HIGH FREQUENCY ASYMPTOTICS FOR WAVELET-BASED TESTS FOR GAUSSIANITY AND ISOTROPY ON THE TORUS

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Abstract

We prove a multivariate CLT for skewness and kurtosis of the wavelets coefficients of a stationary field on the torus. The results are in the framework of the fixed-domain asymptotics, i.e. we refer to observations of a single field which is sampled at higher and higher frequencies. We consider also studentized statistics for the case of an unknown correlation structure. The results are motivated by the analysis of high-frequency financial data or cosmological data sets, with a particular interest towards testing for Gaussianity and isotropy.

Running title High Frequency Asymptotics on the Torus.

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1 Introduction

In recent years, a growing interest has been drawn by infill (or fixed-domain, or high-resolution) asymptotics, i.e., the investigation of the limiting behaviour of statistics of a single observation of a stochastic processes on a fixed time span or of a random field on a compact space, observed at a greater and greater resolution as the sample size increases (see (Stein 1999b) e.g.).

Such interest was mainly stimulated by various fields of applications. For stochastic processes defined on subsets of $\mathbb{R}$, the leading motivations have come from finance, where it is now customary to observe high-frequency or ultra-high frequency data sets collected in a short amount of time (even a single day). In such cases, clearly, the

standard asymptotic framework, envisaging an ever-increasing time span of observations, can be highly misleading and a fixed-domain approach seems a valid alternative. Many important contributions have focussed on asymptotic statistical inference on discretely observed diffusions, see for instance Kessler and Sørensen (Kessler and Sørensen 1999), Sørensen and Uchida (Sørensen and Uchida 2003).

Our interest here, however, is not on diffusions, but rather on stationary processes, as for instance in Stein (Stein 1999a) and (Stein 2002). In this area, strong interest has been brought in by the cosmological and astrophysical literature (see e.g. (Angers and Kim 2005) and (Meinshausen and Rice 2006)), a particularly active field being the analysis of Cosmic Microwave Background radiation. The latter can be viewed as a relic of the Big Bang which provides a snapshot of the Universe some 13,7 billion years ago; as such it is considered a goldmine of information on fundamental physics. A widely debated issue relates to the probability law of such radiation: this has fuelled considerable activity on testing for Gaussianity on isotropic random fields defined on $\mathbb{S}^2$, the sphere in three-dimensional Euclidean space. Wavelet analysis has also been proposed here, for instance by Vielva et al. (Vielva, Martínez-González, Barreiro, Sanz and Cayón 2004), Jin et al. (Jin, Starck, Donoho, Aghanim and Forni 2005), Cabella et al. (Cabella, Hansen, Marinucci, Pagano and Vittorio 2004). See also (Marinucci 2006b) for rigorous results on non Gaussianity testing, but based on the angular bispectrum.

Wavelet analysis has proved to be a powerful tool in this kind of problems, owing to their good localization properties. To the best of our knowledge, however, no theoretical analysis exists so far in the literature on the asymptotic (high-resolution) behaviour of the random wavelets coefficients for random fields on bounded domains. In this paper we focus on isotropic random fields on the torus $\mathbb{S}^1$; we define a suitable wavelet expansion and we derive the correlation structure of the random wavelets coefficients. The wavelet setting that is exploited here is based on the Littlewood-Paley construction. It produces a frame, instead of an orthonormal basis, but it is very stable and expands on the Fourier basis using only a finite number of coefficients. Therefore it is by essence localized in frequency. It is also localized in the time domain. This property is extremely useful in the presence of gaps in the observations.

We use these results to establish asymptotic independence of the coefficients, leading to a multivariate central limit theorem for the skewness and kurtosis statistics. These are useful to investigate the Gaussianity of the field. As explained before, the asymptotic theory is clearly developed in the high-resolution sense. The results are then extended to the case where the correlation structure of the field is unknown and estimated from the data.

One of the main interests of this construction is also that it generalizes very easily to the d-dimensional case and, although we present the calculation on the circle, it trivially extends to the d-dimensional torus. It is certainly desirable to extend our results, for instance, to bounded subsets of $\mathbb{R}$ or to $\mathbb{S}^2$; such extensions are currently under investigation. However also the present results are of a practical interest. Indeed, random fields on the circle are the natural environment for many geophysical models, for instance concerning atmospheric data (Niu and Tiao 1995). Also, it is not unusual in the CMB literature to approximate $\mathbb{S}^2$ as the union of copies of $\mathbb{S}^1$, the so-called ring-torus approach (Wandelt and Hansen 2003). Finally, our assumptions can be used for models of stationary time series on a fixed domain.

The plan of the paper is as follows. In §2 we review the Petrushev construction of needlets on general spaces; in §3 we introduce random fields on the torus and the associ-

ated random wavelets coefficients; a fundamental bound is established on the covariances of the latter. §5 is devoted to the multivariate Central Limit Theorem for the Skewness and Kurtosis statistics of these random wavelets coefficients. This result is extended to studentized statistics in §6, whereas §7 is devoted to the investigation of the aliasing effect, that is, the discretized sampling of the continuous random field of interest. §8 presents some Monte Carlo evidence. In the sequel, we use c to denote a positive constant, which need not be the same from line to line.

2   Petrushev construction of Needlets

Frames were introduced in the 1950's (Duffin and Schaeffer 1952) to represent functions via over-complete sets. Frames including tight frames arise naturally in wavelet analysis on $\mathbb{R}^d$. Tight frames which are very close to orthonormal bases are especially useful in signal and image processing.

We shall see that the following construction has the advantage of being easily computable and of producing well localized tight frames constructed on a specified orthonormal basis.

Definition 2.1 *Let $\mathbb{H}$ be a Hilbert space, and (e_n) a sequence in $\mathbb{H}$; (e_n) is a tight frame (with constant 1) if :*

$$\forall f \in \mathbb{H}, \ \|f\|^2 = \sum_n |\langle f, e_n \rangle|^2$$

Let now $\mathscr{Y}$ be a metric space endowed with a finite measure μ. Assume that the following decomposition holds

$$\mathbb{L}_2(\mathscr{Y}, \mu) = \bigoplus_{l=0}^{\infty} H_l$$

where the H_l's are finite dimensional spaces. For the sake of simplicity, we suppose that H_0 is reduced to the constants. Let L_l be the orthogonal projection on H_l :

$$\forall f \in \mathbb{L}_2(\mathscr{Y}, \mu), \qquad L_l(f)(x) = \int_{\mathscr{Y}} f(y) L_l(x, y) d\mu(y)$$

where

$$L_l(x, y) = \sum_{i=1}^{m_l} e_i^l(x) \bar{e}_i^l(y)$$

m_l is the dimension of H_l and $(e_i^l)_{i=1,\dots,m_l}$ an orthonormal basis of H_l. Let us observe that we have the following property of the projection operators:

$$\int L_l(x, y) L_m(y, z) \, d\mu(y) = \delta_{l,m} L_l(x, z) \ . \tag{2.1}$$

The following construction, also inspired by (Frazier, Jawerth and Weiss 1991), is based on two fundamental steps: Littlewood-Paley decomposition and discretization, which are summarized in the two following subsections.

2.1 Littlewood -Paley decomposition

Let ϕ be a C^∞ function supported in $|\xi| \leq 1$, such that $1 \geq \phi(\xi) \geq 0$ and $\phi(\xi) = 1$ if $|\xi| \leq \frac{1}{2}$. Let us define:

$$a^2(\xi) = \phi(\tfrac{\xi}{2}) - \phi(\xi) \geq 0$$

so that

$$\forall |\xi| \geq 1, \qquad \sum_j a^2(\tfrac{\xi}{2^j}) = 1 . \tag{2.2}$$

Actually in the previous sum all middle terms cancel telescopically. Let us define the operator

$$\Lambda_j = \sum_{l \geq 0} a^2(\tfrac{l}{2^j}) L_l$$

and the associated kernel

$$\Lambda_j(x, y) = \sum_{l \geq 0} a^2(\tfrac{l}{2^j}) L_l(x, y) = \sum_{2^{j-1} < l < 2^{j+1}} a^2(\tfrac{l}{2^j}) L_l(x, y).$$

Then it holds:

Proposition 2.2

$$\forall f \in \mathbb{H}, \qquad f = \lim_{J \to \infty} L_0(f) + \sum_{j=0}^{J} \Lambda_j(f) \tag{2.3}$$

and, if $M_j(x, y) = \sum_l a(\tfrac{l}{2^j}) L_l(x, y)$,

$$\Lambda_j(x, y) = \int M_j(x, z) M_j(z, y) d\mu(z) . \tag{2.4}$$

Proof.

$$L_0(f) + \sum_{j=0}^{J} \Lambda_j(f) = L_0(f) + \sum_{j=0}^{J} \sum_l a^2(\tfrac{l}{2^j}) L_l(f) = \sum_l \phi(\tfrac{l}{2^{J+1}}) L_l(f). \tag{2.5}$$

Therefore, as $\phi(\tfrac{l}{2^{J+1}}) = 1$ as soon as $2^J \geq l$,

$$\Big\| \sum_l \phi(\tfrac{l}{2^{J+1}}) L_l(f) - f \Big\|^2 = \sum_{l \geq 2^{J+1}} \| L_l(f) \|^2 + \sum_{2^J \leq l < 2^{J+1}} \big\| L_l(f)(1 - \phi(\tfrac{l}{2^{J+1}})) \big\|^2$$

$$\leq \sum_{l \geq 2^J} \| L_l(f) \|^2 \longrightarrow 0, \text{ as } J \to \infty.$$

(2.4) is a simple consequence of (2.1).

$\square$

2.2 Discretization

Let us define

$$\mathcal{K}_l = \bigoplus_{m=0}^{l} H_m,$$

and let us assume that some additional assumptions are true:

1. $f \in \mathscr{K}_l \Rightarrow \bar{f} \in \mathscr{K}_l$
2. $f \in \mathscr{K}_l, \ g \in \mathscr{K}_l \Rightarrow fg \in \mathscr{K}_{l+l}$
3. Quadrature formula: for every $l \in \mathbb{N}$ there exists a finite subset $\mathscr{X}_l \subset \mathscr{Y}$ and positive real numbers $\lambda_\eta > 0$, $\eta \in \mathscr{X}_l$, such that

$$\forall f \in \mathscr{K}_l, \quad \int f d\mu = \sum_{\eta \in \mathscr{X}_l} \lambda_\eta f(\eta). \tag{2.6}$$

Then the operator M_j defined in Proposition 2.2 is such that $M_j(x,z) = \overline{M_j(z,x)}$ and

$$z \mapsto M_j(x,z) \in \mathscr{K}_{2^{j+1}-1},$$

so that

$$z \mapsto M_j(x,z)M_j(z,y) \in \mathscr{K}_{2^{j+2}-2},$$

and we can write:

$$\Lambda_j(x,y) = \int M_j(x,z)M_j(z,y)\,d\mu(z) = \sum_{\eta \in \mathscr{X}_{2^{j+2}-2}} \lambda_\eta M_j(x,\eta)M_j(\eta,y).$$

This implies:

$$\Lambda_j f(x) = \int \Lambda_j(x,y)f(y)\,d\mu(y) = \int \sum_{\eta \in \mathscr{X}_{2^{j+2}-2}} \lambda_\eta M_j(x,\eta)M_j(\eta,y)f(y)\,d\mu(y)$$

$$= \sum_{\eta \in \mathscr{X}_{2^{j+2}-2}} \sqrt{\lambda_\eta} M_j(x,\eta) \int \overline{\sqrt{\lambda_\eta} M_j(y,\eta)} f(y)\,d\mu(y).$$

This can be summarized in the following way, if we set

$$\mathscr{X}_{2^{j+2}-2} = \mathscr{Z}_j, \qquad \psi_{j,\eta} := \sqrt{\lambda_\eta} M_j(x,\eta)$$

then

$$\Lambda_j f(x) = \sum_{\eta \in \mathscr{Z}_j} \langle f, \psi_{j,\eta} \rangle \psi_{j,\eta}(x).$$

Proposition 2.3 *The family* $(\psi_{j,\eta})_{j\in\mathbb{N},\eta\in\mathscr{Z}_j}$ *is a tight frame*

Proof. As

$$f = \lim_{J\to\infty} \left(L_0(f) + \sum_{j\leq J} \Lambda_j(f) \right)$$

$$\|f\|^2 = \lim_{J\to\infty} \left(\langle L_0(f), f \rangle + \sum_{j\leq J} \langle \Lambda_j(f), f \rangle \right)$$

but

$$\langle \Lambda_j(f), f \rangle = \sum_{\eta \in \mathscr{Z}_j} \langle f, \psi_{j,\eta} \rangle \langle \psi_{j,\eta}, f \rangle = \sum_{\eta \in \mathscr{Z}_j} |\langle f, \psi_{j,\eta} \rangle|^2$$

and if ψ_0 is a normalized constant such that $\langle L_0(f), f \rangle = |\langle f, \psi_0 \rangle|^2$, then

$$\|f\|^2 = |\langle f, \psi_0 \rangle|^2 + \sum_{j\in\mathbb{N},\eta\in\mathscr{Z}_j} |\langle f, \psi_{j,\eta} \rangle|^2.$$

$\square$

2.3 Localization properties

Petrushev and coauthors have analysed the previous construction proving that very nice localization properties hold.

In the case of the sphere of $\mathbb{R}^{d+1}$, where the spaces H_l are spanned by spherical harmonics, it is proved in (Narcowich, Petrushev and Ward 2006) the following localisation property: for any k there exists a constant c_k such that :

$$|\psi_{j\eta}(\xi)| \leq \frac{c_k 2^{dj/2}}{(1 + 2^j \arccos\langle\eta,\xi\rangle)^k} \ .$$

In the case of Jacobi polynomials on $[-1,1]$ with respect to the Jacobi weight, it is proved (Petrushev and Xu 2005) the following localization property: for any k there exist constants C, c such that :

$$|\psi_{j\eta}(\cos\theta)| \leq \frac{c 2^{j/2}}{(1 + (2^j |\theta - \arccos\eta|)^k \sqrt{w_{\alpha\beta}(2^j, \cos\theta)}} \ .$$

where $w_{\alpha\beta}(n,x) = (1 - x + n^{-2})^{\alpha+1/2}(1 + x + n^{-2})^{\beta+1/2}, -1 \leq x \leq 1$ if $\alpha > -\frac{1}{2}$, $\beta > -\frac{1}{2}$. In the following section, we consider the case, which is our framework, where $\mathscr{Y}$ is the torus, (e_k) is the Fourier basis, and $H_m = \text{Span}\{e_m\}$.

2.4 Quadrature Formula on the torus

Proposition 2.4 *Assume that $\mathscr{Y} = \mathbb{T}$, the torus. If, for $m \in \mathbb{N}$,*

$$\mathscr{K}_m = \Big\{ \sum_{|k| \leq m} a_k e^{ikx}, a_k \in \mathbb{C} \Big\},$$

the quadrature formula (2.6), holds for

$$\mathscr{X}_m = \Big\{ \frac{2l\pi}{m+1}, \ l \in \{0, \ldots, m\} \Big\}, \quad \lambda_{\frac{2l\pi}{m+1}} = \frac{1}{m+1}.$$

Proof. Let $\mathbb{T}$ be the torus, identified with $[0, 2\pi]$ and endowed with the measure $\int_{\mathbb{T}} f \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx$. Let $f : \mathbb{T} \to \mathbb{C}$ with the following expansion on the trigonometric basis,

$$f = \sum_k a_k e^{ikx} \ .$$

For $m \in \mathbb{N}$ let us define,

$$T_{m+1}(f) = \frac{1}{m+1} \sum_{l=0}^{m} f(x + \tfrac{2l\pi}{m+1}) \ .$$

It is obvious that $T_{m+1}f$ is periodic with period $\frac{2l\pi}{m+1}$. Therefore $T_{m+1}f$ has the expansion

$$\frac{1}{2\pi} \int_{\mathbb{T}} T_{m+1}f(x) \, e^{-ikx} \, dx = 0 \ \text{ if } \ k \neq 0 (\text{mod } m+1) \ ,$$

$$\frac{1}{2\pi} \int_{\mathbb{T}} T_{m+1}(f)(x) \, e^{-ik(m+1)x} \, dx = a_{k(m+1)} \ .$$

Hence,

$$T_{m+1}f(x) = \sum_k a_{k(m+1)} e^{ik(m+1)x} \ .$$

If f is a trigonometric polynomial of degree smaller than or equal to m, we have

$$T_{m+1}f(x) = \frac{1}{m+1}\sum_{l=0}^m f(x + \tfrac{2l\pi}{m+1}) = a_0 = \frac{1}{2\pi}\int_{\mathrm{T}} f(u)du \ .$$

Therefore, if $f \in \mathcal{K}_m$,

$$\frac{1}{m+1}\sum_{l=0}^m f(\tfrac{2l\pi}{m+1}) = \frac{1}{2\pi}\int_{\mathrm{T}} f(u)du \ .$$

$\square$

2.5 Localization properties for trigonometric series

Following the steps of the sections above, we have for $\eta \in \mathscr{X}_{2^{j+2}-1}$, that is $\eta = \frac{2k\pi}{2^{j+2}}$, $k \in \{0,\ldots,2^{j+2}-1\}$,

$$\Lambda_j(x,y) = \sum_{l\neq 0} a^2(\tfrac{l}{2^j}) e^{il(x-y)}, \qquad \psi_{j\eta}(x) = \frac{1}{2^{j/2+1}}\sum_{l\neq 0} a(\tfrac{l}{2^j}) e^{il(x-\eta)}. \qquad (2.7)$$

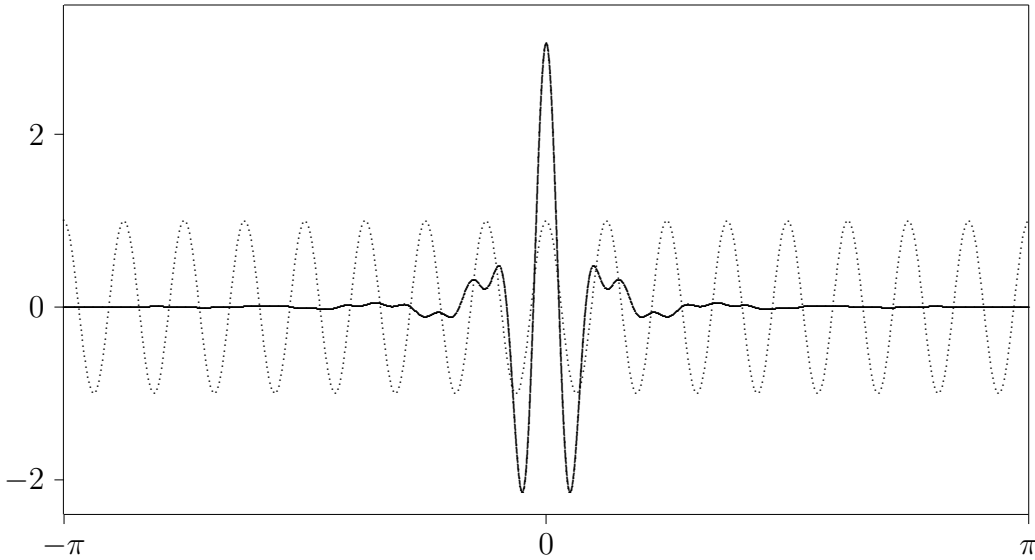We prove now at the same time the concentration of the wavelet and of Λ_j. Let A be

Figure 1: Comparison between the wavelet ψ_{j0} (solid) and the trigonometric function at comparable frequency (dashes). Here $j = 4$. The local character of the wavelet is apparent. In this picture the wavelet has been normalized in order to have the same L^2 norm (otherwise it would be smaller).

a a continuous compactly supported function and let us consider, associated to A and j, the function

$$\xi_j(x) = \sum_{l\neq 0} A(\tfrac{l}{2^j}) e^{ilx} \ .$$

We denote by $W_1^k = W_1^k(\mathbb{R})$ the Besov space of functions with integrable weak derivative of order k.

Theorem 2.5 *Let A be a continuous, compactly supported function such that $A \in W_1^k(\mathbb{R})$ for some $k \geq 2$. Then, for all $m \in \mathbb{N}$ there exists a constant $c_{m,k}$ such that*

$$|D^m \xi_j(x)| \leq \frac{c_{m,k} 2^{(m+1)j}}{(1 + |2^j x|)^k}. \tag{2.8}$$

Proof. Clearly $\xi_j(x)$ is a trigonometric polynomial. Moreover let us put

$$B(x) = \bar{\mathscr{F}}(A)(x) = \frac{1}{2\pi} \int A(y) e^{iyx} \, dy.$$

If $B \in \mathbb{L}_1(\mathbb{R})$, by the Poisson summation formula :

$$\xi_j(x) = \sum_k A(\tfrac{k}{2^j}) e^{ikx} = \sum_k \hat{B}(\tfrac{k}{2^j}) e^{ikx} = 2\pi \sum_{l \in \mathbb{Z}} 2^j B(2^j(x - 2\pi l))$$

and more generally if $D^m(B) \in \mathbb{L}_1(\mathbb{R})$

$$D^m(\xi_j)(x) = 2\pi \sum_{l \in \mathbb{Z}} 2^{(m+1)j} D^m(B)(2^j(x - 2\pi l)) \ .$$

But

$$D^m(B)(x) = D^m \bar{\mathscr{F}}(A)(x) = \bar{\mathscr{F}}((iy)^m A(y))(x)$$

is bounded (and, by the Lebesgue-Riemann lemma, even vanishes at infinity). Furthermore,

$$(-ix)^k D^m(B)(x) = (-ix)^k \bar{\mathscr{F}}((iy)^m A(y))(x) = \bar{\mathscr{F}}(D^k\{(iy)^m A(y)\})(x) =$$

$$= i^m \sum_{l=0}^k \bar{\mathscr{F}}(\{D^{k-l} y^m D^l(A)(y)\})(x)$$

and this function is bounded as $A \in W_1^k$. Therefore

$$|D^m(B)(x)| \leq c_{m,k} \frac{1}{1 + |x|^k} \leq c_{m,k} \frac{1}{(1 + |x|)^k}.$$

Hence

$$|D^m \xi_j(x)| \leq c_{m,k} 2^{(m+1)j} \sum_{l \in \mathbb{Z}} \frac{1}{(1 + |2^j(x - 2\pi l)|)^k}.$$

The result is now a consequence of the following lemma.

Lemma 2.6 *For $k \geq 2$*

$$\theta_j(x) = \sum_{l \in \mathbb{Z}} \frac{1}{(1 + |2^j(x - 2\pi l)|)^k}$$

is a 2π-periodic function such that

$$\theta_j(x) \leq \frac{5}{(1 + |2^j x|)^k}.$$

Proof. Let $|x| \leq \pi$. Then

$$\sum_{l \in \mathbb{Z}} \frac{1}{(1 + |2^j(x - 2\pi l)|)^k} = \frac{1}{(1 + |2^j x|)^k} + \sum_{l \neq 0} \frac{1}{(1 + |2^j(x - 2\pi l)|)^k}.$$

Since

$$\sum_{l \neq 0} \frac{1}{(1 + |2^j(x - 2\pi l)|)^k} \leq 2 \sum_{l > 0} \frac{1}{(1 + 2^j(2l - 1)\pi)^k} \leq$$

$$\leq \frac{2}{(1 + 2^j \pi)^k} \left(1 + \sum_{l \geq 2} \left(\frac{1 + 2^j \pi}{1 + 2^j(2l - 1)\pi} \right)^k \right) \leq$$

$$\leq \frac{2}{(1 + 2^j \pi)^k} \left(1 + \int_2^\infty \left(\frac{1 + 2^j \pi}{1 + 2^j x \pi} \right)^k dx \right) \leq \frac{4}{(1 + 2^j \pi)^k}$$

one gets finally

$$\theta_j(x) \leq \frac{1}{(1 + |2^j x|)^k} + \frac{4}{(1 + 2^j \pi)^k} \leq \frac{5}{(1 + |2^j x|)^k}.$$

$\square$

3 Assumptions and random wavelet coefficients

3.1 Assumptions on the model

Consider the random field

$$X(\vartheta) = \sum_{l = -\infty}^{\infty} w_l e^{il\vartheta}, \quad \vartheta \in [0, 2\pi],$$

where

$$w_0 = 0, \ Ew_l = 0, \ E|w_l|^2 = C_l, \ l = 1, 2, \ldots, \sum_{l = -\infty}^{\infty} C_l < \infty.$$

Throughout this paper, we introduce the following regularity condition on the behaviour of the angular power spectrum.

Assumption A1 There exists a function $g : \mathbb{N} \to [c_1, c_2]$ such that $g \in W_1^M$ for some $M \geq 0$ and

$$C_l = g(l)l^{-\alpha} \text{ for all } l \in \mathbb{N}, \ \alpha > 1.$$

For some results to follow, this assumption is strengthened to

Assumption A2 A1 holds and there exists a sequence of functions $h_N(u) : [\frac{1}{2}, 2] \to [\frac{c_1}{c_2}, \frac{c_2}{c_1}]$ such that

$$h_N\left(\tfrac{4l}{N}\right) := \frac{g(l)}{g(\frac{N}{4})}, \qquad \frac{N}{8} \leq l \leq \frac{N}{2}, \ N = 8, 16, 32, \ldots$$

and

$$\sup_N \sup_{1/2 \leq u \leq 2} |h_N^{(M)}(u)| \leq C, \text{ some } C > 0, \text{ some } M \in \mathbb{N}, \tag{3.9}$$

where $h_N^{(M)}$ denotes the M-th order weak derivative of h_N.

Remark 3.1 Condition A1 is a very mild regularity condition; it is implied, for instance, if $g(l)$ is any trigonometric polynomial bounded away from zero. The requirement $\alpha > 1$ is necessary to ensure the sequence C_l to be summable, which in turn is a consequence of the finite variance of the field. Condition A2 is a slightly stronger smoothness condition, which implies that $h_N \in W_1^M$.

3.2 Random wavelet coefficients

We recall the frame introduced in (2.7), namely

$$\psi_{j\eta}(x) = \frac{1}{2^{1+j/2}} \sum_{l \neq 0} a(\tfrac{l}{2^j}) e^{il(x-\eta)}, \quad \eta = \frac{2k\pi}{2^{j+2}}, \; k \in \{0, \dots, 2^{j+2}-1\}.$$

The notations will be shortened into the following way. For $j \in \mathbb{N}$, we put $N = 2^{j+2}$,

$$\psi_{Nk}(t) = \psi_N(t - k\tau), \quad \tau = \frac{2\pi}{N}, \; k \in \{0, \dots, N-1\}$$

$$\psi_N(t) = \frac{1}{\sqrt{N}} \sum_{l=-\infty}^{\infty} a(\tfrac{4l}{N}) e^{ilt}$$

where a is a $\mathscr{C}^{\infty}$ function, compactly supported in $[\frac{1}{2}, 2]$. Hence, $\widehat{\psi}_N(l)$ has support in $(\frac{N}{8}, \frac{N}{2})$; indeed

$$\widehat{\psi}_N(l) = \frac{1}{\sqrt{N}} a(\tfrac{4l}{N}) \; .$$

We have also

$$\widehat{\psi}_{Nk}(l) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi_N(\vartheta + k\tau) e^{il\vartheta} \, d\vartheta = e^{-ilk\tau} \widehat{\psi}_N(l) \; .$$

We now define the associated wavelets coefficients of the process X

$$\beta_{Nk} := \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\vartheta) \psi_{Nk}(\vartheta) \, d\vartheta = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\vartheta) \psi_N(\vartheta + k\tau) \, d\vartheta, \quad k = 0, 1, \dots, N-1.$$

It is immediate to see that $E\beta_{Nk} = 0$; also

$$\begin{aligned}
E(\beta_{Nk_1} \beta_{Nk_2}) &= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} EX(\vartheta) X(\vartheta') \psi_{Nk_1}(\vartheta) \psi_{Nk_2}(\vartheta') \, d\vartheta \, d\vartheta' = \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} EX(\vartheta) X(\vartheta') \psi_{Nk_1}(\vartheta) \psi_{Nk_2}(\vartheta') \, d\vartheta \, d\vartheta' = \\
&= \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{l=-\infty}^{\infty} C_l e^{il(\vartheta-\vartheta')} \psi_N(\vartheta + k_1\tau) \psi_N(\vartheta' + k_2\tau) \, d\vartheta \, d\vartheta' = \\
&= \sum_{l=-\infty}^{\infty} C_l \widehat{\psi}_{Nk_1}(l) \widehat{\psi}_{Nk_2}(-l) = \sum_{l=-\infty}^{\infty} C_l |\widehat{\psi}_N(l)|^2 e^{il\tau(k_1-k_2)} = \\
&= \frac{1}{N} \sum_{l=-\infty}^{\infty} C_l a(\tfrac{4l}{N})^2 e^{il\tau(k_1-k_2)}.
\end{aligned} \tag{3.10}$$

Next we study the asymptotics of the correlation coefficient, defined as

$$\text{Corr}\,(\beta_{Nk_1}, \beta_{Nk_2}) = \frac{\sum_{N/8 \leq |l| \leq N/2} C_l a^2(\tfrac{4l}{N}) e^{il\frac{2\pi}{N}(k_1-k_2)}}{\sum_{N/8 \leq |l| \leq N/2} C_l a^2(\tfrac{4l}{N})} \; .$$

Lemma 3.2 *Under Assumption A2,*

$$|\mathrm{Corr}(\beta_{Nk_1}, \beta_{Nk_2})| \leq \frac{2c_M}{(1 + [k_1 - k_2]_{N/2})^M}$$

for some $c_M > 0$ where $[a]_b$ means $a(\mathrm{mod})b$.

Proof. Write

$$\mathrm{Corr}(\beta_{Nk_1}, \beta_{Nk_2}) = \frac{\frac{1}{N}\sum_{N/8 \leq |l| \leq N/2} \frac{C_l}{C_{N/4}} a^2(\frac{l}{N}) e^{il\frac{2\pi}{N}(k_1 - k_2)}}{\frac{1}{N}\sum_{N/8 \leq |l| \leq N/2} \frac{C_l}{C_{N/4}} a^2(\frac{4l}{N})},$$

and note that under Assumption A2 it holds for the denominator

$$c_1 2^{-\alpha} \leq \frac{1}{N} \sum_{N/8 \leq |l| \leq N/2} \frac{C_l}{C_{N/4}} a^2(\tfrac{4l}{N}) \leq c_2 2^{\alpha} \text{ , as } N \to \infty .$$

Thus we focus on

$$\left| \frac{1}{N} \sum_{N/8 \leq |l| \leq N/2} \frac{C_l}{C_{N/4}} a^2(\tfrac{4l}{N}) e^{il\frac{2\pi}{N}(k_1 - k_2)} \right| \leq$$

$$\leq \frac{1}{N} c_2 2^{\alpha} \left| \sum_{N/8 \leq |l| \leq N/2} \frac{g(l)}{g(N/4)} (\tfrac{4l}{N})^{-\alpha} a^2(\tfrac{4l}{N}) e^{il\frac{2\pi}{N}(k_1 - k_2)} \right|.$$

To complete the argument, we use Theorem 2.5. To apply the theorem, we notice that for all N

$$A_N(\xi) := h_N(\xi)(\xi)^{-\alpha} a^2(\xi)$$

is the (sampling of) the Fourier transform of an infinitely differentiable expression, so we obtain

$$\left| \frac{1}{N} \sum_{N/8 \leq |l| \leq N/2} \frac{g(l)}{g(N/4)} (\tfrac{4l}{N})^{-\alpha} a^2(\tfrac{4l}{N}) e^{il\frac{2\pi}{N}(k_1 - k_2)} \right| =$$

$$= \left| \frac{1}{N} \sum_{N/8 \leq |l| \leq N/2} h_N(\tfrac{4l}{N})(\tfrac{4l}{N})^{-\alpha} a^2(\tfrac{4l}{N}) e^{il\frac{2\pi}{N}(k_1 - k_2)} \right| \leq$$

$$\leq \frac{1}{N} \frac{2c_M N}{(1 + N(\frac{2\pi}{N}[k - k']_{N/2}))^M} \leq \frac{2c_M}{(1 + [k - k']_{N/2})^M} ,$$

which gives the required bound; note that c_M does not depend on N, in view of (3.9). $\qquad\square$

Remark 3.3 Lemma 3.2 highlights a quite remarkable property of random wavelet coefficients. Indeed it entails that wavelet coefficients located at finite distance are asymptotically (with respect to the frequency $N = 2^{j+2}$) uncorrelated.

We write

$$\sigma_N^2 := \frac{1}{N} \sum_{N/8 \leq |l| \leq N/2} C_l a^2(\tfrac{4l}{N}) = \frac{2}{N} \sum_{N/8 \leq l \leq N/2} C_l a^2(\tfrac{4l}{N}) \qquad (3.11)$$

$$\widehat{\beta}_{Nk} := \frac{\beta_{Nk}}{\sigma_N}. \qquad (3.12)$$

so that $E\widehat{\beta}_{Nk}^2 = 1$.

Remark 3.4 It holds, as $N \to \infty$,

$$\frac{1}{N} \sum_{N/8 \leq l \leq N/2} a^2(\tfrac{4l}{N}) \longrightarrow \int_{1/2}^{2} a^2(t)\, dt$$

therefore, as under Assumption A2 $0 < c_1 \leq C_l/C_{N/4} \leq c_2 < +\infty$, there exist constants $0 \leq c_1' \leq c_2'$ such that

$$c_1' \leq \frac{\sigma_N^2}{C_{N/4}} \leq c_2'$$

for every $N \geq 0$.

In view of the asymptotic results of next section, we shall always focus on the Gaussian case, as motivated by our statistical applications.

Assumption B The field X is Gaussian.

4 Asymptotics of the wavelet statistics

4.1 The sample mean

Our first aim in this Section is to investigate the asymptotic behaviour of the sample mean for the random wavelet coefficients. More precisely, define

$$M_N := \frac{1}{N} \sum_{k=1}^{N} \widehat{\beta}_{Nk} \;;$$

we have immediately $EM_N \equiv 0$. Under Assumption A1, it is also simple to show that

$$
\begin{aligned}
E[M_N^2] &= \frac{1}{N^2} \sum_{k_1,k_2=1}^{N} E[\widehat{\beta}_{Nk_1}\widehat{\beta}_{Nk_2}] = \\
&= \frac{1}{N^2} \sum_{N/8 \leq |l| \leq N/2} \frac{C_l}{\sigma_N^2} a^2(\tfrac{4l}{N}) \frac{1}{N} \Big| \sum_{k=1}^{N} e^{i\frac{2\pi l}{N}k} \Big|^2 = 0,
\end{aligned}
\tag{4.13}
$$

from the well-known properties of the Dirichlet kernel

$$\sum_{k=1}^{N} e^{i\frac{2\pi l}{N}k} = 0 \text{ for all } l \in \mathbb{N} \text{ such that } \tfrac{2\pi l}{N} \neq 2k\pi \;,\; k = 0, \pm 1, \pm 2, \dots \;.$$

It follows immediately that $M_N = 0$ with probability one. It is interesting to realize what happens here. Given the fast decay of the covariances established in the previous Section, we might have expected standard asymptotics to go through, i.e. a Central Limit Theorem for the normalized sample mean. This is not the case because the variance is degenerate; intuitively, this is due to the wavelet transform which is 'overdifferencing' the random field. Put in another way, if we view the wavelet coefficients as a discrete time periodic random process, then this process as a zero spectral density at the origin. This complicated dependence structure does not prevent, however, the Central Limit Theorem to hold for higher-order statistics, as we shall show in the sequel.

4.2   Skewness and Kurtosis

Motivated by testing for non-Gaussianity on random fields on the torus, we introduce here the Skewness and Kurtosis statistics of the wavelet coefficients. More precisely, we consider (recall (3.11) and (3.12))

$$S_N := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \widehat{\beta}_{Nk}^3 \quad \text{and} \quad U_N := \frac{1}{\sqrt{N}} \sum_{k=1}^{N} (\widehat{\beta}_{Nk}^4 - 3) \ .$$

We have immediately

$$E[S_N] = E[U_N] = 0 \ .$$

The variance of these statistics is more complicated; in view of the following Lemma, it is convenient to introduce Fejér's kernel, defined by

$$K_N(t) := \frac{1}{2\pi N} \Big| \sum_{k=1}^{N} e^{itk} \Big|^2 = \frac{1}{2\pi N} \frac{\sin^2(\frac{1}{2} Nt)}{\sin^2(\frac{1}{2} t)}.$$

Fejér's kernel vanishes at the Fourier frequencies $\frac{2\pi}{N} l$, unless $l = kN$, $k \in \mathbb{Z}$; moreover, if $l = kN$, $K_N(\frac{2\pi}{N} l) = \frac{N}{2\pi}$. We define also

$$\sigma_{S_N}^2 := \frac{12\pi}{\sigma_N^6} \frac{1}{N^3} \sum_{l_1 l_2 l_3} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) K_N((l_1 + l_2 + l_3)\tau), \tag{4.14}$$

and $\sigma_{U_N}^2 = \sigma_{1U_N}^2 + \sigma_{2U_N}^2$, where

$$\sigma_{1U_N}^2 := \frac{72}{\sigma_N^4} \frac{2\pi}{N^2} \sum_{l_1 l_2} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) K_N((l_1 + l_2)\tau) \ , \tag{4.15}$$

$$\sigma_{2U_N}^2 := \frac{24}{\sigma_N^8} \frac{48\pi}{N^4} \sum_{l_1 l_2 l_3 l_4} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) C_{l_4} a^2(\tfrac{4l_4}{N}) \times \\ \times K_N((l_1 + l_2 + l_3 + l_4)\tau). \tag{4.16}$$

Remark 4.1 As Fejér's kernel vanishes at the Fourier frequencies $\frac{2\pi}{N} l$, $l \neq kN$, $k \in \mathbb{Z}$, in the previous expressions most terms vanish. Taking into account the fact that $a^2(x) = 0$ unless $\frac{1}{2} \leq |x| \leq 2$, one can write in a more computationally tractable way

$$\sigma_{S_N}^2 = \frac{6}{\sigma_N^6 N^2} \Big\{ \sum_{l_1 l_2} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{-l_1-l_2} a^2(-\tfrac{4l_1+4l_2}{N}) + $$

$$+ \sum_{l_1 l_2} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{N-l_1-l_2} a^2(4\tfrac{N-l_1-l_2}{N}) + $$

$$+ \sum_{l_1 l_2} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{-N-l_1-l_2} a^2(-4\tfrac{N+l_1+l_2}{N}) \Big\}$$

Likewise

$$\sigma_{1U_N}^2 := \frac{72}{\sigma_N^4} \frac{1}{N} \sum_{l_1} C_{l_1}^2 a^4(\tfrac{4l_1}{N}).$$

A similar, a bit more complicated, expression can easily be derived also for $\sigma_{2U_N}^2$.

Lemma 4.2 *Under Assumptions A1 and B*

$$E[S_N^2] = \sigma_{S_N}^2 \ , \tag{4.17}$$

$$E[U_N^2] = \sigma_{1U_N}^2 + \sigma_{2U_N}^2 \ . \tag{4.18}$$

Proof. For (4.17), we note that, by the diagram formula (see (Adler 1981), p.108 e.g.)

$$E[S_N^2] = \frac{1}{N\sigma_N^6} \sum_{k_1,k_2=1}^N E[\beta_{Nk_1}^3 \beta_{Nk_2}^3] =$$

$$= \frac{1}{N\sigma_N^6} \sum_{k_1,k_2=1}^N \left\{ 9 E[\beta_{Nk_1}\beta_{Nk_1}] E[\beta_{Nk_2}\beta_{Nk_2}] E[\beta_{Nk_1}\beta_{Nk_2}] + 6 E[\beta_{Nk_1}\beta_{Nk_2}]^3 \right\} =$$

$$= \frac{9}{N\sigma_N^2} \sum_{k_1,k_2=1}^N E[\beta_{Nk_1}\beta_{Nk_2}] + \frac{6}{N\sigma_N^2} \sum_{k_1,k_2=1}^N \left\{ \frac{1}{N} \sum_l C_l a^2(\tfrac{4l}{N}) e^{il\tau(k_1-k_2)} \right\}^3 .$$

We have seen already in (4.13) that the first term is equal to zero. As for the second one, we obtain (recall that $\tau = \frac{2\pi}{N}$)

$$\frac{6}{N\sigma_N^6} \sum_{k_1,k_2=1}^N \left\{ \frac{1}{N} \sum_l C_l a^2(\tfrac{4l}{N}) e^{il\tau(k_1-k_2)} \right\}^3 =$$

$$= \frac{6}{N\sigma_N^6} \sum_{k_1,k_2=1}^N \left\{ \frac{1}{N^3} \sum_{l_1 l_2 l_3} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) e^{i\tau(k_1-k_2)(l_1+l_2+l_3)} \right\} =$$

$$= \frac{12\pi}{\sigma_N^6} \frac{1}{N^3} \sum_{l_1 l_2 l_3} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) K_N((l_1+l_2+l_3)\tau) \ ,$$

whence (4.17) follows. Likewise, for (4.18), we have

$$E[(\widehat{\beta}_{Nk_1}^4 - 3)(\widehat{\beta}_{Nk_2}^4 - 3)] = E[\widehat{\beta}_{Nk_1}^4 \widehat{\beta}_{Nk_2}^4] - 9.$$

Again by the diagram formula and recalling that $E[\widehat{\beta}_{Nk_1}^2] = 1$,

$$E[\widehat{\beta}_{Nk_1}^4 \widehat{\beta}_{Nk_2}^4] = 24 E[\widehat{\beta}_{Nk_1}\widehat{\beta}_{Nk_2}]^4 + 72 E[\widehat{\beta}_{Nk_1}\widehat{\beta}_{Nk_2}]^2 + 9$$

so that

$$EU_N^2 = \frac{1}{N} \sum_{k_1,k_2=1}^N \left(E[\widehat{\beta}_{Nk_1}^4 \widehat{\beta}_{Nk_2}^4] - 9 \right) =$$

$$= \frac{24}{N\sigma_N^8} \sum_{k_1,k_2=1}^N E[\beta_{jk_1}\beta_{jk_2}]^4 + \frac{72}{N\sigma_N^4} \sum_{k_1,k_2=1}^N E[\beta_{Nk_1}\beta_{Nk_2}]^2.$$

Now

$$\frac{1}{N\sigma_N^8} \sum_{k_1,k_2=1}^N E[\beta_{Nk_1}\beta_{Nk_2}]^4 = \frac{1}{N^5\sigma_N^8} \sum_{k_1,k_2=1}^N \left(\sum_l C_l a^2(\tfrac{4l}{N}) e^{il\tau(k_1-k_2)} \right)^4 =$$

$$= \frac{1}{N^5\sigma_N^8} \sum_{k_1,k_2=1}^N \sum_{l_1 l_2 l_3 l_4} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) C_{l_4} a^2(\tfrac{4l_4}{N}) e^{i\tau(k_1-k_2)(l_1+l_2+l_3+l_4)} =$$

$$= \frac{2\pi}{N^4\sigma_N^8} \sum_{l_1 l_2 l_3 l_4} C_{l_1} a^2(\tfrac{4l_1}{N}) C_{l_2} a^2(\tfrac{4l_2}{N}) C_{l_3} a^2(\tfrac{4l_3}{N}) C_{l_4} a^2(\tfrac{4l_4}{N}) K_N((l_1+l_2+l_3+l_4)\tau) \ .$$

On the other hand

$$\frac{1}{N\sigma_N^4}\sum_{k_1,k_2=1}^{N}E[\beta_{Nk_1}\beta_{Nk_2}]^2 = \frac{1}{N^3\sigma_N^4}\sum_{k_1,k_2=1}^{N}\left(\sum_l C_l a^2(\tfrac{4l}{N})e^{i\tau l(k_1-k_2)}\right)^2 =$$

$$= \frac{2\pi}{N^2\sigma_N^4}\sum_{l_1 l_2}C_{l_1}a^2(\tfrac{4l_1}{N})C_{l_2}a^2(\tfrac{4l_2}{N})K_N((l_1+l_2)\tau) \ .$$

$\square$

Remark 4.3 By the same arguments of Remark 3.4 it is immediate that, as $N \to \infty$,

$$\frac{1}{N^3}\sum_{l_1 l_2 l_3}a^2(\tfrac{4l_1}{N})a^2(\tfrac{4l_2}{N})a^2(\tfrac{4l_3}{N})K_N(\tfrac{2\pi}{N}(l_1+l_2+l_3)) \ \to$$

$$\int_{1/2}^{2}\int_{1/2}^{2}\int_{1/2}^{2}a^2(t_1)a^2(t_2)a^2(t_3)\frac{t_1+t_2+t_3}{2\sin^2(t_1+t_2+t_3)}\,dt_1\,dt_2\,dt_3 \ .$$

Therefore under Assumptions A1 and B we have

$$0 < c_1 \le \sigma_{S_N}^2 \le c_2 \tag{4.19}$$

for some constants $0 \le c_1 \le c_2$. By the same argument we see that also $\sigma_{U_N}^2$ is bounded and bounded away from 0, for every $N > 0$

5 The Central Limit Theorem

This Section is devoted to the Central Limit Theorem for our statistics of interest. The idea is to prove the results by the method of moments. To analyze the behaviour of higher order moments, we shall extensively use the already mentioned diagram formula, that states that, for a multivariate zero-mean Gaussian vector $(X_1, \ldots, X_{2k})$, it holds

$$E(X_1 X_2 \ldots X_{2k}) = \sum E(X_{i_1}X_{i_2})\ldots E(X_{i_{2k-1}}X_{i_{2k}}) \ . \tag{5.20}$$

Consider the cartesian product $I \times J$, where I, J are sets of positive integers of cardinality $\#(I) = P$, $\#(J) = Q$; it is convenient to visualize these elements in a $P \times Q$ matrix with P rows. A *diagram* γ is any partition of the $P \cdot Q$ elements into pairs like $\{(i_1, j_1), (i_2, j_2)\}$; these pairs are called the *edges* of the diagram. We label $\Gamma(I, J)$ the family of these diagrams. It can be checked that, for given I, J, there exist $(P \cdot Q - 1)!!$ different diagrams, each of them composed by $\frac{1}{2}P \cdot Q$ pairs; we recall that $(2p-1)!! := (2p-1) \cdot (2p-3) \cdots 1$ for $p = 1, 2, \ldots$

To any diagram we can associate a *graph* having I as the set of vertices (or nodes) and connecting any two of these vertices, $i_k, i_{k'}$, by an arc whenever in the diagram an edge of the type $\{(i_k, j_k), (i_{k'}, j_{k'})\}$ is present. This graph is not directed, that is, (i_1, i_2) and (i_2, i_1) identify the same arc; however, we do allow for repetitions of edges (two rows may be linked twice). We shall use some result on graphs below; with a slight abuse of notation, we denote the graph γ with the same letter as the corresponding diagram.

We say that

a) A diagram has a *flat edge* if there is at least a pair $\{(i_1, j_1), (i_2, j_2)\}$ with $i_1 = i_2$. we write $\gamma \in \Gamma_F(I, J)$ for a diagram with at least a flat edge, and $\gamma \in \Gamma_{\overline{F}}(I, J)$ otherwise.

A graph corresponding to a diagram with a flat edge includes an edge $i_k i_k$ which arrives in the same vertex where it started.

b) A diagram $\gamma \in \Gamma_{\overline{F}}(I, J)$ is *connected* if it is not possible to partition the corresponding graph into two sets A, B such that there are no arcs connecting the nodes in A with nodes in B. We write $\gamma \in \Gamma_C(I, J)$ for connected diagrams, $\gamma \in \Gamma_{\overline{C}}(I, J)$ otherwise.

c) A diagram $\gamma \in \Gamma_{\overline{F}}(I, J)$ is *paired* if, given any two sets of edges $\{(i_1, j_1), (i_2, j_2)\}$ and $\{(i_3, j_3), (i_4, j_4)\}$, then $i_1 = i_3$ implies $i_2 = i_4$; in words, the rows are completely coupled two by two. We write $\gamma \in \Gamma_P(I, J)$ for paired diagrams.

Obviously if $P > 2$ a paired diagram cannot be connected. Note that if Q is odd, paired diagrams cannot have flat edges, so that the assumption $\gamma \in \Gamma_{\overline{F}}(I, J)$ becomes redundant. If I or J (or both) can be simply taken as the set of the first p or q natural numbers, i.e. $I = \{1, \ldots, p\}$, $J = \{1, \ldots, q\}$ we shall occasionally write $\Gamma(I, q), \Gamma p, J)$ or $\Gamma(p, q)$ for $\Gamma(I, J)$. For a nice and comprehensive discussion on the diagram formula, we refer to (Surgailis 2003).

Theorem 5.1 *Under Assumptions A1 and B, as $N \to \infty$,*

$$\lim_{N \to \infty} E\left[\left(\frac{S_N}{\sigma_{S_N}}\right)^{p_1} \left(\frac{U_N}{\sigma_{U_N}}\right)^{p_2}\right] = \begin{cases} (2p_1 - 1)!! \cdot (2p_2 - 1)!! & \text{if } p_1, p_2 \text{ are even} \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{pmatrix} \frac{1}{\sigma_{S_N}} S_N \\ \frac{1}{\sigma_{U_N}} U_N \end{pmatrix} \xrightarrow[N \to \infty]{\mathscr{D}} N(0, I_2) .$$

Proof. For brevity's sake and notational simplicity, we focus on the case $p_2 = 0$; the general argument can be pursued under the same lines. The idea is to use (5.20) and partition the various summands in this expression according to the nature of the associated diagrams/graphs. To this aim, we visualize our coefficients as positioned on a diagram with $I = 2p_1 = 2p$ rows and $J = 3$ columns; we obtain

$$ES_N^{2p} = \frac{1}{N^p \sigma_N^{6p}} \sum_{k_1, \ldots, k_{2p}} E[\beta_{Nk_1}^3 \ldots \beta_{Nk_{2p}}^3] =$$

$$= \frac{1}{N^p \sigma_N^{6p}} \sum_{k_1, \ldots, k_{2p}} E\left[\prod_{k \in \{k_1, \ldots, k_{2p}\}} \left\{\frac{1}{\sqrt{N}} \sum_{N/8 \leq |l| \leq N/2} w_l \, a\left(\frac{4l}{N}\right) e^{il\tau k}\right\}^3\right] =$$

$$= \frac{1}{N^{4p} \sigma_N^{6p}} \sum_{k_1, \ldots, k_{2p}} E\left[\prod_{k \in \{k_1, \ldots, k_{2p}\}} \prod_{j=1}^{3} \sum_{N/8 \leq |l| \leq N/2} w_l \, a\left(\frac{4l}{N}\right) e^{il\tau k}\right] =$$

$$= \frac{1}{N^{4p} \sigma_N^{6p}} \sum_{k_1, \ldots, k_{2p}} \sum_{\gamma \in \Gamma(2p,3)} \prod_{\{(m,j)(m',j')\} \in \gamma} \left\{\sum_{N/8 \leq |l|, |l'| \leq N/2} E[w_l w_{l'}] a\left(\frac{4l}{N}\right) a\left(\frac{4l'}{N}\right) e^{il\tau k_m} e^{il'\tau k_{m'}}\right\}$$

$$= \frac{1}{N^{4p} \sigma_N^{6p}} \sum_{k_1, \ldots, k_{2p}} \sum_{\gamma \in \Gamma(2p,3)} \prod_{\{(m,j)(m',j')\} \in \gamma} \left\{\sum_{N/8 \leq |l| \leq N/2} C_l \, a^2\left(\frac{4l}{N}\right) e^{il\tau k_m} e^{-il\tau k_{m'}}\right\} ,$$

where we used the fact that $E[w_l w_{l'}] = 0$ unless $l' = -l$. For fixed $k_1, \ldots, k_{2p}$, let us concentrate on the contribution of a single diagram $\gamma \in \Gamma(2p, 3)$.

$$\prod_{\{(m,j)(m',j')\} \in \gamma} \left\{\sum_{N/8 \leq |l| \leq N/2} C_l \, a^2\left(\frac{4l}{N}\right) e^{il\tau k_m} e^{-il\tau k_{m'}}\right\} =$$

$$\sum_{l_{11}, \ldots, l_{13}, \ldots, l_{2p,1}, \ldots, l_{2p,3}} \prod_{\{(m,j)(m',j')\} \in \gamma} C_{l_{m,j}} a^2\left(\frac{4l_{m,j}}{N}\right) e^{il_{m,j}\tau k_m} e^{-il_{m,j}\tau k_{m'}} \delta_{l_{m,j}, l_{m',j'}}$$

where the indices $l_{m,j}$, $m = 1, \ldots, 2p$, $j = 1, 2, 3$ vary between $N/8$ and $N/2$. Now, as every k_m appears exactly three times in the diagram,

$$\prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N}) e^{il_{m,j}\tau k_m} e^{-il_{m,j}\tau k_{m'}} \delta_{l_{m,j},l_{m',j'}} =$$

$$= \Big(\prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N}) \delta_{l_{m,j},l_{m',j'}}\Big) \Big(\prod_{\{(m,j)(m',j')\}\in\gamma} e^{il_{m,j}\tau k_m} e^{-il_{m,j}\tau k_{m'}}\Big) =$$

$$= \Big(\prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N}) \delta_{l_{m,j},l_{m',j'}}\Big) \Big(\prod_{m=1}^{2p} e^{i(l_{m,1;\gamma}+l_{m,2;\gamma}+l_{m,3;\gamma})\tau k_m}\Big)$$

where $l_{m,j;\gamma} = l_{m,j}$ if (m, j) is a departing point in γ, $l_{m,j;\gamma} = -l_{m,j}$ if (m, j) is an arrival point. Summing on the possible values of $k_1, \ldots, k_{2p}$, we get

$$\sum_{k_1,\ldots,k_{2p}} \prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N}) e^{il_{m,j}\tau k_m} e^{-il_{m,j}\tau k_{m'}} \delta_{l_{m,j},l_{m',j'}} =$$

$$= \Big(\prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N}) \delta_{l_{m,j},l_{m',j'}}\Big) \Big(\prod_{m=1}^{2p} D_N([l_{m,1;\gamma}+l_{m,2;\gamma}+l_{m,3;\gamma}]\tau)\Big) =$$

$$= \underbrace{\Big(\prod_{\{(m,j)(m',j')\}\in\gamma} \delta_{l_{m,j},l_{m',j'}}\Big)}_{=\delta(\gamma;l_{1,1},\ldots,l_{2p,3})} \Big(\prod_{\{(m,j)(m',j')\}\in\gamma} C_{l_{m,j}} a^2(\tfrac{4l_{m,j}}{N})\Big) \times$$

$$\times \Big(\prod_{m=1}^{2p} D_N([l_{m,1;\gamma}+l_{m,2;\gamma}+l_{m,3;\gamma}]\tau)\Big) =$$

$$= \delta(\gamma; l_{1,1}, \ldots, l_{2p,3}) \prod_{m=1}^{2p} \Big(C_{l_{m,1}} a^2(\tfrac{4l_{m,1}}{N}) C_{l_{m,2}} a^2(\tfrac{4l_{m,2}}{N}) C_{l_{m,3}} a^2(\tfrac{4l_{m,3}}{N})\Big)^{1/2} \times$$

$$\times D_N([l_{m,1;\gamma}+l_{m,2;\gamma}+l_{m,3;\gamma}]\tau).$$

Let us define

$$X_{l_{m,1},l_{m,2},l_{m,3};\gamma} =$$

$$= \frac{1}{N^2\sigma_N^3}\Big(C_{l_{m,1}} a^2(\tfrac{4l_{m,1}}{N}) C_{l_{m,2}} a^2(\tfrac{4l_{m,2}}{N}) C_{l_{m,3}} a^2(\tfrac{4l_{m,3}}{N})\Big)^{1/2} D_N([l_{m,1;\gamma}+l_{m,2;\gamma}+l_{m,3;\gamma}]\tau) .$$

In conclusion

$$ES_N^{2p} = \sum_{\gamma\in\Gamma(2p,3)} \sum_{l_{1,1},\ldots,l_{2p,3}} \delta(\gamma; l_{1,1}, \ldots, l_{2p,3}) \prod_{m=1}^{2p} X_{l_{m,1},l_{m,2},l_{m,3};\gamma}.$$

Recall that $\delta(\gamma; l_{1,1}, \ldots, l_{2p,3}) = 0$ unless $l_{m,j} = l_{m',j'}$ for every $(m, j)(m', j')\} \in \gamma$. The proof is done by proving that the leading contribution to ES_N^{2p} is given by paired diagrams whereas the non paired ones are negligible in the asymptotics. This is made explicit in the two following lemmas.

Lemma 5.2 *For the terms corresponding to the paired diagrams $\gamma \in \Gamma_P(2p, 3)$ we have*

$$\sum_{\gamma\in\Gamma_P(2p,3)} \sum_{l_{11},\ldots,l_{2p,3}} \delta(\gamma; l_{1,1}, \ldots, l_{2p,3}) \prod_{m=1}^{2p} X_{l_{m,1}l_{m,2}l_{m,3};\gamma} = (2p-1)!!\sigma_{S_N}^{2p} .$$

Proof. Remark first that the number of possible ways of partitioning the $2p$ rows of the diagram into subsets of cardinality 2 is exactly $(2p-1)!!$. Also, in every diagram $\gamma \in \Gamma_P(2p,3)$ the contribution of two paired rows is exactly $\sigma^2_{S_N}$.

□

To conclude the proof, we need only to show that the terms corresponding to all remaining diagrams $\gamma \notin \Gamma_P(2p,3)$ are asymptotically of smaller order.

Lemma 5.3 *All terms corresponding to diagrams with connected components of order larger than 2 ($\gamma \notin \Gamma_P(2p,3)$) are of order $O(\frac{\log N}{\sqrt{N}})$.*

Proof. We focus on any two nodes that are connected but not paired; such two nodes certainly exist, otherwise $\gamma \in \Gamma_P$. We consider the case where there is a single edge linking these two nodes; the proof in the remaining case is entirely analogous. Without loss of generality, we label edges and vertices in such a way that the edge connecting these two nodes is labelled $\{(1,1),(2,1)\}$.

As in Lemma 3.1 of (Marinucci 2006a), we apply iteratively the Cauchy-Schwartz inequality to show that

$$\left| \sum_{l_{11},\ldots,l_{2p,3}} \delta(\gamma; l_{1,1},\ldots,l_{2p,3}) \prod_{m=1}^{2p} X_{l_{m1},l_{m2},l_{m3};\gamma} \right| \leq$$

$$\leq \prod_{m=1}^{2p} \left(\sum_{l_{m,1}l_{m,2}l_{m,3}} X^2_{l_{m,1}l_{m,2}l_{m,3};\gamma} \right)^{1/2} \times \left(\sum_{l_{1,2}l_{1,3}l_{2,2}l_{2,3}} Y^2_{l_{1,2}l_{1,3}l_{2,2}l_{2,3};\gamma} \right)^{1/2} ,$$

where

$$Y_{l_{1,2}l_{1,3}l_{2,2}l_{2,3};\gamma} = \sum_{l_{1,1},l_{2,1}} \delta_{l_{1,1},-l_{2,1}} X_{l_{1,1}l_{1,2}l_{1,3};\gamma} X_{l_{2,1}l_{2,2}l_{2,3};\gamma} =$$

$$= \frac{1}{N^3\sigma^6_N} \sqrt{C_{l_{1,2}}a^2(\tfrac{4l_{1,2}}{N}) C_{l_{1,3}}a^2(\tfrac{4l_{1,3}}{N}) C_{l_{2,2}}a^2(\tfrac{4l_{2,2}}{N}) C_{l_{2,3}}a^2(\tfrac{4l_{2,3}}{N})} \times$$

$$\times \frac{1}{N} \sum_{l_{1,1}} C_{l_{1,1}}a^2(\tfrac{4l_{1,1}}{N}) D_N((l_{1,1;\gamma} + l_{1,2;\gamma} + l_{1,3;\gamma})\tau) D_N((-l_{1,1;\gamma} + l_{2,2;\gamma} + l_{2,3;\gamma})\tau) .$$

Now

$$\sum_{l_{m,1}l_{m,2}l_{m,3}} X^2_{l_{m,1}l_{m,2}l_{m,3};\gamma} =$$

$$= \frac{1}{N^3\sigma^6_N} \sum_{l_{m,1}l_{m,2}l_{m,3}} C_{l_{m,1}}a^2(\tfrac{4l_{m,1}}{N}) C_{l_{m,2}}a^2(\tfrac{4l_{m,2}}{N}) C_{l_{m,3}}a^2(\tfrac{4l_{m,3}}{N}) \times$$

$$\times \frac{1}{N} D_N([l_{m,1;\gamma} + l_{m,2;\gamma} + l_{m,3;\gamma}]\tau)^2 =$$

$$= O\left(\frac{1}{N^3} \sum_{l_{m,1;\gamma}l_{m,2;\gamma}l_{m,3;\gamma}} K_N([l_{m,1;\gamma} + l_{m,2;\gamma} + l_{m,3;\gamma}]\tau) \right) = O(1) .$$

On the other hand

$$\sum_{l_{1,2}l_{1,3}l_{2,2}l_{2,3}} Y^2_{l_{1,2}l_{1,3}l_{2,2}l_{2,3};\gamma} =$$

$$= \frac{1}{N^4} \sum_{l_{1,2}l_{1,3}l_{2,2}l_{2,3}} \frac{C_{l_{1,2}}a^2(\frac{4l_{1,2}}{N})C_{l_{1,3}}a^2(\frac{4l_{1,3}}{N})C_{l_{2,2}}a^2(\frac{4l_{2,2}}{N})C_{l_{2,3}}a^2(\frac{4l_{2,3}}{N})}{\sigma_N^8} \times$$

$$\times \left[\frac{1}{N}\sum_{l_{1,1}} \frac{C_{l_{1,1}}a^2(\frac{4l_{1,1}}{N})}{\sigma_N^2} \frac{D_N((l_{1,1;\gamma}+l_{1,2;\gamma}+l_{1,3;\gamma})\tau)D_N((-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau)}{N} \right]^2.$$

Now we observe that

$$D_N\big((l_{1,1;\gamma}+l_{1,1;\gamma}+l_{1,3;\gamma})\tau\big)D_N\big((-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau\big) =$$

$$= \sum_{k_1=1}^{N} e^{(l_{1,1;\gamma}+l_{1,2;\gamma}+l_{1,3;\gamma})\tau k_1} \sum_{k_2=1}^{N} e^{(-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau k_2}$$

$$= \sum_{k_1=1}^{N} e^{(l_{1,1;\gamma}+l_{1,2;\gamma}+l_{1,3;\gamma})\tau k_1} \sum_{u=1}^{N} e^{(l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau(k_1-u)}$$

$$= D_N\big((l_{1,2;\gamma}+l_{1,3;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau\big)D_N\big((-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau\big),$$

whence

$$\left| \frac{1}{N^2\sigma_N^2}\sum_{l_{1,1}} C_{l_{1,1}}a^2(\tfrac{4l_{1,1}}{N})D_N((l_{1,1;\gamma}+l_{1,2;\gamma}+l_{1,3;\gamma})\tau)D_N((-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau) \right| \le$$

$$\le \left| \frac{1}{N}D_N((l_{1,2;\gamma}+l_{1,3;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau) \right| \times$$

$$\times \left| \frac{1}{N}\sum_{l_{1,1}} \frac{1}{\sigma_N^2}C_{l_{11;\gamma}}a^2(\tfrac{4l_{1,1}}{N})D_N((-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma})\tau) \right| \le$$

$$\le \frac{C}{|l_{1,2;\gamma}+l_{1,3;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma}|+1}\frac{1}{N}\sum_{l_{1,1;\gamma}} \frac{N}{|-l_{1,1;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma}|+1} \le$$

$$\le C\frac{\log N}{|l_{1,2;\gamma}+l_{1,3;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma}|+1}.$$

Thus we can conclude that

$$\left(\sum_{l_{1,2}l_{1,3}l_{2,2}l_{2,3}} Y^2_{l_{1,2}l_{1,3}l_{2,2}l_{2,3};\gamma} \right)^{1/2} \le$$

$$\le c\left(\frac{1}{N^4}\sum_{l_{1,2}l_{1,3}l_{2,2}l_{2,3}} \frac{\log^2 N}{(|l_{1,2;\gamma}+l_{1,3;\gamma}+l_{2,2;\gamma}+l_{2,3;\gamma}|+1)^2} \right)^{1/2} \le c\frac{\log N}{\sqrt{N}}.$$

$$\square$$

6 Studentized statistics

6.1 Estimation of σ_N^2.

The statistics described in the previous sections can be impossible to compute in practice, as they depend on the correlation structure of the field, which is in general unknown. In

this Section, we show how asymptotic variances can be consistently estimated from the data, in the presence of observations at higher and higher resolution. We start from the variance of the wavelets coefficients, which we recall is given by

$$\sigma_N^2 := \frac{2}{N} \sum_{N/8 \leq l \leq N/2} C_l a^2(\tfrac{4l}{N}) \ .$$

An obvious estimator is provided by

$$\widehat{\sigma}_N^2 := \frac{2}{N} \sum_{N/8 \leq l \leq N/2} |w_l|^2 a^2(\tfrac{4l}{N}) \ .$$

Of course, $\widehat{\sigma}_N^2$ is unbiased and mean square consistent for σ_N^2, in the trivial sense that both converge to zero as N diverges. The following result is stronger.

Lemma 6.1 *Under assumptions A1 and B, as $N \to \infty$, we have*

$$\lim_{N \to \infty} \frac{\widehat{\sigma}_N^2}{\sigma_N^2} = 1 \qquad in \ L^2 \ .$$

Proof. It is immediate to see that

$$E\Big(\frac{\widehat{\sigma}_N^2}{\sigma_N^2}\Big) = E\Big(\frac{\sum_{N/8 \leq l \leq N/2} |w_l|^2 a^2(\tfrac{4l}{N})}{\sum_{N/8 \leq l \leq N/2} C_l a^2(\tfrac{4l}{N})}\Big) \equiv 1$$

and

$$\mathrm{Var}(\widehat{\sigma}_N^2) = \sum_{N/8 \leq l \leq N/2} \mathrm{Var}(|w_l|^2) a^4(\tfrac{4l}{N}) = 2 \sum_{N/8 \leq l \leq N/2} C_l^2 a^4(\tfrac{4l}{N}) \ .$$

It is sufficient now to note that, under Assumption A1,

$$0 < c_1 \leq \frac{C_l}{C_{N/4}} \leq c_2 < \infty, \qquad \text{for } N = 2^{j+2} \text{ and for all } l \in [N/8, N/2] \ ,$$

whence

$$\mathrm{Var}\Big(\frac{\widehat{\sigma}_N^2}{\sigma_N^2}\Big) = \frac{\sum_l C_l^2 a^4(\tfrac{4l_1}{N})}{\{\sum_l C_l a^2(\tfrac{4l}{N})\}^2} = O\Big(\frac{\sum_l a^4(\tfrac{4l}{N})}{\{\sum_l a^2(\tfrac{4l}{N})\}^2}\Big) \ .$$

Remark now that $\sum_l a^4(\tfrac{4l}{N}) \sim N \int_{1/2}^1 a^4(t)\,dt$, $\sum_l a^2(\tfrac{4l}{N}) \sim N \int_{1/2}^1 a^2(t)\,dt$, so that the left-hand term tends to 0 as $\frac{1}{N}$.

$\square$

6.2 Estimation of the variance for Skewness and kurtosis

We now go on with the estimation for the sample variance for the statistics S_N and U_N. We note first that under Gaussianity $(|w_l|^2 / C_l)_l$ is a sequence of independent and identically distributed exponential random variables with mean unity; we define for all $p \in \mathbb{N}$

$$\delta_{l_1 l_2 \dots l_p} := E\Big(\prod_{l=l_1}^{l_p} \frac{|w_l|^2}{C_l}\Big) \ ;$$

to be quite explicit we have for instance

$$\delta_{l_1 l_2} = \begin{cases} 1 & \text{if } |l_1| \neq |l_2| \\ 2 & \text{if } |l_1| = |l_2| \end{cases}$$

$$\delta_{l_1 l_2 l_3} = \begin{cases} 1 & \text{if } |l_1|, |l_2|, |l_3| \text{ are distinct} \\ 2 & \text{if among } |l_1|, |l_2|, |l_3| \text{ two are equal and the third is different} \\ 6 & \text{if } |l_1| = |l_2| = |l_3| \end{cases}$$

In view of (4.14)–(4.16), a natural proposal is to consider for the Skewness

$$\widehat{\sigma}_{S_N}^2 := \frac{12\pi}{N^3 \widehat{\sigma}_N^6} \sum_{l_1 l_2 l_3} \frac{1}{\delta_{l_1 l_2 l_3}} |w_{l_1}|^2 a^2(\tfrac{4l_1}{N}) |w_{l_2}|^2 a^2(\tfrac{4l_2}{N}) |w_{l_3}|^2 a^2(\tfrac{4l_3}{N}) K_N(\tfrac{2\pi}{N}(l_1 + l_2 + l_3)) \quad (6.21)$$

and for the Kurtosis

$$\widehat{\sigma}_{U_N}^2 := \widehat{\sigma}_{1U_N}^2 + \widehat{\sigma}_{2U_N}^2 , \tag{6.22}$$

$$\widehat{\sigma}_{1U_N}^2 := \frac{72}{\widehat{\sigma}_N^4} \frac{2\pi}{N^2} \sum_{l_1 l_2} \frac{1}{\delta_{l_1 l_2}} |w_{l_1}|^2 a^2(\tfrac{4l_1}{N}) |w_{l_2}|^2 a^2(\tfrac{4l_2}{N}) K_N(\tfrac{2\pi}{N}(l_1 + l_2)) , \tag{6.23}$$

$$\widehat{\sigma}_{2U_N}^2 := \frac{24}{\widehat{\sigma}_N^8} \frac{2\pi}{N^4} \sum_{l_1 l_2 l_3 l_4} \frac{1}{\delta_{l_1 l_2 l_3 l_4}} \left\{ \prod_{l=l_1}^{l_4} |w_l|^2 a^2(\tfrac{4l}{N}) \right\} K_N(\tfrac{2\pi}{N}(l_1 + l_2 + l_3 + l_4)). \tag{6.24}$$

Remark 6.2 Using the properties of Fejér's kernel recalled in §4.2, in the summations above most terms vanish. From a computational point of view more tractable expressions can be derived in the spirit of §4.2. In particular it holds

$$\widehat{\sigma}_{1U_N}^2 := \frac{72}{\widehat{\sigma}_N^4} \frac{1}{N} \sum_l |w_l|^4 a^4(\tfrac{4l}{N}) . \tag{6.25}$$

Lemma 6.3 *Under Assumptions A1 and B, as $N \to \infty$, we have*

$$\frac{\widehat{\sigma}_{S_N^2}}{\sigma_{S_N^2}} \xrightarrow[N \to \infty]{P} 1, \qquad \frac{\widehat{\sigma}_{1U_N^2}}{\sigma_{1U_N}^2} \xrightarrow[N \to \infty]{P} 1, \qquad \frac{\widehat{\sigma}_{2U_N^2}}{\sigma_{2U_N}^2} \xrightarrow[N \to \infty]{P} 1 .$$

Proof. We give the proof for $\widehat{\sigma}_{1U_N}^2, \widehat{\sigma}_{2U_N}^2$ only, as the remaining case is entirely analogous (indeed slightly simpler). Let us denote $\widetilde{\sigma}_{1U_N}^2 = \widehat{\sigma}_{1U_N}^2 \cdot \widehat{\sigma}_N^4 / \sigma_N^4$, that is the same as in (6.21) with $\widehat{\sigma}_N$ replaced by σ_N. As $E\left(\frac{1}{\delta_{l_1 l_2}} |w_{l_1}|^2 |w_{l_2}|^2\right) = C_{l_1} C_{l_2}$ for every l_1, l_2, it is clear that

$$E\left(\frac{\widetilde{\sigma}_{1U_N}^2}{\sigma_{1U_N}^2}\right) = 1. \tag{6.26}$$

Moreover, by the alternate expression (6.25) and in view of Remark 3.4,

$$\text{Var}(\widetilde{\sigma}_{1U_N}^2) = \frac{72^2}{\sigma_N^8} \frac{1}{N^2} \sum_l \text{Var}(|w_l|^4) a^8(\tfrac{4l}{N}) = \frac{c_0}{N} \frac{C_{N/4}^4}{\sigma_N^8} \frac{1}{N} \sum_l a^8(\tfrac{4l}{N}) \sim \frac{c_1}{N}.$$

As we know that under Assumption A1 $\sigma_{1U_N}^2$ is bounded away from zero (see Remark 4.3), this implies that $\text{Var}(\widetilde{\sigma}_{1U_N}^2 / \sigma_{1U_N}^2) \to 0$ as $N \to \infty$.

The argument for $\widehat{\sigma}^2_{2U_N}$ is similar; indeed if we define $\widetilde{\sigma}^2_{2U_N}$ in analogy with $\widetilde{\sigma}^2_{1U_N}$, it is immediate that $E[\widehat{\sigma}^2_{2U_N}/\sigma^2_{2U_N}] = 1$. On the other hand, note that the summands in $\widetilde{\sigma}^2_{2U_N}$ have a martingale-difference structure on the lattice $\mathbb{Z}^3$ (see Poghosyan and Rœlly (Poghosyan and Rœlly 1998) e.g.), whence, in view of Assumption A

$$\text{Var}\left(\frac{\widehat{\sigma}^2_{2U_N}}{\sigma^2_{2U_N}}\right) =$$

$$= O\left(\text{Var}\left\{\frac{24}{\sigma^8_N}\frac{2\pi}{N^4}\sum_{l_1 l_2 l_3 l_4}\frac{1}{\delta_{l_1 l_2 l_3 l_4}}\left[\prod_{l=l_1}^{l_4}|w_l|^2 a^2(\tfrac{4l}{N})\right]K_N(\tfrac{2\pi}{N}(l_1 + l_2 + l_3 + l_4))\right\}\right)$$

$$= O\left(\frac{1}{N^8}\sum_{l_1 l_2 l_3 l_4}K_N^2(\tfrac{2\pi}{N}(l_1 + l_2 + l_3 + l_4))\text{Var}\left\{\prod_{l=l_1}^{l_4}\frac{|w_l|^2}{\sigma^2_N}\right\}\right)$$

$$= O\left(\frac{1}{N^6}\sum_{l_1 l_2 l_3 l_4}\text{Var}\left\{\prod_{l=l_1}^{l_4}\frac{|w_l|^2}{\sigma^2_N}\right\}\right) = O(\tfrac{1}{N^2}) = o(1) .$$

We have thus proved that $\widetilde{\sigma}^2_{1U_N}/\sigma^2_{1U_N} \to 1$ and $\widetilde{\sigma}^2_{2U_N}/\sigma^2_{2U_N} \to 1$ as $N \to \infty$ in L^2. Therefore, in view of Lemma 6.1, $\widetilde{\sigma}^2_{1U_N}/\sigma^2_{1U_N} \to 1$ and $\widetilde{\sigma}^2_{2U_N}/\sigma^2_{2U_N} \to 1$ as $N \to \infty$ in probability. The rest of the proof is quite similar.

$\square$

As an immediate consequence of Theorem 5.1 and Lemma 6.3 we have the following.

Theorem 6.4 *Under the assumptions A1 and B, as $N \to \infty$*

$$\begin{pmatrix}\frac{1}{\boldsymbol{b}_{S_N}}S_N \\ \frac{1}{\boldsymbol{b}_{U_N}}U_N\end{pmatrix} \xrightarrow[N\to\infty]{\mathscr{D}} N(0, I_2) .$$

7 Aliasing

The tests provided in the sections above are based on the wavelet coefficients

$$\beta_{Nk} := \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\vartheta)\psi_{Nk}(\vartheta)d\vartheta, \quad k = 0, 1, \ldots, N-1$$

In practice, β_{Nk} will be approximated by the sums

$$\widetilde{\beta}_{Nk} = \frac{1}{M}\sum_{m=0}^{M-1} X(\tfrac{2\pi}{M} m)\psi_{Nk}(\tfrac{2\pi}{M} m) .$$

The purpose of this section is to prove that, if our data have enough high frequencies, this approximation does not affect the test.

We need to strengthen our previous assumptions as follows.

Assumption C As $l \to \infty$ we have

$$C_l = L(l)l^{-\alpha}, \ \alpha > 1$$

where $L(l)$ denotes a slowly varying function (Bingham, Goldie and Teugels 1989), which we assume to be bounded and bounded away from zero.

Assumption D N is such that

$$\frac{1}{N} + \frac{N}{M^{\alpha/(\alpha+1)}} \to 0 \text{ as } M \to \infty .$$

Assumption C is mild, entailing simply a regular behaviour of the angular power spectrum at infinity. Assumption D is a sort of bandwidth condition, suggesting that the frequencies that we can use fruitfully for statistical inference must grow more slowly than the sampling rate. The condition become less and less tight the faster the angular correlation function decays to zero: for instance if $\alpha = 4$ we must impose $N = o(M^{4/5})$. In practice α can be estimated from the data; the most cautious choice can be $N = o(\sqrt{M})$, as $\alpha > 1$ is implied by the finite variance of the field.

We have the following result.

Proposition 7.1 *Under assumptions A1,B,C and D the result of Theorem 5.1 remains true when replacing the β_{Nk}'s by the $\widetilde{\beta}_{Nk}$'s.*

Proof. It holds,

$$\beta_{Nk} - \widetilde{\beta}_{Nk} = \sum_{|\ell|>M-\frac{N}{2}} w_\ell \Big(\frac{1}{M} \sum_{m=0}^{M-1} \psi_{Nk}(\tfrac{2\pi m}{M}) e^{i2\pi m \frac{\ell}{M}} \Big) .$$

Therefore,

$$E\big|\beta_{Nk} - \widetilde{\beta}_{Nk}\big|^2 = E\Big(\sum_{|\ell|>M-\frac{N}{2}} |w_\ell|^2 \frac{1}{M^2} \Big| \sum_{m=0}^{M-1} \psi_{Nk}(\frac{2\pi m}{M}) e^{i2\pi m \frac{\ell}{M}} \Big|^2 \Big) = \sum_{|\ell|>M-\frac{N}{2}} C_l \big|A_{M(l)}\big|^2 ,$$

where

$$A_{M(l)}^2 = \frac{1}{M^2}\Big| \sum_{m=0}^{M-1} \psi_{Nk}(\tfrac{2\pi m}{M}) e^{i2\pi m \frac{l}{M}} \Big|^2 \leq \frac{c_k}{|1 + [l]_M|^k} \text{ for all } k > 0 \text{ , some } c_k > 0 ,$$

in view of Theorem (2.5), which implies

$$\Big| \sum_{m=0}^{M-1} \psi_{Nk}(\tfrac{2\pi m}{M}) e^{i2\pi m \frac{l}{M}} \Big| \leq \frac{c_k M}{|1 + M[l/M]_{2\pi}|^k} .$$

Under Assumptions C and D we have easily

$$\sum_{|\ell|>M-\frac{N}{2}} C_l \big|A_{M(l)}\big|^2 = O\Big(C_M \sum_{|\ell|>M-\frac{N}{2}} \frac{C_l}{C_M} \big|A_{M(l)}\big|^2 \Big)$$

$$= O\Big(C_M \sum_{u=1}^{\infty} \sum_{v=-M}^{M} \frac{C_{uM+v}}{C_M} \big|A_{M(uM+v)}\big|^2 \Big) =$$

$$= O\Big(M^{-\alpha} \sum_{u=1}^{\infty} u^{-\alpha} \sum_{v=-M}^{M} \frac{1}{(1+|v|)^k} \Big) = O\left(M^{-\alpha} \right)$$

and

$$\frac{E(|\beta_{Nk} - \widetilde{\beta}_{Nk}|^2)}{E|\beta_{Nk}|^2} \leq c \frac{M^{-\alpha}}{\sigma_N^2} \leq c\Big(\frac{M}{N}\Big)^{-\alpha}. \tag{7.27}$$

The result then follows from the following lemma.

Lemma 7.2 *For X_{in} and Y_{in} mutually Gaussian centered random variables, set*

$$c_n := \frac{E[(X_{in} - Y_{in})^2]}{E[X_{in}^2]}.$$

Let us assume that the sequence

$$\frac{\sum_{i=1}^n X_{in}^3}{\sqrt{\operatorname{Var}\left(\sum_{i=1}^n X_{in}^3\right)}} \tag{7.28}$$

converges in distribution to a variable X, where for $\gamma_1, \gamma_2 > 0$

$$\gamma_1 \leq \sqrt{\frac{1}{n}\operatorname{Var}\left\{\sum_{i=1}^n X_{in}^3\right\}} \leq \gamma_2 \ \text{ and } c_n = o\left(\frac{1}{n}\right). \tag{7.29}$$

Then

$$\frac{\sum_{i=1}^n Y_{in}^3}{\sqrt{\operatorname{Var}\left\{\sum_{i=1}^n X_{in}^3\right\}}} \tag{7.30}$$

also converges in distribution to X. The same result is true if we replace in (7.28),(7.29) X_{in}^3 by $X_{in}^4 - EX_{in}^4$ and replace in (7.30), Y_{in}^3 by $Y_{in}^4 - EY_{in}^4$.

Proof. We shall actually prove a stronger result, namely

$$\lim_{n\to\infty} E\left|\frac{\sum_{i=1}^n X_{in}^3}{\sqrt{\operatorname{Var}\left\{\sum_{i=1}^n X_{in}^3\right\}}} - \frac{\sum_{i=1}^n Y_{in}^3}{\sqrt{\operatorname{Var}\left\{\sum_{i=1}^n X_{in}^3\right\}}}\right| = 0.$$

Using $(x - y)^3 = x^3 - y^3 - 3x(x - y)^2 + 3x^2(x - y)$, we get :

$$E|X_{in}^3 - Y_{in}^3| \leq E|X_{in} - Y_{in}|^3 + 3E|X_{in} - Y_{in}|^2|X_{in}| + 3E|X_{in} - Y_{in}||X_{in}|^2 \leq$$
$$\leq E|X_{in} - Y_{in}|^3 + 3[E|X_{in} - Y_{in}|^4]^{\frac{1}{2}}[E|X_{in}|^2]^{\frac{1}{2}} + 3[E(X_{in} - Y_{in})^2]^{\frac{1}{2}}[E|X_{in}|^4]^{\frac{1}{2}}.$$

Now, when Z is a Gaussian random variable, for $h \in \mathbb{N}^*$,

$$E|Z|^h = \sigma_h[E|Z|^2]^{\frac{h}{2}},$$

where σ_h is the h-moment of the standard gaussian distribution, centered and with variance 1. Therefore, we have $E|X_{in} - Y_{in}|^k \leq \sigma_h c_n^{\frac{h}{2}}[EX_{in}^2]^{\frac{h}{2}}$, and

$$\sum_{i=1}^n E|X_{in}^3 - Y_{in}^3| \leq \sigma_3 c_n^{\frac{3}{2}} \sum_{i=1}^n [EX_{in}^2]^{\frac{3}{2}} + 3\sigma_4^{\frac{1}{2}} c_n [\sum_{i=1}^n [EX_{in}^2]^{\frac{3}{2}} + 3c_n^{\frac{1}{2}}\sigma_4^{\frac{1}{2}} \sum_{i=1}^n [EX_{in}^2]^{\frac{3}{2}}$$

$$\leq \sum_{i=1}^n [EX_{in}^2]^{\frac{3}{2}}\{\sigma_3 c_n^{\frac{3}{2}} + 3\sigma_4^{\frac{1}{2}} c_n^{\frac{1}{2}} + 3\sigma_4^{\frac{1}{2}} c_n\}.$$

Therefore

$$\frac{\sum_{i=1}^n E|X_{in}^3 - Y_{in}^3|}{\sqrt{\operatorname{Var}\left\{\sum_{i=1}^n X_{in}^3\right\}}} \leq C\frac{\sum_{i=1}^n [EX_{in}^2]^{\frac{3}{2}} c_n^{\frac{1}{2}}}{\sqrt{n}} \leq C\sqrt{nc_n} = o(1), \qquad \text{as } n \to \infty.$$

This proves the result for the third power. As for the forth one, we proceed in the same way, and prove using the same path,

$$|x^4 - y^4| \le 4|x|^3|x - y| + 6x^2|x - y|^2 + 4|x||x - y|^3 + |x - y|^4$$

$$E|X_{in}^4 - Y_{in}^4| \le C\sqrt{c_n}[EX_{in}^2]^2$$

$$\frac{\sum_{i=1}^n E|X_{in}^4 - Y_{in}^4|}{\sqrt{\mathrm{Var}\left\{\sum_{i=1}^n X_{in}^4\right\}}} \le C\sqrt{c_n n} = o(1) \ .$$

$\square$

Our final result in this Section extends the analysis of the aliasing effect to studentized statistics. In particular, in the previous section the variances of Skewness and Kurtosis where estimated on the basis of the spectral coefficients $\{w_l\}$, which are obtained as

$$w_l = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\vartheta)e^{-il\vartheta}\, d\vartheta \ , \ l = 1, 2, \dots, N \ .$$

As before, in practice these Fourier coefficients will be approximated by interpolations sums, i.e. for $M \ge N$ we have to consider

$$\widetilde{w}_l = \frac{1}{M} \sum_{m=0}^{M-1} X(\tfrac{2\pi m}{M})e^{-i\frac{2\pi m}{M}l} \ .$$

We note that

$$\widetilde{w}_l = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{k=1}^{\infty} w_k e^{\frac{2i\pi}{M}mk} e^{-\frac{2i\pi}{M}ml} = \frac{1}{M} \sum_{m=0}^{M-1} \sum_{k=1}^{\infty} w_k e^{\frac{2i\pi}{M}(k-l)m}$$

$$= \frac{1}{M} \sum_{k=1}^{\infty} w_k D_M(\tfrac{2\pi}{M}(k-l)) = w_l + \sum_{k=1}^{\infty} w_{l+kM}. \tag{7.31}$$

It follows, using Assumption D, that

$$\frac{E|w_l - \widetilde{w}_l|^2}{C_l} = \frac{1}{C_l} \sum_{k=1}^{\infty} C_{l+kM} \le c\frac{M^{-\alpha}}{C_l} \sum_{k=1}^{\infty} k^{-\alpha} \le c\left(\frac{M}{N}\right)^{-\alpha} = o(\tfrac{1}{N}). \tag{7.32}$$

Proposition 7.3 *Under Assumptions A1,B, C and D the result of Theorem 6.4 remains true when replacing the w_l's by the $\widetilde{w}_l$'s.*

Proof. It is clearly enough to show that

$$\lim_{N\to\infty} \frac{\widetilde{\sigma}_{S_N}^2 - \widehat{\sigma}_{S_N}^2}{\widehat{\sigma}_{S_N}^2} = 0 \qquad \lim_{N\to\infty} \frac{\widetilde{\sigma}_{U_N}^2 - \widehat{\sigma}_{U_N}^2}{\widehat{\sigma}_{U_N}^2} = 0 \tag{7.33}$$

in probability, where $\widehat{\sigma}_{S_N}^2, \widehat{\sigma}_{U_N}^2$ are defined in (6.21) -(6.22),

$$\widetilde{\sigma}_{S_N}^2 := \frac{12\pi}{N^3\widehat{\sigma}_N^6} \sum_{l_1 l_2 l_3} \frac{1}{\delta_{l_1 l_2 l_3}} |\widetilde{w}_{l_1}|^2 a^2(\tfrac{4l_1}{N}) |\widetilde{w}_{l_2}|^2 a^2(\tfrac{4l_2}{N}) |\widetilde{w}_{l_3}|^2 a^2(\tfrac{4l_3}{N}) K_N(\tfrac{2\pi}{N}(l_1 + l_2 + l_3))$$

and $\widetilde{\sigma}_{U_N}^2 = \widetilde{\sigma}_{1U_N}^2 + \widetilde{\sigma}_{2U_N}^2$, where

$$\widetilde{\sigma}_{1U_N}^2 := \frac{72}{\sigma_N^4} \frac{2\pi}{N^2} \sum_{l_1 l_2} \frac{1}{\delta_{l_1 l_2}} |\widetilde{w}_{l_1}|^2 a^2(\tfrac{4l_1}{N}) |\widetilde{w}_{l_2}|^2 a^2(\tfrac{4l_2}{N}) K_N(\tfrac{2\pi}{N}(l_1 + l_2)) \ ,$$

$$\widetilde{\sigma}_{2U_N}^2 := \frac{24}{\sigma_N^8} \frac{2\pi}{N^4} \sum_{l_1 l_2 l_3 l_4} \frac{1}{\delta_{l_1 l_2 l_3 l_4}} \left\{ \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) \right\} K_N(\tfrac{2\pi}{N}(l_1 + l_2 + l_3 + l_4)) \ .$$

We focus on $\widetilde{\sigma}^2_{2U_N}$ as the other cases are strictly analogous, indeed slightly simpler. We have

$$\frac{1}{\widehat{\sigma}^2_{2U_N}} = O_p(1)$$

and

$$E[\widetilde{\sigma}^2_{2U_N} - \widehat{\sigma}^2_{2U_N}] \leq$$

$$\leq \frac{24}{\sigma^8_N} \frac{2\pi}{N^4} \sum_{l_1 l_2 l_3 l_4} \frac{1}{\delta_{l_1 l_2 l_3 l_4}} K_N\big(\tfrac{2\pi}{N}(l_1 + l_2 + l_3 + l_4)\big) E\Big| \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) - \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) \Big| \leq$$

$$\leq C\Big[\max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \frac{1}{\sigma^8_N} E\Big| \Big\{ \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) \Big\} - \Big\{ \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) \Big\} \Big| \Big] .$$

Now notice that

$$x_1 x_2 x_3 x_4 - y_1 y_2 y_3 y_4 = x_1 x_2 x_3 (x_4 - y_4) + x_1 x_2 (x_3 - y_3) y_4 + x_1 (x_2 - y_2) y_3 y_4 + (x_1 - y_1) y_2 y_3 y_4$$

whence

$$\max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \Big\{ \frac{1}{\sigma^8_N} E\Big| \prod_{l=l_1}^{l_4} |\widetilde{w}_l|^2 a^2(\tfrac{4l}{N}) - \prod_{l=l_1}^{l_4} |w_l|^2 a^2(\tfrac{4l}{N}) \Big| \Big\} \leq$$

$$\leq \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \Big\{ \Big(\prod_{l=l_1}^{l_4} a^2(\tfrac{4l}{N}) \Big) E\Big[\frac{|\widetilde{w}_{l_4}|^2 - |w_{l_4}|^2}{\sigma^2_N} \prod_{l=l_1}^{l_3} \frac{|\widetilde{w}_l|^2}{\sigma^2_N} \Big] \Big\} +$$

$$+ \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \Big\{ \Big(\prod_{l=l_1}^{l_4} a^2(\tfrac{4l}{N}) \Big) E\Big| \frac{|\widetilde{w}_{l_3}|^2 - |w_{l_3}|^2}{\sigma^2_N} \frac{|w_{l_4}|^2}{\sigma^2_N} \prod_{l=l_1}^{l_2} \frac{|\widetilde{w}_l|^2}{\sigma^2_N} \Big| \Big\} +$$

$$+ \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \Big\{ \Big(\prod_{l=l_1}^{l_4} a^2(\tfrac{4l}{N}) \Big) E\Big| \frac{|\widetilde{w}_{l_1}|^2}{\sigma^2_N} \frac{\{|\widetilde{w}_{l_2}|^2 - |w_{l_2}|^2\}}{\sigma^2_N} \frac{|w_{l_3}|^2 |w_{l_4}|^2}{\sigma^4_N} \Big| \Big\} +$$

$$+ \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \Big\{ \Big(\prod_{l=l_1}^{l_4} a^2(\tfrac{4l}{N}) \Big) E\Big| \frac{\{|\widetilde{w}_{l_1}|^2 - |w_{l_1}|^2\}}{\sigma^2_N} \frac{|w_{l_2}|^2 |w_{l_3}|^2 |w_{l_4}|^2}{\sigma^6_N} \Big| \Big\} \leq$$

$$\leq \frac{C}{\sigma^6_N} \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} \big\{ E|\widetilde{w}_l|^2 + E|w_l|^2 \big\}^3 \times \frac{1}{\sigma^2_N} \max_{N/8 \leq l_1, l_2, l_3, l_4 \leq N/2} E\big| |\widetilde{w}_l|^2 - |w_l|^2 \big| \leq$$

$$\leq C\Big(\frac{M}{N} \Big)^{-\alpha} = o(1) .$$

$\square$

Remark 7.4 It is evident from the proof that, in order to establish (7.33), it is sufficient to impose the minimal bandwidth condition

$$\frac{1}{N} + \frac{N}{M} \to 0 \text{ as } M \to \infty .$$

which is weaker than Assumption D. This is intuitively due to peculiar form that the aliasing effect assumes for Fourier transforms in the discrete case, see (7.31).

8 Monte Carlo evidence

In a way of confirmation of the CLT, simulations has been performed. It has been chose to put $c_l = l^{-4}$. 1600 fields were simulated on the torus with these specifications and for each of them the skewness and kurtosis statistics were computed for $N = 2^{12}$. The values obtained were then normalized dividing the by the theoretical standard deviations of these statistics, computed using formulas (3.11), (4.14), (4.15) and (4.16). Histograms are presented in Figures 2 and 3, showing a good accordance with Gaussianity.
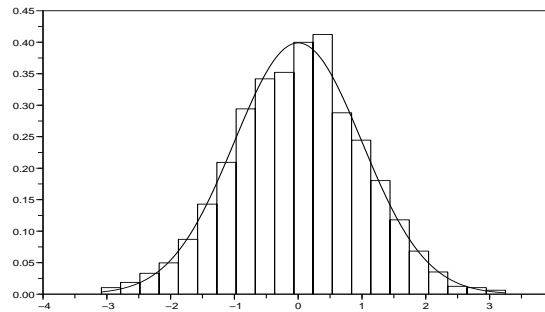

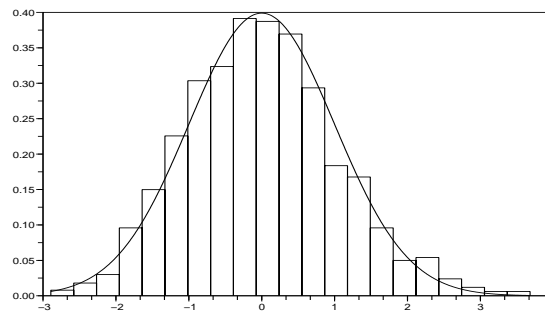Figure 2: Histogram of the skewness statistics over 1600 simulated fields. Here $N = 2^{12}$.

Figure 3: Histogram of the kurtosis statistics over the same 1600 simulated fields.

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