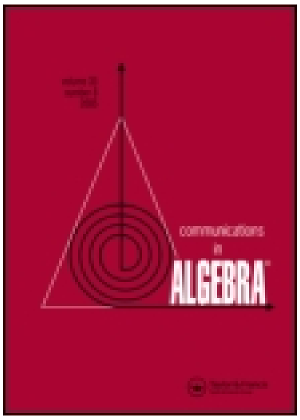

Communications in Algebra

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lagb20

Lines in Quantum Grassmannians

Elisabetta Strickland [a]

[a] Dipartimento di Matematica, Università di Roma, Tor Vergata, Roma, Italy

Available online: 03 Sep 2006

To cite this article: Elisabetta Strickland (2006): Lines in Quantum Grassmannians, Communications in Algebra, 34:4, 1279-1289

To link to this article: http://dx.doi.org/10.1080/00927870500454489

Communications in Algebra®, 34: 1279–1289, 2006
Copyright © Taylor & Francis Group, LLC
ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870500454489

Taylor & Francis
Taylor & Francis Group

LINES IN QUANTUM GRASSMANNIANS

Elisabetta Strickland
Dipartimento di Matematica, Università di Roma, Tor Vergata, Roma, Italy

We study quantum lines in quantum grassmannians and prove that there are only finitely many corresponding to lines in usual grassmannians fixed by a maximal torus.

Key Words: Flag varieties; Grassmannians; Quantum groups.

Mathematics Subject Classification: Primary 20G42; Secondary 16W35.

1. INTRODUCTION

Let V be a vector space over a field k. Consider the Grassmann variety $Gr(h, V)$ of subspaces in V of dimension h, $1 \leq h \leq \dim V - 1$. If we consider the Plücker embedding $Gr(h, V) \to \mathbb{P}(\wedge^h V)$, we have that the variety of lines in $Gr(h, V)$ i.e., the variety of lines in $P(\wedge^h V)$ which lie in $Gr(h, V)$, equals $G(2, V)$ if $h = 1$, $G(n-2, V)$ if $h = n-1$, while, if $2 \leq h \leq n-2$, is the variety $\mathcal{F}(h-1, h+1, V)$ of flags $U_1 \subset U_2 \subset V$, where $\dim U_1 = h-1$ and $\dim U_2 = h+1$. The line corresponding to such a flag consists of those h-dimensional spaces U such that $U_1 \subset U \subset U_2$. The purpose of this article is to show that in the quantum case the situation is drastically different. Indeed in this case we shall see that the quantum $Gr(h, V)$ "contains" only a finite number of lines and these lines are in fact naturally identified with the set of T-fixed points in the variety of lines of a non-quantum $Gr(h, V)$, T being a maximal torus in $Gl(V)$.

In Cohen (1998), Landsberg and Manivel (2003), and Strickland (2002), one studies in full generality the lines in a variety of type G/P, G a semisimple algebraic group, P a parabolic subgroup. In particular it is shown that in most cases (and always if G is simply laced), whenever one fixes a projective embedding $G/P \to \mathbb{P}^N$, the G-variety of lines in $\mathbb{P}^N$ lying in G/P and representing a given homology class is, if not empty, again a variety G/Q for a suitable parabolic subgroup Q.

Received September 15, 2004; Revised October 15, 2005. Communicated by M. Rosso.
Address correspondence to Elisabetta Strickland, Dipartimento di Matematica, Università di Roma, Tor Vergata, Via Della Ricerca Scientifica, I-00133 Roma, Italy; Fax: +39-6-72594599; E-mail: strickla@math.uniroma2.it

In view of the result we obtain here for quantum grassmannians, it is then natural to conjecture, at least in the simply laced case, that an analogous statement will hold in general, namely that lines in a quantum G/P will correspond to T-fixed points of the variety of lines in the classical G/P. However, the method we use in the present article to obtain our results relies on an explicit description of the defining relations of the projective coordinate ring of the quantum grassmannian, and such a description is not available in the general case.

2. DETERMINANTS

Let k be a field, $k(q)$ the field of rational functions in the indeterminate q and $\mathscr{A}$ a $k(q)$-algebra. Given a square $n \times n$ matrix, $B = (b_{i,j})$ of elements in $\mathscr{A}$, we can define the row q-determinant

$$rdet_q(B) := \sum_{w \in S_n} (-q)^{\ell(w)} b_{1,w(1)} \cdots b_{n,w(n)}$$

and its column q-determinant

$$cdet_q(B) := \sum_{w \in S_n} (-q)^{\ell(w)} b_{w(1),1} \cdots b_{w(n),n},$$

where S_n denotes the symmetric group on n letters and ℓ denotes as usual the length function of S_n with respect to the standard generators $s_i = (i, i+1)$, $i = 1, \ldots, n$. We shall say that a non-necessarily square matrix $C = (c_{i,j})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$ is a q-matrix (Parshall and Wang, 1991), if the following relations among the $c_{i,j}$'s are satisfied:

$$\begin{aligned}
&c_{i,j}c_{i,k} = qc_{i,k}c_{i,j}, && \text{for } 1 \leq i \leq m, \quad 1 \leq j < k \leq n \\
&c_{i,j}c_{k,j} = qc_{k,j}c_{i,j}, && \text{for } 1 \leq i < k \leq m, \quad 1 \leq j \leq n \\
&c_{i,j}c_{h,k} = c_{h,k}c_{i,j}, && \text{for } 1 \leq i < h \leq m, \quad 1 \leq k < j \leq n \\
&c_{i,j}c_{h,k} = c_{h,k}c_{i,j} + (q - q^{-1})c_{h,j}c_{i,k}, && \text{for } 1 \leq i < h \leq m, \quad 1 \leq j < k \leq n.
\end{aligned}$$

In the case in which our square matrix B is also a q-matrix, it is not hard to see (Parshall and Wang, 1991) that

$$rdet_q(B) = cdet_q(B) := det_q(B)$$

and this will be called the quantum determinant of B. Let us make a few remarks. For any $1 \leq h \leq n-1$, let us denote by $\mathscr{S}_{h,n}$ the set of permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) \cdots < \sigma(h)$; $\sigma(h+1) < \sigma(h+2) \cdots < \sigma(n)$.

Moreover if we choose indices $1 \leq i_1 < \cdots i_h \leq m$, $1 \leq j_1 < \cdots j_h \leq n$, we shall denote by $[i_1, \ldots, i_h \mid j_1, \ldots, j_h]_q^r$ (respectively $[i_1, \ldots, i_h \mid j_1, \ldots, j_h]_q^c$) the row (respectively column) q-determinants of the minor of our matrix C with the specified row and column indices. We then have the following version of Laplace expansion, whose proof, identical to the usual one, we leave to the reader.

Lemma 2.1. *Let $B = (b_{i,j})$ be a $n \times n$ matrix with coefficients in the algebra $\mathcal{A}$. Then, for every $1 \leq h \leq n-1$,*

$$rdet_q(B) = \sum_{\sigma \in \mathcal{S}_{h,n}} (-q)^{\ell(\sigma)} [1, \dots, h \mid \sigma(1), \dots, \sigma(h)]_q^r [h+1, \dots, n \mid \sigma(h+1), \dots, \sigma(n)]_q^r$$

$$cdet_q(B) = \sum_{\sigma \in \mathcal{S}_{h,n}} (-q)^{\ell(\sigma)} [\sigma(1), \dots, \sigma(h) \mid 1, \dots, h]_q^c [\sigma(h+1), \dots, \sigma(n) \mid h+1, \dots, n]_q^c.$$

Let us now use the above expansion to deduce a few facts about our determinants. We shall state the results in the case of the column expansions, but the reader can easily state and prove the completely analogous results in the row case. Assume that each column of our square matrix $B = (b_{i,j})$ is a column q-vector, i.e., that for each $1 \leq j \leq n$ the $n \times 1$ matrix $(b_{i,j})$, $1 \leq i \leq n$ is a q-matrix or equivalently that $b_{i,j} b_{i+t,j} = q b_{i+t,j} b_{i,j}$ for all $i, j, 0 < t \leq n-i$. We have the following proposition.

Proposition 2.2. 1) *If two consecutive columns of B are equal, then*

$$cdet_q(B) = 0.$$

2) *If for some $1 \leq i < n$ the $n \times 2$ matrix formed by the ith and $i+1$th columns is a q-matrix, then if B' is the matrix obtained from B by exchanging these columns,*

$$cdet_q(B') = (-q) cdet_q(B).$$

Proof. Using Lemma 2.1, we can clearly assume that our two columns are the first. Again using Lemma 2.1, we can expand with respect to the first two columns and we are reduced to prove our claims for 2×2 matrices. To show 1) we have just to remark that if $ab = qba$, then

$$cdet_q\left(\begin{pmatrix} a & a \\ b & b \end{pmatrix} \right) = ab - qba = 0.$$

To prove 2) we have to see that if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a q-matrix,

$$\begin{aligned} cdet_q\left(\begin{pmatrix} b & a \\ d & c \end{pmatrix} \right) &= bc - qda = bc - q(ad - (q - q^{-1})bc) \\ &= -q(ad - qbc) = (-q) cdet_q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right). \end{aligned}$$

□

Remark 2.3. Notice that in our proof we have only used the fact that the two columns involved are q-columns.

Corollary 2.4. 1) *Assume that the square matrix $B = (b_{i,j})$ is a q-matrix. Let $w \in S_n$ and let B^w be the matrix obtained applying the permutation w to the columns of B. Then*

$$cdet_q(B^w) = (-q)^{\ell(w)} cdet_q(B).$$

2) *Assume that there is a permutation $w \in S_n$ such that in B^w two distinct consecutive (not necessarily adjacent) columns which are not equal, form an $n \times 2$ q-matrix, but B has two columns which are equal. Then*

$$cdet_q(B) = 0.$$

3. QUANTUM GRASSMANNIANS AND PLÜCKER RELATIONS

We define the ring of functions on quantum $m \times n$ matrices as the $k(q)$-algebra $k_q[x_{i,j}]$, with generators the elements $x_{i,j}$, $i = 1, \dots m$, $j = 1, \dots, n$ subject to the relations:

$$\begin{aligned}
x_{i,j}x_{i,k} &= qx_{i,k}x_{i,j}, \qquad \text{for } 1 \le i \le m, \quad 1 \le j < k \le n \\
x_{i,j}x_{k,j} &= qx_{k,j}x_{i,j}, \qquad \text{for } 1 \le i < k \le m, \quad 1 \le j \le n \\
x_{i,j}x_{h,k} &= x_{h,k}x_{i,j}, \qquad \text{for } 1 \le i < h \le m, \quad 1 \le k < j \le n \\
x_{i,j}x_{h,k} &= x_{h,k}x_{i,j} + (q - q^{-1})x_{h,j}x_{i,k}, \qquad \text{for } 1 \le i < h \le m, \quad 1 \le j < k \le n.
\end{aligned}$$

We shall call the matrix $X = (x_{i,j})$, which is a q-matrix, the generic $m \times n$ q-matrix. It is well known (Parshall and Wang, 1991) that $k_q[x_{i,j}]$ is a domain and it is a q-deformation of the commutative polynomial ring in mn variables. Now assume $m > n$. Set, using the notations of the previous section,

$$[i_1, \dots, i_n] := [i_1, \dots, i_n \mid 1, \dots, n],$$

$1 < i_1 < \cdots < i_n \le m$.

Let R be the subalgebra of $k_q[x_{i,j}]$ generated by the collection of the elements $[i_1, \dots, i_n]$. We grade R by giving each element $[i_1, \dots, i_n]$ degree 1. This algebra, which obviously depends on m and n, will be called the projective coordinate ring of the quantum grassmannian $Gr_q(n, m)$ (Lenagan and Rigal, 2004). It is a quantum deformation of the usual projective coordinate ring of the Grassmann variety of n-dimensional linear subspaces in an m-dimensional space with the respect to the Plücker embedding. We shall call the elements $[i_1, \dots, i_n] \in R$ *Plücker coordinates*. We are now going to describe a set of relations satisfied by the generators which are completely analogous to the usual Plücker relations. To state the result, let us define for any subset $\{i_1, \dots, i_n\} \subset \{1, \dots, n\}$, $[i_1, \dots, i_n] = (-q)^{\ell(w)}[i_{w(1)}, \dots, i_{w(n)}]$, where $w \in S_n$ is the unique permutation such that $i_{w(1)} < \cdots < i_{w(n)}$. Moreover, if $i_1, \dots i_n$ is a sequence with at least a repetition, we set $[i_1, \dots, i_n] = 0$.

Proposition 3.1. *Let $1 \le i_1, \dots, i_{n+1} \le m$. Then for any two sequences $1 \le r_1 < \cdots < r_{h-1} \le m$, $1 \le p_1 < \cdots < p_{n-h} \le m$ we have*

$$\sum_{\sigma \in \mathcal{S}_{n-h+1,n+1}} (-q)^{\ell(\sigma)}[r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)}] \cdot [i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)}, p_1, ., p_{n-h}] = 0. \quad (1)$$

Proof. By our definitions and Corollary 2.4, we deduce that for every $\sigma \in \mathcal{S}_{n-h+1,n+1}$,

$$[r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)}] = [r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)} \mid 1, \dots, n]_q^r,$$

and similarly for $[i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)}, p_1, ., p_{n-h}]$. Thus, using Lemma 2.1, we get

$$\begin{aligned}
&[r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)}] \\
&\quad = \sum_{\tau \in \mathcal{S}_{h-1,n}} (-q)^{\ell(\tau)} [r_1, ., r_{h-1} \mid \tau(1), ., \tau(h-1)][i_{\sigma(1)}, ., i_{\sigma(n-h+1)} \mid \tau(h), ., \tau(n)]
\end{aligned}$$

and

$$\begin{aligned}
&[i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)}, p_1, ., p_{n-h}] \\
&\quad = \sum_{\phi \in \mathcal{S}_{h,n}} (-q)^{\ell(\phi)} [i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)} \mid \phi(1), ., \phi(h)][p_1, ., p_{n-h} \mid \phi(h+1), ., \phi(n)].
\end{aligned}$$

Notice that we have implicitly used the fact that all q-determinants involved are q-determinants of q-matrices. Substituting, we get

$$\begin{aligned}
&\sum_{\sigma \in \mathcal{S}_{n-h+1,n+1}} (-q)^{\ell(\sigma)} [r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)}][i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)}, p_1, ., p_{n-h}] \\
&= \sum_{\sigma \in \mathcal{S}_{n-h+1,n+1}} (-q)^{\ell(\sigma)} \Bigg(\sum_{\tau \in \mathcal{S}_{h-1,n}} (-q)^{\ell(\tau)} [r_1, ., r_{h-1} \mid \tau(1), ., \tau(h-1)] \\
&\qquad\qquad \cdot [i_{\sigma(1)}, ., i_{\sigma(n-h+1)} \mid \tau(h), ., \tau(n)] \Bigg) \\
&\quad \cdot \Bigg(\sum_{\phi \in \mathcal{S}_{h,n}} (-q)^{\ell(\phi)} [i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)} \mid \phi(1), ., \phi(h)] \cdot [p_1, ., p_{n-h} \mid \phi(h+1), ., \phi(n)] \Bigg) \\
&= \sum_{\tau \in \mathcal{S}_{h-1,n}, \phi \in \mathcal{S}_{h,n}} (-q)^{\ell(\tau)+\ell(\phi)} [r_1, ., r_{h-1} \mid \tau(1), ., \tau(h-1)] \\
&\quad \cdot \Bigg(\sum_{\sigma \in \mathcal{S}_{n-h+1,n+1}} (-q)^{\ell(\sigma)} [i_{\sigma(1)}, ., i_{\sigma(n-h+1)} \mid \tau(h), ., \tau(n)] \\
&\qquad \cdot [i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)} \mid \phi(1), ., \phi(h)] \Bigg) \cdot [p_1, ., p_{n-h} \mid \phi(h+1), ., \phi(n)].
\end{aligned}$$

But now notice that for every choice of $\tau(h), ., \tau(n), \phi(1), ., \phi(h)$,

$$\sum_{\sigma \in \mathcal{S}_{n-h+1,n+1}} (-q)^{\ell(\sigma)} [i_{\sigma(1)}, ., i_{\sigma(n-h+1)} \mid \tau(h), ., \tau(n)] \cdot [i_{\sigma(n-h+2)}, ., i_{\sigma(n+1)} \mid \phi(1), ., \phi(h)]$$

clearly equals $cdet_q(S)$, where S is the matrix having as rows the rows of X of indices $i_1, \dots i_{n+1}$ and as columns the columns of X of indices $\tau(h), ., \tau(n), \phi(1), ., \phi(h)$. Since X has only n columns and we have $n+1$ indices, we get that S has two equal columns. On the other hand, the requirements of Corollary 2.4 are clearly satisfied, so that we deduce that $cdet_q(S) = 0$ and hence our claim. $\square$

From now on, in order to simplify our notations, we shall set for any sequence $T = (t_1, \ldots, t_n)$, with $1 \leq t_1, \ldots, t_n \leq n$, $p_T := [t_1, \ldots, t_n]$. Given two such sequences $T = (t_1, \ldots, t_n)$ and $S = (s_1, \ldots, s_n)$, we shall say that $T \leq S$ if, for each $1 \leq h \leq n$, $t_h \leq s_h$. Having given this definition, we have, as in the classical case the following corollary.

Corollary 3.2. *Let $T = (t_1 < \cdots < t_n)$ and $S = (s_1 < \cdots < s_n)$ be two sequences. Assume that $t_i \leq s_i$ for $i < h$, while $t_h > s_h$. Then the product $p_T p_S$ can be expressed as a linear combination of products $p_{T'} p_{S'}$ with $T' < T$, $S' > S$.*

Proof. Apply Proposition 3.1 with $(i_1 \ldots, i_{n+1}) = (s_1, \ldots, s_h, t_h, \ldots, t_n)$, $(r_1, \ldots, r_{h-1}) = (t_1, ., t_{h-1})$ and $(p_1, \ldots, p_{n-h}) = (s_{h+1}, \ldots, s_n)$.

Now remark that, except for the unique $\sigma \in \mathscr{S}_{n-h+1,n+1}$ with $\sigma(1) = h+1, \ldots, \sigma(n-h+1) = n+1$, for which $(r_1, ., r_{h-1}, \sigma(1), \ldots, \sigma(n-h+1)) = T$ and $(\sigma(n-h+2), \ldots, \sigma(n+1), p_1, \ldots p_{n-h}) = S$, for all other $\sigma \in \mathscr{S}_{n-h+1,n+1}$ one has that $(r_1, ., r_{h-1}, \sigma(1), \ldots, \sigma(n-h+1)) < T$ and $(\sigma(n-h+2), \ldots, \sigma(n+1), p_1, \ldots p_{n-h}) > S$. $\square$

Corollary 3.3. *Let $T = (t_1 < \cdots < t_n)$ and $S = (s_1 < \cdots < s_n)$ be two sequences. Assume that $t_i = s_i$ for $i < h$ (respectively $i > h$), while $s_h < t_h$. Then the product $p_T p_S$ can be expressed as a linear combination of products $p_{T'} p_{S'}$ with $S' = (s_1, \ldots, s_{h-1}, t_p, s_{h+1}, \ldots s_n)$ for some $p \geq h$ (respectively $T' = (t_1, \ldots, t_{h-1}, s_p, t_{h+1}, \ldots t_n)$ for some $p \leq h$).*

Proof. We shall give the proof in the case in which $t_i = s_i$ for $i < h$, the proof in the other case being completely analogous. In our case, let us apply Proposition 3.1 with $(i_1 \ldots, i_{n+1}) = (s_1, \ldots, s_h, t_h, \ldots, t_n)$, $(r_1, \ldots, r_{h-1}) = (t_1, ., t_{h-1})$ and $(p_1, \ldots, p_{n-h}) = (s_{h+1}, \ldots, s_n)$. Notice that if $\sigma \in \mathscr{S}_{n-h+1,n+1}$ is such that for at least one $2 \leq \ell \leq n-h+1$, we have $\sigma(\ell) < h$. So $[r_1, ., r_{h-1}, i_{\sigma(1)}, ., i_{\sigma(n-h+1)}] = 0$, having two equal indices. Using this, the claim follows immediately. $\square$

As in the classical case, the above relations allow us to get a basis for the ring R. We only sketch the proof, which is identical to the one given classically (see Hodge, 1943 and also Lenagan and Rigal, 2004). Of course this is just a special case of the standard monomial theory of Littelmann (1998), but one should remark that as for the usual coordinate ring of a grassmannian, the above relations furnish a straightening algorithm. To state this let us define a monomial $p_{T_1} \ldots p_{T_r}$ to be standard, if $T_1 \leq T_2 \leq \cdots \leq T_r$.

Theorem 3.4. *The standard monomials are a $k(q)$-basis of R.*

Furthermore the ideal of relations among the generators p_T's is generated by the Plücker relations of Proposition 3.1.

Proof. The fact that every monomial in the p_T's can be written as a linear combination of standard monomials, using only the Plücker relations, is an immediate consequence of Corollary 3.2 and a simple induction.

So what remains to show is the linear independence. To see this, let $\sum a_i M_i = 0$, with $a_i \in k(q)$, M_i standard monomials in the p_T's, be a linear relation. Multiplying by a suitable power of $(q-1)$ and using the fact that $q-1$ is not a zero divisor, we can assume that none of the a_i's has a pole at $q=1$ and at least one is nonzero modulo $q-1$. So, we can specialize at $q=1$ and, using the linear independence in the classical case, deduce that $a_i \equiv 0$ modulo $q-1$ for each i—a contradiction. $\square$

Remark 3.5. Notice that the Plücker relations have coefficients in $\mathbb{Z}[q, q^{-1}]$, so that we can easily define a $\mathbb{Z}[q, q^{-1}]$ form $\mathcal{R}$ of our ring R. We leave to the reader the immediate verification that the above theorem holds verbatim for the $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{R}$.

4. LINES IN QUANTUM GRASSMANNIANS

In accord to our definition of a quantum grassmannian, the quantum projective line is the graded $k(q)$-algebra $k_q[x, y]$ generated by the two degree 1 elements x, y subject to the relation

$$xy = qyx.$$

Usually (Manin, 1988) this algebra is called the quantum plane, but here we want to stress the fact that, with its grading, it is a q-analogue of the homogeneous coordinate ring of $\mathbb{P}^1$.

Definition 1. Let R be, as in the previous section, the coordinate ring of the quantum grassmannian $Gr_q(n, m)$. A line in $Gr_q(n, m)$ is a graded ideal $I \subset R$ such R/I is isomorphic to $k_q[x, y]$ as a graded ring.

Before analyzing the lines in $Gr_q(n, m)$, let us make a few simple remarks.

Lemma 4.1. *Let $\phi : k_q[x, y] \to k_q[x, y]$ be a (graded) automorphism. Then there exist nonzero constants $\alpha, \beta \in k(q)$ such that*

$$\phi(x) = \alpha x, \qquad \phi(y) = \beta y.$$

Proof. We have $\phi(x) = \alpha x + \gamma y$ and $\phi(y) = \delta x + \beta y$, with $\alpha, \gamma, \delta, \beta \in k(q)$ and $\alpha\beta - \gamma\delta \neq 0$. So

$$\begin{aligned}(\alpha x + \gamma y)(\delta x + \beta y) &= \alpha\delta x^2 + (\alpha\beta + q^{-1}\gamma\delta)xy + \gamma\beta y^2 \\ &= q(\delta x + \beta y)(\alpha x + \gamma y) = q\alpha\delta x^2 + (\alpha\beta + q\gamma\delta)xy + q\gamma\beta y^2.\end{aligned}$$

In particular $(1-q)\alpha\delta = (1-q)\beta\gamma = (q - q^{-1})\gamma\delta = 0$. This clearly implies that $\gamma = \delta = 0$, as desired. $\square$

Let us now determine the lines in a $m-1$-dimensional quantum projective space. In this case, R is the ring $k_q[x_1, \ldots, x_m]$, with $x_i x_i = q x_j x_i$ if $i < j$.

Proposition 4.2. *Let $I \subset R$ be a line in the $m-1$-dimensional quantum projective space. Then there exists two indices $1 \leq i < j \leq m$ such that I is the ideal generated by the elements x_h, $h \neq i, j$. Furthermore, one can fix the isomorphism $\phi : R/I \to k_q[x, y]$ in such a way that $\bar{\phi}(x_i) = x$, $\bar{\phi}(x_j) = y$ where $\bar{\phi}$ is the composition of ϕ with the projection $R \to R/I$.*

Proof. Let us fix an isomorphism $\phi : R/I \to k_q[x, y]$ and let $\bar{\phi} : R \to k_q[x, y]$ denote the composition of the natural projection with ϕ. Since $\bar{\phi}$ is surjective and graded, there must be two distinct indices $i < j$ such that $\bar{\phi}(x_i)$ and $\bar{\phi}(x_j)$ span the degree one component of $k_q[x, y]$. Now we have that $x_i x_j = q x_j x_i$, so by Lemma 4.1 we deduce that $\bar{\phi}(x_i) = \alpha x$, $\bar{\phi}(x_j) = \beta y$ for two nonzero constants α and β. Composing with an automorphism of $k_q[x, y]$, we then deduce that we can assume that $\bar{\phi}$ carries x_i into x and x_j into y. Let us now take $h \neq i, j$ and assume $\bar{\phi}(x_h) \neq 0$. If $h < i$, then necessarily $\bar{\phi}(x_h) = \alpha x$ and $\bar{\phi}(x_i) = \beta y$, a contradiction. Similarly, if $h > j$ one deduces that $\bar{\phi}(x_j) = \beta x$. Finally, if $i < h < j$, one deduces on the one hand that $\bar{\phi}(x_h) = \alpha x$, on the other that $\bar{\phi}(x_h) = \beta y$, again a contradiction. This proves our claim. □

Notice that Proposition 4.2 shows our main claim in the case of projective spaces. Indeed consider k^n with basis $e_1, \dots e_m$, and take the maximal torus T in $Gl(n)$ consisting of diagonal matrices. The set of T-fixed lines in $\mathbb{P}(K^n)$, consists exactly of the lines $L_{i,j}$, $1 \leq i < j \leq m$ where $L_{i,j}$ is the line joining the points $[e_i]$ and $[e_j]$. Hence, Proposition 4.2 tells us that the only lines which can be quantized are exactly the T-fixed lines.

We pass now to the general case of the quantum $G_q(m, n)$. We can assume that $1 < n < m - 1$, since otherwise we are in the case of projective spaces, which we have already discussed in Proposition 4.2. First let us make some considerations on the usual Grassmann variety. In this case a line in $G(m, n)$, consists (Strickland, 2002) of the set of n-dimensional subspaces in k^m, which contain a given $n-1$-dimensional subspace H and are contained in a given $n+1$-dimensional subspace K. Thus, as a variety, the lines in $G(m, n)$ are just the variety of flags $\{H \subset K \subset k^m \mid \dim H = n-1, \dim K = n+1\}$. A fix point in this variety, under the maximal torus of diagonal matrices, is nothing else that the set of lines containing a given $n-1$-dimensional coordinate subspace, say the one generated by the basis vectors $e_{i_1}, \dots, e_{i_{n-1}}$, $1 \leq i_1 < \cdots < i_{n-1} \leq m$ and contained in the $n+1$-dimensional subspace K spanned by H and two more basis vectors, say e_h, e_k. The next Proposition shows that these points survive in the quantum case.

Proposition 4.3. *Let R be the the projective coordinate ring of the quantum grassmannian $Gr_q(n, m)$. Let $1 \leq i_1 < \cdots < i_{t-1} < h < i_t < \cdots i_{s-1} < k < i_s < \cdots i_{n-1} \leq m$. Set $\mathcal{I} = \{i_1, \dots, i_{n-1}\}$ and $H = \{h, k\}$. Then there is a homomorphism*

$$\psi_{\mathcal{I},H} : R \to k_q[x, y]$$

such that

$$\psi_{\mathcal{I},H}([i_1, \dots, i_{t-1}, h, i_t, \dots, i_{n-1}]) = x,$$
$$\psi_{\mathcal{I},H}([i_1, \dots, i_{s-1}, k, i_s, \dots, i_{n-1}]) = y,$$

while for any other Plücker coordinate $[j_1, \ldots, j_n]$,

$$\psi_{\mathcal{I},H}([j_1, \ldots, j_n]) = 0.$$

Proof. Consider the free algebra T with generators x_J, with J any subset with n elements in $\{1, \ldots, m\}$. We have an obvious surjective morphism $\pi : T \to R$ defined by $\pi(x_J) = [j_1, \ldots, j_n]$, for each $J = \{j_1 < \cdots < j_n\}$, whose kernel is the ideal generated by the Plücker relations

$$\sum_{\sigma \in \mathcal{S}_{h,n+1}} (-q)^{\ell(\sigma)} x_{\{r_1,..,r_{h-1},i_{\sigma(1)},..,i_{\sigma(n-h+1)}\}} x_{\{i_{\sigma(n-h+2)},..,i_{\sigma(n+1)},p_1,..,p_{n-h}\}}.$$

Also, we clearly have a surjective morphism $\psi'_{\mathcal{I},H} : T \to k_q[x, y]$ defined by setting $\psi'_{\mathcal{I},H}(x_J) = x$ if $J = \{i_1, \ldots, i_{t-1}, h, i_t, \ldots, i_{n-1}\}$, $\psi'_{\mathcal{I},H}(x_J) = y$ if $J = \{i_1, \ldots, i_{s-1}, k, i_s, \ldots, i_{n-1}\}$, $\psi'_{\mathcal{I},H}(x_J) = 0$ for all other Js. Thus in order to show the existence of our $\psi_{\mathcal{I},H}$, what we have to show is that each of the Plücker relations lies in the kernel of $\psi'_{\mathcal{I},H}$. It is clear from the nature of these relations, that the only case we need to consider is the case in which the subsets $\{i_1, \ldots, i_{t-1}, h, i_t, \ldots, i_{n-1}\}$ and $\{i_1, \ldots, i_{s-1}, k, i_s, \ldots, i_{n-1}\}$ are involved in the relation. In this case we get exactly one relation, namely

$$x_{\{i_1,..,i_{t-1},h,i_t,..,i_{n-1}\}} x_{\{i_1,..,i_{s-1},k,i_s,..,i_{n-1}\}} - q x_{\{i_1,..,i_{s-1},k,i_s,..,i_{n-1}\}} x_{\{i_1,..,i_{t-1},h,i_t,..,i_{n-1}\}}.$$

Applying $\psi'_{\mathcal{I},H}$ we get

$$xy - qyx = 0$$

so everything follows. □

Finally, we are going to see that the set of quantum lines in the quantum grassmannian $Gr_q(n, m)$ consists only of the points we have already constructed. Namely, we have the following theorem.

Theorem 4.4. *Let* $\psi : R \to k_q[x, y]$ *be a graded surjective homomorphism. Let* I *denote its kernel. Then there exist two subsets in* $\{1, \ldots, m\}$, $\mathcal{I} = \{i_1, \ldots, i_{n-1}\}$ *and* $H = \{h, k\}$ *with* $H \cap \mathcal{I} = \emptyset$, *such that* $I = ker\, \psi_{\mathcal{I},H}$.

Furthermore, if $i_1 < \cdots < i_{t-1} < h < i_t < \cdots < i_{s-1} < k < i_s < \cdots i_{n-1}$, *there exist two nonzero constants* $\alpha, \beta \in k(q)$ *such that*

$$\psi([i_1, \ldots, i_{t-1}, h, i_t, \ldots, i_{n-1}]) = \alpha x,$$
$$\psi([i_1, \ldots, i_{s-1}, k, i_s, \ldots, i_{n-1}]) = \beta y,$$

Proof. Since ψ is not zero, it is clear that there exist a sequence $T = (t_1, \ldots, t_n)$ such that p_T does not lie in I. Suppose that if a sequence $T' = (t'_1, \ldots, t'_n)$ differs from T only by one index, then $p_{T'} \in I$. On the other hand ψ is surjective, so that there must be another element p_S, $S = (s_1, \ldots, s_n)$, different from T and not lying in I. By our assumption the two elements differ at least by 2 indices.

Assume that h is the largest index such that $t_h \neq s_h$. Up to exchanging T with S, we can also assume $t_h > s_h$. Then applying Corollary 3.3, we deduce that the product $p_T p_S$ can be written as a linear combination of products $p_{T'} p_{S'}$ with $T' = (t_1, \ldots, t_{h-1}, s_p, t_{h+1}, \ldots t_n)$ for some $p \leq h$. Then, applying our assumptions, we deduce that $\psi(p_T p_S) = 0$. Since $k_q[x, y]$ is a domain, this contradicts the fact that both p_T and p_S do not lie in I. □

At this point we have found that there must exist $1 \leq i_1 < \cdots < i_{t-1} < h < i_t < \cdots i_{s-1} < k < i_s < \cdots i_{n-1} \leq m$ such that, setting $T = (i_1, \ldots, i_{t-1}, h, i_t \ldots, i_{n-1})$ and $S = (i_1, \ldots, i_{s-1}, k, i_s \ldots, i_{n-1})$,

$$\psi(p_T) \neq 0 \quad \text{and} \quad \psi(p_S) \neq 0.$$

Since we have that

$$p_T p_S = q p_S p_T,$$

we also obtain, by Lemma 4.1, that there exist two nonzero constants $\alpha, \beta \in k(q)$ such that

$$\psi(p_T) = \alpha x \quad \text{and} \quad \psi(p_S) = \beta y.$$

So the only thing we still have to show is that, for any ordered sequence $V = (v_1 < \cdots < v_n)$ not equal either to T or to S, we have that $\psi(p_V) = 0$.

To see this, let us start by remarking that if V differs from T only by one index, and if furthermore the product $p_V p_T$ is standard, then, since

$$p_V p_T = q p_T p_V,$$

we deduce using Lemma 4.1 that $\psi(p_V) = 0$. In a completely analogous way we see that if V differs from S only by one index, and if furthermore the product $p_S p_V$ is standard, $\psi(p_V) = 0$.

Now assume that $p_V p_T$ is standard. Let $j \leq n$ be the largest integer such that v_j is different from the jth index in T. Again, applying Corollary 3.3, we deduce that $p_T p_V$ can we written as a linear combination of elements $p_{T'} p_{V'}$ with T' obtained from T by substituting the jth index with an index v_r, $r \leq j$. Thus applying ψ and using the above remarks, we deduce that $\psi(p_T p_V) = 0$. Since $k_q[x, y]$ is a domain, we then get that $p_V \in I$. In a completely analogous fashion, we can show that if $p_S p_V$ is standard, then $\psi(p_V) = 0$.

Assume now that both $p_T p_V$ and $p_V p_T$ are not standard. Applying Corollary 3.2, we can write $p_T p_V$ as a linear combination of products $p_{T'} p_{V'}$ with $p_{T'} p_T$ standard. We deduce in this case too that $\psi(p_T p_V) = 0$, and hence $\psi(p_V) = 0$. Similarly we see that, if neither $p_S p_V$ and $p_V p_S$ are standard, then $\psi(p_V) = 0$.

Thus it remains only to deal with the case in which $p_T p_V p_S$ is standard. In this case, we can repeat all our previous reasoning and deduce that if $\psi(p_V) \neq 0$, either

$$\psi(p_V) = \gamma x \quad \text{or} \quad \psi(p_V) = \gamma y,$$

with γ a nonzero constant in $k(q)$. In the first case, our previous argument gives, since $p_T p_V$ is standard, that $\psi(p_T) = 0$, a contradiction. In the second, that $\psi(p_S) = 0$, again a contradiction. So $\psi(p_V) = 0$, and everything follows. $\square$

REFERENCES

Cohen, A. M., Cooperstein, B. N. (1998). Lie incidence systems for projective varieties. *Proc. Amer. Math. Soc.* 126:2095–2102.

Hodge, W. V. D. (1943). Some enumerative results in the theory of forms. *Proc. Cambridge Philos. Soc.* 39:22–30.

Landsberg, J. M., Manivel, L. (2003). On the projective geometry of homogeneous varieties. *Comment. Math. Helv.* 78:65–100.

Lenagan, T. H., Rigal, L. (2004). Quantum graded algebras with a straightening law and the AS-Cohen-Macaulay property for quantum determinantal rings and quantum grassmannians. QA/0403021.

Littelmann, P. (1998). Contracting modules and standard monomial theory for symmetrizable Kac-Moody algebras. *J. Amer. Math. Soc.* 11:551–567.

Manin, Y. I. (1988). Quantum Groups and Noncommutative Geometry. Universitè de Montrèal, Centre de Recherches Mathématiques, Montreal, QC, vi+91 pp.

Parshall, B., Wang, J. P. (1991). Quantum linear groups. *Mem. Amer. Math. Soc.* 89(439):vi+157 pp.

Strickland, E. (2002). Lines in G/P. *Mathematische Zeitschrift*. 242(2):227–240.