

New Results on Periodic Symbolic Sequences of Second Order Digital Filters with Two's Complement Arithmetic

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SUMMARY

In this article, the second order digital filter with two's complement arithmetic in [1] is considered. Necessary conditions for the symbolic sequences to be periodic after a number of iterations are given when the filter parameters are at $b = a + 1$ and $b = -a + 1$. Furthermore, for some particular values of a , even when one of the eigenvalues is outside the unit circle, the system may behave as a linear system after a number of iterations and the state vector may toggle between two states or converge to a fixed point at the steady state. The necessary and sufficient conditions for these phenomena are given in this article.

KEY WORDS: second order digital filter; two's complement arithmetic; symbolic sequences; eigenvalues

1. INTRODUCTION

A nonlinear behavior may occur on a second order digital filter when the filter is implemented using a two's complement arithmetic for the addition operation. To analyze such a behavior, a symbolic analysis was proposed and the admissibility of the symbolic sequences was studied for the special case when $b=-1$ and $|a|<2$ [1-4, 7]. However, even when the symbolic sequence is admissible, there are many possibilities. In order to study the various possibilities, the set of admissible sequences can be partitioned into three subsets: One set contains periodic symbolic sequences. The second set contains symbolic sequences that are periodic after a number of iterations. The third set contains symbolic sequences that are never periodic. Some results on the first two sets were obtained for $b=-1$ and $|a|<2$ [1-4, 7]. These results have been extended for other real values of a , while the value of b is still equal to -1 [5].

However, will those existing results remain valid if the filter parameter $b \neq -1$? Specifically, we are interested in the nonlinear behavior which may occur when $b = a+1$ and $b = -a+1$, and the answer to the following questions: Under what conditions will the symbolic sequence be periodic? If the symbolic sequence is periodic, under what conditions will the system behave as a linear system after a number of iterations and the state vector toggles among several states or converges to a fixed point? In this article, we focus on both the cases of $b = a+1$ and $b = -a+1$. In section 2, we will present the notations used in the existing literatures [1-7], and this article will also employ the same set of notations. In section 3, some new results on the above problems are presented. Finally, a conclusion is summarized in section 4.

2. NOTATIONS

The notations used in [1-7] are adopted as follows:

The system is defined as:

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = F\left(\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}\right) = \begin{bmatrix} x_2(k) \\ f(b \cdot x_1(k) + a \cdot x_2(k)) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k \quad (1)$$

$$\text{where } \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2 \equiv \{(x_1, x_2) : -1 \leq x_1 < 1, -1 \leq x_2 < 1\} \quad (2)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \quad (3)$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad (4)$$

$s_k \in \{-m, \dots, -1, 0, 1, \dots, m\}$ where m is the minimum integer satisfying

$$-2 \cdot m - 1 \leq b \cdot x_1 + a \cdot x_2 < 2 \cdot m + 1 \quad (5)$$

and $f(x) = x - 2 \cdot n$ such that $2 \cdot n - 1 \leq x < 2 \cdot n + 1$ and $n \in \mathbb{Z}^+ \cup \{0\}$ (6)

Given an initial condition $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2$, a symbolic sequence $s = (s_0, s_1, \dots) \in \Sigma$ can

be generated by the map $S: I^2 \rightarrow \Sigma$, and a sequence s in Σ is admissible if

$$\exists \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in I^2 \text{ such that } S\left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}\right) = s.$$

The set Σ can be partitioned into three subsets: $\Sigma_\alpha = \{s = (s_0 s_1 s_2 \dots) : s \text{ is periodic}\}$,

$\Sigma_\beta = \{s = (s_0 s_1 s_2 \dots) : s \text{ is periodic after a number of iterations}\}$ and

$$\Sigma_\gamma = \Sigma \setminus (\Sigma_\alpha \cup \Sigma_\beta).$$

3. NEW RESULTS FOR THE PERIODIC SYMBOLIC SEQUENCES

This section presents several conditions for the state vector to be periodic after a number of iterations. These conditions can be stated in the following lemmas, theorems and remarks, where the stability of these periodic orbits is stated in the observations:

Lemma 1

For $b = a + 1$, if $\exists M \in \mathbb{Z}^+$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k + M) \\ x_2(k + M) \end{bmatrix}$ for $k \geq k_0$, then

$$(x_1(0) + x_2(0)) \cdot ((a+1)^M - 1) + 2 \cdot (2 \cdot (a+1)^M - 1) \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} + 2 \cdot (a+1)^M \cdot \sum_{j=k_0}^{k_0+M-1} \frac{s_j}{(a+1)^{j+1}} = 0.$$

Proof

For $b = a + 1$, the nonlinear system in (1) can be represented by the state equation

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cdot s_k. \text{ Hence, the solution of the system is:}$$

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a+2} \cdot \begin{bmatrix} (a+1)^k + (a+1) \cdot (-1)^k & (a+1)^k - (-1)^k \\ (a+1)^{k+1} - (a+1) \cdot (-1)^k & (a+1)^{k+1} + (-1)^k \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2} \cdot \sum_{j=0}^{k-1} \begin{bmatrix} (a+1)^{k-1-j} - (-1)^{k-1-j} \\ (a+1)^{k-j} + (-1)^{k-1-j} \end{bmatrix} \cdot s_j \quad (7)$$

If $\exists M \in \mathbb{Z}^+$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k + M) \\ x_2(k + M) \end{bmatrix}$ for $k \geq k_0$, then:

$$\begin{aligned} & \frac{1}{a+2} \cdot \begin{bmatrix} (a+1)^{k_0} + (a+1) \cdot (-1)^{k_0} & (a+1)^{k_0} - (-1)^{k_0} \\ (a+1)^{k_0+1} - (a+1) \cdot (-1)^{k_0} & (a+1)^{k_0+1} + (-1)^{k_0} \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2} \cdot \sum_{j=0}^{k_0-1} \begin{bmatrix} (a+1)^{k_0-1-j} - (-1)^{k_0-1-j} \\ (a+1)^{k_0-j} + (-1)^{k_0-1-j} \end{bmatrix} \cdot s_j \\ & = \frac{1}{a+2} \cdot \begin{bmatrix} (a+1)^{k_0+M} + (a+1) \cdot (-1)^{k_0+M} & (a+1)^{k_0+M} - (-1)^{k_0+M} \\ (a+1)^{k_0+M+1} - (a+1) \cdot (-1)^{k_0+M} & (a+1)^{k_0+M+1} + (-1)^{k_0+M} \end{bmatrix} \cdot \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} \begin{bmatrix} (a+1)^{k_0+M-1-j} - (-1)^{k_0+M-1-j} \\ (a+1)^{k_0+M-j} + (-1)^{k_0+M-1-j} \end{bmatrix} \cdot s_j \end{aligned} \quad (8)$$

Hence,

$$\left\{ \begin{aligned} & \left((a+1)^M - 1 \right) \cdot \frac{(a+1)^{k_0}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) + \left((-1)^M - 1 \right) \cdot \frac{(-1)^{k_0}}{a+2} \cdot \left((a+1) \cdot x_1(0) - x_2(0) - 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(-1)^{j+1}} \right) \\ & + \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot \left((a+1)^{k_0+M-1-j} - (-1)^{k_0+M-1-j} \right) = 0 \end{aligned} \right. \quad (9)$$

$$\left\{ \begin{aligned} & \left((a+1)^M - 1 \right) \cdot \frac{(a+1)^{k_0+1}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) + \left((-1)^M - 1 \right) \cdot \frac{(-1)^{k_0}}{a+2} \cdot \left(2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(-1)^{j+1}} - ((a+1) \cdot x_1(0) - x_2(0)) \right) \\ & + \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot \left((a+1)^{k_0+M-j} + (-1)^{k_0+M-1-j} \right) = 0 \end{aligned} \right. \quad (10)$$

Let

$$t_1 = \left((a+1)^M - 1 \right) \cdot \frac{(a+1)^{k_0}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) \quad (11)$$

$$t_2 = \left((-1)^M - 1 \right) \cdot \frac{(-1)^{k_0}}{a+2} \cdot \left((a+1) \cdot x_1(0) - x_2(0) - 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(-1)^{j+1}} \right) \quad (12)$$

$$t_3 = \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (a+1)^{k_0+M-1-j} \quad (13)$$

$$t_4 = \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (-1)^{k_0+M-1-j} \quad (14)$$

Then we have:

$$\begin{cases} t_1 + t_2 + t_3 - t_4 = 0 \\ (a+1) \cdot t_1 - t_2 + (a+1) \cdot t_3 + t_4 = 0 \end{cases} \quad (15)$$

which

$$\Rightarrow a = -2 \quad \text{or} \quad t_1 = -t_3 \quad (16)$$

For $a \neq -2$, we have $t_1 = -t_3$, that is:

$$\begin{aligned} & \left((a+1)^M - 1 \right) \cdot \frac{(a+1)^{k_0}}{a+2} \cdot \left(x_1(0) + x_2(0) + 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} \right) = - \frac{2}{a+2} \cdot \sum_{j=0}^{k_0+M-1} s_j \cdot (a+1)^{k_0+M-1-j} \quad (17) \\ & \Rightarrow (x_1(0) + x_2(0)) \cdot \left((a+1)^M - 1 \right) + 2 \cdot \left(2 \cdot (a+1)^M - 1 \right) \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a+1)^{j+1}} + 2 \cdot (a+1)^M \cdot \sum_{j=k_0}^{k_0+M-1} \frac{s_j}{(a+1)^{j+1}} = 0 \quad \blacksquare \end{aligned}$$

Remark 1

If, after a number of iterations, the state vector is periodic with period M , then the symbolic sequence will also be periodic with the same period, that is, $s \in \Sigma_\beta$. Hence, this lemma gives a necessary condition for a symbolic sequence to be periodic after a number of iterations.

However, s_k is an integer in $\{-m, \dots, -1, 0, 1, \dots, m\}$ and the periodicity of the symbolic sequence is M . So there are $(2 \cdot m + 1)^M$ possibilities. In the following

theorem, the case of $s_k = 0$ for $k \geq k_0$ is discussed.

Theorem 1

For $b = a + 1$ and $|a + 1| > 1$, $s_k = 0$ for $k \geq k_0$ if and only if $\exists k_0 \in \mathbb{Z}^+ \cup \{0\}$ such that $x_1(k_0) = -x_2(k_0)$.

Proof

For the *if* part, since $b = a + 1$ and $s_k = 0$ for $k \geq k_0$, we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a + 2} \cdot \begin{bmatrix} (a + 1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0)) + (-1)^{k-k_0} \cdot ((a + 1) \cdot x_1(k_0) - x_2(k_0)) \\ (a + 1)^{k-k_0+1} \cdot (x_1(k_0) + x_2(k_0)) - (-1)^{k-k_0} \cdot ((a + 1) \cdot x_1(k_0) - x_2(k_0)) \end{bmatrix} \quad (18)$$

Since $|a + 1| > 1$, $(a + 1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ diverges as $k \rightarrow +\infty$ if

$x_1(k_0) + x_2(k_0) \neq 0$. However, as $s_k = 0$ for $k \geq k_0$, this implies that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \in I^2$

and that $(a + 1)^{k-k_0} \cdot (x_1(k_0) + x_2(k_0))$ is bounded for $k \geq k_0$. Hence,

$x_1(k_0) + x_2(k_0) = 0$ and this proves the *if* part.

For the *only if* part, since $x_1(k_0) = -x_2(k_0)$, we have:

$$\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f((a + 1) \cdot x_1(k_0) - a \cdot x_1(k_0)) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ f(x_1(k_0)) \end{bmatrix}. \text{ Since } |x_1(k_0)| < 1, \text{ we have}$$

$$\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} -x_1(k_0) \\ x_1(k_0) \end{bmatrix} \text{ and } s_{k_0} = 0.$$

Similarly, we have $\begin{bmatrix} x_1(k_0 + 2) \\ x_2(k_0 + 2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ -x_1(k_0) \end{bmatrix}$ and $s_{k_0+1} = 0$. Since

$\begin{bmatrix} x_1(k_0 + 2) \\ x_2(k_0 + 2) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix}$, we have $s_k = 0$ for $k \geq k_0$ and this proves the *only if* part. ■

Remark 2

The eigenvalues of matrix \mathbf{A} is -1 and $a + 1$. Since $|a + 1| > 1$, one of the

eigenvalues is outside the unit circle. However, the system may behave as a linear system when $s_k = 0$ for $k \geq k_0$. Theorem 1 gives the necessary and sufficient condition for the nonlinear system to behave as a linear system after a number of iterations. It is interesting to note that the system may behave as a linear system after a number of iterations if and only if the state vector toggles between two points on a particular straight line of the phase portrait.

Example 1

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

$a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.9003 \\ -0.5377 \end{bmatrix}$. Figure 1 shows the phase portrait of the system. It

can be seen from the figure that the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait.

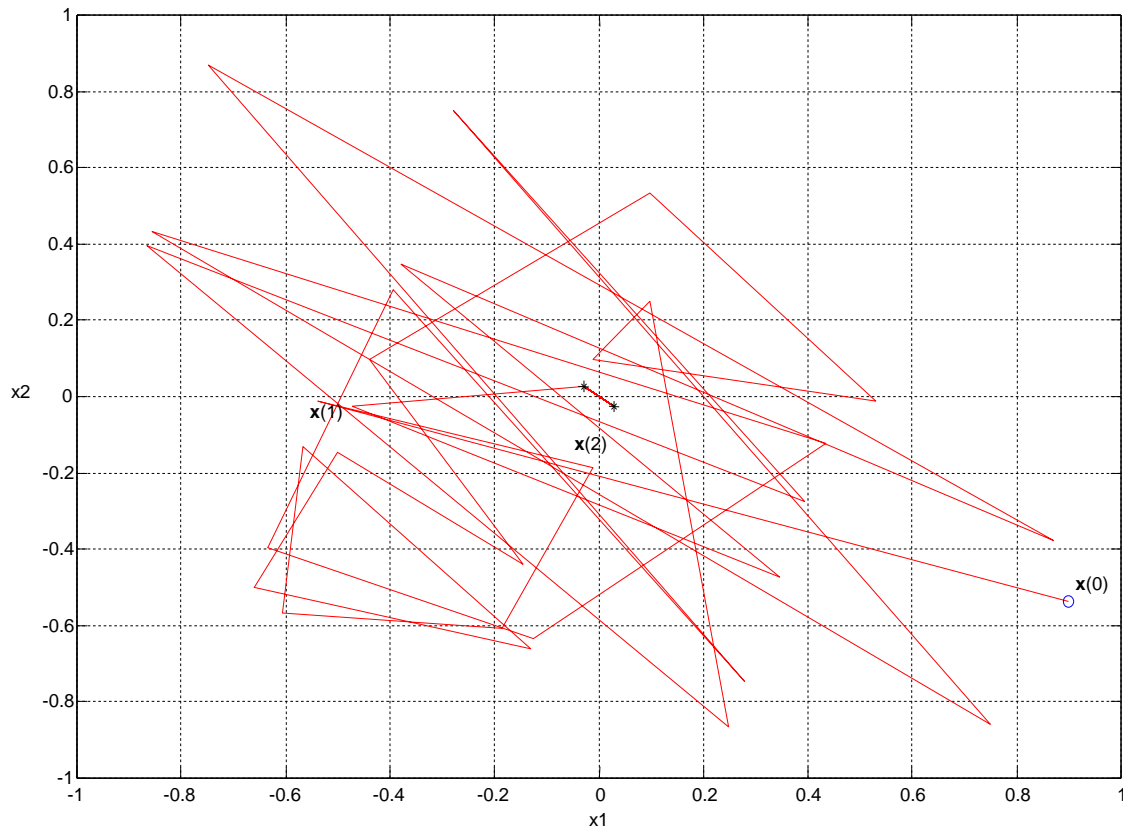


Figure 1. The phase portrait of the second order digital filter with two's complement arithmetic. The points $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(2)$ are as annotated, and the points with '*' denote the 'steady states' of \mathbf{x} .

We have discussed the case when $s_k = 0$ for $k \geq k_0$ in theorem 1. What happens when $s_k \neq 0$ for $k \geq k_0$? We will present an interesting result in the following theorem:

Theorem 2

For $b = a + 1$ and a being an odd integer, $\exists k_0 \in \mathbb{Z}^+ \cup \{0\}$ such that

$$x_1(k_0) = x_2(k_0) = -1 \text{ if and only if } s_k = a \text{ and } \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} \text{ for } k \geq k_0.$$

Proof

For the *if* part, since $\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a + 1) \cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}$, if $\exists k_0 \in \mathbb{Z}^+ \cup \{0\}$

such that $x_1(k_0) = x_2(k_0) = -1$, then we have $\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} -1 \\ f(-(2 \cdot a + 1)) \end{bmatrix}$. Since a

is an odd integer, we have $\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ and $s_{k_0} = a$. As

$$\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \text{ we have } s_k = a \text{ and } x_1(k) = x_2(k) = -1 \text{ for } k \geq k_0.$$

For the *only if* part, since $\begin{bmatrix} x_1(k_0 + 1) \\ x_2(k_0 + 1) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a + 1) \cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}$, if

$x_1(k) = x_1(k_0)$ and $x_2(k) = x_2(k_0)$ for $k \geq k_0$, then we have

$$\begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} = \begin{bmatrix} x_2(k_0) \\ f((a + 1) \cdot x_1(k_0) + a \cdot x_2(k_0)) \end{bmatrix}, \text{ which implies } x_1(k_0) = x_2(k_0) \text{ and}$$

$x_2(k_0) = f((a + 1) \cdot x_1(k_0) + a \cdot x_2(k_0))$. Since $s_k = a$ for $k \geq k_0$, we have

$x_1(k_0) = (a + 1) \cdot x_1(k_0) + a \cdot x_1(k_0) + 2 \cdot a$, which implies $2 \cdot a \cdot (x_1(k_0) + 1) = 0$. As a is an

odd integer, so $a \neq 0$. As a result, we have $x_1(k_0) = -1$ and we prove the *only if* part. ■

Remark 3

Theorem 2 states the necessary and sufficient condition for the state vector to stay at a fixed point after a number of iterations when the parameter a is an odd integer. It is interesting to note that this fixed point is $(-1,-1)$.

Example 2

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ a+1 & a \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$,

$a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.1875 \\ -0.25 \end{bmatrix}$. Figure 2 shows the phase portrait of the system. It can

be seen from the figure that the state vector converges to a fixed point

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

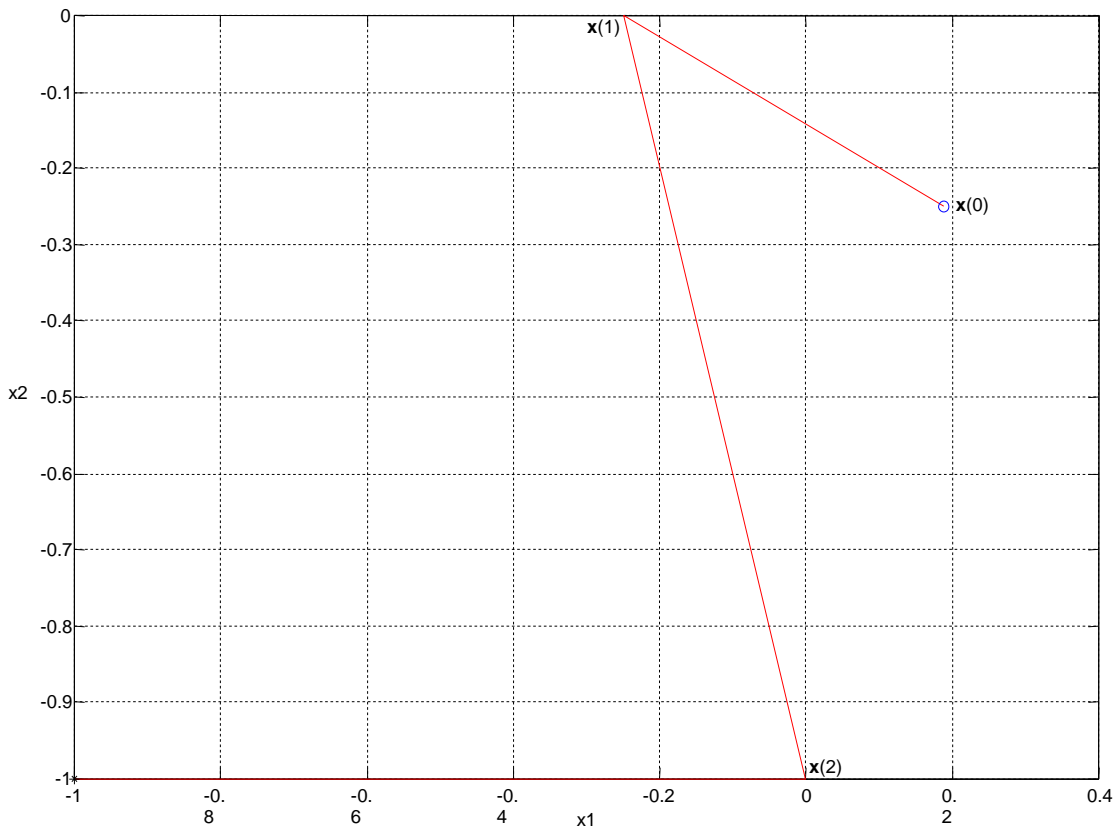


Figure 2. The phase portrait of the second order digital filter with two's complement arithmetic. The points $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(2)$ are as annotated, and the point with '*' denotes the 'steady state' of \mathbf{x} .

Remark 4

The above results are obtained when $b = a + 1$. What happens when $b = -a + 1$? We can show that results similar to lemma 1 and theorem 1 can be obtained as lemma 2

and theorem 3, while the results of theorem 2 can be modified to that of theorem 4.

Remark 5

Since one of the eigenvalues is unstable, one may predict that the periodic orbits are unstable. However, a counter-intuitive result is found that the periodic orbits are stable if a is an odd integer. The result is stated in observation 1 below:

Observation 1

When $b = a + 1$ and a is an odd integer, the state vector toggles between two states at the steady state on a particular straight line $x_1 = -x_2$ of the phase portrait or

converges to a fixed point $\mathbf{x}^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ for all the initial conditions in I^2 .

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2 , it can be shown in figure 3a that the state converges to a period-2 signal.

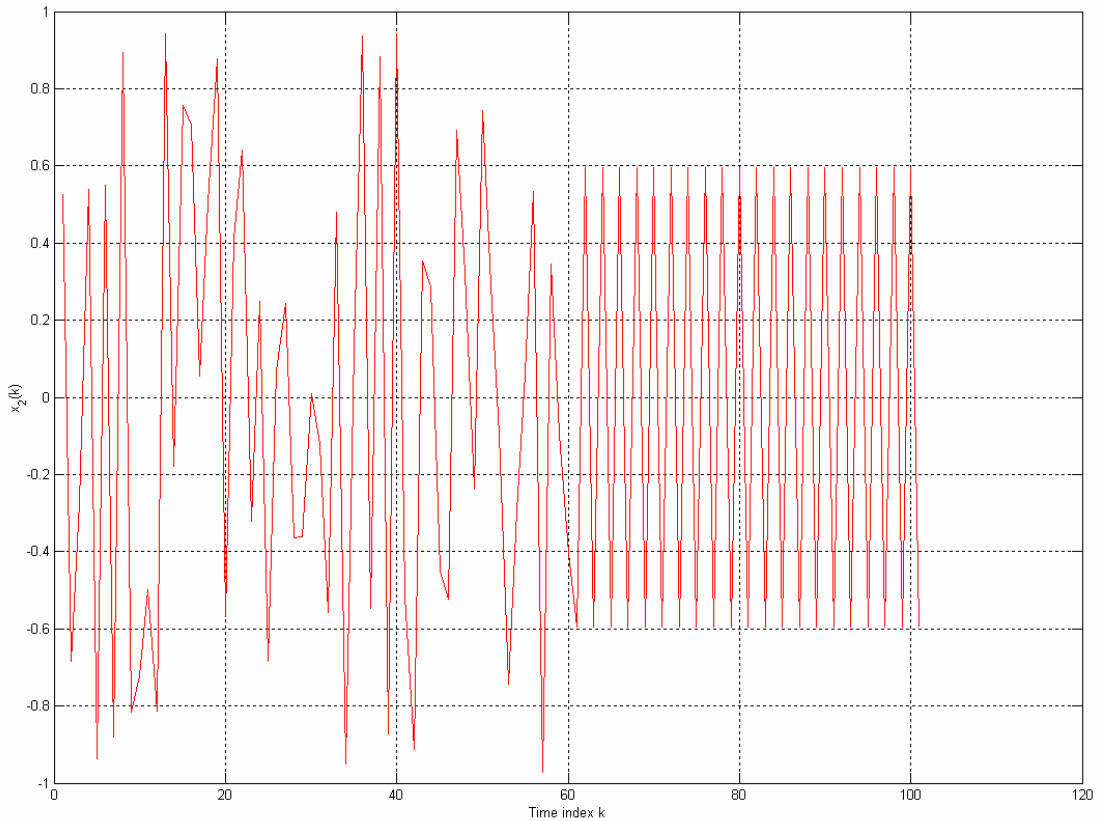


Figure 3a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition $\mathbf{x}(0)=[0.7826 \ 0.5242]^T$ is generated randomly. When $a=5$ and $b=6$, the state converges to a period-2 signal.

However, when a deviates from an odd integer a little bit, the state neither converges to a periodic signal nor a fixed point, as shown in figure 3b.

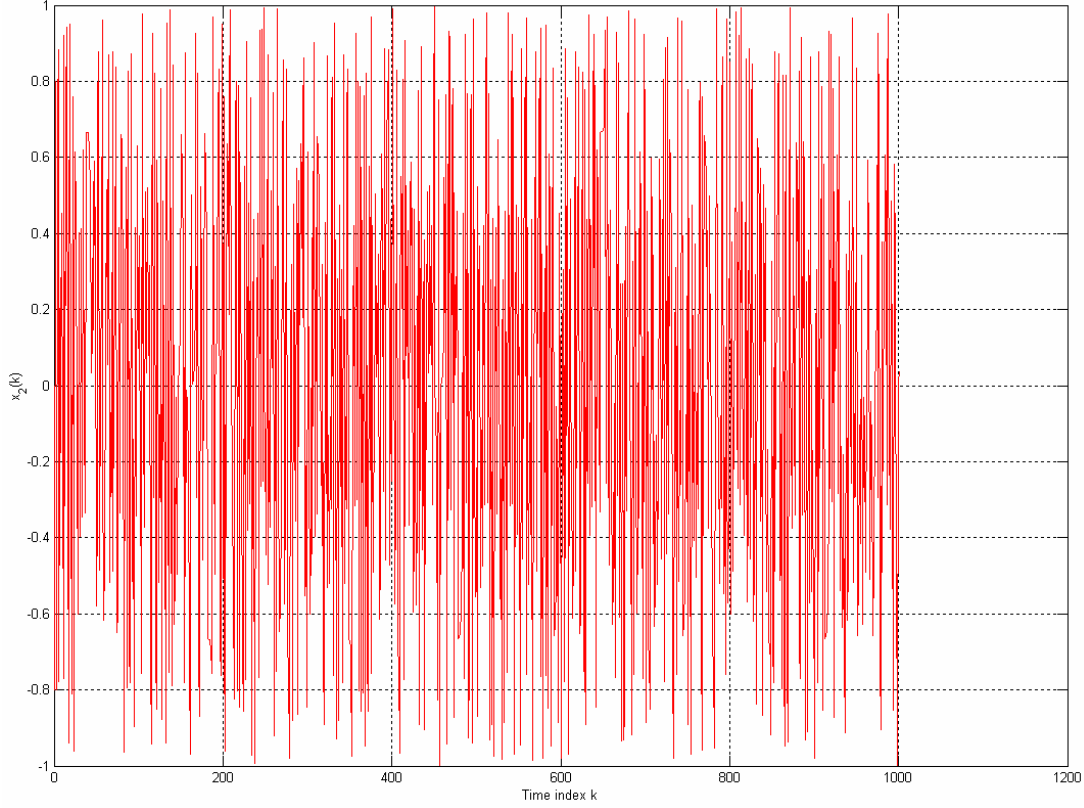


Figure 3b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.8 \ -0.7999]^T$, $a=3.001$ and $b=4.001$, the state neither converges to a periodic signal nor a fixed point.

Lemma 2

For $b = -a + 1$, if $\exists M \in \mathbb{Z}^+$ such that $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} x_1(k+M) \\ x_2(k+M) \end{bmatrix}$ for $k \geq k_0$, then

$$(1 - (a-1)^M) \cdot \left(x_1(0) - x_2(0) - 2 \cdot \sum_{j=0}^{k_0-1} \frac{s_j}{(a-1)^{j+1}} \right) + 2 \cdot \sum_{j=k_0}^{k_0+M-1} (a+1)^{M-j-1} \cdot s_j = 0.$$

Theorem 3

For $b = -a + 1$ and $|a-1| > 1$, $s_k = 0$ for $k \geq k_0$ if and only if $\exists k_0 \in \mathbb{Z}^+ \cup \{0\}$ such that $x_1(k_0) = x_2(k_0)$.

Example 3

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a = 3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -0.1875 \\ -0.1234 \end{bmatrix}$. Figure 4 shows the phase portrait of the

system. It can be seen from the figure that the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait.

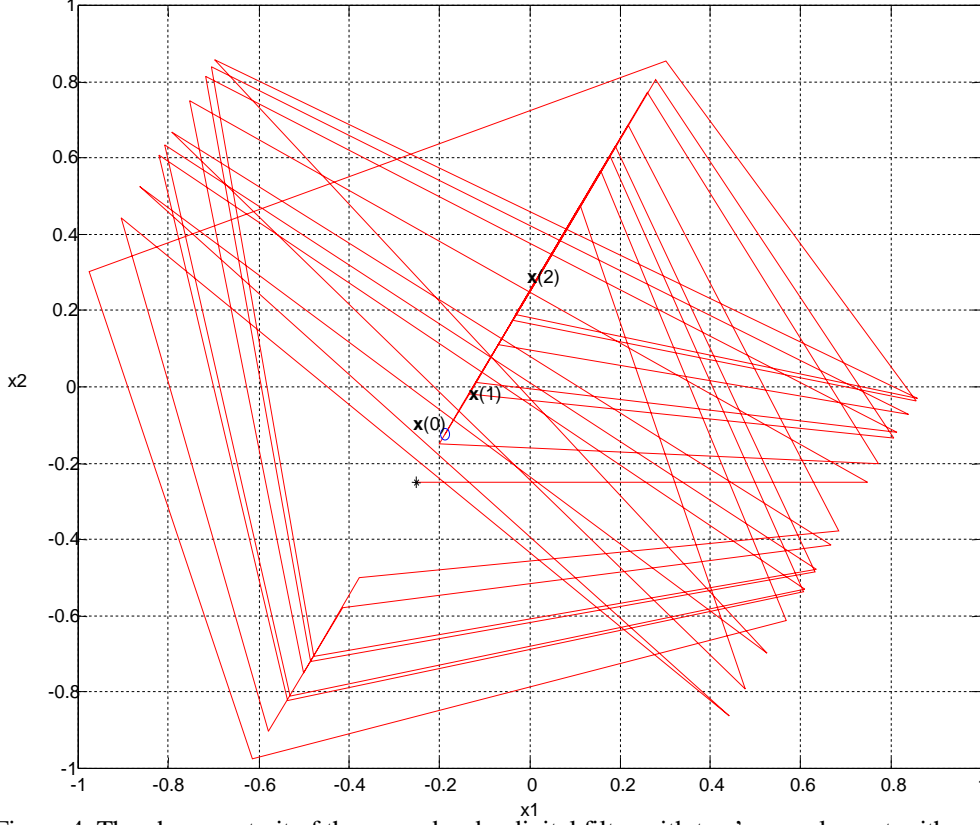


Figure 4. The phase portrait of the second order digital filter with two's complement arithmetic. The points $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(2)$ are as annotated, and the point with '*' denotes the 'steady state' of \mathbf{x} .

Theorem 4

For $b = -a + 1$ and a being an odd integer, there does not exist $k_0 \in \mathbb{Z}^+$ such that

$$s_k = a \text{ for } k \geq k_0.$$

Proof

Since $\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \mathbf{A}^{k-k_0} \cdot \begin{bmatrix} x_1(k_0) \\ x_2(k_0) \end{bmatrix} + \sum_{j=k_0}^{k-1} \mathbf{A}^{k-1-j} \cdot \mathbf{B} \cdot s_j$ for $k > k_0$, if $s_k = a$ for $k \geq k_0$,

we have:

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \frac{1}{a-2} \cdot \begin{bmatrix} (a-1)^{k-k_0} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a}{2-a} - 2 \cdot a \cdot (k-k_0) \\ (a-1)^{k-k_0+1} \cdot \left(x_2(k_0) - x_1(k_0) - \frac{2 \cdot a}{2-a} \right) + (a-1) \cdot x_1(k_0) - x_2(k_0) + \frac{2 \cdot a \cdot (a-1)}{2-a} - 2 \cdot a \cdot (k-k_0) \end{bmatrix} \quad (19)$$

As $k \rightarrow +\infty$, $k - k_0 \rightarrow +\infty$, so $\lim_{k \rightarrow +\infty} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \notin I^2$. Hence, there does not exist

$k_0 \in \mathbb{Z}^+$ such that $s_k = a$ for $k \geq k_0$, and this proves the theorem. ■

Example 4

Consider the system $\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \mathbf{B} \cdot s_k$, where $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a+1 & a \end{bmatrix}$,

$\mathbf{B} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $a=3$ and $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.7826 \\ 0.5242 \end{bmatrix}$. Figure 5 shows the phase portrait of the

system.

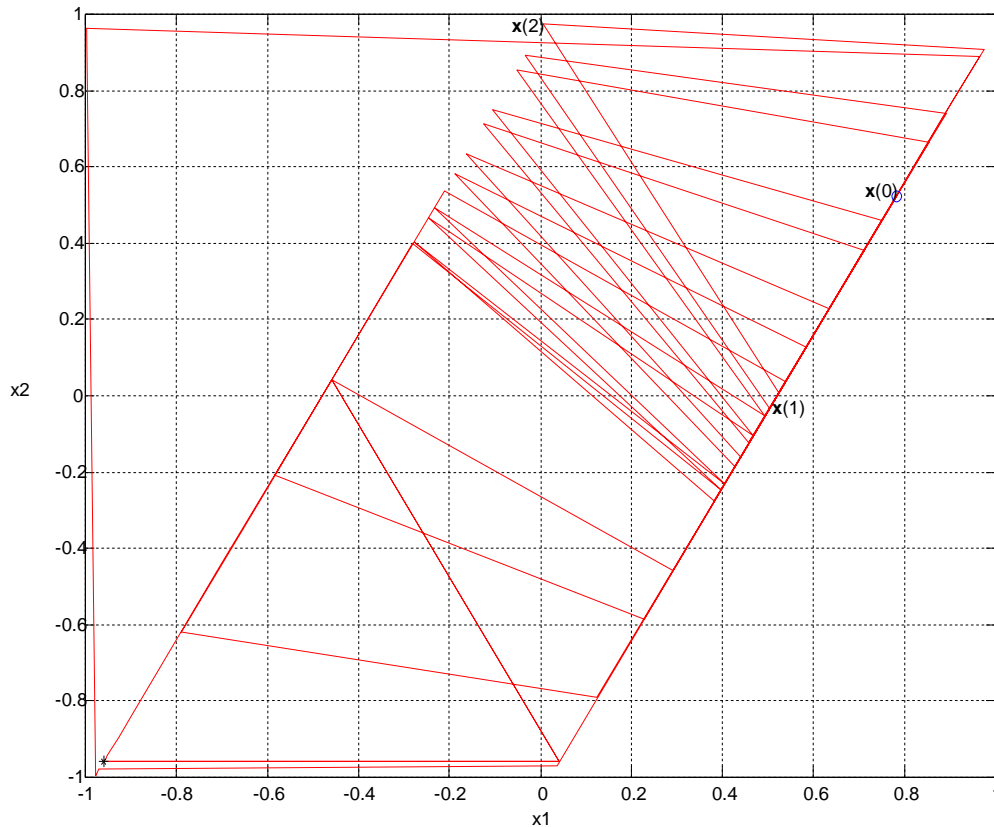


Figure 5. The phase portrait of the second order digital filter with two's complement arithmetic. The points $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(2)$ are as annotated, and the point with '*' denotes the 'steady state' of \mathbf{x} .

Observation 2

When $b = -a+1$ and a is an odd integer, the state vector converges to a fixed point on a particular straight line $x_1 = x_2$ of the phase portrait for all the initial conditions in I^2 .

To demonstrate this phenomenon, a random initial condition $\mathbf{x}(0)$ is generated in I^2 , it can be shown in figure 6a that the state converges to a fixed point.

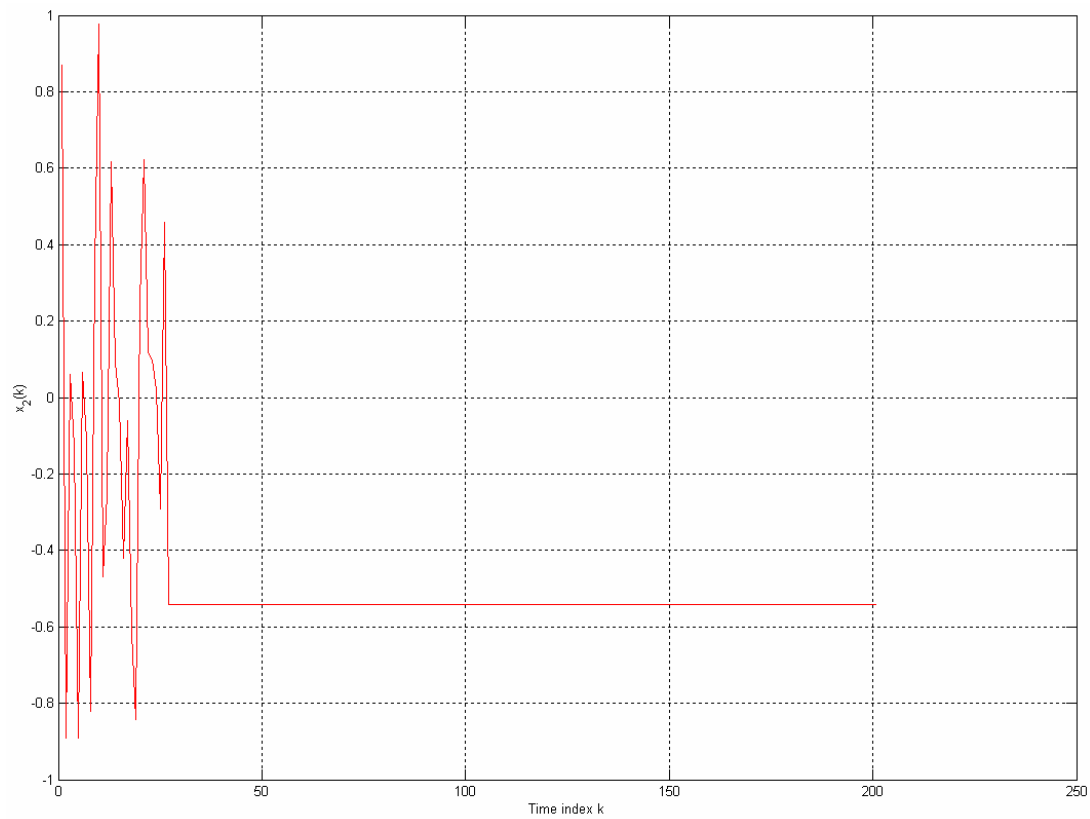


Figure 6a. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[-0.1886 \ 0.87909]^T$, $a=5$ and $b=-4$, the state converges to a fixed point.

However, when a deviates from an odd integer a little bit, the state does not converge to a fixed point, as shown in figure 6b.

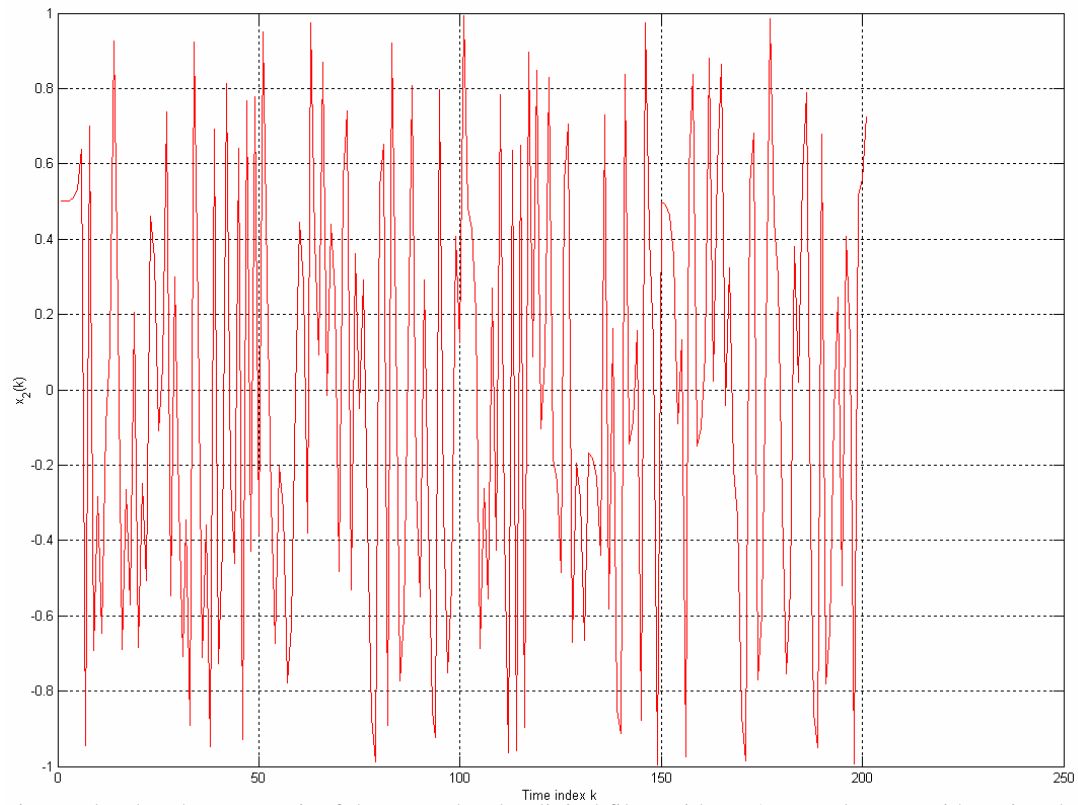


Figure 6b. The phase portrait of the second order digital filter with two's complement arithmetic. The initial condition is $\mathbf{x}(0)=[0.5 \ 0.5001]^T$, $a=5.01$ and $b=-4.01$, the state does not converge to a fixed point.

4. CONCLUSIONS

In this article, some interesting behaviors of second-order digital filters with two's complement arithmetic are explored. The cases of $b = a + 1$ and $b = -a + 1$ are analyzed and some necessary conditions for the symbolic sequences to be periodic after a number of iterations are given. The necessary and sufficient conditions for the system to behave as a linear system after a number of iterations and the state vector to toggle among several states or converge to a fixed point are given.

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