# Sthenic incompatibilities in rigid bodies motion

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**Abstract.** When a rigid body slides with friction on a surface, hopping motion is observed: this is an everyday phenomenon. In rigid bodies mechanics, this phenomenon appears when it is no longer possible to compute the reaction contact forces. The difficulty is overcome by a motion theory involving velocity discontinuities. Velocity discontinuities may result either from an obstacle which makes impossible to compute the acceleration: this is *a cinematic incompatibility* or from the impossibility to compute the reaction forces: this is a *sthenic incompatibility.* We describe two examples: the Klein and Painlevé sthenic incompatibilities.

## 1 Introduction

When a rigid body collides with a rigid plane, it is no longer possible to solve the smooth equations of motion because it is impossible to compute the acceleration. This is a *cinematic incompatibility.* Collision theory, assumes a time discontinuity of the velocity. This assumption associated with the basic laws of mechanics, i.e., equations of motions and constitutive laws satisfying the laws of thermodynamics, gives a solution to this problem by producing a predictive theory which takes into account the cinematic incompatibilities [2], [1].

The motion of rigid bodies may involve friction which introduces reaction forces. These forces depend on the velocities through algebraic or differential equations. It may happen that these equations have no solution whereas there is no cinematic incompatibility. Again it is no longer possible to solve the equations of motion. What occurs? May the predictive theory cope with this unexpected situation? Is it too schematic and has more sophistication to be added? In this case, one may think that the rigidity assumption has to be removed. We show that the above mentioned collision theory is rich enough to provide a solution and that there is no necessity to get rid of the rigidity assumption. We call this kind of incompatibility, a *sthenic incompatibility.* We describe two examples: the Klein and Painlevé sthenic incompatibilities, [4], [6], [2].

## 2 The Klein sthenic incompatibility

Let us consider a bar with length $2l$, mass $m$, the ends of which are moving in two slides which are fixed to a massive rigid support, (Fig. 1). One slide has a Coulomb friction, the other one is without friction. The state of the *system bar-slides* depends on a unique parameter: the abscissa $x(t)$ of the bar center of mass $G$. An horizontal force $F$ is applied at distance $b = GA$ of the center of mass ($b$ is positive downward, the $b$ of Fig. 1 is positive). A torque $\hat{C}$ is also applied. The angle of the bar with the inferior slide is $\theta \in ]0, \pi[$. Let the horizontal initial velocity $U = dx/dt$ be given and let us find the bar motion. For some values of the data, $U$, $F$ and $\hat{C}$ it is not possible to solve the equations of motions, more precisely it is not possible to find the reaction forces of the slides, [4], [2]. What occurs? We show that it is possible to overcome the difficulty within the rigid body theory by describing carefully what occurs when velocities are discontinuous, [2], [1].

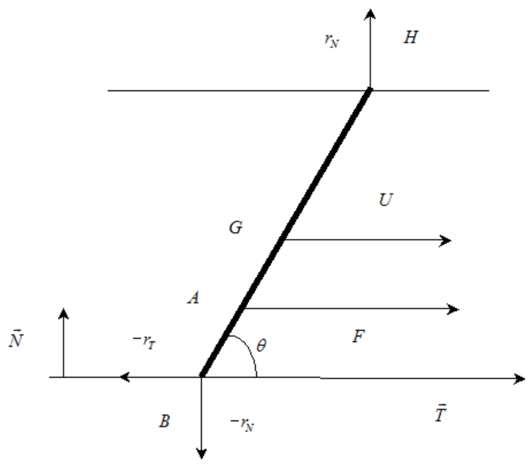


**Fig. 1.** Ends $B$ and $H$ of a bar are guided by two slides with normal vector $\mathbf{N}$ (direction $N$ is normal to the slides, direction $T$ is tangential). The superior slide is without friction. The inferior slide has a Coulomb friction. Force $F$ is applied at distance $b = GA$ from center of mass $G$, $\theta$ is the angle between the bar and the horizontal axis. Inferior slide applies reaction force $(-r_T, -r_N)$ to the bar and superior slide force $(0, r_N)$.

### 2.1 The constitutive laws

The 2-D system is made of the rigid bar and of the two horizontal rigid slides which are fixed to a massive rigid support in such a way that we may assume their velocities are 0. Contact is bilateral in each slide. The superior slide is without friction. The inferior slide has a Coulomb friction. The interior forces of the system are the reaction force $(r_T, r_N)$ of the slide at point $B$ together with the reaction percussions $(P_T, P_N)$ in case the velocities are discontinuous, i.e., there is a collision or a velocity jump. We assume Coulomb friction at point $B$. The Coulomb friction in case of discontinuity of velocity, involves the quantity $(U^+ + U^-)$ as shown in [2] and [1], ($U^+$ and $U^-$ are the velocities after and before the velocity discontinuity). We choose the same friction coefficient $\mu$ for the smooth and non-smooth evolutions due to experimental evidence and theoretical results, [1].

### 2.2 The equations

The equations of motion and constitutive laws give the following equations [2]:

**Smooth evolution**

*Almost everywhere in time*

$$m\frac{dU}{dt} = -r_T + F, \tag{1}$$

$$0 = -r_T l \sin\theta + 2r_N l \cos\theta + Fb\sin\theta + C, \tag{2}$$

by denoting $C = \hat{C} - bF_N \cos\theta$. Second equation is due to the zero angular velocity. Constitutive law for the normal reaction force is

$$r_N \in \partial I_0(0) = \mathbf{R},$$

where $I_0$ is the indicator function of the origin of $\mathbf{R}$, [5]. Normal reaction force can be positive or negative because the contact is bilateral. The Coulomb constitutive law is

$$r_T \in \partial I_{r_N}(U), \tag{3}$$

with

$$I_{r_N}(x) = \mu\,|r_N|\,|x|\,.$$

**Non smooth evolution**

*At any time*

$$m\,[U] = -P_T, \tag{4}$$

$$0 = -P_T l \sin\theta + 2P_N l \cos\theta, \tag{5}$$

where $[U] = U^+ - U^-$. Second equation is due to the zero angular velocity. Constitutive law for the normal reaction percussion is

$$P_N \in \partial I_0(0) = \mathbf{R}.$$

Normal reaction percussion is positive or negative because the contact is bilateral. The dissipation function for the tangential velocity is

$$I_{P_N}(x) = \mu \left| P_N \right| \left| x \right|,$$

which gives the constitutive law, [2], [1]

$$P_T \in \partial I_{P_N}(U^+ + U^-). \tag{6}$$

### 2.3 An example of sthenic incompatibility

In a smooth evolution, the reaction forces or the interior forces satisfy two algebraic equations (2) and (3), when velocity $U$ is known. If these equations have no solution, it is impossible to solve the differential equation (1), thus to solve the smooth equations of motion: for instance, in case

$$U > 0,\ bF + \frac{C}{\sin\theta} < 0,\ \ |\mathrm{tg}\theta| > \frac{2}{\mu},$$

as shown in Fig. 2. Impossibility does not result from the impossibility to compute the acceleration as when a solid collides a rigid plane. It results from the impossibility to compute the interior forces. We have a *sthenic incompatibility* whereas we have a cinematic incompatibility when a solid collides a rigid plane.

Let us note that with such initial velocity $U$, force $F$ and torque $C$, the smooth evolution we are expecting because there is no obstacle, cannot exist. A difficulty seems to prevent to solve the equations of motion. It may be shown that this is not the case and that this situation is completely normal. The difficulty is overcome by the system by having a velocity discontinuity because in case

$$U^+ + U^- = 0 \text{ and } \ |\mathrm{tg}\theta| > \frac{2}{\mu},$$

there is an unique solution $U^+ < 0$ of the three algebraic equations (4),(5) and (6). Indeed, motion may go on with this new initial condition because in case

$$U < 0,\ bF + \frac{C}{\sin\theta} < 0,\ \ |\mathrm{tg}\theta| > \frac{2}{\mu},$$

there are two possible reaction forces $r_T$ as it is shown in Fig.3.

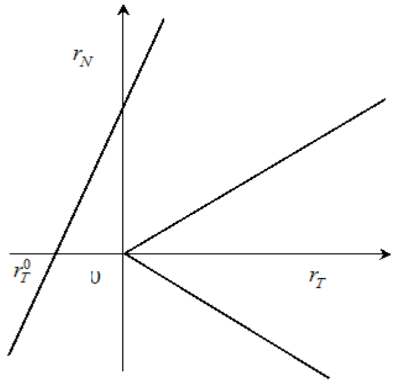


**Fig. 2.** For $U > 0$, $bF + \frac{C}{\sin\theta} < 0$, and $|\mathrm{tg}\theta| > \frac{2}{\mu}$, there is no possible reaction force because half-lines $r_T = \mu |r_N|$ and line $0 = -r_T l \sin\theta + 2lr_N \cos\theta + bF \sin\theta + C$ do not intersect. This is a sthenic incompatibility: it is impossible to find reaction forces which satisfy the equations of motion.

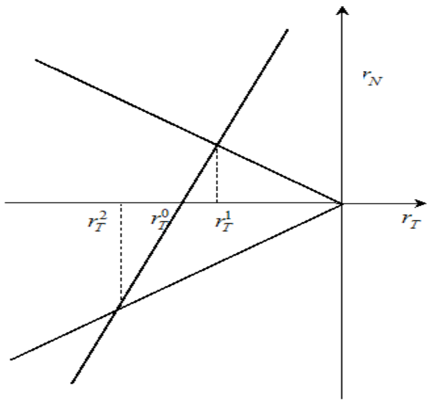


**Fig. 3.** For $U < 0$, $bF + \frac{C}{\sin\theta} < 0$, $|\mathrm{tg}\theta| > \frac{2}{\mu}$, there are two reaction forces at intersection of half-lines $r_T = -\mu |r_N|$ and line $0 = -r_T l \sin\theta + 2lr_N \cos\theta + bF \sin\theta + C$.

## 3 The Painlevé sthenic incompatibility

Let us consider a rigid slender bar which is sliding with friction on an horizontal plane, as illustrated in Fig. 4. The mass of the bar is m, its length is $2l$, its mass moment of inertia is $I$. The coefficient of friction is $\mu$. The bar, being pointed in the direction of motion, is sliding towards the left. The velocity of the center of mass is $(U_T, U_N)$. Velocity $U_T$ is the horizontal or tangential velocity and velocity $U_N$ is the vertical or normal velocity. The velocity of the contact point $A$ is $(V_T, V_N) = (U_T + \omega l sin\theta,\ U_N - \omega l cos\theta)$, where $\theta$ is the angle of the bar with respect to the horizontal and $\omega = d\theta/dt$ is the angular velocity. In some configurations, the sliding motion becomes impossible when the contact forces diverge to infinity.

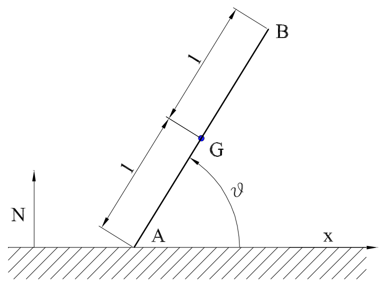


**Fig. 4.** Painlev's example

### 3.1 The equations of motion

It is easy to get, *almost everywhere in time*:

$$m\frac{dU_N}{dt} = -r_N - g,\ m\frac{dU_T}{dt} = -r_T,$$
$$I\frac{d\omega}{dt} = l(-r_T sin\theta + r_N cos\theta), \quad (7)$$

where $-(r_T, r_N)$ is the reaction of the plane and $(0, -g)$ is the gravity force, and *at any time*:

$$m[U_N] = -P_N,\ m[U_T] = -P_T,$$
$$I[\omega] = l(-P_T sin\theta + P_N cos\theta), \quad (8)$$

where $-(P_T, P_N)$ is the percussion reaction of the plane on the bar. The constitutive laws for the reaction forces and percussions are still the Coulomb friction law.

We suppose that the bar is sliding towards the left: $y_a = 0$, $V_N = 0$, $V_T < 0$. We suppose that at the beginning of the motion $\omega^2 ml sin\theta - g < 0$ and $I - ml^2 cos\theta(\mu sin\theta - cos\theta) > 0$. The normal reaction is

$$r_N = \frac{I(\omega^2 ml sin\theta - g)}{I - ml^2 cos\theta(\mu sin\theta - cos\theta)} \geq 0. \quad (9)$$

We suppose that the evolution is such that either the denominator or both the numerator and the denominator of $r_N$ go to zero with $r_N$ going to infinity. In this situation the smooth evolution is no longer possible. What happens? It may be shown that, depending on the cinematic and geometric conditions, the bar may leave the plane either smoothly (i.e., without velocity discontinuity) or non smoothly (i.e. with a velocity discontinuity). This property is given by the algebraic and differential equations resulting from the non smooth and smooth Coulomb constitutive laws and equations of motion (8) and (7).

Consider now the data $m = l = 1, I = 1/12,\ \mu = 0.9,\ g = 1$, with the initial conditions $y_A(0) = 0, \theta(0) = 0.85, V_T(0) = -7, V_N(0) = 0, \omega(0) = 0$. The reaction $r_N$ diverges to infinity. It can be proved that a discontinuity of velocity occurs, [2]. The future velocities $(U_T^+, U_N^+, \omega^+)$ depending on $(U_T^-, U_N^-, \omega^-)$ are given by the algebraic equations (8) and the Coulomb constitutive law. In this configuration there is *not uniqueness* of the solution. The angular velocity $\omega^+$ is indeterminate. It depends on the parameter $[\omega]$ which verifies, [2]:

$$0 \leq [\omega] \leq \frac{-2V_T^-}{l(\mu cos\theta + sin\theta)} = [\omega]_{max}. \quad (10)$$

Figs. 5 and 6 show the effect of the sthenic incompatibility for $[\omega] = 2$.

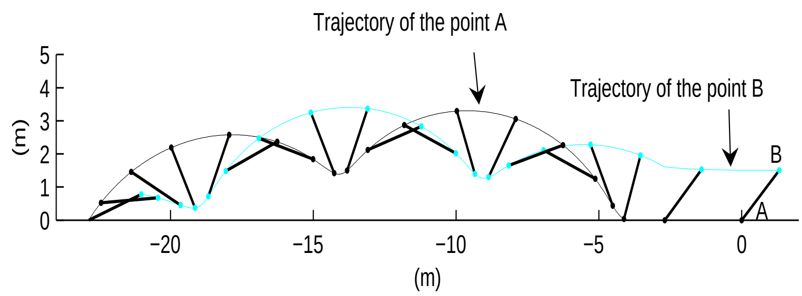


**Fig. 5.** Sthenic incompatibility for $[\omega] = 2 < [\omega]_{max} = 7.4398$. The sthenic incompatibility is responsible for the jump of the bar moving towards the left.

## 4 Conclusion

The predictive motion theory involving velocity discontinuities, takes into account both *cinematic and sthenic incompatibilities*, [2]. The velocity discontinuities result from two different reasons: the best-known *cinematic incompatibilities*, when it is impossible to compute the acceleration and the less-known

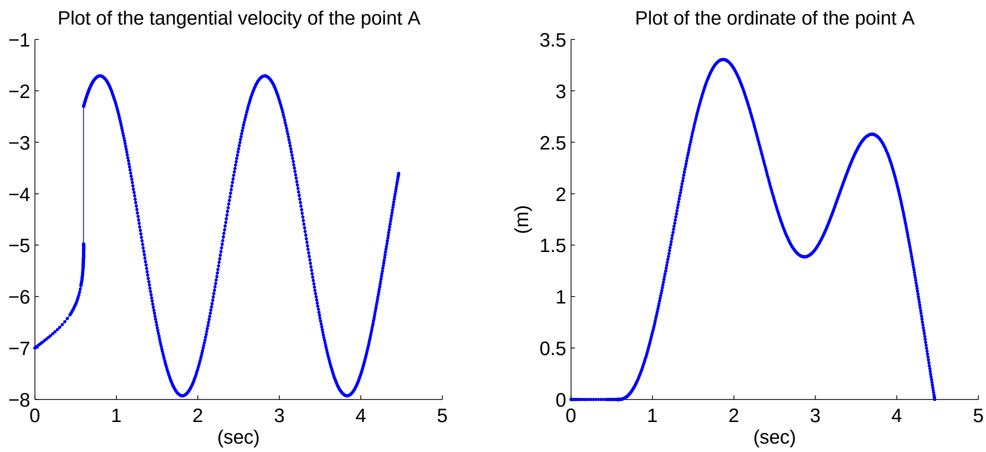


**Fig. 6.** Sthenic incompatibility for $[\omega] = 2$. After the jump of the horizontal velocity due to the sthenic incompatibility, the bar flies, makes two turns and falls again on the plane.

*sthenic incompatibilities* when it is impossible to compute the reaction or interior forces. These two incompatibilities are equivalent: they are overcome by velocity discontinuities determined by the theory. The difficulties in modelling the frictional hopping motion disappear if one uses this collision theory that satisfies the basic requirements of mechanics. We prove that when a smooth evolution is not possible, a velocity discontinuity occurs. The converse is also true. Let us also note that as it is usual with Coulomb friction law, the solutions of the Painlevé and Klein problems are not always unique.

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