Large deviation estimates of the crossing probability for pinned Gaussian processes

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31st March 2008

Abstract. The paper deals with the asymptotic behavior of the bridge of a Gaussian process conditioned to stay in n fixed points at n fixed past instants. In particular, functional large deviation results are stated for small time. Several examples are considered: integrated or not fractional Brownian motion, m-fold integrated Brownian motion. As an application, the asymptotic behavior of the exit probability is studied and used for the practical purpose of the numerical computation, via Monte Carlo methods, of the hitting probability up to a given time of the unpinned process.

Keywords: conditioned Gaussian processes; reproducing kernel Hilbert spaces; large deviations; exit time probabilities; Monte Carlo methods.

2000 MSC: 60F10, 60G15, 65C05.

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1 Introduction

Simple formulas for the crossing probability in small time for pinned processes have been recently investigated in the literature, because of their use in improving the performance of the numerical simulation of processes to be killed when a prescribed boundary is reached. The idea underlying the application is simple. In fact, consider the generic step of the simulation procedure: one has generated the process of interest, say X, at some $n \geq 1$ fixed instants $0 < T_1 < \cdots < T_n$, observing the positions $X_{T_1} = x_1, \ldots, X_{T_n} = x_n$. In order to get the exit time, one simulates again the process, at time $T_n + \varepsilon$, and if the observed position $X_{T_n+\varepsilon} = y$ reaches the boundary, then the crossing is achieved. This gives rise to an overestimate of the exit time, which can dramatically bring to a significant error, as observed by many authors. One way to overcome this difficulty is to compute the crossing probability of the pinned process, that is for the conditional process $(X_{T_n+\varepsilon t})_{0\leq t\leq 1}$ given all the past observations $X_{T_1} = x_1, \ldots, X_{T_n} = x_n$ and the present one $X_{T_n+\varepsilon} = y$, and to use it in order to decide if the boundary has been breached or not. Let us stress that in the general case, no closed formulas are available, so that such a procedure is carried out with an approximation (by large deviations, as $\varepsilon \to 0$) of the exit probability.

In the case of diffusion processes, the Markov property allows one to work with the bridge-process between the observations at times T_n and $T_n+\varepsilon$. This case has been widely studied in the literature, see e.g. Baldi and Caramellino [3] and the references quoted therein. This approach obviously fails if a non-Markovian process is studied, so that one has to consider all the past observations and to handle the bridge of the conditional process.

The present paper deals with the large deviation asymptotic behavior of the exit probability of such a pinned process whenever the original one is a (continuous) Gaussian process (and in particular, not necessarily a Markovian one). Our wide class of examples can be split in two main sets.

First, we consider the fractional Brownian motion, which is widely used in risk theory modelling (see e.g. Baldi and Pacchiarotti [4]). As a consequence, we can handle the semimartingale process resulting from a linear combination between a fractional Brownian motion with Hurst index greater than 3/4 and a standard Brownian motion, independent each other, a promising tool to set up a non-Markovian model in Mathematical Finance (see Cheridito [6]).

Secondly, we can deal with an integrated Gaussian process, that is a process defined as the integral w.r.t. the Lebesgue measure of a Gaussian one. As an example, we obtain the integrated fractional Brownian motion, which is linked to fractal properties of solutions of the inviscid Burgers equation. Notice that the law of its maximum, analyzed e.g. in Molchan and Khokhlov [13], is strictly connected to level crossing probabilities. Furthermore, we can consider the m-fold iterated Brownian motion (see e.g. Chen and Li [5]) and in particular the integrated Brownian motion, having interesting applications in nonparametric estimating in Statistics (see e.g. Groeneboom, Jongbloed and Wellner [11] and references quoted therein) and used in metrology as a model for the atomic clock error (whose precision and re-synchronization are strictly related to the level crossing, see e.g. Galleani, Sacerdote, Tavella and Zucca [9]).

The paper is organized as follows. After a brief recall of some well known results related to large deviations for Gaussian processes (Section 2), we first get a functional large deviation result, for small time, for Gaussian processes conditioned to stay in n fixed positions $x_1, \dots, x_n$ at n fixed instants $T_1 < \cdots < T_n$ (Section 3). In a second moment (Section 4), we state a functional large deviation principle for the bridge of such conditional processes. Let us stress that, surprisingly, we obtain a degenerate kind of large deviations for Gaussian processes having a quite smooth covariance function (e.g. for integrated Gaussian processes), so that we give some refined results allowing to handle also these cases. In particular, we obtain examples of an interesting and non-trivial asymptotic behavior, in which the (non-degenerate) large deviation speed is different according to the conditional process or its bridge: the speed associated to the bridge can be much faster than the one turning out for the conditional process. Finally (Section 5), we are able to give the asymptotic behavior, in terms of large deviations, of the probability of crossing one or two possibly time dependent levels, and we propose some numerical results concerning the fractional Brownian motion.

2 Large deviations for Gaussian processes

We briefly recall here some main facts related to the large deviation theory for Gaussian processes we are going to use. There are many references in the literature on this topics, where all details and proofs may be found; let us here recall some classical references: Azencott [1], Deuschel and Strook [8], Dembo and Zeitouni [7]. Without loss of generality, we can consider centered Gaussian processes.

Throughout the paper, $C([0,1])$ will denote the set of the continuous paths on $[0,1]$, endowed with the topology induced by the sup-norm. Moreover, $\mathscr{M}[0,1]$ will be its dual, i.e. the set of the signed Borel measures on $[0,1]$, and for any $\lambda \in \mathscr{M}[0,1]$, $\langle \lambda, \cdot \rangle$ will stand for the associated linear functional: $\langle \lambda, h \rangle = \int_0^1 h_t \lambda(dt)$, $h \in C([0,1])$.

A continuous process $U = (U_t)_{t\in[0,1]}$, defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, is a centered Gaussian process if for any $\lambda \in \mathscr{M}[0,1]$, $\langle \lambda, U \rangle = \int_0^1 U_t \lambda(dt)$ is a centered Gaussian r.v. taking value on $\mathbb{R}$. The associated continuous covariance function $k(t,s) = \mathrm{Cov}(U_t, U_s)$, $t, s \in [0,1]$, plays a crucial role. For example, one has

$$\mathrm{Var}(\langle \lambda, U \rangle) = \int_0^1 \int_0^1 k(t,s)\lambda(dt)\lambda(ds), \quad \text{for any } \lambda \in \mathscr{M}[0,1].$$

In addition to k, another important instrument for handling Gaussian processes is the associated reproducing kernel Hilbert space. It is a Hilbert space in $C([0,1])$ which is usually defined through the following (dense w.r.t. a suitable norm) subset:

$$\mathscr{L} = \Big\{ h \in C([0,1]) \,:\, h_t = \int_0^1 k(t,s)\lambda(ds), \text{ with } \lambda \in \mathscr{M}[0,1] \Big\}.$$

Let us be a little bit more precise about $\mathscr{H}$. First, let μ denote the measure induced by the Gaussian process U: $\mu(A) = \mathbb{P}(U \in A)$ for any Borel set A in $C([0,1])$. Let $\Gamma \subset L^2(\mu)$ be defined as the following set of (real) Gaussian r.v.'s:

$$\Gamma = \{ Y \,:\, Y(\cdot) = \langle \lambda, \cdot \rangle, \text{ with } \lambda \in \mathscr{M}[0,1] \}.$$

It immediately follows that for $Y_1, Y_2 \in \Gamma$, with $Y_i(\cdot) = \langle \lambda_i, \cdot \rangle$ as $i = 1, 2$, then

$$\mathrm{Cov}(Y_1, Y_2) = (Y_1, Y_2)_{L^2(\mu)} = \int_0^1 \int_0^1 k(t,s)\lambda_1(dt)\lambda_2(ds),$$

where, from now on, the symbol $(\cdot, \cdot)$ denotes an inner product. We define now

$$H = \overline{\Gamma}^{\,\|\cdot\|_{L^2(\mu)}}.$$

Obviously, H is a closed subspace of $L^2(\mu)$ and is indeed a set of Gaussian r.v.'s taking values on $\mathbb{R}$. Moreover, it becomes a Hilbert space if endowed with the inner product

$$(Y_1, Y_2)_H = (Y_1, Y_2)_{L^2(\mu)}, \quad Y_1, Y_2 \in H.$$

We now set the following map:

$$\begin{array}{rrl} \mathcal{S}: & H \to & C([0,1]) \\ & Y \mapsto & (SY)_t = \displaystyle\int x_t Y(x)\mu(dx) \equiv \mathbb{E}(U_t Y). \end{array}$$

It can be shown that $\mathcal{S}$ is a linear, one-to-one and continuous map, so that $\mathcal{S}^{-1} : \mathcal{S}H \to H$ is a well posed continuous and linear map. The reproducing kernel Hilbert space $\mathscr{H}$ is defined as the image of H through $\mathcal{S}$:

$$\mathscr{H} = \mathcal{S}H \equiv \{h \in C([0,1]) \,:\, h_t = (SY)_t, Y \in H\}.$$

Finally, setting

$$(h_1, h_2)_{\mathscr{H}} = (\mathcal{S}^{-1}h_1, \mathcal{S}^{-1}h_2)_H \equiv (\mathcal{S}^{-1}h_1, \mathcal{S}^{-1}h_2)_{L^2(\mu)}, \quad h_1, h_2 \in \mathscr{H},$$

then $(\cdot, \cdot)_{\mathscr{H}}$ is an inner product on $\mathscr{H}$, which in turn makes $\mathscr{H}$ a Hilbert space. This is the rigorous definition of the reproducing kernel Hilbert space associated to a (centered) Gaussian process. Finally, it immediately follows that

$$\mathscr{H} = \overline{\mathscr{L}}^{\,\|\cdot\|_{\mathscr{H}}}, \quad \text{with } \mathscr{L} = \{x \in C([0,1]) \,:\, x_t = \int_0^1 k(t,s)\lambda(ds), \text{ with } \lambda \in \mathscr{M}[0,1]\}.$$

In the sequel, we will speak about "the reproducing kernel Hilbert space associated to the covariance function $k(t,s)$". In fact, given a continuous, symmetric and positive definite function $k(t,s)$ defined on $[0,1] \times [0,1]$, one can build a centered and continuous Gaussian process $U = (U_t)_{t \in [0,1]}$ having k as its covariance function. Now, the associated reproducing kernel Hilbert space is naturally defined.
The main property we are going to use is related to the Cramèr transform (see e.g. Deuschel and Strook [8]):

Theorem 2.1. **[Cramèr transform]** *Let I denote the Cramèr transform, that is*

$$I(x) = \sup_{\lambda \in \mathscr{M}[0,1]} \Big(\langle \lambda, x \rangle - \log \mathbb{E}(e^{\langle \lambda, U \rangle}) \Big) = \sup_{\lambda \in \mathscr{M}[0,1]} \Big(\langle \lambda, x \rangle - \frac{1}{2} \int_0^1 \int_0^1 k(t,s)\lambda(dt)\lambda(ds) \Big).$$

Then,

$$I(x) = \begin{cases} \dfrac{1}{2}\,\|x\|^2_{\mathscr{H}} & \text{if } x \in \mathscr{H} \\ +\infty & \text{otherwise.} \end{cases}$$

Suppose now to have a family of continuous Gaussian processes $\{U^\varepsilon\}_\varepsilon$: is it possible to determine a large deviation principle? Because of the special form of the Laplace transform for Gaussian measures, a large deviation principle can be stated if a nice asymptotic behavior holds for the Laplace transforms, as summarized in the following

Theorem 2.2. *Let $\{U^\varepsilon\}_\varepsilon$ be a family of continuous Gaussian processes. Let γ_ε be an infinitesimal function, i.e. $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$, and suppose that, for any $\lambda \in \mathscr{M}[0,1]$,*

$$0 = \lim_{\varepsilon\to 0}\mathbb{E}(\langle\lambda, U^\varepsilon\rangle) \quad \text{and} \quad \Lambda(\lambda) = \lim_{\varepsilon\to 0}\frac{\mathrm{Var}(\langle\lambda, U^\varepsilon\rangle)}{\gamma_\varepsilon^2} \equiv \int_0^1\int_0^1 \bar{k}(t,s)\lambda(dt)\lambda(ds),$$

for some continuous, symmetric and positive definite function $\bar{k}$. Then, $\{U^\varepsilon\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and (good) rate function

$$I(h) = \begin{cases} \dfrac{1}{2}\|h\|^2_{\bar{\mathscr{H}}} & \text{if } h \in \bar{\mathscr{H}} \\ +\infty & \text{otherwise,} \end{cases} \qquad (1)$$

where $\bar{\mathscr{H}}$ and $\|\cdot\|_{\bar{\mathscr{H}}}$ denote, respectively, the reproducing kernel Hilbert space and the related norm associated to the covariance function $\bar{k}$.

Let us recall, once for all, that the sentence "$\{U^\varepsilon\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$ with inverse speed γ_ε^2 and (good) rate function I" means: $\lim_{\varepsilon\to 0}\gamma_\varepsilon = 0$; the set $\{I \le a\}$ is compact in $C([0,1])$, for any fixed a; the following inequalities hold:

- for any open set G in $C([0,1])$, $\liminf_{\varepsilon\to 0}\gamma_\varepsilon^2\log\mathbb{P}(U^\varepsilon \in G) \ge -\inf_{h\in G} I(h)$;
- for any closed set F in $C([0,1])$, $\limsup_{\varepsilon\to 0}\gamma_\varepsilon^2\log\mathbb{P}(U^\varepsilon \in F) \le -\inf_{h\in F} I(h)$.

For the sake of convenience, Theorem 2.2 is written for a non-centered family of Gaussian processes, even if it requires that the expected path weakly converges to zero. The idea of the proof of Theorem 2.2 is the following. It is well known (e.g. by applying the Gärtner-Ellis Theorem, see e.g. Dembo and Zeitouni [7]) that a large deviation principle holds if the hypotheses of Theorem 2.2 are satisfied, and the rate function is given by the Legendre transform of

$$\bar{\Lambda}(\lambda) = \frac{1}{2}\int_0^1\int_0^1 \bar{k}(t,s)\lambda(dt)\lambda(ds), \quad \lambda \in \mathscr{M}[0,1].$$

In view of Theorem 2.1, one immediately obtains formula (1).

3 Large deviations for the conditional process

Let $X = (X_t)_{t\geq 0}$ be a Gaussian, centered process with continuous covariance function

$$k(t,s) = \mathrm{Cov}(X_t, X_s). \tag{2}$$

For a fixed $n \in \mathbb{N}$ and $j = 1, \dots, n$, let $X^j = (X^j_t)_{t\geq 0}$ stand for the process giving the conditional behavior of X given that it assumes the values $x_1, \dots, x_j$ at the j times $0 < T_1 < \dots < T_j$ respectively. Since the original process X is Gaussian, the process $X^j = (X^j_t)_{t\geq 0}$ is equal in law to

$$X^j_t = X^{j-1}_t - \alpha_j(t)(X^{j-1}_{T_j} - x_j), \tag{3}$$

where

$$\alpha_j(t) = \frac{k_{j-1}(t, T_j)}{k_{j-1}(T_j, T_j)} \tag{4}$$

and also k_j, giving the covariance function associated to X^j, is recursively defined as

$$\begin{aligned} k_j(t,s) = \mathrm{Cov}(X^j_t, X^j_s) = &\ k_{j-1}(t,s) - \alpha_j(t) k_{j-1}(s, T_j) \\ = &\ k_{j-1}(t,s) - \alpha_j(s) k_{j-1}(t, T_j). \end{aligned} \tag{5}$$

Obviously, the case $j = 0$ is related to the original process and its covariance function, that is $X^0 \equiv X$ and $k_0 \equiv k$.
Our first aim is to study the behavior of the covariance function of the original process X in order to get a functional large deviation principle for the n-fold conditional process X^n for small time, that is for $\{X^n_{T_n+\varepsilon\cdot}\}_\varepsilon$ as $\varepsilon \to 0$.
Let us consider an infinitesimal function γ_ε ($\gamma_\varepsilon \to 0$ as $\varepsilon \to 0$), whose square will play the role of the inverse speed of the large deviation principles we are going to study, and let us introduce the following hypothesis.

Assumption 3.1. *There exists the asymptotic covariance function* $\bar{k}(t,s)$, *defined as*

$$\begin{aligned} \bar{k}(t,s) = &\ \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_{T_n+\varepsilon s} - X_{T_n})}{\gamma_\varepsilon^2} \\ = &\ \lim_{\varepsilon\to 0} \frac{k(T_n+\varepsilon t, T_n+\varepsilon s) - k(T_n+\varepsilon t, T_n) - k(T_n, T_n+\varepsilon s) + k(T_n, T_n)}{\gamma_\varepsilon^2} \end{aligned} \tag{6}$$

uniformly as $(t,s) \in [0,1] \times [0,1]$.

Assumption 3.2. *For any fixed* $T > 0$, *the following limit exists:*

$$\bar{\rho}(t,T) = \lim_{\varepsilon\to 0} \frac{\mathrm{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_T)}{\gamma_\varepsilon} = \lim_{\varepsilon\to 0} \frac{k(T_n+\varepsilon t, T) - k(T_n, T)}{\gamma_\varepsilon}, \tag{7}$$

uniformly as $t \in [0,1]$.

Intuitively, Assumption 3.1 defines a "local process". In fact, it says that locally, as $\varepsilon \to 0$, the process $(X_{T_n+\varepsilon t} - X_{T_n})_{t\in[0,1]}$ behaves as a Gaussian process with covariance function given by $\gamma_\varepsilon^2 \bar{k}(t,s)$. Assumption 3.2 is set in order to describe the influence of a distant value on the local process.
Let us discuss some simple but useful consequences of the assumptions introduced above. As an immediate application of Theorem 2.2 (take $U_t^\varepsilon = X_{T_n+\varepsilon t} - X_{T_n}$), Assumption 3.1 implies that the family $\{(X_{T_n+\varepsilon t} - X_{T_n})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and good rate function given by

$$J(h) = \begin{cases} \frac{1}{2}\|h\|^2_{\bar{\mathscr{H}}} & \text{if } h \in \bar{\mathscr{H}} \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

where $\bar{\mathscr{H}}$ is the reproducing kernel Hilbert space associated to the covariance function $\bar{k}(t,s)$ and the symbol $\|\cdot\|_{\bar{\mathscr{H}}}$ denotes the usual norm defined on $\bar{\mathscr{H}}$.
Now, in order to achieve a large deviation principle for the n-fold conditional process X^n, we have to investigate the behavior of the functions k_j, defined through (5), in a small time interval of length ε. This can be done by means of Assumption 3.2, as follows.

Lemma 3.3. *(i) Under Assumption 3.2, as $j = 1, \dots, n$ one has*

$$\lim_{\varepsilon\to 0} \frac{\alpha_j(T_n + \varepsilon t) - \alpha_j(T_n)}{\gamma_\varepsilon} = \bar{\alpha}_j(t), \quad \textit{uniformly as } t \in [0,1],$$

where

$$\bar{\alpha}_j(t) = \frac{\bar{\rho}_{j-1}(t, T_j)}{k_{j-1}(T_j, T_j)} \tag{9}$$

k_{j-1} being defined in (5), $\bar{\rho}_0 \equiv \bar{\rho}$ and

$$\begin{aligned} \bar{\rho}_j(t,T) = \lim_{\varepsilon\to 0} \frac{k_j(T_n + \varepsilon t, T) - k_j(T_n, T)}{\gamma_\varepsilon} &= \bar{\rho}_{j-1}(t,T) - \bar{\alpha}_j(t) k_{j-1}(T, T_j) \\ &= \bar{\rho}_{j-1}(t,T) - \alpha_j(T)\bar{\rho}_{j-1}(t, T_j), \end{aligned} \tag{10}$$

the above limit being uniformly as $t \in [0,1]$.

(ii) Under Assumptions 3.1 and 3.2, as $j = 1, \dots, n$ one has

$$\lim_{\varepsilon\to 0} \mathbb{E}(X^j_{T_n+\varepsilon t} - X^j_{T_n}) = 0, \quad \textit{uniformly as } t \in [0,1] \tag{11}$$

and

$$\lim_{\varepsilon\to 0} \frac{\mathrm{Cov}(X^j_{T_n+\varepsilon t} - X^j_{T_n}, X^j_{T_n+\varepsilon s} - X^j_{T_n})}{\gamma_\varepsilon^2} = \bar{k}_j(t,s), \quad \textit{uniformly as } t,s \in [0,1],$$

with

$$\bar{k}_j(t,s) = \bar{k}(t,s) - \sum_{\ell=1}^{j} k_{\ell-1}(T_\ell, T_\ell)\bar{\alpha}_\ell(t)\bar{\alpha}_\ell(s), \tag{12}$$

with $\bar{\alpha}_\ell$ defined in (9).

Proof. (*i*) From Assumption 3.2 and (4), one immediately has

$$\bar{\alpha}_1(t) = \frac{\bar{\rho}(t, T_1)}{k(T_1, T_1)}.$$

Therefore, by using (5), there exists, uniformly as $t \in [0, 1]$,

$$\begin{aligned} \bar{\rho}_1(t, T) = \lim_{\varepsilon \to 0} \frac{k_1(T_n + \varepsilon t, T) - k_1(T_n, T)}{\gamma_\varepsilon} &= \bar{\rho}_0(t, T) - \bar{\alpha}_1(t) k_0(T, T_1) \\ &= \bar{\rho}_0(t, T) - \alpha_1(T) \bar{\rho}_0(t, T_1) \end{aligned}$$

where, as usual, we have set $\bar{\rho}_0 \equiv \bar{\rho}$ and $k_0 \equiv k$. This ensures the existence of $\bar{\alpha}_2$. The statement now follows by iteration.

The proof of (*ii*) is a straightforward application of Assumption 3.1 and part (*i*). □

Notice that in particular, since $X^n_{T_n} = x_n$, one has, again uniformly for $t, s \in [0, 1]$,

$$x_n = \lim_{\varepsilon \to 0} \mathbb{E}(X^n_{T_n + \varepsilon t}) \tag{13}$$

$$\bar{k}_n(t, s) = \lim_{\varepsilon \to 0} \frac{\mathrm{Cov}(X^n_{T_n + \varepsilon t}, X^n_{T_n + \varepsilon s})}{\gamma^2_\varepsilon} = \bar{k}(t, s) - \sum_{\ell=1}^{n} k_{\ell-1}(T_\ell, T_\ell) \bar{\alpha}_\ell(t) \bar{\alpha}_\ell(s). \tag{14}$$

We are now ready to prove the main large deviation result of this section:

Theorem 3.4. *Under Assumption 3.1 and 3.2, the family $\{(X^n_{T_n + \varepsilon t})_{t \in [0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0, 1])$, with inverse speed γ^2_ε and good rate function*

$$J_n(h) = \begin{cases} \frac{1}{2} \|h - x_n\|^2_{\bar{\mathscr{H}}_n} & \text{if } h_0 = x_n \text{ and } h - x_n \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise.} \end{cases} \tag{15}$$

$\bar{\mathscr{H}}_n$ being the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_n(t, s) = \bar{k}(t, s) - \sum_{j=1}^{n} k_{j-1}(T_j, T_j) \bar{\alpha}_j(t) \bar{\alpha}_j(s) \tag{16}$$

where $\bar{k}(\cdot, \cdot)$, $k_j(\cdot, \cdot)$ and $\bar{\alpha}_j(\cdot)$ are defined through (6), (5) and (9) respectively.

Proof. We start by showing that $\{(X^1_{T_n + \varepsilon t} - X^1_{T_n})_{t \in [0,1]}\}_\varepsilon$, X^1 being defined in (3) with $j = 1$, satisfies a large deviation principle. By (11), it follows that

$$\lim_{\varepsilon \to 0} \mathbb{E}(\langle \lambda, X^1_{T_n + \varepsilon \cdot} - X^1_{T_n} \rangle) = \lim_{\varepsilon \to 0} \int_0^1 \mathbb{E}(X^1_{T_n + \varepsilon t} - X^1_{T_n}) \lambda(dt) = 0$$

and, recalling that $\mathrm{Var}(X_{T_1}) = k(T_1, T_1) \equiv k_0(T_1, T_1)$, by (12)

$$\lim_{\varepsilon \to 0} \frac{\mathrm{Var}(\langle \lambda, X^1_{T_n + \varepsilon \cdot} - X^1_{T_n} \rangle)}{\gamma^2_\varepsilon} = \lim_{\varepsilon \to 0} \int_0^1 \lambda(dt) \int_0^1 \lambda(ds) \frac{\mathrm{Cov}(X^1_{T_n + \varepsilon t} - X^1_{T_n}, X^1_{T_n + \varepsilon s} - X^1_{T_n})}{\gamma^2_\varepsilon}$$

$$= \int_0^1 \lambda(dt) \int_0^1 \lambda(ds) \Big(\bar{k}(t,s) - k(T_1,T_1)\bar{\alpha}_1(t)\bar{\alpha}_1(s) \Big).$$

By using Theorem 2.2 one gets the large deviation principle. Now, iterating the same procedure up to n, one would achieve the following (recall that $\mathrm{Var}(X^{j-1}_{T_j}) = k_{j-1}(T_j,T_j)$):

$$\lim_{\varepsilon\to 0} \mathbb{E}(\langle \lambda, X^n_{T_n+\varepsilon\cdot} - X^n_{T_n}\rangle) = 0$$

$$\lim_{\varepsilon\to 0} \frac{\mathrm{Var}(\langle \lambda, X^n_{T_n+\varepsilon\cdot} - X^n_{T_n}\rangle)}{\gamma_\varepsilon^2} = \int_0^1 \lambda(dt) \int_0^1 \lambda(ds) \bar{k}_n(t,s)$$

with

$$\bar{k}_n(t,s) = \Big(\bar{k}(t,s) - \sum_{j=1}^n k_{j-1}(T_j,T_j)\bar{\alpha}_j(t)\bar{\alpha}_j(s) \Big)$$

Notice that, by (14), $\bar{k}_n$ is a continuous covariance function, being the (uniform) limit of a continuous, symmetric and positive definite function. Therefore, we can assert that $\{(X^n_{T_n+\varepsilon t} - X^n_{T_n})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and good rate function

$$H_n(\varphi) = \begin{cases} \frac{1}{2}\,\|\varphi\|^2_{\bar{\mathscr{H}}_n} & \text{if } \varphi \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise} \end{cases}$$

Finally, since $X^n_{T_n+\varepsilon t} = x_n + (X^n_{T_n+\varepsilon t} - X^n_{T_n})$, the large deviation principle as in the statement follows by contraction and the associated rate function is actually given by (15). □

Before to continue with the asymptotic behavior of the n-fold conditional bridge process, let us give some examples of applications of last Theorem 3.4 to the fractional Brownian motion and integrated Gaussian processes.

3.1 Fractional Brownian motion

The following result holds as a consequence of Theorem 3.4:

Theorem 3.5. *Let X be a fractional Brownian motion, with Hurst index $H \in (0,1)$, and let X^n denote the n-fold conditional process as in (3). Then, the family of processes $\{(X^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed ε^{2H} and good rate function*

$$J_n(h) = \begin{cases} \frac{1}{2}\,\|h - x_n\|^2_{\mathscr{H}_H} & \text{if } h_0 = x_n \text{ and } h - x_n \in \mathscr{H}_H \\ +\infty & \text{otherwise} \end{cases} \tag{17}$$

$\mathscr{H}_H$ being the reproducing kernel Hilbert space associated to the fractional Brownian motion itself.

Let us recall that a fractional Brownian motion X with Hurst index $H \in (0,1)$ is a continuous, non-Markovian unless $H = 1/2$, centered, Gaussian process whose covariance function is

$$k_H(t,s) = \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}.$$

Proof of Theorem 3.5. We show that both Assumption 3.1 and 3.2 do hold. First, one has

$$\frac{\mathrm{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_{T_n+\varepsilon s} - X_{T_n})}{\varepsilon^{2H}} = \mathrm{Cov}(X_t, X_s)$$

because of the homogeneity and self-similarity properties holding for the fractional Brownian motion, so that the limit in (6) trivially exists and $\bar{k}(t,s) = k_H(t,s)$. Concerning Assumption 3.2, straightforward computations (using Taylor expansion) allow easily to state that

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,1]} \frac{|k_H(T_n+\varepsilon t, T) - k_H(T_n, T)|}{\varepsilon^H} = 0,$$

for any $T > 0$, so that $\bar{\rho} \equiv 0$. This in turn implies that $\bar{\alpha}_j(t) = 0$, for any $t \in [0,1]$ and $j = 1, \dots, n$, as an immediate consequence of what developed in Lemma 3.3 (i). Then $\bar{k}_n \equiv k$ and the statement now follows from Theorem 3.4. □

Notice that the n-fold conditional fractional Brownian motion satisfies a large deviation principle with the same rate function as the non-conditioned process. This means that the asymptotic behavior of the n-fold conditional process does not depend on the past, although X is not Markovian unless $H = 1/2$. However, such a local independence property has been recently observed by several authors, e.g. Mandjes, Mannersalo, Norros and van Uitert [12], as well as Norros and Saksman [14].

Example 3.6. As an example, let us consider the process

$$X_t = cB_t + c_H B_t^H,$$

in which c and c_H are non-null real numbers, B stands for a standard Brownian motion and B^H denotes a fractional Brownian motion with Hurst index $H \neq 1/2$. Suppose moreover that B and B^H are independent. Such a process has been studied by Cheridito [6], who proved that X is a semimartingale if and only if $H \in (3/4, 1)$, a property allowing to get interesting applications in finance. The covariance function associated to X is given by

$$k(t,s) = c^2 k_{1/2}(t,s) + c_H^2 k_H(t,s).$$

Then, by using arguments similar to the ones developed in the proof of Theorem 3.5, one can state a large deviation principle for $\{(X^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ on $C([0,1])$, with inverse speed $\varepsilon^{2(H\wedge 1/2)}$ and good rate function associated to the covariance function

$$\bar{k}_n(t,s) = \sigma_H^2\, k_{H\wedge 1/2}(t,s), \quad \text{with } \sigma_H^2 = \begin{cases} c^2 & \text{if } H > 1/2 \\ c_H^2 & \text{if } H < 1/2 \end{cases} \tag{18}$$

where $k_{H\wedge 1/2}$ denotes the covariance function associated to a fractional Brownian motion with Hurst index $H \wedge 1/2$. By contraction, the constant σ_H^2 can be put inside the rate function, which becomes:

$$J_n(h) = \begin{cases} \frac{1}{2\sigma_H^2} \|h - x_n\|^2_{\mathscr{H}_{H\wedge 1/2}} & \text{if } h_0 = x_n \text{ and } h - x_n \in \mathscr{H}_{H\wedge 1/2} \\ +\infty & \text{otherwise} \end{cases}$$

$\mathscr{H}_{H\wedge 1/2}$ being the reproducing kernel Hilbert space associated to a fractional Brownian motion with Hurst index $H \wedge 1/2$.

3.2 Integrated Gaussian Process

Let Z be a centered Gaussian process with covariance function $\kappa(t,s)$ and let X be the integrated process, i.e.,

$$X_t = \int_0^t Z_u du. \tag{19}$$

X is a continuous, centered Gaussian process whose covariance function k is given by

$$k(t,s) = \int_0^t \int_0^s \kappa(u,v) du dv. \tag{20}$$

As a consequence of Theorem 3.4 one has:

Theorem 3.7. *Let X be an integrated Gaussian process as in (3.2), with $\kappa(t,s)$ continuous, and let X^n denote the n-fold conditional process as in (3). Then, the family $\{(X^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed ε^2 and good rate function*

$$J_n(h) = \begin{cases} \frac{1}{2} \|h - x_n\|^2_{\bar{\mathscr{H}}_n} & \text{if } h_0 = x_n \text{ and } h - x_n \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise} \end{cases} \tag{21}$$

$\bar{\mathscr{H}}_n$ being the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_n(t,s) = a_n^2 \cdot ts, \quad \text{where} \quad a_n^2 = \kappa(T_n,T_n) - \sum_{j=1}^n \frac{d_{j-1}(T_j)^2}{k_{j-1}(T_j,T_j)}$$

and $d_{j-1}(T)$ is recursively defined as: $d_0(T) = \int_0^T \kappa(T_n,u)du$ and as $i = 1,2,\dots,n-1$,

$$d_i(T) = d_{i-1}(T) - \alpha_i(T) d_{i-1}(T_i)$$

(recall that k_j and α_j are defined through (5) and (4) respectively).

Proof. Let us first show that Assumption 3.1 holds, with $\gamma_\varepsilon = \varepsilon$ and $\bar{k}(t,s) = ts\,\kappa(T_n,T_n)$. In fact,

$$\left| \frac{1}{\varepsilon^2} \operatorname{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_{T_n+\varepsilon s} - X_{T_n}) - ts\,\kappa(T_n,T_n) \right| \leq$$

$$\leq \frac{1}{\varepsilon^2}\int_{T_n}^{T_n+\varepsilon} du \int_{T_n}^{T_n+\varepsilon} dv |\kappa(u,v)-\kappa(T_n,T_n)| \leq \sup_{u,v\in[T_n,T_n+\varepsilon]} |\kappa(u,v)-\kappa(T_n,T_n)|$$

and the last term goes to zero as $\varepsilon \to 0$ because κ is continuous, thus uniformly continuous on compact sets. Similarly, one proves that also Assumption 3.2 holds, with $\bar\rho(t,T) = t\int_0^T \kappa(T_n,v)dv$. The large deviation principle is now an immediate application of Theorem 3.4. Finally, in order to give the above more explicit expression for $\bar k_n$, we need the functions $\bar\alpha_j$. By (9), it is sufficient to show that

$$\bar\rho_j(t,T) = d_j(T)\, t$$

where $d_0(T) = \int_0^T \kappa(T_n,u)du$ and as $j = 1,2,\dots,n-1$,

$$d_j(T) = d_{j-1}(T) - \alpha_j(T) d_{j-1}(T_i).$$

We have already seen that $\bar\rho_0(t,T) = \bar\rho(t,T) = d_0(T)\, t$, so that by (10),

$$\bar\rho_1(t,T) = \bar\rho_0(t,T) - \alpha_1(T)\bar\rho_0(t,T_1) = d_1(T)\, t,$$

with $d_1(T) = d_0(T) - \alpha_1(T)d_0(T_1)$. By iteration, the statement holds. □

Remark 3.8. It follows the law of an n-fold conditional integrated Gaussian process behaves asymptotically as $a_n U\, t$, being U a standard Gaussian random variable. Moreover, a deeper view to the proof of Theorem 3.7 shows that this kind of "degenerate" behavior can be stated for any Gaussian process whose covariance function $k(t,s)$ is quite smooth, in particular if both the first and the mixed second derivatives exist, the latter being continuous on the diagonal points (T,T). In fact, in this case the asymptotic covariance $\bar k_n(t,s)$ for X^n is again of the type $a_n^2 \cdot ts$.

Example 3.9. [**m-fold integrated Brownian motion**] Suppose that X is defined as

$$X_t = \int_0^t du\Big(\int_0^u du_{m-1}\cdots\int_0^{u_2} du_1 W_{u_1}\Big),$$

where W denotes a standard Brownian motion. It is known that X is a centered, Gaussian process with covariance function

$$k(t,s) = \frac{1}{(m!)^2}\int_0^{s\wedge t}(s-\xi)^m(t-\xi)^m\, d\xi = \int_0^t\int_0^s \kappa(u,v)\, du\, dv,$$

where

$$\kappa(t,s) = \frac{1}{((m-1)!)^2}\int_0^{t\wedge s}(t-\xi)^{m-1}(s-\xi)^{m-1}\, d\xi$$

(for details, see Chen and Li [5]). Then, Theorem 3.7 applies to X. Notice that, for $T \leq T_n$ and $m \geq 1$,

$$d_0(T) = \frac{m}{(m!)^2}\int_0^T (T_n-\xi)^{m-1}(T-\xi)^m\, d\xi.$$

Example 3.10. [Integrated fractional Brownian motion] Suppose $X_t = \int_0^t Z_u du$, where Z denotes a fractional Brownian motion with Hurst index H. Then, the associated covariance function is

$$k(t,s) = \int_0^t \int_0^s \kappa_H(u,v)\,du\,dv, \quad \text{with } \kappa_H(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Again, Theorem 3.7 immediately applies to X. Here, for $T \le T_n$, one has,

$$d_0(T) = \frac{1}{2}\Big[T_n^{2H} T + \frac{1}{2H+1}(T^{2H+1} - T_n^{2H+1} - (T_n - T)^{2H+1})\Big].$$

4 Large deviations for the bridge of the conditional process

Let $(X_t^n)_{t\ge 0}$ be the n-fold conditional process defined in Section 3 and let us now consider the process Y^n defined as the *bridge* of the process X^n, i.e, the process X^n conditioned to be in y at the future time $T_n + \varepsilon$. Then, in law one has,

$$Y^n_{T_n+\varepsilon t} = X^n_{T_n+\varepsilon t} - \beta^\varepsilon_{T_n+\varepsilon t}(X^n_{T_n+\varepsilon} - y), \tag{22}$$

where

$$\beta^\varepsilon_{T_n+\varepsilon t} = \frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)}{k_n(T_n+\varepsilon, T_n+\varepsilon)}. \tag{23}$$

Now, in order to achieve a large deviation principle for $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$, one needs a nice asymptotic behavior for $\beta^\varepsilon_{T_n+\varepsilon\cdot}$. In fact, one has

Lemma 4.1. *Let Assumptions 3.1 and 3.2 be satisfied. Then there exists the limit*

$$\lim_{\varepsilon\to 0} \beta^\varepsilon_{T_n+\varepsilon t} = \frac{\bar{k}_n(t,1)}{\bar{k}_n(1,1)} =: \bar{\beta}_t, \quad \textit{uniformly as } t \in [0,1]. \tag{24}$$

Proof. One has,

$$|\beta^\varepsilon_{T_n+\varepsilon t} - \bar{\beta}_t| = \Big|\frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)}{k_n(T_n+\varepsilon, T_n+\varepsilon)} - \frac{\bar{k}_n(t,1)}{\bar{k}_n(1,1)}\Big|$$

$$\le \frac{\gamma_\varepsilon^2}{k_n(T_n+\varepsilon, T_n+\varepsilon)}\Big|\frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)}{\gamma_\varepsilon^2} - \bar{k}_n(t,1)\Big| + |\bar{k}(t,1)|\Big|\frac{\gamma_\varepsilon^2}{k_n(T_n+\varepsilon, T_n+\varepsilon)} - \frac{1}{\bar{k}_n(1,1)}\Big|$$

From (14),

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,1]}\Big|\frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)}{\gamma_\varepsilon^2} - \bar{k}_n(t,1)\Big| = 0 \text{ and } \lim_{\varepsilon\to 0}\Big|\frac{\gamma_\varepsilon^2}{k_n(T_n+\varepsilon, T_n+\varepsilon)} - \frac{1}{\bar{k}_n(1,1)}\Big| = 0,$$

so that the statement holds. □

It is now easy to prove a first large deviation principle. But, as we will see, there are cases in which the next, immediate, result turns out to be degenerate in some sense. So, let us split this Section in two part, the former containing a first result and the latter developing some refinements.

4.1 A first large deviation result for the bridge

Theorem 4.2. *Let Y^n be the bridge of the n-fold conditional process X^n, as defined in (22). Under Assumptions 3.1 and 3.2, the family of processes $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and good rate function*

$$J_Y(h) = \begin{cases} \frac{1}{2}\|h-\bar{m}\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h_0 = x_n, h_1 = y, h-\bar{m}\in\bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases} \tag{25}$$

where $\bar{m}_t = x_n + \bar{\beta}_t(y-x_n)$ and $\bar{\mathscr{H}}_Y$ is the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_Y(t,s) = \bar{k}_n(t,s) - \bar{\beta}_s\bar{k}_n(t,1) = \bar{k}_n(t,s) - \frac{\bar{k}_n(t,1)\bar{k}_n(s,1)}{\bar{k}_n(1,1)}.$$

Proof. First, let us set

$$U^n_{T_n+\varepsilon t} = Y^n_{T_n+\varepsilon t} - \bar{m}_t, \quad \text{where } \bar{m}_t = x_n + \bar{\beta}_t(y-x_n) = \lim_{\varepsilon\to 0}\mathbb{E}(Y^n_{T_n+\varepsilon t})$$

and notice that, by (22), (13) and (24), the above limit holds uniformly as $t\in[0,1]$. We will start by showing a large deviation principle for $\{U^n_{T_n+\varepsilon\cdot}\}_\varepsilon$, by using again Theorem 2.2. In fact,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big(\langle\lambda, U^n_{T_n+\varepsilon\cdot}\rangle\Big) = \int_0^1 \lambda(dt)\mathbb{E}(U^n_{T_n+\varepsilon t}) = 0,$$

for any $\lambda\in\mathscr{M}[0,1]$. Moreover, from (14) and Lemma 4.1, one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(U^n_{T_n+\varepsilon t}, U^n_{T_n+\varepsilon s})}{\gamma_\varepsilon^2} &= \lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(Y^n_{T_n+\varepsilon t}, Y^n_{T_n+\varepsilon s})}{\gamma_\varepsilon^2} = \\ = \lim_{\varepsilon\to 0}&\frac{\mathrm{Cov}(X^n_{T_n+\varepsilon t}, X^n_{T_n+\varepsilon s}) - \beta^\varepsilon_{T_n+\varepsilon s}\mathrm{Cov}(X^n_{T_n+\varepsilon t}, X^n_{T_n+\varepsilon})}{\gamma_\varepsilon^2} = \\ &= \bar{k}_n(t,s) - \bar{\beta}_s\bar{k}_n(t,1) =: \bar{k}_Y(t,s),\end{aligned}$$

uniformly as $s,t\in[0,1]$, so that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{\mathrm{Var}\Big(\langle\lambda, U^n_{T_n+\varepsilon\cdot}\rangle\Big)}{\gamma_\varepsilon^2} \\ &= \lim_{\varepsilon\to 0}\frac{1}{\gamma_\varepsilon^2}\int_0^1\lambda(dt)\int_0^1\lambda(ds)\,\mathrm{Cov}(U^n_{T_n+\varepsilon t}U^n_{T_n+\varepsilon s}) = \int_0^1\lambda(dt)\int_0^1\lambda(ds)\,\bar{k}_Y(t,s),\end{aligned}$$

for any $\lambda\in\mathscr{M}[0,1]$. We can then assert that the family of processes $\{(U^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ does satisfy a large deviation principle on $C([0,1])$, with inverse speed γ_ε^2 and good rate function

$$J_U(\varphi) = \begin{cases} \frac{1}{2}\|\varphi\|^2_{\bar{\mathscr{H}}_Y} & \text{if } \varphi_0 = \varphi_1 = 0, \varphi\in\bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases}$$

being $\bar{\mathscr{H}}_Y$ the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_Y(t,s) = \bar{k}_n(t,s) - \bar{\beta}_s \bar{k}_n(t,1) = \bar{k}_n(t,s) - \frac{\bar{k}_n(t,1)\bar{k}_n(s,1)}{\bar{k}_n(1,1)}.$$

Let us stress that the condition $\varphi_0 = \varphi_1 = 0$ is trivially satisfied if $\varphi \in \bar{\mathscr{H}}_Y$ (it immediately follows from the fact that $k_Y(0,s) = k_Y(1,s) = 0$ for any s), but we have chosen to write it for the sake of clearness. Now, since $Y^n_{T_n+\varepsilon t} = U^n_{T_n+\varepsilon t} + \bar{m}_t$, by contraction one immediately obtains the large deviation principle for $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ on $C([0,1])$, with inverse speed γ_ε^2 and good rate function as in (25). □

Remark 4.3. The rate function J_Y given by (25) can also be written in the following way:

$$J_Y(h) = \begin{cases} \frac{1}{2}\left(\|h - x_n\|^2_{\bar{\mathscr{H}}_n} - \frac{(y-x_n)^2}{\bar{k}_n(1,1)}\right) & \text{if } h_0 = x_n, h_1 = y, h - x_n \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise} \end{cases} \tag{26}$$

$\bar{\mathscr{H}}_n$ being the reproducing kernel Hilbert space associated to the covariance function $\bar{k}_n$ defined in (16). Such a representation agrees with well known formulas, for example whenever X is a standard Brownian motion (see e.g. Baldi, Caramellino and Iovino [2]). The proof of (26) is postponed to Appendix A.

Example 4.4. [Fractional Brownian motion] Following Section 3.1, let X be a fractional Brownian motion with Hurst index H and let X^n be the associated n-fold conditional process. As seen in Theorem 3.5, both Assumption 3.1 and 3.2 hold and the asymptotic covariance function $\bar{k}_n(t,s)$ coincides with the original one $k_H(t,s)$. By applying Theorem 4.2, the bridge process Y^n satisfies a functional large deviation principle for small time, with inverse speed ε^{2H} and good rate function

$$J_Y(h) = \begin{cases} \frac{1}{2}\|h - \bar{m}\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h_0 = x_n, h_1 = y, h - x_n \in \bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases} \tag{27}$$

$\bar{\mathscr{H}}_Y$ being the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_Y(t,s) = k_H(t,s) - k_H(t,1)k_H(1,s).$$

By using (26), J_Y can be written also in terms of the reproducing kernel Hilbert space $\mathscr{H}_H$ associated to the original fractional Brownian motion X:

$$J_Y(h) = \begin{cases} \frac{1}{2}\Big(\|h - x_n\|^2_{\mathscr{H}_H} - (y-x_n)^2\Big) & \text{if } h_0 = x_n, h_1 = y, h - x_n \in \mathscr{H}_H \\ +\infty & \text{otherwise} \end{cases} \tag{28}$$

Whenever $H = 1/2$, that is X is a standard Brownian motion, then the above result is well known and widely applied in the literature. Moreover, formula (28) confirms that, as a consequence of the "local" independence of the n-fold conditional fractional Brownian motion, also its bridge satisfies a large deviation principle which is independent of all the past except for what happens at time T_n.

Example 4.5. **[Cheridito process]** Let X be the process as in Example 3.6: $X_t = cB_t + c_H B_t^H$, with $c, c_H \neq 0$ constant numbers, B and B^H are independent, with B a Brownian motion and B^H a fractional Brownian motion with Hurst index $H \neq 1/2$. By developing arguments similar to the ones in Example 4.4, one obtains that the bridge of the associated n-fold conditional process satisfies a large deviation principle. By taking into account the results in Example 3.6 and formula (26), one easily obtains that the inverse speed is equal to $\varepsilon^{2(H\wedge 1/2)}$ and the good rate function is given by the following formula:

$$J_Y(h) = \begin{cases} \dfrac{1}{2\sigma_H^2}\Big(\|h - x_n\|^2_{\mathscr{H}_{H\wedge 1/2}} - (y - x_n)^2\Big) & \text{if } h_0 = x_n, h_1 = y, h - x_n \in \mathscr{H}_{H\wedge 1/2} \\ +\infty & \text{otherwise} \end{cases} \tag{29}$$

where σ_H^2 is given by (18).

Example 4.6. **[Integrated Gaussian processes]** Following Section 3.2, let X^n be the n-fold conditional process when X is an integrated Gaussian process as in (19). Under the hypotheses of Theorem 3.7, Assumptions 3.1 and 3.2 hold, and a functional large deviation principle for X^n follows, with asymptotic covariance function $\bar{k}_n(t,s) = a_n^2 \cdot ts$ for a suitable constant a_n^2. Now, by applying Theorem 4.2, one obtains a functional large deviation principle for the bridge process Y^n as well, but unfortunately one gets a degenerate asymptotic behavior because the associated rate function turns out to be

$$J_Y(h) = \begin{cases} 0 & \text{if } h = \bar{m} \\ +\infty & \text{otherwise.} \end{cases}$$

This follows from the fact that, since $\bar{\mathscr{H}}_n$ is "spanned" by the covariance function $\bar{k}_n(t,s) = a_n^2 \cdot ts$, it contains only the paths running at constant speed. Then, J_Y is finite only for h such that $h - \bar{m} = ct$. Since here $\bar{m}_t = x_n + (y - x_n)t$, the additional constraints $h_0 = x_n$ and $h_1 = y$ give the unique path $h = \bar{m}$.

Notice that Theorem 4.2 gives an unsatisfactory large deviation result not only for integrated Gaussian processes but also for Gaussian processes whose (original) covariance function is smooth enough: as observed in Remark 3.8, in this case the asymptotic covariance function is $const \cdot ts$ as well, and the same degenerate behavior holds for the rate function. This motivates next section, in which we study some refinements allowing one to state non-trivial large deviation estimates, or more precisely the right large deviation speed.

4.2 Faster large deviations for the bridge

In this Section we prove a refined version of Theorem 4.2: we study here the exact (faster) speed giving a non-trivial rate function whenever the covariance is smooth. With the same notation of Section 3, Assumptions 3.1 and 3.2 must be here strengthened as follows:

Assumption 4.7. *For some* $\alpha \in (0, 1]$, (*i*) *there exist a function* $\bar{\varphi}(t,s)$*, a constant* a^2 *and a remaining term* $\mathscr{R}^1_\varepsilon(t,s)$ *(depending on* T_n*) such that*

$$\begin{aligned} &\mathrm{Cov}(X_{T_n+\varepsilon t}-X_{T_n},X_{T_n+\varepsilon s}-X_{T_n}) = \varepsilon^2\left[a^2\,ts+\bar{\varphi}(t,s)\varepsilon^\alpha+\mathscr{R}^1_\varepsilon(t,s)\right] \\ &\text{with } \lim_{\varepsilon\to 0}\sup_{s,t\in[0,1]}\frac{|\mathscr{R}^1_\varepsilon(t,s)|}{\varepsilon^\alpha}=0; \end{aligned} \tag{30}$$

(*ii*) *for any fixed* $T>0$*, there exist a function* $\bar{\psi}(t,T)$*, a constant* $c(T)$ *and a remaining term* $\mathscr{R}^2_\varepsilon(t;T)$ *(depending on* T_n*) such that*

$$\begin{aligned} &k(T_n+\varepsilon t,T)-k(T_n,T) = \varepsilon\left[c(T)\,t+\bar{\psi}(t;T)\varepsilon^\alpha+\mathscr{R}^2_\varepsilon(t;T)\right] \\ &\text{with } \lim_{\varepsilon\to 0}\sup_{t\in[0,1]}\frac{|\mathscr{R}^2_\varepsilon(t;T)|}{\varepsilon^\alpha}=0. \end{aligned} \tag{31}$$

As a consequence of Assumption 4.7, by using the same arguments as in Lemma 3.3 one immediately proves the following

Lemma 4.8. *For* $j=1,\ldots,n$

$$k_j(T_n+\varepsilon t)-k_j(T_n,T)=\varepsilon[c_j(T)\,t+\bar{\psi}_j(t;T)\varepsilon^\alpha+\mathscr{R}^2_\varepsilon(t;T)], \tag{32}$$

where setting, $c_0\equiv c$*, and* $\bar{\psi}_0\equiv\bar{\psi}$*,* c_j *and* $\bar{\psi}_j$ *are given by*

$$c_j(T)=c_{j-1}(T)-\alpha_j(T)c_{j-1}(T_j) \quad \text{and} \quad \bar{\psi}_j(t;T)=\bar{\psi}_{j-1}(t;T)-\alpha_j(T)\bar{\psi}_{j-1}(t;T_j).$$

Moreover

$$\mathrm{Cov}(X^j_{T_n+\varepsilon t}-X^j_{T_n},X^j_{T_n+\varepsilon s}-X^j_{T_n})=\varepsilon^2[a_j^2\,ts+\bar{\varphi}_j(t,s)\varepsilon^\alpha+\mathscr{R}^{1,j}_\varepsilon(t,s)],$$

where $\mathscr{R}^{1,j}_\varepsilon(t,s)\to 0$ *as* $\varepsilon\to 0$ *uniformly on* $[0,1]\times[0,1]$*,* $a_j=a-\sum_{\ell=1}^j c^2_{\ell-1}(T_\ell)$ *and*

$$\bar{\varphi}_j(t,s)=\bar{\varphi}(t,s)-\sum_{\ell=1}^j\frac{c_{\ell-1}(T_\ell)}{k_{\ell-1}(T_\ell,T_\ell)}(\bar{\psi}_{\ell-1}(t;T_\ell)\,s+\bar{\psi}_{\ell-1}(s;T_\ell)\,t). \tag{33}$$

In particular, since $X^n_{T_n}=x_n$*,*

$$k_n(T_n+\varepsilon t,T_n+\varepsilon s)=\varepsilon^2[a_n^2\,ts+\bar{\varphi}_n(t,s)\varepsilon^\alpha+\mathscr{R}^{1,n}_\varepsilon(t,s)],\ \text{with } \lim_{\varepsilon\to 0}\sup_{t,s\in[0,1]}\frac{|\mathscr{R}^{1,n}_\varepsilon(t,s)|}{\varepsilon^\alpha}=0. \tag{34}$$

Then, one has

Theorem 4.9. *Let* Y^n *be the bridge of the n-fold conditional process* X^n*, as defined in (22). If Assumption 4.7 holds, then the family of processes* $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ *satisfies a large deviation principle on* $C([0,1])$*, with inverse speed* $\varepsilon^{2+\alpha}$ *and good rate function*

$$J_Y(h)=\begin{cases}\frac{1}{2}\,\|h-\bar{m}\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h-\bar{m}\in\bar{\mathscr{H}}_Y\\ +\infty & \text{otherwise}\end{cases} \tag{35}$$

where $\bar m_t = x_n + \bar\beta_t(y - x_n)$ and $\mathscr{H}_Y$ is the reproducing kernel Hilbert space associated to the covariance function

$$\bar k_Y(t,s) = \bar\varphi_n(t,s) + ts\,\bar\varphi_n(1,1) - t\,\bar\varphi_n(1,s) - s\,\bar\varphi_n(t,1). \tag{36}$$

Proof. The proof is the same as Theorem 4.2. It is enough to observe that in this case one has, from (34),

$$\begin{aligned}
\mathrm{Cov}(Y^n_{T_n+\varepsilon t}, Y^n_{T_n+\varepsilon s}) &= k_n(T_n+\varepsilon t, T_n+\varepsilon s) - \frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)k_n(T_n+\varepsilon, T_n+\varepsilon s)}{k_n(T_n+\varepsilon, T_n+\varepsilon)}\\
&= \varepsilon^2\Big[a_n^2\, ts + \bar\varphi_n(t,s)\varepsilon^\alpha - \frac{(a_n^2\, t + \bar\varphi_n(t,1)\varepsilon^\alpha)(a_n^2\, s + \bar\varphi_n(1,s)\varepsilon^\alpha)}{a_n^2\; + \bar\varphi_n(1,1)\varepsilon^\alpha + \mathscr{R}^{1,n}_\varepsilon(1,1)} + \mathscr{R}^{1,n}_\varepsilon(t,s)\Big]
\end{aligned}$$

$$= \frac{a_n^2\varepsilon^2}{a_n^2\; + \bar\varphi_n(1,1)\varepsilon^\alpha + \mathscr{R}^{1,n}_\varepsilon(1,1)}\Big((\bar\varphi_n(t,s) + ts\,\bar\varphi_n(1,1) - t\,\bar\varphi_n(1,s) - s\,\bar\varphi_n(t,1))\varepsilon^\alpha + \mathscr{R}^{1,n}_\varepsilon(t,s)\Big).$$

Therefore

$$\lim_{\varepsilon\to 0}\frac{\mathrm{Cov}(Y^n_{T_n+\varepsilon t}, Y^n_{T_n+\varepsilon s})}{\varepsilon^{2+\alpha}} = \bar\varphi_n(t,s) + ts\,\bar\varphi_n(1,1) - t\,\bar\varphi_n(1,s) - s\,\bar\varphi_n(t,1), \tag{37}$$

uniformly as $s,t \in [0,1]$ and the thesis holds. □

Remark 4.10. Notice that $\bar\varphi_n$ is symmetric and continuous whereas it is not positive definite in general, so that it is not necessarily a covariance function. Nevertheless, k_Y given by (36) does represent a covariance function, as an immediate consequence of (37). However, if $\bar\varphi_n$ was a covariance function, a curious effect would happen: the asymptotic behavior of the bridge is regulated by a covariance function which coincides with the one associated to what is usually called “the false bridge”, that is a process of the type $Z_t - tZ_1$, being Z a Gaussian process with covariance $\bar\varphi_n$.

Now, if the function $k(t,s)$ is more regular, then Theorem 4.9 would give again a degenerate behavior. In fact, suppose that k has continuous derivatives up to the third order. Then, since k is symmetric, by straightforward computations one obtains

$$\bar\varphi(t,s) = \frac{1}{3!}(3\partial^3_{tts}k(T_n,T_n)t^2 s + 3\partial^3_{tss}k(T_n,T_n)ts^2) = \frac{1}{2}\partial^3_{tts}k(T_n,T_n)ts(t+s)$$

and by using (33) one arrives to show that $\bar\varphi_n(t,s) = b_n ts(t+s)$, for a suitable constant b_n. Therefore, by (36) one has

$$\bar k_Y(t,s) = b_n ts(t+s) + 2b_n ts - b_n(t^2+t)s - b_n(s+s^2)t \equiv 0,$$

and again a trivial large deviation principle holds for the bridge of the conditional process. Let us refine further on the hypothesis.

Assumption 4.11. *For some $\alpha \in (0,1]$,* (*i*) *there exist a function* $\bar{\varphi}(t,s)$*, constants* a^2 *and* b *and a remaining term* $\mathscr{R}^1_\varepsilon(t,s)$ *(depending on* T_n*) such that*

$$\begin{aligned}&\mathrm{Cov}(X_{T_n+\varepsilon t}-X_{T_n},X_{T_n+\varepsilon s}-X_{T_n})=\\&\quad=\varepsilon^2[a^2\,ts+b(t^2s+ts^2)\varepsilon+\bar{\varphi}(t,s)\varepsilon^{1+\alpha}+\mathscr{R}^1_\varepsilon(t,s)]\\&\quad\textit{with}\ \lim_{\varepsilon\to0}\sup_{s,t\in[0,1]}\frac{|\mathscr{R}^1_\varepsilon(t,s)|}{\varepsilon^{1+\alpha}}=0;\end{aligned}\tag{38}$$

(*ii*) *for any fixed* $T>0$*, there exist a function* $\bar{\psi}(t,T)$*, constants* $c(T)$ *and* $d(T)$ *and a remaining term* $\mathscr{R}^2_\varepsilon(t;T)$ *(depending on* T_n*) such that*

$$\begin{aligned}&k(T_n+\varepsilon t,T)-k(T_n,T)=\varepsilon[c(T)\,t+d(T)\,t^2\varepsilon+\bar{\psi}(t;T)\varepsilon^{1+\alpha}+\mathscr{R}^2_\varepsilon(t;T)]\\&\textit{with}\ \lim_{\varepsilon\to0}\sup_{t\in[0,1]}\frac{|\mathscr{R}^2_\varepsilon(t;T)|}{\varepsilon^{1+\alpha}}=0.\end{aligned}\tag{39}$$

Let us remark that if $k(t,s)$ is smooth enough then one immediately has $a^2=\partial^2_{ts}k(T_n,T_n)$, $b=\frac{1}{2}\partial^3_{tts}k(T_n,T_n)$, $c(T)=\partial_t k(T_n,T)$ and $d(T)=\frac{1}{2}\partial^2_{tt}k(T_n,T)$.
Moreover, as an immediate consequence of Assumption 4.11, by using the same arguments as in Lemma 3.3 and Lemma 4.8, one proves that

Lemma 4.12. *For* $j=1,\dots,n$

$$k_j(T_n+\varepsilon t)-k_j(T_n,T)=\varepsilon[c_j(T)\,t+d_j(T)\,t^2\varepsilon+\bar{\psi}_j(t;T)\varepsilon^{1+\alpha}+\mathscr{R}^{2,j}_\varepsilon(t;T)]\tag{40}$$

where setting, $c_0(T)=c(T)$*,* $d_0(T)=d(T)$ *and* $\bar{\psi}_0\equiv\bar{\psi}$*,* $c_j(T)$*,* $d_j(T)$ *and* $\bar{\psi}_j$ *are defined in the following way,*

$$\begin{gathered}c_j(T)=c_{j-1}(T)-\alpha_j(T)c_{j-1}(T_j),\quad d_j(T)=d_{j-1}(T)-\alpha_j(T)d_{j-1}(T_j),\\\bar{\psi}_j(t;T)=\bar{\psi}_{j-1}(t;T)-\alpha_j(T)\bar{\psi}_{j-1}(t;T_j).\end{gathered}$$

Moreover

$$\mathrm{Cov}(X^j_{T_n+\varepsilon t}-X^j_{T_n},X^j_{T_n+\varepsilon s}-X^j_{T_n})=\varepsilon^2[a_j^2\,ts+b_j(t^2s+ts^2)\varepsilon+\bar{\varphi}_j(t,s)\varepsilon^{1+\alpha}+\mathscr{R}^{1,j}_\varepsilon(t,s)],$$

where $a_j^2=a^2-\sum_{\ell=1}^j c^2_{\ell-1}(T_\ell)$*,* $b_j=b-\sum_{\ell=1}^j c_{\ell-1}(T_\ell)d_{\ell-1}(T_\ell)$ *and*

$$\bar{\varphi}_j(t,s)=\begin{cases}\bar{\varphi}(t,s)-\displaystyle\sum_{\ell=1}^j\frac{c_{\ell-1}(T_\ell)}{k_{\ell-1}(T_\ell,T_\ell)}(\bar{\psi}_{\ell-1}(t;T_\ell)\,s+\bar{\psi}_{\ell-1}(s;T_\ell)\,t) & \textit{for}\,\alpha<1\\ \bar{\varphi}(t,s)-\displaystyle\sum_{\ell=1}^j\frac{c_{\ell-1}(T_\ell)}{k_{\ell-1}(T_\ell,T_\ell)}(\bar{\psi}_{\ell-1}(t;T_\ell)\,s+\bar{\psi}_{\ell-1}(s;T_\ell)\,t)+ & \\ \qquad-\displaystyle\sum_{\ell=1}^j\frac{d^2_{\ell-1}(T_\ell)}{k_{\ell-1}(T_\ell,T_\ell)}t^2s^2 & \textit{for}\,\alpha=1.\end{cases}\tag{41}$$

In particular since $X^n_{T_n}=x_n$*, one has*

$$k_n(T_n+\varepsilon t,T_n+\varepsilon s)=\varepsilon^2[a_n\,ts+b_n(t^2s+ts^2)\varepsilon+\bar{\varphi}_n(t,s)\varepsilon^{1+\alpha}+\mathscr{R}^{1,n}_\varepsilon(t,s)].\tag{42}$$

Let us stress that in Lemma 4.12, the notation $\mathscr{R}_\varepsilon$ (with some suitable superscript) stands for a generical remaining term, which uniformly converges to 0 as $\varepsilon \to 0$.
Then, one has

Theorem 4.13. *Let Y^n be the bridge of the n-fold conditional process X^n, as defined in (22). If Assumption 4.11 holds, then the family of processes $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed $\varepsilon^{3+\alpha}$ and good rate function*

$$J_Y(h) = \begin{cases} \frac{1}{2}\,\|h-\bar{m}\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h-\bar{m}\in\bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases} \tag{43}$$

where $\bar{m}_t = x_n + \bar{\beta}_t(y-x_n)$ and $\bar{\mathscr{H}}_Y$ is the reproducing kernel Hilbert space associated to the covariance function

$$\bar{k}_Y(t,s) = \begin{cases} \bar{\varphi}_n(t,s) + ts\,\bar{\varphi}_n(1,1) - t\,\bar{\varphi}_n(1,s) - s\,\bar{\varphi}_n(t,1) & \text{if } \alpha<1 \\ b_n^2(ts^2+t^2s-t^2s^2-st)+ & \\ +\bar{\varphi}_n(t,s) + ts\,\bar{\varphi}_n(1,1) - t\,\bar{\varphi}_n(1,s) - s\,\bar{\varphi}_n(t,1) & \text{if } \alpha=1 \end{cases} \tag{44}$$

Proof. The proof is the same as Theorem 4.2. It is enough to observe that

$$\begin{aligned}\mathrm{Cov}(Y^n_{T_n+\varepsilon t}, Y^n_{T_n+\varepsilon s}) = k_n(T_n+\varepsilon t, T_n+\varepsilon s) - \frac{k_n(T_n+\varepsilon t, T_n+\varepsilon)k_n(T_n+\varepsilon, T_n+\varepsilon s)}{k_n(T_n+\varepsilon, T_n+\varepsilon)} = \\ \varepsilon^2\Big[a_n^2\,ts + b_n(t^2s+ts^2)\varepsilon + \bar{\varphi}_n(t,s)\varepsilon^{1+\alpha} \\ -\frac{(a_n^2\,t + b_n(t^2+t)\varepsilon + \bar{\varphi}_n(t,1)\varepsilon^{1+\alpha})(a_n^2\,s + b_n(s+s^2)\varepsilon + \bar{\varphi}_n(1,s)\varepsilon^{1+\alpha})}{a_n^2\, + 2b_n\varepsilon + \bar{\varphi}_n(1,1)\varepsilon^{1+\alpha} + \mathscr{R}^1_\varepsilon(1,1)} + \mathscr{R}^1_\varepsilon(t,s)\Big] = \\ \frac{a_n^2\varepsilon^2}{a_n^2\, + 2b_n\varepsilon + \bar{\varphi}_n(1,1)\varepsilon^{1+\alpha} + \mathscr{R}^1_\varepsilon(1,1)}\Big((\bar{\varphi}_n(t,s) + ts\,\bar{\varphi}_n(1,1) - t\,\bar{\varphi}_n(1,s) \\ -s\,\bar{\varphi}_n(t,1))\varepsilon^{1+\alpha} + b_n^2(ts^2+t^2s-t^2s^2-st)\varepsilon^2 + \mathscr{R}^1_\varepsilon(t,s)\Big),\end{aligned}$$

therefore the thesis holds. □

Let us observe that if the covariance function $k(t,s)$ is more regular, that is $C^{4+\beta}$ for some $\beta\geq 0$, then Theorem 4.13 continues to hold and the associated asymptotic covariance $\bar{k}_Y$, given by (44), is not in general degenerate. In fact, since the fourth derivatives exist, one gets $\bar{\varphi}_n(t,s) = e_n(t^3s+ts^3)+f_nt^2s^2$. Therefore, tedious but straightforward computations will give $\bar{k}_Y(t,s) = const\ ts(1-t)(1-s)$.

Let us now come back to the example suggesting to refine our first result for the bridge of the n-fold conditional process, that is the integrated Gaussian process:

$$X_t = \int_0^t Z_u du,$$

Z being a centered Gaussian process with covariance function $\kappa(t,s)$. We are looking for conditions on κ so that Assumption 4.7 or 4.11 is satisfied and then a large deviation principle as in Theorem 4.9 or 4.13 does hold. One has

Proposition 4.14. *1. Suppose that for some $\alpha \in (0,1]$,*

$$\begin{aligned}\kappa(T_n+\varepsilon u, T_n+\varepsilon v) &= \kappa(T_n,T_n)+\varepsilon^{\alpha}\hat{g}(u,v)+\hat{\mathscr{R}}_\varepsilon(u,v)\\ \int_0^T \kappa(T_n+\varepsilon u, v)dv &= \int_0^T dv\,\kappa(T_n,v)+\varepsilon^{\alpha}\tilde{g}(u;T)+\tilde{\mathscr{R}}_\varepsilon(u;T), \quad T>0,\end{aligned}$$

(the above functions and remaining terms may all depend on T_n) with $\hat{g} \in L^1([0,1]^2)$, $\tilde{g}(\cdot;T) \in L^1([0,1])$ and

$$\lim_{\varepsilon\to 0}\varepsilon^{-\alpha}\,\|\hat{\mathscr{R}}_\varepsilon(\cdot,\cdot)\|_{L^1([0,1]^2)}=0, \quad \lim_{\varepsilon\to 0}\varepsilon^{-\alpha}\,\|\tilde{\mathscr{R}}_\varepsilon(\cdot;T)\|_{L^1([0,1])}=0.$$

Then, Assumption 4.7 holds, with

$$\bar{\varphi}(t,s)=\int_0^t du\int_0^s dv\,\hat{g}(u,v) \quad \text{and} \quad \bar{\psi}(t,T)=\int_0^t du\,\tilde{g}(u;T).$$

2. Suppose that for some $\alpha \in (0,1]$,

$$\begin{aligned}\kappa(T_n+\varepsilon u, T_n+\varepsilon v) &= \kappa(T_n,T_n)+\varepsilon e\cdot(u+v)+\varepsilon^{1+\alpha}\hat{g}(u,v)+\hat{\mathscr{R}}_\varepsilon(u,v)\\ \int_0^T \kappa(T_n+\varepsilon u, v)dv &= \int_0^T dv\,\kappa(T_n,v)+\varepsilon u f(T)+\varepsilon^{1+\alpha}\tilde{g}(u;T)+\tilde{\mathscr{R}}_\varepsilon(u;T), T>0,\end{aligned}$$

(the above functions, remaining terms and the constants e and $f(T)$ may all depend on T_n), with $\hat{g} \in L^1([0,1]^2)$, $\tilde{g}(\cdot;T) \in L^1([0,1])$ and

$$\lim_{\varepsilon\to 0}\varepsilon^{-(1+\alpha)}\,\|\hat{\mathscr{R}}_\varepsilon(\cdot,\cdot)\|_{L^1([0,1]^2)}=0, \quad \lim_{\varepsilon\to 0}\varepsilon^{-(1+\alpha)}\,\|\tilde{\mathscr{R}}_\varepsilon(\cdot;T)\|_{L^1([0,1])}=0.$$

Then, Assumption 4.11 holds, with

$$\bar{\varphi}(t,s)=\int_0^t du\int_0^s dv\,\hat{g}(u,v) \quad \text{and} \quad \bar{\psi}(t,T)=\int_0^t du\,\tilde{g}(u;T).$$

The proof is straightforward and postponed to Appendix B.

Example 4.15. [**m-fold integrated Brownian motion**] Let us come back to Example 4.6, with X as the m-fold integrated Brownian motion:

$$X_t=\int_0^t du\Big(\int_0^u du_{m-1}\cdots\int_0^{u_2}du_1 W_{u_1}\Big),$$

where W denotes a standard Brownian motion. Recall that the covariance function is here

$$k(t,s)=\int_0^t\int_0^s \kappa(u,v)\,du\,dv, \quad \text{with } \kappa(t,s)=\frac{1}{((m-1)!)^2}\int_0^{t\wedge s}(t-\xi)^{m-1}(s-\xi)^{m-1}\,d\xi.$$

Things are slightly different according to $m=1$ or $m\geq 2$. Let us consider only $m\geq 2$, the case $m=1$ being contained in next Example 4.16. Straightforward computations allow to show that

$$\begin{aligned}\kappa(T_n+\varepsilon u, T_n+\varepsilon v) &= \kappa(T_n,T_n)+\varepsilon\tfrac{1}{2}T_n^{2m-2}(u+v)+\\ &+\varepsilon^2\frac{(m-1)}{2m-3}T_n^{2m-3}\Big[\frac{(m-2)}{2}(u+v)^2+uv\Big]+\mathscr{O}(\varepsilon^3)\end{aligned} \tag{45}$$

and

$$
\begin{aligned}
\int_0^T \kappa(T_n+\varepsilon u, v)\,dv = \int_0^T \kappa(T_n, v)\,dv + \varepsilon u \frac{m-1}{m}\int_0^T (T_n-x)^{m-2}(T-x)^m\,dx + \\
+\varepsilon^2 u^2 \frac{(m-2)(m-1)}{m}\int_0^T (T_n-x)^{m-3}(T-x)^m\,dx + \mathscr{O}(\varepsilon^3),
\end{aligned} \tag{46}
$$

in which $\mathscr{O}(\varepsilon^3)$ denotes a function going to 0 as $\varepsilon \to 0$ in the right L^1 space at speed ε^3. Therefore, thanks to (45), (46) and Proposition 4.14, Assumption 4.11 does hold with

$$
\begin{aligned}
\bar\varphi(t,s) &= \frac{(m-1)}{4(2m-3)}T_n^{2m-3}[(m-2)(t^2 s + t s^2) + t^2 s^2] \\
\bar\psi(t,T) &= \Big(\frac{1}{2}\frac{(m-2)(m-1)}{m}\int_0^T (T_n-x)^{m-3}(T-x)^m\,dx\Big)t^2.
\end{aligned}
$$

By using Theorem 4.13, we can assert that the bridge process Y satisfies a (non-degenerate) large deviation principle with inverse speed ε^4 and asymptotic covariance as in (44), with $\alpha = 1$.

Example 4.16. [Integrated fractional Brownian motion] Let $X_t = \int_0^t Z_u du$, with Z a fractional Brownian motion with Hurst index H. This is a quite interesting example because, according to $H \le 1/2$ or $H > 1/2$, one has both the cases studied in Proposition 4.14. In fact, straightforward computations allow to state that

$$
\kappa_H(T_n+\varepsilon u, T_n+\varepsilon v) = \kappa_H(T_n,T_n) + HT_n^{2H-1}(u+v)\varepsilon - \frac{1}{2}|u-v|^{2H}\varepsilon^{2H} + \mathscr{O}(\varepsilon^2) \tag{47}
$$

and

$$
\int_0^T \kappa_H(T_n+\varepsilon u, v)dv =
$$

$$
= \begin{cases}
\int_0^T \kappa_H(T_n,v)dv + \varepsilon\Big[(HT_n^{2H-1}T - \frac{1}{2}T_n^{2H} + \frac{1}{2}(T_n-T)^{2H})u\Big] + \mathscr{O}(\varepsilon), & H < \frac{1}{2} \\
T^2/2, & H = \frac{1}{2} \\
\int_0^T \kappa_H(T_n,v)dv + \varepsilon\Big[(HT_n^{2H-1}T - \frac{1}{2}T_n^{2H} + \frac{1}{2}(T_n-T)^{2H})u\Big] + \mathscr{O}(\varepsilon^2), & H > \frac{1}{2}
\end{cases} \tag{48}
$$

Therefore, the asymptotic behavior can be resumed as follows.

(a) If $H < 1/2$, part 1. in Proposition 4.14 holds, with $\alpha = 2H$ and

$$
\hat g(u,v;T_n) = -\frac{1}{2}|u-v|^{2H} \qquad \tilde g(u;T_n,T) \equiv 0.
$$

(b) If $H = 1/2$, again part 1. in Proposition 4.14 holds, with $\alpha = 1$ and

$$
\hat g(u,v;T_n) = \frac{1}{2}((u+v) - |u-v|) = u \wedge v \qquad \tilde g(u;T_n,T) \equiv 0.
$$

(c) If $H > 1/2$, part 2. in Proposition 4.14 holds, with $\alpha = 2H - 1$ and

$$\hat{g}(u, v; T_n) = -\frac{1}{2}|u - v|^{2H} \qquad \tilde{g}(u; T_n, T) = 0$$

In conclusion, by suitably applying Theorem 4.9 and 4.13, the family of bridges $(Y^n_{T_n+\varepsilon\cdot})_\varepsilon$ satisfies a large deviation principle on $C([0,1])$, with inverse speed ε^{2+2H} and asymptotic covariance function given by

$$\bar{k}_Y(t, s) = \bar{\varphi}_H(t, s) + ts\,\bar{\varphi}_H(1, 1) - t\,\bar{\varphi}_H(1, s) - s\,\bar{\varphi}_H(t, 1),$$

where

$$\bar{\varphi}_H(t, s) \equiv \bar{\varphi}_n(t, s) = \begin{cases} \dfrac{(|t-s|^{2H+2} - t^{2H+2} - s^{2H+2})}{2(2H+1)(2H+2)} & H \neq 1/2 \\ \dfrac{(t \wedge s)^3}{3} + \dfrac{(t \wedge s)^2}{2}|t - s| & H = 1/2. \end{cases}$$

Let us add some further remarks. In the case $H = 1/2$, it is immediate to check that $\bar{\varphi}_H(t, s) = \int_0^t \int_0^s \kappa_H(u, v)\,dudv$. In other words, $\bar{\varphi}_{1/2}$ turns out to be the covariance function of the process X. Then, by taking into account Remark 4.10, the large deviations associated to the bridge of the n-fold integrated Brownian motion behave as "the false bridge", even if with a faster speed (in fact, in this case the inverse speed is ε^3, while the inverse speed of the non-conditioned n-fold process is given by ε^2).

5 The asymptotic behavior of the crossing probability

In this section, the previous results are applied in order to state the large deviation asymptotic behavior of the hitting probability, the underlying process of interest being the bridge of an n-fold conditional Gaussian process. The already collected results can be resumed in the following

Hypothesis 5.1. 1. The family of the n-fold conditional processes $\{(X^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle with inverse speed γ_ε^2 and rate function

$$J_n(h) = \begin{cases} \dfrac{1}{2}\,\|h - x_n\|^2_{\bar{\mathscr{H}}_n} & \text{if } h_0 = x_n \text{ and } h - \bar{m} \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise} \end{cases}$$

$\bar{\mathscr{H}}_n$ being the reproducing kernel Hilbert space associated to a suitable covariance function $\bar{k}_n$.

2. The family of the bridges of the n-fold conditional processes $\{(Y^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle with inverse speed η_ε^2 and rate function

$$J_Y(h) = \begin{cases} \dfrac{1}{2}\,\|h - \bar{m}\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h_0 = x_n, h_1 = y \text{ and } h - \bar{m} \in \bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases}$$

$\bar{\mathscr{H}}_Y$ being the reproducing kernel Hilbert space associated to a suitable covariance function $\bar{k}_Y$ and $\bar{m}_t = x_n + \bar{\beta}_t(y - x_n) \equiv x_n + \bar{k}_n(t, 1)(y - x_n)/\bar{k}_n(1, 1)$.

Throughout this section, we assume that Hypothesis 5.1 always holds.

Now, let us first focus on the upper barrier case, the same arguments will apply for lower barriers.

Let $U : \mathbb{R} \to \mathbb{R}$ be a continuous function standing for an upper barrier, and consider the probability that $Y^n_{T_n+\varepsilon\cdot}$ reaches the barrier U up to the final time 1, that is

$$\mathbb{P}(\tau^U_\varepsilon \leq 1), \quad \text{with } \tau^U_\varepsilon = \inf\{t > 0 \,:\, Y^n_{T_n+\varepsilon t} \geq U_{T_n+\varepsilon t}\}$$

The above probability is negligible if $Y^n_{T_n} = x_n < U_{T_n}$ and $Y^n_{T_n+\varepsilon t} = y < U_{T_n+\varepsilon t}$ for any ε close to 0, that is $y \leq U_{T_n}$. As we will see, the case $y = U_{T_n}$ will give a non-relevant estimate, so that we can assume both x_n and y are less than U_{T_n}. So, if $x_n, y < U_{T_n}$, one has

$$\eta^2_\varepsilon \log \mathbb{P}(\tau^U_\varepsilon \leq 1) \cong -I^U_Y,$$

as $\varepsilon \cong 0$, with $I^U_Y > 0$. Let us now see what I^U_Y is. Set $Z^n_{T_n+\varepsilon t} = Y^n_{T_n+\varepsilon t} - U_{T_n+\varepsilon t}$. Since $\lim_{\varepsilon\to 0} U_{T_n+\varepsilon t} = U_{T_n}$ uniformly for $t \in [0,1]$, by contraction it immediately follows that $\{(Z^n_{T_n+\varepsilon t})_{t\in[0,1]}\}_\varepsilon$ satisfies a large deviation principle as well, with the same inverse speed and rate function

$$J_Z(h) = J_Y(h + U_{T_n}).$$

Then, one has

$$\lim_{\varepsilon\to 0} \eta^2_\varepsilon \log \mathbb{P}(\tau^U_\varepsilon \leq 1) = -\inf_{\gamma\in\Gamma_U} J_Y(\gamma + U_{T_n}) = -I^U_Y,$$

being $\Gamma_U = \{\gamma \,:\, \sup_{t\in[0,1]} \gamma_t \geq 0\}$.

If a (continuous) lower barrier L_t were considered, then the same arguments would apply, giving

$$\lim_{\varepsilon\to 0} \eta^2_\varepsilon \log \mathbb{P}(\tau^L_\varepsilon \leq 1) = -\inf_{\gamma\in\Gamma_L} J_Y(\gamma + L_{T_n}) = -I^L_Y,$$

where $\tau^L_\varepsilon = \inf\{t > 0 \,:\, Y^n_{T_n+\varepsilon t} \leq L_{T_n+\varepsilon t}\}$ and $\Gamma_L = \{\gamma \,:\, \inf_{t\in[0,1]} \gamma_t \leq 0\}$, and this is interesting when $x_n, y > L_{T_n}$. Finally, in the double barrier case, with $L_t \leq U_t$ for any t, then the hitting probability behaves as follows:

$$\lim_{\varepsilon\to 0} \eta^2_\varepsilon \log \mathbb{P}(\tau^{L,U}_\varepsilon \leq 1) = -I^{L,U}_Y,$$

where $\tau^{L,U}_\varepsilon = \tau^L_\varepsilon \wedge \tau^U_\varepsilon$ is the first time at which $Y^n_{T_n+\varepsilon\cdot}$ reaches at least one barrier and $I^{L,U}_Y$ is a suitable quantity, which is strictly positive if $x_n, y \in (L_{T_n}, U_{T_n})$.

The quantities I^U_Y, I^L_Y and $I^{L,U}_Y$ are computed in next

Proposition 5.2. *Suppose that L and U are continuous functions, with $L_t \leq U_t$ for any $t \in [0,1]$. Then,*

$$\begin{aligned}
I^U_Y &= \inf_{t\in[0,1]} \frac{\Big((U_{T_n} - x_n)(1-\bar\beta_t) + \bar\beta_t(U_{T_n} - y)\Big)^2}{2\,\bar k_Y(t,t)} && \text{if } x_n, y < U_{T_n} \\
I^L_Y &= \inf_{t\in[0,1]} \frac{\Big((x_n - L_{T_n})(1-\bar\beta_t) + \bar\beta_t(y - L_{T_n})\Big)^2}{2\,\bar k_Y(t,t)} && \text{if } x_n, y > L_{T_n} \\
I^{L,U}_Y &= \min\Big(I^L_Y, I^U_Y\Big) && \text{if } x_n, y \in (L_{T_n}, U_{T_n})
\end{aligned}$$

Proof. Consider the first equality (single upper barrier case). We have to show that

$$\inf_{\gamma\in\hat\Gamma_U}\frac{1}{2}\,\|\gamma+U_{T_n}-\bar m\|^2_{\bar{\mathscr H}_Y}=\inf_{t\in[0,1]}\frac{\Big((U_{T_n}-x_n)(1-\bar\beta_t)+\bar\beta_t(U_{T_n}-y)\Big)^2}{2\,\bar k_Y(t,t)},$$

being $\hat\Gamma_U=\{\gamma:\gamma+U_{T_n}-\bar m\in\bar{\mathscr H}_Y,\ \sup_{t\in[0,1]}\gamma_t\geq 0\}$. Setting $\hat\Gamma_{t,U}=\{\gamma:\gamma+U_{T_n}-\bar m\in\bar{\mathscr H}_Y,\ \gamma_t=0\}$, one has that $\hat\Gamma=\cup_{0<t<1}\hat\Gamma_t$, so that we simply need that

$$\inf_{\gamma\in\hat\Gamma_{t,U}}\frac{1}{2}\,\|\gamma+U_{T_n}-\bar m\|^2_{\bar{\mathscr H}_Y}=\frac{\Big((U_{T_n}-x_n)(1-\bar\beta_t)+\bar\beta_t(U_{T_n}-y)\Big)^2}{2\,\bar k_Y(t,t)},$$

As already seen (see Section 2), a set of paths which is dense in $\bar{\mathscr H}_Y$ is the one formed by those which are the barycenters of the random variable belonging to the dual space of $C([0,1])$, that is,

$$\gamma_u+U_{T_n}-\bar m_u=\int_0^1\bar k_Y(u,v)\lambda(dv),$$

as λ varies in $\mathscr M[0,1]$. Since for such kind of paths

$$\|\gamma+U_{T_n}-\bar m\|^2_{\bar{\mathscr H}_Y}=\int_0^1\int_0^1\bar k_Y(u,v)\lambda(du)\lambda(dv),$$

it is enough to minimize the r.h.s. above with respect to λ, with the additional constraint $\gamma_t=0$, which becomes

$$\bar m_t-U_{T_n}+\int_0^1\bar k_Y(t,v)\lambda(dv)=0.$$

This is a constrained extremum problem: using Lagrange multipliers, λ must satisfy

$$\int_0^1\bar k_Y(u,v)\lambda(dv)-\alpha\bar k_Y(t,u)=0,\quad\text{for any } u\in[0,1],$$

for some $\alpha\in\mathbb R$. Taking care of the constraint one finds

$$\alpha=\frac{U_{T_n}-\bar m_t}{\bar k_Y(t,t)},\qquad \lambda(dv)=\frac{U_{T_n}-\bar m_t}{\bar k_Y(t,t)}\,\delta_{\{t\}}(dv).$$

$\delta_{\{t\}}$ standing for the Dirac mass in t. Therefore,

$$\inf_{w\in\hat\Gamma_{t,U}}\frac{1}{2}\int_0^1\int_0^1\bar k_Y(u,v)\lambda(u)\lambda(dv)=\frac{(U_{T_n}-\bar m_t)^2}{2\bar k_Y(t,t)}$$

and the statement immediately follows by recalling that $\bar m_t=x_n+\bar\beta_t(y-x_n)$.
Concerning the second equality, it follows by developing analogous arguments. As for the final one, it is standard in large deviation theory (see e.g. the discussion in the proof of Theorem 2.2 in Baldi and Caramellino [4]). □

Let us stress that the barriers U and/or L can be also piecewise continuous, in which case the previous machinery runs again if the jump times coincide with some of the conditional times $T_1, \ldots, T_n$.

Before to develop some examples, let us recall that $\bar\beta_t = \bar k_n(t,1)/\bar k_n(1,1)$, where $\bar k_n$ is defined in (16) and represents the asymptotic covariance function associated to the n-fold conditional process. When our first set of large deviation results for the bridge holds (as in Section 4.1), one has

$$\bar k_Y(t,s) = \bar k_n(t,s) - \frac{\bar k_n(t,1)\bar k_n(s,1)}{\bar k_n(1,1)},$$

$\bar k_Y$ being more complicated if it turns out following Section 4.2. Therefore, the minimization problem as required to compute I_Y^U and I_Y^L has not a closed form in general, so that for practical purposes one might be forced to use some numerical method (e.g. the Newton method).

Example 5.3. [Integrated Brownian motion] Following Example 4.16 (with $H = 1/2$), let us consider $X_t = \int_0^t B_s\,ds$, with B as a standard Brownian motion. Such a process has interesting applications in metrology, where it is used as a model for the atomic clock error and the exit from a fixed boundary means that the clock error exceeds an allowed limit so that it has to be re-synchronized. Here, we are in the second set of our large deviation estimates: the bridge of the n-fold conditional process satisfies a large deviation principle at inverse speed ε^3 and with rate function associated to the asymptotic covariance function

$$\begin{aligned}&\bar k_Y(t,s) = \bar\varphi(t,s) + ts\,\bar\varphi(1,1) - t\,\bar\varphi(1,s) - s\,\bar\varphi(t,1),\\ &\text{with}\quad \bar\varphi(t,s) = \frac{(t\wedge s)^3}{3} + \frac{(t\wedge s)^2}{2}\,|t-s|.\end{aligned}$$

Since $\bar k_n(t,s) = a_n^2\,ts$, one has $\bar\beta_t = t$, so that $I_Y^U = g(U_{T_n})$ and $I_Y^L = g(L_{T_n})$, with

$$g(a) = \inf_{t\in[0,1]} \frac{\Big(|a-x_n|(1-t) + t|a-y|\Big)^2}{2\,t^2(1-t)^2/3}.$$

The solution is simple to find:

$$g(a) = \frac{3}{2}\left(|a-x_n|^{1/2} + |a-y|^{1/2}\right)^4.$$

Example 5.4. [Fractional Brownian motion] Following Section 3.1 and Example 4.4, let us consider a fractional Brownian motion X with Hurst index H, in which one has

$$\bar k_n(t,s) = k_H(t,s) = \frac{t^{2H} + s^{2H} - |t-s|^{2H}}{2}.$$

So, in order to compute I_Y^U and I_Y^L, giving the asymptotic behavior of the hitting probability of the bridge Y^n, by Proposition 5.2 one should be able to compute

$$g_H(a) = \inf_{t\in[0,1]} \frac{\Big((a-x_n)(1-k_H(t,1)) + k_H(t,1)(a-y)\Big)^2}{2\left(k_H(t,t) - k_H^2(t,1)\right)},$$

either with $a > x_n, y$ or $a < x_n, y$. In fact, one has $I_Y^U = g_H(U_{T_n})$ and $I_Y^L = g_H(L_{T_n})$. As far as we know, the exact solution can be computed only when $H = 1/2$, that is when a standard Brownian motion is taken into account, in which case one has

$$g_{1/2}(a) = 2(a - x_n)(a - y),$$

which agrees with well known formulas (see e.g. Baldi and Caramellino [3]).

In relation to the above Example 5.4, we have performed some numerical experiments concerning the fractional Brownian motion. In particular, we have estimated via Monte Carlo the probability of crossing the upper barrier $U = 1$ up to time 1 in two different ways: by crude simulations, in which the exit is reached if a simulated position is greater than $U = 1$, and by means of the corrected procedure as recalled in the Introduction, for which at each step the crossing is decided by using the large deviation approximation for the exit probability of the pinned process. In all the experiments, the exit probability is numerically computed through 10^5 simulations. The results are given in terms of the method (corrected/crude) and of the step size ($\varepsilon = 0.01, 0.002, 0.001$), for varying values of the Hurst index H, which is set equal to 0.3, 0.5 and 0.7. Whenever $H = 0.5$, everything is known (exit probability= 0.31732), including the fact that the crude approach works very poorly, so it has been considered to asses the procedure and for comparison purposes. The choices $H = 0.3$ and $H = 0.7$ have been taken to compare the results when $H < 1/2$ (short memory, more irregular paths) and $H > 1/2$ (long memory, less irregular paths). The results, given in Table 1, show how much is the sensitivity w.r.t. the method (corrected/crude) when H decreases, that is when the irregularity of the path tends to be higher. This is not surprising because the inverse speed of the large deviations for the bridge is in fact ε^{2H}, so that the correction works more when H decreases.

Method	Step	$H = 0.3$	$H = 0.5$	$H = 0.7$
corrected	0.01	0.60876 $(0.60573, 0.61178)$	0.31820 $(0.31531, 0.32109)$	0.20564 $(0.20313, 0.20814)$
corrected	0.002	0.61841 $(0.61540, 0.62142)$	0.31980 $(0.31691, 0.32269)$	0.20274 $(0.20025, 0.20523)$
crude	0.01	0.47909 $(0.47599, 0.48219)$	0.28918 $(0.28637, 0.29199)$	0.19884 $(0.19637, 0.20131)$
crude	0.002	0.54114 $(0.53805, 0.544230)$	0.30496 $(0.30211, 0.30781)$	0.20222 $(0.19973, 0.20471)$
crude	0.001	0.56082 $(0.55774, 0.56390)$	0.30878 $(0.30592, 0.31164)$	0.20251 $(0.20002, 0.20500)$

Table 1: Fractional Brownian motion: Monte Carlo estimated probability of crossing the upper barrier $U = 1$ up to time 1, for varying values of the Hurst index H. In brackets, the associated 95% confidence interval.

Let us give a final remark. Our simulation scheme relies on the assumption that the past pinned instants are fixed, although they are all of order ε, and only the step size of the current time interval is considered negligible. In fact, the past observations have been considered as a datum, that is fixed, and only the bridge's length is supposed to be small enough to approximate the exit probability with its large deviation estimate. As remarked by an unknown referee, it would be interesting to see what happens when the size of all the past time intervals goes to zero, a case in which the "local independence" fails. We are now working to this case and it will be nice to measure the sensitivity of the method w.r.t. the independence of the past, by comparing the new results with the ones reported in Table 1.

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Appendix A: proof of (26)

This Appendix is devoted to the proof of representation (26). Let J_Y be the rate function given by Theorem 4.2, i.e.

$$J_Y(h) = \begin{cases} \frac{1}{2}\,\|h-\bar m\|^2_{\bar{\mathscr{H}}_Y} & \text{if } h_0 = x_n, h_1 = y, h-\bar m \in \bar{\mathscr{H}}_Y \\ +\infty & \text{otherwise} \end{cases}$$

where $\bar m_t = x_n + \bar\beta_t(y-x_n)$ and $\bar{\mathscr{H}}_Y$ is the reproducing kernel Hilbert space associated to the covariance function

$$\bar k_Y(t,s) = \bar k_n(t,s) - \bar\beta_s \bar k_n(t,1) = \bar k_n(t,s) - \frac{\bar k_n(t,1)\bar k_n(s,1)}{\bar k_n(1,1)},$$

$\bar k_n$ being defined in (16). We claim that J_Y can be written as

$$J_Y(h) = \begin{cases} \frac{1}{2}\Big(\|h-x_n\|^2_{\bar{\mathscr{H}}_n} - \dfrac{(y-x_n)^2}{\bar k_n(1,1)}\Big) & \text{if } h_0 = x_n, h_1 = y, h-x_n \in \bar{\mathscr{H}}_n \\ +\infty & \text{otherwise} \end{cases}$$

$\bar{\mathscr{H}}_n$ being the reproducing kernel Hilbert space associated to the covariance function $\bar k_n$. Let us observe that this can be done in two ways: by large deviation arguments (in particular, by using contraction type properties allowing to transfer large deviation principles) or by handling reproducing kernel Hilbert spaces. Here, we follow the second way.

First, let us prove that the sets where the two functionals are finite are the same, that is $\mathscr{K}_1 = \mathscr{K}_2$, being

$$\mathscr{K}_1 = \{h : h_0 = x_n, h_1 = y, h-\bar m \in \bar{\mathscr{H}}_Y\}$$
$$\mathscr{K}_2 = \{h : h_0 = x_n, h_1 = y, h-\bar m \in \bar{\mathscr{H}}_n\}.$$

If we set

$$\mathscr{D}_1 := \{h \in \mathscr{K}_1 : h_t - \bar m_t = \int_0^1 \bar k_Y(t,s)\alpha(ds), \text{ for some } \alpha \in \mathscr{M}[0,1]\}$$
$$\mathscr{D}_2 := \{h \in \mathscr{K}_2 : h_t - \bar m_t = \int_0^1 \bar k_n(t,s)\gamma(ds), \text{ for some } \gamma \in \mathscr{M}[0,1]\},$$

then the statement would follow from

$$\mathscr{D}_1 = \mathscr{D}_2 \quad \text{and} \quad \|h-\bar m\|_{\bar{\mathscr{H}}_Y} = \|h-\bar m\|_{\bar{\mathscr{H}}_n}, \text{ for any } h \in \mathscr{D}_1 = \mathscr{D}_2. \tag{49}$$

Indeed, since $\overline{\mathscr{D}_1}^{\|\cdot\|_{\mathscr{H}_Y}} = \mathscr{K}_1$ and $\overline{\mathscr{D}_2}^{\|\cdot\|_{\mathscr{H}_n}} = \mathscr{K}_2$, it immediately will follow that $\mathscr{K}_1 = \mathscr{K}_2$. So, let us show that (49) does hold.
If one takes $h \in \mathscr{D}_1$, then

$$\begin{aligned} h_t - \bar{m}_t = & \int_0^1 \bar{k}_Y(t,s)\alpha(ds) = \int_0^1 \Big(\bar{k}_n(t,s) - \frac{\bar{k}_n(1,t)\bar{k}_n(1,s)}{\bar{k}_n(1,1)}\Big)\alpha(ds) \\ = & \int_0^1 \bar{k}_n(t,s)\Big(\alpha(ds) - \frac{\int_0^1 \bar{k}_n(1,u)\alpha(du)}{\bar{k}_n(1,1)}\delta_{\{1\}}(ds)\Big) \end{aligned}$$

where $\delta_{\{1\}}$ denotes the Dirac mass, and then $h \in \mathscr{D}_2$. Conversely, if $h \in \mathscr{D}_2$, then $h_t - \bar{m}_t = \int_0^1 \bar{k}_n(t,s)\gamma(ds)$, and in particular it must be

$$0 = h_1 - \bar{m}_1 = \int_0^1 \bar{k}_n(1,s)\gamma(ds).$$

Therefore,

$$h_t - \bar{m}_t = \int_0^1 \bar{k}_n(t,s)\gamma(ds) = \int_0^1 \Big(\bar{k}_n(t,s) - \frac{\bar{k}_n(1,t)\bar{k}_n(1,s)}{\bar{k}_n(1,1)}\Big)\gamma(ds) = \int_0^1 \bar{k}_Y(t,s)\gamma(ds)$$

and $h \in \mathscr{D}_1$. Finally,

$$\|h - \bar{m}\|^2_{\mathscr{H}_Y} = \int_0^1\int_0^1 \bar{k}_Y(t,s)\alpha(ds)\alpha(dt) = \int_0^1\int_0^1 \bar{k}_n(t,s)\gamma(ds)\gamma(dt) = \|h - \bar{m}\|^2_{\bar{\mathscr{H}}_n}$$

where α and γ denote the measures representing $h - \bar{m}$ in $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively, so that (49) is completely proved.
Now, we need to prove that for any $h \in \mathscr{D}_2$ one has $\|h - \bar{m}\|^2_{\bar{\mathscr{H}}_n} = \|h - x_n\|^2_{\bar{\mathscr{H}}_n} - (y - x_n)^2/\bar{k}_n(1,1)$. This follows from the fact that $\bar{m} - x_n$ belongs to the reproducing kernel Hilbert space $\bar{\mathscr{H}}_n$, because

$$\bar{m}_t - x_n = \frac{\bar{k}_n(t,1)}{\bar{k}_n(1,1)}(y - x_n) = \int_0^1 \bar{k}_n(t,s)\,\frac{y - x_n}{\bar{k}_n(1,1)}\,\delta_{\{1\}}(ds).$$

Moreover, it holds

$$\|\bar{m} - x_n\|^2_{\bar{\mathscr{H}}_n} = \frac{(y - x_n)^2}{\bar{k}_n(1,1)}.$$

Take now $h \in \mathscr{D}_2$. In particular, for some measure γ one has $h_t - \bar{m}_t = \int_0^1 \bar{k}_n(t,s)\gamma(ds)$. Then, the measure $\hat{\gamma}(ds) = \gamma(ds) + (y - x_n)\delta_{\{1\}}(ds)/\bar{k}_n(1,1)$ is such that $h_t - x_n = \int_0^1 \bar{k}_n(t,s)\,\hat{\gamma}(ds)$ and

$$\begin{aligned} \langle h - x_n, \bar{m} - x_n\rangle_{\bar{\mathscr{H}}_n} &= \int_0^1\int_0^1 \bar{k}_n(t,s)\Big(\gamma(ds) + \frac{y - x_n}{\bar{k}_n(1,1)}\,\delta_{\{1\}}(ds)\Big)\,\frac{y - x_n}{\bar{k}_n(1,1)}\,\delta_{\{1\}}(dt) \\ &= \frac{y - x_n}{\bar{k}_n(1,1)}\int_0^1 \bar{k}_n(1,s)\gamma(ds) = \frac{y - x_n}{\bar{k}_n(1,1)}(h - x_n)_1 = \frac{(y - x_n)^2}{\bar{k}_n(1,1)} \end{aligned}$$

Therefore,

$$\|h-\bar{m}\|^2_{\bar{\mathscr{H}}_n} = \|h-x_n\|^2_{\bar{\mathscr{H}}_n} + \|\bar{m}-x_n\|^2_{\bar{\mathscr{H}}_n} - 2\langle h-x_n, \bar{m}-x_n\rangle_{\bar{\mathscr{H}}_n} = \|h-x_n\|^2_{\bar{\mathscr{H}}_n} - \frac{(y-x_n)^2}{\bar{k}_n(1,1)},$$

and the statement finally holds. □

Appendix B: proof of Proposition 4.14

Proof. 1. Since X is an integrated Gaussian process, one has

$$\mathrm{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_{T_n+\varepsilon s} - X_{T_n}) = \int_{T_n}^{T_n+\varepsilon t} du \int_{T_n}^{T_n+\varepsilon t} dv \kappa(u,v) =$$
$$\varepsilon^2 \int_0^t du \int_0^s dv\, \kappa(T_n+\varepsilon u, T_n+\varepsilon v).$$

Therefore, one has

$$\mathrm{Cov}(X_{T_n+\varepsilon t} - X_{T_n}, X_{T_n+\varepsilon s} - X_{T_n}) = \varepsilon^2 \int_0^t du \int_0^s dv \Big[\kappa(T_n,T_n) + \varepsilon^\alpha \hat{g}(u,v) + \hat{\mathscr{R}}_\varepsilon(u,v)\Big] =$$
$$= \varepsilon^2 \Big(\kappa(T_n,T_n)\, ts + \varepsilon^\alpha \int_0^t du \int_0^s dv\, \hat{g}(u,v) + \int_0^t du \int_0^s dv\, \hat{\mathscr{R}}_\varepsilon(u,v)\Big),$$

so that (i) of Assumption 4.7 is satisfied with $\bar{\varphi}(t,s) = \int_0^t du \int_0^s dv\, \hat{g}(u,v)$.
Moreover, since

$$k(T_n+\varepsilon t, T) - k(T_n,T) = \int_{T_n}^{T_n+\varepsilon t} du \int_0^T dv\, \kappa(u,v) = \varepsilon \int_0^t du \int_0^T dv\, \kappa(T_n+\varepsilon u, v),$$

one obtains

$$k(T_n+\varepsilon t, T) - k(T_n,T) = \varepsilon\Big(\int_0^t du\Big[\int_0^T dv\, \kappa(T_n,v) + \varepsilon^\alpha \tilde{g}(u;T) + \tilde{\mathscr{R}}_\varepsilon(u;T)\Big]\Big) =$$
$$= \varepsilon\Big(t\int_0^T dv\, \kappa(T_n,v) + \varepsilon^\alpha \int_0^t du\, \tilde{g}(u;T) + \int_0^t du\, \tilde{\mathscr{R}}_\varepsilon(u;T)\Big).$$

Then, also (ii) of Assumption 4.7 is satisfied with $\bar{\psi}(t,T) = \int_0^t du\, \tilde{g}(u;T)$.
The proof of part 2. follows exactly the same lines as for part 1. □