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Explicit computation of second-order moments of importance sampling estimators for fractional Brownian motion

PAOLO BALDI* and BARBARA PACCHIAROTTI**

*Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Roma, Italy. E-mail: *baldi@mat.uniroma2.it; **pacchiar@mat.uniroma2.it*

We study a family of importance sampling estimators of the probability of level crossing when the crossing level is large or the intensity of the noise is small. We develop a method which gives explicitly the asymptotics of the second-order moment. Some of the results apply to fractional Brownian motion, some are more general. The main tools are refined versions of classical large-deviations results.

Keywords: Gaussian processes; importance sampling; large deviations; ruin probabilities

1. Introduction

In this paper we address the problem of computing the asymptotics of the second-order moment of a class of importance sampling estimators arising naturally when dealing with Gaussian processes. We consider two questions. Let $X = (X_t)_{t\geqslant 0}$ be a fractional Brownian motion. A classical problem in risk theory is the investigation of the ruin probability

$$P\left(\sup_{t>0}(X_r - \varphi_t) > B\right). \tag{1.1}$$

Under suitable assumptions on the function φ (typically $\varphi_t \to +\infty$ as $t \to +\infty$) this probability is very small and its computation by a naive simulation requires a large number of iterations in order to achieve a reasonable precision. A natural technique is then *importance sampling*, that is, the simulation of the process under a new probability Q, for which the event considered is not rare, and to compensate by taking into account the density of P with respect to Q.

A closely related question is the computation, as $\varepsilon \to 0$, of the probability

$$p_\varepsilon \stackrel{\text{def}}{=} P\left(\sup_{0<t\leqslant 1}(\varepsilon X_t - \varphi_t) > 1\right). \tag{1.2}$$

More precisely, we consider, for the computation of the level crossing probability (1.2), the class of admissible importance sampling estimators of the form

$$1_{\{\tau_\varepsilon\leqslant 1\}} Z_{\tau_\varepsilon} \tag{1.3}$$

where the process X is simulated according to a distribution Q_ε, obtained by translating P with a path $\varepsilon^{-1}\,\gamma$, γ belonging to the reproducing kernel Hilbert space (RKHS) of X. Here τ_ε is the time at which level 1 is attained (possibly $\tau_\varepsilon = +\infty$) and Z_{τ_ε} is the density of P with respect to Q_ε up to time τ_ε. For the ruin problem (1.1) the class of admissible importance sampling estimators is similar, (see Section 7).

For the probabilities (1.1) and (1.2) there exist in the literature classical results providing the asymptotics (see Hüsler and Piterbarg 1999). We point out, however, that our goal here is mainly to investigate the existence and behaviour of importance sampling estimators in this context, which is something of importance in itself. The interested reader can refer to Glasserman and Kou (1995), Glasserman and Wang (1997) and Sadowsky (1996) for results on existence and/or counterexamples for importance sampling distributions. Also one should be aware that the above mentioned asymptotics are actually of little help in practice, as the formulae contains constants (e.g. Pickand's constant) whose value should be evaluated numerically.

In our situation the probability p_ε of (1.2) has a large-deviation limit,

$$\lim_{\varepsilon\to 0}\varepsilon^2\log p_\varepsilon = -i_\varphi.$$

The importance sampling estimator (1.3) is *asymptotically efficient* if, for its second-order moment,

$$\lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{E}^{Q_\varepsilon}[1_{\{\tau_\varepsilon\leqslant 1\}}Z^2_{\tau_\varepsilon}] = -2i_\varphi. \tag{1.4}$$

Actually if (1.4) is not satisfied, in order to achieve a given precision, the number of iterations increases exponentially fast. Conversely, if (1.4) holds, the number of iterations may increase, but not at an exponential rate.

As stated above, we investigate the existence and properties of asymptotically efficient importance sampling estimators for the probabilities (1.2) and (1.1). The paper is organized as follows. In Section 3 we deal with the asymptotics of the probability (1.2) as $\varepsilon\to 0$. These results are well known, but we have chosen to include them because we develop a method that also works for the computation of the second-order moment. This is the object of Section 4. It should be pointed out that the results in Sections 3 and 4 hold for general Gaussian processes: X only needs to be Gaussian, continuous, centred and starting at 0. We must also make an assumption on X (see Assumption 4.1), which is satisfied for many Gaussian processes of interest (including fractional Brownian motion).

In Sections 5 and 6 we discuss the existence of asymptotically efficient estimators for the probability (1.2) for Brownian motion and fractional Brownian motion. In particular, we prove that if the path φ is of the form $\varphi_t = kt^\alpha$, then no asymptotically efficient importance sampling estimator of the form considered exists for fractional Brownian motion. We prove also that, for Brownian motion, an asymptotically efficient change of probability exists only if $\alpha > \frac{3}{4}$.

From the results of Sections 3 and 4 we derive in Section 7 the behaviour of the second-order moment for the importance sampling estimators for the ruin problem for fractional Brownian motion and for φ of the form $\varphi_t = kt^\alpha,\ k>0$. For fractional Brownian motion with $H\neq\frac{1}{2}$ we prove that an asymptotically efficient change of probability does not exist in the class of importance sampling distributions considered. We show, however, that if $H > \frac{1}{2}$, then there exist changes of probability which are close to asymptotic efficiency. This negative result leads to two questions: how to enlarge the class of admissible distributions, in order to have existence of an importance sampling distribution; and how to compute the most efficient change of probability in the class considered.

In this context Michna (1999) proposed an importance sampling estimator. We are able to compute exactly the behaviour of its second-order moment and show the lack of asymptotic efficiency (see Section 8).

This kind of phenomenon (non-existence of asymptotically efficient changes of probability, and in particular lack of efficiency by translating with the most likely path) have already been pointed out in the literature; see Glasserman and Kou (1995), Glasserman and Wang (1997) and Sadowsky (1996).

From a technical point of view, the main tool is Varadhan's lemma from large-deviation theory, but we have had to overcome some technical problems in order to use it (see Section 9).

2. The main setting

In this section we introduce the setting that we are going to consider throughout the paper.

Let $\mathcal{C}_t = \mathcal{C}([0, t], \mathbb{R}^m)$ be the Banach space of real continuous paths on $[0, t]$ vanishing at 0, endowed with the uniform norm $|w|_{t,\infty} = \sup_{0\leqslant s\leqslant t}|w_s|$. Its dual $\mathcal{C}'_t$ is formed by the signed measures on $[0, t]$ with finite variation, through the duality

$$\langle \alpha, w\rangle = \int_0^t w_s \, \mathrm{d}\alpha_s.$$

A probability P on $\mathcal{C}_t$ is said to be Gaussian if the canonical random variables (r.v.s) $X_s : \mathcal{C}_t \to \mathbb{R}$, $s \leqslant t$, defined by $X_s(w) = w_s$ form a Gaussian family. In the following we assume that P is *full*, that is, that it gives strictly positive probability to every open set of $\mathcal{C}_t$. Then $(X_s)_s$ is a (continuous) Gaussian process and we denote by $K(u, v) = \operatorname{cov}(X_u, X_v)$ its *covariance function*. On $\mathcal{C}_t$ we consider the filtration $\mathcal{F}_t = \sigma(X_u, u \leqslant t)$ generated by the process. If $\alpha \in \mathcal{C}'_t$, then the r.v.s $X_\alpha(w) = \langle \alpha, w\rangle$ are also Gaussian. Let $\mathcal{C}'_{t,P}$ be the completion in $L^2(\mathcal{C}_t, P)$ of $\mathcal{C}'_t$. It is a closed vector space of $L^2(\mathcal{C}_t, P)$, whose elements form a Gaussian family. For every r.v. $Z \in \mathcal{C}'_{t,P}$ let us define, for $s \leqslant t$,

$$w_s = \mathrm{E}(X_s Z). \tag{2.1}$$

Then w is a continuous path, that is, $w \in \mathcal{C}_t$. It is easy to prove that the application $Z \to w$ is one to one. We denote by $\mathcal{H}_t$ the set of the paths w of this form. Endowed with the norm

$$|w|_{\mathcal{H}_t} \stackrel{\text{def}}{=} \|Z\|_{L^2(\mathcal{C}_t,P)},$$

$\mathcal{H}_t$ is a Hilbert space, the RKHS of P. It is useful to remark that the r.v.s X_r, $r \leqslant t$, certainly belong to $\mathcal{C}'_{t,P}$ and that the corresponding paths are

$$w_s = \mathrm{E}(X_s X_r) = K(r, s).$$

Thus $K(r, \cdot)$ always belongs to $\mathcal{H}_t$ and $|K(r, \cdot)|_{\mathcal{H}_t} = \mathrm{E}[X_r^2]^{1/2} = K(r, r)^{1/2}$. More generally, if Z is of the form $Z = \int_0^t X_s \, d\alpha_s$, then the corresponding path in the RKHS is

$$w_s = \mathrm{E}(X_s Z) = \int_0^t K(s, u) \, d\alpha_u. \tag{2.2}$$

By construction these paths are dense in $\mathcal{H}_t$. In the following we write $\mathcal{C}$, $\mathcal{C}_P$ and $\mathcal{H}$ instead of $\mathcal{C}_1$, $\mathcal{C}'_{1,P}$ and $\mathcal{H}_1$, respectively.

3. Estimates of level crossing

We now compute the limit $\lim_{\varepsilon\to 0}\varepsilon^2 \log p_\varepsilon$ for a fixed continuous path $\varphi \in \mathcal{C}$. This result is not new, but we include its proof for completeness' sake and because the method we use is quite similar to that developed in the next section for the second-order moment estimation. The computation is actually simple as the process $t \to \varepsilon X_t - \varphi_t$ satisfies a large deviation principle, so that, denoting by I its rate function,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log p_\varepsilon = -\inf_{w\in\mathcal{A}} I(w), \tag{3.1}$$

where $\mathcal{A} = \{w \in \mathcal{C};\ \sup_{0<t\leqslant 1} w_t - \varphi_t \geqslant 1\}$ (see Section 9). The rate function I is $I(w) = \frac{1}{2}|w|^2_{\mathcal{H}}$ if w belongs to the RKHS $\mathcal{H}$ and $I(w) = +\infty$ otherwise.

Proposition 3.1. *The infimum in* (3.1) *is equal to*

$$\inf_{0<t\leqslant 1} \frac{(1+\varphi_t)^2}{2K(t, t)},$$

that is,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log P\Big(\sup_{0<t\leqslant 1} (\varepsilon X_t - \varphi_t) \geqslant 1\Big) = -\inf_{0<t\leqslant 1} \frac{(1+\varphi_t)^2}{2K(t, t)} \stackrel{def}{=} -i_\varphi. \tag{3.2}$$

Proof. Let $\mathcal{A}_t = \{w \in \mathcal{C};\ w_t = 1 + \varphi_t\}$, so that $\mathcal{A} = \bigcup_{0<t\leqslant 1}\mathcal{A}_t$, and

$$\inf_{w\in\mathcal{A}} I(w) = \inf_{0<t\leqslant 1}\ \inf_{w\in\mathcal{A}_t} I(w).$$

A set of paths which is dense in $\mathcal{H}$ is that formed by those which are barycentres of the r.v.s belonging to $\mathcal{C}'$. That is,

$$w_t = \mathrm{E}\Big(X_t \int_0^1 X_s \, d\alpha(s)\Big) = \int_0^1 K(t, s) \, d\alpha(s). \tag{3.3}$$

Since $|w|^2_{\mathcal{H}}$, is equal to the variance of the centred Gaussian r.v. $\int_0^1 X_t \, d\alpha(t)$,

$$|w|^2_{\mathcal{H}} = \int_0^1\int_0^1 K(u, v) \, d\alpha(u) \, d\alpha(v) \stackrel{def}{=} V(\alpha).$$

Recalling that $I(w) = \frac{1}{2}|w|^2_{\mathcal{H}}$,

$$\inf_{w \in \mathcal{A}_t} I(w) = \inf V(\alpha) = \inf \frac{1}{2} \int_0^1 \int_0^1 K(u, v) \, \mathrm{d}\alpha(u) \, \mathrm{d}\alpha(v),$$

the infimum on the right-hand side being taken among all $\alpha \in \mathcal{C}'$ such that $\int_0^1 K(t, s) \, \mathrm{d}\alpha(s) = 1 + \varphi_t$. This is a constrained extremum problem. Using Lagrange multipliers, we find that α must satisfy

$$\int_0^1 K(u, v) \, \mathrm{d}\alpha(v) + \lambda K(t, u) = 0, \qquad \text{for every } u, \, 0 < u \leqslant 1,$$

for some $\lambda \in \mathbb{R}$. Bearing in mind the constraint, one finds that

$$-\lambda = \frac{1 + \varphi_t}{K(t, t)}, \qquad \alpha_0 = \frac{1 + \varphi_t}{K(t, t)} \delta_t.$$

Therefore

$$\inf_{w \in \mathcal{A}_t} I(w) = \frac{1}{2} \int_0^t \int_0^t K(u, v) \, \mathrm{d}\alpha_0(u) \, \mathrm{d}\alpha_0(v) = \frac{1}{2} \frac{(1 + \varphi_t)^2}{K(t, t)}.$$

□

Remark 3.1. A closer look at the above proof shows that the minimizing path $w \in \mathcal{A}$ is

$$w_s = \frac{1 + \varphi_t^*}{K(t^*, t^*)} K(t^*, s), \tag{3.4}$$

where

$$t^* = \underset{0 < t \leqslant 1}{\operatorname{argmin}} \frac{(1 + \varphi_t)^2}{2K(t, t)};$$

in particular, whatever the path φ, the minimizing w is of the form const $\cdot \, K(t^*, \cdot)$, for some $t^* \in]0, 1]$.

Example 3.1 Fractional Brownian motion. If $\varphi_t = kt^\alpha$ and X is a fractional Brownian motion, the infimum in (3.1) becomes

$$\inf_{0 < t \leqslant 1} \frac{1}{2} \frac{(1 + kt^\alpha)^2}{t^{2H}}.$$

Its computation is straightforward, as the derivative is negative for $t \leqslant t^*$ and positive for $t > t^*$, where

$$t^* = \left(\frac{H}{k(\alpha - H)} \right)^{1/\alpha}. \tag{3.5}$$

If $t^* \leqslant 1$ the infimum is attained at t^* and, from (3.4), the minimizing path is

$$w_s = a^* K\big(t^*, s\big), \tag{3.6}$$

where a^* is such that $a^* K(t^*, t^*) = 1 + \varphi_{t^*}$, that is $a^* = (1 + kt^{*\alpha})/t^{*2H}$. If $t^* > 1$, then the infimum is attained at $t = 1$. The fact that the most likely path is as in (3.6) was obtained in O'Connell and Procissi (1998) by a different approach.

4. Importance sampling

In this section we consider a family of importance sampling estimators of the level crossing probability (1.2) and give explicit estimates for their second-order moments. This will enable us to study the existence of asymptotically efficient importance sampling estimators.

Let $\gamma \in \mathcal{H}$ and $Z \in \mathcal{C}'_P$ the corresponding r.v. By Girsanov's theorem, with respect to the probability $\mathrm{d}Q_\varepsilon = \exp(\varepsilon^{-1}Z - (2\varepsilon^2)^{-1}|\gamma|^2_{\mathcal{H}})\,\mathrm{d}P$, the process εX has the same distribution as $t \to \varepsilon X_t + \gamma_t$ under P. Let $\varphi : [0, 1] \to \mathbb{R}$ be a continuous path and define the stopping times

$$\tau(\omega) = \inf\{t;\, \omega_t - \varphi_t \geqslant 1\}, \qquad \tau_\varepsilon(\omega) = \tau(\varepsilon\omega)$$

with the usual understanding that $\inf\varnothing = +\infty$. In the following, E and $\mathrm{E}^{Q_\varepsilon}$ denote the expectations with respect to P and Q_ε respectively. Let $(Z^\varepsilon_t)_t$ be the right continuous version of the martingale

$$Z^\varepsilon_t = \frac{\mathrm{d}P}{\mathrm{d}Q_\varepsilon}|_{\mathcal{F}_t}.$$

Then, under Q_ε, the r.v.

$$1_{\{\tau_\varepsilon \leqslant 1\}} Z^\varepsilon_{\tau_\varepsilon} \tag{4.1}$$

is an unbiased estimator of p_ε. Actually, by conditioning with respect to $\mathcal{F}_{\tau_\varepsilon}$,

$$p_\varepsilon = P\bigg(\sup_{0<t\leqslant 1}(\varepsilon X_t - \varphi_t) > 1\bigg) = P(\tau_\varepsilon \leqslant 1) = \mathrm{E}^{Q_\varepsilon}[1_{\{\tau_\varepsilon\leqslant 1\}}Z^\varepsilon_t] = \mathrm{E}^{Q_\varepsilon}[1_{\{\tau_\varepsilon\leqslant 1\}}Z^\varepsilon_{\tau_\varepsilon}].$$

We now investigate the existence of asymptotically efficient importance sampling estimators in the class (4.1). Let us set $m_2(\varepsilon) = \mathrm{E}^{Q_\varepsilon}(1_{\{\tau_\varepsilon\leqslant 1\}}(Z^\varepsilon_{\tau_\varepsilon})^2)$. Thus this importance sampling estimator is asymptotically efficient (see (1.4) for the definition) if

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log m_2(\varepsilon) = -2i_\varphi. \tag{4.2}$$

Remark that in (4.2) the left-hand side is always equal to or greater than the right-hand side, as the second-order moment cannot be smaller than the square of the mean. Remark that if $\zeta_t = \mathrm{E}(Z\,|\,\mathcal{F}_t)$, then $(\zeta_t)_t$ is a Gaussian P-martingale (the σ-field $\mathcal{F}_t$ is generated by $(X_s)_{s\leqslant t}$ and $((X_s)_{s\leqslant t}, Z)$ is a Gaussian family). Hence it has independent increments and, if $v_t = \mathrm{E}(\zeta_t^2)$, $\tilde{Z}^\varepsilon_t = \exp(\varepsilon^{-1}\zeta_t - (2\varepsilon^2)^{-1}v_t)$ is also a P-martingale; since $\tilde{Z}^\varepsilon_1 = \exp(\varepsilon^{-1}Z - (2\varepsilon^2)^{-1}|\gamma|^2_{\mathcal{H}})$, then $\tilde{Z}^\varepsilon_t = \mathrm{E}(\exp(\varepsilon^{-1}Z - (2\,\varepsilon^2)^{-1}|\gamma|^2|\,\mathcal{F}_t)$. Thus $\tilde{Z}^\varepsilon_t = (\mathrm{d}Q_\varepsilon/\mathrm{d}P)\,|\,_{\mathcal{F}_t}$ and

$$Z^\varepsilon_t = (\tilde{Z}^\varepsilon_t)^{-1} = \exp(-\varepsilon^{-1}\zeta_t + (2\varepsilon^2)^{-1}v_t).$$

The second-order moment of the estimator (4.1) is therefore

$$\begin{aligned} m_2(\varepsilon) &= \mathrm{E}^{Q_\varepsilon}(1_{\{\tau_\varepsilon\leqslant 1\}}(Z^\varepsilon_{\tau_\varepsilon})^2) = \mathrm{E}^P(1_{\{\tau_\varepsilon\leqslant 1\}}Z^\varepsilon_{\tau_\varepsilon}) = \mathrm{E}^P(1_{\{\tau_\varepsilon\leqslant 1\}}\exp(-\varepsilon^{-1}\zeta_{\tau_\varepsilon} + (2\varepsilon^2)^{-1}v_{\tau_\varepsilon})) \\ &= \int_{\{\tau_\varepsilon\leqslant 1\}} \exp(-\varepsilon^{-1}\zeta_{\tau_\varepsilon}(\omega) + (2\varepsilon^2)^{-1}v_{\tau_\varepsilon(\omega)})\,\mathrm{d}P(\omega) \\ &= \int 1_{\{\tau(\varepsilon\omega)\leqslant 1\}}\exp\left(\varepsilon^{-2}\left(-\zeta_\tau(\varepsilon\omega) + \frac{1}{2}v_{\tau(\varepsilon\omega)}\right)\right)\mathrm{d}P(\omega), \end{aligned}$$

as ζ_t is a linear function of ω.

Assumption 4.1. *We say that the probability P satisfies the continuity of Gaussian martingales (CGM) property if, for every r.v. $Z \in \mathcal{C}'_P$, the Gaussian martingale $(\zeta_t)_t$ admits a continuous version.*

The results of this section hold under Assumption 4.1. This assumption is satisfied for many Gaussian processes of interest and in particular for Brownian motion (of course) and fractional Brownian motion.

It is possible to find examples of continuous Gaussian processes that do not satisfy Assumption 4.1. In Section 9 we prove that, under Assumption 4.1, the functional

$$-\zeta_\tau + \tfrac{1}{2}v_\tau \tag{4.3}$$

satisfies the assumptions of Varadhan's lemma (Theorem 9.2). This gives

$$\begin{aligned} &\lim_{\varepsilon\to 0}\varepsilon^2\log m_2(\varepsilon) \\ &\quad= \lim_{\varepsilon\to 0}\varepsilon^2\log \mathrm{E}^P(1_{\{\tau_\varepsilon\leqslant 1\}}\exp(-\varepsilon^{-1}\zeta_{\tau_\varepsilon} + (2\varepsilon^2)^{-1}v_{\tau_\varepsilon})) = \sup_{\omega,\tau(\omega)\leqslant 1}\{-\zeta_\tau(w) + \tfrac{1}{2}v_r - I(w)\} \\ &\quad= \sup_{t\leqslant 1}\sup_{w\in\mathcal{H};w_t-\varphi_t=1}\{-\zeta_t(w) + \tfrac{1}{2}v_t - \tfrac{1}{2}|w|^2_{\mathcal{H}}\} = -\inf_{t\leqslant 1}\inf_{w\in\mathcal{H};w_t-\varphi_t=1}\{\zeta_t(w) - \tfrac{1}{2}v_t + \tfrac{1}{2}|w|^2_{\mathcal{H}}\}. \end{aligned}$$

In the next statements we investigate the infima on the right-hand side.

Proposition 4.1. *Let* $t \in [0, 1]$. *Then*

$$\inf_{w\in\mathcal{H};w_t-\varphi_t=1}\{\zeta_t(w) - \tfrac{1}{2}v_t + \tfrac{1}{2}|w|^2_{\mathcal{H}}\} = -v_t + \frac{1}{2K(t,\,t)},\,(1+\varphi_t+\gamma_t)^2. \tag{4.4}$$

Proof. $\zeta_t = \mathrm{E}(Z\,|\,\mathcal{F}_t)$ is a Gaussian r.v. and the corresponding path in the RKHS is $\gamma^{(t)}_s = \mathrm{E}(X_s\zeta_t)$. Obviously the paths $\gamma^{(t)}$ and γ coincide up to time t. If $w \in \mathcal{H}$, $\zeta_t(w) = \langle\gamma^{(t)}, w\rangle_{\mathcal{H}}$. Thus we are looking for the minimum of the functional $F(w) = \langle\gamma^{(t)}, w\rangle_{\mathcal{H}} + \frac{1}{2}|w|^2_{\mathcal{H}}$ on the hyperplane $w_t = 1 + \varphi_t$. This is, again, a constrained extremum to be handled by Lagrange multipliers. As F is convex and tends to $+\infty$ as $|w|_{\mathcal{H}} \to +\infty$, it is sufficient to look for a critical point. For $h \in \mathcal{H}$, $DF(w)h = \langle\gamma^{(t)}, h\rangle_{\mathcal{H}} + \langle w, h\rangle_{\mathcal{H}}$ and, if $G(w) = w_t$, $DG(w)h = h_t$. Thus the Lagrange multipliers relation for a critical point is

$$\langle \gamma^{(t)}, h\rangle_{\mathcal{H}} + \langle w, h\rangle_{\mathcal{H}} + \lambda h_t = 0.$$

The only vector $y \in \mathcal{H}$ such that $\langle y, h\rangle_{\mathcal{H}} = \lambda h_t$ for every $h \in \mathcal{H}$ is $y = K(t, \cdot)$. Thus the relation $\langle \gamma^{(t)} + w, h\rangle_{\mathcal{H}} = -\lambda h_t$ for every $h \in \mathcal{H}$ implies $\gamma^{(t)} + \mathrm{w} = -\lambda K(t, \cdot)$, that is, $w_s = -\gamma_s^{(t)} - \lambda K(t, s)$. The constraint $w_t = 1 + \varphi_t$ implies $-\lambda = (1 + \varphi_t + \gamma_t)K(t, t)^{-1}$ (recall that $\gamma_t^{(t)} = \gamma_t$). Thus, the critical path $\tilde{w}$ is

$$\tilde{w}_s = -\gamma_s^{(t)} + \frac{1 + \varphi_t + \gamma_t}{K(t, t)} K(t, s).$$

Replacing the value of $\tilde{w}$ and using the relationships

$$\langle \gamma^{(t)}, \gamma^{(t)}\rangle_{\mathcal{H}} = \mathrm{E}(\zeta_t^2) = \upsilon_t, \qquad \langle \gamma^{(t)}, K(t, \cdot)\rangle_{\mathcal{H}} = \gamma_t, \qquad \langle K(t, \cdot), K(t, \cdot)\rangle_{\mathcal{H}} = K(t, t),$$

one obtains

$$\zeta_t(\tilde{w}) = \langle \tilde{w}, \gamma^{(t)}\rangle_{\mathcal{H}} = -\upsilon_t + \gamma_t \frac{1 + \varphi_t + \gamma_t}{K(t, t)},$$

$$|\tilde{w}|^2_{\mathcal{H}} = \upsilon_t + 2\lambda\gamma_t + \lambda^2 K(t, t) = \upsilon_t + 2\gamma_t \frac{1 + \varphi_t + \gamma_t}{K(t, t)} + \frac{(1 + \varphi_t + \gamma_t)^2}{K(t, t)}$$

and, finally, putting things together gives (4.4). □

Let us define

$$H_\gamma(t) := -\upsilon_t + \frac{1}{2K(t, t)}(1 + \varphi_t + \gamma_t)^2. \tag{4.5}$$

We shall call H_γ the *master function* associated with the translation γ. Let

$$\rho(\gamma) := \inf_{0 \leqslant t \leqslant 1} H_\gamma(t). \tag{4.6}$$

Since, by Varadhan's lemma,

$$\lim_{\varepsilon \to 0} \varepsilon \log m_2(\varepsilon) = -\inf_{t \leqslant 1} H_\gamma(t) = -\rho(\gamma), \tag{4.7}$$

γ is asymptotically efficient if and only if $\rho(\gamma) = 2i_\varphi$. The next statement is useful in determining whether $\gamma \in \mathcal{H}$ is asymptotically efficient.

Proposition 4.2. *For every* $> \gamma \in \mathcal{H}$ *and* $> 0 < t \leqslant 1$,

$$H_\gamma(t) \leqslant \frac{(1 + \varphi_t)^2}{K(t, t)}. \tag{4.8}$$

Moreover, if γ *is asymptotically efficient, the infimum in* (4.6) *is attained at every*

$$t^* \in \operatorname*{argmin}_{0 < s \leqslant 1} \frac{(1 + \varphi_t)^2}{2K(t, t)},$$

and γ *must be of the form* $\gamma_s = a^* K(t^*, s)$ *for* $s \leqslant t^*$, *where* $a^* = (1 + \varphi_{t^*})/K(t^*, t^*)$.

Proof. For every t, the master function can be rewritten as

$$H_\gamma(t) = -\left(v_t - \frac{1}{K(t,\,t)}\gamma_t^2\right) - \frac{1}{2K(t,\,t)}(1+\varphi_t-\gamma_t)^2 + \frac{(1+\varphi_t)^2}{K(t,\,t)}. \tag{4.9}$$

If Z is the r.v. associated with γ in $\mathcal{C}'_P$, then $\gamma_t = \mathrm{E}(X_t Z) = \mathrm{E}(X_t \mathrm{E}(Z\,|\,\mathcal{F}_t))$, so that

$$\gamma_t^2 \leqslant \mathrm{E}(\mathrm{E}(Z\,|\,\mathcal{F}_t)^2)\mathrm{E}({X_t}^2) = v_t K(t,\,t), \tag{4.10}$$

which implies

$$v_t - \frac{1}{K(t,\,t)}\gamma_t^2 \geqslant 0.$$

Therefore the first two terms on the left-hand side of (4.9) are both negative, which proves (4.8). Rewriting (4.9) for $t = t^*$ gives

$$H_\gamma(t^*) = -\left(v_{t^*} - \frac{1}{K(t^*,\,t^*)}\gamma_{t^*}^2\right) - \frac{1}{2K(t^*,\,t^*)}(1+\varphi_{t^*}-\gamma_{t^*})^2 + \underbrace{\frac{(1+\varphi_{t^*})^2}{K(t^*,\,t^*)}}_{=2i_\varphi}.$$

Therefore a necessary condition for γ to be asymptotically efficient is that both the relations

$$v_{t^*} - \frac{1}{K(t^*,\,t^*)}\gamma_{t^*}^2 = 0, \qquad 1+\varphi_{t^*}-\gamma_{t^*} = 0$$

hold. The second inequality implies $\gamma_{t^*} = 1+\varphi_{t^*}$. As for the first one, a closer look at (4.10) shows that the inequality is strict unless the r.v.s $\mathrm{E}(Z\,|\,\mathcal{F}_{t^*})$ and X_{t^*} are collinear, that is, unless $\gamma_{t^*} = \lambda \mathrm{E}(X_{t^*}^2) = \lambda K(t^*,\,t^*)$. The condition $\gamma_{t^*} = 1+\varphi_{t^*}$ implies $\lambda = (1+\varphi_{t^*})/K(t^*,\,t^*) = a^*$. Thus, if $s \leqslant t^*$, then

$$\gamma_s = \mathrm{E}(X_s Z) = \mathrm{E}[X_s \mathrm{E}(Z\,|\,\mathcal{F}_{t^*})] = a^* \mathrm{E}(X_s X_{t^*}) = a^* K(t^*,\,s).$$

□

Remark 4.1. It is useful to point out the two key features of Proposition 4.2. First, any asymptotically efficient translation γ must be of the form $\gamma_s = a^* K(t^*,\,s)$ for $s \leqslant t^*$. No condition is stated concerning the behaviour of γ after time t^*. Second, as $H_\gamma(s) \leqslant (1+\varphi_s)^2/K(s,\,s)$ for every $0 < s \leqslant 1$ and $i_\varphi = (1+\varphi_{t^*})^2/(2K(t^*,\,t^*))$, if γ is asymptotically efficient then t^* must necessarily be a point of absolute minimum of H_γ and $H_\gamma(t^*) = (1+\varphi_{t^*})^2/K(t^*,\,t^*)$.

In the following we apply the results of this section and compute the asymptotics of the second moment of the importance sampling estimator in two natural situations.

5. Application: Brownian motion

In this section we assume that P is the Wiener measure. Thus X is a continuous Brownian motion and $K(t,\,s) = t \wedge s$. Proposition 4.2 states that an asymptotically efficient translation γ must necessarily be such that $\gamma_s^* = a^* s$ for $s \leqslant t^* = \operatorname{argmin}(1+\varphi_t)^2/t$. This is only a necessary condition for asymptotic efficiency and an asymptotically efficient translation may not exist. There are, however, a number of situations in which it can be shown that the translation $\gamma_s^* = a^* s$, $0 \leqslant s \leqslant 1$, is asymptotically efficient.

Let us first make a couple of remarks. Let $t^* \in]0, 1[$ be a time at which $r(t) = (1+\varphi_t)^2/2t$ has a local minimum. The relation $r'(t^*) = 0$ straightforwardly implies easily $\varphi'_{t^*} = (1+\varphi_{t^*})/2t^*$. If $\gamma_s = as$ for some $a \in \mathbb{R}$, then

$$H_\gamma(t) = -a^2 t + \frac{1}{2t}(1+\varphi_t + at)^2 = -\frac{1}{2}a^2 t + \frac{(1+\varphi_t)^2}{2t} + a\varphi_t \tag{5.1}$$

and

$$H'_\gamma(t^*) = -\frac{1}{2}a^2 + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\frac{(1+\varphi_t)^2}{2t}\bigg|_{t=t^*}}_{=0} + a\varphi'_{t^*} = -\frac{1}{2}a^2 + \frac{a}{2t^*}(1+\varphi_{t^*}) = \frac{a}{2t^*}(1+\varphi_{t^*} - at^*).$$

Thus, if $a = a^* = (1+\varphi_{t^*})/t^*$, t^* is also a critical point for H_γ. Moreover, with $a = a^*$ in (5.1), we can immediately see that $H_{\gamma^*}(t^*) = 2i_\varphi$. Therefore $\gamma^*(t) = a^* t$ is asymptotically efficient if and only if t^* is an absolute minimum for H_{γ^*} (see Figures 1 and 2). This is immediate if H_{γ^*} is convex,

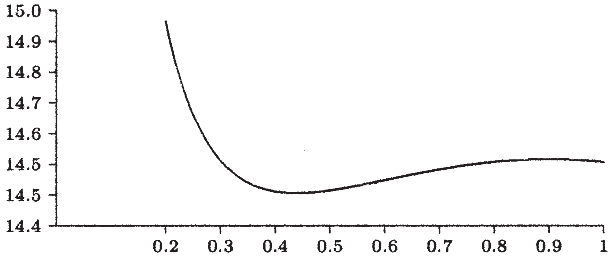

Figure 1. Graph of H_{γ^*} (recall that $\gamma_t^* = a^* t$) for $\alpha = 0.83$ and $k = 3$. H_{γ^*} is not convex, but the minimum is attained at $t^* = 0.439$ and γ^* is asymptotically efficient.

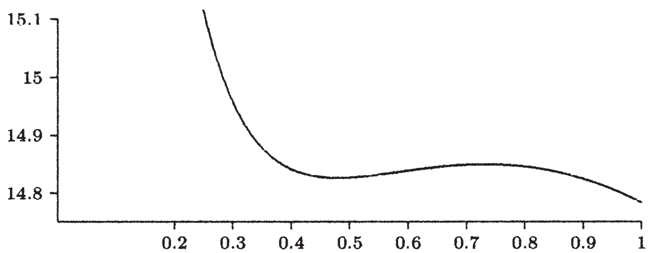

Figure 2. Graph of H_{γ^*} for $\alpha = 0.8$ and $k = 3$. Now the minimum is attained at $t = 1$ and $t^* = 0.48$ is only a local minimum. γ^* is no longer asymptotically efficient.

$$H''_{\gamma^*}(t) = \varphi''_t a + \frac{\varphi''_t(1+\varphi_t)}{t} + \frac{(t\varphi'_t - 1 - \varphi_t)^2}{t^3},$$

and it is easy to check that, if φ is convex, then H_{γ^*} itself is a convex function of t. Thus, if φ is convex, $\gamma^*_s = a^* s$ provides a translation which is asymptotically efficient.

Proposition 5.1. *Let* $> \varphi_t = kt^\alpha$. *If* $> \alpha \geqslant 1$ *then the path* $> \gamma^*_s = a^* s$ *is asymptotically efficient. If* $> 0 < \alpha \leqslant \frac{3}{4}$, *then an asymptotically efficient path does not exist.*

Proof. The previous arguments settle the question $\alpha \geqslant 1$. Otherwise, by Proposition 4.2, an asymptotically efficient translation γ^* must satisfy $\gamma^*_s = a^* s$ for $s \leqslant t^*$. Simple calculations show that $H''_{\gamma^*}(t^*) = C(4\alpha - 3)$, with $C > 0$. Thus, if $\alpha < \frac{3}{4}$, then H_{γ^*} does not have a minimum at t^* and, by the remark at the beginning of this section, γ^* cannot be asymptotically efficient (Figure 3). Finally, for $\alpha = \frac{3}{4}$, $H''_{\gamma^*}(t^*) = 0$ but $H'''_{\gamma^*}(t^*) \neq 0$; therefore t^* is a point of inflection and γ^* cannot be asymptotically efficient. □

Remark 5.1. For $\alpha > \frac{3}{4}$, t^* is only a local minimum. It may not be an absolute minimum (Figure 2), but let us consider, under Q^ε, the estimator

$$1_{\{\sup_{t^*-\delta \leqslant t \leqslant t^*+\delta}(\varepsilon W_t - \varphi_t) > 1\}} Z_{\tau_\varepsilon}.$$

Since

$$\mathrm{E}^{Q^{*\varepsilon}}\left[1_{\{\sup_{t^*-\delta \leqslant t \leqslant t^*+\delta}(\varepsilon W_t - \varphi_t) > 1\}} Z_{\tau_\varepsilon}\right] = P\left(\sup_{t^*-\delta \leqslant t \leqslant t^*+\delta}(\varepsilon W_t - \varphi_t) > 1\right),$$

a repetition of the arguments in Section 2 shows that this estimator is biased, but its bias tends to zero exponentially faster than the probability to be estimated. The same arguments as in Proposition 4.1 allow us to state that the logarithm of its second-order moment goes to $-\infty$ as

$$-\inf_{t^*-\delta \leqslant t \leqslant t^*+\delta} H_{\gamma^*}(t)\frac{1}{\varepsilon^2}.$$

If δ is small enough, the infimum is attained at $t = t^*$ and the estimator is asymptotically efficient.

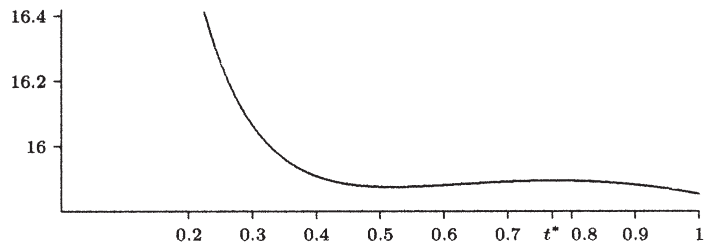

Figure 3. Graph of H_{γ^*} for $\alpha = 0.7$ and $k = 3$. There is no minimum at t^*, and γ^* is not asymptotically efficient.

Finally, as for the uniqueness, it is easy to see that, if $\gamma_t^* = a^* t$ is asymptotically efficient, then the path that is defined by $\tilde{\gamma}_t = a^* t$ if $t \leqslant r$ and $\tilde{\gamma}_t = a^* r$ if $t \geqslant r$ is also asymptotically efficient if r is chosen close enough to 1.

6. Application: fractional Brownian motion

Assume now that $(\mathcal{C}, (X_t)_{t\leqslant 1}, P)$ is a fractional Brownian motion. Norros *et al.* (1999) (see also Molchan and Golosov 1969) proved that, if

$$w(t, s) = \begin{cases} c_1 s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}, & \text{if } 0 \leqslant s < t, \\ 0 & \text{if } s \geqslant t \end{cases} \tag{6.1}$$

where $c_1 = \left(2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)\right)^{-1}$, then

$$M_t = \int_0^t w(t, s)\, \mathrm{d}X_s$$

is a Gaussian martingale (the *fundamental martingale*) such that $\mathrm{E}(M_t^2) = c_2^2 t^{2-2H}$, where

$$c_2^2 = \frac{\Gamma(\frac{3}{2}-H)}{2H(2-2H)\Gamma(H+\frac{1}{2})\Gamma(2-2H)}. \tag{6.2}$$

Figure 4 shows c_2^2 as a function of H, its behaviour being relevant in what follows. Norros *et al.* (1999) prove two important properties of the fundamental martingale, namely, the representation formula

$$X_t = \int_0^t \tilde{z}(t, s)\, \mathrm{d}M_s, \tag{6.3}$$

where the integral is an ordinary stochastic integral with respect to the continuous square integrable martingale $(M_t)_t$, and

$$\tilde{z}(t, s) = 2H\left(t^{H-1/2}(t-s)^{H-1/2} - (H-\tfrac{1}{2})\int_s^t u^{H-3/2}(u-s)^{H-1/2}\, \mathrm{d}u\right). \tag{6.4}$$

The fundamental martingale generates the filtration $(\mathcal{F}_t)_t$ and every r.v. in the Gaussian space $\mathcal{C}'_P$ can be written in the form

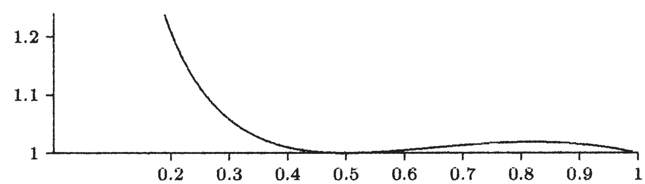

Figure 4. Behaviour of c_2^2 as H varies. For $H > \frac{1}{2}$, the values remain very close to 1.

$$\int_0^1 g(s)\,\mathrm{d}M_s \tag{6.5}$$

for g in the Hilbert space L_H^2 of functions such that $\int_0^1 g(s)^2\,\mathrm{d}\langle M\rangle_s = c_2^2(2H-2)\int_0^1 g(s)^2 s^{1-2H}\,\mathrm{d}s < +\infty$. The representation formula (6.5) implies that the CGM property of Assumption 4.1 is satisfied for fractional Brownian motion. Actually, if $Z \in \mathcal{C}'_P$, then it is of the form (6.5), so that

$$\zeta_t = \mathrm{E}(Z\,|\,\mathcal{F}_t) = \int_0^t g(s)\,\mathrm{d}M_s.$$

The term υ_t appearing in the master function is now, for $t \leqslant t^*$,

$$\upsilon_t = a^{*2}\mathrm{E}(\mathrm{E}(X_{t^*}\,|\,\mathcal{F}_t)^2) = \mathrm{E}\left[\mathrm{E}\left(\left(\int_0^{t^*}\tilde{z}(t^*,u)\,\mathrm{d}M_u|\mathcal{F}_t\right)^2\right]\right.$$

$$= \mathrm{E}\left[\left(\int_0^t \tilde{z}(t^*,u)\,\mathrm{d}M_u\right)^2\right] = a^{*2}(2-2H)c_2^2\int_0^t \tilde{z}(t^*,s)^2 s^{1-2H}\,\mathrm{d}s. \tag{6.6}$$

Proposition 6.1. *If $\varphi_t = kt^\alpha$, $\alpha > 0$, then an asymptotically efficient path does not exist.*

Proof. We first consider the case $H > \frac{1}{2}$. Proposition 4.2 states that the corresponding translation γ should be of the form $\gamma_t = a^*K(t^*,t) = a^*\mathrm{E}(X_tX_{t^*})$ for $t \leqslant t^*$, where t^* is a point at which the infimum in (3.2) is attained. We now prove that the associated master function H_γ has a minimum that is strictly smaller than $2i_\varphi$. This will follow from the fact that $H'_\gamma(t^*-) = 0$, $H''_\gamma(t^*-) = -\infty$, thus H_γ is increasing in a left neighbourhood of t^*, and therefore has a minimum that is strictly smaller than $H_\gamma(t^*) = 2i_\varphi$. The computation of these derivatives is somewhat involved, mostly because of the term υ_t, which is to be handled by derivation under the integral sign using (6.4) and (6.6). Since

$$H'_\gamma(t) = -\upsilon'_t + \frac{(1+\varphi_t+\gamma_t)[(\varphi'_t+\gamma'_t)t - H(1+\varphi_t+\gamma_t)]}{t^{2H+1}},$$

one obtains $H'_\gamma(t^*-) = 0$, as $\gamma'_{t^*} = \varphi'_{t^*} = H(1+\varphi_{t^*})/t^*$ and $\tilde{z}(t^*,t^*) = 0$. As for the second-order derivative, denoting by $\ldots$ the terms that have a finite limit in t^*,

$$H''_\gamma(t) = -\upsilon''_t + \frac{(1+\varphi_t+\gamma_t)}{t^{2H}}\gamma''$$

$$+ \ldots \simeq 2(2H-1)Ha^{*2}(t^*-t)^{2H-2}\underbrace{\left(\frac{\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)} - 1\right)}_{:=K_H}.$$

The quantity K_H is strictly negative for every $H \neq \frac{1}{2}$ (see Figure 5). Therefore $H''_\gamma(t^*-) = -\infty$ and the master function is increasing in a left neighbourhood of t^*. Similar computations show that, if $H < \frac{1}{2}$, the derivative of the master function, H_γ, for

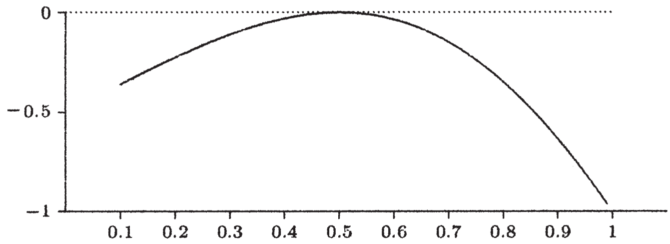

Figure 5. Behaviour of K_H as H varies. K_H is strictly negative unless $H = \frac{1}{2}$.

$\gamma_t = a^* K(t^*, t)$, converges to $+\infty$ as $t \uparrow t^*$. Thus, in this case an asymptotically efficient translation also cannot exist. □

Remark 6.1. The computations of the this section raise two questions. First, concerning fractional Brownian motion, there is no importance sampling distribution which is asymptotically efficient *in the class considered*. Is there a reasonable, larger, class of importance sampling distributions in which an asymptotically efficient one can be found? Second, if an asymptotically efficient importance sampling distribution does not exist, is it possible to find an importance sampling distribution that is optimal in the class considered (even if it fails to be asymptotically efficient)? In other words, if $J(\gamma) = \inf_{0<t\leq 1} H_\gamma(t)$, does the functional J attain a maximum as γ varies in $\mathcal{H}$? If it does, what are these maxima?

7. The ruin probability

In this section we apply the results of Sections 2 and 3 and compute the asymptotics of the second moment for the importance sampling estimator of the ruin probability (1.1) as $B \to +\infty$. Whereas the results of Sections 3 and 4 hold for general Gaussian processes (albeit centred and continuous and satisfying Assumption 4.1), in this section we deal with fractional Brownian motion.

Let $\tilde{X} = (\tilde{X}_s)_{s\geq 0}$ be a fractional Brownian Motion and $c > 0$, $\alpha > 0$. We investigate the behaviour, as $B \to +\infty$, of

$$P\Big(\sup_{s>0} (\tilde{X}_s - cs^\alpha) > B\Big). \tag{7.1}$$

Proposition 7.1. *If $\alpha > H$, up to an exponentially negligible term, the ruin probability* (7.1) *is equivalent to $p_\varepsilon = P(\sup_{0<t\leq 1}(\varepsilon X_t - kt^\alpha) > 1)$, where X is a fractional Brownian motion with Hurst exponent H, $\varepsilon = B^{-1+H/\alpha} d^{H/\alpha}$ and $k = cd$, if d is large enough.*

Proof. Let d be a positive number. By scaling,

$$P\Bigg(\sup_{0<s\leq (Bd)^{1/\alpha}} (\tilde{X}_s - cs^\alpha) > B\Bigg) = P\Bigg(\sup_{0<t\leq 1} (B^{-1+H/\alpha} d^{H/\alpha} X_t - cdt^\alpha) > 1\Bigg), \tag{7.2}$$

where $X_t = (Bd)^{-H/\alpha}\tilde{X}t(Bd)^{1/\alpha}$ is again a fractional Brownian motion with Hurst exponent equal to H. If $\alpha > H$, $\varepsilon = B^{-1+H/\alpha}d^{H/\alpha}$ and $k = cd$, the large-deviations asymptotics of the probability (7.2) can be deduced by the results of the previous sections: if

$$s^* = \left(\frac{H}{c(\alpha - H)}\right)^{1/\alpha}$$

and d is such that $d > s^{*\alpha}$, then

$$\lim_{B\to+\infty} B^{-2+2H/\alpha} \log P\left(\sup_{0<s\leqslant(Bd)^{1/\alpha}} (\tilde{X}_s - cs^\alpha) > B\right)$$
$$= d^{-2H/\alpha} \lim_{\varepsilon\to 0} \varepsilon^2 \log P\left(\sup_{0<t\leqslant 1} (\varepsilon X_t - kt^\alpha) > 1\right) = d^{-2H/\alpha} \inf_{0<t\leqslant 1} \frac{(1 + kt^\alpha)^2}{2t^{2H}} \stackrel{\text{def}}{=} -\tilde{i}. \quad (7.3)$$

With the assumptions made on d,

$$t^* = \left(\frac{H}{cd(\alpha - H)}\right)^{1/\alpha} < 1.$$

Thus the critical point t^* belongs to]0, 1[and

$$\tilde{i} = d^{-2H/\alpha}\frac{1}{2}\left(\frac{\alpha}{\alpha - H}\right)^2\left(\frac{k(\alpha - H)}{H}\right)^{2H/\alpha} = \frac{1}{2}\left(\frac{\alpha}{\alpha - H}\right)^2\left(\frac{c(\alpha - H)}{H}\right)^{2H/\alpha} \quad (7.4)$$

does not depend on d (provided that $d > s^{*\alpha}$). In order to complete the proof it is sufficient to show that

$$\limsup_{B\to+\infty} B^{-2+2H/\alpha} \log P\left(\sup_{s>(Bd)^{1/\alpha}} (\tilde{X}_s - cs^\alpha) > B\right) < -\tilde{i}.$$

This is a straightforward application of Fernique's theorem, as outlined in Duffield and O'Connell (1995). □

Let us denote by $\mathcal{H}$ and $\tilde{\mathcal{H}}_B$ the RKHS of $(X_t)_{t\leqslant 1}$ and $(\tilde{X}_t)_{t\leqslant(Bd)^{1/\alpha}}$ respectively. If $\tilde{\gamma} \in \tilde{\mathcal{H}}_B$, then $\tilde{\gamma}_t = \mathrm{E}(\tilde{Z}\tilde{X}_t)$, where $\tilde{Z}$ belongs to the Gaussian space generated by the r.v.s $(\tilde{X}_t)_{t\leqslant(Bd)^{1/\alpha}}$. As pointed out in Section 6, this Gaussian space is well described using the fundamental martingale of Norros *et al.* (1999). Let

$$\tilde{M}_t = \int_0^t w(t, s)\,\mathrm{d}\tilde{X}_s,$$

where w is as in (6.1); every r.v. belonging to the Gaussian space generated by $(\tilde{X}_t)_{t\leqslant(Bd)^{1/\alpha}}$ is of the form

$$\tilde{Z} = \int_0^{(Bd)^{1/\alpha}} \tilde{g}(s)\,\mathrm{d}\tilde{M}_s$$

for some function $\tilde{g}$ such that $\int_0^{(Bd)^{1/\alpha}} \tilde{g}(s)^2 \mathrm{d}\langle \tilde{M}\rangle_s < +\infty$. We denote by $\tilde{L}^2_H$ the Hilbert space of these functions, endowed with the scalar product

$$\langle \tilde{g}, \tilde{h}\rangle_{\tilde{\mathcal{H}}} = \int_0^{(Bd)^{1/\alpha}} \tilde{g}(s)\tilde{h}(s)\,\mathrm{d}\langle \tilde{M}\rangle_s = (2H-2)c_2^2 \int_0^{(Bd)^{1/\alpha}} \tilde{g}(s)\tilde{h}(s)s^{1-2H}\,\mathrm{d}s.$$

With every function $\tilde{g} \in \tilde{L}^2_H$ one can therefore associate a path $\tilde{\gamma} \in \tilde{\mathcal{H}}_B$ by setting

$$\tilde{\gamma}_s = \mathrm{E}\left(\tilde{X}_s \int_0^{(Bd)^{1/\alpha}} \tilde{g}(u)\,\mathrm{d}\tilde{M}_u\right) = \int_0^s \tilde{z}(s, u)\tilde{g}(u)\,\mathrm{d}\langle \tilde{M}\rangle_u. \tag{7.5}$$

Moreover, the relation (7.5) between $\tilde{L}^2_H$ and $\mathcal{H}$ is one to one. We remark that the Gaussian spaces generated by $(X_t)_{t\leqslant 1}$ and $(\tilde{X}_t)_{t\leqslant (Bd)^{1/\alpha}}$ obviously coincide. Thus, if Y is a Gaussian r.v. belonging to this common Gaussian space, one may associate with it the paths

$$\gamma_t = \mathrm{E}[X_t Y], \qquad 0 \leqslant t \leqslant 1,$$

$$\tilde{\gamma}_B(s) = \mathrm{E}[\tilde{X}_s \mathrm{Y}], \qquad 0 \leqslant s \leqslant (Bd)^{1/\alpha}.$$

Of course $\gamma \in \mathcal{H}$ and $\tilde{\gamma}_B \in \tilde{\mathcal{H}}_B$.

The relationship between γ and $\tilde{\gamma}_B$, can be made explicit. Actually

$$\tilde{\gamma}_B(s) = B\mathrm{E}[X_{s(Bd)^{-1/\alpha}} Y] = B\gamma(s(Bd)^{-1/\alpha}). \tag{7.6}$$

We consider the family of importance sampling estimators for the ruin probability of the form

$$1_{\{\tilde{\tau}_B \leqslant (Bd)^{1/\alpha}\}} \tilde{Z}_{\tilde{\tau}_B} \tag{7.7}$$

where the r.v.s $\tilde{X}_s$ are simulated under the distribution, $\tilde{Q}_B$, of a fractional Brownian motion translated with a path $\tilde{\gamma}_B$ of the form (7.6) for some $\gamma \in \mathcal{H}$, and $(Z_t)_t$ is the martingale (under $\tilde{Q}_B$) that gives the density of the change of probability, that is,

$$\tilde{Z}_t = \exp\left(-\int_0^t \tilde{g}_B(s)\,\mathrm{d}M_s + \frac{1}{2}\int_0^t \tilde{g}_B(s)^2\,\mathrm{d}\langle M\rangle_s\right),$$

where $\tilde{g}_B \in \tilde{L}^2_H$ represents the path $\tilde{\gamma}_B$ as in (7.5). The second-order moment of this estimator is

$$\tilde{m}_2(B) = \mathrm{E}\left[1_{\{\tilde{\tau}_B \leqslant (Bd)^{1/\alpha}\}} \exp\left(-\int_0^t \tilde{g}_B(s)\,\mathrm{d}M_s + \frac{1}{2}\int_0^t \tilde{g}_B(s)^2\,\mathrm{d}\langle M\rangle_s\right)\right].$$

Similarly to the definition given in Section 4, we say that the estimators defined through a family of paths $(\tilde{\gamma}_B)_B$ is *asymptotically efficient* if

$$\limsup_{B\to+\infty} B^{-2+2H/\alpha} \log \tilde{m}_2(B) = -2\tilde{i}.$$

Lemma 7.2. *Let X be a fractional Brownian motion and $\tilde{X}_t = \lambda^{-H} X_{\lambda t}$, so that $\tilde{X}$ is still a fractional Brownian motion. If $g \in L^2_H([0, T])$, then*

$$\int_0^T g(s)\,\mathrm{d}X_s = \lambda^H \int_0^{T/\lambda} g(\lambda s)\,\mathrm{d}\tilde{X}_s \qquad \text{almost surely.}$$

Proof. As all the r.v.s considered belong to the Gaussian space generated by $(X_t)_t$, it is sufficient to prove that the two r.v.s above have the same covariance with respect to each of the r.v.s X_t, $t > 0$, which is straightforward. □

From this lemma one obtains the relation between the fundamental martingales of X and $\tilde{X}$. These are defined by

$$M_t = \int_0^t w(t, s)\,\mathrm{d}X_s, \qquad \tilde{M}_t = \int_0^t w(t, s)\,\mathrm{d}\tilde{X}_s.$$

Thus $M_t = \lambda^{1-H}\tilde{M}_{t/\lambda}$ and

$$\int_0^t g(s)\,\mathrm{d}M_s = \lambda^{1-H}\int_0^{t/\lambda} g(\lambda s)\,\mathrm{d}\tilde{M}_s \tag{7.8}$$

for every function $g \in L^2(\mathrm{d}\langle M\rangle_s)$. Let us define

$$\tau_\varepsilon = \inf\{t;\ \varepsilon X_t - kt^\alpha > 1\}, \qquad \tilde{\tau}_B = \inf\{s;\ \tilde{X}_s - cs^\alpha > B\},$$

with $\varepsilon = B^{-1+H/\alpha}d^{H/\alpha}$, $k = cd$. It immediately follows that $\tau_\varepsilon = (Bd)^{-1/\alpha}\tilde{\tau}_B$. Moreover, thanks to formula (7.8) with $\lambda = (Bd)^{-1/\alpha}$, for every $g \in L^2(\mathrm{d}\langle M\rangle_s)$,

$$\int_0^{\tau_\varepsilon} g(t)\,\mathrm{d}M_t = \int_0^{(Bd)^{-1/\alpha}\tilde{\tau}_B} g(t)\,\mathrm{d}M_t = (Bd)^{-(1-H)/\alpha}\int_0^{\tilde{\tau}_B} g(t(Bd)^{-1/\alpha})\,\mathrm{d}\tilde{M}_t,$$

whereas

$$\int_0^{\tau_\varepsilon} g(t)^2\,\mathrm{d}\langle M\rangle_t = (Bd)^{-(2-2H)/\alpha}\int_0^{\tilde{\tau}_B} g(s(Bd)^{-1/\alpha})^2\,\mathrm{d}\langle\tilde{M}\rangle_s.$$

Therefore, recalling that $\varepsilon = B^{-1+H/\alpha}d^{H/\alpha}$,

$$\begin{aligned}
&-\frac{1}{\varepsilon}\int_0^{\tau_\varepsilon} g(t)\,\mathrm{d}M_t + \frac{1}{2\varepsilon^2}\int_0^{\tau_\varepsilon} g(t)^2\,\mathrm{d}\langle M\rangle_t \\
&\quad = -B^{1-1/\alpha}d^{-1/\alpha}\int_0^{\tilde{\tau}_B} g(s(Bd)^{-1/\alpha})\,\mathrm{d}\tilde{M}_s + \frac{1}{2}B^{2-2/\alpha}d^{-2/\alpha}\int_0^{\tilde{\tau}_B} g(s(Bd)^{-1/\alpha})^2\,\mathrm{d}\langle\tilde{M}\rangle_s.
\end{aligned}$$

Thus, if we define $\tilde{g}_B(s) = B^{1-1/\alpha}d^{-1/\alpha}g(s(Bd)^{-1/\alpha})$, the two r.v.s

$$\begin{gathered}
1_{\{\tau_\varepsilon \leqslant 1\}}\exp\left(-\frac{1}{\varepsilon}\int_0^{\tau_\varepsilon} g(t)\,\mathrm{d}M_t + \frac{1}{2\varepsilon^2}\int_0^{\tau_\varepsilon} g(t)^2\,\mathrm{d}\langle M\rangle_t\right), \\
1_{\{\tilde{\tau}_B \leqslant (Bd)^{1/\alpha}\}}\exp\left(-\int_0^{\tilde{\tau}_B}\tilde{g}_B(s)\,\mathrm{d}\tilde{M}_s + \frac{1}{2}\int_0^{\tilde{\tau}_B} g_B(s)^2\,\mathrm{d}\langle\tilde{M}\rangle_s\right)
\end{gathered} \tag{7.9}$$

are P-a.s. equal. Thanks to (7.9), we have $\tilde{m}_2(B) = m_2(\varepsilon)$ and

$$\lim_{B\to\infty} B^{2-2H/\alpha} \log \tilde{m}_2(B) = \mathrm{d}^{2H/\alpha} \lim_{\varepsilon\to 0} \varepsilon^2 \log m_2(\varepsilon).$$

Comparing with (7.3) and (7.4) we have proved the following proposition:

Proposition 7.3. *The family of translations* $(\tilde{\gamma}_B)_B$ *is asymptotically efficient for the probability* (1.1) *if and only if* γ *is asymptotically efficient for the probability* (1.2).

8. Asymptotic efficiency for the ruin problem

Michna (1999) investigates the efficiency of the importance sampling estimator for the ruin probability $P(\sup_{0<t\leqslant 1}(X_t - ct) > B)$ obtained by a translation with a path of the form $\gamma_t = at$. He studied, using extensive simulations, the efficiency of importance sampling distributions of this kind. This choice corresponds, in the notation introduced above, to $\tilde{g}_B \equiv 1$ and $g \equiv 1$. Asymptotic efficiency is achieved if and only if

$$\inf_{0<t\leqslant 1} H_\gamma(t) = \inf_{0<t\leqslant 1} \frac{(1+kt)^2}{t^{2H}}.$$

Thanks to Proposition 4.2 we know that this cannot be (see Figure 6) as, in order to produce an asymptotically efficient translation, the path γ must be of the form $\gamma_t = a^* K(t^*, t)$, for $t \leqslant t^*$. One can, however, search for the best value of a and check whether this estimator is far from asymptotical efficiency.

If $\gamma_t = at$, $v_t = a^2 \mathrm{E}(\mathrm{E}(M_1 \mid \mathcal{F}_t)^2) = a^2 \mathrm{E}(M_t^2) = a^2 c_2^2 t^{2-2H}$. Therefore the master function is

$$H_\gamma(t) = -a^2 c_2^2 t^{2-2H} + \frac{1}{2t^{2H}}(1+(k+a)t)^2$$

and one must determine the value of a such that $\inf_{0<t\leqslant 1} H_\gamma(t)$ is largest. It can be shown that in this case sup and inf can be interchanged and

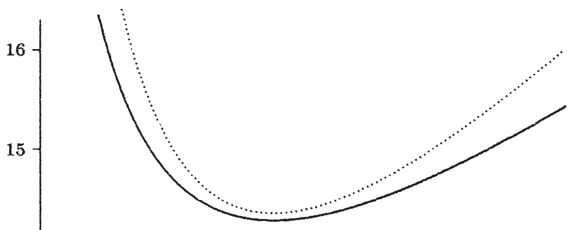

Figure 6. Graphs of $t \to (1+kt)^2 t^{-2H}$ (dotted line) and of H_γ (solid line), with $\gamma_t = at$. It is apparent that the infimum of the latter is strictly smaller. Thus the translation is not asymptotically efficient. Here $H = 0.6$, $k = 3$, $a = (1+kt^*)/((2c_2^2 - 1)t^*) = 4.95$; the infimum of H_γ is equal to 14.29 instead of 14.36 for asymptotic efficiency.

$$\max_{a>0}\left\{-a^2c_2^2t^{2-2H}+\frac{1}{2t^{2H}}(1+(k+a)t)^2\right\}=\frac{c_2^2}{2c_2^2-1}\frac{(1+kt)^2}{t^{2H}} \tag{8.1}$$

the maximum being attained at $a=a(t)=(1+kt)/((2c_2^2-1)t)$. The infimum of the right-hand side of (8.1) is attained at $t^*=H/(c(\alpha-H))$ and the value of the infimum is

$$\frac{c_2^2}{2c_2^2-1}\frac{(1+kt^*)^2}{t^{*2H}}=\frac{c_2^2}{2c_2^2-1}2i_\varphi.$$

This estimator would be asymptotically efficient but for the presence of the factor $c_2^2(2c_2^2-1)^{-1}$. The graph of c_2^2, in Figure 4 suggests that, for $\frac{1}{2}\leqslant H\leqslant 1$, c_2^2 is very close to 1, as well as $c_2^2(2c_2^2-1)^{-1}$, and the graph in Figure 7 confirms this fact. Therefore, for $\frac{1}{2}< H\leqslant 1$, this importance sampling estimator, while not asymptotically efficient, is not far from being so (see also Figure 6). If $H=0.6$, $c_2^2(2c_2^2-1)^{-1}=0.995$. It might, however, be interesting to check numerically whether this estimator is useful in practice for the usual values of the level B. Its performance is much poorer for values of H smaller than $\frac{1}{2}$.

The computations above allow us to determine the best value of a. Actually

$$a^*=\frac{1+kt^*}{(2c_2^2-1)t^*}=\frac{k}{(2c_2^2-1)H}.$$

The path $\gamma_t=a^*t$ has the lowest second-order moment among the linear paths for the simulation of the probability (1.2). The corresponding path for the simulation of the probability (1.1) is therefore, using (7.6), $\tilde{\gamma}_B(s)=c((2c_2^2-1)H)^{-1}s$. Numerical evidence in Michna (1999) indicates that, for $H=0.6$ and $c=1$, the best path of this form is $\gamma_t=at$ with a between 1.5 and 1.9 – beware that a corresponds to $1+a$ in Michna (1999). The computation above gives $a=((2c_2^2-1)H)^{-1}=1.65$.

Remark 8.1. If the translation $\gamma_t=at$ is not asymptotically efficient, one can investigate whether it is optimal, in the sense of the second point of Remark 6.1. The answer to this question is also negative, as we are able to produce a translation γ such that $\inf_{0<t\leqslant 1}H_\gamma(t)$ is larger: let γ be a path associated to an r.v. of the form $Z=a^*X_{t^*}+Y$, with Y independent of $\mathcal{F}_{t^*}$. Thus Y is of the form $Y=\int_{t^*}^1 g(s)\,\mathrm{d}M_s$ with $g\in L_H^2$. Z being of the form above,

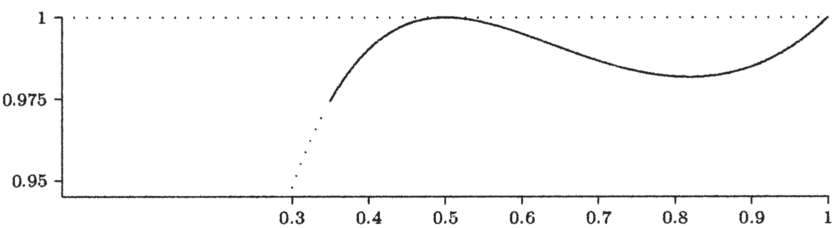

Figure 7. Graph of $c_2^2(2c_2^2-1)^{-1}$ as a function of H. Between $\frac{1}{2}$ and 1 the minimum is attained at $\simeq 0.82$ and takes a value $\simeq 0.9817$.

$$\gamma_t = \mathrm{E}[X_t(a^* X_{t^*} + Y)] = \begin{cases} a^* \mathrm{E}(X_t X_{t^*}) = a^* K(t^*, t), & \text{if } t \leqslant t^*, \\ a^* K(t^*, t) + \int_{t^*}^t \tilde{z}(t, s) g(s)\, \mathrm{d}\langle M \rangle_s, & \text{if } t > t^*. \end{cases}$$

Assuming for simplicity that $H > \frac{1}{2}$, for $t \geqslant t^*$,

$$\begin{aligned} \gamma_t &= a^* K(t^*, t) + 2H(2-2H)(H-\tfrac{1}{2})c_2^2 \int_{t^*}^t g(s) s^{1-2H}\, \mathrm{d}s \int_s^t u^{H-1/2}(u-s)^{H-3/2}\, \mathrm{d}u \\ &= a^* K(t^*, t) + 2H(2-2H)(H-\tfrac{1}{2})c_2^2 \int_{t^*}^t u^{H-1/2} \int_{t^*}^u g(s) s^{1-2H}(u-s)^{H-3/2}\, \mathrm{d}s. \end{aligned}$$

The corresponding conditional variance $v_t = \mathrm{E}(\mathrm{E}(Z \mid \mathcal{F}_t)^2)$ takes the form

$$v_t = \begin{cases} a^*(2-2H)c_2^2 \int_0^t \tilde{z}(t, s)^2 s^{1-2H}\, \mathrm{d}s, & \text{if } t \leqslant t^*, \\ v_t = a^* K(t^*, t^*) + (2-2H)c_2^2 \int_{t^*}^t g_s^2 s^{1-2H}\, \mathrm{d}s, & \text{if } t \geqslant t^*. \end{cases}$$

If g is of the form $g(s) = c_0(s-t^*)^\beta s^{2H-1}$ for some values of c_0, β to be chosen later, one obtains

$$\gamma_t = a^* K(t^*, t) + \frac{c_0 \Gamma(\beta+1)\Gamma(\frac{3}{2}-H)}{\Gamma(H+\beta+\frac{1}{2})\Gamma(2-2H)} \int_{t^*}^t u^{H-1/2}(u-t^*)^{H+\beta-1/2}\, \mathrm{d}u \tag{8.2}$$

for $t \geqslant t^*$, whereas, for the same values of t,

$$v_t = a^* K(t^*, t^*) + (2-2H)c_0^2 c_2^2 \int_{t^*}^t (s-t^*)^{2\beta} s^{2H-1}\, \mathrm{d}s.$$

The master function H_γ is easily computed numerically. Figures 8 and 9 show that, for a good choice of c_0 and β, γ is closer to asymptotic efficiency than Michna's estimator.

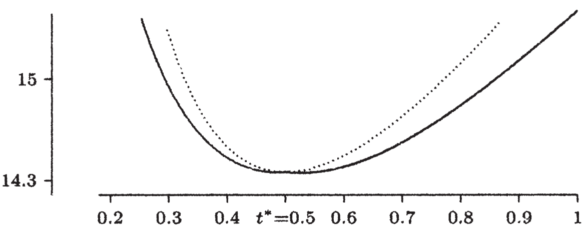

Figure 8. Graphs of $t \to (1+kt)^2 t^{-2H}$ (dotted) and H_γ (solid), again for $H = 0.6$, $k = 3$, with $\gamma = a^* K(t^*, t)$ for $t \leqslant t^*$ and defined as in (8.2) for $t > t^*$. Here $c = 4.7$ and $\beta = 0.01$. The infimum of H_γ is now equal to 14.3527, which is very close to asymptotic efficiency (the required value is still 14.3588). The minimum is attained at $t = 0.526$.

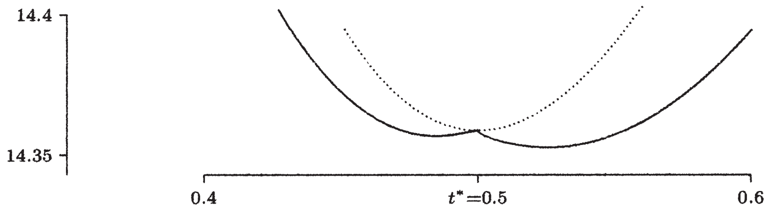

Figure 9. Enlargement of Figure 8, showing the behaviour near t^*. Recall that, at the left of t^*, the path is $\gamma_t = \alpha^* K(t^*, t)$.

9. Some basic facts

In this section we recall some well-known facts about continuous Gaussian processes. Many references are available in the literature; we refer mainly to Lifshits (1995). We prove also an extension of Varadhan's lemma that is needed in the proofs in Section 4.

9.1. Girsanov's theorem

If $w \in \mathcal{H}_t$ and $Z \in \mathcal{C}'_{t,P}$ is the corresponding r.v., then the r.v.

$$\mathrm{e}^{Z - \frac{1}{2}|w|^2_{\mathcal{H}_t}} = \mathrm{e}^{Z - \frac{1}{2}\|Z\|^2_{L^2}}$$

has mean 1 with respect to P, so that $\mathrm{d}Q = \mathrm{e}^{Z - \frac{1}{2}|w|^2_{\mathcal{H}_t}}\,\mathrm{d}P$ is a probability measure on $\mathcal{C}_t$. Moreover, if we denote by P_w the probability obtained by a translation by w, then it is easy to see that $Q = P_w$.

9.2. Restriction of an RKHS path

If a path γ belongs to $\mathcal{H}_T$ and $t \leqslant T$, then one can consider its restriction to $[0, t]$ and inquire whether it belongs to $\mathcal{H}_t$.

Proposition 9.1. *Let $\gamma \in \mathcal{H}_T$ and $t \leqslant T$, and denote by $\tilde{\gamma}$ the restriction of γ to* $[0, t]$. *Then $\tilde{\gamma} \in \mathcal{H}_t$ and there exists a unique path $\gamma^{(t)} \in \mathcal{H}_T$ coinciding with γ up to time t and such that*

$$|\gamma^{(t)}|^2_{\mathcal{H}_T} = |\tilde{\gamma}|^2_{\mathcal{H}_t}, \tag{9.1}$$

$$|\gamma^{(t)}|^2_{\mathcal{H}_T} = \inf |w|^2_{\mathcal{H}_T}, \tag{9.2}$$

the infimum being taken over all paths $w \in \mathcal{H}_T$ whose restriction to $[0, t]$ *coincides with $\tilde{\gamma}$. Moreover, $\gamma^{(t)}$ is the unique path at which this infimum is attained.*

Proof. Let $Z \in \mathcal{C}'_{T,P}$ be the r.v. corresponding to $\gamma \in \mathcal{H}_T$. Thus $\gamma_s = \mathrm{E}(X_s Z)$. The r.v. $\zeta_t = \mathrm{E}(Z \mid \mathcal{F}_t)$ is still Gaussian, as $\{Z, X_u, u \leqslant t\}$ is a Gaussian family. The corresponding path, $\gamma^{(t)}$ say, enjoys the property that, for $s \leqslant t$,

$$\gamma^{(t)}_s = \mathrm{E}(X_s \tilde{Z}) = \mathrm{E}(X_s Z) = \gamma_s,$$

that is, it coincides with γ up to time t. However, it immediately follows that $\zeta_t \in \mathcal{C}'_{P,t}$ and that the path in $\mathcal{H}_t$ corresponding to ζ_t is the restriction $\tilde{\gamma}$ of γ to $[0, t]$. This immediately gives (9.1), as both sides are equal to $\mathrm{E}(\zeta_t^2)$. Moreover, if $w \in \mathcal{H}_T$ is any path coinciding with γ up to time t, by repeating the previous argument, since its corresponding Gaussian r.v. Z_w satisfies $w_s = \mathrm{E}(Z_w X_s)$, we have, for $s \leqslant t$, $\mathrm{E}(Z_w \mid \mathcal{F}_t) = \mathrm{E}(Z \mid \mathcal{F}_t) = \zeta_t$; therefore $Z_w = \tilde{\zeta}_t + W$, for some r.v. $W \in \mathcal{C}'_{T,P}$ independent of $\mathcal{F}_t$. Thus

$$|\gamma^{(t)}|^2_{\mathcal{H}_T} = \mathrm{E}(\zeta_t^2) \leqslant \mathrm{E}(\tilde{Z}^2) + \mathrm{E}(W^2) = \mathrm{E}(Z_w^2) = |w|^2_{\mathcal{H}_T}, \tag{9.3}$$

which gives (9.2). Moreover, in (9.3) equality holds if and only if $W = 0$, that is, if and only if $w = \gamma^{(t)}$. □

9.3. Large deviations

Let $(\mathcal{C}_T, \mathcal{F}_T, P)$ be a Gaussian process as in Section 3 and let P^ε be the probability obtained by scaling with parameter ε: $P^\varepsilon(A) = P(\varepsilon^{-1}A)$. Then the following large-deviations result is well known: for every closed set $F \subset \mathcal{C}_t$ and every open set $G \subset \mathcal{C}_t$,

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon(F) \leqslant -\inf_{w\in \mathrm{F}} I(w), \qquad \liminf_{\varepsilon\to 0} \varepsilon^2 \log P^\varepsilon(G) \geqslant -\inf_{w\in G} I(w), \tag{9.4}$$

where $I(w) = \frac{1}{2}|w|^2_{\mathcal{H}_t}$ if $w \in \mathcal{H}_t$ and $I(w) = +\infty$ otherwise (see Lifshits 1995: Chapter 12).

9.4. An extension of Varadhan's lemma

Varadhan's lemma states that if the family of probability measures $(\mu_\varepsilon)_\varepsilon$ on the complete metric space (E, d) satisfies a large-deviation principle with rate function I and $f : E \to \mathbb{R}$ is a continuous function satisfying suitable tail properties, then, for a closed set F,

$$\lim_{\varepsilon\to 0} \varepsilon \log \int_F \mathrm{e}^{f(x)/\varepsilon}\, \mathrm{d}\mu_\varepsilon \leqslant \sup_{x\in F} [f(x) - I(x)].$$

In Section 4 we applied this result to the functional $-\zeta_\tau + \frac{1}{2}v_\tau$, which is not continuous in general. We now prove an extension of Varadhan's lemma and then show that actually the functional $-\zeta_\tau + \frac{1}{2}v_\tau$ satisfy its assumptions. The extension is quite natural and the proofs are straightforward revisitations of the classical ones (see Dembo and Zeitouni 1998: 139).

Definition 9.1. *Let $(\mu_\varepsilon)_\varepsilon$ be a family of probability measures on the complete metric space (E, d) satisfying a large-deviation principle with rate function I with speed ε as $\varepsilon \to 0$. A function $\Phi : E \to \mathbb{R}$ is said to be quasi-upper semicontinuous (quasi-lower semicontinuous) with respect to $(\mu_\varepsilon)_\varepsilon$ if for every $\rho > 0$, $R > 0$, $\eta > 0$ there exist $\delta > 0$, $\varepsilon_0 > 0$ such that for every $\varepsilon \leqslant \varepsilon_0$,*

$$\mu_\varepsilon(x;\ \Phi(x) > \Phi(\gamma) + \eta,\ d(x, \gamma) < \delta) \leqslant \mathrm{e}^{-R/\varepsilon}$$

$$(\mu_\varepsilon(x;\ \Phi(x) < \Phi(\gamma) - \eta,\ d(x, \gamma) < \delta) \leqslant \mathrm{e}^{-R/\varepsilon}) \tag{9.5}$$

uniformly for $I(\gamma) \leqslant \rho$. Φ is said to be quasi-continuous if it is quasi-upper semicontinuous and quasi-lower semicontinuous.

The extension of Varadhan's lemma is the following.

Theorem 9.2. *Assume that the family of probability measures $(\mu_\varepsilon)_\varepsilon$ on the complete metric space (E, d) satisfies a large-deviation principle with rate function I and speed ε as $\varepsilon \to 0$. Let $F \subset E$ be a closed ($G \subset E$ an open) set and $\Phi : \mathrm{E} \to \mathbb{R}$ a function such that:*

(i) *For every $R > 0$ there exist $M > 0$, $\varepsilon_0 > 0$ such that for every $\varepsilon \leqslant \varepsilon_0$,*

$$\int \mathrm{e}^{\Phi(x)/\varepsilon} 1_{\{|\Phi(x)|>M\}} \, \mathrm{d}\mu_\varepsilon \leqslant \mathrm{e}^{-R/\varepsilon}. \tag{9.6}$$

(ii) *Φ is quasi-upper (quasi-lower) semicontinuous with respect to $(\mu_\varepsilon)_\varepsilon$.*

Then

$$\limsup_{\varepsilon\to 0} \varepsilon \log \int \mathrm{e}^{\Phi(x)/\varepsilon} 1_F \, d\mu_\varepsilon \leqslant \sup_{x\in F} [\Phi(x) - I(x)]$$

$$\left(\liminf_{\varepsilon\to 0} \varepsilon \log \int \mathrm{e}^{\Phi(x)/\varepsilon} 1_G \, d\mu_\varepsilon \geqslant \sup_{x\in G} [\Phi(x) - I(x)] \right) \tag{9.7}$$

We now prove that the functional $-\zeta_\tau + \frac{1}{2} v_\tau$ appearing in (4.3) satisfies the assumptions of Theorem 9.2 with respect to the family $(P^\varepsilon)_\varepsilon$. First, we take care of (9.6). The term v_τ is bounded. If $\zeta^* = \sup_{0\leqslant t\leqslant 1} |\zeta_t|$, then obviously $\zeta^* \geqslant -\zeta_\tau$, and it is sufficient to prove (9.6) for $\Phi = \zeta^*$. Since $(\zeta_t)_t$ is a continuous Gaussian process, the r.v. ζ^* is finite and Borell's inequality (see van der Vaart and Wellner 1996) gives

$$P(\zeta^* \geqslant r) \leqslant 2\mathrm{e}^{-c^* r^2}$$

for some $c^* > 0$ (actually one can choose $c^* = (8\mathrm{E}[(\zeta^*)^2])^{-1}$).

Lemma 9.3. *For every $R > 0$ there exists $M > 0$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon^2 \log \int_{\mathcal{C}} \exp(\varepsilon^{-2}\zeta(x)) 1_{\{x;\zeta^*(x)>M\}} \, \mathrm{d}P^\varepsilon(x) \leqslant -R.$$

Proof. Observe, first, that the distribution of ζ^* under P_ε is the same as the distribution of $\zeta^*(\varepsilon\cdot)$ under P, and that $\zeta^*(\varepsilon x) = \varepsilon\zeta^*(x)$ P-a.s. Therefore,

$$\int_{\mathcal{C}} \exp(\varepsilon^{-2}\zeta^*(x)) 1_{\{x;\zeta^*(x)>M\}} \, \mathrm{d}P^\varepsilon(x) = \int_{\mathcal{C}} \exp(\varepsilon^{-1}\zeta^*(x)) 1_{\{x;\varepsilon\zeta^*(x)>M\}} \, \mathrm{d}P(x)$$

$$\leqslant P\left(\zeta^* \geqslant \frac{M}{\varepsilon}\right)^{1/2} \mathrm{E}(\exp(2\varepsilon^{-1}\zeta^*))^{1/2}. \tag{9.10}$$

By Borell's inequality,

$$
\begin{aligned}
\mathrm{E}(\exp(2\varepsilon^{-1}\zeta^*)) &= \int_0^{+\infty} P(\exp(2\varepsilon^{-1}\zeta^*) \geqslant r)\,\mathrm{d}r \leqslant 1 + \int_1^{+\infty} P(\exp(2\varepsilon^{-1}\zeta^*) \geqslant r)\mathrm{d}r \\
&= 1 + \int_0^{+\infty} P(\exp(2\varepsilon^{-1}\zeta^*) \geqslant \mathrm{e}^t)\mathrm{e}^t\,\mathrm{d}t = 1\int_0^{+\infty} P\Big(\zeta^* \geqslant \frac{\varepsilon}{2}t\Big)\mathrm{e}^t\,\mathrm{d}t \\
&\leqslant 1 + \int_0^{+\infty} \exp\Big(-\frac{c^*\varepsilon^2}{4}t^2 + t\Big)\,\mathrm{d}t \\
&\leqslant 1 + \exp\Big(\frac{1}{c^*\varepsilon^2}\Big)\int_{-\infty}^{+\infty} \exp\bigg(-\frac{c^*\varepsilon^2}{4}\Big(t - \frac{2}{c^*\varepsilon^2}\Big)^2\bigg)\,\mathrm{d}t \\
&= 1 + \frac{1}{\varepsilon}\exp\Big(\frac{1}{c^*\varepsilon^2}\Big)\sqrt{\frac{\pi}{c^*}}.
\end{aligned}
$$

Going back to (9.10),

$$
\int_{\mathcal{C}} \exp(\varepsilon^{-2}\zeta^*(x))1_{\{x;\zeta^*(x)>M\}}\,\mathrm{d}P^\varepsilon(x) \leqslant \bigg(1 + \frac{1}{\varepsilon}\exp\Big(\frac{1}{c^*\varepsilon^2}\Big)\sqrt{\frac{\pi}{c^*}}\bigg)^{1/2}\exp\Big(-\frac{c^*M^2}{2\varepsilon^2}\Big),
$$

which gives

$$
\limsup_{\varepsilon\to 0}\varepsilon^2\log\int_{\mathcal{C}} \exp(\varepsilon^{-2}\zeta^*(x))1_{\{x;\zeta^*(x)>M\}}\,\mathrm{d}P^\varepsilon(x) \leqslant \frac{1}{2c^*} - \frac{c^*M^2}{2},
$$

allowing us to conclude. □

Let us now check that the functional $-\zeta_\tau + \frac{1}{2}v_\tau$ is quasi-continuous. v_τ is continuous almost surely, since τ is continuous P^ε-a.s. for every $\varepsilon > 0$, as proved below in Proposition 9.7. The following statement gives an elementary, consequence of the CGM property (see Assumption 4.1).

Lemma 9.4. *If the CGM property is satisfied, then $t \to \gamma^{(t)}$ is continuous from $[0, T]$ to $\mathcal{H}_T$.*

Proof. It is sufficient to remark that, for $s \leqslant t$, $|\gamma^{(t)} - \gamma^{(s)}|^2_{\mathcal{H}_T} = \mathrm{E}[(\mathrm{E}(Z\,|\,\mathcal{F}_t) - \mathrm{E}(Z\,|\,\mathcal{F}_s))^2]$ and use the fact that for Gaussian r.v.s almost sure convergence implies convergence in L^2. □

The quasi-continuity of ζ_τ follows from the following statement. Recall that $\zeta_t = \mathrm{E}(Z\,|\,\mathcal{F}_t)$, where Z is the Gaussian r.v. associated with the path $\gamma \in \mathcal{H}$ and that we assume that $(\zeta_t)_t$ is continuous.

Proposition 9.5. *For every $\rho > 0$, $R > 0$, $\eta > 0$ there exist $\varepsilon_0 > 0$, $\delta > 0$ such that, for every $w \in \mathcal{H}$,*

$$P\left(x;\ \sup_{0\leqslant t\leqslant 1}|\varepsilon\zeta_t(x)-\langle w,\gamma^{(t)}\rangle_{\mathcal{H}}|\geqslant\eta,\ \|\varepsilon x-w\|_\infty\leqslant\delta\right)\leqslant \mathrm{e}^{-R/\varepsilon^2},\tag{9.11}$$

for every $\varepsilon\leqslant\varepsilon_0$, uniformly for $|w|_{\mathcal{H}}\leqslant\rho$.

Proof. The two-dimensional process $(X_t,\zeta_t)_t$ takes its values in $\mathcal{C}([0,1],\mathbb{R}^2)$ and is Gaussian. It is easy to check that its RKHS $\tilde{\mathcal{H}}$ is formed by the paths (w,g), where $w\in\mathcal{H}$ and $g_t=\langle w,\gamma^{(t)}\rangle_{\mathcal{H}}$, with the norm $|(w,g)|_{\tilde{\mathcal{H}}}=|w|_{\mathcal{H}}$. The rest of the proof consists in majorizing the left-hand side in (9.11) using large deviations. Actually (9.11) is more or less equivalent to the the large-deviations estimates for $(X_t,\zeta_t)_t$, as developed in Baldi and Sanz (1991), whose arguments we reproduce here. The estimates (9.4) give

$$\limsup_{\varepsilon\to 0}P\left(x;\ \sup_{0\leqslant t\leqslant 1}|\varepsilon\zeta_t(x)-\langle w,\gamma^{(t)}\rangle_{\mathcal{H}}|\geqslant\eta,\ \|\varepsilon x-w\|_\infty\leqslant\delta\right)\leqslant -\inf\tfrac{1}{2}|(y,g)|^2_{\tilde{\mathcal{H}}},\tag{9.12}$$

the infimum being taken among those paths (y,g) such that $\|y-w\|_\infty\leqslant\delta$ and $\|g-\langle w,\gamma^{(\cdot)}\rangle_{\mathcal{H}}\|_\infty\geqslant\eta$. Let us fix $R>0$. Thanks to Lemma 9.6 below, for every $\eta>0$ there exists $\delta>0$ such that, if $|\tilde{w}|^2_{\mathcal{H}}<2R$ and $\|\tilde{w}-w\|_\infty\leqslant\delta$, then $\|\langle\tilde{w},\gamma^{(\cdot)}\rangle_{\mathcal{H}}-\langle w,\gamma^{(\cdot)}\rangle_{\mathcal{H}}\|_\infty\leqslant\eta$. With this choice of δ, if (y,g) satisfies $\|y-w\|_\infty\leqslant\delta$ and $\|g-\langle w,\gamma^{(\cdot)}\rangle_{\mathcal{H}}\|_\infty\geqslant\eta$, then $|(y,g)|^2_{\tilde{\mathcal{H}}}\geqslant 2R$ and the right-hand side in (9.12) is smaller than $-R$, which completes the proof. □

Lemma 9.6. *Let $K_R=\{w;\ I(w)\leqslant R\}$. If $w,w_n\in K_R$, $n\geqslant 1$ and $\|w_n-w\|_\infty\to 0$, then $w_n\to w$ in the weak topology of $\mathcal{H}$, and the map $w\to(t\to\langle w,\gamma^{(t)}\rangle_{\mathcal{H}})$ from K_R to $\mathcal{C}$, both endowed with the $\|\cdot\|_\infty$ topology, is continuous.*

Proof. The statement follows by the Lebesgue theorem if $w_n,w,\gamma\in\mathcal{H}$ are of the form

$$w_n(t)=\int K(t,s)\,\mathrm{d}\alpha_n(s),\qquad w(t)=\int K(t,s)\,\mathrm{d}\alpha(s),\qquad \gamma(t)=\int K(t,s)\,\mathrm{d}\beta(s),\tag{9.13}$$

with $\alpha,\alpha_n,\beta\in\mathcal{C}'$. Then remark that paths of the form (9.13) are dense.

As for the second part of the statement, let $(w_n)_n\subset K_R$ such that $\|w_n-w\|_\infty\to 0$. It is straightforward that the set $(\langle w_n,\gamma^{(\cdot)}\rangle_{\mathcal{H}})_n$ is equi-bounded and equi-continuous and, by the Arzelà–Ascoli theorem, relatively compact. Since we already know that $\langle w_n,\gamma^{(t)}\rangle_{\mathcal{H}}\to\langle w,\gamma^{(t)}\rangle_{\mathcal{H}}$ for every t, all the limit points in $\mathcal{C}$ of the sequence $(\langle w_n,\gamma^{(\cdot)}\rangle_{\mathcal{H}})_n$ must coincide with $\langle w,\gamma^{(\cdot)}\rangle_{\mathcal{H}}$ and the proof is complete. □

Finally, let, for $a>0$, $\tau_a(w)=\inf\{t\geqslant 0;\ X_t(w)\geqslant a\}$.

Proposition 9.7. *τ_a is a functional of the path w which is P^ε-a.s. continuous for every $\varepsilon>0$.*

Proof. τ_a is not a continuous functional of the path. It is, however, lower semi-continuous. Similarly, the stopping time $\tau'_a(w)=\inf\{t\geqslant 0;\ X_t(w)>a\}$ is upper semi-continuous. The statement then follows if $\tau_a=\tau'_a$ P^ε-a.s. Let $q>0$, then

$$P^\varepsilon(\tau_a \leqslant q < \tau'_a) = P^\varepsilon\left(\sup_{s \leqslant q} X_s = a\right) = 0$$

as, by Tsyrelson's theorem (see Lifshits 1995: 136) the r.v. $\sup_{s\leqslant q} X_s$ has a density. Thus the event $\{\tau_a < \tau'_a\}$ is negligible, being the countable union of the events $\{\tau_a \leqslant q < \tau'_a\}$, $q \in \mathbb{Q}$. □

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