circ (R^* \circ R)_{|_\omega}$. ∎

Remark 3.4. The above inequality is not optimal.

Remark 3.5. A posteriori we see that the fibre spaces $(\rho,\rho)/I_\omega$ are finite dimensional, thus the completion in the Hilbert and the C^*-norm was superfluous. They are finite dimensional (thus unital) C^*-algebras.

Corollary 35. *For each* (ρ,σ) *and for each point of the base space* Ω *the associated Hilbert fibre space* $(\rho,\sigma)_\omega$ *is finite dimensional.*

Proof That $(\sigma,\rho)_\omega \circ (\rho,\sigma)_\omega$ is finite dimensional follows from the preceding proposition and remark by immerging it into $(\rho,\rho)_\omega$. But the map $(\rho,\sigma)_\omega \to (\rho,\sigma)_\omega \circ (\sigma,\rho)_\omega$: $S_\omega \to S^*_\omega \circ S_\omega$, $S_\omega \in (\rho,\sigma)_\omega$ is injective. Thus $(\rho,\sigma)_\omega$ is finite dimensional as well. ∎

Lemma 36. *Let* $T \in (B \overset{\rho}{\leftarrow} A, B \overset{\sigma}{\leftarrow} A)$, *with* ρ *and* σ *indecomposable* 1*-arrows. Then* $\forall \omega \in \Omega_A \mid T_\omega \neq 0$ *the homeomorphisms between* Ω_A *and* Ω_B *induced by* ρ *and* σ *coincide, i.e.* $\theta_\rho^{*-1}(\omega) = \theta_\sigma^{*-1}(\omega)$.

Proof For any $w \in (\iota_A, \iota_A)$ we have

$$T \otimes w = T \circ (1_\rho \otimes w) = T \circ (\theta_\rho(w) \otimes 1_\rho) = \theta_\rho(w) \otimes T$$

and in the same way we have

$$T \otimes w = (1_\sigma \otimes w) \circ T = \theta_\sigma(w) \otimes 1_\sigma) \circ T = \theta_\sigma(w) \otimes T.$$

In particular

$$(\theta_\rho(w) \otimes T)_{|_\omega} = ((\theta_\sigma(\omega) \otimes T)_{|_\omega}$$

which by proposition 25 is equivalent to

$$\theta_\rho(w)_{|_{\theta_\rho^{*-1}(\omega)}} \otimes T_{|_\omega} = \theta_\sigma(w)_{|_{\theta_\rho^{*-1}(\omega)}} \otimes T_{|_\omega},$$

which in turn gives $\theta_\rho(w)_{|_{\theta_\rho^{*-1}(\omega)}} = \theta_\sigma(w)_{|_{\theta_\rho^{*-1}(\omega)}}$. As this must hold for any $w \in (\iota_A, \iota_A)$ we have $\theta_\rho^{*-1}(\omega) = \theta_\sigma^{*-1}(\omega)$. ∎

We show now that starting from a 2-C^*-category $\mathcal{C}$, for each $\omega_0 \in \Omega_A$, where A is an arbitrary object, it is possible to construct a 2-C^*-category, which we will indicate by $\mathcal{C}^{\omega_0,A}$, with simple units.

Consider the full sub 2-C^*-category of $\mathcal{T}$ of C generated by indecomposable 1-arrows, their products and their sub1-arrows. This way to each $B \xleftarrow{\rho} A \in \mathcal{T}$ will be associated a homeomorphism $\theta_\rho^{*-1} : \Omega_A \to \Omega_B$.

Now define

- as objects the set $\{\omega_B \in \Omega_B,\ \forall B \in \mathcal{C}\}$ such that there exists a $B \xleftarrow{\rho} A \in \mathcal{T}$ with $\omega_B = \theta_\rho^{*-1}(\omega_0)$,
- as 1-arrows for any ω_B, ω_C as above a $\omega_C \xleftarrow{\sigma} \omega_B$ in correspondence to any $C \xleftarrow{\sigma} B$ verifying $\theta_\sigma^{*-1}(\omega_B) = \omega_C$
- as 2-arrows, for any $\omega_C \xleftarrow{\sigma} \omega_B$, $\omega_C \xleftarrow{\eta} \omega_B$ as above, $(\omega_C \xleftarrow{\sigma} \omega_B, \omega_C \xleftarrow{\eta} \omega_B)$ to be the set $\{T_{|_{\omega_B}},\ \forall T \in (\sigma, \eta)\}$.

Remark 3.6. By the preceding lemma we see that for $\omega_C \xleftarrow{\sigma} \omega_B$, $\omega'_C \xleftarrow{\eta} \omega_B$, $\omega_C \neq \omega'_C$ $\forall T \in (\sigma, \eta)$ $T_{|_{\omega_B}} = 0$, i.e. we don't loose any information by considering ω_C and ω'_C as distinct objects in the new category.

We define the $\circ$ and $\otimes$ products of 29 as the operations of our new category and we endow the spaces of 2-arrows with the fibre C^*-norm introduced above, thus obtaining a $2 - C^*$-category. For each object ω_B we have the 1-unit $\omega_B \xleftarrow{\iota_{\omega_B}} \omega_B$ corresponding to ι_B. For each 1-arrow $\omega_C \xleftarrow{\sigma} \omega_B$ we have the 2-unit $1_{\sigma|_{\omega_B}}$. As

$$(\omega_B \xleftarrow{\iota_{\omega_B}} \omega_B, \omega_B \xleftarrow{\iota_{\omega_B}} \omega_B) = (\iota_B, \iota_B)_{\omega_B} = \mathbf{C},$$

we see that each 1-unit is simple. Corollary 31 ensures that this category is closed under conjugation. Closure with respect to projections is not automatically ensured. For example, if $P^{\omega_B} \in (\omega_C \xleftarrow{\sigma} \omega_B, \omega_C \xleftarrow{\sigma} \omega_B)$ is a projection, there does not necessarily exist a projection (thus a corresponding 1-arrow)

P in (σ,σ) such that $P_{|_{\omega_B}} = P^{\omega_B}$. We will consider the completion under projections of the above category, and denote it by $\mathcal{C}^{\omega_0,A}$.

Note that in $\mathcal{C}^{\omega_0,A}$ several points $\omega_B, \omega'_B \dots$ of Ω_B may appear as distinct objects. In fact, as the various maps $\theta_\rho^{*-1} : \Omega_A \to \Omega_B$ form a groupoid, we see that each point ω_0 determines an orbit in the spaces $\Omega_A, \Omega_B, \dots$, and that the construction leading to $\mathcal{C}^{\omega_0,A}$ depends only on the choice one of these (disjoint) orbits.

We consider a particular case as an example. Let $B \xleftarrow{\rho} A, A \xleftarrow{\bar\rho} B$, be a pair of indecomposable, conjugate 1-arrows. We can consider the full sub $2-C^*$-category generated by these two elements, i.e. their compositions $\rho,\ \rho\otimes\bar\rho, \rho\otimes\bar\rho\otimes\rho\dots$ and their sub-1-arrows. This is the categorical analogue of Jones' basic construction. We can take the sequence of algebras

$$(\rho,\rho),\ (\rho\otimes\bar\rho, \rho\otimes\bar\rho),\ (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho), \dots$$

Each of these algebras can be embedded injectively into another:

$$(\rho,\rho) \to (\rho,\rho)\otimes 1_{\bar\rho} \subset (\rho\otimes\bar\rho, \rho\otimes\bar\rho),$$

$$(\rho\otimes\bar\rho, \rho\otimes\bar\rho) \to (\rho\otimes\bar\rho, \rho\otimes\bar\rho)\otimes 1_\rho \subset (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho), \dots$$

We choose $\omega_0 \in \Omega_A$ and construct $\mathcal{C}^{\omega_0,A}$ as above. In this case we have only two objects, namely ω_0 and $\theta_\rho^{*-1}(\omega_0)$, as $\theta_{\bar\rho}^* = \theta_\rho^{*-1}$, which implies $\theta_{\rho\otimes\bar\rho}^{*-1} = \theta_\rho^{*-1}\circ\theta_{\bar\rho}^{*-1} = id$. Thus, for example,

$$\theta^{*-1}_{\rho\otimes\bar\rho\otimes\rho}(\omega_0) \xleftarrow{\rho\otimes\bar\rho\otimes\rho} \omega_0 = \ \theta_\rho^{*-1}(\omega_0) \xleftarrow{\rho\otimes\bar\rho\otimes\rho} \omega_0,$$

and analogously for the other 1-arrows of $\mathcal{A}^{\omega_0,A}$.

As in the case of subfactors, we have a sequence of inclusions of finite dimensional C^*-algebras:

$$(\rho,\rho)_{\omega_0} \to (\rho,\rho)_{\omega_0}\otimes 1_{\bar\rho|_{\theta_\rho^{*-1}(\omega_0)}} \subset (\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega_0)}$$

$$(\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega_0)} \to (\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega_0)}\otimes 1_{\rho|_{\omega_0}} \subset (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho)_{\omega_0}\dots$$

4 Standard solutions

In this section we will introduce a particular class of solutions to the conjugation equations.

We begin by recalling the analogous definition and some basic facts concerning the case of simple units, contained in [16]. So, for the moment, we suppose that ρ is a 1-arrow going from A to B and that $(\iota_A, \iota_A) = \mathbf{C}$, $(\iota_B, \iota_B) = \mathbf{C}$.

Let $\bar{\rho}$ be a conjugate 1-arrow going from B to A. The main difference from (and simplification) the general case is that the algebra (ρ, ρ) and its isomorphic $(\bar{\rho}, \bar{\rho})$ are finite dimensional C^*-algebras, i.e. a finite direct sum of matrix algebras. It is thus possible to decompose the unit arrow 1_ρ as a sum of minimal projections e_i in the algebra (ρ, ρ). To each of these minimal projections will correspond an irreducible ρ_i, i.e. $(\rho_i, \rho_i) = \mathbf{C}$, and we think of ρ as a direct sum $\oplus_i \rho_i$, i.e. there exists a complete family of isometric 2-arrows $W_i \in (\rho_i, \rho)$ s.t. $W_i \circ W_i^* = e_i$, $W_i^* \circ W_i = 1_{\rho_i}$. Two projections e_i, e_j dominated by the same minimal central projection in (ρ, ρ) will lead to equivalent irreducible 1-arrows, i.e. there will exist a unitary $V_{i,j} \in (\rho_i, \rho_j)$.

For irreducible ρ, i.e. $(\rho, \rho) = \mathbf{C}$, we give the following definition (notice that by Frobenius reciprocity $\bar{\rho}$ is irreducible too):

Definition 37. *Let $\rho, \bar{\rho}$ be irreducible. $R, \bar{R}$ are said to be a standard solution if $R^* \circ R = \bar{R}^* \circ \bar{R}$.*

Let now ρ be not necessarily irreducible and $\oplus_i \rho_i$ its decomposition into irreducibles. Let $\{ \bar{\rho}_i \}$ be irreducibles, each conjugate to its corresponding ρ_i and R_i, $\bar{R}_i$ standard solutions for each of these couples .

Then $\oplus_i \bar{\rho}_i$ and ρ are conjugate and $\oplus_i R_i, \oplus_i \bar{R}_i$ is a solution.

Definition 38. *Let $\rho = \oplus \rho_i$ and $\bar{\rho} = \oplus_i \bar{\rho}_i$ be conjugate. A solution of the form $\oplus_i R_i$, $\oplus_i \bar{R}_i$, where the R_i and $\bar{R}_i$ are irreducible and standard as defined above, is called a standard solution.*

As we will see, standard solutions are uniquely defined up to a unitary in (ρ, ρ).

In this context standard solutions always exist:

Proposition 39. *Let ρ and $\bar{\rho}$ be conjugate 1-arrows between objects A and B, with $(\iota_A, \iota_A) = \mathbf{C}$ and $(\iota_B, \iota_B) = \mathbf{C}$. Then standard solutions always exist.*

Proof. Choose an arbitrary solution $R', \bar{R}'$. Decompose ρ into a sum of irreducibles $\oplus_i \rho_i$ by means of a complete family of orthogonal projections $\{e_i\}$ in (ρ, ρ). The corresponding elements $e_i^\bullet \in (\bar{\rho}, \bar{\rho})$ will be a complete family of disjoint idempotents(i.e. $e_i^\bullet e_j^\bullet = \delta_{i,j} e_i^\bullet$), as the $\bullet$ operation is an anti-isomorphism between the algebras (ρ, ρ) and $(\bar{\rho}, \bar{\rho})$. But they will fail to be self adjoint. Nevertheless there will exist an invertible $A \in (\bar{\rho}, \bar{\rho})$ such that $\{\bar{e}_i := A e_i^\bullet A^{-1}\}$ is a complete family of orthogonal projections. As before, we can decompose $\bar{\rho} = \oplus_i \bar{\rho}_i$, where each $\bar{\rho}_i$ corresponds to the projection $\bar{e}_i$.

Taking now $R'' := (A \otimes 1_\rho) \circ R'$ and $\bar{R}'' := (1_\rho \otimes A^{-1*}) \circ \bar{R}'$ as solutions, we see that each pair ρ_i and $\bar{\rho}_i$ is a couple of conjugates, with $R''_i := (\bar{e}_i \otimes e_i) \circ R''$, $\bar{R}''_i := (e_i \otimes \bar{e}_i) \circ \bar{R}''$ as solutions.

Furthermore each pair $R''_i, \bar{R}''_i$ is determined up to a constant as ρ_i and $\bar{\rho}_i$ are irreducible. We rescale each couple by a constant so to have the equality $R_i^{''*} \circ R''_i = \bar{R}_i^{''*} \circ \bar{R}''_i$. This amounts to multiplying the invertible element $A \in (\bar{\rho}, \bar{\rho})$ by a diagonal element D in the same algebra (in fact A is determined up to diagonal elements and permutations of the $\bar{e}_i$ belonging to the same central projection). We set $R := (D \otimes 1_\rho) \circ R''$ and $\bar{R} := (1_\rho \otimes D^{-1}) \circ \bar{R}''$ and $R_i := (\bar{e}_i \otimes e_i) \circ R$, $\bar{R}_i := (e_i \otimes \bar{e}_i) \circ \bar{R}$. It is easily verified that $R = \oplus_i R_i$ and $\bar{R} = \oplus_i \bar{R}_i$. ∎

The number $R_i^* \circ R_i = \bar{R}_i^* \circ \bar{R}_i$ is called the dimension of the irreducible ρ_i and depends only on the equivalence class of the ρ_i. In fact, suppose ρ_i and ρ_j are equivalent, i.e. there exists a partial isometry $V_{i,j} \in (\rho, \rho)$ such that $V^* \circ V = e_i$ and $V \circ V^* = e_j$ (in other words, e_i and e_j have same central support). Then, by Frobenius reciprocity, the same will hold for $\bar{\rho}_i$ and $\bar{\rho}_j$, with a partial isometry $\bar{V}_{i,j} \in (\bar{\rho}, \bar{\rho})$.

Take $(\bar{V}_{i,j} \otimes V_{i,j}) \circ R_i$ and $(V_{i,j} \otimes \bar{V}_{i,j}) \circ \bar{R}_i$ as a solution for ρ_j and $\bar{\rho}_j$. This solution can differ from $R_j, \bar{R}_j$ only by a invertible element in $(\rho_j, \rho_j) = \mathbf{C}$, i.e. $(\bar{V}_{i,j} \otimes V_{i,j}) \circ R_i = \lambda R_j$ and $(V_{i,j} \otimes \bar{V}_{i,j}) \circ \bar{R}_i = \lambda^{-1} \bar{R}_j$. But this implies $(R_i^* \circ R_i)(\bar{R}_i^* \circ \bar{R}_i) = (R_j^* \circ R_j)(\bar{R}_j^* \circ \bar{R}_j)$ and , because of the normalization we have chosen above, $R_i^* \circ R_i = R_j^* \circ R_j$.

The number $R^* \circ R = \sum_i R_i^* \circ R_i$ is the statistical dimension of the object ρ. It is additive with respect to direct sums.

The class of standard solution defines a trace on the algebra (ρ, ρ) (and in an analogous way on the algebra $(\bar{\rho}, \bar{\rho})$).

Take an element $S \in (\rho, \rho)$. Thinking of (ρ, ρ) as a direct sum of matrix algebras, we indicate by $S_{i,j}$ the matrix elements of S corresponding to a representation given by the projectors $\{e_i\}$.

Thus $(\bar{R}^* \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}) = (\sum_i S_{i,i} \bar{R}_i^* \circ \bar{R}_i) = (\sum_i S_{i,i} R_i^* \circ R_i) = R^* \circ (1_{\bar{\rho}} \otimes S) \circ R$ which is a trace, as the dimensions $R_i^* \circ R_i$ depend only on the central supports of the e_i.

Notice that we have shown that for a standard solution the following holds:

$$R^* \circ (1_{\bar{\rho}} \otimes S) \circ R = \bar{R}^* \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R} \quad \forall S \in (\rho, \rho).$$

This can be used as an equivalent definition for standardness (see [16]).

Remark. We can summarize the situation as follows: fix a faithful normalized trace tr (i.e. $tr(1) = 1$) on the algebra (ρ, ρ). A generic solution $R', \overline{R}'$ to the conjugation relations will induce faithful functionals $\Phi'(S) := R' \circ (1_\rho \otimes S) \circ R' = tr(DKS)$, $\Psi'(S) := \bar{R}' \circ (S \otimes 1_\rho) \circ R' = tr(DK^{-1}S)$, where D is a positive invertible central element in (ρ, ρ), whose trace is the dimension of ρ, and K is a positive invertible element of (ρ, ρ). Taking $R := (1_{\bar{\rho}} \otimes K^{-1}) \circ R'$ and $\bar{R} := (K \otimes 1_{\bar{\rho}}) \circ \bar{R}'$ will give a standard solution. Notice that the element

K is uniquely defined by the original choice $R', \bar{R}'$. The arbitrariness of the choice of the trace is expressed by the central element D.

We now drop the hypothesis of simple units and give the general definition of standardness.

Let R and $\bar{R}$ be solutions of the conjugation equation for ρ and $\bar{\rho}$. Consider a complete set $\{P_i\}$ of orthogonal central projections in (ρ, ρ), i.e. $\sum_i P_i = 1_\rho$ and $P_i P_j = 0$ for $i \neq j$. Then also $\{P_i^\bullet\}$ is a complete set of central projections in $(\overline{\rho}, \overline{\rho})$. This implies $R^* \circ R = \sum_i R_i^* \circ R_i$ where $R_i := (P_i^\bullet \otimes P_i) \circ R$. The same way $\bar{R} = \sum \bar{R}_i$, $\bar{R}_i := (P_i \otimes P_i^\bullet) \circ \bar{R}$ and $\bar{R}^* \circ \bar{R} = \sum_i \bar{R}_i^* \circ \bar{R}_i$. To each orthogonal projection P_i we associate a sub-1-arrow ρ_i.

Remark 4.1. Each choice of the set $\{P_i\}$ is bound to be finite.

Proof In fact

$$(\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ R) = (\sum_i (\bar{R}_i^* \circ \bar{R}_i)) \otimes (\sum_j P_j) \otimes (\sum_k (R_k^* \circ R_k))$$

$$\geq \sum_i (\bar{R}_i^* \circ \bar{R}_i) \otimes P_i \otimes (R_i^* \circ R_i).$$

Taking the (ι_B, ι_B) product of this last term with the identity

$$_{(\iota_B,\iota_B)} < \sum_i (\bar{R}_i^* \circ \bar{R}_i) \otimes P_i \otimes (R_i^* \circ R_i), 1_\rho >^{(\rho,\rho)}$$

we get a positive bounded function in $C(\Omega_B)$, i.e. $\sum_i (\bar{R}_i^* \circ \bar{R}_i) \theta_{\rho_i} (R_i^* \circ R_i)$. By corollary (12) we have $(\bar{R}_i^* \circ \bar{R}_i) \otimes P_i \otimes (R_i^* \circ R_i) \geq P_i$, which implies $(\bar{R}_i^* \circ \bar{R}_i) \theta_{\rho_i} (R_i^* \circ R_i) \geq 1$. Thus the sum above is finite. ∎

We can consider the finite central decomposition of $1_\rho = \oplus_i 1_{\rho_i}$, where, as before, ρ_i is the 1-arrow corresponding to the central projection P_i.

Definition 40. *We define a solution to be standard if* $\forall X \in (\rho, \rho)$ *the following holds :*

$$\oplus_i \; 1_{\rho_i} \otimes (R_i^* \circ (1_{\bar{\rho}} \otimes X) \circ R_i) = \oplus_i \; (\bar{R}_i^* \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_i) \otimes 1_{\rho_i}.$$

Remark 4.2. The following is an equivalent condition

$$(R_i^* \circ (1_{\bar{\rho}} \otimes X) \circ R_i)_{|\omega} = (\bar{R}_i^* \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_i)_{|_{\theta_{\rho_i}^{-1*}(\omega)}} \quad \forall X \in (\rho, \rho), \; \forall \omega \in \Omega, \; \forall i.$$

The following lemma shows that another appropriate name could have been "minimal".

Lemma 41. *Let* $R, \bar{R}$ *be standard solutions for* $\rho, \bar{\rho}$. *Then for any other solution* $R', \bar{R}'$ *we have:* $\oplus_i (\bar{R}'^*_i \circ \bar{R}'_i) \otimes 1_{\rho_i} \otimes (R'^*_i \circ R'_i) \geq \oplus_i (\bar{R}_i{}^* \circ \bar{R}_i) \otimes 1_{\rho_i} \otimes (R_i^* \circ R_i)$.

Proof. We may suppose, without loss of generality, that ρ and $\bar{\rho}$ are indecomposable. We have $R' = (1_{\bar{\rho}} \otimes X) \circ R$ and $\bar{R}' = (X^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$ for some invertible $X \in (\rho, \rho)$. Thus

$$(\bar{R}'^* \circ \bar{R}') \otimes 1_\rho \otimes (R'^* \circ R') =$$

$$(\bar{R}^* \circ (X^{-1} \circ X^{-1*}) \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) =$$

$$1_\rho \otimes ((R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) \circ (R^* \circ 1_{\bar{\rho}} \otimes (X^{-1} \circ X^{-1*}) \circ R)).$$

The claim is implied by the inequality

$$(R^* \circ 1_{\bar{\rho}} \otimes (X^* \circ X) \circ R) \circ (R^* \circ 1_{\bar{\rho}} \otimes (X^{-1} \circ X^*) \circ R) \geq (R^* \circ R)^2,$$

which can be written equivalently

$$< X, X >_{(\iota_A, \iota_A)} < X^{-1*}, X^{-1*} >_{(\iota_A, \iota_A)} \geq 1$$

(where we have used the $< \, , \, >_{(\iota_A, \iota_A)}$ inner product defined by R and $\bar{R}$).

It is sufficient to prove this inequality for each $\omega \in \Omega_A$, which is easily done by means of a Cauchy-Schwarz argument: first rewrite $< X, X >_{(\iota_A, \iota_A)}$ and $< X^{-1*}, X^{-1*} >_{(\iota_A, \iota_A)}$ as $< (X \circ X^*)^{\frac{1}{2}}, (X \circ X^*)^{\frac{1}{2}} >_{(\iota_A . \iota_A)}$ and $< (X \circ X^*)^{-\frac{1}{2}}, (X \circ X^*)^{-\frac{1}{2}} >_{(\iota_A, \iota_A)}$ respectively. Then

$$< (X \circ X^*)^{\frac{1}{2}}, (X \circ X^*)^{\frac{1}{2}} >_{(\iota_A, \iota_A)} < (X \circ X^*)^{-\frac{1}{2}}, (X \circ X^*)^{-\frac{1}{2}} >_{(\iota_A, \iota_A)}$$

$$\geq \quad (< 1_\rho, 1_\rho >_{(\iota_A, \iota_A)})^2 = 1. \quad \blacksquare$$

Lemma 42. *Suppose that $R, \bar{R}$ and $R', \bar{R}'$ are two pairs of standard solutions. Then there exists a unitary $U \in (\rho, \rho)$ such that $R' = (1_{\bar{\rho}} \otimes U) \circ R$ and $\bar{R}' = (U \otimes 1_{\bar{\rho}}) \circ \bar{R}$.*

Proof. We suppose for simplicity (and without loss of generality) ρ to be indecomposable. As we have seen, there exists an invertible U satisfying $R' = (1_{\bar{\rho}} \otimes U) \circ R$ and $\bar{R}' = (U^{-1*} \otimes 1_{\bar{\rho}}) \circ \bar{R}$. We must prove that U is unitary.

By the definition of standardness we have

$$(\bar{R} \circ A \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho = 1_\rho \otimes (R^* \circ 1_\rho \otimes A \circ R), \ \forall A \in (\rho, \rho).$$

This implies $(\bar{R} \circ (U^{-1} A U^{-1}) \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho = 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes (U^* A U^{-1*}) \circ R)$. As R' and $\bar{R}'$ are standard solutions as well, we have

$$(\bar{R} \circ (U^{-1} A U^{-1*}) \otimes 1_{\bar{\rho}} \circ \bar{R}) \otimes 1_\rho = 1_\rho \otimes (R^* \circ 1_{\bar{\rho}} \otimes (U^* A U) \circ R), \ , \ \forall \in (\rho, \rho).$$

This implies

$$< U, AU >_{(\iota_A, \iota_A)} = < U^{-1*}, AU^{-1*} >_{(\iota_A, \iota_A)} \ \ \forall A \in (\rho, \rho).$$

In particular $< U, U >_{(\iota_A, \iota_A)} = < U^{-1*}, U^{-1*} >_{(\iota_A, \iota_A)}$. R' and $\bar{R}'$ also satisfy the minimality condition of the preceding lemma, i.e. $(\bar{R}'^* \circ R'^*) \otimes 1_\rho \otimes (R'^* \circ$

$R') = (\bar{R}^* \circ \bar{R}) \otimes 1_\rho \otimes (R^* \circ R)$. In other words we have $< U, U >_{(\iota_A, \iota_A)} = < U^{-1*}, U^{-1*} >_{(\iota_A, \iota_A)} = 1$ (where we are using as before the $<, >_{(\iota_A, \iota_A)}$ inner product relative to the $R, \bar{R}$ solution). Noticing that

$$< U^*U, U^*U >_{(\iota, \iota_A)} = < UU^*U, U >_{(\iota_A, \iota_A)}$$

$$= < U^{-1*}U^*U, U^{-1*} >_{(\iota_A, \iota_A)} = < 1_\rho, 1_\rho >_{(\iota_A, \iota_A)}$$

we see that the product $< (1 - U^*U), (1 - U^*U) >_{(\iota_A, \iota_A)}$ is zero.

thus $U = U^{-1*}$, as the (ι_A, ι_A) inner product is faithful. ∎

Remark 4.3. In an analogous way one can show that there exists a unitary $\bar{U} \in (\bar{\rho}, \bar{\rho})$ such that $R' = \bar{U} \otimes 1_\rho \circ R$ and $\bar{R}' = 1_\rho \otimes \bar{U} \circ \bar{R}$.

Proposition 43. *Let $R, \bar{R}$ be a standard solution for $\rho, \bar{\rho}$. Then the associated inner product $<, ., >_{(\iota_A, \iota_A)}$ is tracial, i.e.*

$$< ST, 1_\rho >_{(\iota_A, \iota_A)} = < TS, 1_\rho >_{(\iota_A, \iota_A)} \quad \forall S, T \in (\rho, \rho).$$

Proof It suffices to prove

$$< U^*SU, 1_\rho >_{(\iota_A, \iota_A)} > = < S, 1_\rho >_{(\iota_A, \iota_A)}$$

for any unitary $U \in (\rho, \rho)$, $\forall S \in (\iota_A, \iota_A)$. We have

$$< U^*SU, 1_\rho >_{(\iota_A, \iota_A)} = (R^* \circ R)^{-1} \circ R^*(1_{\bar{\rho}} \otimes U^*SU) \circ R$$

$$= (R^* \circ R)^{-1} \circ R^* \circ (1_{\bar{\rho}} \otimes U^*) \circ (1_{\bar{\rho}} \otimes S) \circ (1_{\bar{\rho}} \otimes U) \circ R.$$

But $R' := (1_{\bar{\rho}} \otimes U) \circ R$, $\bar{R}' := (U \otimes 1_{\bar{\rho}}) \circ \bar{R}$ is still a standard solution, so there exists a unitary $\bar{U} \in (\bar{\rho}, \bar{\rho})$ such that $R' = (\bar{U} \otimes 1_\rho) \circ R$ and $\bar{R}' = (1_\rho \otimes \bar{U}) \circ \bar{R}$. Thus

$$< USU^*, 1_\rho >_{(\iota_A, \iota_A)} = (R^* \circ R)^{-1} \circ R^* \circ (\bar{U}^* \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes S) \circ (\bar{U} \otimes 1_\rho) \circ R$$

$$= (R^* \circ R)^{-1} R^* \circ (1_{\bar{\rho}} \otimes S) \circ R = < S, 1_\rho >_{(\iota_A, \iota_A)} . \quad ∎$$

Remark 4.4. Analogously one proves that the $_{(\iota_B, \iota_B)} <, ., >$ inner product is tracial too.

Lemma 44. *Let ρ and σ be two indecomposable 1-arrows with standard solutions $R_\rho, \bar{R}_\rho$ and $R_\sigma, \bar{R}_\sigma$ respectively. Then the product solution is standard.*

Proof. The definition of the product solution for $\sigma \otimes \rho$ is $R_{\sigma \otimes \rho} := 1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho \circ R_\rho$, $\bar{R}_{\sigma \otimes \rho} := 1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}} \circ \bar{R}_\sigma$.

For every $A \in (\sigma \otimes \rho, \sigma \otimes \rho)$ we have :

$$1_{\sigma \otimes \rho} \otimes (R^*_{\sigma \otimes \rho} \circ (1_{\bar{\rho} \otimes \bar{\sigma}} \otimes A) \circ R_{\sigma \otimes \rho}) =$$

$$1_\sigma \otimes 1_\rho \otimes (R^*_\rho \circ (1_{\bar{\rho}} \otimes R^*_\sigma \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes 1_{\bar{\sigma}} \otimes A) \circ (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho).$$

Using the standard solution property of $R_\rho, \bar{R}_\rho$ on the element $R_\sigma^* \otimes 1_\rho \circ (1_{\bar{\sigma}} \otimes A) \circ R_\sigma \otimes 1_\rho \in (\rho, \rho)$ we get:

$$1_\sigma \otimes 1_\rho \otimes (R_\rho^* \circ (1_{\bar{\rho}} \otimes R_\sigma^* \otimes 1_\rho) \circ (1_{\bar{\rho}} \otimes 1_{\bar{\sigma}} \otimes A) \circ (1_{\bar{\rho}} \otimes R_\sigma \otimes 1_\rho) \circ R_\rho) =$$

$$1_\sigma \otimes (\bar{R}_\rho^* \circ ((R_\sigma^* \otimes 1_\rho \circ 1_{\bar{\sigma}} \otimes A \circ R_\sigma \otimes 1_\rho) \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho) \otimes 1_\rho =$$

$$1_\sigma \otimes (R_\sigma^* \otimes \bar{R}_\rho^* \circ (1_{\bar{\sigma}} \otimes A \otimes 1_\rho) \circ R_\sigma \otimes \bar{R}_\rho) \otimes 1_\rho.$$

The same reasoning applied to the solution $R_\sigma, \bar{R}_\sigma$ and the element $\bar{R}_\rho^* \circ (A \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho \in (\sigma, \sigma)$ gives :

$$1_\sigma \otimes (R_\sigma^* \otimes \bar{R}_\rho^* \circ (1_{\bar{\sigma}} \otimes A \otimes 1_\rho) \circ R_\sigma \otimes \bar{R}_\rho) \otimes 1_\rho =$$

$$(\bar{R}_\sigma^* \circ (1_\sigma \otimes \bar{R}_\rho^* \otimes 1_{\bar{\sigma}}) \circ (A \otimes 1_{\bar{\rho}} \otimes 1_{\bar{\sigma}}) \circ (1_\sigma \otimes \bar{R}_\rho \otimes 1_{\bar{\sigma}}) \circ \bar{R}_\sigma) \otimes 1_\sigma \otimes 1_\rho.$$

Thus

$$1_{\sigma\otimes\rho} \otimes (R_{\sigma\otimes\rho}^* \circ (1_{\bar{\rho}\otimes\bar{\sigma}} \otimes A) \circ R_{\sigma\otimes\rho}) = (\bar{R}_{\sigma\otimes\rho}^* \circ (A \otimes 1_{\sigma\otimes\rho}) \circ \bar{R}_{\sigma\otimes\rho}) \otimes 1_{\sigma\otimes\rho}$$

and analogous equations hold for each element $(\sigma \otimes \rho)_i$ of the central decomposition, i.e.

$$1_{(\sigma\otimes\rho)_i} \otimes (R_{(\sigma\otimes\rho)_i}^* \circ (1_{(\bar{\rho}\otimes\bar{\sigma})_i} \otimes A) \circ R_{(\sigma\otimes\rho)_i})$$
$$= (\bar{R}_{(\sigma\otimes\rho)_i}^* \circ (A \otimes 1_{(\sigma\otimes\rho)_i}) \circ \bar{R}_{(\sigma\otimes\rho)_i}) \otimes 1_{(\sigma\otimes\rho)_i}. \quad \blacksquare$$

Proposition 45. *The class of standard solutions is stable under the operations of direct sum, tensor product, projections and conjugation.*

Proof . Closure with respect to direct sums, conjugation and sub-objects is obvious from the definition of standard solutions. Closure with respect to the tensor product follows from the above lemma. $\blacksquare$

The natural question is whether a choice of standard solution is availble for all 1-arrows. In the rest of this chapter we will give some partial answers.

As already mentioned, the notion of Banach bundle is more general than the more familiar notion of locally trivial bundle, even in the case of bundles with finite dimensional fibres.

The following example shows the necessity for considering such a notion.

Example.

Consider the following tensor C^*-category (i.e. 2-C^*-category with one object): take $SU_n \times [0,1]$, the trivial group bundle. $\Gamma(SU_n \times [0,1])$ the group of continuous sections and $\mathcal{U} := \{\xi \in \Gamma(SU_n \times [0,1])\ s.t.\ \xi_\omega = 1_n$ (the identity of SU_n), $\forall \omega \in [\frac{1}{2}, 1]\}$, a closed subgroup. Consider the trivial bundle $H \times [0,1]$, where $H = \mathbb{C}^n$, together with the natural action of $\mathcal{U}$ on it and denote it by ρ. Denote by $\rho^{\otimes n}$ the nth tensor product (fibre tensor product over the base space $[0,1]$) of the same bundle with itself, with the natural action of $\mathcal{U}$. For

$n = 0$ let $\rho^0 = \iota =: \mathbb{C} \times [0,1]$, the trivial line bundle with the trivial action of $\mathcal{U}$.

The powers of ρ induce a C^* tensor category, where (ρ^n, ρ^m) are sections of intertwining operators, i.e. continuous sections $S \in (H^n, H^m) \times [0,1]$ such that $S\rho^n(g)\xi = \rho^m(g)S\xi$ $,\forall \xi \in \Gamma(H^n \times [0,1])$, $\forall g \in \mathcal{U}$. It is easy to see that ι is the unit object in this tensor category. The conjugate object $\overline{\rho}$ is the conjugate fibre $\overline{H} \times [0,1]$ with the conjugate action of $\mathcal{U}$. A standard solution for the conjugation equations is given by $R \in (\iota, \overline{\rho}\rho) := \sum_i \overline{e_i} \otimes e_i$; $\overline{R} \in (\iota, \rho\bar{\rho}) := \sum_i e_i \otimes \overline{e_i}$, where e_i and $\bar{e}_i$ are the constant sections given by the canonical basis for H and $\bar{H}$ respectively.

Then $(\rho, \rho) = \{S \in \Gamma(M_n \times [0,1])$ such that $S_\omega \in \mathbb{C}1_n \forall \omega \in [0, \frac{1}{2}]\}$. Thus the fibres of the Banach bundle (ρ, ρ) are of two types, $\mathbb{C}$ for $\omega \in [0, \frac{1}{2}]$ and M_n for $\omega \in (\frac{1}{2}, 1]$.

We recall some facts about the structure of C^*-algebra bundles.

We already know from the definition that the fibre (ρ, ρ) is full, i.e. $\forall \omega \in \Omega$, $\forall K^\omega \in (\rho, \rho)_\omega$, $\exists A \in (\rho, \rho)$ such that $A_{|_\omega} = K^\omega$. In other words, each element of a single fibre can be extended to a continuous section defined on the whole base space. More can be said. The following lemma (whose proof can be found for example in [7]) will be useful in the sequel.

Lemma 46. *Let F be a finite dimensional C^*-algebra, Ξ a C^*-algebra bundle over a normal topological space Ω and $\Omega \times F$ be the product (trivial) C^*-algebra bundle with fibre F. Let $\Phi : A \times F \to \Xi_{|_A}$ be a C^*-algebra bundle embedding of the reduced bundles over a subset $A \subset \Omega$. Then there exists an open subset $U \supset A$ and an embedding $\Psi : U \times F \to \Xi_{|_U}$ which extends Φ.*

This enables us to prove the following (we suppose here that the base space is a normal topological space):

Lemma 47. *Let K^ω be a positive invertible element in the finite dimensional algebra $(\rho, \rho)_\omega$. Then there exists an invertible $A \in (\rho, \rho)$ and an open neighbourhood of ω U_ω such that $A_{|\omega} = K^\omega$ and $A_{|_{\omega'}} = 1_{\rho|_{\omega'}}$ $\forall \omega' \notin U_\omega$.*

Proof Take an open set $U_\omega \ni \omega$ and a C^*algebra bundle embedding $\Psi : U_\omega \times (\rho, \rho)_\omega \to (\rho, \rho)_{|_{U_\omega}}$ that extends the identity bundle embedding. Take a positive invertible section H in the product bundle $U_\omega \times (\rho, \rho)_\omega$ which extends K^ω (for example the constant section). Now take a second open set $W \ni \omega$ such that $U_\omega \supset \bar{W}$ (we can do so, as the base space is normal) and a continuous complex valued function f defined on U_ω such $f = 0$ out of W and $f(\omega) = 1$. Then $\Psi(\exp(f \ln H))$ will be a continuous section with value K^ω in ω and value $1_{\rho|_{\omega'}}$ for ω' out of W. Extending it with the identity section on the rest of the base space Ω we get a globally defined continuous section with the desired property. ∎

We can now state a first result concerning standardness :

Proposition 48. *For each $\omega \in \Omega_A$ there exists a solution to the conjugation equations $\overset{\omega}{R}, \overset{\omega}{\bar{R}} \in (\rho, \rho)$ such that $\oplus_i \, (1_{\rho_i} \otimes (\overset{\omega}{R_i}{}^* \circ (1_{\bar{\rho}_i} \otimes X) \circ \overset{\omega}{R_i}))_{|\omega} = \oplus_i \, ((\overset{\omega}{\bar{R}_i}{}^* \circ (X \otimes 1_{\rho_i}) \circ \overset{\omega}{\bar{R}_i}) \otimes 1_{\rho_i})_{|\omega}$.*

Remark. As before we have indicated with R_i^{ω} and $\bar{R}_i^{\omega}$ the elements of the central decomposition, defined by $R_i^{\omega} := (P_i^{\bullet} \otimes P_i) \circ R^{\omega}$ and $\bar{R}_i^{\omega} := (P_i \otimes P_i^{\bullet}) \circ \bar{R}^{\omega}$.

Proof. Take a general solution $R, \bar{R}$. As corollary 31 points out, we can, mutatis mutandis, repeat the argument of the simple unit case for the algebras $(\rho, \rho)_{\omega}$ and $(\bar{\rho}, \bar{\rho})_{\theta_{\rho}^{-1*}(\omega)}$ and the local solutions $R'_{|_{\omega}}, \bar{R}'_{|_{\theta_{\rho}^{-1*}(\omega)}}$. By the remarks after proposition 39 we see that there exists a positive invertible $K^{\omega} \in (\rho, \rho)_{\omega}$ such that

$$R^*_{|_{\omega}} \circ (1_{\bar{\rho}|_{\theta_{\rho}^{*-1}(\omega)}} \otimes (K^{\omega -1} \circ S_{|_{\omega}} \circ K^{\omega -1}) \circ R_{|_{\omega}} =$$

$$\bar{R}^*_{|_{\theta_{\rho}^{-1*}(\omega)}} \circ ((K^{\omega *} \circ S_{|_{\omega}} \circ K^{\omega} \otimes 1_{\bar{\rho}|_{\theta_{\rho}^{-1*}(\omega)}}) \circ \bar{R}_{|_{\theta_{\rho}^{-1*}(\omega)}}$$

where we have used the $\circ$ and $\otimes$ products of fibre elements introduced in the preceding section.

By the above lemma we can choose a positive invertible element $K \in (\rho, \rho)$ such that $K_{|_{\omega}} = K^{\omega}$. Let $R^{\omega} := (1_{\bar{\rho}} \otimes K^{-1}) \circ R$ and $\bar{R}^{\omega} := (K \otimes 1_{\bar{\rho}}) \circ \bar{R}$. ∎

Remark. This implies that for each $\omega \in \Omega$ $R^{\omega *}_{|_{\omega}} \circ (1_{\bar{\rho}|_{\omega}} \otimes X_{|_{\omega}}) \circ R)_{|_{\omega}}$ is a uniquely defined (non normalized) trace on $(\rho, \rho)_{\omega}$. It would be tempting to use this trace as a definition for a standard (ι_A, ι_A) valued trace. Unfortunately in the general case the section K^{ω} is not a priori continuous, thus the above formula does not give a continuous trace, but only an upper semicontinuous one, as it is the inferior limit of a family of continuous functionals.

Nevertheless, we have the following

Proposition 49. *Suppose (ρ, ρ) is a locally trivial bundle. Then a standard solution exists.*

Proof. If (ρ, ρ) is locally trivial it has constant fibre, i.e. for each point of the base space ω, $(\rho, \rho)_{\omega}$ is isomorphic to a finite dimensional algebra F. We can choose a finite atlas of local charts, i.e. maps $\Theta_{\alpha} := U_{\alpha} \times F \to (\rho, \rho)_{|_{U_{\alpha}}}$ which are local isomorphisms of the trivial bundle $U_{\alpha} \times F$ onto the restriction of (ρ, ρ) over the open space $U_{\alpha} \subset \Omega$. The sets U_{α} form an open covering of Ω. Where two maps overlap we have transition functions, i.e. when for example $U_{\alpha} \cap U_{\beta} \neq \emptyset$ we have unitary sections in $W_{\alpha,\beta} \in (U_{\alpha} \cap U_{\beta}) \times F$ such that $\Theta_{\alpha|_{U_{\alpha} \cap U_{\beta}}} = W_{\alpha,\beta} \circ \Theta_{\beta|_{U_{\alpha} \cap U_{\beta}}} \circ W^*_{\alpha,\beta}$.

Fix a faithful trace tr on the algebra F. This defines a trace on each local chart Θ_{α}. As the transition functions are unitary sections, these local traces paste together into a continuous trace defined on the whole bundle. Take a generic solution $R', \bar{R}'$ of the conjugation equations. As shown in the remark following proposition 39 for each point ω of the base space we have $(R' \circ (1_{\bar{\rho}} \otimes S) \circ R')_{|_{\omega}} = tr(D^{\omega} K^{\omega} S_{|_{\omega}})$ and $(\bar{R}' \circ (S \otimes 1_{\bar{\rho}}) \circ \bar{R}')_{|_{\theta_{\rho}^{-1*}(\omega)}} = tr(D^{\omega} K^{\omega^{-1}} S_{|_{\omega}})$

for a positive invertible K^{ω} and a positive invertible central D^{ω} in the fibre $(\rho, \rho)_{\omega}$. Essentially we only have to prove that the section realized by these K^{ω} is a continuous section. Then there will be a corresponding element in (ρ, ρ) fulfilling our requirements.

But as the left hand sides of the above equations are continuous, and as the trace is continuous, this implies that both the sections $D^\omega K^\omega$ and $D^\omega K^{\omega^{-1}}$ are continuous sections in the locally trivial bundle (ρ,ρ). Thus K^ω is a continuous section, and a corresponding element $K \in (\rho,\rho)$ exists.

$R := (1_{\bar\rho} \otimes K^{-1}) \circ R'$, $\bar R := (K \otimes 1_{\bar\rho}) \circ \bar R'$ will be a standard solution ∎.

5 Bundles of Hopf algebras

Finite irreducible subfactors of depth two are characterized by the action of finite dimensional Hopf algebras (see f.e. [15], [22]).

This situation corresponds, in the context of $2-C^*$-categories with simple units, to an irreducible 1-arrow ρ (i.e. $(\rho,\rho) = \mathbf{C}1_\rho$) generating a $2-C^*$-category of depth two (see below). We show that in the general case of non simple units a similar example can be realized, obtaining a continuous bundle of finite dimensional Hopf algebras in duality. We will follow the exposition given in [18], as it is closer to our context.

We begin by considering a 1-arrow $B \xleftarrow{\rho} A$ such that $(\rho,\rho) = 1_\rho \otimes (\iota_A, \iota_A)$ (we will call such ρ "irreducible") together with its conjugate $A \xleftarrow{\bar\rho} B$. We denote by $\mathcal{C}$ the $2-C^*$-category generated by compositions of ρ and $\bar\rho$ (i.e. $\rho, \rho\otimes\bar\rho, \rho\otimes\bar\rho\otimes\rho \dots$) and their projections.

As we have seen we have a sequence of inclusions of C^*-algebras

$$(\rho,\rho) \to (\rho,\rho)\otimes 1_{\bar\rho} \subset (\rho\otimes\bar\rho, \rho\otimes\bar\rho),$$

$$(\rho\otimes\bar\rho, \rho\otimes\bar\rho) \to (\rho\otimes\bar\rho, \rho\otimes\bar\rho)\otimes 1_\rho \subset (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho), \dots$$

For each $\omega \in \Omega_A$ we consider the category $\mathcal{C}^{\omega,A}$ introduced in section 3 and obtain the sequence of finite dimensional C^*-algebras

$$(\rho,\rho)_\omega \to (\rho,\rho)_\omega \otimes 1_{\bar\rho|_{\theta_\rho^{*-1}(\omega)}} \subset (\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega)}$$

$$(\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega)} \to (\rho\otimes\bar\rho, \rho\otimes\bar\rho)_{\theta_\rho^{*-1}(\omega)} \otimes 1_{\rho|_\omega} \subset (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho)_\omega \dots$$

In the case of categories with simple units, there is a well established notion of finite depth: the $2-C^*$-category generated by ρ and $\bar\rho$ has finite depth n if the number of isomorphism classes of 1-arrows is finite (i.e. it's rational) and all of them appear as sub-1-arrows of the first n products $\rho, \rho\otimes\bar\rho \dots$.

For the sequence of inclusions $(\rho,\rho) \subset (\rho\otimes\bar\rho, \rho\otimes\bar\rho) \subset \dots$ this means that the corresponding principal part of the Bratteli diagram is finite, with depth n.

In the general case of non simple units we say that ρ has finite depth n if for each $\omega \in \Omega_A$ the associated to $\mathcal{C}^{\omega,A}$ has finite depth and the maximum depth among them is n.

In particular we will be interested in the case of an irreducible ρ of depth two.

As ρ is irreducible, it has a standard solution. The same will be true for the products $\rho\otimes\bar\rho, \rho\otimes\bar\rho\otimes\rho \dots$, as the product solutions of standard solutions are still standard.

In order to simplify the notation, we set

$$\mathcal{B} := (\rho\otimes\bar\rho, \rho\otimes\bar\rho),\ \mathcal{A} := (\bar\rho\otimes\rho, \bar\rho\otimes\rho),\ \mathcal{C} := (\rho\otimes\bar\rho\otimes\rho, \rho\otimes\bar\rho\otimes\rho).$$

$\mathcal{A}$ and $\mathcal{B}$ are in a natural way $C(\Omega_A)$ (respectively, $C(\Omega_B)$) bimodules, where left and right actions coincide (as $\theta_{\rho\otimes\bar{\rho}}$ and $\theta_{\bar{\rho}\otimes\rho}$ are the identity endomorphisms).

As we have standard solutions $R_{\bar{\rho}\otimes\rho}, \bar{R}_{\bar{\rho}\otimes\rho}$ for $\bar{\rho}\otimes\rho$, the right and left $(\iota_A, \iota_A) \cong C(\Omega_A)$ valued inner products coincide as well and give a faithful (non normalized) trace Tr_A on $\mathcal{A}$:

$$a \in \mathcal{A}, \quad Tr_A(a) := \ d_A \otimes < a, 1_{\bar{\rho}\otimes\rho} >_{(\iota_A,\iota_A)} = \ R^*_{\bar{\rho}\otimes\rho} \circ (1_{\bar{\rho}\otimes\rho} \otimes a) \circ R_{\bar{\rho}\otimes\rho}.$$

where by d_A we indicate $R^*_{\bar{\rho}\otimes\rho} \circ R_{\bar{\rho}\otimes\rho}$. The same way we have a $(\iota_B, \iota_B) \cong C(\Omega_B)$ valued trace on $\mathcal{B}$:

$$b \in \mathcal{B}, \quad Tr_B(b) := \ d_B \otimes < b, 1_{\rho\otimes\bar{\rho}} >_{(\iota_B,\iota_B)} = \ R^*_{\rho\otimes\bar{\rho}} \circ (1_{\rho\otimes\bar{\rho}} \otimes b) \circ R_{\rho\otimes\bar{\rho}}.$$

One defines a Fourier transform $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ as the linear map defined by

$$\mathcal{F}(a) := 1_{\rho\otimes\bar{\rho}} \otimes \bar{R}^*_\rho \circ (1_\rho \otimes a \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho \otimes 1_{\rho\otimes\bar{\rho}}.$$

and analogously $\hat{\mathcal{F}} : B \to A$ as

$$\hat{\mathcal{F}}(b) := 1_{\bar{\rho}\otimes\rho} \otimes R^*_\rho \circ (1_{\bar{\rho}} \otimes b \otimes 1_\rho) \circ R_\rho \otimes 1_{\bar{\rho}\otimes\rho}$$

The maps $\mathcal{S} := \hat{\mathcal{F}} \circ \mathcal{F} : \mathcal{A} \to \mathcal{A}$ and $\hat{\mathcal{S}} := \mathcal{F} \circ \hat{\mathcal{F}} : \mathcal{B} \to \mathcal{B}$ are the antipodes, and are antimultiplicative (this is an easy consequence of the conjugation relations).

Proposition 50. *The Fourier transform preserves the inner product given by the traces Tr_A and Tr_B in the following sense:*

$$\forall a, a' \in \mathcal{A}, \quad Tr_A(a'^* \circ a) = \theta_{\bar{\rho}}(Tr_B(\mathcal{F}(a')^* \circ \mathcal{F}(a))),$$

$$\forall b, b' \in \mathcal{B}, \quad Tr_B(b'^* \circ b) = \theta_\rho(Tr_A(\hat{\mathcal{F}}(b')^* \circ \hat{\mathcal{F}}(b))).$$

Proof We give only a sketch of the proof and skip the tedious exposition of all the equalities.

We have $Tr_A(\hat{\mathcal{F}}(b')^* \circ \hat{\mathcal{F}}(b)) = R^*_\rho \circ (1_{\bar{\rho}} \otimes X) \circ R_\rho$, where the expression for $X \in (\rho, \rho)$ is

$$(\bar{R}^*_\rho \otimes 1_\rho) \circ (1_\rho \otimes R^*_\rho \otimes 1_{\bar{\rho}} \otimes 1_\rho) \circ (1_\rho \otimes 1_{\bar{\rho}} \otimes b'^* \otimes 1_\rho) \circ$$
$$(1_{\rho\otimes\bar{\rho}\otimes\rho} \otimes (R_\rho \circ R^*_\rho)) \circ (1_\rho \otimes 1_{\bar{\rho}} \otimes b \otimes 1_\rho) \circ (1_\rho \otimes R_\rho \otimes 1_{\bar{\rho}} \otimes 1_\rho) \circ (\bar{R}_\rho \otimes 1_\rho)$$

As $R_\rho, \bar{R}_\rho$ is standard, we have $R^*_\rho \circ (1_{\bar{\rho}} \otimes X) \circ R_\rho = \theta^{-1}_\rho(\bar{R}^*_\rho \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho)$. One checks that $\bar{R}^*_\rho \circ (X \otimes 1_{\bar{\rho}}) \circ \bar{R}_\rho = Tr_{\mathcal{B}}(b'^* \circ b)$, which proves the second statement. The first statement is proved analogously. ∎

We also have the following proposition, which is a consequence of standardness. We omit the proof (see f.e. [18])

Proposition 51. *The following relations hold*

$$\mathcal{S} \circ \mathcal{S} = id_{\mathcal{A}} \ ; \ \hat{\mathcal{S}} \circ \hat{\mathcal{S}} = id_{\mathcal{B}}$$

where by $id_{\mathcal{A}}$ and $id_{\mathcal{B}}$ we indicate the identity endomorphisms of $\mathcal{A}$ and $\mathcal{B}$ respectively.

We can define "convolution" products on $\mathcal{A}$ and $\mathcal{B}$ the following way:

$$a, a' \in A, \quad a \star a' := \mathcal{F}^{-1}(\mathcal{F}(a)\mathcal{F}(a')) \ ; \ b, b' \in B, \quad b \star b' := \hat{\mathcal{F}}^{-1}(\hat{\mathcal{F}}(b)\hat{\mathcal{F}}(b').$$

We restrict our attention for a moment to the case $(\iota_A, \iota_A) = \mathbf{C}$, $(\iota_B, \iota_B) = \mathbf{C}$.

Then $\mathcal{A}$ and $\mathcal{B}$ are finite dimensional algebras and we are able to define a bilinear pairing between $\mathcal{A}$ and $\mathcal{B}$, i.e. a faithful linear form $< .,. >: \mathcal{A} \otimes_{\mathbf{C}} \mathcal{B} \to \mathbf{C}$, by $< a, b > := d_\rho^{-1} Tr_A(a\mathcal{F}^{-1}(b))$, thus establishing a duality (as linear spaces) between $\mathcal{A}$ and $\mathcal{B}$. This duality enables us to define coproducts $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, $\hat{\Delta} : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$ by

$$< \Delta(a), x \otimes y > := < a, xy >, \quad a \in \mathcal{A}, x, y \in \mathcal{B}, \tag{5.1}$$

$$< a \otimes b, \hat{\Delta}(x) > := < ab, x >, \quad a, b \in \mathcal{A}, x \in \mathcal{B}. \tag{5.2}$$

Coassociativity of Δ and $\hat{\Delta}$ are implied by associativity of the multiplications m and $\hat{m}$ of $\mathcal{A}$ and $\mathcal{B}$ respectively.

We can also define counits

$$\varepsilon(a) := < a, 1 >, \quad a \in \mathcal{A}; \quad \hat{\varepsilon}(b) := < 1, b >, \quad b \in \mathcal{B}. \tag{5.3}$$

These operations endow the algebras $\mathcal{A}$ and $\mathcal{B}$ with the structure of Hopf algebras. We quote the following lemma and the following proposition from [18]:

Lemma 52. *(cfr. [18], Lemma 6.18) The maps*

$$\Phi_1 : \mathcal{A} \otimes_{\mathbf{C}} \mathcal{B} \to \mathcal{C}, \quad a \otimes b \to 1_\rho \otimes a \circ b \otimes 1_\rho,$$

$$\Phi_2 : \mathcal{A} \otimes_{\mathbf{C}} \mathcal{B} \to C, \quad a \otimes b \to b \otimes 1_\rho \circ 1_\rho \otimes a$$

are bijections.

Proposition 53. *(cfr. [18], Proposition 6.19) Let $\varepsilon, \hat{\varepsilon}, \Delta, \hat{\Delta}$ be defined as above, Then*

- *$\varepsilon, \hat{\varepsilon}$ are multiplicative,*
- *$\Delta, \hat{\Delta}$ are multiplicative,*
- *$S, \hat{S}$ are coinverses, i.e. $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\varepsilon$, etc. (where η is the unit map : $c \in \mathbf{C} \to c1_{\mathcal{A}} \in \mathcal{A}$).*

- *$\mathcal{A}$ and $\mathcal{B}$ are finite dimensional Hopf algebras in duality, and $\mathcal{C}$ is the Weyl algebra (in the sense of [19]) of $\mathcal{A}$.*

The fact that the $2-C^*$-category $\mathcal{C}$ is of depth two means that there is only one isomorphism class of irreducible 1-arrows connecting A to B, the one determined by ρ. The same holds for $\bar{\rho}$. This implies that for any sub1-arrow X of $\bar{\rho}\otimes\rho$ we have $\rho\otimes X \;\cong \oplus_{i=1,\dots,n}\,\rho$, the direct sum of n copies of ρ. As the dimension is additive, we have $d_{\rho\otimes X} = n\ d_\rho$. But the dimension is multiplicative as well, i.e. $d_{\rho\otimes X} = d_X d_\rho$. This implies that the dimension d_X of any sub 1-arrow X of $\bar{\rho}\otimes\rho$ is an integer. Such elements $\{X$ sub1-arrow of $\bar{\rho}\otimes\rho\}$ form a rational tensor category. In fact, as $\bar{\rho}\otimes\rho\otimes\bar{\rho}\otimes\rho \cong \oplus_{1,\dots,d_{\bar{\rho}\otimes\rho}}\,\bar{\rho}\otimes\rho$, all isomorphism classes of 1-arrows in $\mathcal{C}$ connecting A to A appear in this set. One can construct a faithful tensor functor from this category into the category of finite Hilbert spaces assigning to each X the complex Hilbert space of dimension d_X. The natural transformations of this functor have the structure of a Hopf algebra, and one can show that this is exactly the Hopf algebra $\mathcal{A}$ introduced above (see f.e. [18], prop. 6.20).

Now let's return to the general case. We fix an $\omega\in\Omega_A$. Then the 1-arrows $\theta_\rho^{*-1}(\omega) \stackrel{\rho}{\leftarrow} \omega$, $\omega \stackrel{\bar{\rho}}{\leftarrow} \theta_\rho^{*-1}(\omega)$ in the category $\mathcal{C}^{\omega,A}$ satisfy all the conditions of the above propositions. In particular $(\rho\otimes\bar{\rho},\rho\otimes\bar{\rho})_{\theta_\rho^{*-1}(\omega)} = \mathcal{B}_{\theta_\rho^{*-1}(\omega)}$ and $(\bar{\rho}\otimes\rho,\bar{\rho}\otimes\rho)_\omega = \mathcal{A}_\omega$ are finite dimensional Hopf algebras in duality.

Depth two of $\theta_\rho^{*-1}(\omega) \stackrel{\rho}{\leftarrow} \omega$ in the category $\mathcal{C}^{\omega,A}$ implies that the associated dimension $d_{\mathcal{A}|_\omega}$ has integer values and its square coincides with the dimension (as a vector spaces) of the algebra $\mathcal{A}_{|_\omega}$. The same conclusion applies to $\mathcal{B}_{\theta_\rho^{*-1}(\omega)}$. As this function is continuous with respect to ω, we conclude that $d_\mathcal{A}$ is a constant function on Ω_A. Thus the dimensions (as vector spaces) of the fibre algebras $\mathcal{B}_{\theta_\rho^{*-1}(\omega)}$ and $\mathcal{A}_\omega$ are constant respect to ω. Lemma 46 tells us that for each $\omega\in\Omega$ we can find a neighbourhood U and an algebraic embedding of $U\times\mathcal{A}_{|_\omega}$ into $\mathcal{A}_{|_U}$. This embedding is actually surjective, as the fibre algebras of $\mathcal{A}_{|_U}$ have all the same finite dimension. Thus all fibre algebras $\mathcal{A}_\omega$ are isomorphic. i.e. $\mathcal{A}$ is a locally trivial bundle. The same conclusion applies to $\mathcal{B}$.

Thus we have the following: $\mathcal{A}$ and $\mathcal{B}$ are locally trivial C^*-algebra bundles over Ω_A and Ω_B, with fibres isomorphic to finite dimensional algebras $\mathcal{A}^0$ and $\mathcal{B}^0$ respectively.

We can view $\mathcal{B}$ as a $C(\Omega_A)$ valued Hilbert module by means of the isomorphism θ_ρ. Thus for $f\in C(\Omega_A)$, we have $\theta_\rho(f)b = b\theta_\rho(f)\in\mathcal{B}$, $\forall b\in\mathcal{B}$. The $C(\Omega_A)$ valued inner product is given by $\theta_\rho^{-1}(<,.,>_{(\iota_B,\iota_B)})$.

Thus we can form the tensor product $\mathcal{A}\otimes_{C(\Omega_A)}\mathcal{B}$, where for example $a\otimes_{C(\Omega_A)}\theta_\rho(f)b = af\otimes_{C(\Omega_A)} b$, for any $a\in A$, $b\in B$, $f\in C(\Omega_A)$. This actually corresponds to the usual tensor product of the fibre bundles relative to $\mathcal{A}$ and $\mathcal{B}$.

Analogously, we can define a $C(\Omega_A)$ valued linear faithful form

$$<,.,>: \mathcal{A}\otimes_{C(\Omega_A)}\mathcal{B}\to C(\Omega_A), \quad <a,b>:= d_\rho^{-1} Tr_\mathcal{A}(a\mathcal{F}^{-1}(b)).$$

This form is well defined, as one checks from the definition of $\mathcal{F}^{-1}$ that $f \otimes \mathcal{F}^{-1}(b) = \mathcal{F}^{-1}(\theta_\rho(f) \otimes b),\ \forall f \in C(\Omega_A),\ \forall b \in \mathcal{B}$ holds.

If we would like to think of $\mathcal{A}, \mathcal{B}$ as continuous bundles of Hopf algebras, the natural candidate for a continuous comultiplication Δ would be defined by

$$a \in \mathcal{A},\ x, y \in \mathcal{B},\quad \Delta : a \to \Delta(a) \in \mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A},$$

such that

$$< \Delta(a), x \otimes_{C(\Omega_A)} y > = < a, xy > .$$

As mentioned above, for each point $\omega \in \Omega_A$ we can consider the category $\mathcal{C}^{\omega, A}$. The evaluation of the bilinear form $<,.,>$ in ω gives the pairing between the Hopf algebras $\mathcal{A}_{|_\omega}$ and $\mathcal{B}_{|_{\theta_\rho^{*-1}(\omega)}}$:

$$b_{\theta_\rho^{*-1}(\omega)} \in \mathcal{B}_{\theta_\rho^{*-1}(\omega)},\ a_\omega \in \mathcal{A}_\omega,\quad < a_\omega, b_{\theta_\rho^{*-1}(\omega)} > := < a, b >_{|_\omega}$$

where $a \in A$ is such that $a_{|_\omega} = a_\omega$, i.e. a is a continuous section in the fibre bundle associated to the (bi-)module $\mathcal{A}$ whose value corresponding to the base point ω is the element a_ω of the finite dimensional fibre algebra $\mathcal{A}^0$, and the same way $b \in B$ such that $b_{|_{\theta^{*-1}(\omega)}} = b_{\theta^{*-1}(\omega)}$.

Thus the above defined $\Delta(a)$ indeed defines a section in the fibre bundle associated to the bimodule $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$. What we have to prove is that this section is continuous, i.e. it belongs to $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$.

More precisely we will show the following

Lemma 54. *Let $a \in \mathcal{A}$. Then $\forall \epsilon > 0$ there exist a finite set of $a_1^k, a_2^k \in \mathcal{A}$ such that*

$$|(\sum_k < a_1^k, b >< a_2^k, c > - < a, bc >)_{|_\omega}| < \epsilon,\ \forall \omega \in \Omega_A,\ \forall b, c \in \mathcal{B},\ \|b\| \le 1, \|c\| \le 1.$$

Proof For our convenience we will choose for $\mathcal{A}$ and $\mathcal{B}$ the norms given by the inner product, i.e. the $C(\Omega_A)$ (resp. $C(\Omega_B)$) valued traces $Tr_{\mathcal{A}}$ (resp. $Tr_{\mathcal{B}}$).

We choose locally trivial algebraic maps $\Phi_i : \mathcal{A}_{|_{U_i'}} \to U_i' \times \mathcal{A}^0$, $\Psi_i : \mathcal{B}_{|_{V_i'}} \to V_i' \times \mathcal{B}^0$, where U_i' and V_i' are open coverings of Ω_A and Ω_B. Without loss of generality, we suppose that $V_i' = \theta_\rho^{*-1}(U_i')$.

For each $x \in \mathcal{A}^0$ we can consider the corresponding constant section in $U_i' \times \mathcal{A}^0$, which we will indicate with the same symbol. Then $\Phi_i(x)^{-1}$ will be an element in $\mathcal{A}_{|_{U_i'}}$. The same way, for $y \in \mathcal{B}^0$, $\Psi_i^{-1}(y)$ will be an element of $\mathcal{B}_{|_{V_i'}}$.

Then we have the following linear form on $\mathcal{A}^0 \otimes_{\mathbf{C}} \mathcal{B}^0$ defined as

$$< \Phi_i^{-1}(x), \Psi_i^{-1}(y) >_{|_\omega},\quad \omega \in U_i'.$$

This linear form depends continuously on ω.

We define [2]

$$\Delta^\omega : \mathcal{A}^0 \to \mathcal{A}^0 \otimes \mathcal{A}^0, \quad x \in \mathcal{A}^0, \ \Delta^\omega(x) := \sum_l x_1^l \otimes x_2^l$$

such that for any pair $y, z \in \mathcal{B}^0$

$$(\sum_l < \Phi_i^{-1}(x_1^l), \Psi_i^{-1}(y) >< \Phi_i^{-1}(x_2^l), \Psi_i^{-1}(z) >)_{|\omega} =< \Phi_i^{-1}(x), \Psi_i^{-1}(yz) >_{|\omega}$$

As $\mathcal{A}^0$ is finite dimensional, all norms give equivalent topologies. For our convenience we will choose the following as norm

$$\|x\| := sup_{\omega \in U_i'} (Tr_{\mathcal{A}} \Phi_i^{-1}(x^* x))^{\frac{1}{2}}_{|\omega}, \quad x \in \mathcal{A}^0, \ \forall U_i'.$$

We may suppose that we have chosen the sets U_i' such that

$$\|\Phi_i(a)_{|\omega} - \Phi_i(a)_{|\omega'}\| \ < \epsilon, \quad \omega, \omega' \in U_i' \tag{5.4}$$

as $\Phi_i(a)$ is a continuous section $U_i' \times \mathcal{A}^0$.

We choose an ω_i in U_i'. Then, as the map Δ^ω is continuous, we may also suppose that we have chosen a set W_i such that

$$\|\Delta^\omega(\Phi_i(a)_{|\omega_i}) - \Delta^{\omega'}(\Phi_i(a)_{|\omega_i})\| < \epsilon, \ \omega, \omega' \in W_i \tag{5.5}$$

where $\Delta^\omega(\Phi_i(a)_{|\omega_i}), \Delta^{\omega'}(\Phi_i(a)_{|\omega_i})$ are elements of $\mathcal{A}^0 \otimes \mathcal{A}^0$ with the norm specified just above.

Letting ω_i vary arbitrarily in U_i', the W_i form an open cover of Ω_A. We choose a finite refinement of the two open covers of Ω_A given by the $\{U_i'\}$ and $\{W_i\}$ such that both (5.4) and (5.5) hold. We will denote with little abuse of notation these sets by $\{U_i\}$ and by $\{\Phi_i\}$ the relative local charts.

Remark We note that for any $b \in \mathcal{B}, \ \|b\| = \|Tr_{\mathcal{B}}(b^* b)^{\frac{1}{2}}\|^{C(\Omega_B)} < 1$ and $a \in \mathcal{A}, \ \|a\| = \|Tr_{\mathcal{A}}(a^* a)^{\frac{1}{2}}\|^{C(\Omega_A)} < 1$ one has $| < a, b >_{|\omega} | < \|d_\rho\|$ by the definition of the bilinear form and proposition 50.

Now let's consider the difference

$$| < \Delta(a) - (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega_i}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} | \quad =$$

$$| < \Delta(a) - < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega}$$

$$+ < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} - < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega_i}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} |$$

$$\leq \ | < (\Delta(a), b \otimes c >_{|\omega} - < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} |$$

$$+| < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} - < (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega_i}(\Phi_i(a)_{|\omega_i}), b \otimes c >_{|\omega} |$$

[2] To be precise, we should write Δ_i^ω, as the map depends on the choice of the local charts Φ_i and Ψ_i as well.

evaluated in $\omega \in U_i$. Notice that $< \Delta(a), b\otimes c >_{|_\omega} =< a, bc >_{|_\omega}$ (by definition of Δ) $=< \Phi_i^{-1}(\Phi_i(a)_{|_\omega}), bc >_{|_\omega}$. Also $< (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^\omega(\Phi_i(a)_{|_{\omega_i}}), b \otimes c >_\omega =< \Phi_i^{-1}(\Phi_i(a)_{|_{\omega_i}}), bc >_\omega$. It follows from 5.4 and the remark above that the first summand satisfies $| < \Phi_i^{-1}(\Phi_i(a)_{|_\omega}), bc >_\omega - < \Phi_i^{-1}(\Phi_i(a)_{\omega_i}), bc >_\omega | < \|d_\rho\|\epsilon$.

Analogously, from continuity of Δ^ω respect to ω and 5.5 the second summand is $< \|d_\rho\|^2\epsilon$.

Thus we have proven that

$$| < \Delta(a) - (\Phi_i^{-1}\otimes\Phi_i^{-1})\circ\Delta^{\omega_i}(\Phi_i(a)_{|_{\omega_i}}), b\otimes c >_{|_\omega} | < (\|d_\rho^{-1}\| + \|d_\rho^{-1}\|^2)\epsilon, \ \forall\omega \in U_i.$$

Now take a partition of unity $\{f_i\}$ subordinate to the open covering of Ω_A realized by the U_i. Then $\sum_i f_i \ (\Phi_i^{-1} \otimes \Phi_i^{-1}) \circ \Delta^{\omega_i}(\Phi_i(a)_{|_{\omega_i}})$ is a sum $\sum_k a_1^k \otimes_{C(\Omega_A)} a_2^k$ of elements of $\mathcal{A} \otimes_{C(\Omega_A)} \mathcal{A}$ which satisfies the claim of the lemma (modulo the constant $(\|d_\rho\| + \|d_\rho\|^2)$). ∎

An analogous result holds for $\mathcal{B}$. Thus we see that the expressions 5.1, 5.2. 5.3 make sense even in the case $(\iota_A, \iota_A) \neq \mathbf{C}$. We can think of 5.3 as a (ι_A, ι_A) valued counit (resp. (ι_B, ι_B) valued counit), or as continuous sections of counits for the fibre algebras. Continuity is obvious as in the definition only continuous objects are involved. The only non trivial part is to check continuity of the maps $\Delta : \ \mathcal{A} \to \mathcal{A}$ and $\hat{\Delta} : \ \mathcal{B} \otimes_{C(\Omega_B)} \mathcal{B}$, which has been established by the previous lemma.

We have thus the following analogue of 53

Proposition 55. *Let $\mathcal{A}, \mathcal{B}, \varepsilon, \hat{\varepsilon}, \Delta, \hat{\Delta}$ be defined as above, then*

- *$\mathcal{A}$ and $\mathcal{B}$ are locally trivial bundles of Hopf algebras, with fibres the finite dimensional algebra $\mathcal{A}^0$ and $\mathcal{B}^0$ respectively,*
- *$\varepsilon, \hat{\varepsilon}$ are multiplicative,*
- *$\Delta, \hat{\Delta}$ are multiplicative,*
- *$S, \hat{S}$ are coinverses, i.e. $m(S \otimes id)\Delta = m(id \otimes S)\Delta = \eta\varepsilon$, etc.*
- *for every $\omega \in \Omega_A$, $\mathcal{A}_{|_\omega}$ and $\mathcal{B}_{|_{\theta_\rho^{*-1}(\omega)}}$ are finite dimensional Hopf algebras in duality, and $\mathcal{C}_{|_\omega}$ is the Weyl algebra of $\mathcal{A}_{|_\omega}$.*

Proof All algebraic properties follow by applying 53 to each fibre algebra. Continuity of the maps follows from the above remarks and the above lemma. ∎

Remark 5.1. A notion of $C(X)$ Hopf algebra with a continuous field of coproducts over the topological space X was introduced in [3]. This is a more general definition dealing with fields (i.e. bundles) of infinite and not necessarily unital C^*-algebras over a locally compact space X. It is fairly easy to see that our example fits this definition as well.

6 Frobenius algebras and Q-systems

In this section we would like to make some remarks concerning Frobenius algebras and Q-systems. The notion of a Q-systems was introduced in the context of von Neumann algebras in [15], where in some sense it plays the role of a formalization of a sub-factor inclusion, and subsequently for general tensor C^*-categories the notion of abstract Q-system was introduced in [16]. The notion of Frobenius algebra in a tensor category is more general. We will follow the exposition given in [18] where not only the C^* case is treated but more general categories are considered and a general correspondence between Frobenius algebras and pairs of conjugate $1-$arrows in $2-$categories is studied.

Definition 56. *(cfr. [18], definition 3.1.) Let $\mathcal{A}$ be a strict tensor category. A Frobenius algebra in $\mathcal{A}$ is a quintuple (λ, V, V', W, W'), where λ is an object in $\mathcal{A}$ and $V : \iota \to \lambda$, $V' : \lambda \to \iota$, $W : \lambda \to \lambda^2$, $W' : \lambda^2 \to \lambda$ are morphisms satisfying the following conditions:*

$$W \otimes 1_\lambda \circ W \ = \ 1_\lambda \otimes W \circ W \tag{6.1}$$

$$W' \circ W' \otimes 1_\lambda \ = \ W' \circ 1_\lambda \otimes W' \tag{6.2}$$

$$V' \otimes 1_\lambda \circ W \ = 1_\lambda = \ 1_\lambda \otimes V' \circ W \tag{6.3}$$

$$W' \circ V \otimes 1_\lambda \ = \ 1_\lambda \ = \ W' \circ 1_\lambda \otimes V \tag{6.4}$$

$$W' \otimes 1_\lambda \circ 1_\lambda \otimes W \ = \ W \circ W' \ = \ 1_\lambda \otimes W' \circ W \otimes 1_\lambda . \tag{6.5}$$

Definition 57. *(cfr. [18], Definition 3.3) Two Frobenius algebras (λ, V, V', W, W'), $(\tilde{\lambda}, \tilde{V}, \tilde{V}', \tilde{W}, \tilde{W}')$ in the strict tensor category $\mathcal{A}$ are isomorphic if there is an isomorphism $S : \lambda \to \tilde{\lambda}$ such that*

$$S \circ V = \tilde{V}, \quad V' = \tilde{V}' \circ S, \quad S \otimes S \circ W = \tilde{W} \circ S, \quad S \circ W' = \tilde{W}' \circ S \otimes S.$$

We recall the definition of conjugation in the case of a (not necessarily C^*) 2-category $\mathcal{C}$ (the term "duality" is often used instead of "conjugation"):

Definition 58. *A 2-category $\mathcal{C}$ is said to have left (right) duals if for every 1-arrow $\rho : B \leftarrow A \in \mathcal{C}$ there are ${}^{-}\!\rho : A \leftarrow B$ ($\bar{\rho} : A \leftarrow B$) together with 2-arrows $e_\rho \in (\iota, \rho \otimes {}^{-}\!\rho)$, $d_\rho \in ({}^{-}\!\rho \otimes \rho, \iota)$ ($\varepsilon_\rho \in (\iota, \bar{\rho} \otimes \rho)$, $\eta_\rho \in (\rho \otimes \bar{\rho}, \iota)$ satisfying*

$$1_\rho \otimes d_\rho \circ e_\rho \otimes 1_\rho = 1_\rho \ , \quad d_\rho \otimes 1_{{}^{-}\!\rho} \circ 1_{{}^{-}\!\rho} \otimes e_\rho = 1_{{}^{-}\!\rho}$$

$$\eta_\rho \otimes 1_{\bar{\rho}} \circ 1_{\bar{\rho}} \otimes \varepsilon_\rho = 1_{\bar{\rho}} \ , \quad 1_\rho \otimes \eta_\rho \circ \varepsilon_\rho \otimes 1_\rho = 1_\rho$$

If ${}^{-}\!\rho = \bar{\rho}$, ${}^{-}\!\rho$ is said to be a two sided dual, and we indicate it by $\bar{\rho}$.

We will assume in the sequel duals to be two sided. Duals are automatically two sided in a $*$-category. It is easy to see that the above definition of duality (i.e. conjugation) reduces to the one already introduced for the C^* case.

Lemma 59. *The object λ of a Frobenius algebra is self conjugate.*

Proof. Set $e_\lambda := W \circ V$, $d_\lambda := V' \circ W'$. It is easy to see that they satisfy the claimed relations.∎

The following is an important example:

Lemma 60. *(cfr. [18], Lemma 3.4) Let $\rho : B \leftarrow A$ be a 1-arrow in a 2-category $\mathcal{E}$ and let $\bar{\rho} : A \leftarrow B$ be a two sided dual with duality 2-morphisms $d_\rho, e_\rho, \varepsilon_\rho, \eta_\rho$. Positing $\lambda = \bar{\rho} \otimes \rho : A \leftarrow A$ there are V, V', W, W' such that (λ, V, V', W, W') is a Frobenius algebra in the tensor category $\mathcal{A} = HOM_{\mathcal{E}}(A, A)$.*

It suffices to choose

$$V := \varepsilon_\rho,\ V' := d_\rho,\ W := 1_{\bar{\rho}} \otimes e_\rho \otimes 1_\rho,\ W' := 1_{\bar{\rho}} \otimes \eta_\rho \otimes 1_\rho.$$

It is not difficult to check that the Frobenius algebra relations hold. In the sequel we will prove explicitly a similar result in the C^* case.

The following propositions show to which extent a generic Frobenius algebra can be realized as a couple of conjugate 1-arrows in a 2-category as in the example above.

Definition 61. *An almost-2-category is defined as a 2-category except that we do not require the existence of a unit 1-arrow ι_σ for every object σ.*

Proposition 62. *(cfr. [18], Proposition 3.8) Let $\mathcal{A}$ be a strict tensor category and $\lambda = (\lambda, V, V', W, W')$ a Frobenius algebra in $\mathcal{A}$. Then there is an almost-2-category $\mathcal{E}_0$ satisfying:*

- *Obj $\mathcal{E}_0 = \{A, B\}$.*
- *There is an isomorphism $I : \mathcal{A} \to HOM_{\mathcal{E}_0}(A, A)$ of tensor categories.*
- *There are 1-arrows $\rho : B \leftarrow A$ and $\bar{\rho} : A \leftarrow B$ such that $\bar{\rho} \otimes \rho = I(\lambda)$.*

If $\mathcal{A}$ is $\mathbf{F}$-linear then so is $\mathcal{E}_0$. Isomorphic Frobenius algebras give rise to isomorphic almost-2-categories.

Definition 63. *A strict tensor category $\mathcal{A}$ is said to be (ι, ι)-linear if $z \otimes S = S \otimes z$ for all $z \in (\iota, \iota)$ and $S \in (\sigma, \sigma')$.*

Theorem 64. *(cfr. [18], Theorem 3.11) Let $\mathcal{A}$ be a strict tensor category and $\lambda = (\lambda, V, V', W, W')$ a Frobenius algebra in $\mathcal{A}$. Assume that one of the following conditions is satisfied:*

- *$W' \circ W = 1_\lambda$.*
- *A is (ι, ι)-linear and*
$$W' \circ W = z_1 \otimes 1_\lambda,$$
where z_1 is an invertible element of the commutative monoid (ι, ι).

Then the completion $\mathcal{E} = \mathcal{E}^P$ of the $\mathcal{E}_0$ defined in Proposition 62 is a bicategory such that

- *$Obj\mathcal{E} = \{A, B\}$.*
- *There is a fully faithful tensor functor $I : \mathcal{A} \to HOM_{\mathcal{E}}(A, A)$ such that for every $Y \in HOM_{\mathcal{E}}(A, A)$ there is $X \in \mathcal{A}$ such that Y is a retract (i.e. sub-1-arrow) of $I(X)$.*
- *There are 1-arrows $\rho : B \leftarrow A$ and $\bar{\rho} : A \leftarrow B$ such that there are 2-arrows*

  $$e_\rho : \iota_B \to \rho \otimes \bar{\rho},\ \varepsilon_\rho : \iota_A \to \bar{\rho} \otimes \rho,\ d_\rho : \bar{\rho} \otimes \rho \to \iota_A,\ \eta_\rho : \rho \otimes \bar{\rho} \to \iota_B$$

  satisfying the conjugation (i.e. duality) relations.
- *We have the identity*

  $$I(\lambda, V, V', W, W') = (\bar{\rho} \otimes \rho, e_\rho, \eta_\rho, 1_\rho \otimes \varepsilon_\rho \otimes 1_{\bar{\rho}}, 1_\rho \otimes d_\rho \otimes 1_{\bar{\rho}})$$

  of Frobenius algebras in $HOM_{\mathcal{E}}(A, A)$.
- *If $\mathcal{A}$ is a preadditive category, then $\mathcal{E}$ is a preadditive 2-category.*
- *If $\mathcal{A}$ has direct sums then $\mathcal{E}$ has direct sums of 1-arrows.*

Isomorphic Frobenius algebras $\lambda, \tilde{\lambda}$ give rise to isomorphic bicategories $\mathcal{E}, \tilde{\mathcal{E}}$.

Remark 6.1. As we have seen, a generic Frobenius algebra can be realized as the product of a couple of 1-arrows in a bicategory. In order for these 1-arrows to be conjugate, in theorem 64 additional hypotheses were required. With further requirements one can prove the universality of this construction.

Definition 65. *(cfr. [18], Definition 3.13) Let $\mathcal{A}$ be an (ι, ι)-linear category. A Frobenius algebra (λ, V, V', W, W') in $\mathcal{A}$ is "strongly separable" iff*

$$W' \circ W = z_1 \otimes 1_\rho,$$

$$V' \circ V = z_2,$$

where $z_1, z_2 \in (\iota, \iota)$ are invertible. λ is said to be normalized if $z_1 = z_2$.

Theorem 66. *(cfr. [18], Theorem 3.17) Let $\mathcal{A}$ be (ι, ι)-linear and (λ, V, V', W, W') a strongly separable Frobenius algebra in $\mathcal{A}$. Let $\mathcal{E}$ be as constructed in Theorem 64 and $\tilde{\mathcal{E}}$ be any bicategory such that:*

- *Obj $\tilde{\mathcal{E}} = \{A, B\}$.*
- *Idempotent 2-arrows in $\tilde{\mathcal{E}}$ split.*
- *There is a fully faithful tensor functor $\tilde{I} : \mathcal{A} \to HOM_{\tilde{\mathcal{E}}}(A, A)$ such that every object of $HOM_{\tilde{\mathcal{E}}}(A, A)$ is a retract of $\tilde{I}(X)$ for some $X \in \mathcal{A}$.*

- *There are mutually two-sided dual 1-arrows $\tilde{\rho} : B \leftarrow A$, $\tilde{\bar{\rho}} : A \leftarrow B$ and an isomorphism $\tilde{S} : I(\lambda) \to \tilde{\bar{\rho}} \otimes \tilde{\rho}$ between the Frobenius algebras $I(\lambda, V, V', W, W')$ and $(\tilde{\bar{\rho}} \otimes \tilde{\rho}, \tilde{e}_{\tilde{\rho}}, \dots)$ in $HOM_{\tilde{\mathcal{E}}}(A, A)$.*

Then there is an equivalence $E : \mathcal{E} \to \tilde{\mathcal{E}}$ of bicategories such that there is a tensor isomorphism between the tensor functors $\tilde{I}$ and $(E|_{HOM_{\tilde{\mathcal{E}}}(A,A)}) \circ I$.

We recall the notion of a Q-system:

Definition 67. *Let $\mathcal{A}$ be a (ι, ι)-linear tensor $*-$ category. A Q-system in $\mathcal{A}$ is a triple (λ, S, T) where λ is an object in $\mathcal{A}$ and $T \in (\iota, \lambda)$, $S \in (\lambda, \lambda^2)$ are arrows satisfying the following relations :*

$$T^* \otimes 1_\lambda \circ S = 1_\lambda = 1_\lambda \otimes T^* \circ S \tag{6.6}$$

$$S^* \circ S = 1_\lambda \tag{6.7}$$

$$S \otimes 1_\lambda \circ S = 1_\lambda \otimes S \circ S \tag{6.8}$$

$$S^* \otimes 1_\lambda \circ 1_\lambda \otimes S = S \circ S = 1_\lambda \otimes S^* \circ S \otimes 1_\lambda. \tag{6.9}$$

Remark 6.2. Alternatively (as in [18]) one could define define a Q-system as a strongly separable Frobenius algebra $(\lambda, T, T^*, S, S^*)$. This definition is equivalent up to isomorphism to the one just given: it suffices to renormalize S and T by the invertible element z_1 in order to turn S into an isometry. $(\iota, \iota)-$linearity was not required in the original definition in [16] as it was supposed $(\iota, \iota) = \mathbf{C}$.

Remark 6.3. In the C^* case the above four relations are not independent. Having assumed 6.6 to hold, we can choose any couple of 6.7, 6.8 , 6.9 and the remaining relation will follow (up to isomorphism). That 6.6, 6.7, 6.8 imply 6.9 was proven in [16]. That 6.6, 6.7, 6.9 imply 6.8 was proven in [10]. It is not necessary to suppose $(\iota, \iota) = \mathbf{C}$ and (ι, ι)-linearity plays no role in the proof.

We first show the following lemma (we recall that we assume the spectra Ω associated to $(\iota, \iota) \cong C(\Omega)$ to be connected).

Lemma 68. *Let (λ, S, T) be a Q-system in a tensor C^*-category. Then $T^* \circ S^* \circ S \circ T$ and $T^* \circ T$ are positive invertible elements of (ι, ι).*

$S \circ T$, $T^* \circ S^*$ satisfy the conjugation relations for λ and this implies that $T^* \circ S^* \circ S \circ T$ is invertible. The inequality

$$\|S^* \circ S\| \;\; T^* \circ T \;\geq T^* \circ S^* \circ S \circ T$$

implies that $T^* \circ T$ is invertible as well. ▮

Proposition 69. *Let λ, S, T be as in 67 in a (not necessarily (ι, ι)-linear) tensor C^*-category $\mathcal{A}$. Assume 6.6 to hold. Then we have the following implications*

- (6.7), (6.8) $\to$ (6.9)
- (6.7), (6.9) $\to$ (6.8)
- (6.8), (6.9) $\to$ (6.7) *for* S', T' *isomorphic to* S, T.

Proof

(6.7), (6.8) $\to$ (6.9). Consider the inequality

$$(S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (S \circ S^*) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) \leq$$
$$\|S \circ S^*\| (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) =$$
$$(S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda)$$

(as $\|S \circ S^*\| = 1$). The difference $X := (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) - (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) \circ (S \circ S^*) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda)$ is a positive element in (λ, λ). Checking that $< 1_\lambda, X >^{(\lambda,\lambda)}_{(\iota,\iota)} = 0$ one concludes that the inequality is actually an equality, as the above product is faithful. We put $X' := (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) - S \circ S^*$. Using the preceding result, one checks that $X'^* \circ X' = 0$, which is a restatement of (6.9).

(6.7), (6.9) $\to$ (6.8) One has

$$(1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) = (1_\lambda \otimes S) \circ (S \circ S^*).$$

On the other hand

$$(1_\lambda \otimes S) \circ (1_\lambda \otimes S^*) \circ (S \otimes 1_\lambda) = (1_\lambda \otimes (S \circ S^*)) \circ (S \otimes 1_\lambda) =$$
$$(1_\lambda \otimes ((S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S))) \circ (S \otimes 1_\lambda) = ((1_\lambda \otimes S^* \otimes 1_\lambda) \circ (S \otimes S) =$$
$$(1_\lambda \otimes S^* \otimes 1_\lambda) \circ (S \otimes 1_\lambda \otimes 1_\lambda) \circ (1_\lambda \otimes S) = ((S \circ S^*) \otimes 1_\lambda) \circ (1_\lambda \otimes S) =$$
$$(S \otimes 1_\lambda) \circ (S^* \otimes 1_\lambda) \circ (1_\lambda \otimes S) = (S \otimes 1_\lambda) \circ (S \circ S^*).$$

Thus $(1_\lambda \otimes S) \circ (S \circ S^*) = (S \otimes 1_\lambda) \circ (S \circ S^*)$, which is equivalent to (6.8), as S is an isometry.

(6.8), (6.9) $\to$ (6.7). Consider the positive element $H := S^* \circ S \in (\lambda, \lambda)$. We show that

$$S \circ H = H \otimes 1_\lambda \circ S = 1_\lambda \otimes H \circ S$$

holds:

$$H \otimes 1_\lambda \circ S = (S^* \circ S) \otimes 1_\lambda \circ S = S^* \otimes 1_\lambda \circ (S \otimes 1_\lambda \circ S) = S^* \otimes 1_\lambda \circ (1_\lambda \otimes S \circ S) =$$
$$(S^* \otimes 1_\lambda \circ 1_\lambda \otimes S) \circ S = (S \circ S^*) \circ S = S \circ H.$$

The same way

$$H \otimes 1_\lambda \circ S = (S^* \circ S) \otimes 1_\lambda \circ S = S^* \otimes 1_\lambda \circ (S \otimes 1_\lambda \circ S) = S^* \otimes 1_\lambda \circ (1_\lambda \otimes S \circ S) =$$

$$(S^* \otimes 1_\lambda \circ 1_\lambda \otimes S) \circ S = (1_\lambda \otimes S^* \circ S \otimes 1_\lambda) \circ S =$$
$$1_\lambda \otimes S^* \circ (S \otimes 1_\lambda \circ S) = 1_\lambda \otimes S^* \circ 1_\lambda \otimes S \circ S = 1_\lambda \otimes H \circ S.$$

Now we show that H is invertible. We have

$$1_\lambda = S^* \circ T \otimes 1_\lambda \circ T^* \otimes 1_\lambda \circ S \leq S^* \circ ((T^* \circ T) \otimes 1_\lambda \otimes 1_\lambda) \circ S = (T^* \circ T) \otimes H$$

where we have used 6.6 and the inequality $T \circ T^* \leq (T^* \circ T) \otimes 1_\lambda$ which follows by the considerations at the end of the proof of 10. As $T^* \circ T$ is invertible, we can write

$$(T^* \circ T)^{-1} \otimes 1_\lambda \leq H.$$

Thus H is positive and greater or equal to a positive invertible element, and this implies that H is invertible as well.

We can define

$$S' := H^{-\frac{1}{2}} \otimes 1_\lambda \circ S = \; 1_\lambda \otimes H^{-\frac{1}{2}} \circ S = \; S \circ H^{-\frac{1}{2}} = \; H^{-\frac{1}{2}} \otimes H^{-\frac{1}{2}} \circ S \circ H^{\frac{1}{2}}$$

and $T' := H^{\frac{1}{2}} \circ T$.

Then (λ, S', T') is a Q-system isomorphic to (λ, S, T) and S' is an isometry. ∎

Corollary 70. *Let $\mathcal{C}$ be a tensor C^*-category. The conclusions of theorem 64 are valid without assuming anyone of the two conditions stated in the hypothesis.*

As a last step in the sequence of propositions dealing with the relationship between Q-systems and 2-categories, we quote the following, which shows that starting from a Q-system one can extend the $*$ operation and, when starting with a Q-system in a tensor C^*-category, actually recover a $2-C^*-$category.

Proposition 71. *(cfr. [18], Proposition 5.5) Let $\mathcal{A}$ be a tensor *-category and λ a Q-system in $\mathcal{A}$. Then $\mathcal{E}_0$ has a positive *-operation which extends the given one on $\mathcal{A}$. Let $\mathcal{E}_*$ be the full sub-bicategory of $\mathcal{E}$ whose 1-arrows are (X, P), where X is an object in $\mathcal{A}$ and where $P = P \circ P = P^*$. Then $\mathcal{E}_*$ is equivalent to $\mathcal{E}$ with positive involution *.*

The following proposition answers positively a question left open in [16], namely, whether a couple of conjugate elements $\rho, \bar{\rho}$ in a tensor C^*-category does give rise to a Frobenius algebra $(\bar{\rho} \otimes \rho, S, T)$ such that S is an isometry also in the case $(\iota, \iota) \neq \mathbf{C}$.

Proposition 72. *Let $B \xleftarrow{\rho} A$ be a 1-arrow in a $2-C^*$-category $\mathcal{C}$ and let $A \xleftarrow{\bar{\rho}} B$ be a conjugate for ρ. Let $\lambda := \bar{\rho} \otimes \rho$. Then there exist S, T such that (λ, S, T) satisfy the defining relations for a Q-system (6.6), (6.7), (6.8), (6.9) in the C^* tensor category $\mathcal{HOM}(A, A)$ generated by 1-arrows connecting the object A to itself.*

Proof. As ρ and $\bar{\rho}$ are conjugate, there exist $R_\rho, \bar{R}_\rho$ satisfying the conjugation equations. Defining $T' := R_\rho, S' := (1_{\bar{\rho}} \otimes \bar{R}_\rho \otimes 1_\rho) \circ R_\rho$ we see that

$T' \in (\iota, \lambda)$ and $S' \in (\lambda, \lambda \otimes \lambda)$. (λ, S', T') satisfy the relations (6.6), (6.8), (6.9) (we leave to the reader the easy proof, which relies on the conjugation relations for $\rho, \bar{\rho}$). Defining $S := S' \circ (S^* \circ S)^{-\frac{1}{2}}$, $T := (S^* \circ S)^{\frac{1}{2}} \circ T'$, proposition 69 tells us that (λ, S, T) satisfy all four of (6.6)(6.7)(6.8)(6.9). ∎

Remark 6.4. A different choice of solutions $R, \bar{R}$ or of the conjugate $\bar{\rho}$ changes the Frobenius algebra only up to isomorphism. The Frobenius algebra (λ, S, T) fails in general to be a Q-system (according to definition 67) as we have not supposed (ι, ι)-linearity of the category $\mathcal{HOM}(A, A)$.

Remark 6.5. As $R_\rho^* \circ R_\rho$ and $\bar{R}_\rho^* \circ \bar{R}_\rho$ are invertible, the elements

$$E_\rho := R_\rho \circ (R_\rho^* \circ R_\rho)^{-1} \circ R_\rho^*, \ \bar{E}_\rho := \bar{R}_\rho \circ (\bar{R}_\rho^* \circ \bar{R}_\rho)^{-1} \circ \bar{R}_\rho^*.$$

are easily seen to be projections. If we suppose ρ to be indecomposable and assumption 1 to hold (thus we have (ι, ι) linearity of the category generated by $\bar{\rho} \otimes \rho$) we can also suppose to have chosen R_ρ and $\bar{R}_\rho$ such that $\bar{R}_\rho^* \circ \bar{R}_\rho = \theta_\rho(R_\rho^* \circ R_\rho)$ (it suffices to renormalize R_ρ and $\bar{R}_\rho$ by tensoring with $(R_\rho^* \circ R_\rho)^{-\frac{1}{4}} \otimes \theta_\rho^{-1}((\bar{R}_\rho^* \circ \bar{R}_\rho^*)^{\frac{1}{4}})$ and $(\bar{R}_\rho^* \circ \bar{R}_\rho)^{-\frac{1}{4}} \otimes \theta_\rho(R_\rho^* \circ R_\rho)^{\frac{1}{4}})$.

Then the following relations hold (cfr. [16]):

$$1_\rho \otimes (R_\rho \circ R_\rho^*) \circ (\bar{R}_\rho \circ \bar{R}_\rho^*) \otimes 1_\rho \circ 1_\rho \otimes (R_\rho^* \circ R_\rho) = 1_\rho \otimes (R_\rho^* \circ R_\rho) = \theta_\rho(R_\rho^* \circ R_\rho) \otimes 1_\rho.$$

This implies

$$1_\rho \otimes E_\rho \circ \bar{E}_\rho \otimes 1_\rho \circ 1_\rho \otimes E_\rho = \theta_\rho(R_\rho^* \circ R_\rho)^{-2} \otimes 1_\rho \otimes E_\rho = (\bar{R}_\rho^* \circ \bar{R}_\rho)^{-2} \otimes 1_\rho \otimes E_\rho.$$

Analogously one obtains:

$$1_{\bar{\rho}} \otimes \bar{E}_\rho \circ E_\rho \otimes 1_\rho \circ 1_{\bar{\rho}} \otimes \bar{E}_\rho = (R_\rho^* \circ R_\rho)^{-2} \otimes 1_{\bar{\rho}} \otimes \bar{E}_\rho.$$

These are the Jones relations which in the case $(\iota, \iota) = \mathbf{C}$ lead to a representation of the Temperley-Lieb algebra related to the parameter $R_\rho^* \circ R_\rho$. The difference here is that $R_\rho^* \circ R_\rho$ is, in general, a positive invertible function in the C^*-algebra $(\iota, \iota) \cong C(\Omega)$. Obviously if the function $R_\rho^* \circ R_\rho$ takes values in the discrete part of the spectrum of the Jones index, i.e. $(R_\rho^* \circ R_\rho)^2(\omega) \in \{4cos^2\pi/k, \ k \in \mathbf{N}, \ k \geq 3\}$, then it is a constant function, as it has to be continuous on the connected space Ω.

Lemma 73. *Let $\mathcal{A}$ be a tensor C^*-category and $(\lambda, S, S^*, T, T^*)$ a Frobenius algebra in $\mathcal{A}$. Suppose that assumption 1 holds in $\mathcal{A}$. Then the tensor C^*-category generated by λ (i.e. its tensor powers and their sub-objects) is (ι, ι)-linear.*

Proof. It suffices to prove (ι, ι)-linearity only for λ, i.e. that for any $z \in (\iota, \iota)$, $z \otimes 1_\lambda = 1_\lambda \otimes z$. The same relation for powers of λ and sub-objects follows immediately.

Let's suppose for the moment λ to be indecomposable. By assumption 1 there exists a $w \in (\iota, \iota)$ such that $z \otimes 1_\lambda = 1_\lambda \otimes w$. Thus we have:

$$(1_\lambda \otimes T^*) \circ (1_\lambda \otimes z \otimes 1_\lambda) \circ S = (1_\lambda \otimes T^*) \circ (1_\lambda \otimes 1_\lambda \otimes w) \circ S.$$

For the left hand side the following hold:

$$(1_\lambda \otimes T^*) \circ (1_\lambda \otimes z \otimes 1_\lambda) \circ S = (1_\lambda \otimes z \otimes T^*) \circ S = (1_\lambda \otimes (z \circ T^*) \circ S =$$

$$(1_\lambda \otimes z) \circ (1_\lambda \otimes T^*) \circ S = 1_\lambda \otimes z.$$

For the right hand side we have:

$$(1_\lambda \otimes T^*) \circ (1_\lambda \otimes 1_\lambda \otimes w) \circ S = ((1_\lambda \otimes T^*) \circ S) \otimes w = 1_\lambda \otimes w.$$

Thus $1_\lambda \otimes z = 1_\lambda \otimes w$, i.e. $z = w$.

Now consider a decomposable $\lambda = \oplus_i \lambda_i$, where each λ_i is indecomposable. Then for any z in (ι, ι) we have $z \otimes 1_\lambda = z \otimes (\oplus_i 1_{\lambda_i}) = \oplus_i (1_{\lambda_i} \otimes \theta_{\lambda_i}(z))$. But we have just seen that each θ_{λ_i} is trivial, i.e. $\theta_{\lambda_i}(z) = z$ for any z in (ι, ι).

Thus $z \otimes 1_\lambda = 1_\lambda \otimes z$. ∎

Corollary 74. *Let $\mathcal{A}$ be a tensor C^*-category for which assumption 1 holds. Then each Frobenius algebra $\lambda \in \mathcal{A}$ is a Q-system in the tensor C^*-category generated by the tensor powers of λ itself.*

7 Conclusions

In all of the present work a fundamental role has been played by the C^*-property of the norm and the conjugation relations. A bundle structure of the spaces of 2-arrows appears, whereas in preceding works it was (implicitly or explicitly) a starting hypothesis. The fact that the bundle structure is preserved by the $\circ$ composition (proposition 24) is a direct consequence of the conjugation equations. In order to give a reasonable description of the behaviour of this structure under the $\otimes$ composition, we have introduced an additional hypothesis, i.e. assumption 1. This has enabled us to describe our initial 2-C^*-category as a collection of 2-C^*-categories with simple units indexed by the elements of a compact topological space (the categories $C^{\omega_0 \mathcal{A}}$ at the end of section 3), a structure resembling that of fibre bundles. Properties analogous to the case of simple units have been proven (as, for example, the finite dimensionality of the fibres), as well continuity of the sections in this "fibre picture". Assumption 1 seems general enough to handle many interesting cases. It would be interesting to study to which extent one can consider this hypothesis as valid, or give some counter examples.

A second open question is that of the existence of standard solutions. We gave some partial answers, i.e. a "weak" positive answer (proposition 48) in the general case and a "global" positive answer in the case of locally trivial bundles (proposition 49). We don't know the answer for the general case (nor do we have any counter examples).

The bundle approach seems to have proven itself fruitful giving an example of bundles of Hopf algebras in section 5.

Finally, the remarks in section 6 show that, once more, the conjugation relations together with the C^*-property of the norm may hide more structure than what appears at first glance.

Acknowledgements

I would like to thank J. E. Roberts for suggestions, comments, observations and encouragement. His handwritten notes were fundamental for the initial settings of this work. I also would like to thank E. Vasselli for fruitful discussions and R. Schrader for kind hospitality at the Department of Physics - Freie Universität Berlin where part of this work was written.

References

[1] H. Baumgärtel, F. Lledó: An Application of the Doplicher-Roberts Duality for Compact Groups to Endomorphisms Categories of C^*-algebras with Nontrivial Center, in *Mathematical Physics in Mathematics and Physics - Quantum and Operator Algebraic Aspects*, *Fields Institute Communications* **30** (2001), 1-10.

[2] H. Baumgärtel, F. Lledó: Duality of Compact Groups and Hilbert C^*-systems for C^*-algebras with a non Trivial Center, *Int. J. Math.* **15** (2004), 759-812.

[3] E. Blanchard: Déformations de C^*-algèbres de Hopf, *Bull. Soc. math. France* **124** (1996), 141-215.

[4] J. Dixmier, A. Douady: Champs Continus d'Espaces Hilbertiens et de C^*-Algèbres, *Bull. Soc. math. France* **91** (1963), 227-284.

[5] S. Doplicher, J. E. Roberts: A New Duality Theory for Compact Groups, *Inv. Math.* **98** (1989), 157-218.

[6] S. Doplicher, J. E. Roberts: Endomorphisms of C^*-algebras, Cross Products and Duality for Compact Groups, *Ann. Math.* **130** (1989), 75-119.

[7] M. J. Dupré: The Classification and Structure of C^*-Algebra Bundles, *Mem. of the A.M.S.* **Vol. 21 , n. 222** (1979).

[8] F. Fidaleo, T. Isola: Minimal Expectations for Inclusions with Atomic Centres, *Int. J. Math.* **3** (1996), 307-327.

[9] M.Izumi: Inclusions of Simple C^*-algebras, *J. Reine Angew. Math.* **547** (2000), 97-138.

[10] M. Izumi, H. Kosaki: On a Subfactor Analogue of the Second Cohomology, *Rev. Math. Phys.* **Vol. 14, n. 7** (2002), 733-757.

[11] T. Kajiwara, C. Pinzari, Y. Watatani: Jones Index Theory for Hilbert C^*-Bimodules and its Equivalence with Conjugation Theory, *J. Funct. Anal.* **215** (2004), 1-49.

[12] E. Kirchberg, S. Wassermann: Operations on Continuous Bundles of C^*-algebras, *Math. Ann.* **303** (1995), 677–697.

[13] T. Leinster: Higher Operads, Higher Categories, *Lond. Math. Soc. L.N.S.* **298** (2004).

[14] R. Longo: Index of Subfactors and Statistics of Quantum Fields I, *Comm. Math. Phys.* **126**, (1989), 217-247; R. Longo: Index of Subfactors and Statistics of Quantum Fields II, *Comm. Math. Phys.* **130**, (1990), 285-309.

[15] R. Longo: A Duality for Hopf Algebras and for Subfactors, *Comm. Math. Phys.* **159**, No. 1, (1994), 133-150.

[16] R. Longo, J.E. Roberts: A Theory of Dimension, *K-Theory* **11** (1997), 133-159.

[17] S. Mac Lane, *Categories for the Working Mathematician* (2nd edition), Springer, Berlin, 1998 .

[18] M. Müger: From Subfactors to Categories and Topology I: Frobenius Algebras in and Morita Equivalence of Tensor Categories, *J. Pure Appl. Alg.* **180** (2003), 81-157.

[19] F. Nill: Weyl Algebras, Fourier Transformations and Integrals on Finite-dimensional Hopf Algebras, *Rev. Math. Phys.* **6** (1994), 149-166.

[20] J. E. Roberts: *Unpublished Manuscript.*

[21] K. Szlachányi, Weak Hopf Algebras Symmetries of C^*-algebra Inclusions, *In : S. Coen (Ed.), Geometry Seminars* (2000), 229-246, Univ. Stud. Bologna, 2001.

[22] W. Szymanski: Finite Index Subfactors and Hopf Algebras Crossed Products, *Proc. Am. Math. Soc.* **120** No. 2, (1994), 519-528.

[23] E. Vasselli: Continuous Fields of C^*-Algebras Arising from Extensions of Tensor C^*-categories, *J. Funct. Anal.* **199** (2003), 122-152.

[24] E. Vasselli: Crossed Products by Endomorphisms, Vector Bundles and Group Duality, arxiv:math.OA/0301214.

[25] Y. Watatani: Index for C^*-subalgebras, *Memoirs of the Am. Math. Soc.* **424**, 1990.