Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 9, No. 3 (2006) 379–390

ENTANGLED MARKOV CHAINS ARE INDEED ENTANGLED

LUIGI ACCARDI

Centro V. Volterra, University of Roma Tor Vergata,
Roma 00133, Italy
accardi@volterra.mat.uniroma2.it

TAKASHI MATSUOKA

Department of Business Administration and Information,
Tokyo University of Science Suwa,
Chino City, Nagano 391-0292, Japan
matsuoka@rs.suwa.tus.ac.jp

MASANORI OHYA

Department of Information Science, Tokyo University of Science,
Noda City, Chiba 278-8510, Japan
ohya@rs.noda.tus.ac.jp

Received 17 March 2006
Communicated by O. Smolyanov

Entangled Markov chains (EMC) were so baptized on the basis of the conjecture that they provide examples of states, on infinite tensor products of matrix algebras, which are in some sense "entangled".[2] In this paper we introduce the notion of multiple (or "many-body") entanglement and extend the two-body criterion of entanglement obtained in Ref. 17 to this case. We then apply this extension to EMC and prove that "generically" they satisfy the entanglement conditions.

Keywords: Quantum Markov chain; multiple entanglement; degree of entanglement.

AMS Subject Classification: 60J99, 81Q10, 94A17

1. Introduction and Preparation

In the recent development of quantum information many people have discussed the problem of finding a satisfactory quantum generalization of the classical random walks. The relevance of this problem for quantum information has been emphasized in a large number of papers, e.g. see Refs. 7, 8, 12–14 and 19. However, these proposals introduce some features which are not completely satisfactory. Motivated by such situation Accardi and Fidaleo introduced the notion of entangled Markov chains which includes that of quantum random walk.[2] They listed the requirement that should be fulfilled by any candidate definition of a quantum random walk.

(1) It should be a quantum Markov chain in the sense of Ref. 1 (locality),

(2) it should be purely generated in the sense of Ref. 9 (pure entanglement),

(3) its restriction on at least one maximal Abelian subalgebra, should be a classical random walk (quantum extension property),

(4) it should be uniquely determined, up to arbitrary phases, by its classical restriction (amplitude condition).

In order to give an intuitive idea of the connection of their construction with entanglement, let us note that the key characteristic of entanglement is the superposition principle and the corresponding interpretation of the amplitudes as "complex square roots of probabilities". This suggests an approach in which, given a homogeneous classical Markov chain with finite state space S, determined by a stochastic matrix T and an initial distribution described by a row vector P, one can construct such a quantum Markov chain. The construction is as follows.

Let $S = \{1, 2, \ldots, d\}$ be a state space of cardinality $|S| = d(< \infty)$. We consider a classical Markov chain (S_n) with state space S, initial distribution $P = (p_j)$ and transition probability matrix $T = (t_{ij})$.

Fix the orthonormal basis (ONB for short) $\{|e_i\rangle\}_{i \le d}$ of $\mathbb{C}^{|S|}$ and fix a vector $|e_0\rangle$ in this basis. We consider the infinite tensor product Hilbert space

$$\mathcal{H} \equiv \bigotimes_{\mathbb{N}}^{(|e_0\rangle)} \mathbb{C}^{|S|} . \tag{1.1}$$

Let $T = (t_{ij})$ be any stochastic matrix (i.e. $t_{ij} \ge 0$, $\sum_j t_{ij} = 1$) and let $\sqrt{t_{ij}}$ be any complex square root of t_{ij} (i.e. $|\sqrt{t_{ij}}|^2 = t_{ij}$). Define the vector

$$|\Psi_n\rangle = \sum_{j_0,\ldots,j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{t_{j_\alpha j_{\alpha+1}}} |e_{j_0}, \ldots, e_{j_n}\rangle , \tag{1.2}$$

where $|e_{j_0}, \ldots, e_{j_n}\rangle \equiv (\otimes_{\alpha\in[0,n]} |e_{j_\alpha}\rangle) \otimes (\otimes_{\alpha\in[0,n]^c} |e_0\rangle)$.

Let $M_{|S|}$ denote the $|S| \times |S|$ (i.e. $d \times d$) complex matrix algebra and let $\mathcal{A} \equiv M_{|S|} \otimes M_{|S|} \otimes \cdots = \bigotimes_{\mathbb{N}} M_{|S|}$ be the $\mathbf{C}^*$-infinite tensor product of $\mathbb{N}$-copies of $M_{|S|}$.

Definition 1.1. An element $A_\Lambda \in \mathcal{A}$ (observable) will be said to be localized in a finite region $\Lambda \subseteq \mathbb{N}$ if there exists an operator $\bar{A}_\Lambda \in \otimes_\Lambda M_{|S|}$ such that

$$A_\Lambda = \bar{A}_\Lambda \otimes 1_{\Lambda^c} .$$

We denote $\mathcal{A}_\Lambda$ the local algebra at Λ and in the following we will identify $A_\Lambda = \bar{A}_\Lambda$.

The basic property of $|\Psi_n\rangle$ is that, although the limit $\lim_{n\to\infty} |\Psi_n\rangle$ will not exist, the following result holds:

Lemma 1.1. *For every local observable* $A \in \mathcal{A}_{[0,k]}$, $(k \in \mathbb{N})$ *one has*

$$\langle \Psi_{k+1}, A\Psi_{k+1}\rangle = \lim_{n\to\infty} \langle \Psi_n, A\Psi_n\rangle =: \varphi(A) . \tag{1.3}$$

Accardi and Fidaleo showed that the state φ defined by (1.3) is a quantum Markov chains in the sense of Ref. 1 and they called "entangled Markov chains" the family of quantum Markov chains that can be obtained by the above construction.

Definition 1.2. A state φ is a (homogeneous) quantum Markov chain (QMC for short) with initial state φ_0 over $M_{|S|}$ and transition expectation $\mathcal{E} : M_{|S|} \otimes M_{|S|} \mapsto M_{|S|}$ if

$$\begin{aligned}&\varphi(A_0 \otimes A_1 \otimes \cdots \otimes A_n \otimes 1 \otimes 1 \otimes \cdots)\\ &\quad = \varphi_0[\mathcal{E}(A_0 \otimes \cdots \mathcal{E}(A_{n-2} \otimes \mathcal{E}(A_{n-1} \otimes \mathcal{E}(A_n \otimes 1))) \cdots)] \,. \end{aligned} \tag{1.4}$$

For entangled Markov chains the transition expectation $\mathcal{E}$ is expressed in terms of the following linear map:

Definition 1.3. Define the linear map $V_n : \mathcal{H}_n \to \mathcal{H}_n \otimes \mathcal{H}_{n+1}$ by linear extension of

$$V_n|e_{j_n}\rangle = \sum_{j_{n+1}\in S} \sqrt{t_{j_n j_{n+1}}}|e_{j_n}\rangle \otimes |e_{j_{n+1}}\rangle \,, \tag{1.5}$$

where $\mathcal{H}_n = \mathcal{H}_{n+1} = \mathbb{C}^{|S|}$ for each $n \in \mathbb{N}$.

It is easy to show that $V_n^* V_n = 1$. Moreover, $\mathcal{E}_n(\cdot) \equiv V_n^* \cdot V_n$: $\mathcal{A}_n \otimes \mathcal{A}_{n+1} \to \mathcal{A}_n$ becomes a transition expectation and its dual $\mathcal{E}_n^*$: $\mathcal{A}_n^* \to (\mathcal{A}_n \otimes \mathcal{A}_{n+1})^*$ becomes a linear lifting in the sense of Ref. 3, where $\mathcal{A}_n = \mathcal{A}_{n+1} = M_{|S|}$ for each $n \in \mathbb{N}$.

Now let us extend V_n to an isometry still denoted with the same symbol

$$V_n : \otimes_{\alpha\in[0,n]}\mathcal{H}_\alpha \to \otimes_{\alpha\in[0,n+1]}\mathcal{H}_\alpha \tag{1.6}$$

by the prescription

$$V_n \otimes_{\alpha\in[0,n]} |e_{j_\alpha}\rangle \equiv (\otimes_{\alpha\in[0,n-1]}|e_{j_\alpha}\rangle) \otimes V_n|e_{j_n}\rangle \,. \tag{1.7}$$

It is easily shown that for each $j_0 \in S$ one has

$$|\Psi_n\rangle = \sum_{j_0,\ldots,j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{t_{j_\alpha j_{\alpha+1}}}|e_{j_0},\ldots,e_{j_n}\rangle = \sum_{j_0} \sqrt{p_{j_0}} V_{n-1}\cdots V_0|e_{j_0}\rangle \,. \tag{1.8}$$

We give an initial pure state φ_0 as

$$\varphi_0(\cdot) = \mathrm{tr}_{\mathcal{H}}(|\Psi_0\rangle\langle\Psi_0|\cdot) = \langle\Psi_0| \cdot |\Psi_0\rangle \,, \tag{1.9}$$

where $|\Psi_0\rangle = \sum_{j_0} \sqrt{p_{j_0}}|e_{j_0}\rangle$. Then from (1.7) and (1.8) we define a pure state φ_n over $\otimes_{j\in[0,n]}\mathcal{A}_j$ by using the isometric lifting $\mathcal{E}_n^*$ given by

$$\mathcal{E}_n^*(|\Psi_n\rangle\langle\Psi_n|) \equiv V_n|\Psi_n\rangle\langle\Psi_n|V_n^* \,, \tag{1.10}$$

$$\varphi_n(\cdot) \equiv \mathrm{tr}(\mathcal{E}_{n-1}^*(\mathcal{E}_{n-2}^*(\cdots(\mathcal{E}_1^*(\mathcal{E}_0^*(|\Psi_0\rangle\langle\Psi_0|)))\cdots))(\cdot)) \,. \tag{1.11}$$

Definition 1.4. An entangled Markov chain is a quantum Markov chain $\varphi \equiv \{\varphi_0, \mathcal{E}\} \equiv \{(p_j), (t_{ij}), \{|e_j\rangle\}\}$ over $\mathcal{A}$ where (i) φ_0 is a pure state over $M_{|S|}$, (ii) the transition expectation $\mathcal{E}(\cdot) \equiv V^* \cdot V$ is given by (1.5) for some stochastic matrix $T = (t_{ij})$ and for some fixed ONB $\{|e_j\rangle\}$.

Accardi and Fidaleo did not prove that the states given by Lemma 1.1 are entangled. In this paper we will analyze the entanglement of φ_n and also that of

φ. For our purpose we give the definitions of the entangled compound state in the following three cases:

Definition 1.5. Let $\mathcal{A}_j$ $(j \in \{1,2\})$ be $\mathbf{C}^*$-algebra, then $\omega \in \mathcal{S}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ is separable if

$$\omega \in \overline{\mathbf{Conv}}\{\omega_1 \otimes \omega_2; \omega_j \in \mathcal{S}(\mathcal{A}_j), j = 1,2\}\,.$$

Definition 1.6. Let $\mathcal{A}_j$ $(j \in \{1,2,\ldots,n\})$ be $\mathbf{C}^*$-algebra, then $\omega \in \mathcal{S}(\otimes_{j=1}^n \mathcal{A}_j)$ is separable if

$$\omega \in \overline{\mathbf{Conv}}\left\{\bigotimes_{j=1}^{n} \omega_j; \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in \{1,2,\ldots,n\}\right\}.$$

Definition 1.7. Let $\mathcal{A}_j$ $(j \in \{1,2,\ldots,\infty\})$ be $\mathbf{C}^*$-algebra, then $\omega \in \mathcal{S}(\otimes_{j=1}^\infty \mathcal{A}_j)$ is separable if

$$\omega \in \overline{\mathbf{Conv}}\left\{\bigotimes_{j=1}^{\infty} \omega_j; \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in \{1,2,\ldots,\infty\}\right\}.$$

A non-separable state is called entangled. Notice that a separable pure state must be a product of pure states. We introduce the notion of multiple entanglement as follows:

Definition 1.8. Let $\mathcal{A} = \otimes_{j=1}^n \mathcal{A}_j$ be the finite tensor product of $\mathbf{C}^*$-algebra. A state $\omega \in \mathcal{S}(\mathcal{A})$ is called 2-separable if

$$\omega \in \overline{\mathbf{Conv}}\{\omega_{k]} \otimes \omega_{(k}; \omega_{k]} \in \mathcal{S}(\mathcal{A}_{k]}), \omega_{(k} \in \mathcal{S}(\mathcal{A}_{(k})\}, \forall\, k \in \{1,2,\ldots,n\}\,,$$

where $\mathcal{A} = \mathcal{A}_{k]} \otimes \mathcal{A}_{(k} := \mathcal{A}_{[1,k]} \otimes \mathcal{A}_{(k,n]}$.

A state $\omega \in \mathcal{S}(\mathcal{A})$ is called 2-entangled if

$$\omega \notin \overline{\mathbf{Conv}}\{\omega_{k]} \otimes \omega_{(k}; \omega_{k]} \in \mathcal{S}(\mathcal{A}_{k]}), \omega_{(k} \in \mathcal{S}(\mathcal{A}_{(k})\}, \forall\, k \in \{1,2,\ldots,n\}\,.$$

Lemma 1.2. *If $\omega \in \mathcal{S}(\mathcal{A})$ is 2-entangled, then ω is entangled.*

Proof. It is clear from the definition. □

According to Definition 1.8 we extend the degree of entanglement defined by Belavkin and Ohya[5,6] to entangled Markov chains (EMC for short).

Entanglement degree for mixed states has been studied by some entropic measures such as quantum relative entropy and quantum mutual entropy. An example of such degree was defined in Ref. 10 using the relative entropy $S(\theta,\theta_0) \equiv \operatorname{tr}\theta(\log\theta - \log\theta_0)$ for a density operator θ as

$$D(\theta) = \min\{S(\theta, \theta_0); \theta_0 \in \mathcal{D}\}, \tag{1.12}$$

where $\mathcal{D}$ is the set of all separable densities.

Since, to compute this measure, one has to take the minimum over all separable state, it is difficult to compute it analytically. Thus another degree of entanglement was introduced by Belavkin and Ohya[5,6] using the quantum quasi-mutual entropy defined in Ref. 16.

Definition 1.9. Let θ be a density matrix on $\mathcal{A}_1 \otimes \mathcal{A}_2$ with marginal densities ρ and σ where $\mathcal{A}_1$, $\mathcal{A}_2$ are arbitrary (not necessarily finite dimensional) matrix algebras

(1) The quantum quasi-mutual entropy of ρ and σ w.r.t. θ is defined by $I_\theta(\rho, \sigma) \equiv \operatorname{tr} \theta(\log \theta - \log \rho \otimes \sigma)$.

(2) $D_{\mathrm{EN}}(\theta; \rho, \sigma) \equiv \frac{1}{2}\{S(\rho) + S(\sigma)\} - I_\theta(\rho, \sigma)$ is called the degree of entanglement (DEN for short), where $S(\cdot)$ is the von-Neumann entropy.

(3) θ_1 has stronger entanglement than θ_2 if

$$D_{\mathrm{EN}}(\theta_1; \rho, \sigma) < D_{\mathrm{EN}}(\theta_2; \rho, \sigma). \tag{1.13}$$

Using the degree of entanglement D_{EN} Ohya and Matsuoka gave the following characterization of pure entangled states on $\mathcal{A}_1 \otimes \mathcal{A}_2$ in Ref. 17 (see the Appendix).

Theorem 1.1. *For a pure state θ with marginal states ρ and σ,*

(1) *ω is separable iff $D_{\mathrm{EN}}(\theta; \rho, \sigma) = 0$,*

(2) *ω is not separable, i.e. entangled iff $D_{\mathrm{EN}}(\theta; \rho, \sigma) < 0$.*

In Sec. 2 it will be shown that the EMC generated by a deterministic stochastic matrix is 2-separable if and only if it is separable. In Sec. 3 we will prove that 2-entangled EMC generated by stochastic matrix with strictly positive elements is characterized by the entropy of density matrix associated with its stochastic matrix and that the EMC of a unitarily implementable matrix has the strongest possible entanglement.

In Ref. 15 the localized state of EMC was considered by taking a partial trace and it was shown that the conjecture is established in the case $d = 2$ by means of the Horodeckies, Peres entanglement criterion[11,18] which is applicable only to the $d = 2$ case. In this paper we do not take such a partial trace and will consider a general state in the Hilbert space with the dimensional d by applying the above criterion (i.e. Theorem 1.1).

2. Entangled Markov Chains Generated by a Deterministic Stochastic Matrix

Throughout this paper we analyze the EMC with stochastic matrix $T = (t_{ij})$ and associated invariant measure $P = (p_i)$, i.e. $\sum_i p_i t_{ij} = p_j$ is satisfied for each j. If the EMC with an invariant measure is restricted to Abelian subalgebra, then it gives a classical stationary Markov chain.

In this section we consider an invariant measure $P = (p_i)$ of the stochastic matrix $T = (t_{ij})$ with only 0, 1 as entries. Such stochastic matrices are called *deterministic*.

Let $\mathcal{J}_S$ be a set of all maps π from S into S (i.e. $\pi(S) \subseteq S$). Note that if $\pi(S) = S$, then π is a permutation. Let us denote $\pi(S)$ by S_π. In general there exists an integer k such that, denoting $S_\pi \equiv \pi^k(S)$, the restriction of π on $\mathcal{S}_\pi$ is a permutation on S_π.

Definition 2.1. An orbit of the dynamical system $\pi : S \to S$ is a minimal π-invariant subset of S. $R \subseteq S$ is called a minimal π-invariant subset of S if:

(i) $\pi(R) = R$

(ii) $R \neq \phi$ and if $Q \subseteq R$ is such that $\pi(Q) = Q$, then $Q = R$.

Definition 2.2. To every $\pi \in \mathcal{J}_S$ we associate the stochastic matrix T_π with elements $t_{jk} \equiv \delta_{k,\pi(j)}$, i.e. T_π is deterministic.

Remark 2.1. Given T_π as above, there exists an ONB $\{|e_j\rangle\}$ in a Hilbert space $\mathcal{H}$ whose dimension is the cardinality of S such that

$$T_\pi|e_j\rangle = \sum_k t_{jk}|e_k\rangle = |e_{\pi(j)}\rangle\,. \tag{2.1}$$

From the following well-known theorem we know that such a matrix T_π has many invariant measures.

Theorem 2.1. *Let π be in $\mathcal{J}_\pi$ and let T_π be the associated stochastic matrix. Then the set of T_π-invariant measures is precisely the set of probability measures of the form*:

$$p_k = \frac{1}{|S_l|} q_{S_l}, \forall\ k \in S_l, \forall\ l = 1, \ldots, m\,, \tag{2.2}$$

where $\{S_1, S_2, \ldots, S_m\}$ is an arbitrary partition of S into π-invariant subsets.

Proof. A T_π-invariant measure is characterized by the identity:

$$p_j = \sum_{i \in S} p_i t_{ij} = \sum_{i \in S} p_i \delta_{\pi(j),i} = p_{\pi(j)} \tag{2.3}$$

which shows that subsets of S where the map $j \mapsto p_j$ is constant are π-invariant. Conversely, if $\{S_1, S_2, \ldots, S_m\}$ is any partition of S into π-invariant sets and $S_l \mapsto q_{S_l}$ $(l = 1, \ldots, m)$ is any probability measure, then the probability measure on S defined by $p_j \equiv \frac{q_{S_l}}{|S_l|}$ $(j \in S_l)$ satisfies (2.3) and therefore it is T_π-invariant. □

We will show that the entanglement of φ_n can be measured by the DEN and using Theorem 1.1 the 2-separability condition of φ_n is shown. For any $k \in [1, n]$ let $\mathcal{H}_{k]}$ be the tensor product Hilbert space given by $\mathcal{H}_{k]} = \otimes_{j \in [0,k]} \mathcal{H}_j$ and $\mathcal{H}_{(k}$ be the tensor product Hilbert space given by $\mathcal{H}_{(k} = \otimes_{j \in (k,n]} \mathcal{H}_j$. Then the pure state

φ_n can be recognized as compound state w.r.t. the composite system $\mathcal{H}_{k]} \otimes \mathcal{H}_{(k}$. Theorem 2.1 means that $|\Psi_n\rangle$ is given as

$$\begin{aligned}|\Psi_n\rangle &= \sum_l^m \sum_{i_0 \in S_l} \sqrt{p_{i_0}} |e_{i_0}, e_{\pi(i_0)}, \dots, e_{\pi^n(i_0)}\rangle \\ &= \sum_l^m \sum_{i_0 \in S_l} \sqrt{\frac{q_{S_l}}{|S_l|}} |e_{i_0}, e_{\pi(i_0)}, \dots, e_{\pi^n(i_0)}\rangle \,. \end{aligned} \tag{2.4}$$

Then the marginal densities of φ_n are given as

$$\begin{aligned}\rho_{k]} &= \mathrm{tr}_{\mathcal{H}_{(k}} |\Psi_n\rangle\langle\Psi_n| \\ &= \sum_l^m \sum_{i_0 \in S_l} \frac{q_{S_l}}{|S_l|} |e_{i_0}, e_{\pi(i_0)}, \dots, e_{\pi^k(i_0)}\rangle\langle e_{i_0}, e_{\pi(i_0)}, \dots, e_{\pi^k(i_0)}| \,, \end{aligned} \tag{2.5}$$

$$\begin{aligned}\sigma_{(k} &= \mathrm{tr}_{\mathcal{H}_{k]}} |\Psi_n\rangle\langle\Psi_n| \\ &= \sum_l^m \sum_{i_0 \in S_l} \frac{q_{S_l}}{|S_l|} |e_{\pi^{k+1}(i_0)}, \dots, e_{\pi^n(i_0)}\rangle\langle e_{\pi^{k+1}(i_0)}, \dots, e_{\pi^n(i_0)}| \,. \end{aligned} \tag{2.6}$$

Note that both decompositions (2.5), (2.6) are Schatten decompositions, i.e. the spectral decompositions of $\rho_{k]}$, $\sigma_{(k}$ into one-dimensional orthogonal projectors. From the purity of φ_n one can compute its DEN as follows (see the Appendix):

$$\begin{aligned}&D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k}) \\ &\quad = -S(\sigma_{(k})(\mathrm{or} = -S(\rho_{k]})) \\ &\quad = \sum_l^m \sum_{i_0 \in S_l} \frac{q_{S_l}}{|S_l|} \log \frac{q_{S_l}}{|S_l|} = \sum_l^m q_{S_l} \log q_{S_l} - \sum_l^m q_{S_l} \log |S_l| \le 0 \,. \end{aligned} \tag{2.7}$$

$D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k})$ has the same value for any $k \in [1, m]$, so that let denote $D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k})$ by $D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n|)$. Then $D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n|) = 0$ if and only if there exists a number $l \in [1, m]$ such that $q_{S_l} = |S_l| = 1$. Since $q_{S_l} \log q_{S_l} \le 0$ and $-q_{S_l} \log |S_l| \le 0$ for any $l \in [1, n]$. According to Theorem 1.1 we know that this condition gives the 2-separability condition of φ_n. In this case the 2-separability condition of φ_n is equivalent to the separability condition of φ_n, i.e. φ_n can be represented by

$$\varphi_n(\cdot) = \mathrm{tr}_{\otimes_{j \in [1,n]} \mathcal{H}_j} \left(\bigotimes_{j \in [1,n]} |e_l\rangle\langle e_l| \cdot \right) .$$

Summarizing the above argument we have the following theorem:

Theorem 2.2. (i) *For any $n < \infty$ the state φ_n is a pure separable state if and only if the T_π-invariant measure is extreme (i.e. the probabilities are concentrated on a single point of S).*

(ii) *For any $n < \infty$ the state φ_n is pure entangled state if and only if the T_π-invariant measure is not extreme.*

In general, we define

$$\underline{D_{\mathrm{EN}}}(|\Psi_n\rangle\langle\Psi_n|) \equiv \inf_{k\in(0,n]} D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k}) \tag{2.8}$$

and we call it the DEN of the restriction of the EMC φ to the interval $[0, n]$. Then the following definition introduces to a natural way to measure analytically the strength of entanglement of an EMC.

Definition 2.3. Let φ be the EMC defined by (1.3). The DEN of φ is defined by

$$D_{\mathrm{EN}}(\varphi) \equiv \lim_{n\to\infty} \underline{D_{\mathrm{EN}}}(|\Psi_n\rangle\langle\Psi_n|)\,. \tag{2.9}$$

Theorem 2.3. *Let $|\Psi_n\rangle$ be given by (2.4). Then $\underline{D_{\mathrm{EN}}}(|\Psi_n\rangle\langle\Psi_n|) = D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n|)$. Moreover,*

(1) *the DEN of φ is equal to*

$$D_{\mathrm{EN}}(\varphi) = \sum_{l=1}^{m} q_{S_l} \log q_{S_l} - \sum_{l=1}^{m} q_{S_l} \log |S_l| \le 0\,. \tag{2.10}$$

(2) *When π acts transitively on S (so that $p_i = \frac{1}{d}$ for $\forall\ i \in S$), the strongest DEN is given by*

$$D_{\mathrm{EN}}^{\text{strongest}}(\varphi) = -\log d\,.$$

3. Entangled Markov Chains Generated by Unitarily Implementable Matrix

In this section we consider the particular case in which the transition matrix T has strictly positive elements, i.e. $t_{ij} > 0$ for any $i, j \in S$ and is unitarily implementable, i.e. there exists a unitary matrix $U = (u_{ij})$ such that $|u_{ij}|^2 = t_{ij}$ for any i and j. We show that, in this case, the associated EMC φ has the strongest possible entanglement. We start from the (unique) measure $P = (p_i)$ of the stochastic matrix $T = (t_{ij})$. Then the following theorem holds.

Theorem 3.1. *To the stochastic matrix T we associate the density matrix σ_T given as*

$$\sigma_T \equiv \sum_i p_i |f_i\rangle\langle f_i|\,, \tag{3.1}$$

where $|f_i\rangle = \sum_k \sqrt{t_{ik}}|e_k\rangle$. Then

(1) *The state φ_n is a pure 2-separable state for any $n < \infty$ iff $S(\sigma_T) = 0$.*

(2) *The state φ_n is a pure 2-entangled state for any $n < \infty$ iff $S(\sigma_T) > 0$.*

(3) *There always exists the DEN of φ such that*

$$-H(P) \le D_{EN}(\varphi) = -S(\sigma_T) \le 0\,,$$

where $H(P)$ is the Shannon entropy of the probability measure P.

Before giving the proof of Theorem 3.1 we will show the following lemma.

Lemma 3.1. *Put* $\sigma(1) \equiv \sigma_T$ *and, with* $\mathcal{E}_m^*$ *given by* (1.10), *define the density matrix* $\sigma(m)$ *as*

$$\sigma(m) \equiv \mathcal{E}_{m-1}^*(\mathcal{E}_{m-2}^*(\cdots(\mathcal{E}_1^*(\sigma(1)))\cdots)).$$

Then

$$S(\sigma(1)) = S(\sigma(m)), \forall\ m \in [1, \infty).$$

Proof. It is known that von Neumann entropy of a density matrix ρ is preserved under isometric transformation ($\rho \mapsto V\rho V^*$, $V^*V = 1$) and, from (1.5) we know that each $\mathcal{E}_m^*$ is implemented by an isometry. □

Proof. Now φ_n is given by

$$\varphi_n(\cdot) = \mathrm{tr}(|\Psi_n\rangle\langle\Psi_n|\cdot),$$

where $|\Psi_n\rangle = \sum_{j_0,j_1,j_2,\ldots,j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{t_{j_\alpha j_{\alpha+1}}} |e_{j_0}, e_{j_1}, \ldots, e_{j_n}\rangle$. From the purity of φ_n one has

$$D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k}) = -S(\sigma_{(k}), \forall\ k \in [1, n]. \tag{3.2}$$

Since P is the invariant measure of T, the marginal density $\sigma_{(k}$ is computed as

$$\begin{aligned}
\sigma_{(k} &= \mathrm{tr}_{\mathcal{H}_{k]}} |\Psi_n\rangle\langle\Psi_n| \\
&= \sum_{\substack{j_0,j_1,\ldots,j_{k-1}, \\ j_k,j_{k+1},\ldots,j_n,l_{k+1},\ldots,l_n}} p_{j_0} \prod_{\alpha=0}^{k-1} t_{j_\alpha j_{\alpha+1}} \sqrt{t_{j_k l_{k+1}}}^* \cdots \sqrt{t_{l_{n-1} l_n}}^* \\
&\quad \sqrt{t_{j_k j_{k+1}}} \cdots \sqrt{t_{j_{n-1} j_n}} |e_{j_{k+1}}, \ldots, e_{j_n}\rangle\langle e_{l_{k+1}}, \ldots, e_{l_n}| \\
&= \sum_{\substack{i,j_{k+1},\ldots,j_n, \\ l_{k+1},\ldots,l_n}} p_i \sqrt{t_{il_{k+1}}}^* \cdots \sqrt{t_{l_{n-1} l_n}}^* \sqrt{t_{ij_{k+1}}} \cdots \sqrt{t_{j_{n-1} j_n}} \\
&\quad |e_{j_{k+1}}, \ldots, e_{j_n}\rangle\langle e_{l_{k+1}}, \ldots, e_{l_n}| \\
&\equiv \sum_i p_i |f_i(n-k)\rangle\langle f_i(n-k)|,
\end{aligned} \tag{3.3}$$

where $|f_i(n-k)\rangle = \sum_{j_{k+1},\ldots,j_n} \sqrt{t_{ij_{k+1}}} \cdots \sqrt{t_{j_{n-1} j_n}} |e_{j_{k+1}}, \ldots, e_{j_n}\rangle$.

It is easily checked that the norm of $|f_i(n-k)\rangle$ is equal to 1 but the set $\{|f_i(n-k)\rangle\}$ is not orthogonal in general because the set $\{|f_i\rangle\}$ is not. Therefore an exact calculation of $S(\sigma_{(k})$ is not easy. However, one can estimate the entropy of $\sigma_{(k}$ as follows:

$$0 \le S(\sigma_{(k}) \le -\sum p_i \log p_i = H(P), \tag{3.4}$$

where $S(\sigma_{(k}) = H(P)$ holds if $\{|f_i(n-k)\rangle\}$ is an orthogonal set (i.e. it is an ONB).

In the case of $k = n-1$ one has

$$\sigma_{(n-1} = \sum_i p_i |f_i(1)\rangle\langle f_i(1)| = \sum_i p_i |f_i\rangle\langle f_i| = \sigma_T\,. \tag{3.5}$$

According to the notation of Lemma 3.1 $\sigma_{(n-m}$ can be represented as

$$\sigma_{(n-m} = \sigma(m)\,.$$

Lemma 3.1 means that

$$D_{\mathrm{EN}}(|\Psi_n\rangle\langle\Psi_n| : \rho_{k]}, \sigma_{(k}) = -S(\sigma_T)\,.$$

From Theorem 1.1 and the definition of $D_{\mathrm{EN}}(\varphi)$ the statements of this theorem hold. □

In the above theorem if the stochastic matrix $T = (t_{ij})$ is unitarily implementable and we take $\sqrt{t_{ij}} = u_{ij}$, then the set $\{|f_i\rangle\}$ giving the decomposition of σ_T by (3.1) becomes an ONB, i.e.

$$\langle f_j, f_i\rangle = \sum_k u_{jk}^* u_{ik} = (UU^*)_{ij} = \delta_{i,j}\,.$$

Thus the following theorem holds.

Theorem 3.2. *If EMC φ has an invariant measure P of a unitarily implementable matrix T, then the DEN of φ exists and is equal to*:

$$D_{\mathrm{EN}}(\varphi) = -H(P)\,. \tag{3.6}$$

Appendix A

If θ on $\mathcal{H} \otimes K$ is an entangled pure state with marginal states ρ, σ, then von Neumann entropy $S(\theta) = 0$. Moreover, from the Araki–Lieb inequality[4]:

$$|S(\rho) - S(\sigma)| \le S(\theta) \le S(\rho) + S(\sigma)\,, \tag{A.1}$$

the purity of θ implies that $S(\rho) = S(\sigma)$. It follows

$$\begin{aligned} I_\theta(\rho,\sigma) &= \operatorname{tr}\theta(\log\theta - \log\rho\otimes\sigma) \\ &= \operatorname{tr}\theta\log\theta - \operatorname{tr}\theta\log\rho\otimes I - \operatorname{tr}\theta\log I\otimes\sigma \\ &= S(\rho) + S(\sigma) - S(\theta) \\ &= 2S(\rho)\,. \end{aligned}$$

Then the proof of Theorem 1.1 is given by the following:

Proof. In the case of a pure state θ, $D_{\mathrm{EN}}(\theta;\rho,\sigma)$ can be computed as

$$\begin{aligned} D_{\mathrm{EN}}(\theta;\rho,\sigma) &\equiv \frac{1}{2}\{S(\rho) + S(\sigma)\} - I_\theta(\rho,\sigma) \\ &= S(\rho) - 2S(\rho) \\ &= -S(\rho)(\text{or} = -S(\sigma))\,. \end{aligned} \tag{A.2}$$

If $D_{\mathrm{EN}}(\theta; \rho, \sigma) < 0$, then $S(\rho) = S(\sigma) > 0$ which means that ρ and σ are mixture states. Therefore ρ can be written as $\rho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ where $\{|x_i\rangle\}$ is an ONB in $\mathcal{H}$ and $\sum_i \lambda_i = 1$, $0 \le \lambda_i \le 1$ and at least two λ_i are strictly positive. Then due to the Schmidt decomposition there exists an ONB $\{|y_i\rangle\}$ of $\mathcal{K}$ such that θ is given by $\theta = |\Psi\rangle\langle\Psi|$ where

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i} |x_i\rangle \otimes |y_i\rangle \, .$$

Since at least two λ_i are strictly positive, this implies that θ is a pure entangled state. The converse statement obviously holds.

If $D_{\mathrm{EN}}(\theta; \rho, \sigma) = 0$, then $S(\rho) = S(\sigma) = 0$ which means that ρ and σ are pure states respectively. Thus θ is a pure state whose marginals are pure states. This implies that ω is a product of pure states. Conversely, if θ is pure and separable, then it is the product of two pure states, hence $D_{\mathrm{EN}}(\theta; \rho, \sigma) = 0$. □

References

1. L. Accardi, Topics in quantum probability, *Phys. Rep.* **77** (1981) 169–192.
2. L. Accardi and F. Fidaleo, Entangled Markov chains, *Ann. Mat. Pura Appl.* (2004).
3. L. Accardi and M. Ohya, Compound channels, transition expectations, and liftings, *J. Appl. Math. Optim.* **39** (1999) 33–59.
4. H. Araki and E. H. Lieb, Entropy inequalities, *Commun. Math. Phys.* **18** (1970) 160–170.
5. V. P. Belavkin and M. Ohya, Quantum entropy and information in discrete entangled state, *Infin. Dim. Anal. Quantum Probab. Relat. Top.* **4** (2001) 137–160.
6. V. P. Belavkin and M. Ohya, Entanglement, quantum entropy and mutual information, *Proc. R. Soc. London A* **458** (2002) 209–231.
7. T. A. Brun, H. A. Carteret and A. Ambainis, Quantum walks driven by many coins, *Phys. Rev. A* **67** (2003) 052317–17.
8. A. M. Childs, E. Farhi and S. Gutmann, An example of the difference between quantum and classical random walks, *Quantum Inf. Process.* **1** (2002) 35–43.
9. M. Fannes, B. Nachtergaele and R. F. Werner, Finitely correlated pure states, *J. Funct. Anal.* **120** (2002) 511–534.
10. L. Henderson and V. Vedral, Information, relative entropy of entanglement, and irreversibility, *Phys. Rev. Lett.* **84** (2000) 2263–2266.
11. M. Horodecki, P. Horodecki and R. Horodecki, Separability of mixed states: Necessary and sufficient condition, *Phys. Lett. A* **223** (1996) 1–8.
12. N. Konno, Quantum random walks in one dimensions, *Quantum Inf. Process.* **1** (2003) 345–354.
13. T. D. Mackey, S. D. Bartlett, L. T. Stephanson and B. C. Sanders, Quantum walks in higher dimensions, *J. Phys. A* **36** (2003) 241–253.
14. D. A. Meyer, From quantum cellular automata to quantum lattice gases, *Stat. Phys.* **85** (1996) 551–574.
15. T. Miyadera, Entangled Markov chains generated by symmetric channels, *Infin. Dim. Anal. Quantum Probab. Relat. Top.* **8** (2005) 497–504.
16. M. Ohya, Some aspects of quantum information theory and their applications to irreversible process, *Rep. Math. Phys.* **27** (1989) 19–47.
17. M. Ohya and T. Matsuoka, Quantum entangled state and its characterization, *Found. Probab. Phys. 3* **750** (2005) 298–306.

18. A. Peres, Separability criterion for density matrices, *Phys. Rev. Lett.* **77** (1996) 1413–1415.
19. B. C. Travaglione and G. J. Milburn, Implementing the quantum random walk, *Phys. Rev. A* **65** (2002) 032310.